NBER WORKING PAPER SERIES

TRADING WITH THE UNBORN: A NEW PERSPECTIVE ON CAPITAL INCOME TAXATION

Kent A. Smetters

Working Paper 9412
http://www.nber.org/papers/w9412

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
December 2002

Helpful comments were received from seminar participants at the 1998 Brown Macroeconomics Workshop; the 1999 IMF seminar series; the 2000 Harvard Monetary and Fiscal Policy Workshop; The Centre for Economic Policy Research workshop held in Tilburg, Holland, 2000; the 2000 Wharton Macro-Finance workshop, the 2000 Wharton Insurance and Risk Management Workshop, as well as Laurence Kotlikoff, Greg Mankiw, and Jan Walliser. The views expressed herein are those of the authors and not necessarily those of the National Bureau of Economic Research.

Trading with the Unborn: A New Perspective on Capital Income Taxation
Kent A. Smetters
NBER Working Paper No. 9412
December 2002
JEL No. D9, E6, H2, H3

ABSTRACT

Security markets between generations are incomplete due to a "biological trading constraint" that prevents living generations from negotiating contingent contracts with the unborn. This paper shows, however, that government policy can be used to replicate the trades that would have occurred if these generations could trade. Specifically, for the class of linear securities, these trades can be replicated using a Domar-Musgrave capital income tax that is similar to the U.S. capital income tax.

It is then proven that the Replicating Tax Rate (RTR) in the replicating capital income tax system is positive in a production economy if wage and capital returns are uncorrelated (i.e., only depreciation is stochastic). The sign of the RTR is ambiguous, however, if wage and capital returns are perfectly correlated (i.e., only productivity is stochastic). But, in this case, if we also assume that (i) production takes the Cobb-Douglas form, (ii) depreciation per period is less than 100 percent, and (iii) the inter-temporal substitution elasticity (IES) is unity, then the RTR is actually negative.

Since completing a missing market is not necessarily pareto improving in the presence of general-equilibrium effects, this paper also investigates whether the Replicating Tax increases efficiency. In the case in which the RTR can be signed as negative, efficiency is proven to increase. While this result is one of the first derivations of efficiency gains associated with completing a missing market in a production economy with an endogenous equity return distribution, this result is still restrictive. Simulation evidence, therefore, is reported for more realistic cases in which both productivity and depreciation are stochastic; a calibrated value for the IES parameter is also used and other realistic features of the U.S. economy are incorporated. Welfare results, corresponding to a change in the RTR, are reported for both transition and steady-state generations using a recursive technique that accommodates a state space that expands rapidly over time.

Kent A. Smetters
The Wharton School
The University of Pennsylvania
Philadelphia, PA 19104-6218
and NBER
smetters@wharton.upenn.edu

I. Introduction

Figure 1 helps illustrate the incomplete market that exists between generations in the market economy.[1] Generation t agents are alive today at time t and generation t+1 agents will be born at time t+1. Generation t works and invests in capital during their first period of life at time t in order to afford second-period retirement at time t+1. The process repeats for generation t+1 who earns labor income at time t+1. National income at time t+1, therefore, is divided between stochastic capital income received by generation t and stochastic wage income received by the generation t+1.

At time t, generations t and t+1 could share the time-(t+1) risks but a "biological" trading constraint – i.e., generation t+1 is not yet born – prevents them. When generation t+1 arrives at time t+1, there is no more time-(t+1) uncertainty and so the risk sharing opportunity is lost.

But suppose that these generations *could* trade risk-sharing contracts at time t. In particular, we could think of a "court appointed executor" that negotiates with generation t on generation t+1's behalf. Both generations trade securities conditional on the information that is available at time t. Generation t, therefore, trades from an "interim" position, i.e., conditional on their first-period wages that have already been realized at time t. Generation t+1, however, trades more from an "ex-ante" position since, at time t, they don't know their first-period wages yet. This thought experiment, therefore, is different than that of Rawls, which would require *each* generation to negotiate from behind a "veil of ignorance," interpreted herein as not knowing their first-period wage.[2] Moreover, the experiment herein is different from the standard "interim" perspective, which would, effectively, require a separate "court appointed executor" for each possible state of first-period wages that

[1] The story is kept simple for now. In the analysis below, e.g., generation t+1 simultaneously negotiates with generation t+2, *ad infinitum*, i.e., sequential one-period contracts are introduced.

[2] Rangel and Zeckhauser (2001) clearly explain how finite-horizon lives produces inefficiencies in a Rawlsian economy. Similarly, Ball and Mankiw (2001) consider a linear economy. These more tractable economies represent a natural setting for the Rawlsian experiment since it is not clear how his theory applies with endogenous capital returns.

generation t+1 could face. Under the interim criterion, only the contract corresponding to the executor for the state that is actually realized at time t+1 would be enforced.[3] The experiment herein, therefore, can probably be best thought of as simply relaxing a biological trading restriction.

So if generations t and t+1 could trade, how would risk at time t+1 be shared, and at what equilibrium price? At this price, would generation t+1, for example, want to hold a long position in equity returns that are realized on their birthday at time t+1 in order to gain exposure to stock returns before time t+2? Or would they maybe want a short position in order to hedge shocks to their first-period wages? And, would these trades actually improve *efficiency* with endogenous prices?

This thought experiment, though, would be nothing more than a theoretical *curio so* unless there were something that could actually be done to replicate these markets. Hence, probably the most important question is: can the government actually *replicate* these inter-generational trades?

Why might the government be able to produce securities that the private market cannot? The reason is that sharing risks across generations requires the ability to pre-commit future generations to accept negative transfers in some states of the world into which they are born. But most societies do not allow individual citizens to make these sorts of "negative bequests."[4] Collectively, though, societies can pre-commit the unborn using the government's tax authority. Whether the government *would* actually replicate these markets is a political-economy issue outside the scope of the paper; Rangel (2000) examines the provision of inter-generational goods in a political-economy setting.

---

[3] The potential strength of the interim concept is that trades that improve efficiency under the interim criterion will naturally do likewise under the Rawlsian perspective (Mas-Collel, Whinston and Green, 1995, Section 23.F). However, trades that improve efficiency *ex post* (i.e., after all the risks facing both generations are realized) would improve interim efficiency and, hence, Rawlsian-based efficiency (*Ibid*), thereby seemingly making the ex-post criterion even more attractive. But, if taken seriously, the *ex post* criterion implies that *observable* insurance markets (e.g., automobile insurance) do *not* improve efficiency, even at *fixed* prices. Similarly, many fiscal policies that could improve the expected utility of future generations today are ruled out under the interim concept, although to a lesser extent.

[4] This fact is not changed with "infinitely-lived" corporations since current generations cannot force future generations to purchase corporations with negative net worth. See also Gale (1990) and Allen and Gale (1997).

Main Findings of This Paper

This paper characterizes the trades for a linear class of securities that would exist between generations if they could trade. It then shows that a Domar-Musgrave (1944) type of capital income tax, which is fairly similar to the type of capital income tax that is used in the United States, can be used by the government to replicate these trades. Risk sharing occurs through the government's budget constraint: the more [less] tax revenue that is raised from taxing generation-t's capital income at time t+1, the smaller [larger] the wage tax must be on generation t+1 at time t+1.

Whether the Replicating Tax Rate (RTR) in the replicating capital income tax system is positive or even negative in a production economy depends on the main source of uncertainty.

If only depreciation is stochastic then capital income returns received by generation t at time t+1 are uncorrelated with the wages received by generation t+1 at time t+1. Hence, generation t+1 wants a *long* position in generation t's capital returns, which is provided with a positive RTR: larger [smaller] capital income returns produce more tax revenue from generation t, allowing generation t+1 to keep a larger [smaller] fraction of their pre-tax wages for personal consumption.

If only factor productivity is stochastic, then capital income returns received by generation t at time t+1 are perfectly correlated with the wages received by generation t+1 at time t+1. In this case, the sign of the RTR is generally ambiguous. But if we further assume that (i) production takes the Cobb-Douglas form, (ii) the depreciation rate less than 100 percent, and (iii) the inter-temporal substitution elasticity (IES) is one, then the RTR is actually negative. A negative capital income tax rate gives generation t+1 a *short* position in generation t's stock returns: *smaller* [larger] capital income returns raise more tax revenue from generation t, allowing generation t+1 to keep a larger [smaller] fraction of their wages. Now generation t+1 can hedge its first-period wage at time t+1.

However, completing a missing market is not necessarily pareto improving in the presence of general-equilibrium price effects. This paper, therefore, investigates whether the Replicating Tax increases efficiency. In the case in which the RTR can be signed as negative, efficiency is proven to increase. While this result is one of the first derivations of efficiency gains associated with completing a missing market in a production economy with an endogenous equity return distribution, this result is still restrictive. Simulation evidence, therefore, is reported for more realistic cases in which both productivity and depreciation are stochastic. In these simulations, a calibrated value for the IES parameter is also used, and other realistic features of the U.S. economy are incorporated. Changes in welfare, corresponding to a change in the RTR, are reported for both transition and steady-state generations. The simulations use a recursive technique along a dense lattice that accommodates a state space that expands rapidly over time.

Outline

Section II describes what we'll refer to as the "Inter-generational Trading Economy" (ITE) where trading between generations is allowed and the capital income tax is set to zero. Section III outlines what we'll refer to as the "Market and Tax Economy" (MTE), which corresponds to the actual market economy where the biological trading constraint is enforced and the government can impose a capital income tax. Section III then demonstrates how capital income taxes can be chosen in the MTE in order to replicate the ITE. Section IV derives the sign of the Replicating Tax Rate in two boundary cases where closed-form solutions can be found. Section V generalizes the results to a sequential trading economy where generation t+1 trades not only with generation t but also with generation t+2, which, in turn, trades with generation t+3, ad infinitum. Since introducing a new market is not necessarily pareto improving with endogenous prices, Section VI investigates whether the introduction of the Replicating Tax actually improves efficiency. Section VII concludes.

II. The Inter-generational Trading Economy (ITE)

The Inter-generational Trading Economy is composed of three sectors: households, firms and a primitive government that raises revenue using a wage tax. Although not strictly needed, the government sector motivates a net supply of bonds in a two-period model.

Generation-*t* Households

Agents live for two periods.[5] In the first-period, generation-*t* consumers decide how much to save in government debt, s^B, and unleveraged capital, s^K, to maximize their homothetic expected lifetime utility over first-period consumption, c_1, and second-period consumption, c_2. Conditional on the state of the economy at time *t*, generation-*t*'s problem, therefore, is as follows:

$$\max_{s_t^B, s_t^K, q_{t,t+1}} E_t U(c_{1,t}, c_{2,t+1}) = u(c_{1,t}) + \beta E_t u(c_{2,t+1}) \tag{1}$$

s.t.

$$c_{1,t} + s_t^K + s_t^B = w_t(1-\tau_t^W) \tag{2}$$

$$\begin{aligned} c_{2,t+1} = {} & s_t^K \cdot (1+e_{t+1}) + s_t^B \cdot (1+r_{t+1}) \\ & - p_{t,t+1} \cdot q_{t,t+1} + q_{t,t+1} \cdot (e_{t+1} - r_{t+1}) \end{aligned} \tag{3}$$

where the function, $u(c)$, satisfies $\partial u(c)/\partial c > 0$, $\partial^2 u(c)/\partial c^2 < 0$, and $\lim_{c \to 0} \partial u(c)/\partial c = \infty$. w is the wage rate known at time *t*; τ_t^W is the tax rate on wage income earned at time *t*; e_{t+1} and r_{t+1} are the second-period risky return to private capital and the risk-free return, respectively.

$q_{t,t+1}$ is a linear forward contract traded between generation *t* and generation *t* + 1 at time *t*. This contract pays the *realized* equity risk premium, $e_{t+1} - r_{t+1}$, at time *t* + 1, and trades at price $p_{t,t+1}$. Payment for the contract is also made at time *t*+1 and when both generations overlap. $q_{t,t+1}$ equals the number of contracts purchased by generation *t* and $q_{t+1,t}$ is the number of units purchased by generation *t*+1. Inter-generation contracts are in zero net supply: $q_{t,t+1} = - q_{t+1,t}$.

---

[5] The two-period assumption is discussed in more detail later.

A couple comments on the inter-generational contract are in order. First, the payoff to the contract is net of the risk-free rate in order to both simplify the algebra and to make the connection with Domar-Musgrave capital income taxes. However, since the risk-free return is already known at time t, no additional risk-sharing between generations can be accomplished by including its value in the forward contract's index. This simplification, therefore, is immaterial.

