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ABSTRACT

This paper explores the relationship between the closely linked concepts of E-stability and least-squares learnability, featured in recent work by Evans and Honkapohja (1999, 2001), and the minimum-state-variable (MSV) solution defined by McCallum (1983) and used by many researchers for rational expectations (RE) analysis. It is shown that the MSV solution, which is unique by construction, is E-stable--and therefore LS learnable when nonexplosive--in all linear RE models that satisfy conditions for being "well formulated." The latter property involves two requirements. The first is that values of the model's parameters are restricted so as to avoid any infinite discontinuity, of the steady state values of endogenous variables, in response to small changes in these parameters. (It is expressed in terms of the eigenvalues of a matrix that is the sum of those attached to the one-period-ahead and one-period-lagged values of the endogenous variables in a first-order vector formulation of the model.) The second, which is needed infrequently, is that the parameters are restricted to prevent any infinite discontinuities in the MSV response coefficients.

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1. Introduction

Much recent research in economics, especially in monetary economics, has focused on issues involving analytical indeterminacy—multiplicity of stable rational expectations solutions—often in dynamic general equilibrium models based on optimizing behavior by individual agents.[1] In this context, the recent appearance of major publications by Evans and Honkapohja (1999, 2001) has stimulated new interest in the relationship between the linked concepts of E-stability and least-squares learnability, featured in their work,[2] and the minimum-state-variable solution concept promoted by McCallum (1983, 1999), which identifies a unique rational expectations solution in all linear models.[3] In the present paper it will be argued that the connection is much tighter, at least for linear models, than is generally believed. Indeed, it will be shown that, for all linear models (within a very wide class) that satisfy the property of being well formulated, the unique minimum-state-variable (MSV) solution is E-stable and therefore least-squares (LS) learnable.[4] There are a few existing examples of MSV solutions that are not E-stable, which have been mentioned and given some emphasis by Evans and Honkapohja (1992, 2001). But these examples pertain to models that are not plausible economically for the relevant range of parameter values; restrictions on these parameters that are necessary for the models to be well formulated have the effect of ruling out the range that permits E-instability. Thus the paper's results provide a newly developed reason for believing that the MSV solution will generally be the one that is

[1] In monetary economics such issues include indeterminacy under inflation forecast targeting (Woodford, 1994; Bernanke and Woodford, 1997; King, 2000), deflationary traps (Benhabib, Schmitt-Grohe, and Uribe, 2001), the fiscal theory of the price level (Sims, 1994; Woodford, 1995; Cochrane, 1998; Kotcherlakota and Phelan, 1999; McCallum, 2001), and the validity of the "Taylor Principle" (Woodford, 2002). For a useful overview of several related points, see Bullard and Mitra (2002).
[2] Evans and Honkapohja (1999) is an extensive survey article in the Taylor-Woodford Handbook of Macro-economics, whereas their (2001) is an ambitious treatise published by Princeton University Press.
[3] The point that the MSV solution is unique by construction is reviewed in Section 2.

economically relevant. In this sense, the paper amounts to a strong, albeit delayed, response to Lucas's (1986, p. S419) observation that "it is unclear what the behavioral rationale for this principle [i.e., general adoption of the MSV solution] is." In particular, the paper shows that the learnability rationale suggested in Lucas's paper provides considerable support for the MSV solution concept.[5]

In exploring the featured relationship, it is necessary to be unambiguous about the concept of a MSV solution. Throughout that term will be used to designate the unique solution—unique by construction—described in McCallum (1983, 1999). This is the way that the term was used by Evans (1986, 1989) and Evans and Honkapohja (1992) but differs from the terminology in the latter's more recent publications (1999, p. 496; 2001, p. 194), which permits multiple solutions to be given the MSV adjective. Either terminology could be used, of course, but the one adopted here is more appropriate for the issue at hand. In particular, it is more convenient (as well as more aesthetically pleasing) to refer to the pertinent solution as "the MSV solution" rather than as "the MSV solution that is singled out by the procedure defined by McCallum (1983, 1999)."

The outline of the paper is as follows. In Section 2, the concepts of E-stability, LS learnability, and MSV solutions are briefly reviewed. Then in Section 3 the main existing example of a linear model in which the MSV solution is not E-stable is examined, and it is argued that the range of parameter values necessary for this result renders the model implausible economically. That argument is rather ad hoc in nature, however, so Section 4 proposes some general requirements for a model to be regarded as well formulated.

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[4] The specific requirement for being well formulated is explained below. E-stability implies LS learnability in most cases, but the implication is not entirely general, especially when the solution is explosive.
[5] The paper does not present any claims, however, concerning the presence or absence of E-stability for non-MSV solutions.

Basically, these requirements rule out parameter configurations that permit infinite discontinuities in the model's properties. The first two applications of these requirements are developed in Section 5. There it is shown that, for two leading univariate models featured by Evans and Honkapohja (1999, 2001), the proposed requirements for being well formulated suffice to guarantee that the MSV solution is E-stable. Then Section 6 turns to multivariate models and develops an extension of the argument that applies quite generally to linear RE models. Finally, Section 7 provides a brief summary and conclusion.

2. Concepts and Issues

It will be useful to begin with a short review of E-stability, LS learnability, and MSV solutions. Evans (1985, 1986), building upon a result of DeCanio (1979), developed iterative E-stability as a selection criterion for RE models with multiple solutions.[6] The basic presumption is that individual agents will not be endowed with perfect knowledge of the economic system's structure, so it must be considered whether plausible correction mechanisms are convergent. Consider the example model:

(1) $y_t = \alpha + aE_ty_{t+1} + cy_{t-1} + u_t,$

where $u_t = \rho u_{t-1} + \varepsilon_t$ with $|\rho| < 1$ and ε_t being white noise.[7] With this specification, the usual "fundamentals" RE solution will be of the form

(2) $y_t = \phi_0 + \phi_1 y_{t-1} + \phi_2 u_t,$

but suppose that agents do not "initially" know the exact values of the ϕ_j parameters. If at any date t the agents' prevailing belief is that their values are $\phi_0(n)$, $\phi_1(n)$, and $\phi_2(n)$—where n indexes iterations—so that the perceived law of motion (PLM) is

[6] The emphasis is usually on cases with multiple solutions that are dynamically stable (i.e., non-explosive) since explicit or implicit transversality conditions will often rule out explosive paths as solutions.
[7] Evans and Honkapohja (1999, 2001) use the symbols β and δ instead of a and c.

(3) $y_t = \phi_0(n) + \phi_1(n)y_{t-1} + \phi_2(n)u_t$,

then the implied unbiased expectation of y_{t+1} will be

(4) $\phi_0(n) + \phi_1(n)y_t + \phi_2(n)\rho u_t$.

Using this last expression in place of $E_t y_{t+1}$ in (1)—which implies that we have temporarily abandoned RE—gives

(5) $y_t = \alpha + a[\phi_0(n) + \phi_1(n)y_t + \phi_2(n)\rho u_t] + cy_{t-1} + u_t$

or, rearranging,

(6) $y_t = [1-a\phi_1(n)]^{-1} [\alpha + a\phi_0(n) + a\phi_2(n)\rho u_t + cy_{t-1}] + u_t$

as the system's actual law of motion (ALM). Now imagine a sequence of iterations from the PLM to the ALM. Writing the left-hand side of (6) in the form (3) for iteration n+1 then implies that

(7a) $\phi_0(n+1) = [1 - a\phi_1(n)]^{-1}[\alpha + a\phi_0(n)]$

(7b) $\phi_1(n+1) = [1 - a\phi_1(n)]^{-1}c$

(7c) $\phi_2(n+1) = [1 - a\phi_1(n)]^{-1}[a\phi_2(n)\rho + 1]$.

