

NBER WORKING PAPER SERIES

CONSISTENT EXPECTATIONS, RATIONAL EXPECTATIONS,
MULTIPLE-SOLUTION INDETERMINACIES,
AND LEAST-SQUARES LEARNABILITY

Bennett T. McCallum

Working Paper 9218
<http://www.nber.org/papers/w9218>

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
September 2002

This paper has been prepared for the European Monetary Forum conference in honor of Sir Alan Walters, held in Cardiff on May 10-12, 2002. I am indebted to Huberto Ennis, Andreas Hornstein, Andrew Levin, and Edward Nelson for helpful discussions. The views expressed herein are those of the authors and not necessarily those of the National Bureau of Economic Research.

© 2002 by Bennett T. McCallum. All rights reserved. Short sections of text, not to exceed two paragraphs, may be quoted without explicit permission provided that full credit, including © notice, is given to the source.

Consistent Expectations, Rational Expectations, Multiple-Solution Indeterminacies,

and Least-Squares Learnability
Bennett T. McCallum
NBER Working Paper No. 9218
September 2002
JEL No. C32, E00, C63

ABSTRACT

After some historical discussion of the rational expectations (RE) solution procedures of John Muth, Alan Walters, and Robert Lucas, this paper considers the relevance for actual economies of issues stemming from the existence of multiple RE equilibria. In all linear models, the minimum state variable (MSV) solution—as defined by the author (JME, 1983)—is unique by construction. While it might be argued that the MSV solution warrants special status as the bubble-free solution, the focus in this paper is on its adaptive, least-squares learnability by individual agents, as discussed extensively in important recent publications by George Evans and Seppo Honkapohja.

Although the MSV solution is learnable and the main alternatives are not, in most standard models, Evans and Honkapohja have stressed an example in which the opposite is true. The present paper shows, however, that parameter values yielding that result are such that the model is not well formulated, in a specified sense (one that avoids implausible discontinuities). More generally, analysis of a pair of prominent univariate specifications, featured by Evans and Honkapohja, shows that the MSV solution is invariably learnable in these structures, if they are well formulated.

Bennett T. McCallum
Graduate School of Industrial Administration
Carnegie Mellon University
Pittsburgh, PA 15213 USA
Tel. (412) 268-2347
and NBER
Email: bm05@andrew.cmu.edu

1. Introduction

It is not widely known, I believe, that the first publication to present a rational expectations analysis of a complete macroeconomic/monetary model was authored by A. A. Walters (1971). This paper, “Consistent Expectations, Distributed Lags, and the Quantity Theory,” appeared somewhat earlier in the year than Thomas Sargent’s (1971) justly influential “A Note on the Accelerationist Controversy,” and furthermore the latter did not feature the explicit solution of a full macroeconomic model.¹ Robert Lucas’s first two money/macro papers with rational expectations (1972a, 1972b) had been presented at conferences in 1970-71 but had not yet appeared in print.

Of course Walters termed his expectational hypothesis “consistent expectations,” rather than rational expectations (RE), and refers to John Muth’s (1961) seminal paper only briefly, in a footnote.² But that does not diminish the insightfulness of Walters’s analysis. Indeed, this reader is left with the feeling that his expectational hypothesis and method of analysis were worked out independently of previous writings, with knowledge of Muth’s paper perhaps arriving rather late in the publication process.

In the 30-plus years since 1971 a lot of activity has taken place in the area of RE money/macro analysis, to put it mildly. Consequently, I have no intention of trying to survey the many developments that have taken place. But I would like to take up some particular issues concerning solution concepts and the problem of “indeterminacy,” or multiple solutions, in RE models. I will begin in Section 2 by outlining Walters’s solution procedure and contrasting it with the one used by Muth (1961). Then, in Section 3, I will outline

¹ Sargent’s paper, like Walters’s, emphasizes that fixed distributed-lag formulas for expectations can be consistently incorrect, since they fail to reflect policy processes.

² Where Muth is given his brother’s first name, Richard. Incredibly, the same mistake appears over 20 years later in Krugman (1994, p. 49).

Lucas's (1972b) procedure and turn to the topic of multiple solutions, which has been active for many years and recently has become increasingly prominent. My own "minimum-state-variable" interpretation and extension of Lucas's procedure, developed in McCallum (1983), is also discussed and the dependence of several recent controversies on the solution concept is emphasized. Next, Section 4 describes an approach to selection among multiple solutions, based on the criteria of E-stability and adaptive learnability, that was initiated in the 1980s by George Evans and recently treated comprehensively in major publications by Evans and Honkapohja (1999, 2001). Section 5 examines an example featured by those authors in which their criterion conflicts with my own, and argues that this conflict occurs only with parameter values that make the model economically implausible. That argument is rather ad hoc in nature, however, so Section 6 proposes some general requirements for a model to be regarded as plausible or "well formulated." The paper's main result is in Section 7, which shows that for an important class of well formulated models, the unique MSV solution is invariably learnable. Finally, Section 8 provides a brief summary and conclusion.