Second, while most derivative securities (e.g., forwards, futures, swaps and options) are single-indexed contracts, even more risk sharing can be accomplished with a contract that is also a function of wages at time t+1. In the analytical results below, though, we consider two boundary cases. In the first case, only depreciation is stochastic and so wages are non-stochastic and, hence, they are fully predictable. In the second case, only productivity is stochastic and so capital returns and wages are perfectly correlated. In each case, no improvement in inter-generational risk-sharing could be achieved by making the payoff of the inter-generational contract dependent on wages at time t+1. But if capital returns and wages are both stochastic *and* imperfectly correlated, the single-index assumption does limit risk sharing somewhat. Double-indexed contracts in this case, though, would substantially complicate matters, and so we stick to the traditional focus on single-indexed contracts. We leave dual-indexed contracts as possible future research.

The first-order conditions for the demand for bonds and equities for given tax parameters are,

(4) $$\beta E\left[\frac{u'(c_{2,t+1})}{u'(c_{1,t})}\left(1+r_{t+1}\right)\right]=1 \quad , \qquad \text{and,}$$

(5) $$\beta E\left\{\frac{u'(c_{2,t+1})}{u'(c_{1,t})}\left(1+e_{t+1}\right)\right\}=1$$

(6) $$E_t\left[u'\left(c_{2,t+1}\right)\cdot\left(e_{t+1}-r_{t+1}\right)\right]-E_t\left[u'\left(c_{2,t+1}\right)\right]\cdot p_{t,t+1}=0$$

Generation-(t+1) Households

Generation t+1 optimizes over each possible state at time t+1, $\sigma_{t+1} \in \{A_{t+1}, k_{t+1}, \delta_{t+1}, D_{t+1}\}$, conditional on the state at time t, to determine its demand for the contract with generation t, $q_{t+1,t}$:

$$\max_{s^B_{t+1}, s^K_{t+1}, q_{t+1,t}} E_t\left\{E_{t+1}\left[U\left(c_{1,t+1}(\sigma_{t+1}), c_{2,t+2}(\sigma_{t+1})\right)\right]\right\} \\ = E_t\left\{u\left(c_{1,t+1}(\sigma_{t+1})\right) + \beta E_{t+1} u\left(c_{2,t+2}(\sigma_{t+1})\right)\right\} \tag{7}$$

s.t.

$$c_{1,t+1}(\sigma_{t+1}) + s^K_{t+1}(\sigma_{t+1}) + s^B_{t+1}(\sigma_{t+1}) + p_{t,t+1} \cdot q_{t+1,t} = \\ w_{t+1}(\sigma_{t+1}) \cdot \left(1 - \tau^W_{t+1}\right) + q_{t+1,t} \cdot \left(e_{t+1}(\sigma_{t+1}) - r_{t+1}\right) \tag{8}$$

$$c_{2,t+2}(\sigma_{t+1}, \sigma_{t+2}) = s^K_{t+1}(\sigma_{t+1}) \cdot \left(1 + e_{t+2}\right) + s^B_{t+1}(\sigma_{t+1}) \cdot (1 + r_{t+2}) \tag{9}$$

$w_{t+1} \in W^*_{t+1|t}$ is unknown at time t but $\inf\left(W^*_{t+1|t}\right) > 0$ since $A_{t+1} > 0$ and so the program (7) - (9) is well defined. (The state index is omitted on time-(t+2) factor prices and policy functions to save notation.) Multiple expectation operators are used (instead of combining them) to emphasize the differences in state spaces over which generation t+1 must optimize relative to generation t.

Generation t+1's first-order conditions are:

$$E_t\left[u'\left(c_{1,t+1}(\sigma_{t+1})\right)\right] = \beta \cdot E_t\left\{E_{t+1}\left[u'\left(c_{2,t+2}(\sigma_{t+1})\right) \cdot \left(1 + e_{t+2} - (e_{t+2} - r_{t+2})\tau^K_{t+2}\right)\right]\right\} \tag{10}$$

$$E_t\left[u'\left(c_{1,t+1}(\sigma_{t+1})\right)\right] = \beta \cdot E_t\left\{E_{t+1}\left[u'\left(c_{2,t+2}(\sigma_{t+1})\right) \cdot \left(1 + r_{t+2}\right)\right]\right\} \tag{11}$$

$$E_t\left[u'\left(c_{1,t+1}(\sigma_{t+1})\right)\right] \cdot p_{t,t+1} = E_t\left[u'\left(c_{1,t+1}(\sigma_{t+1})\right) \cdot \left(e_{t+1} - r_{t+1}\right)\right] \tag{12}$$

Production

Net output at time t takes the Cobb-Douglas form and is produced using capital, K, and labor, L. It is also determined by the economy's level of productivity, A, and the depreciation rate, δ:

$$F\left(K_t, L_t\right) = A_t K_t^{\alpha} L_t^{1-\alpha} - \delta_t K_t \quad , \tag{13}$$

or, in intensive (per labor unit) form,

(14) $$f(k_t) = A_t k_t^{\alpha} - \delta_t k_t$$

where $k \equiv K / L$ is the capital-labor ratio. Both A and δ are stochastic to allow for an imperfect correlation between wage and capital returns, as in Bohn (1999). Let $A_t = (1 + a_t)A_{t-1}$ where a is a positive bounded i.i.d. random variable, $\underline{a} < a_t < \overline{a}$, with trend λ. Moreover, let $\delta_t = \hat{\delta} + \xi_t$ where $\hat{\delta}$ is a constant and ξ is i.i.d. with a zero mean. Stochastic factor prices are neoclassic,

(15) $$w_t = A_t(1 - \alpha)k_t^{\alpha}$$

(16) $$e_t = A_t \alpha k_t^{\alpha - 1} - \delta_t$$

The inequality, $-1 < a$, ensures positive productivity.

Government

Tax revenue in the ITE is collected only from wage income,

(17) $$T_{t+1} = \tau_{t+1}^{W} w_{t+1}$$

where the size of the workforce relative to retirees is stationary, $L_t = 1 \ \forall\, t$.

Government debt evolves as

(18) $$D_{t+2} = G_{t+1} - T_{t+1} + (1 + r_{t+1})D_{t+1}$$

where $G_t = G_0 \cdot \left(f(k_t) \big/ f(k_0) \right)$ is government spending. Period $t = 0$ could represent, e.g., the start of the new policy. Scaling government spending is required to prevent a diverging debt-capital ratio.

Wage taxes must be stochastic since, even for a small G_t, the debt-capital ratio can diverge with enough bad shocks. Tax rates adjust to target a capital-debt ratio, $D_t \big/ K_t \equiv \overline{\tilde{d}}$, by the end of each generation. The modified model in Section V, though, allows risks to be shared across multiple generations. The wage tax rate at time t+1 that stabilizes the debt-capital ratio $\tilde{d}_{t+2}$ at $\overline{\tilde{d}}$ equals,

(19) $$\tau_{t+1}^{W} = \frac{G_{t+1} + (1 + r_{t+1}) \cdot \overline{\tilde{d}} \cdot k_{t+1} - \overline{\tilde{d}} \cdot k_{t+2}}{w_{t+1}}$$

Equation (19) is derived from (18) with $\tilde{d}_{t+1} = \tilde{d}_{t+2} = \overline{\tilde{d}}$.

Capital Market Clearing

Market clearing in general equilibrium requires that the capital stock at time t equals the capital saving by private agents. Similarly, government debt must equal bonds held by the public.

(20) $$k_{t+1} = s_t^k$$

(21) $$D_{t+1} = s_t^B$$

Substituting equation (20) into (16) shows that stochastic equity returns depend on the level of capital saving, s_t^k. Since the equity return, equation (16), degenerates to infinity as k approaches zero, a short sale constraint would never bind. Moreover, via equation (19), wage tax rates are lower at higher levels of capital saving as smaller wage taxes are needed to stabilize the debt-capital ratio.

General Equilibrium

A general equilibrium at time t is composed of a set of household policy rules, $\left\{c_{1,t}(\cdot), c_{2,t+1}(\cdot), s_t^K(\cdot), s_t^B(\cdot), q_{t,t+1}(\cdot), q_{t+1,t}(\cdot)\right\}$, the risk-free rate, r_{t+1}, and the equity return distribution for e_{t+1}, $\Xi\left(\bar{e}; A_t, \lambda, k_{t+1}, \delta_t, D_t\right)$, given $\left\{A_t, k_t, r_t, \delta_t, G_0, \bar{\bar{d}}\right\}$, satisfying:

1. The household's maximization problem, equations (2) - (6), holds.
2. The conditional equity return distribution for e_{t+1} satisfies,

   $$\Xi\left(\bar{e}; A_t, \lambda, k_{t+1}, \delta_t, D_t\right) = \Pr\left(\left\{a_{t+1}, \xi_{t+1}\right\} \in \Theta^a \times \Theta^\xi \middle| e\left(A_{t+1}, k_{t+1}, \delta_{t+1}\right) < \bar{e}\right)$$

   where Θ^a and Θ^ξ are the sets of possible values that the shock terms a and ξ can take. The expression $\Theta^a \times \Theta^\xi$ is the set of all possible states (i.e., the sigma field), and $\Pr(\cdot)$ gives the probability measure of state $\left\{a_{t+1}, \xi_{t+1}\right\}$.
3. The wage tax at time t, τ_t^W, satisfies equation (19).
4. The inter-generational market clears: $q_{t,t+1} = - q_{t+1,t}$.
5. The capital market clearing conditions, (20) and (21), hold.

III. The Market and Tax Economy (MTE)

The actual Market and Tax Economy (MTE) is also composed of three sectors: households, firms and government. Now, however, the realistic biological trading constraint is enforced, thereby removing the inter-generational security market. Instead, capital income taxes are operative.

Households

With capital income taxes, the household problem of generation-t is as follows:

(22)
$$\max_{s_t^B, s_t^K, q_{t,t+1}} E_t U(c_{1,t}, c_{2,t+1}) = u(c_{1,t}) + \beta E_t u(c_{2,t+1})$$

$s.t.$

(23)
$$c_{1,t} + s_t^K + s_t^B = w_t(1 - \tau_t^W)$$

(24)
$$c_{2,t+1} = s_t^K \cdot \left[1 + e_{t+1} - (e_{t+1} - r_{t+1})\tau_{t+1}^K\right] + s_t^B \cdot (1 + r_{t+1})$$

where τ_{t+1}^K is a Domar-Musgrave (1944) type of capital income tax. In particular, it is a *symmetric* tax on the *risky component* of capital income. The D-M tax is similar to that actually used in the United States with its fairly generous backward-looking and forward-looking loss offset rules.[6] Bond returns are untaxed; this assumption is immaterial (outside of calibration) since taxing debt would not change the after-tax risk-free return under the no-arbitrage first-order conditions below.[7]

The first-order conditions for the demand for bonds and equities for given tax parameters are,

(25)
$$\beta E\left[\frac{u'(c_{2,t+1})}{u'(c_{1,t})}(1 + r_{t+1})\right] = 1 \quad , \qquad \text{and,}$$

(26)
$$\beta E\left\{\frac{u'(c_{2,t+1})}{u'(c_{1,t})}\left[1 + e_{t+1} - (e_{t+1} - r_{t+1})\tau_{t+1}^K\right]\right\} = 1$$

[6] A Domar-Musgrave tax has two key features: (i) the risk-free component of investments is untaxed and (ii) the tax is symmetric so that the government shares in investment risk. The Domar-Musgrave tax has been explored in many papers, including Mossin (1968), Stiglitz (1969), and Sandmo (1969, 1977, 1985). Gordon (1985) showed the U.S. tax system is approximately Domar-Musgrave; see also, Bradford (1995), Zodrow (1995), and Hubbard and Gentry (1999). To the extent that conditions (i) and (ii) do not hold exactly, they would need to be part of the Replicating Tax.

[7] Taxing the risk-free component of the *capital* return is, however, material, as discussed below.