The issue, then, is whether iterations defined by (7) are such that the $\phi_j(n)$ converge to the ϕ_j values in an expression of form (2) as $n \rightarrow \infty$. If they do, then that solution (2) is said to be iteratively E-stable. Evans (1986) found that in several prominent and controversial examples the MSV solution—to be discussed shortly—is iteratively E-stable.

On the basis of results by Marcet and Sargent (1989), Evans (1989) and Evans and Honkapohja (1992) switched attention to E-stability without the "iterative" qualification, defined as follows. Conversion of equations (7) to a continuous form, appropriate as the iteration interval approaches zero,[8] results in

(8a) $d\phi_0(n)/dn = [1 - a\phi_1(n)]^{-1}[\alpha + a\,\phi_0(n)] - \phi_0(n)$

(8b) $d\phi_1(n)/dn = [1 - a\phi_1(n)]^{-1}c - \phi_1(n)$

(8c) $d\phi_2(n)/dn = [1 - a\phi_1(n)]^{-1}[a\,\phi_2(n)\rho + 1] - \phi_2(n)$.

If the differential equation system (8) is such that $\phi_j(n) \rightarrow \phi_j$ for all j, the solution (2) is E-stable.[9] An important feature of this continuous version of the iterative process is that it is intimately related to an adaptive learning process that is modeled as taking place in real time.[10] [11] For most models of interest, that is, values of parameters analogous to the ϕ_j in (2) that are estimated by LS regressions on the basis of data from periods t−1, t−2, …, 1 and used to form expectations in period t, will converge to the actual values in (2) as time passes if equations (8) converge to those values and (2) is dynamically stable (non-explosive). Also, such convergence will not occur if equations (8) do not converge. Thus E-stability and LS learnability typically go hand in hand. This result, which is discussed extensively by Evans and Honkapohja (1999, 2001), is useful because it is technically much easier, in many cases, to establish E-stability than to establish LS learnability. The latter concept is arguably the more important, in a fundamental sense, as learnability of some type might be regarded as a necessary condition for the relevance of a RE equilibrium.[12]

To this point the discussion of equation (1) has presumed that expectations are not

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[8] There is also a positive speed-of-adjustment coefficient in each of equations (8), but its magnitude is irrelevant for the convergence issue so is usually (as here) set equal to 1. See, e.g., Evans (1989, p. 299).

[9] Throughout the paper we shall be discussing weak E-stability, rather than the more demanding concept of strong E-stability. For the distinction, see footnote 27 below and E&H (2001, pp. 41-42).

[10] The E-stability process is itself conceived of as taking place in notional time (meta time).

[11] An influential early analysis regarding learning of RE solutions was provided by Bray (1982).

[12] In this regard, note that the LS learning process assumes that (i) agents are collecting an ever-increasing number of observations on all relevant variables while (ii) the structure is remaining unchanged. Furthermore, (iii) the agents are estimating the relevant unknown parameters with an appropriate estimator in (iv) a properly specified model. Thus if a proposed RE solution is not learnable by the process in question, it would seem implausible that it could prevail in practice.

formed rationally, i.e., that $E_t y_{t+1}$ in (1) has been replaced with some approximation such as (4). Now we return to the RE assumption and consider the nature of the MSV solution to model (1). One requirement of the MSV solution is that it be real and linear. Another is that it not include any extraneous state variables, so for model (1) it will be of form (2). That form implies that $E_t y_{t+1} = \phi_0 + \phi_1(\phi_0 + \phi_1 y_{t-1} + \phi_2 u_t) + \phi_2 \rho u_t$ so substitution of the latter and (2) into (1) implies that the undetermined coefficients, the ϕ_js, must be real and satisfy the following equations:

(9a) $\phi_0 = \alpha + a\phi_0 + a\phi_1\phi_0$

(9b) $\phi_1 = a\,\phi_1^2 + c$

(9c) $\phi_2 = a\phi_1\phi_2 + a\rho\phi_2 + 1.$

Clearly, the second of these yields two potential values for ϕ_1, namely, $[1 \pm \sqrt{1-4ac}\,]/2a$. These expressions should be considered as two different functions of a and c, which therefore define two different RE solutions. But the MSV solution is unambiguously provided by use of the ϕ_1 function with the minus sign, for that is the one that implies $\phi_1 = 0$ in the special case in which $c = 0$. (In this case y_{t-1} does not appear in the model so ϕ_1 must equal zero to avoid inclusion of an extraneous state variable in the solution.) Then with ϕ_1 uniquely determined, the other two ϕ_j values are given unambiguously by the remaining two equations in (9).[13] For more general models, the MSV solution is found in an analogous manner: different solutions correspond to different groupings of eigenvalues and eigenvectors relating to matrix counterparts of the quadratic equation (9b).[14] The grouping is chosen that excludes extraneous state variables from the solution in a special case, namely, one that includes no

[13] This discussion presumes that the roots to the quadratic are real. If they are complex, then by definition the MSV solution does not exist. For some parameter values this will be the case.

predetermined variables in the model's structure.

For the system of our example (1), then, the issue of principal interest is whether the MSV solution possesses E-stability, i.e., whether the differential equations (8) are locally stable at the MSV values for the ϕ_j. Necessary and sufficient conditions for E-stability of this system are given by Evans and Honkapohja (2001, p. 202) as follows: $a(1-a\phi_1)^{-1} < 1$, $ca(1-a\phi_1)^{-2} < 1$, $\rho a(1-a\phi_1)^{-1} < 1$. These will be utilized below.

3. Questionable Example[15]

In various places, Evans and Honkapohja (henceforth, E&H) have indicated that MSV solutions may or may not have the property of E-stability (and LS learnability). Here the agenda is to describe and reconsider the main example put forth by E&H (1992, pp. 9-10; 1999, pp. 496-7; 2001, p. 197) as representing a case in which the MSV solution is not E-stable. The relevant model's reduced form can be written as

(10) $$y_t = \alpha + \gamma E_{t-1}y_t + \zeta E_{t-1}y_{t+1} + \delta y_{t-1} + \varepsilon_t,$$

with $\delta \neq 0$, $\zeta \neq 0$, and ε_t white noise. The MSV solution will be of the form

(11) $$y_t = \phi_0 + \phi_1 y_{t-1} + \phi_2 \varepsilon_t,$$

and ϕ_1 will be determined by a quadratic equation with the MSV solution given by the ϕ_1 root that equals zero when $\delta = 0$. The other root gives a non-MSV "bubble" solution and there are also bubble solutions of a form that includes additional terms involving y_{t-2} and ε_{t-1} on the right-hand side of (11).