2. Consistent and Rational Expectations

Walters (1971) analyzed price level behavior in a model that is fairly similar to the standard workhorse for monetary RE analysis, which includes the Cagan (1956) money demand function and a policy process represented in terms of money supply. Walters's money-demand equation is written as

$$(1) \quad p_t = \alpha m_{t-1} + \beta(p_t^e - p_{t-1}) + \varepsilon_t,$$

with $\alpha > 0$ and $0 < \beta < 1$. Here the dating of variables differs from the version that has become standard and, for some reason, p_t and m_t represent the price level and the money stock, rather than their logarithms. The expectational variable is p_t^e , the expectation of p_t

formed at time $t-1$. The shock term ε_t is taken to be purely random (i.e., white noise) so its expectation at $t-1$ is zero and thus we have $p_t^e = \alpha m_{t-1} + \beta(p_t^e - p_{t-1})$. Consequently, we can solve out p_t^e and obtain the solution expression

$$(2) \quad p_t = [\alpha/(1-\beta)] m_{t-1} - [\beta/(1-\beta)] p_{t-1} + \varepsilon_t.$$

It will be noted that the foregoing solution procedure—of taking expectations, solving for p_t^e , and substituting out the latter—cannot be used when p_{t+1}^e enters the system.

Walters (1971) considers the implied paths of p_t , and representations of p_t^e , for three different money supply processes. The paper's main message is that the p_t^e representations usually do not satisfy the adaptive expectations formula, $p_t^e = (1-\lambda)[p_{t-1} + \lambda p_{t-2} + \lambda^2 p_{t-3} + \dots]$, that was very widely used at the time. Indeed, any fixed distributed-lag formula for expectations will be systematically incorrect unless it happens to reflect the money supply process. This important conclusion, which was also the main message of Sargent (1971), is a precursor of the famed Lucas (1976) critique. Two limitations of Walters's analysis are that (i) the effect of shocks to the money supply is not considered and (ii) the model is not extended to include structural equations of a more standard macroeconomic system with sluggish price adjustments of the expectational Phillips-curve type.

Walters (1971, p. 273; 1988, p. 290) has expressed the view that the term “consistent expectations” is preferable to rational expectations, and I would not strongly disagree. I would argue, however, that the related term “model-consistent expectations” is somewhat undesirable.³ The reason is that it leads to easily into an anti-RE argument such as “it is implausible that all of an economy's agents would believe in the particular model of the

³ This term has been used by many writers including Brayton et. al. (1997) and Isard, Laxton, and Eliasson (1999).

economy being used by the researcher.”⁴ My objection (McCallum, 1999b) is that this statement does not represent the assumption that is actually required for the basic version of RE. The proper assumption is that agents form expectations so as to avoid systematic expectational errors in actuality, which implies that each agent behaves as if he knew the structure of the actual economy. Then expectations will agree with the researcher’s model, but the reason is that the latter is by design his best attempt to depict the true structure of the actual economy—for if it were not, he would adopt a different model. There is no assumption that agents consciously create explicit models at all, only that they manage their own private affairs so as to avoid systematic expectational errors in actuality.

From here on I will use $E_{t|z_{t+j}}$ to denote $E(z_{t+j}|\Omega_t)$, where Ω_t is the information set at t , typically (but not necessarily) taken to include all variables dated t and earlier. Using this notation, the first of Muth’s (1961) two models can be written as

$$(3) \quad -\beta p_t = \gamma E_{t-1} p_t + u_t,$$

where p_t is a market price and u_t is a random shock term. If the latter is white noise, the same

solution procedure as Walters’s could be used, but Muth generalizes to permit $u_t = \sum_{i=0}^{\infty} w_i \varepsilon_{t-i}$

where ε_t is white noise. Then to obtain a solution he essentially applies an undetermined coefficient approach to the moving-average solution form

$$(4) \quad p_t = \sum_{i=0}^{\infty} W_i \varepsilon_{t-i}$$

in order to evaluate the W_i s in terms of β , γ , and the w_i s. That same strategy is adequate, and is used, with Muth’s second and more complex model. The latter, which recognizes

⁴ A variant is the claim that it is implausible that all agents would believe in the same model of the economy. But, first, this is an objection to macroeconomics, not rational expectations, and second, there are some RE models in which agents’ expectations are not all alike.

inventory speculation, can be expressed as

$$(5) \quad -\beta p_t + I_t = \gamma E_{t-1} p_t + u_t + I_{t-1}$$

$$(6) \quad I_t = \alpha(E_t p_{t+1} - p_t)$$

where I_t is inventory holdings at the end of t , $-\beta p_t$ is consumption demand in t , and $\gamma E_{t-1} p_t + u_t$ is production. Substituting (6) into (5), one obtains an equation involving p_t , $E_{t-1} p_t$, $E_t p_{t+1}$, and p_{t-1} as well as u_t . Again the solution procedure of undetermined coefficients (henceforth, UC) in terms of the moving average representation of the solution (i.e., in terms of ε_t , ε_{t-1} , ...) is applicable, but now it leads to a quadratic characteristic equation. Muth selects between the two roots on the grounds of boundness—i.e., non-explosiveness or dynamic stability—of the resulting solution. This same procedure could be applied if additional exogenous shocks were included in the model, so we see that Muth's (1961) paper developed a solution procedure—and an implicit solution concept—for a rather wide class of linear models.⁵