Combining equations (25) into (26) gives the following equation (derived in the Appendix):

$$\beta E\left[\frac{u'(c_{2,t+1})}{u'(c_{1,t})}\left(1+e_{t+1}\right)\right]=1 \tag{26'}$$

(26') is the same first-order condition that one would get without capital income taxes, equation (5). The tax on capital income falls out due to symmetry around the risk-free rate. The absence of the tax in (26') implies that first-period consumption, $c_{1,t}$, and total saving, $s_t^K + s_t^B$, for second-period consumption, $c_{2,t}$, is unaffected by the tax. Of course, the tax is still present in the budget constraint, (24). Neutrality of consumption and total saving is achieved through a portfolio shift:

Lemma 1.[8] *Let* $s_t^{K^*}$ *and* $s_t^{B^*}$ *equal generation t's desired saving in capital and bonds at the tax rate* $\tau_{t+1}^{K^*}$. *A new tax rate of* $\tau_{t+1}^{K^{**}}$ *is announced at time t and let* $s_t^{K^{**}}$ *and* $s_t^{B^{**}}$ *equal generation t's new desired saving in capital and bonds. At given prices,* $s_t^{K^{**}} = \left[(1-\tau_{t+1}^{K^*})\big/(1-\tau_{t+1}^{K^{**}})\right]\cdot s_t^{K^*}$ *and* $s_t^{B^{**}} = s_t^{B^*} - \left(s_t^{K^{**}} - s_t^{K^*}\right)$. *The value of* $c_{1,t}$ *and the policy function* $c_{2,t+1}(\bullet)$, *are unchanged.*

Lemma 1 follows immediately from equations (23) - (26). For example, if the government increases the capital income tax rate from zero to 50 percent, agents respond by doubling their capital saving, while reducing their bond holdings an equivalent amount. Consumption is unchanged. Portfolio re-balancing allows the investor to maintain the portfolio's Sharp ratio with no wealth effect. If the equity premium is positive (i.e., $E_t(e_{t+1}) > r_{t+1}$), the government's expected revenue also increases which compensates the government for "its" share of the risk reflected in the premium.

It is important to note that a change in the capital income tax rate is not Ricardian (neutral)

[8] Sandmo (1977) shows this type of result extends to an arbitrary number of risky assets. In particular, $\frac{\partial s^j}{\partial \tau^j} = \frac{s^j}{1-\tau^j}$ and $\frac{\partial s^i}{\partial \tau^j} = 0$ where i and j are two different risky assets.

in the MTE. Portfolio re-balancing occurs because the risk in capital income tax revenue is passed to the next generation under the government's budget constraint.[9] In particular, higher [lower] capital income taxes collected from generation t at time t+1 requires less [more] wage income taxes to be collected from generation t+1. This change in risk sharing will have price effects in the MTE.

Government

Total revenue now includes taxes on capital income,

(27)
$$T_{t+1} = \tau^W_{t+1} w_{t+1} + \tau^K_{t+1}(e_{t+1} - r_{t+1}) s^K_t$$

The wage tax rate at time t+1 that stabilizes the debt-capital ratio $\tilde{d}_{t+2}$ at $\bar{\tilde{d}}$ now equals,

(28)
$$\tau^W_{t+1} = \frac{G_{t+1} + (1 + r_{t+1}) \cdot \bar{\tilde{d}} \cdot k_{t+1} - \bar{\tilde{d}} \cdot k_{t+2} - \tau^K_{t+1}(e_{t+1} - r_{t+1}) s^K_t}{w_{t+1}}$$

General Equilibrium

The description of the production sector as well as the conditions for capital market clearing in the MTE are the same as those shown for the ITE. The description of general equilibrium is also similar, except for the omission of the inter-generational contracts and the presence of the capital income taxes. In particular, a general equilibrium at time t in the MTE is the set of household policy rules, $\left\{c_{1,t}, c_{2,t+1}(\cdot), s^K_t, s^B_t\right\}$, the risk-free rate and equity return distribution for e_{t+1}, $\left\{r_{t+1}, \Xi\left(\bar{e}; A_t, \lambda, k_{t+1}, \delta_t, D_t\right)\right\}$, given $\left\{A_t, k_t, r_t, \delta_t, G_0, \bar{\tilde{d}}\right\}$, satisfying the following conditions:

1. The household's maximization problem, (23) - (26), holds.

2. The conditional equity return distribution for e_{t+1} satisfies,

$$\Xi\left(\bar{e}; A_t, \lambda, k_{t+1}, \delta_t, D_t\right) = \Pr\left(\left\{a_{t+1}, \xi_{t+1}\right\} \in \Theta^a \times \Theta^\xi \middle| e\left(A_{t+1}, k_{t+1}, \delta_{t+1}\right) < \bar{e}\right)$$

[9] In Gordon (1985), for example, portfolio reallocation does not occur because, in his model, the government uses a lump-sum rebate that compensates each retiree exactly the amount they paid in capital income taxes. In the OLG model herein, this rebate must be age discriminatory to avoid the type of inter-generational transfer considered herein.

where Θ^{a} and Θ^{ξ} are the sets of possible values that the shock terms a and ξ can take. The expression $\Theta^{a} \times \Theta^{\xi}$ is the set of all possible states (i.e., the sigma field), and $\Pr(\cdot)$ gives the probability measure of state $\{a_{t+1}, \xi_{t+1}\}$.

3. The wage tax at time t, τ_t^W, satisfies equation (28).

4. The market clearing conditions, (20) and (21), hold.

Existence of a globally unique stochastic stationary equilibrium can be proven á la Wang (1993).[10]

Replicating the Inter-generational Trading Economy (ITE)

The following proposition shows how the market economy outlined in this section can be used to replicate the ITE. From equations (2) and (3) or (8) and (9) we get:

Proposition 1. *Let* $s_t^{K^*}$, $q_{t,t+1}^*$, $p_{t,t+1}^*$, r_{t+1}^*, *and* $\tau_t^{W^*}$ *be equilibrium values in the ITE at time t, given a wage rate,* w_t. *The equilibrium is identical to that in the MTE with the following tax rates:* $\tau_{t+1}^{K^*} = {-q_{t,t+1}^*}/{s_t^{K^*}}$ *and* $\tau_t^W = \tau_t^{W^*} + \Delta\tau_t^W$ *where* $\Delta\tau_t^W = \dfrac{p_{t,t+1}^* \cdot q_{t,t+1}^*}{\left(1+r_{t+1}^*\right)\cdot w_t}$.

Definition 1. $\tau_{t+1}^{K^*} = {-q_{t,t+1}^*}/{s_t^{K^*}} = {q_{t+1,t}^*}/{s_t^{K^*}}$ *is the Replicating Tax in the MTE.*

In words, a symmetric tax on risk can be chosen to share the same risk between generations as the inter-generational contract.[11] The change in the wage tax, $\Delta\tau_t^W$, in the MTE corresponds to the payment for the inter-generation contract in the ITE. Since contracts are in zero net supply ($q_{t,t+1} = -q_{t+1,t}$), Proposition 1 shows that the sign of the Replicating Tax Rate is the same sign as generation t+1's demand for the inter-generational contract, $q_{t+1,t}^*$, at price $p_{t,t+1}^*$. We now sign the RTR.

---

[10] Existence in the ITE as well immediately follows once the equivalence with the MTE is proven below.

[11] Smetters (2001) shows that investing the Social Security trust fund in equities inside a defined-benefit system is equivalent to implementing a Domar-Musgrave tax. As a result, either policy can be used to replicate the ITE.

IV. The Sign of the Replicating Tax Rate (RTR)

The sign of the Replicating Tax Rate (RTR) is now derived analytically for two boundary cases. In the first case, depreciation is stochastic and productivity is not (i.e., zero correlation between wages and capital income). In the second case, productivity is stochastic and depreciation is not (i.e., perfect correlation). Both of these variables are allowed to be stochastic in Section VI.

The Equilibrium Inter-generational Contract Price, $p^*_{t,t+1}$

The equilibrium price of the inter-generational contract, $p^*_{t,t+1}$, in the ITE sets the demand for the contract by generation t+1 equal to minus the demand by generation t: $q^*_{t+1,t}\left(p^*_{t,t+1}\right) = -q^*_{t,t+1}\left(p^*_{t,t+1}\right)$. The households first-order (no-arbitrage) conditions produce a zero price.

Lemma 2. $p^*_{t,t+1} = 0$ *and* $\Delta\tau^W_t = 0$.

Equations (4) and (5) $\Rightarrow E_t\left[u'\left(c_{2,t+1}\right)\cdot\left(e_{t+1} - r_{t+1}\right)\right] = 0 \Rightarrow p^*_{t,t+1} = 0$ by (6). The equality $\Delta\tau^W_t = 0$ follows immediately from Proposition 1. Q.E.D.

Intuitively, the equilibrium price for the inter-generational contract in the ITE must be zero in order for no arbitrage opportunity to exist between generation t's stock-bond portfolio mix and the inter-generational contract $q_{t,t+1}$. If, for example, $p^*_{t,t+1} > 0$ then generation t could guarantee a profit by selling short the contract ($q_{t,t+1} < 0$) while investing more in stocks and less bonds.

Signing the Equilibrium Value of $q_{t+1,t}$

Denote the equilibrium value of $q_{t+1,t}$ at $p_{t,t+1} = 0$, as $q^*_{t+1,t}\big|_{p_{t,t+1}=0}$. Then we have,

Lemma 3. $sign\left(q^*_{t+1,t}\big|_{p_{t,t+1}=0}\right) = sign\left\{E_t\left[u'\left(c_{1,t+1}(\sigma_{t+1})\right)\cdot\left(e_{t+1} - r_{t+1}\right)\right]\right\}$

To prove Lemma 3, let λ_1 be the Lagrangian multiplier for the constraint $q_{t+1,t} \geq 0$ and let λ_2 be associated with the constraint $q_{t+1,t} \leq 0$. Equation (12) becomes,

(29) $$E_t\left[u'\left(c_{1,t+1}(\sigma_{t+1})\right)\cdot\left(e_{t+1} - r_{t+1}\right)\right] + \lambda_1 - \lambda_2 = 0$$

where $\lambda_1 = 0$ if $\lambda_2 > 0$ and $\lambda_2 = 0$ if $\lambda_1 > 0$. If $\lambda_1 > 0$ then $q^*_{t+1,t}\big|_{p_{t,t+1}=0} < 0$ and $E_t[\bullet] < 0$. Similarly, if $\lambda_2 > 0$ then $q^*_{t+1,t}\big|_{p_{t,t+1}=0} > 0$ and $E_t[\bullet] > 0$. It follows that,

(30) $$sign\left(q^*_{t+1,t}\big|_{p_{t,t+1}=0}\right) = sign\left\{E_t\left[u'\left(c_{1,t+1}(\sigma_{t+1})\right)\cdot\left(e_{t+1} - r_{t+1}\right)\right]\right\}$$ Q.E.D.

By Definition 1, the right-hand side of equation (30) also gives the sign of the Replicating Tax, $\tau^{K^*}_{t+1}$.

The Replicating Tax Rate (RTR) is Positive When Only Depreciation is Stochastic

By equations (15) and (16), wages and capital income returns are *uncorrelated* if depreciation is stochastic but productivity is not ($a_t = \lambda$). We arrive at the following result.

Proposition 2. *The sign of the RTR is positive (* $sign\left(\tau^{K^*}_{t+1}\right) > 0$ *) if depreciation is stochastic (* $\xi > 0$ *) and productivity is not (* $a_t = \lambda$ *)* $\forall t$.

To prove this result, we need to show that generation t+1 holds a long position in the q-contract at the equilibrium contract price (i.e., $q^*_{t+1,t}\big|_{p_{t,t+1}=0} > 0$). Decompose $E_t[\bullet]$ in (30) as:

(31) $$E_t\left[u'\left(c_{1,t+1}(\sigma_{t+1})\right)\cdot\left(e_{t+1} - r_{t+1}\right)\right] = E_t\left[u'\left(c_{1,t+1}(\sigma_{t+1})\right)\right]\cdot E_t\left[\left(e_{t+1} - r_{t+1}\right)\right] + \operatorname{cov}\left[u'\left(c_{1,t+1}(\sigma_{t+1})\right),\left(e_{t+1} - r_{t+1}\right)\right]$$

The first "expected return" term on the right-hand side of equation (31) is always positive by non-satiation ($u' > 0$) and since $E_t\left[\left(e_{t+1} - r_{t+1}\right)\right] > 0$ if equity returns are risky ($\xi > 0$) and agents are risk averse ($u''<0$). The second "risk" term is zero since wages are not stochastic in this case and, hence, are uncorrelated with equities. So $sign\left\{E_t[\bullet]\right\} > 0 \Rightarrow sign\left(q^*_{t+1,t}\Big|_{p_{t,t+1}=0}\right) > 0$. Q.E.D.