Necessary conditions for E-stability of a solution of form (11) are (E&H, 1992, p. 6)

(12) $$\gamma + \zeta - 1 + \zeta\phi_1 < 0 \quad \text{and} \quad \gamma - 1 + 2\zeta\phi_1 < 0.$$

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[14] See McCallum (1983, pp. 165-6; 1999, eqns. (21)-(26)).
[15] Sections 3 and 4 are based on McCallum (2002).

On the basis of these, E&H (1992, pp. 9-10) show that the non-MSV solution of form (11) is E-stable, and the MSV solution is E-unstable, when $\gamma = -\zeta > 1$ and $\delta > 0$. Also, on p. 5 they show that the bubble solutions with additional terms are E-stable if $\gamma > 1$, $\delta\zeta > 0$, and $\zeta < 0$. If such parameter values were economically sensible, these results would constitute explicit counter-examples to my suggestion that MSV solutions are invariably E-stable.

Let us, however, reconsider the economic model that E&H (1992) use to motivate the reduced form equation (10). It is a log-linear "model of aggregate demand and supply with wealth effects in aggregate demand, money demand, and aggregate supply" (1992, p. 9). Letting y_t, m_t, and p_t be the logs of output, money, and the price level with i_t a nominal interest rate, E&H write:[16]

(13a) $y_t = -g_1(i_t - E_{t-1}(p_{t+1} - p_t)) + g_2(m_t - p_t) + v_{1t}$

(13b) $y_t = f(m_t - p_t) + v_{2t}$

(13c) $m_t - p_t = y_t - a_1 i_t + a_2(m_t - p_t) + v_{3t}$

(13d) $m_t = d\, p_{t-1} + v_{4t}$.

The fourth equation "is a monetary policy reaction function…." (1992, p. 9). Solving these four equations for the reduced form expression for p_t gives

(14) $p_t = d\, p_{t-1} + h\, E_{t-1}(p_t - p_{t+1}) + u_t$

with $h = g_1[f - g_2 + g_1(a_2 + f - 1)a_1^{-1}]^{-1}$ and u_t a linear combination of the (white noise) v_{it} terms. Consequently, the model is of form (10) with y_t in the latter representing p_t in the model and with $\gamma = h$, $\zeta = -h$, and $\delta = d$.

It follows, then, that the condition $\gamma = -\zeta > 1$ requires $h > 1$. In that regard, note first that if real-balance terms are excluded, i.e., if $g_2 = f = a_2 = 0$, then $h = -a_1$ is negative. Thus sizeable real-balance effects are needed. Second, note that a_2 should arguably be specified as negative, not positive, since the latter would imply a money demand function with income elasticity greater than 1.0, in contrast with most empirical estimates. But with $a_2 < 0$, f would have to be quite large to generate $h > 1$, if $g_2 > 0$. Thus $h > 1$ seems highly implausible in the context of an IS-LM model of the type utilized.

In addition, the condition $\delta > 0$ implies $d > 0$ in (14d), implying that the money supply is increased by the monetary authority when the price level is higher than average in the previous period. That represents, at least arguably, a somewhat perverse form of policy behavior.

An alternative way of interpreting the reduced-form equation (10), not mentioned by E&H, is as a microeconomic supply-demand model. Suppose we have demand and supply functions

(15a) $q_t = \beta_0 + \beta_1 p_t + \beta_2 E_{t-1}(p_{t+1} - p_t) + v_{1t}$

(15b) $q_t = \alpha_0 + \alpha_1 p_t + \alpha_2 E_{t-1} p_t + v_{2t}$

where the disturbance terms include effects of exogenous variables such as demanders' income and the price of inputs to production. Here we presume that $\beta_1 < 0$ and $\beta_2 > 0$, to reflect downward sloping demand with respect to the current price and a speculative demand motive. Also, let $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$ to reflect upward sloping supply with respective to relevant prices. Then the reduced form is

(16) $p_t = (\alpha_1 - \beta_1)^{-1} [(\beta_0 - \alpha_0) + \beta_2 E_{t-1} p_{t+1} - (\alpha_2 + \beta_2) E_{t-1} p_t + v_{1t} - v_{2t}]$.

In terms of equation (10), this specification suggests $\zeta > 0$, $\gamma < 0$, and $\delta = 0$. But the first two of these are just opposite in sign to the requirements for the E&H example. Furthermore, it is

[16] It is my impression that E&H intend for all parameters in (13) to be interpreted as non-negative.

conceivable that p_{t-1} would appear instead of $E_{t-1}p_t$ in the supply equation. But then its coefficient in the reduced form would be negative, and therefore inconsistent with the $\delta > 0$ assumption in the E&H case under discussion.

In sum, I would argue that the specification used most prominently by E&H, as an example featuring the absence of E-stability for the MSV solution, is unappealing in terms of basic economic theory. It will be recognized, however, that this argument of mine is quite specific and rather ad hoc in nature. Some more general position is needed, so one will be developed in the next section.

4. Requirements for Well Formulated Models

In this section I propose conditions necessary for important classes of linear models to be well formulated. Consider first the single-variable specification (1), which is reproduced here for convenience:

(17) $y_t = \alpha + aE_ty_{t+1} + cy_{t-1} + u_t,$

where $u_t = \rho u_{t-1} + \varepsilon_t$ with ε_t white noise. Thus u_t is an exogenous forcing variable with an unconditional mean of zero. Applying the unconditional expectation operator to (17) yields

(18) $E\, y_t = \alpha + aEy_{t+1} + cEy_{t-1} + 0.$

If y_t is covariance stationary, we then have[17]

(19) $E\, y_t = \alpha / [1-(a + c)].$

But from the latter, it is clear that as a + c approaches 1.0 from above, the unconditional mean of y_t approaches $-\infty$ (assuming without loss of generality that $\alpha > 0$), whereas if a + c approaches 1.0 from below, the unconditional mean approaches $+\infty$. Thus there is an infinite discontinuity at a + c = 1.0. This implies that a tiny change in a + c could alter the average (i.e., steady state) value Ey_t from an arbitrarily large positive number to an arbitrarily large negative number. Such a property seems highly implausible and therefore unacceptable for a well-formulated model.[18]

In light of the preceding observations, my contention is that, to be considered well formulated, the model at hand needs to include a restriction on its admissible parameter values that rules out $a + c = 1$, and yet admits a large open set of values that includes $(a, c) = (0, 0)$. In the case at hand, the appropriate restriction is $a + c < 1$. Of course, $a + c > 1$ would serve just as well mathematically to avoid the infinite discontinuity, but it is clear that $a + c < 1$ is vastly more appropriate from an economic perspective since it includes the region around $(0, 0)$. It should be clear, in addition, that the foregoing argument could be easily modified to apply to y_t processes that are trend stationary, rather than strictly (covariance) stationary.

Generalizing, suppose that y_t in (17) is a $m\times1$ vector of endogenous variables, so that α is $m\times1$ while a and c become $m\times m$ matrices A and C. Then the counterpart of $1 - (a + c) > 0$ is that the eigenvalues of $[I - (A + C)]$ are all real and positive, which implies that the eigenvalues of $[A + C]$ all have real parts less than 1.0. That requirement is necessary for the multivariate version of (17) to be well formulated.[19] Note that, with A and C being real, it implies that $\det[I - (A + C)] > 0$.[20]

There is, however, a second type of infinite discontinuity that also needs to be ruled

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[17] Note that it is not being assumed that y_t is necessarily covariance stationary. Instead an implication that would hold, if it were, is being used to motivate the assumption that will be made subsequently.