3. Multiple Solutions and the MSV Concept

Lucas (1972a, 1972b) provided the next—enormously influential—publications with RE in money/macro models. The former was the greater piece of work, of course, but for present purposes it will be useful to focus on the simplified linear model in the second. There Lucas's aggregate demand-supply system includes a Phillips-type supply function and a logarithmic nominal income identity, plus a policy rule assumed for simplicity to pertain directly to nominal income. I will not now discuss the model itself, since it includes some questionable features, but will go immediately to the relevant point. This is that Lucas's solution procedure involves a UC calculation not in terms of moving average parameters, but

⁵ To me, writing without the benefit of inside information, it seems possible that recognition of the extent of Muth's achievement may have provided a major reason for Walters to have abstained from additional research in the area during the 1970s. Matthews (1998) suggests that the dominant reason was the attitude taken by the

with respect to the parameters (coefficients) of a conjectured solution form that includes only the variables and shocks recognized to be relevant to the current state of the system, i.e., the relevant state variables.

The importance of this step can be illustrated simply in terms of the following basic, non-specific, model:

$$(7) \quad y_t = \alpha + aE_t y_{t+1} + u_t$$

$$(8) \quad u_t = \rho u_{t-1} + \varepsilon_t.$$

Here $|\rho| < 1$ and ε_t is white noise. Since there are no relevant state variables in sight except u_t ,⁶ it is natural to conjecture a solution of the form

$$(9) \quad y_t = \phi_0 + \phi_1 u_t,$$

and then solve for the coefficients ϕ_0 and ϕ_1 . Since (9) implies $E_t y_{t+1} = \phi_0 + \phi_1 \rho u_t$, substitution into (7) gives $\phi_0 + \phi_1 u_t = \alpha + a(\phi_0 + \phi_1 \rho u_t) + u_t$, which implies

$$(10a) \quad \phi_0 = \alpha + a\phi_0$$

$$(10b) \quad \phi_1 = a\rho\phi_1 + 1.$$

Thus we have $\phi_1 = 1/(1-a\rho)$ and $\phi_0 = \alpha/(1-a)$, the unique solution that is of form (9).

But there are more solutions. Suppose that one enters the apparently extraneous variables y_{t-1} and u_{t-1} into the following candidate solution expression that might be considered instead of (9):

$$(11) \quad y_t = \phi_0 + \phi_1 y_{t-1} + \phi_2 u_t + \phi_3 u_{t-1}.$$

Then proceeding as before leads to the UC equalities

Economic Journal's editor, David Champerone, who was not favorably inclined toward the hypothesis of consistent or rational expectations.

⁶ One could proceed equivalently in terms of u_{t-1} and ε_t , since u_t is AR(1).

$$(12a) \quad \phi_0 = \alpha + a\phi_0 + a\phi_1\phi_0$$

$$(12b) \quad \phi_1 = a\phi_1^2$$

$$(12c) \quad \phi_2 = a\phi_1\phi_2 + a\rho\phi_2 + 1$$

$$(12d) \quad \phi_3 = a\phi_1\phi_3.$$

The second of these is satisfied by $\phi_1^{(-)} = 0$ or by $\phi_1^{(+)} = 1/a$. The first of these roots implies a solution equivalent to the one given previously, but the second leads to the solution

$$(13) \quad y_t = -(\alpha/a) + (1/a)y_{t-1} - (1/a\rho)u_t + \phi_3u_{t-1},$$

which is consistent with all of the model's equations for any value of ϕ_3 . Thus there is an infinity of solutions, if ones of form (11) are considered. In some models based firmly on full optimizing analysis, there will be transversality conditions that exclude explosive solutions, which would eliminate this infinity if $|a| < 1$, as would usually be the case. But there are several notable examples in the literature in which relations such as (13) qualify as solutions under stringent optimizing assumptions.

To many workers, Lucas's procedure of restricting attention to solutions of a form such as (9) will be attractive, since it is capable of generating solutions that are based only on fundamentals—thereby excluding “bubble” components that involve variables that do not enter the model and therefore can appear in the solution only because they are (arbitrarily) expected (by the model's agents) to be relevant. This elimination of bubble solutions does not occur if one adopts a moving average formulation, in the fashion preferred by Muth (1961). Partly for this reason, perhaps, Lucas's approach rapidly gained popularity during the 1970s.