Intuitively, since expected equity returns faced by generation t at time t+1 exceed the risk-free rate, the generation born at time t+1 also wants exposure to equity returns at time t+1 since those returns are uncorrelated with both their first-period wages at time t+1 as well as their second-period equity returns at time t+2. A positive capital income tax effectively gives generation t+1 access to equity returns at time t+1 via the government's budget constraint: more [less] capital tax revenue collected at time t+1 from generation t requires less [more] wage tax revenue from generation t+1.

The Replicating Tax Rate (RTR) is Negative When Only Productivity is Stochastic

By equations (15) and (16), wages and capital income returns are *perfectly correlated* if productivity is stochastic but depreciation is not (ξ = 0). This case is more intricate: the first "expected return" term of (31) is still positive but the second "risk" term is now negative since the marginal utility of consumption is decreasing in wages. The overall sign, therefore, is ambiguous a priori. Signing the RTR generally requires numerical simulation for each case.

A closed-form solution can be obtained, though, for the case of log utility which simplifies the policy functions. We need to also assume that government debt is zero in order to calculate a closed-form solution for the risk-free rate. Both the risk-free rate and the realized equity premium derived below are consistent with these assumptions at a zero capital income tax rate and so the model is internally consistent. To simplify notation, we also set the deterministic rate of depreciation to zero (δ = 0), although we only strictly need δ < 1. We get the following result:

Proposition 3. *If depreciation is certain and less than full ($\xi = 0$, $\delta < 1$) and productivity is stochastic (i.e., a_t is stochastic), the sign of the RTR is ambiguous in general but strictly negative ($sign\left(\tau_{t+1}^{K^*}\right) < 0$) for log utility, zero net debt, and Cobb-Douglas production.*

To prove Proposition 3, we first derive the expression for the risk-free rate, r_{t+1}. Substitute equation (24) with $s_{1,t}^{B} = 0$ into (26). With $u(c) = \ln c$, we get $s_{1,t}^{K} = \dfrac{\beta \cdot w_t}{1+\beta}$ and $c_{1,t} = \dfrac{w_t}{1+\beta}$ via equation (23). Combining these expressions with equations (24) and (25) and simplifying,

(32)
$$r_{t+1} = \left[E_t\left(\frac{1}{1+e_{t+1}} \right) \right]^{-1} - 1 \quad .$$

Now substitute equation (32) along with the expressions for the factor returns, (15) and (16), and the policy function $c_{1,t+1}(\sigma_{t+1}) = \dfrac{w_{t+1}(\sigma_{t+1})}{1+\beta}$ into the expression $E_t[\bullet]$ shown in equation (30):

(33)
$$\begin{aligned} E_t[\bullet] &= E_t\left\{ \frac{1+\beta}{w_{t+1}} \cdot \left(1 + e_{t+1} - \left[E_t\left(\frac{1}{1+e_{t+1}} \right) \right]^{-1} \right) \right\} \\ &= E_t\left\{ \frac{1+\beta}{A_{t+1}(1-\alpha)k_{t+1}^{\alpha}} \cdot \left(1 + A_{t+1}\alpha k_{t+1}^{\alpha-1} - \left[E_t\left(\frac{1}{1+e_{t+1}} \right) \right]^{-1} \right) \right\} \\ &= \left(\frac{1+\beta}{1-\alpha} \right) \cdot \frac{\alpha}{s_t^K} \cdot \left\{ E_t\left(\frac{1}{e_{t+1}} \right) \cdot \left[1 - \left[E_t\left(\frac{1}{1+e_{t+1}} \right) \right]^{-1} \right] + 1 \right\} \end{aligned}$$

Notice that the wage term eventually drops out of equation (33) after some algebraic reduction. This result is due to the perfect correlation between wages and equity returns which allows the sign of $E_t[\bullet]$ to be written as a function of only equity returns at time $t + 1$.

Rearranging the third equality in equation (33) implies,

(34)
$$sign\left(E_t[\bullet] \right) = sign\left\{ E_t\left(\frac{1}{1+e_{t+1}} \right) \left[1 + E_t\left(\frac{1}{e_{t+1}} \right) \right] - E_t\left(\frac{1}{e_{t+1}} \right) \right\} \quad .$$

We now make use of the following lemma which is proven in the Appendix.

Lemma 4. *Let x(ω) be a strictly positive real-valued random variable defined on the probability space (Ω, Ψ,P), $x{:}\Omega \rightarrow R_{++}$. Then* $E\left(\frac{1}{x}\right) \geq E\left(\frac{1}{1+x}\right) \cdot \left[1 + E\left(\frac{1}{x}\right)\right]$*, holding with equality if the space is degenerate (Ω contains a single element); holding with strict inequality otherwise.*

Lemma 4 and equations (30) and (34) imply that $q^*_{t+1,t}\Big|_{p_{t,t+1}=0} < 0$ for random equity returns.[12] The RTR is, therefore, negative by Definition 1 and so Proposition 3 is proven. Q.E.D.

Intuitively, a negative capital income tax provides generation *t*+1 with a hedge to their first-period wage income. A negative tax raises *more* capital income tax revenue from generation *t* after a negative productivity shock and *less* revenue after a positive shock. Required wage taxes on generation *t*+1, therefore, are smaller during bad times and higher during good times, ceteris paribus. Generation *t*+1 prefers this tradeoff, despite the negative expected return when $E_t\left[\left(e_{t+1} - r_{t+1}\right)\right] > 0$.

A non-zero tax rate is somewhat surprising, since the *net* capital return of generation *t* is already perfectly correlated with generation (*t*+1)'s wage income at time *t*+1. Perfect correlation would, at first blush, seem to rule out any potential for risk sharing.[13] However, for there to be no gain from risk sharing, generation *t*'s *gross* capital returns at time *t*+1 must be linear in the wage received by generation *t*+1. But that is not the case if the capital saving by generation *t* has not fully depreciated by the beginning of period *t*+1 (i.e., δ < 1). Only δ = 1 rules out risk sharing gains.[14]

---

[12] If equity returns are certain (i.e., *e* = *r*) then Lemma 2 implies $q^*_{t+1,t}\Big|_{p_{t,t+1}=0} = 0$. Intuitively, there is no value for the Replicating Tax since the tax on risk never raises revenue in this case.

[13] Greg Mankiw helped me think through this particular point. Bohn (1999) is also very informative.

[14] For a depreciation rate of 100 percent (δ = 1) and log utility, $e_t = A_t \alpha k_t^{\alpha-1} - 1 \Rightarrow c_{1,t+1} = \frac{(1-\alpha)}{\alpha\cdot(1+\beta)} \cdot c_{2,t+1}$. So $E\left[u'(c_{1,t+1})\cdot(e_{t+1} - r_{t+1})\right] = \left[\frac{(1-\alpha)}{\alpha\cdot(1+\beta)}\right] \cdot E\left[u'(c_{2,t+1})\cdot(e_{t+1} - r_{t+1})\right] = 0$, i.e, risk sharing offers no value.

V. Generalizing to Sequential Trading

The analysis so far considered risk sharing between two generations: generation t and generation t+1. We now ask the question, is the Replicating Tax Rate still positive [negative] with only depreciation [productivity] shocks if generation t+1 also risk shares with generation t+2, who is, in turn, risk sharing with generation t+3, *ad infinitum*, at time t? The answer is yes.

Let $\sigma_t = \{A_t, k_t, \delta_t, D_t, q_{t,t-1}, p_{t-1,t}\}$ denote the state at time t. Set $Z_{s|t}(\sigma_t)$ holds the possible state vectors at time s, conditional on the state at time t. The associated probability measure is $\pi_{s|t}: Z_{s|t}(\sigma_t) \to [0,1]$. Naturally, the number of vectors contained in $Z_{t|t}(\sigma_t)$, indicated by $\#Z_{t|t}(\sigma_t)$, is one since the time-t state is known at t. If depreciation *or* productivity follows a two-state Markov process, $\#Z_{t+1|t}(\sigma_t) = 2$, $\#Z_{t+2|t}(\sigma_t) = 4$, $\#Z_{t+3|t}(\sigma_t) = 8$, ..., $\#Z_{s|t}(\sigma_t) = 2^{(s-t)}$. If depreciation *and* productivity follow two-state Markov processes then $\#Z_{s|t}(\sigma_t) = 4^{(s-t)}$.

At <u>time t</u>, generation s in the ITE picks policy functions defined over each state in $Z_{s|t}(\sigma_t)$:

(35)
$$\max_{s_s^B, s_s^K, q_{s,s-1}, q_{s,s+1}} E_t\left\{\left[E_s U(c_{1,s}(\sigma_s), c_{2,s+1}(\sigma_s))\right] \middle| Z_{s|t}(\sigma_t)\right\}$$
$$= E_t\left\{\left[u(c_{1,s}(\sigma_s)) + \beta \cdot E_s u(c_{2,s+1}(\sigma_s))\right] \middle| Z_{s|t}(\sigma_t)\right\}$$

s.t.

(36)
$$c_{1,s}(\sigma_s) + s_s^K(\sigma_s) + s_s^B(\sigma_s) + p_{s-1,s}(\sigma_{s-1}) \cdot q_{s,s-1}(\sigma_{s-1}) = w_s(\sigma_s) \cdot (1 - \tau_s^W) + q_{s,s-1}(\sigma_{s-1}) \cdot (e_s(\sigma_s) - r_s)$$

(37)
$$c_{2,s+1}(\sigma_s) + p_{s,s+1}(\sigma_s) \cdot q_{s,s+1}(\sigma_s) = s_s^K(\sigma_s) \cdot [1 + e_{s+1}(\sigma_{s+1})] + s_s^B(\sigma_s) \cdot (1 + r_{s+1}(\sigma_s)) + q_{s,s+1}(\sigma_s) \cdot (e_{s+1}(\sigma_{s+1}) - r_{s+1}(\sigma_s))$$

The $q_{s,s-1}$ contract and price are indexed to the time-(s-1) state and $q_{s,s+1}$ is indexed to the time-s state.

An additional constraint is needed to fully specify the dynamic programming problem: the policy function, $c_{2,T+1}(\sigma_{T+1})$, of generation $T \to \infty$ must be *feasible*. In particular, we must rule out

Ponzi games where each generation tries to sell the subsequent generation an overpriced contract relative to the risk sharing provided. Substitute equation (36) into (37) and re-arranging gives:

(38)
$$q_{s,s+1} = \left[\begin{array}{l} q_{s-1,s} \cdot \left(e_s - r_s - p_{s-1,s}\right) \cdot \left(1 + r_{s+1}\right) - s_s^K \cdot \left(1 + e_{s+1}\right) + \\ \left[w_s \cdot \left(1 - \tau_s^W\right) - s_s^K - c_{1,s}\right] \cdot \left(1 + r_{s+1}\right) + c_{2,s+1} \end{array} \right] \Big/ \left(e_{s+1} - r_{s+1} - p_{s,s+1}\right)$$

where the state index is not shown to reduce notation. Equation (38) is a first-order recurrence relation in q. Integrating (38) forward from date t to T gives the following *realized* value for $q_{T,T+1}$:

(39)
$$q_{T,T+1}\left(\sigma_T\right) = \left[\prod_{i=t}^{T-1} \frac{\phi_1(i) \cdot \left(1 + r_{i+1}\right)}{\phi_1(i+1)}\right] \cdot \left[\sum_{i=t}^{T-1} \frac{\phi_2(i+1)}{\phi_1(i+1) \cdot \left[\prod_{j=t}^{i} \frac{\phi_1(j) \cdot \left(1 + r_{j+1}\right)}{\phi_1(j+1)}\right]} + q_{t-1,t}\left(\sigma_{t-1}\right)\right]$$

where

$$\phi_1(i) = e_i\left(\sigma_i\right) - r_i\left(\sigma_i\right) - p_{i-1,i}\left(\sigma_{i-1}\right)$$

$$\begin{aligned} \phi_2(i+1) = & -s_i^K\left(\sigma_i\right) \cdot \left(1 + e_{i+1}\left(\sigma_{i+1}\right)\right) + \\ & \left[w_i\left(\sigma_i\right) \cdot \left(1 - \tau_i^W\left(\sigma_i\right)\right) - s_i^K\left(\sigma_i\right) - c_{1,i}\left(\sigma_i\right)\right] \cdot \left[1 + r_{i+1}\left(\sigma_i\right)\right] + \\ & c_{2,i+1}\left(\sigma_{i+1}\right) \end{aligned}$$

The transversality condition is $\lim_{T \to \infty} \psi\left(\sigma_T\right) \cdot p_{T,T+1}\left(\sigma_T\right) \cdot q_{T,T+1}\left(\sigma_T\right) = 0$ where $\psi\left(\sigma_T\right)$ is the Lagrangian multiplier for program (35) - (37), with time index T instead of s. Equation (39) implies:

(40)
$$\sum_{i=t}^{\infty} \frac{\phi_2(i+1)}{\phi_1(i+1) \cdot \left[\prod_{j=t}^{i} \frac{\phi_1(j) \cdot \left[1 + r_{j+1}\left(\sigma_j\right)\right]}{\phi_1(j+1)}\right]} = 0$$

along with the initial boundary condition, $q_{t-1,t} = 0$, at date t (i.e., today). (40) restricts the space of admissible *policy functions* chosen at time t so that $c_{2,T+1}\left(\sigma_{T+1}\right)$ is feasible as $T \to \infty$.