[18] If the model has been formulated in terms of percent or fractional deviations from some reference level (e.g., a deterministic steady state) so that there is no explicit constant term, the argument will still apply to the implicit constant term, which will almost always be non-zero.

[19] If some variables enter the model in a fashion that their solutions have unit roots, then the foregoing argument based on unconditional expectations is not directly applicable. In such cases, however, the model can be written entirely in terms of the differences of such variables and the argument applied to the revised system.

[20] That the condition $\det[I - (A + C)] > 0$ is not sufficient to make the model well formulated can be seen by considering a case with diagonal $(A + C)$ in which $I - (A+C)$ has two negative diagonal entries.

out for model (17) to be well formulated. This type pertains to dynamic responses of y_t to the exogenous forcing variable u_t. From (9c) we see that the response coefficient is $\phi_2 = (1 - a\phi_1 - a\rho)^{-1}$ so to rule out an infinite discontinuity the relevant condition is $1 - a\phi_1 - a\rho > 0$. Our particular concern is with the MSV solution, regarding which we note that $1 - a\phi_1 = (1 + d)/2$ is unambiguously positive. Consequently, adoption of the second WF condition $1 - a\phi_1 > a\rho$ implies that $1 > (1 - a\phi_1)^{-1}a\rho$.

To state this second condition directly in terms of a, c, and ρ, we could write it as $(1+d)/2 > a\rho$. For that to fail, it would be required that $d < 2a\rho - 1$. That would imply $1 - 4ac < 4a^2\rho^2 - 4a\rho + 1$, which could prevail only if a and ρ were both negative.[21] In that case the last inequality would imply $-c > a\rho^2 - \rho$ or $\rho > a\rho^2 + c$. This could be ruled out for all $-1 < \rho < 0$ if one were to require $-1 < a + c$, but that condition is more demanding than the second WF condition specified above, which is what we have adopted.

It might be added parenthetically that an alternative way of describing this more demanding version of the second condition is discussed in McCallum (1983, pp. 159-160) under the title of "process consistency." That term was introduced by Flood and Garber (1980), who used the concept in their analysis of monetary reform after the German hyperinflation of 1923, and was developed more fully by E&H (1992, pp. 10-11). A model fails to be process consistent when the procedure of solving out expectational variables, by iteration into the infinite future, is of dubious validity because the implied infinite series does not converge. For model (17) to be process consistent for any $|\rho| < 1$, it must be the case

[21] Clearly $(1+d)/2 > a\rho$ cannot fail unless a and ρ are of the same sign. Further, it is shown below that $(1 - a\phi_1)^{-1}a < 1$ when $a > 0$.

that at least one of the roots to (9b) exceeds 1.0 in absolute value.[22] Clearly, the larger root in absolute value is $\phi_1^{(+)} = (1+d)/2a$, so with $a < 0$, process inconsistency can occur only if $(1+d)/2a > -1$.[23] The latter implies $(1+d) < -2a$ or $d < -(2a+1)$, or $1 - 4ac < 1 + 4a + 4a^2$. Subtracting -1 from each side and dividing by $4a < 0$ yields $-c > 1 + a$ or $-1 > a + c$. Thus to rule out process inconsistency, we would require $a + c > -1$ when $a < 0$. This, of course, is the same condition as the one derived in the previous paragraph; it is a second requirement for model (17) to be well formulated for all $|\rho| < 1$ when $a < 0$, and is one of the sufficient conditions but is not necessary. For multivariate applications, when R is specified to be a stable matrix, the analogous requirement is that at least m of the roots of the system's characteristic equation—e.g., the eigenvalues of matrix M in Section 6 below—exceed 1.0 in absolute value (i.e., in modulus). Adoption of this stronger requirement would rule out cases known in the literature as "irregular" (Binder and Pesaran, 1995, p. 161).

Now let us consider a second model specification that, like (17), is emphasized by E&H. It can be written as

(20) $y_t = \alpha + \beta_0 E_{t-1}y_t + \beta_1 E_{t-1}y_{t+1} + \delta y_{t-1} + u_t$,

where, for simplicity, we take u_t to be white noise.[24] For this case, consider the conditional expectation, $E_{t-1}y_t$:

(21) $E_{t-1}y_t = (1 - \beta_0)^{-1} [\alpha + \beta_1 E_{t-1}y_{t+1} + \delta y_{t-1}]$.

Here it is clear that, for given (predetermined) values of $E_{t-1}y_{t+1}$, y_{t-1}, and u_{t-1}, the expectation $E_{t-1}y_t$ will pass through an infinite discontinuity at $\beta_0 = 1$. Consequently, for basically the

[22] For an extensive discussion of related issues, see Sargent (1987, pp. 176-207 and 305-308).

[23] For reference below, note that $\phi_1^{(+)} = f^{-1}$, where $f = (1-a\phi_1)^{-1}a$, so $|\phi_1^{(+)}| < 1$ is the same condition as $|f| > 1$.

[24] There is no particular need to go into the more general case in which u_t is autoregressive, because generality will be provided in Section 6 by a multivariate extension of model (17).

same reason as outlined above, $\beta_0 < 1$ is necessary for the model to be well formulated. In addition, $\beta_0 + \beta_1 + \delta < 1$ also applies. The multivariate extension for the case in which y_t is a vector yields the requirements that the eigenvalues of $[I - \beta_0]$ and $[I - (\beta_0 + \beta_1 + \delta)]$, where β_0, β_1, and δ are m×m matrices, all have positive real parts.[25]

An application of these criteria to the questionable example of E&H (1992), featured above in Section 3, is immediate. That example's result, of a MSV solution that is not E-stable, requires $\gamma = h > 1$. But, in the notation of (20), that condition implies $\beta_0 > 1$, which is incompatible with our requirement for models of form (20) to be well formulated. Thus the questionable example is discredited on general grounds, in addition to the specific reasons developed in Section 3.[26]

5. Results for Two Leading Univariate Cases

The first result in support of the contention that the MSV solution is E-stable in all well formulated (henceforth, WF) models will be briefly developed for the univariate version of specification (20) with $\delta = 0$. For this model, conditions for E-stability can be found by reference to Figure 1, which is adapted from E&H (1999, p. 492; 2001, p. 191). In these two references, it is derived and reported that the MSV solution is E-stable in regions I, V, and VI but E-unstable in regions II, III, and IV. In regions I and VI, moreover, the MSV solution is reported to be strongly E-stable whereas in V it is weakly E-stable.[27] Reference to our conditions for model (20) to be well formulated (with $\delta = 0$) shows immediately that these conditions obtain only for regions I and VI. Thus in this particular but prominent case, the

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[25] In fact, this type of condition is also applicable to model (17) when $\rho \neq 0$. In that case, transforming the disturbance into white noise by multiplying through by $(1-\rho L)$, where L is the lag operator, brings in a term involving $E_{t-1}y_t$ and leads to the condition $1+a\rho > 0$. That, however, is apparently not a useful restriction.