An issue arises, however, in models that include lagged values of endogenous variables. Suppose that the relevant model includes

$$(14) \quad y_t = \alpha + aE_t y_{t+1} + c y_{t-1} + u_t$$

rather than (7), in addition to (8). Then the solution clearly must include y_{t-1} as well as u_t as a relevant state variable. And then if one searches for a solution of the form

$$(15) \quad y_t = \phi_0 + \phi_1 y_{t-1} + \phi_2 u_t,$$

it will be found that $E_t y_{t+1} = \phi_0 + \phi_1(\phi_0 + \phi_1 y_{t-1} + \phi_2 u_t) + \phi_2 \rho u_t$ and the UC equations become

$$(16a) \quad \phi_0 = \alpha + a\phi_0 + a\phi_1\phi_0$$

$$(16b) \quad \phi_1 = a\phi_1^2 + c$$

$$(16c) \quad \phi_2 = a\phi_1\phi_2 + a\rho\phi_2 + 1.$$

In this case there are two solutions, one based on

$$(17) \quad \phi_1^{(-)} = \frac{1 - \sqrt{1 - 4ac}}{2a}$$

and the other on

$$(18) \quad \phi_1^{(+)} = \frac{1 + \sqrt{1 - 4ac}}{2a},$$

where we use the convention that \sqrt{z} is positive for all $z > 0$. Of course, we shall require that ϕ_1 be real-valued, since complex solutions make no sense for prices or quantities. But whenever there is a real solution there seem to be two—which will often have very different properties—even if we follow the Lucas (1972b) procedure.

A solution concept that provides uniqueness was proposed, however, by McCallum (1983). Clearly, the two expressions (17) and (18) define two different functions and therefore two quite distinct solutions to the model (14)(8). Consequently, consider the special case of (14) in which $c = 0$. In this case y_{t-1} does not enter the model and thus could be considered to be an extraneous state variable, which should not appear in the solution, if it is to include only relevant state variables. Accordingly, McCallum (1981, 1983) proposed

that since $\phi_1^{(-)}$ equals 0 in this special case, and $\phi_1^{(+)}$ does not, then the solution based on $\phi_1^{(-)}$ should be regarded as the relevant solution. His (1983) paper develops a rather general procedure for finding this “bubble-free” solution, which is unique by construction in linear models.⁷ This procedure was given the name “minimum state variable” (MSV) solution by Evans (1986), who referred to the step of choosing between the two roots in the last example as constituting a “subsidiary principle.” In what follows it will be important to be unambiguous about the concept of a MSV solution. Throughout I will be using that term to designate the unique solution—unique by construction—described in McCallum (1983, 1999). This is the way that the term was used by Evans (1986, 1989) and by Evans and Honkapohja (1992), but differs from the terminology in the latter’s more recent publications (1999, p. 496; 2001, p. 194), where their convention permits multiple solutions to be given the MSV adjective. Either terminology could be used, of course, but the one adopted here is more appropriate for the issues at hand.

Recently, the possible occurrence of multiple solutions has assumed new prominence in the area of monetary economics under the heading of “indeterminacies.” Notable topics in which indeterminacy is central to policy issues include (i) inflation forecast targeting [e.g., Woodford (1994), Bernanke and Woodford (1996), King (2000)], (ii) the Taylor Principle [Woodford (2001), King (2000), Clarida, Gali, and Gertler (1997, 1999)], (iii) the zero-lower-bound deflation trap [Benhabib, Schmitt-Grohe, and Uribe (2001), Schmitt-Grohe and Uribe (2000), McCallum (2002), Alstadheim and Henderson (2002)], and (iv) the fiscal theory of the price level [Woodford (1995), Sims (1994), Cochrane (1998), McCallum (2001), Kotcherlakota and Phelan (1999)].

⁷ The MSV solution is required, by definition, to be linear. For a discussion of this and several other points, see

In this context it is important to recognize that the type of indeterminacy present in all of these cases involves multiple RE solutions and accordingly is quite different from the “price level indeterminacy” problem that was discussed extensively in the monetary literature of the 1940s and 1950s by Lange (1942), Patinkin (1949, 1961, 1965), Gurley and Shaw (1960), and Johnson (1962). In particular, the former involves multiple time paths for real variables even with some nominal variable fixed (as a consequence of dynamic expectational behavior) whereas the latter involves the model’s failure to determine any nominal variable despite unique paths for all real variables (occurring as a consequence of the absence of any nominal anchor, a static concept). I have suggested several times that a more constructive terminology would refer to “multiple solutions” and “nominal indeterminacy,” respectively, but thus far have made little headway.

In any event, one’s position on policy issues relating to the four topics (i)-(iv) logically depends on his beliefs concerning the status of multiple RE solutions. Are such multiplicities relevant in principle and empirically for actual economies, or are they theoretical curiosa with little or no relevance to actual economies? The following sections present the outline of an argument in favor of the latter position.

4. E-Stability and Learnability

In a series of articles appearing in the 1980s, George Evans (1985, 1986, 1989) proposed an alternative criterion for designation or “selection” of the economically relevant RE solution in cases in which multiplicity obtains. His initial criterion, now known as iterative E-stability, can be briefly reviewed. The basic presumption is that individual economic agents will not be endowed with perfect knowledge of the economic system’s structure, so it is natural to consider whether plausible error-correction mechanisms are

McCallum (1999).