The collective budget constraint in (40) reflects time-t completeness of the ITE with linear securities: an exhaustion of trades between all future generations s and $s + 1$ ($s > t$) at time t implies that trades between generation s and $s+j$ ($j > 1$) at time t are also exhausted.[15] But (40) must be interpreted with care: agents are "connected" in the ITE but they are not altruistic. For example, an inter-generational lump-sum transfer without a corresponding change in risk sharing is *not* neutral.

Generation s's first-order conditions are:

(41) $$E_t\left[u'(c_{1,s}(\sigma_s))\right] = \beta \cdot E_t\left\{E_s\left[u'(c_{2,s+1}(\sigma_s)) \cdot \left(1 + e_{s+1} - (e_{s+1} - r_{s+1})\tau^K_{s+1}\right)\right]\right\}$$

(42) $$E_t\left[u'(c_{1,s}(\sigma_s))\right] = \beta \cdot E_t\left\{E_s\left[u'(c_{2,s+1}(\sigma_s)) \cdot (1 + r_{s+1})\right]\right\}$$

(43) $$E_t\left[u'(c_{1,s}(\sigma_s)) \cdot p_{s-1,s}(\sigma_{s-1})\right] = E_t\left[u'(c_{1,t+1}(\sigma_{t+1})) \cdot (e_{t+1} - r_{t+1})\right]$$

(44) $$E_t\left\{E_s\left[u'(c_{2,s+1}(\sigma_s))\right] \cdot p_{s,s+1}(\sigma_s)\right\} = E_t\left\{E_s\left[u'(c_{2,s+1}(\sigma_s)) \cdot (e_{s+1} - r_{s+1})\right]\right\}$$

Equations (41), (42), (43) and (44) are generation s's first-order conditions for their state-contingent demand for capital (s^K_s), bonds (s^B_s), inter-generational contract with generation s-1 ($q_{s,s-1}$), and inter-generational contract with generation s+1 ($q_{s,s+1}$), respectively.

<u>Lemma 5.</u> $p_{s,s+1}(\sigma_s) = 0 \;\; \forall\, \sigma_s \in Z_{s|t}(\sigma_t)$ and $\forall\, s \geq t$.

To prove Lemma 5, equations (41) and (42) imply $E_t\left\{E_s\left[u'(c_{2,s+1}(\sigma_s)) \cdot (e_{s+1} - r_{s+1})\right]\right\} = 0$ and so equation (44) implies $p_{s,s+1}(\sigma_s) = 0 \;\; \forall\, \sigma_s \in Z_{s|t}(\sigma_t)$ and $\forall\, s \geq t$. Q.E.D.

Intuitively, similar to before, the zero price for the inter-generational contract rules out arbitrage opportunities, although now on a state-contingent basis. Also, like before, the Replicating Tax Rate at time s+1, therefore, is determined by the sign of $q_{s+1,s}$, or $sign\left\{E_t\left[u'(c_{1,s+1}(\sigma_{s+1})) \cdot (e_{s+1} - r_{s+1})\right]\right\}$. The next proposition follows immediately from these facts.

[15] This result is very similar to the equivalence of consumption allocations that exists between the traditional Arrow-Debreu structure in which all contingent claims are traded at date 0 and the sequential (recursive) economy with one-period Arrow securities. See, for example, Ljungqvist and Sargent (2000).

Proposition 4. *If only depreciation is stochastic, the time-(s+1) Replicating Tax Rate (RTR) is positive at every state at time s. If only productivity is stochastic, the RTR is negative in every state at time s for log utility, zero net debt, C-D production, and not full depreciation (δ<1).*

Intuitively, the fact that generation s+1 now shares risk with generation s+2 does not change the *qualitative* nature of its risk sharing with generation s -- i.e., the *sign* of the RTR – with time-separable utility. Specifically, if only depreciation is stochastic, then for each state at time s, generation s+1 still wants positive exposure to asset returns faced by generation s since those returns are orthogonal to generation s+1's other risks, including its risk sharing with generation s+2. Similarly, with only stochastic productivity, a negative capital income tax still allows generation s+1 to hedge its first-period wage income since wages at time s+1 are perfectly correlated with capital income at time s+1. Since wages at time s+1 are uncorrelated with stock returns at time s+2 (and, hence, the wages of generation s+2), risk sharing with generation s+2 does not alter this fact.

VI. Efficiency

It has been known since Hart (1975) that completing a missing market does not necessarily improve pareto efficiency with endogenous prices. But it is generally difficult to derive closed-form expressions for efficiency gains even for an endowment economy unless preferences are restricted to the CARA form (Willen, 2002); see also Demange and Laroque (1995). This section demonstrates an efficiency gain for our inter-generational q-security in a production economy for the special case in which we can sign the RTR as negative.[16] Simulation evidence is used for a more general setting.

[16] One reason that we can prove efficiency gains is that the "heterogeneity" herein stems only from having young and old agents alive at the same time, where old agents have already selected their portfolios. In contrast, for example, in Willen's paper, all two-period agents are young but differ in earnings and preferences. Including that heterogeneity herein would also prevent us from showing efficiency gains; we instead focus on inter-generational risk sharing.

Perfect Correlation Between Wage and Equity Returns: An Analytical Result

Recall, that in the case of log utility, $u(c) = \ln c$, and zero net bonds, the policy functions for capital saving and first-period consumption by generation t are $s_{1,t}^{K}(w_t) = \frac{\beta \cdot w_t}{1+\beta}$ and $c_{1,t}(w_t) = \frac{w_t}{1+\beta}$, respectively. Notice, in particular, that neither policy function depends on either the shadow risk-free rate, r_{t+1}, nor on the equity return distribution, $\Xi(\bar{e}; A_t, \lambda, k_{t+1}, \delta_t, D_t)$. The reasons stems from the well-know property of log utility: the income and substitution effects exactly cancel. As a result, a change in the Replicating Tax Rate at time t+1 does not impact the capital intensity at time t+1. It, therefore, follows that, in the presence of only productivity shocks, a negative Domar-Musgrave capital income tax at time t+1 is not only desired by generation t+1 (as shown earlier), a negative tax also increases their expected utility at time t without changing generation t's utility.

While this result is one of the first derivations of efficiency gains associated with completing a missing market in a production economy with an endogenous equity return distribution, it is still quite limited. First, we have only shown that a negative capital income tax rate is weakly preferred to a zero rate; we have not determined its size. Second, we required log utility, zero net bonds, and non-stochastic depreciation. We now relax these assumptions. We are now also able to calculate the expected utility of generation t+s ($s \geq 2$) as of time t instead of conditional on the state at time t+s-1.

Imperfect Correlation Between Wage and Equity Returns: Numerical Computations

The model extensions considered in the rest of Section VI include: (i) stochastic productivity *and* depreciation to allow for an imperfect correlation between wages and capital returns; (ii) CRRA utility function with calibrated preferences; (iii) a positive level of government debt; and (iv) a social security system for the sake of generality: a fraction φ of payroll taxes is deposited into a trust fund; the other (1-φ) fraction pays a stochastic wage-indexed pay-as-you-go benefit (the formulae is reported in Smetters, 2001). Sensitivity analysis is also performed.

Each simulation reported below can take up to 20 hours to solve, and so we continue to assume two periods. A many-period production OLG model with *aggregate* uncertainty is difficult to solve due to Bellman's "curse of dimensionality." Two periods allow for exact solutions without resorting to approximations. Any bias in assuming two periods is unclear. More periods could increase the (positive or negative) magnitude of the efficient tax rate: a given tax is less effective at inter-generational risk sharing with more periods.[17] But more periods could decrease the magnitude: with more periods, agents can already risk share with younger living agents. Future computational approximations might provide useful insights (Krueger and Kubler, 2001).

Still, the model herein is more advanced than previous. First, we believe it is the first model to incorporate an exact and endogenous neoclassical equity return distribution; other endogenous variables include capital saving, portfolio choice, risk-free rate, and state-contingent fiscal policies. Second, we also believe the model is the first with aggregate uncertainty to report the exact welfare gain for each generation on the transition path with each calculation measured at the reform date.

<u>Benchmark Calibration</u>

Utility takes the CRRA form, $E_t U_t = \frac{1}{1-\gamma}\left[c_{1,t}^{1-\gamma} + \beta E_t(c_{2,t+1}^{1-\gamma})\right]$, where γ is the level of risk aversion and $\beta = 1/(1+\rho)$ where ρ is time preference. Productivity is a two-state Markov process, $A_t = A_{t-1}\cdot(1+\lambda)\cdot(1+\tilde{a}_t)$, where λ is trend growth and $\tilde{a}_t$ is a mean-zero stochastic shock, $\tilde{a}_t \in \{\chi,-\chi\}$, that can take values χ and $(-\chi)$ with equal probability. Depreciation is stochastic, $\delta = \hat{\delta} + \varepsilon$, with $\varepsilon \in \{\xi,-\xi\}$. The initial economy at time 0 (the reform date) is summarized in Table 1 and explained in the Appendix. Each period represents 30 years. The correlation between pre-tax wage and capital returns is ¾ under the benchmark. The effective tax rate on capital income

[17] The RTR with ∞ horizons is *indeterminate*, not zero (extended Appendix available from author).

is set at 20 percent, based on Auerbach (1996). The vector, $\{k_0, A_0, \delta_0, \lambda, \gamma, \beta, \chi, \xi, \varphi\}$, generating the initial economy is discussed in the Appendix. In the sensitivity analysis below, this vector is re-calculated to generate the same initial economy shown in Table 1 unless specified otherwise.

The Effect of Tax Changes on Macroeconomic Variables

Table 2 reports simulation results along the "mean growth path"[18] where state variables are updated across generations conditional on productivity and depreciation shocks taking their expected values *ex-post*. The table reports the results for two large experiments: reducing the capital income tax from 20 percent to zero as well as doubling its value from 20 percent to 40 percent. Generation 0 are first-period workers alive at the reform, generation 1 are their children born at time 1, etc..

Notice that eliminating capital income taxes reduces the long-run capital stock by 11 percent and output by 3½ percent. The risk-free rate decreases by 20 basis points, the expected return to equities increases by 30 basis points and so the equity premium increases by 50 basis points. These macro changes have two complementary sources. First, smaller capital income taxes reduces risk sharing across generations. By Lemma 1, investors, therefore, shift toward bonds and away from capital, thereby reducing interest rates, increasing equity returns and reducing output. Second, along the mean path, less capital income tax revenue collected from generation 0 requires larger wage tax rates to be levied on future generations; this negative wealth effect reduces long-run capital saving.

Notice that, in contrast, doubling capital income taxes increases the long-run capital stock by 25½ percent and output by 6½ percent. The risk-free rate increases by 180 basis points, the expected return to equities decreases by 60 basis points and the equity premium decreases by 240 basis points. However, the speed of convergence is a little slower for this policy change.

---

[18] This term is analogous to the "constant productivity growth path" terminology used in the real business cycle model literature. The more general terminology used herein is needed since depreciation is also stochastic.

Calculating Welfare Gains

While the real business cycle literature focuses on mean growth paths like in Table 2 herein, that approach does not capture risk. Along the mean path, generation $t+1$ is unexposed to fiscal policy risk created by generation t. The welfare measure in this subsection incorporates that risk.