[26] A second example of a MSV solution that is E-unstable is presented in E&H (1994, pp. 1089-1091). In this case the model is not well formulated as it violates the requirement $[I - (\beta_0 + \beta_1 + \delta)] > 0$.

MSV solution is strongly E-stable if the parameter values are such that the model is well formulated.

Now let us focus on the more important model of equation (17). E-stability regions, as reported by E&H (2001, p. 203) under the assumption $0 \leq \rho < 1$, are shown in Figure 2. In this case, the results reported by E&H indicate that the MSV solution is E-stable in regions I and VII but E-unstable in region IV, while "both solutions [i.e., from both roots of (9b)] are explosive or nonreal" elsewhere (E&H, 2001, p. 203).[28] Specifically, solutions for ϕ_1 are complex-valued in regions III and VI, and both solutions imply explosive behavior in regions II and V. As indicated above, the MSV solution is (with $0 \leq \rho < 1$) well formulated in regions I, V, and VII (being complex in VI). Thus for regions I and VII, the E&H version of Figure 2 supports the hypothesis that the MSV solution is E-stable in any well formulated model.

But what about region V? There the E-stability conditions are in fact met—see below—although in the E&H graphical summary this region is not distinguished from VI because in V the solutions are both dynamically unstable (explosive). But there seems to be no compelling reason to ignore the MSV solution simply because it is explosive; it could be accurately indicating what would happen if (e.g.) extremely unwise policy behavior were imposed on the system at some point in time.[29] For a discussion and rationalization of this position, with a closely related example, see McCallum (1999). In any case we see that this

[27] Strong E-stability occurs in cases in which local convergence to the MSV parameter values occurs even when the function considered includes additional variables (excluded from the MSV specification).

[28] Note that the MSV solution is the AR(1) solution that E&H (2001) refer to as "the $\bar{b}_{-}$ solution." Related univariate results, more general in some respects but without the stochastic forcing variable and developed in terms of adaptive learnability rather than E-stability, have been developed by Gauthier (2003) and Desgranges and Gauthier (2003). Also see Wenzelburger (2002), who suggests that some extension to nonlinear models may be possible.

[29] The same statement does not apply to region II, where the MSV solution is E-stable but explosive, because there the model is not well formulated. This region illustrates that, though sufficient, the WF condition is not necessary for E-stability.

specification, too, conforms to the proposition that MSV solutions are E-stable in all well formulated models. It must be recognized that the usual presumption that E-stability implies LS learnability does not carry over automatically in cases of dynamic instability (which E&H refer to as nonstationarity). E&H (2001, pp. 219-220) indicate, nevertheless, that learnability will prevail in the present case, provided an adjustment is made to permit the shock variance to grow along with the y_t values, when u_t is white noise.

We wish to develop results for the more general case with $|\rho| < 1$, permitting negative values, but let us proceed by first demonstrating algebraically that the E-stability conditions are satisfied by the MSV solution to model (17) when $0 \le \rho < 1$ and the WF restriction $a + c < 1$ is imposed. Then we can go on to the case with $-1 < \rho < 0$ included. The first task, then, is to show that if $1-(a + c) > 0$, then $f = (1-a\phi_1)^{-1}a < 1$ where $\phi_1 = (1-d)/2a$ with $d = \sqrt{1-4ac}$. Note first that $1-a\phi_1 = (1 + d)/2$ so $(1-a\phi_1)^{-1}a = 2a/(1+d)$. For a proof by contradiction, suppose that $2a/(1+d) \ge 1$. Then $a > 0$ and $2a-1 \ge d$. Since both of its sides are positive, the latter implies $4a^2 - 4a + 1 \ge d^2 = 1 - 4ac$. But with $a > 0$ the last inequality reduces to $a - 1 \ge -c$ or $0 \ge 1-(a + c)$, which is the contradiction that proves $f = (1-a\phi_1)^{-1}a < 1$. The latter is the first of the three E-stability conditions listed at the end of Section 2 above. The second results from writing $(1-a\phi_1)^{-2}a\,c = (1-a\phi_1)^{-1}a\phi_1$, which follows because $(1-a\phi_1)^{-1}c = \phi_1$.[30] Since $(1-a\phi_1)^{-1}a\phi_1 = (1-d)/(1+d)$, which is smaller than 1 for all $d > 0$, we have the desired inequality. Finally, with $(1-a\phi_1)^{-1}a < 1$ and ρ non-negative, the third condition also holds.

But if ρ can be negative, which is entirely plausible, it is possible that a fairly large negative ρ together with $(1-a\phi)^{-1}a < -1$ could lead to failure of the third condition. This possibility can be eliminated, however, by adding the second WF requirement that $1 - a\phi_1 - a\rho > 0$. As is shown in Section 4, $1 - a\phi_1 = (1 + d)/2$ for the MSV solution so $1 - a\phi_1 > 0$ and the second WF requirement becomes identical to the third condition for E-stability. Adoption of both WF requirements therefore assures satisfaction of all three of the E&H conditions.

Parenthetically we note that if one were to require process consistency, then he would be requiring that $|f| < 1$ so $|\phi_1^{(+)}| > 1$. Since E&H (2001, p. 203) report that both characteristic roots have absolute values less than 1.0 in region VII, that entire region would be eliminated.

In sum, we have shown that in model (17) with $|\rho| < 1$, the MSV solution is E-stable for all parameter values satisfying our WF conditions. It is this univariate model that will provide a background for an extension of the results to more general multivariate cases.

6. Extension to Multivariate Models

To begin the extension to multivariate models, it will be convenient to consider the specification treated by E&H (2001, p. 236-238) and McCallum (1983, pp. 164-166). With y_t denoting a m×1 vector of endogenous variables, the system is

(22) $y_t = A\, E_t y_{t+1} + C\, y_{t-1} + u_t,$

where $u_t = R\, u_{t-1} + \varepsilon_t$, with R a stable m×m matrix and ε_t a white noise vector.[31] Also, we assume initially that A is nonsingular. That is a strong assumption, which renders the formulation (22) highly inconvenient from a practical perspective, but the implied case provides a useful precursor for the more general analysis that will follow.

[30] The last expression is just a rearrangement of (9b).

[31] A stable matrix has all its eigenvalues less than 1 in modulus. In (22), constant terms have been suppressed for notational simplicity and A and C are of dimension m×m.

In this setting, with or without nonsingular A, the MSV solution will be of the form

(23) $y_t = \Omega\, y_{t-1} + \Gamma\, u_t$,

and the conditions for E-stability (see E&H 2001, p. 238) are that the eigenvalues of the following three matrices all have real parts less than 1.0:

(24a) $(I - A\Omega)^{-1}A$

(24b) $[(I - A\Omega)^{-1}C]' \otimes [(I - A\Omega)^{-1}A]$

(24c) $R' \otimes [(I - A\Omega)^{-1}A]$.