convergent to particular solutions. This can be determined for each of the multiple RE solutions, and the presence or absence of such mechanisms may yield a criterion for selection of one solution as economically relevant. For an illustration, consider again the model (14)(8), which we rewrite for convenience:

$$(14) \quad y_t = \alpha + aE_t y_{t+1} + cy_{t-1} + u_t$$

$$(8) \quad u_t = \rho u_{t-1} + \varepsilon_t.$$

Suppose that the economy's individuals believe that the actual behavior of y_t can be expressed by an equation that includes the same variables as (15), but that they do not know the exact values of the parameters. If at time t the typical agent's belief is that these values are $\phi_0(n)$, $\phi_1(n)$, and $\phi_2(n)$, then the system's perceived law of motion (PLM) will be⁸

$$(19) \quad y_t = \phi_0(n) + \phi_1(n)y_{t-1} + \phi_2(n)u_t.$$

In this case the implied expectation at t of y_{t+1} will be

$$(20) \quad \phi_0(n) + \phi_1(n)y_t + \phi_2(n)\rho u_t.$$

Using that expression in place of $E_t y_{t+1}$ in (14)—which implies that we are temporarily abandoning RE—gives

$$(21) \quad y_t = \alpha + a[\phi_0(n) + \phi_1(n)y_t + \phi_2(n)\rho u_t] + cy_{t-1} + u_t$$

or, rearranging,

$$(22) \quad y_t = [1 - a\phi_1(n)]^{-1} [\alpha + a\phi_0(n) + a\phi_2(n)\rho u_t + cy_{t-1} + u_t]$$

as the system's actual law of motion (ALM). Now imagine a sequence of iterations from the PLM to the ALM. Writing the left-hand side of (22) in the form (19), but for iteration $n+1$, gives $\phi_0(n+1) + \phi_1(n+1)y_{t-1} + \phi_2(n+1)u_t = [1 - a\phi_1(n)]^{-1} [\alpha + a\phi_0(n) + a\phi_2(n)\rho u_t + cy_{t-1} + u_t]$ and therefore implies that

⁸ Here n is being used to index iterations in an educative process of learning that takes place in meta-time.

$$(23a) \quad \phi_0(n+1) = [1 - a\phi_1(n)]^{-1}[\alpha + a\phi_0(n)]$$

$$(23b) \quad \phi_1(n+1) = [1 - a\phi_1(n)]^{-1}c$$

$$(23c) \quad \phi_2(n+1) = [1 - a\phi_1(n)]^{-1}[a\phi_2(n)\rho + 1].$$

The issue, then, is whether iterations defined by (23) are such that the $\phi_j(n)$ converge to the ϕ_j values in (15) as n increases without bound. If they do, then the solution (15) is said to be iteratively E-stable. Evans (1986) found that in several prominent and controversial models the MSV solution is iteratively E-stable.

On the basis of results by Marcat and Sargent (1989), Evans (1989) switched his attention to E-stability without the “iterative” qualification, defined as follows. Conversion of equations (23) to the continuous form, appropriate as the iteration interval approaches zero, yields

$$(24a) \quad d\phi_0(n)/dn = [1 - a\phi_1(n)]^{-1}[\alpha + a\phi_0(n)] - \phi_0(n)$$

$$(24b) \quad d\phi_1(n)/dn = [1 - a\phi_1(n)]^{-1}c - \phi_1(n)$$

$$(24c) \quad d\phi_2(n)/dn = [1 - a\phi_1(n)]^{-1}[a\phi_2(n)\rho + 1] - \phi_2(n).$$

If the differential equation system (24) has $\phi_j(n) \rightarrow \phi_j$ for all j , the solution (15) is E-stable.

An important feature of this continuous version of the iterative process is that it is intimately related to an adaptive learning process that is modelled as taking place in real time.⁹ For most non-explosive models, that is, values of parameters analogous to the ϕ_j in (15), which are estimated by least squares (LS) regressions on the basis of data from periods $t-1, t-2, \dots, 1$ and used to form expectations in period t , will converge to the actual values in (15) as time passes if and only if equations (24) converge to those values. Thus E-stability and LS

⁹ The E-stability process itself is conceived of as taking place in notional time (meta time). For the sake of brevity, the present account omits discussion of several important papers on learning; for many references, see Evans and Honkapohja (1999, 2001).

learnability typically go hand in hand.¹⁰ This result, which is discussed extensively by Evans and Honkapohja (1999, 2001), is useful because it is technically much easier, in most cases, to establish E-stability than to establish LS learnability.¹¹

5. Questionable Example

As mentioned above, Evans's early work indicated that the E-stability/learnability principle often supports the MSV criterion. More recently, however, the implied message has been quite different. Thus in various places Evans and Honkapohja (E&H) have argued that MSV solutions may or may not have the property of E-stability (and LS learnability). It is my belief, however, that this recent message is misleading; that in all or almost all sensible models the MSV solution does possess E-stability. Thus the agenda of this section is to discuss and reconsider the main example put forth by E&H (1992, pp. 9-10; 1999, pp. 496-7; 2001, p. 197) as representing a case in which the MSV solution is not E-stable.