Let a single asterisk (*) denote values before a policy reform. Variables with two asterisks (**) denote values after a policy reform. The *pre-reform* indirect utility of generation s, as calculated at the time of the reform ($t = 0$), is,

(45) $$V^{*}_{s|t}\left(\sigma_{t=0}\right) \equiv \max_{s_s^B, s_s^K, q_{s,s-1}, q_{s,s+1}} E_t\left\{\left[E_s U\left(c_{1,s}\left(\sigma_s, w_s(\sigma_s)\right), c_{2,s+1}(\sigma_s)\right)\right] \middle| Z^{*}_{s|t}\left(\sigma_t\right)\right\}$$

where $w(\cdot)$ is defined by equation (15). Similarly, the *post-reform* indirect utility at $t = 0$ is

(46) $$V^{**}_{s|t}\left(\sigma_{t=0}\right) \equiv \max_{s_s^B, s_s^K, q_{s,s-1}, q_{s,s+1}} E_t\left\{\left[E_s U\left(c_{1,s}\left(\sigma_s, w_s(\sigma_s)\right), c_{2,s+1}(\sigma_s)\right)\right] \middle| Z^{**}_{s|t}\left(\sigma_t\right)\right\}$$

Define the variable $\mu_{s|t}$ as follows:

(47) $$\mu_{s|t} \equiv \mu_{s|t}\left(\sigma_t\right) \equiv \left[\frac{V^{**}_{s|t}(\sigma_t)}{V^{*}_{s|t}(\sigma_t)}\right]^{\frac{1}{1-\gamma}}$$

The following equality can be shown to hold:

(48) $$V^{**}_{s|t}\left(\sigma_{t=0}\right) \equiv \max_{s_s^B, s_s^K, q_{s,s-1}, q_{s,s+1}} E_t\left\{\left[E_s U\left(c_{1,s}\left(\sigma_s, \mu_{s|t} \cdot w_s(\sigma_s)\right), c_{2,s+1}(\sigma_s)\right)\right] \middle| Z^{*}_{s|t}\left(\sigma_t\right)\right\}$$

Notice that V has two asterisks, indicating post-reform expected utility, while the Z term has a single asterisk, indicating the pre-reform state space. Also note the multiplier $\mu_{s|t}$ on wages at time s.

In words, the expected utility of generation s after the policy change equals what their expected utility would be with no policy change if, instead, each possible wage at time s, measured at t ($t \leq s$), is multiplied by $\mu_{s|t}$. When $s = t$, the time-s state is known and $\mu_{s|t}$, therefore, gives the usual Equivalent Variation measure used in single-state models (e.g., Auerbach and Kotlikoff, 1987).

Notice that the welfare measure for each generation – including those born on the transition path and in each steady state – is calculated at the time of the reform, time $t = 0$. As a result, generation $s > 0$ born further into the future faces more uncertainty than a generation $\tilde{s}$ born closer to the policy-reform date, where $s > \tilde{s} > 0$. The nonlinear forms for utility and technology require simulating each path that the economy can take between times t and when each generation s is born, with each path weighted by its probability of occurring. In particular, for discrete Markov processes,

$$
\begin{aligned}
V^*_{s|t}(\sigma_{t=0}) &\equiv \max_{s_s^B, s_s^K, q_{s,s-1}, q_{s,s+1}} E_t\left\{\left[E_s U(c_{1,s}(\sigma_s), c_{2,s+1}(\sigma_s))\right] \middle| Z^*_{s|t}(\sigma_t)\right\} \\
&= \sum_{\sigma_{t+1}\in Z^*_{t+1|t}(\sigma_t)} \; \sum_{\sigma_{t+2}\in Z^*_{t+2|t+1}(\sigma_{t+1})} \cdots \sum_{\sigma_s \in Z^*_{s|s-1}(\sigma_{s-1})} \\
&\quad \left\{ \Pi\left(\sigma_{t+1}, \sigma_{t+2}, \ldots, \sigma_s \middle| \sigma_t\right) \cdot \max_{s_s^B, s_s^K, q_{s,s-1}, q_{s,s+1}} E_s\left[U\left(c_{1,s}\left[\sigma_s, w(\sigma_s)\right], c_{2,s+1}\left[\sigma_s\right]\right)\right]\right\}
\end{aligned} \tag{49}
$$

where $\Pi\left(\{\sigma_i\}_{i=t+1}^{s+1} \middle| \sigma_{t=0}\right)$ is the joint probability of the sequence $\{\sigma_i\}_{i=t+1}^{s+1}$, given the state of the economy at time $t = 0$. (The calculation of V^{**} is similar except with Z^{**} state spaces.) The total number of paths equal $4^{(s-t)}$ for the two-state discrete processes for productivity and depreciation outlined above. So, for example, calculating $V_{5|0}$, the expected utility for the cohort born five generations (150 years) after time 0, requires simulating 1024 (= 4^5) general-equilibrium paths.

The different paths are calculated recursively along a dense lattice structure shown in Figure 2. Each node on the lattice satisfies the conditions for a general equilibrium in the MTE shown in Section III (Social Security adds more conditions). The equilibrium state vector σ calculated at each "parent node" is passed to four (two × two-state Markov processes) "child nodes" where each child represents a possible state the economy can take the next period, conditional on being at that parent. Recursion stops when $\left|\mu_{s+1|t} - \mu_{s|t}\right| < \varepsilon$, where ε is small, which occurs after about five generations (150 years) for the simulations herein. The corresponding lattice contains 5,461 MTE nodes.

Simulation Results

Table 3 reports the percent change in welfare, $[(\mu_{s|t=0} - 1) \cdot 100\%]$, for generations $s \in \{0, 1, ..., 5\}$ corresponding to the two tax reforms considered in Table 2. Recall that generation 0 is a first-period worker alive at the reform time 0, generation 1 is their future children, followed by generation 2, etc. The change in welfare for generation (-1) agents, retired at the time of the reform, is always zero since their portfolio decisions and returns are known at the time of the policy change.

Table 3 shows that removing the capital income tax at t=1 reduces the welfare of future generations by 8½ percent. Convergence is fairly quick. Notice, though, that doubling the tax rate to 40 percent also lowers future welfare, now by 2 percent. Convergence, though, is slower. In other words, these opposite experiments both serve to reduce the welfare of future generations.

To see why, recall that a positive capital income tax gives future generations exposure to previous stock returns through the government's budget constraint: future *expected* after-tax wages increase as do their *volatilities*. Evidentially, even at a large ¾ correlation between pre-tax wages and capital income, wages and capital income are still not correlated enough to warrant a zero tax rate. The reason is that, at a small positive tax rate on capital income, future welfare is influenced more by the expected increase in after-tax wages than by the increased volatility. But, at a large tax rate, future generations are exposed enough to past stock returns and so volatility becomes more important. In sum, future generations don't want a tax rate that is either too low or too high.

Table 3 also shows that increasing the capital income tax rate to 30 percent gets this tradeoff just right. Not only are future generations better off, so is generation 0 – a pareto improvement. To see why, recall that at fixed prices, generation 0 could perfect offset a larger tax by holding more stocks and less bonds (Lemma 1). In general equilibrium, though, this portfolio shift increases the risk-free rate, thereby increasing generation 0's risk-adjusted return to saving.

Sensitivity Analysis

Larger Correlation. When the model is re-calibrated so that the correlation between wage and capital returns along the mean growth path is increased from 0.75 to 0.80 (not shown), the tax rate that maximizes long-run welfare is closer to the 20 percent tax rate used under our baseline rather than 30 percent. This result is robust to many model and parameter changes including: (i) government spending in the utility function;[19] (ii) a reasonable range of values for the risk-free rate and equity premium along the mean growth path; and (iii) starting the economy from a zero tax rate and increasing it to 20 and then 40 percent.[20] Interestingly, 0.80 is almost exactly the point estimate between wage and stock returns for 30-year moving averages since 1929, and so the current effective tax rate in the U.S. appears to maximize long-run expected utility at reasonable parameter values.

The 0.80 correlation estimate, though, is based on two and half unique 30-year periods, and so the standard errors are large; in particular, a correlation of unity cannot be rejected. Table 4, therefore, also reports sensitivity analyses in which the correlation between wages and capital income along the mean growth path is set at unity (i.e., only productivity shocks are operative). Eliminating capital income taxes now appears to be *nearly* pareto improving. The welfare gain to generation 0 is negative due to a smaller risk-free rate but the change in their welfare is so small that it is rounded in Table 4 to zero (i.e., < -0.05 percent). But the welfare gains to all future workers are quite large and equal to one percent. In fact, Table 4 shows that setting the tax rate equal to negative 20 percent improves the welfare of all future generations by even more (1½ percent), with very little impact on current workers (-0.1 percent). Long-run welfare is, in fact, maximized at about - 20 percent. This

[19] Utility with government consumption, G, is $\left[(c_{1,t})^{1-\gamma} + \beta_G (G_t)^{1-\gamma} + \beta E_t\left[c_{2,t+1}\right]^{1-\gamma}\right] / (1-\gamma)$ where β_G is set so that the marginal utility of government spending equals the marginal utility of first-period consumption (and, hence, the expected marginal utility of second-period consumption) calibrated for generation-0 agents: $\beta_G \equiv \left(G_0^* / c_{1,0}^*\right)^{\gamma}$.

[20] The model is always re-calibrated to hit the other aforementioned targets.

result shows that relaxing the assumptions of log utility and zero net bonds does little to undermine our earlier analytical case for a negative tax rate when only productivity shocks are operative.

Relaxing the Domar-Musgrave Assumptions. While this paper focuses on how the government can use a Domar-Musgrave type of capital income tax to replicate inter-generational trading in linear securities, we now examine the importance of the Domar-Musgrave assumptions themselves. This analysis, therefore, addresses the following question: can a nation use its existing capital income tax system to replicate missing inter-generational markets even if its tax system does not allow for symmetry nor excludes the risk-free component? Strictly speaking, the answer, of course, is no. But it is interesting to know how far away – numerically, that is – the economy gets.

Table 5, therefore, reports the results for doubling and eliminating the capital income tax when the Domar-Musgrave conditions are relaxed.[21] In particular, the capital income tax is now levied over the *entire* return to equities, e, and not just $e - r$. Moreover, only the maximum of e and zero, $max(e, 0)$, is now taxed, i.e., no tax symmetry. The wage-equity correlation is again set at ¾. Re-calibration ensures that the initial economy is consistent with Table 1.

Table 5 shows that eliminating the capital income tax still reduces the welfare of future generations, now by about 7½ percent. But doubling the tax now improves their welfare by 5½ percent. Rates too much higher than 40 percent, though, lead to smaller (possibly negative) gains. Hence, abandoning the Domar-Musgrave conditions *seems* to only strengthen our previous case for a positive tax rate at a ¾ correlation. But the underlying mechanisms are now quite different.

Specifically, there two opposing forces at work. On one hand, by taxing the full equity return, and then only if it exceeds zero, the "portfolio effect" outlined in Lemma 1 is undermined. To investigate, the partial-equilibrium response of generation-0 agents was computed by holding the

[21] When these conditions are violated, the effect of a capital income tax on portfolio and saving choices is ambiguous, even in a simple one-period static model, due to competing income and substitution effects (Sandmo, 1985).

return distributions fixed at pre-reform levels. When the risk-free component is taxed but symmetry is still allowed, doubling the capital income tax increased desired capital saving by just 7½ percent, compared to 33⅓ percent before.[22] But when tax symmetry was also abandoned, capital saving *fell* by 21½ percent. This capital reduction leads to lower pre-tax wages for future generations.

On the other hand, when the capital income tax rate is increased, future workers no longer have to subsidize very low capital returns when the Domar-Musgrave assumptions are not in place. As a result, under the government's budget constraint, an increase in capital income taxes redistributes resources from generation 0 to future workers under more states of the world. While capital income taxes still share capital risk over generations, it is substantially less than before.

The second force clearly dominates – but this result must be interpreted carefully. Notice that in sharp contrast to Table 3, generation 0 is now *worse* off after the capital income tax is increased. In fact, Table 5 shows that generation 0's welfare decreases by close to 3 percent. As a result, we can no longer claim that a positive tax rate produces efficiency gains under our benchmark setting. This analysis, therefore, highlights the importance of including transitional generations in the analysis. Had we compared only long-run steady state welfare between Tables 3 and 5, we could have falsely concluded that the Domar-Musgrave assumptions were not important in the analysis.

VII. Conclusions

This paper started with a thought experiment: suppose that living generations could trade risk sharing contracts with the next unborn generation. What would these trades look like? The paper then demonstrated that the government can use fiscal policy to replicate trades in linear securities using a Domar-Musgrave (1944) capital income tax. The corresponding Replicating Tax Rate (RTR)

---

[22] The 33⅓ percent change is consistent with the theory in Section II, i.e., $\left[(1-\tau_{t+1}^{K^*}) / (1-\tau_{t+1}^{K^{**}}) \right] = 0.80/0.60 = 1\frac{1}{3}$.

is positive if depreciation is the only source of uncertainty (i.e., no wage - capital return correlation). In the more complicated case where only productivity is uncertainty (i.e., perfect correlation), the RTR's sign is theoretically ambiguous. But for Cobb-Douglas production, log utility, and incomplete depreciation, the sign is negative. These results generalize to sequential trading.