Our first objective is to show that these conditions are satisfied when Ω is obtained by the MSV procedure, if the eigenvalues of A + C all have real parts less than 1.0 and our second WF requirement holds. Because the eigenvalues of the Kronecker product of two matrices are the products of the eigenvalues of these matrices (E&H, 2001, p. 116), we shall concentrate first on the m×m matrix $[(I - A\Omega)^{-1}A]$, which we now denote as F.[32]

From McCallum (1983, pp. 164-166) we have that the MSV expression for Ω is

(25) $\Omega = -P_{22}^{-1} P_{21}$,

where the P_{ij} are submatrices of P, with P^{-1} defined as the matrix of (right) eigenvectors of the 2m×2m matrix M (assumed diagonalizable) in expression (26) below, which is a first-order way of writing the matrix quadratic equation $A\Omega^2 - \Omega + C = 0$:[33]

$$(26) \quad \begin{bmatrix} \Omega \\ \Omega^2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A^{-1}C & A^{-1} \end{bmatrix} \begin{bmatrix} I \\ \Omega \end{bmatrix}.$$

Since $M = P^{-1}\Lambda P$, with Λ a diagonal matrix with the eigenvalues of M on its diagonal, we can premultiply (26) by P to obtain

[32] The invertibility of $(I - A\Omega)$ will be discussed below.
[33] This quadratic equation is implied by undetermined-coefficient implications of the model, analogous to (9b). There are, of course, many other solutions besides (25) to the quadratic for Ω.

$$(27) \quad \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \Omega \\ \Omega^2 \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} I \\ \Omega \end{bmatrix}.$$

Here Λ_1 and Λ_2 include the eigenvalues of M, with the MSV solution being selected by ordering the eigenvalues so that Λ_1 includes those that approach 0 as C approaches 0. (This ordering implies that the eigenvalues in Λ_2 approach the eigenvalues of A^{-1} as C goes to 0; see McCallum (1983, pp. 165-166).) Since $PM = \Lambda P$, we have that $-P_{22}A^{-1}C = \Lambda_2 P_{21}$ or $P_{21} = -\Lambda_2^{-1}P_{22}A^{-1}C$, so from (25) we obtain the explicit expression

$$(28) \quad \Omega = P_{22}^{-1}\Lambda_2^{-1} P_{22} A^{-1}C,$$

which illustrates that Ω approaches 0 as C approaches 0.[34] (Note that Λ_2 approaches A^{-1} so the product $P_{22}^{-1}\Lambda_2^{-1} P_{22} A^{-1}$ approaches a finite nonzero matrix.)

To show that the eigenvalues of $F = [(I - A\Omega)^{-1}A]$ and Ω meet conditions (24), I will draw on some results of Binder and Pesaran (B&P, 1995). For the moment, assume that A and C commute, i.e., AC = CA. Then A and C are diagonalized by the same matrix, and so too are both Ω and F as a consequence of their definitions. Thus the eigenvalues of F, now denoted λ_F, each satisfy $\lambda_F = (1 - \lambda_A\lambda_\Omega)^{-1}\lambda_A$, as shown by B&P (1995, pp. 157-8). But then the crucial condition $(1 - \lambda_A\lambda_\Omega)^{-1}\lambda_A < 1$ is entirely analogous to the scalar inequality $(1 - a\phi_1)^{-1}a < 1$ used in Section 5. Furthermore, condition (24b) is equivalent to having each λ_F times each λ_Ω be less than 1, and this becomes analogous to the one given in the paragraph on pages 15-16. Consequently, the same argument as developed above applies for conditions (24a) and (24b) if the eigenvalues of A + C all have real parts smaller than 1.0, provided that Ω is determined by the MSV formula (25). What about condition (24c)? Clearly, process consistency implies that all $\lambda_F > -1$, so this condition too would be satisfied if we adopted that requirement—but we shall not do so. The argument based on our weaker version of the second WF requirement will be developed shortly in a more general context.

When A and C do not commute, which will usually be the case, the foregoing reasoning does not hold and a different analytical proof becomes necessary. In this case the first part of the argument can be built around the equality

(29) $(I - A\Omega)(I - F)(I - \Omega) = I - (A + C)$,

which is mentioned by B&P (1995, fn. 34).[35] We are assuming that all eigenvalues of A + C satisfy $\text{Re}(\lambda_{A+C}) < 1$, which implies that $\det[I - (A + C)] > 0$.[36] This suggests that $\det[I - A\Omega] > 0$, $\det[I - F] > 0$, and $\det[I - \Omega] > 0$ but we have to rule out the possibility that two of these three determinants are negative and we have to consider individual eigenvalues. Accordingly, consider a modification of the model in which A and C are each multiplied by a small positive constant ε, with $0 < \varepsilon \leq 1$. Let the resulting MSV values for Ω and F be denoted $\Omega(\varepsilon)$ and $F(\varepsilon)$. Then in place of (29) we would have

(29') $[I - \varepsilon A\Omega(\varepsilon)][I - F(\varepsilon)][I - \Omega(\varepsilon)] = I - \varepsilon(A + C)$.

Clearly, for values of ε close to zero all the eigenvalues of $\Omega(\varepsilon)$ and $F(\varepsilon) = [I - \varepsilon A\Omega(\varepsilon)]^{-1}A\varepsilon$ will be smaller in modulus than 1 and will have real parts less than 1.[37] Also, the determinants of the three left-hand side matrices will all be positive. Then let ε increase continuously to 1 and notice that, with the WF requirement, the value of $\det[I - \varepsilon(A + C)]$ will remain positive throughout this process. Therefore, the value of each

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[34] If instead one uses the ordering that makes Λ_1 the m smallest (in modulus) eigenvalues, the solution would in most cases coincide with the MSV solution. In numerical application, one typically uses the ordering just mentioned, but can modify it, if necessary, as suggested briefly in McCallum (1999, p. 633).

[35] It can be derived by writing out F in (29), rearranging, and inserting C for $\Omega - A\Omega^2$.

[36] In what follows, I will repeatedly use the fact that the eigenvalues of a matrix (I – B) satisfy $\lambda_{I-B} = 1 - \lambda_B$.

left-hand-side determinant must remain positive since these determinants are continuous functions of ε.[38] (Even if two of the determinants were to turn positive at the same point, it would be implied that $\det[I - (A + C)] = 0$, which has been ruled out.) Therefore, at our solution we have $\det[I - A\Omega] > 0$, $\det[I - F] > 0$, and $\det[I - \Omega] > 0$. But how do we know that all eigenvalues of $A\Omega$, F, and Ω will have real parts less than 1? We begin the process with each having real part less than 1. But, clearly, if any real eigenvalue passed through the value 1, it would turn the relevant determinant negative, which has been ruled out. How about complex eigenvalues? They appear, since A and C are real, as conjugate pairs. The product of any conjugate pair is real and strictly larger than the real part. Thus any such product would pass through 1 and turn the relevant determinant negative before the real part were to equal 1, as ε increases. But having a negative value for any of the three determinants has been ruled out, so the real part of no eigenvalue can exceed 1 during the process or at its conclusion. This shows that the MSV solution satisfies condition (24a).