Following E&H (1992), the relevant model's reduced form can be written as

$$(25) \quad y_t = \alpha + \gamma E_{t-1} y_t + \zeta E_{t-1} y_{t+1} + \delta y_{t-1} + \varepsilon_t,$$

with $\delta \neq 0$, $\zeta \neq 0$, and ε_t white noise. The MSV solution will be of the form

$$(26) \quad y_t = \phi_0 + \phi_1 y_{t-1} + \phi_2 \varepsilon_t,$$

and ϕ_1 will be determined by a quadratic equation with the MSV solution given by the ϕ_1 root that equals zero when $\delta = 0$. The other root gives a bubble solution and there are also bubble solutions of a form that includes additional terms involving y_{t-2} and ε_{t-1} on the right-hand side of (26).

Necessary conditions for E-stability of a solution of form (26) are (E&H, 1992, p. 6)

¹⁰ It is interesting to note that a modelling strategy closely related to LS learning is explicitly mentioned by Walters (1971, p. 281).

$$(27) \quad \gamma + \zeta - 1 + \zeta\phi_1 < 0 \quad \text{and} \quad \gamma - 1 + 2\zeta\phi_1 < 0.$$

On the basis of these, E&H show on their pp. 9-10 that the non-MSV solution of form (26) is E-stable, and the MSV solution is E-unstable, when $\gamma = -\zeta > 1$ and $\delta > 0$. Also, on p. 5 they show that the bubble solutions are E-stable if $\gamma > 1$, $\delta\zeta > 0$, and $\zeta < 0$. If such parameter values were economically sensible, these results would constitute explicit counter-examples to my suggestion that MSV solutions are invariably E-stable.

Let us, however, reconsider the economic model that E&H (1992) use to motivate the reduced form equation (25). It is a log-linear “model of aggregate demand and supply with wealth effects in aggregate demand, money demand, and aggregate supply” (1992, p. 9). Letting y_t , m_t , and p_t be the logs of output, money, and the price level, with i_t a nominal interest rate, they write:¹²

$$(28a) \quad y_t = -g_1(i_t - E_{t-1}(p_{t+1} - p_t)) + g_2(m_t - p_t) + v_{1t}$$

$$(28b) \quad y_t = f(m_t - p_t) + v_{2t}$$

$$(28c) \quad m_t - p_t = y_t - a_1 i_t + a_2(m_t - p_t) + v_{3t}$$

$$(28d) \quad m_t = d p_{t-1} + v_{4t}.$$

The fourth equation “is a monetary policy reaction function....” (1992, p. 9). Solving these four equations for a reduced form expression for p_t gives

$$(29) \quad p_t = d p_{t-1} + h E_{t-1}(p_t - p_{t+1}) + u_t$$

with $h = g_1[f - g_2 + g_1(a_2 + f - 1)a_1^{-1}]^{-1}$ and where u_t is a linear combination of the (white noise) v_{it} terms. Consequently, the model is of form (25) with y_t in the latter representing p_t in the model and with $\gamma = h$, $\zeta = -h$, and $\delta = d$.

¹¹ For a notable recent application to monetary policy analysis, see Bullard and Mitra (2000).

¹² Here (28b) is aggregate supply and (28c) is money demand. It is my distinct impression that E&H intend for all parameters to be interpreted as non-negative.

It follows, then, that the condition $\gamma = -\zeta > 1$ requires $h > 1$. In that regard, note first that if real-balance terms are excluded, i.e., if $g_2 = f = a_2 = 0$, then $h = -a_1$ is negative. Thus sizeable real-balance effects are needed. Second, note that a_2 should probably be specified as negative, not positive, since the latter would imply a money demand function with income elasticity greater than 1.0, in contrast with most empirical estimates. But with $a_2 < 0$, the parameter f would have to be quite large to generate $h > 1$. In other words, real money balances would have to enter strongly in the production function for output. Thus $h > 1$ seems highly improbable in the context of the IS-LM model of the type utilized.

In addition, the condition $\delta > 0$ implies $d > 0$ in (28d), implying that the money supply is increased by the monetary authority when the price level is higher than average in the previous period. That represents, it seems clear, a positively perverse form of policy behavior.

An alternative way of interpreting the reduced-form equation (25), not mentioned by E&H, is as a microeconomic supply-demand model. Suppose we have demand and supply functions

$$(30a) \quad q_t = \beta_0 + \beta_1 p_t + \beta_2 E_{t-1}(p_{t+1} - p_t) + v_{1t}$$

$$(30b) \quad q_t = \alpha_0 + \alpha_1 p_t + \alpha_2 E_{t-1} p_t + v_{2t}$$

where the disturbance terms include effects of exogenous variables such as demanders' income and the price of inputs to production. Here we would hypothesize that $\beta_1 < 0$ and $\beta_2 > 0$, to reflect downward sloping demand with respect to the current price and a speculative demand motive. Also, let $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$ to reflect upward sloping supply with respect to relevant prices. Then the reduced form is

$$(31) \quad p_t = (\alpha_1 - \beta_1)^{-1} [(\beta_0 - \alpha_0) + \beta_2 E_{t-1} p_{t+1} - (\alpha_2 + \beta_2) E_{t-1} p_t + v_{1t} - v_{2t}].$$

In terms of equation (25), this specification suggests $\zeta > 0$, $\gamma < 0$, and $\delta = 0$. But the first two of these are just opposite in sign to the requirements for the E&H example. Furthermore, it is plausible that p_{t-1} might appear instead of $E_{t-1}p_t$ in the supply equation (as in the cobweb model). But then its coefficient in the reduced form would be negative, and therefore inconsistent with the $d > 0$ assumption in the E&H case under discussion.