However, completing a missing market is not necessarily pareto improving when price effects are considered. But in the special case discussed above in which the RTR can be unambiguously signed as negative, it was also shown to produce efficiency gains. While this result is one of the first derivations of efficiency gains associated with completing a missing market in a production economy with an endogenous equity return distribution, this result is still limited.

Calibrated simulation evidence, using a fairly detailed general-equilibrium model, was needed to extend beyond this simple setting. After the value of the RTR was changed, welfare values for transitional and steady-state generations were computed recursively using a measure that accommodates a rapidly-expanding state space. Under our benchmark calibration, not only is the efficient capital income tax rate likely to be positive, increasing its value above the current benchmark level for the U.S. could improve the welfare of all generations. But this conclusion is very sensitive to the chosen 30-year correlation between wage and capital income returns. When this correlation is increased from ¾ to unity, shifting to a negative capital income tax rate would likely improve efficiency. These results, however, assumed that the Domar-Musgrave conditions (tax symmetry and the non-taxation of the risk-free investment component) hold. As noted herein, there are some reasons to believe that these conditions describe the U.S. tax system fairly well. But to the extent that these conditions are not part of a country's tax code, they would have to be implemented as part of a tax system that attempts to replicate trades in linear securities between generations.

VIII. Appendix

Derivation of Equation (26')

Equations (26) and (25) imply:

$$\beta E\left\{\frac{u'(c_{2,t+1})}{u'(c_{1,t})}\left[1+e_{t+1}-(e_{t+1}-r_{t+1})\tau^K_{t+1}\right]\right\}=1=\beta E\left[\frac{u'(c_{2,t+1})}{u'(c_{1,t})}\left(1+r_{t+1}\right)\right] \text{ by (25)}$$

$$=> \qquad \beta E\left\{\frac{u'(c_{2,t+1})}{u'(c_{1,t})}\left[1+e_{t+1}-(1+r_{t+1})-(e_{t+1}-r_{t+1})\tau^K_{t+1}\right]\right\}=0$$

$$=> \qquad \beta E\left\{\frac{u'(c_{2,t+1})}{u'(c_{1,t})}\left[(1-\tau^K_{t+1})\cdot(e_{t+1}-r_{t+1})\right]\right\}=0$$

$$=> \qquad \beta E\left\{\frac{u'(c_{2,t+1})}{u'(c_{1,t})}\left[(1+e_{t+1})-(1+r_{t+1})\right]\right\}=0$$

$$=> \qquad \beta E\left\{\frac{u'(c_{2,t+1})}{u'(c_{1,t})}(1+e_{t+1})\right\}-1=0 \qquad \text{by (25).}$$

Proof of Lemma 4[23]

The inequality shown in Lemma 4 clearly holds with equality if x is non-random. If x is random then let $\varsigma \equiv (1/x)$ define the function $g(\varsigma) \equiv \varsigma/(1+\varsigma)$. The expression becomes $E(\varsigma)\geq E\left(\frac{\varsigma}{1+\varsigma}\right)\cdot\left[1+E(\varsigma)\right]$, or, $\frac{E(\varsigma)}{\left[1+E(\varsigma)\right]}\geq E\left(\frac{\varsigma}{1+\varsigma}\right)$, which holds with strict inequality by Jensen's inequality since g is concave. Q.E.D.

Calibration of the Benchmark Economy (Table 1)

The expected annual depreciation equals 5 percent so that 79 percent of the capital stock is *expected* to be depreciated by the end of a 30-year period. The capital share, α, is set at 0.30. The

[23] Thanks to Greg Nini for this succinct proof.

arbitrary scaling parameter A_0 equals unity. Based on Poterba (1998) and Ibbotson data, the annual pre-tax (social) real rate of return to capital equals 8½ percent per year, or 1,056 percent over 30 years, with a coefficient of variation equal to 0.87. The annual risk-free real return, r_1, equals 3 percent, or 143 percent over 30 years, based on historic returns to long-term government securities this century.[24] The annual expected rate of labor-augmenting technological progress is set at 3 percent per year, the average growth rate of the total salaries and wage base since 1929, based on Bureau of Economic Analysis data. The point-estimate correlation between wage and stock returns at a 30-year frequency is about three-quarters. The defended debt-capital ratio, $\bar{\bar{d}}$, is set at 0.25, close to the current ratio of government debt relative to the domestically-owned capital stock as measured in the Federal Reserve Board's Flow of Funds Accounts. The initial tax rate on the generation-0 agent's second-period capital income, τ_1^K, is set at 0.20, following the careful calculations found in Auerbach (1996). The initial proportional tax rate on wage income, τ_0^W, is set at 0.15 which generates a plausible level of tax revenue derived from wages. The Social Security payroll tax is set at 12 percent and the estimated ratio of contributions to the Social Security trust fund divided by benefits paid during the past and next few decades equals about 4 percent.

Calibrating the model involves "inverting" many of the equations presented in Section II to express the parameter vector $\{k_0, A_0, \delta_0, \lambda, \gamma, \beta, \chi, \xi, \varphi\}$ as a function of the economic variables to be targeted at time 0. The resulting parameter vector is unique. The calibrating vector needed to generate the baseline economy described in the previous paragraph is $\{k_0, A_0, \delta_0, \lambda, \gamma, \beta, \chi, \xi, \varphi\}$ = $\{.0056, 1.0, 0.79, 0.860, 0.857, 0.27, 0.61, 6.07, 0.04\}$. The value $\beta = 0.27$ corresponds to an *annual* rate of time preference equal to 4.4 percent. The value of $\gamma = 0.857$ reflects several factors: scaling

[24] The exact choice of the risk-free rate is not so important if the model is properly recalibrated each time. A relatively higher return corresponding to long-term debt, versus short-term debt, is used under our benchmark to avoid potential criticism in the sensitivity analysis where the Domar-Musgrave assumptions are relaxed.

the model to equity returns (rather than consumption data); human capital depreciation in the second period; and a three-quarters correlation of wage-indexed pay-as-you-go Social Security returns with stock returns. The simultaneous equation set outlined in Section II is solved using a Jacobian-based generalized Newton method. Additional details are available from the author.

The model calibration generates additional plausible economic relationships (Table 1). The implied net national saving rate equals a realistic 4.4 percent. The non-Social Security part of government spending equals 15.3 percent which is very close to the value of 15½ percent that the CBO (1999) reports for 1998. Capital income tax revenue equals 4.4 percent of GDP while wage income taxes, not including Social Security payroll taxes, compose 10½ percent of GDP.

References

Allen, Franklin and Douglas Gale. "Financial Markets, Intermediaries, and Intertemporal Smoothing." *Journal of Political Economy*; 105(3), June 1997, pages 523-46.

Auerbach, Alan J., "Tax Reform, Capital Accumulation, Efficiency, and Growth," in Henry J. Aaron and William G. Gale, eds., Economic Effects of Fundamental Tax Reform, The Brookings Institution: Washington, D.C., 1996, 29-81.

Auerbach, Alan and Laurence Kotlikoff. Dynamic Fiscal Policy, Cambridge: Cambridge University Press, 1987.

Ball, Laurence and N. Gregory Mankiw. "Intergenerational Risk Sharing in the Spirit of Arrow, Debreu, and Rawls, with Applications to Social Security Design." NBER WP #8270, 2001.

Bohn, Henning. "Should the Social Security Trust Fund hold Equities? An Intergenerational Welfare Analysis." *Review of Economic Dynamics*, vol.2, no.3, July 1999, 666-697.

Bradford, David. "Consumption Taxes: Some Fundamental Transition Issues." NBER Working Paper No. 5290, 1995.

Congressional Budget Office. "The Economic and Budget Outlook: An Update" July 1, 1999.

Demange, Gabrielle and Guy Laroque. "Optimality of Incomplete Markets." *Journal of Economic Theory*, 65, 1995: 218 - 232.

Domar, E. and R. Musgrave. "Proportional Income Taxation and Risk Sharing." *Quarterly Journal of Economics*, 1944, 58, 388 - 422.

Gale, Douglas. "The Efficient Design on Public Debt." in R. Dornbusch and M. Draghi, Public Debt Management: Theory and History, Cambridge University Press, 1990.

Gordon, Roger. "Taxation of Corporate Capital Income: Tax Revenues versus Tax Distortions." *Quarterly Journal of Economics*, 100, 1985, 1 - 27.

Hart, Oliver. "On the Optimality of Equilibrium when the Market Structure in Incomplete." *Journal of Economic Theory*, 1975, 11: 418-435.

Hubbard, R. Glenn and William Gentry. "Fundamental Tax Reform and Corporate Financial Policy," in James M. Poterba, ed., Tax Policy and the Economy, Volume 12.

Krueger, Dirk and Felix Kubler. "Intergenerational Risk Sharing via Social Security when Financial Markets are Incomplete." Mimeo, Stanford University.

Ljungqvist, Lars and Thomas J. Sargent. Recursive Macroeconomic Theory, MIT Press, 2000.

Mas-Colell, Andreu, Michael D. Whinston, and Jerry R. Green. Microeconomic Theory, Oxford

University Press: New York, 1995.

Mossin, J. "Taxation and Risk-Taking: An Expected Utility Approach," *Economica*, 1968, 35, 74 - 82.

Poterba, James. "The Rate of Return to Corporate Capital and Factor Shares: New Estimates Using Revised National Income Accounts and Capital Stock Data." NBER, #6263, 1998.

Rangel, Antonio and Richard Zeckhauser. "Can Market and Voting Institutions Generate Optimal Intergenerational Risk Sharing?" in J. Campbell and M. Feldstein, eds., Risk Aspects of Investment Based Social Security Reform, Chicago University Press, 2001.

Rangel, Antonio. "Forward and Backward Intergenerational Goods: Why is Social Security Good for the Environment?" NBER WP #7518, 2000. Forthcoming in American Economic Review.

Sandmo, Agnar. "Capital Risk, Consumption, and Portfolio Choice," *Econometrica*, 1969, 37, 586-599.

Sandmo, Agnar. "Portfolio Theory, Asset Demand and Taxation: Comparative Statics with Many Assets," *Review of Economic Studies*, 1977, 44, 369 - 379.

Sandmo, Agnar. "The Effects of Taxation on Savings and Risk Taking." in A.J. Auerbach and M. Feldstein, eds., Handbook of Public Economics, Vol I., 1985, 265 - 311.

Smetters, Kent. "The Equivalence of the Social Security Trust Fund's Portfolio Allocation and Capital Income Tax Policy." NBER Working Paper #8259, 2001.

Stiglitz, Joseph. "The Effects of Income, Wealth and Capital Gains Taxation on Risk-Taking." *Quarterly Journal of Economics*, 1969, 83, 262 - 283.

Wang, Y. "Stationary Equilibria in an Overlapping Generations Economy with Stochastic Production." *Journal of Economic Theory*, 61, 1993, 423 - 435.

Willen, Paul. "New Financial Markets: Who Gains and Who Loses." Mimeo, GSB, U. of Chicago.

Zodrow, George. "Taxation, Uncertainty and the Choice of a Consumption Tax Base." *Journal of Public Economics*, 1995, 58: 257 - 265.