Since it is nevertheless possible that there is some eigenvalue of F with real part less than -1, we need to consider whether our WF requirements guarantee that (24b) and (24c) are satisfied. For this purpose it is useful to consider the identity

(30) $$(I - A\Omega)(I + F)(I + \Omega) = I + (A + C),$$

which is derived in a manner similar to (29). If we consider the argument of the previous paragraph, it will be seen that having some $\mathrm{Re}(\lambda_F) < -1$ is possible, since we have not ruled out the possibility that some $\mathrm{Re}(\lambda_{A+C}) < -1$. Or, an explosive solution with some $\mathrm{Re}(\lambda_\Omega) < -1$ is possible. But the latter case is one in which fewer than m of the eigenvalues of M have

[37] This statement would not be valid for non-MSV values of Ω.

modulus less than 1 whereas the former case has more than m of the eigenvalues of M with modulus less than 1. (This statement regarding F follows from the discussion on pp. 150-151 of B&P (1995), which indicates that for F to have an eigenvalue with modulus greater than 1, there must be more than m of the eigenvalues of M with modulus less than 1. For the case with invertible A matrix, an explicit proof is given in Appendix A.) Thus it is not possible for (24b) to be violated by having eigenvalues with real parts less than −1 for both F and Ω.

Finally, we need to consider condition (24c). But as in the univariate case, the second WF condition—which rules out infinite discontinuities in the response coefficients attached to the exogenous variables in the vector u_t—coincides with this third condition, so the latter does not pose a problem for E-stability.[39] Alternatively, if we were to rule out process inconsistency the current multivariate model would require that all the eigenvalues of F have modulus smaller than 1. Therefore condition (24c) would be satisfied. We do not adopt this stronger assumption but note that, even if we did, we would not be excluding the possibility that Ω could possess an eigenvalue smaller than −1. Thus again dynamic instability is not precluded by our assumption on A and C, although it would be if we assumed that all of the eigenvalues of (A + C) have modulus smaller than 1.

It remains to extend the analysis to a more general class of models. First note that the foregoing argument does not rely upon the nonsingularity of the matrix A. That property is used above only in the paragraph containing equations (25)-(28), whose sole function is to illustrate an explicit formula for calculation of Ω for the MSV solution.[40] But with singular A, the relevant calculation can still be accomplished by various procedures. Among these are

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[38] Note that the present argument implies the existence of $(I - A\Omega)^{-1}$, which E&H (2001, p. 238) and other writers take for granted. B&P (1997, fn.4) observe that it does not require the invertibility of A and state that they have found no cases of a singular $I - A\Omega$ matrix in any well specified model.
[39] This proposition is established in Appendix B.

those of Anderson and Moore (1985), B&P (1995, 1997), Uhlig (1999), King and Watson (1998), and Klein (2001). Thus the foregoing argument does not require that A be nonsingular; it applies to all models of form (22), that is,

(31) $y_t = A\, E_t y_{t+1} + C\, y_{t-1} + u_t$

where $u_t = R\, u_{t-1} + \varepsilon_t$, with R a stable m×m matrix and ε_t a white noise vector.

Furthermore, it is the case that virtually any linear RE model can be written in form (31). To see this, consider the formulation of King and Watson (1998) or Klein (2001), as exposited by McCallum (1999), as follows:

(32) $$\begin{bmatrix} A_{11} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} E_t x_{t+1} \\ k_{t+1} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}\begin{bmatrix} x_t \\ k_t \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}[v_t]$$

Here v_t is an AR(1) vector of exogenous variables (including shocks) with stable AR matrix R while x_t and k_t are $m_1\times 1$ and $m_2\times 1$ vectors of non-predetermined and predetermined endogenous variables, respectively. We assume without loss of generality that B_{11} is invertible and that $G_2 = 0$.[41] Then we can define $y_t = [x_t'\ k_t'\ x_{t-1}'\ k_{t-1}']'$ and write the system in form (31) with the matrices given as follows:

(33) $$A = \begin{bmatrix} B_{11}^{-1}A_{11} & 0 & 0 & -B_{11}^{-1}B_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ B_{21} & B_{22} & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} \quad u_t = \begin{bmatrix} B_{11}^{-1}G_1 v_t \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This representation is important because it is well known that the system (32) permits any finite number of lags, expectational leads, and lags of expectational leads. Also, any higher-order AR process for the exogenous variables can be written in AR(1) form. Thus the only lack of complete generality, for a linear system with stable exogenous forcing variables,

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[40] Also M is defined in (26), but an analogous definition and argument is straightforward.

is that a pure moving-average process for the exogenous variables cannot be included—but of course one can be approximated closely by a sufficiently high order AR process. Thus we have shown that our results of this section are applicable to (32), in which form virtually any linear system can be written. In other words, the basic result described above, that E-stability is assured for the MSV solution if the eigenvalues of (A + C) have real parts less than 1 and the second WF condition obtains, is applicable to virtually any linear RE model.

7. Conclusions

A brief statement of the paper's argument can be developed by adding a few words to its title, as follows. The minimum state variable solution, which is unique by construction, is E-stable—and therefore least squares learnable[42] in real time—in all linear RE models that satisfy conditions for being "well formulated." The latter concept, introduced above and in McCallum (2002), consists of two requirements. The first requirement is that a model's structural parameters are restricted so as to prevent any infinite discontinuity, of the implied (non-stochastic) steady state vector of endogenous variables, in response to small changes in these parameters. (This condition is expressed in terms of the eigenvalues of a matrix, which must be inverted to obtain the steady state solution vector, all of which are required to have positive real parts.) The second requirement is that the parameters are also restricted to prevent an infinite discontinuity in the MSV dynamic response coefficient of any endogenous variable to any of the model's exogenous forcing variables. The first requirement can be expressed quite cleanly in terms of the eigenvalues of a matrix that is the sum of matrices attached to expected future values (one period ahead) and lagged (by one period) values of the endogenous variables. The second requirement involves the relationship between these

[41] If it is desired to include a direct effect of v_t on k_{t+1}, this can be accomplished by definition of another variable in x_t.

matrices and the matrix of coefficients in a first-order autoregressive representation of the exogenous forcing variables. The second requirement is relevant only when one or more of the eigenvalues of the latter matrix is negative. In the univariate case of equation (17), for example, this requirement is relevant only in highly unusual cases involving $f < -1$ and $\rho < 0$.

The second WF requirement could be replaced with the assumption that the model possesses the property of process consistency, which implies that certain infinite series relating to expected future values of exogenous variables must be convergent. This condition is cleaner, but is more restrictive and is not adopted for this paper's argument.

It should be added that the paper does not present any results pertaining to the presence or absence of E-stability for non-MSV solutions, a subject that is being explored by Evans, Honkapohja, Mitra, and others. Finally, it must be recognized that the term "well formulated" is one that some readers might find misleading or even objectionable. I have used it rather than "plausible" because the latter is not sufficiently specific. Perhaps "not implausibly hypersensitive to parameter specification" could be used instead of well formulated. In any case, it is the concept—not its name—that is important.

[42] For explosive MSV solutions, the qualification mentioned on p. 15 is applicable.

Appendix A

The agenda here is to show that the eigenvalues of F are the same as for Λ_2^{-1} in expression (27), i.e., that the eigenvalues of F are the inverses of the m eigenvalues of M that are not included in Λ_1. (The eigenvalues of Λ_1 equal those of $\Omega = - P_{22}^{-1} P_{21}$). This assures us that we cannot have any eigenvalue with real part smaller than -1 for both F and Ω, for the former requires that more than m of the eigenvalues of M are smaller in modulus than 1, while the latter requires that fewer than m of the eigenvalues of M are smaller in modulus than 1.