In sum, I would argue that the specification used most prominently by E&H, to provide an example featuring the absence of E-stability for the MSV solution, is highly unappealing in terms of basic economic theory. It must be admitted, however, that this argument is quite specific and rather ad hoc in nature. Accordingly, I will now turn to a more general line of argument.

6. Well Formulated Models

In this section I propose conditions necessary for important classes of linear models to be “well formulated.” Consider again the single-variable specification (14), which is reproduced once more for convenience:

$$(32) \quad y_t = \alpha + aE_t y_{t+1} + cy_{t-1} + u_t,$$

with $u_t = \rho u_{t-1} + \varepsilon_t$. With ε_t white noise, u_t is an exogenous forcing variable with an unconditional mean of zero. Applying the unconditional expectation operator to (32) yields

$$(33) \quad E y_t = \alpha + aE y_{t+1} + cE y_{t-1} + 0.$$

But if y_t is covariance stationary, we then have¹³

$$(34) \quad E y_t = \alpha / [1 - (a + c)].$$

From the latter, it is clear that as $a + c$ approaches 1.0 from above, the unconditional mean of y_t approaches $-\infty$ (assuming that $\alpha > 0$), whereas if $a + c$ approaches 1.0 from below, the

unconditional mean approaches $+\infty$. Thus there is an infinite discontinuity at $a + c = 1.0$. This implies that a tiny change in $a + c$ could alter the average (i.e., steady state) value of y_t from an arbitrarily large positive number to an arbitrarily large negative number. Such a property is highly implausible and therefore, I suggest, unacceptable for a well-formulated model.

In light of the preceding discussion, my argument is that, to be considered well formulated, the model at hand needs to include a restriction on its admissible parameter values, a restriction that rules out $a + c = 1$ and yet admits a large open set of values that includes $(a, c) = (0, 0)$. In the case at hand, the appropriate restriction is $a + c < 1$. Of course, $a + c > 1$ would serve just as well mathematically to avoid the infinite discontinuity, but it is clear that $a + c < 1$ is vastly more appropriate from an economic perspective since it includes the region around $(0, 0)$. Note that the oft-seen condition $a + c \neq 1$ does not eliminate the unacceptable property. It should be clear, in addition, that the foregoing argument could be easily modified to apply to y_t processes that are trend stationary, rather than strictly (covariance) stationary.¹⁴

Now let us consider a second model specification that, like (32), is emphasized by E&H (1999, 2001). It can be written as

$$(35) \quad y_t = \alpha + \beta_0 E_{t-1}y_t + \beta_1 E_{t-1}y_{t+1} + \delta y_{t-1} + u_t,$$

with $u_t = \rho u_{t-1} + \varepsilon_t$ as before. For this case, consider the conditional expectation, $E_{t-1}y_t$:

$$(36) \quad E_{t-1}y_t = (1 - \beta_0)^{-1} [\alpha + \beta_1 E_{t-1}y_{t+1} + \delta y_{t-1} + \rho u_{t-1}].$$

¹³ Note that it is not being assumed that y_t is necessarily covariance stationary. Instead, an implication that would hold, if it were, is being used to motivate the assumption that will be made subsequently.

¹⁴ Generalizing, suppose that y_t in (32) is a $m \times 1$ vector of endogenous variables, so that α is $m \times 1$ while a and c are $m \times m$ matrices. Then the counterpart of $1 - (a + c) > 0$ is that the eigenvalues of $[I - (a + c)]$ all have positive real parts, i.e., that the eigenvalues of $[a + c]$ all have real parts less than 1.0. That requirement is necessary for the multivariate version of (32) to be well formulated.

Here it is clear that, for given values of $E_{t-1}y_{t+1}$, y_{t-1} , and u_{t-1} , $E_{t-1}y_t$ will pass through an infinite discontinuity at $\beta_0 = 1$. Consequently, for basically the same reason as before, $\beta_0 < 1$ is necessary for the model to be well formulated. In addition, $\beta_0 + \beta_1 + \delta < 1$ continues to apply.¹⁵

An application of these criteria to the questionable example of E&H (1992), featured above in Section 5, is immediate. That example's result, of a MSV solution that is not E-stable, requires $\gamma = h > 1$. But in the notation of (35), that condition is $\beta_0 > 1$, which is incompatible with our requirement for models of form (35) to be well formulated. Thus the questionable example is discredited on general grounds, in addition to the specific reasons developed in Section 5.