Figure 1
The Division of National Income in an Overlapping-Generations Economy

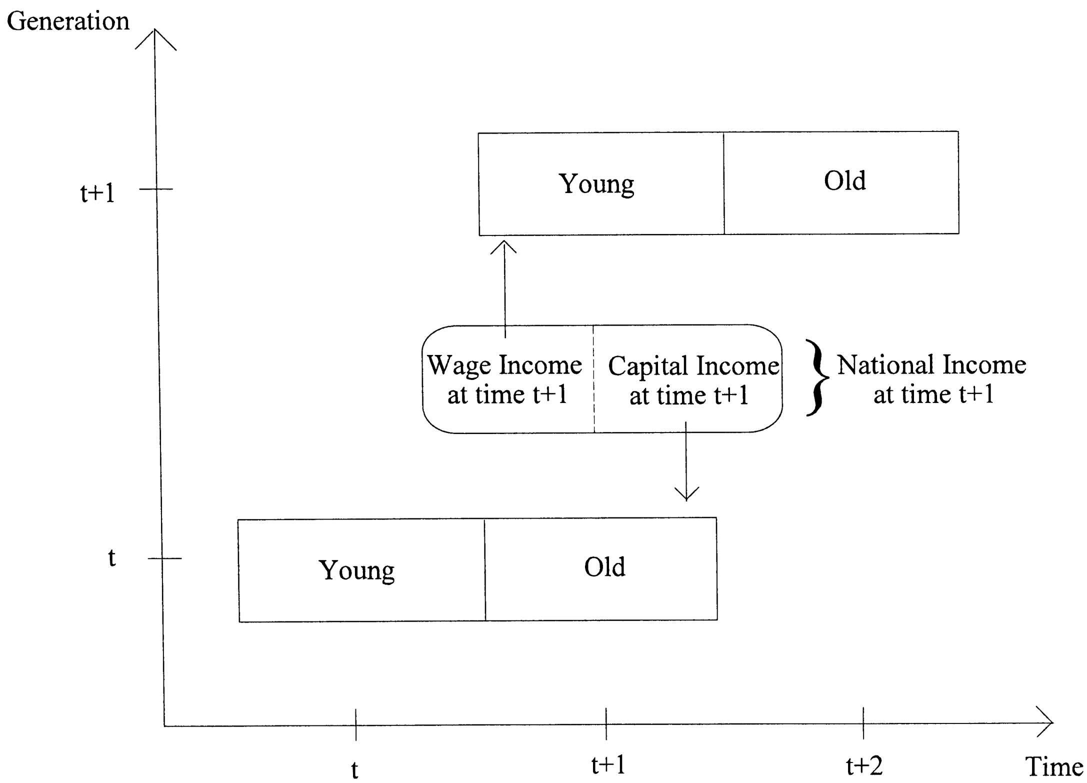


Table 1
Parameters and Implied Values

Variable Description	Value
Exogenous Parameters (same in all simulations, unless indicated otherwise)	
Average annual depreciation rate, $\bar{\hat{\delta}}_{annual}$	5 %
Capital share, α	0.30
Arbitrary Scaling of the Initial Productivity, A_0	1.00
Pre-tax 30-year return to equities on mean path, $\overline{E(e_1 \vert k_1)}$ (Corresponding annual return)	1,056 % (8.5 %)
Coefficient of Variation, $\bar{\sigma}_e / \overline{E(e_1 \vert k_1)}$	0.87
Pre-tax 30-year risk-free real return on mean path, r_1 (Corresponding annual return)	143 % (3 %)
Rate of 30-year labor-augmenting tech. progress on mean path (Corresponding annual return)	143 % (3 %)
Debt-capital ratio, $\bar{\tilde{d}}$	25 %
Tax rate on capital income on constant growth path, τ_1^K	20 %
Social Security pay-as-you-go liabilities tax rate, $\tau_{s\geq 0}^{SS,P}$	11.5 %
Social Security funded portion tax rate, $\tau_{s\geq 0}^{SS,F}$	0.5 %
Implied Endogenous Variables (same in all simulations, unless indicated otherwise)	
Net national saving rate	4.4 %
"On Budget" Spending as a fraction of GDP on mean path, $G_0/[A_0 k_0^{\alpha}]$	15.3 %
Capital income tax revenue as a fraction of GDP on mean path	4.8 %
Non-Social Security wage income tax revenue as a fraction of GDP on mean path	10.5 %
Exogenous Parameter (only for the benchmark)	
Correlation between capital income returns and wages on mean path	0.75

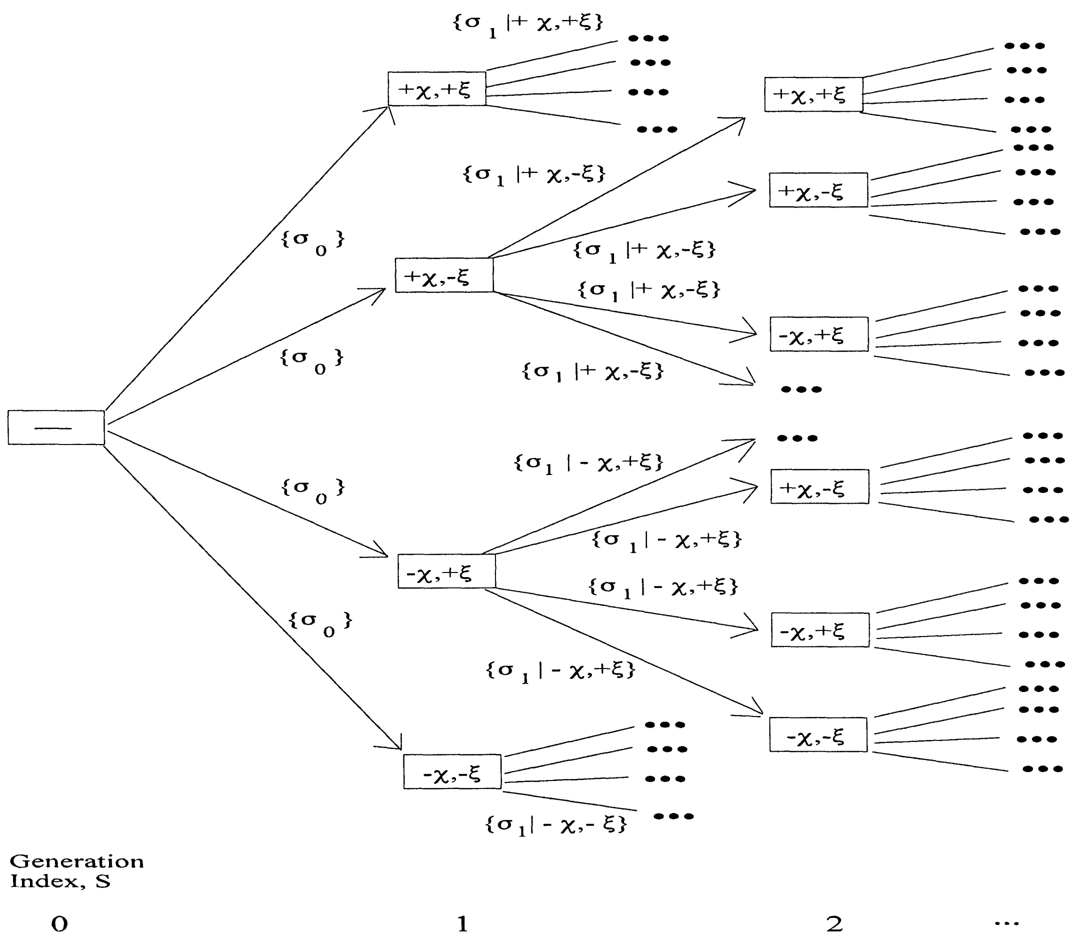
Table 2
Eliminating / Doubling Capital Income Tax Rates:

Changes in Macroeconomic Variables Along the Mean Growth Path[1,2]

Generation Index[3]	Percent Changes: Capital Stock	Percent Changes: Pre-tax Wages	Percent Changes: Post-tax Wages[4]	Percent Changes: National Income	Levels (in percent): Wage Tax Rates	Levels (in percent): Risk-Free Rate (Annual)	Levels (in percent): Expected Return to Equities (Annual)	Levels (in percent): Equity Premium (Annual)[5]
Decrease the Capital Income Tax from 20 to 0 Percent								
0	0.0	0.0	-0.3	0.0	15.2	3.0	8.8	5.8
1	-9.8	-3.1	-11.6	-2.9	21.4	2.0	8.8	6.8
2	-10.9	-3.4	-12.3	-3.2	21.8	2.7	8.8	6.1
3	-11.0	-3.4	-12.4	-3.3	21.8	2.8	8.8	6.0
4	-11.0	-3.4	-12.5	-3.3	21.8	2.8	8.8	6.0
5	-11.0	-3.5	-12.5	-3.3	21.8	2.8	8.8	6.0
Increase the Capital Income Tax from 20 to 40 Percent								
0	0.0	0.0	0.6	0.0	14.6	3.0	8.1	5.1
1	19.6	5.5	5.5	5.2	15.0	5.4	8.0	2.6
2	23.1	6.4	7.8	6.1	14.1	5.1	8.0	2.9
3	24.5	6.8	8.8	6.4	13.6	4.9	7.9	3.0
4	25.1	6.9	9.3	6.6	13.4	4.8	7.9	3.1
5	25.4	7.0	9.5	6.6	13.3	4.8	7.9	3.1

Notes:
1. I.e., state variables are updated between generations conditional on all shocks (both productivity and depreciation) taking their mean values *ex post*.
2. Calculations correspond to the benchmark model and calibration shown in Table 1 discussed in the Appendix.
3. Recall that each generation represents 30 years. Generation 0 is the initial young at the time of the policy change. They are allowed to re-optimize their portfolio and saving decisions in response to a policy change, including announcing a change in the capital income tax rate to be applied at time 1 during their second period of life. Generation -1 agents represent the elderly at the time of the reform and their saving and portfolio decisions and after-tax asset returns have already been determined by the time of the policy change.
4. I.e., after federal and Social Security taxes.
5. The equity premium equals 5.5 percent (annual) along the pre-reform constant growth path, reflecting a pre-reform expected return to equities of 8.5 percent (annual). The equity premium faced by generation-0 agents changes as the asset return distributions change in response to their re-optimization.

Figure 2
Lattice Representation of Two-State Discrete Markov Chains for Productivity and Depreciation [1,2]



Notes:

1. σ_s is the vector of state variables at generation s.
2. Recall that (detrended) productivity can take the shock values $\{+\chi,-\chi\}$ and depreciation can take the shock values $\{+\xi,-\xi\}$.

Table 3
Eliminating / Doubling Capital Income Tax Rates:
–
Risk-Adjusted Changes in Expected Lifetime Resources of Generation[1]

Generation Index, s	Percent Welfare Gain[2] $[(\mu_{s|t=0} - 1) \cdot 100\%]$
Decrease the Capital Income Tax from 20 to 0 Percent	
0	0.2
1	- 8.5
2	- 8.6
3	- 8.4
4	- 8.4
5	- 8.4
Increase the Capital Income Tax from 20 to 40 Percent	
0	3.5
1	- 1.6
2	- 1.3
3	- 1.5
4	- 1.9
5	- 2.1

Notes:
1. Calculations correspond to the benchmark model and calibration shown in Table 1 discussed in the Appendix.
2. I.e., generation s is indifferent between the policy change and a $[(\mu_{s|t=0} - 1) \cdot 100\%]$ percent increase in each possible wage at time s, measured today at time 0. Welfare measures are calculated over all possible paths that the economy can take between the policy reform date (0) and the shown "Generation Index." See the text for more details.

(Table 3 Continued on Next Page)

Table 3 Cont.

Generation Index, s	Percent Welfare Gain $[(\mu_{s\|t=0} - 1) \cdot 100\%]$
Increase the Capital Income Tax from 20 to 30 Percent	
0	1.2
1	1.5
2	2.7
3	2.6
4	2.5
5	2.4

Table 4
Sensitivity Analysis

Correlation Between Wage and Stock Returns is Set Equal to Unity [1]
–
Risk-Adjusted Changes in Expected Lifetime Resources of Generation

Generation Index, *s*	Percent Welfare Gain $[(\mu_{s\|t=0} - 1) \cdot 100\%]$
Decrease the Capital Income Tax from 20 to 0 Percent	
0	0.0
1	0.4
2	0.7
3	0.9
4	1.0
5	1.1
Decrease the Capital Income Tax from 20 to -20 Percent:	
0	- 0.1
1	0.6
2	1.0
3	1.2
4	1.4
5	1.5

Notes:
1. The model is recalibrated to hit this correlation target and the other targets mentioned in the text.

Table 5
Sensitivity Analysis Continued

Abandoning the Domar-Musgrave Assumptions:
Taxing Both the Entire Equity Return with an Asymmetric Tax [1]

–

Risk-Adjusted Changes in Expected Lifetime Resources of Generation

Generation Index, s	Percent Welfare Gain $[(\mu_{s\|t=0} - 1) \cdot 100\%]$
Decrease the Capital Income Tax from 20 to 0 Percent	
0	2.7
1	-6.8
2	-7.6
3	-7.7
4	-7.7
5	-7.7
Increase the Capital Income Tax from 20 to 40 Percent	
0	-2.7
1	4.7
2	5.6
3	5.6
4	5.6
5	5.6

Notes:
1. In particular, the return $max(0,e)$ is taxed.