From $PM = \Lambda P$ we have the submatrix $P_{21} = \Lambda_2 P_{22} - P_{22}A^{-1}$ so an alternative expression for Ω, equivalent to (28), is $-P_{22}^{-1}(\Lambda_2 P_{22} - P_{22}A^{-1})$ and thus we can write

$$F = (I - A\Omega)^{-1}A = (I + AP_{22}^{-1}\Lambda_2 P_{22} - AP_{22}^{-1}P_{22}A^{-1})^{-1}A$$

$$= (AP_{22}^{-1}\Lambda_2 P_{22})^{-1}A = P_{22}^{-1}\Lambda_2^{-1}P_{22}A^{-1}A = P_{22}^{-1}\Lambda_2^{-1} P_{22} .$$

But the latter has the same eigenvalues as Λ_2^{-1}, which is what we set out to establish. Note that this result is the multivariate counterpart, for the case with an invertible A matrix, of the univariate result given in footnote 23.

Appendix B

The purpose here is to show that for model (22) adoption of the second WF requirement, that there be no infinite discontinuity in the response of an element of y_t to an element of u_t, implies satisfaction of the E-stability condition (24c). In this model, the undetermined coefficient equations are (26) and the following:

(A-1) $\Gamma = A\Omega\Gamma + A\Gamma R + I.$

Using $F = (I - A\Omega)^{-1}A$, the latter can be written as

(A-2) $\Gamma = F\Gamma R + (I - A\Omega)^{-1}.$

Using the well-known identity vec $ABC = (C' \otimes A)$ vec B (e.g., E&H 2001, p. 117), we have

(A-3) vec $\Gamma = (R' \otimes F)$ vec Γ + vec $[(I - A\Omega)^{-1}]$

implying

(A-4) vec $\Gamma = [I - (R' \otimes F)]^{-1}$ vec $[(I - A\Omega)^{-1}]$.

Then our WF requirement is that all eigenvalues of $[I - (R' \otimes F)]$ be positive, i.e., that all eigenvalues of $(R' \otimes F)$ be smaller than 1. But this is the same condition as (24c).

Figure 1

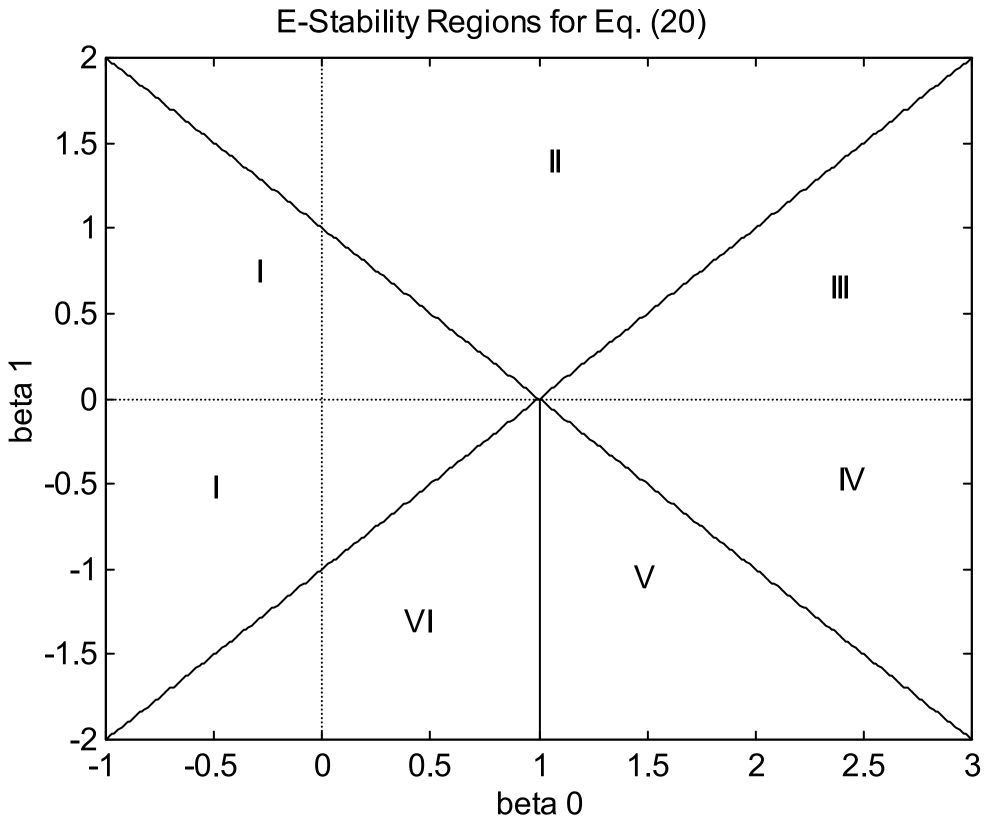
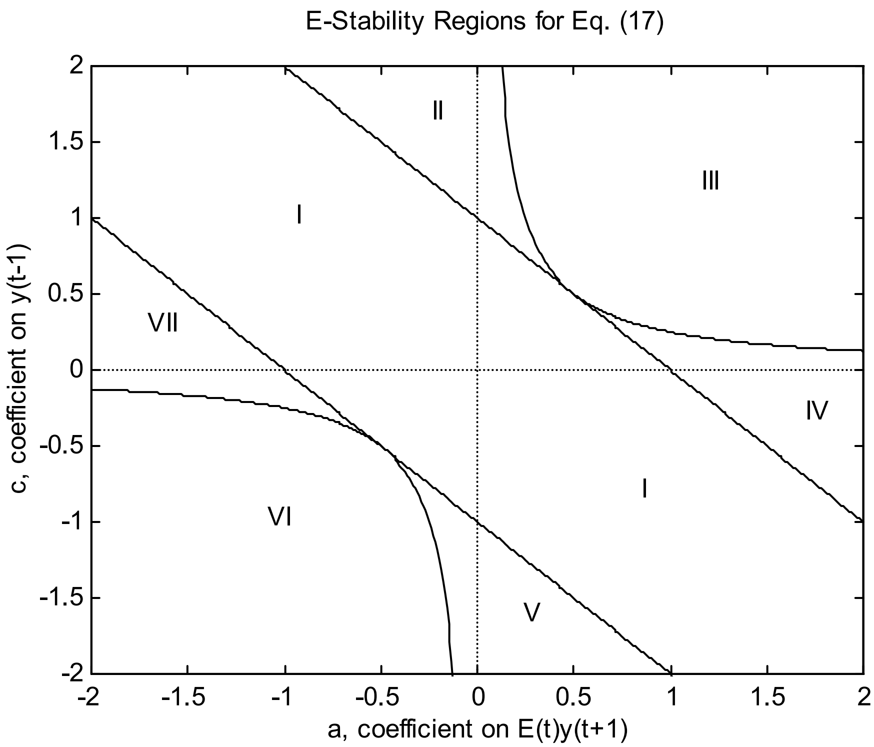

E-Stability Regions for Eq. (20)
beta 1
beta 0
I
II
III
I
IV
V
VI
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1
-0.5
0
0.5
1
1.5
2
2.5
3

Figure 2



E-Stability Regions for Eq. (17)
c, coefficient on y(t-1)
a, coefficient on E(t)y(t+1)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
I
II
III
IV
V
VI
VII
I

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