7. Main Results

We are now prepared to develop a more general version of the foregoing argument. In particular, it will be shown that being well formulated (henceforth, WF) is a sufficient condition for the MSV solution to be E-stable in univariate models of classes (32) and (35). Let us begin with (35), but assuming that $\delta = 0$ since that case has been emphasized by E&H. For this model, conditions for E-stability can be found by reference to Figure 1, which is adapted from the diagram of E&H (1999, p. 492; 2001, p. 191). In the cited references, it is derived and reported that the MSV solution is E-stable in regions I, V, and VI but E-unstable in regions II, III, and IV. In regions I and VI, moreover, the MSV solution is reported to be

¹⁵ The multivariate extension for the case in which y_t is a vector yields the requirements that the eigenvalues of $[I - \beta_0]$ and $[I - (\beta_0 + \beta_1 + \delta)]$ all have positive real parts.

strongly E-stable whereas in V it is weakly E-stable.¹⁶ Reference to our conditions for model (35) to be well formulated (with $\delta = 0$) shows immediately that the condition obtains only for regions I and VI. Thus in this particular but prominent case, the MSV solution is strongly E-stable if the parameter values are such that the model is well formulated.

Next consider the more difficult and important model of equation (32). The issue at hand is whether the MSV solution possesses E-stability, i.e., whether the differential equations (24) are locally stable at the MSV values for the ϕ_j . Necessary and sufficient conditions for this to be true are given by Evans and Honkapohja (2001, p. 202) as follows: $a(1-a\phi_1)^{-1} < 1$, $ca(1-a\phi_1)^{-2} < 1$, $pa(1-a\phi_1)^{-1} < 1$. These will be utilized below, but first it will be useful to examine Figure 2, which again is adapted from E&H (2001, p. 203). There E-stability regions are shown under the assumption $0 \leq \rho < 1$. In this case, the results reported by E&H indicate that the MSV solution is E-stable in regions I and VII but E-unstable in region IV, while “both solutions [i.e., from both roots of (16b)] are explosive or nonreal” elsewhere (E&H, 2001, p. 203).¹⁷ Specifically, solutions for ϕ_1 are complex-valued in regions III and VI, and both solutions feature explosive behavior in regions II and V. As indicated above, the MSV solution is well formulated in regions I, V, and VII (being complex in VI). Thus for regions I and VII, the E&H version of Figure 2 supports the hypothesis that the MSV solution is E-stable in all well formulated models of form (32). But what about region V? There the E-stability conditions are in fact met (E&H, 2001, p. 202). In the E&H graphical summary this region is not distinguished from VI because in V the solutions are both dynamically unstable (explosive). But there is no compelling reason to

¹⁶ Strong E-stability occurs in cases in which local convergence to the MSV parameter values occurs even when the function considered includes additional variables (excluded from the MSV specification). This implies that certain other solutions are not E-stable.

ignore the MSV solution simply because it is explosive; it may be accurately indicating what would happen if (e.g.) extremely unwise policy behavior were imposed on the system.¹⁸ For a discussion and rationalization of this position, with a closely related example, see McCallum (1999). In any case we see that this specification, too, conforms to the proposition that MSV solutions are E-stable in all well formulated models.¹⁹

We wish to have results for the more general case with $|\rho| < 1$, permitting negative values, but let us proceed by first demonstrating algebraically that the E-stability conditions are satisfied by the MSV solution to model (32) when $0 \leq \rho < 1$ and the WF restriction $a + c < 1$ is imposed. Afterwards we can go on to the case with $-1 < \rho < 0$ permitted. The main task, then, is to show that if $1 - (a + c) > 0$, then $(1 - a\phi_1)^{-1}a < 1$ where $\phi_1 = (1-d)/2a$ with $d = \sqrt{1-4ac}$. Note first that $1 - a\phi_1 = (1 + d)/2$ so $(1 - a\phi_1)^{-1}a = 2a/(1+d)$. Then for a proof by contradiction, suppose that $2a/(1+d) > 1$. Then $a > 0$ and $2a - 1 > d$. Since both of its sides are positive, the latter implies $4a^2 - 4a + 1 > d^2 = 1 - 4ac$. But with $a > 0$ the last inequality reduces to $a - 1 > -c$ or $0 > 1 - (a + c)$, which is the contradiction that proves $(1 - a\phi_1)^{-1}a < 1$. The latter is the first of the three E-stability conditions listed in the previous paragraph. The second results from writing $(1 - a\phi_1)^{-2}ac = (1 - a\phi_1)^{-1}a\phi_1$, which follows because $(1 - a\phi_1)^{-1}c = \phi_1$.²⁰ Since $(1 - a\phi_1)^{-1}a\phi_1 = (1-d)/(1+d)$, which is smaller than 1 for all $d > 0$, we have the desired inequality. Finally, with $(1 - a\phi_1)^{-1}a < 1$ and ρ non-negative, the third condition also

¹⁷ Note that the MSV solution is the AR(1) solution that E&H (2001) refer to as “the \bar{b}_- solution.”

¹⁸ The same statement does not apply to region II, where the MSV solution is E-stable but explosive, because there the model is not well formulated. This region illustrates that, though sufficient, the WF condition is not necessary for E-stability.

¹⁹ The usual presumption that E-stability implies LS learnability does not carry over automatically in cases of dynamic instability (explosive solutions). E&H (2001, pp. 219-220) indicate, however, that learnability will

holds.

If ρ can be negative, which is plausible, it is possible that a sufficiently large negative ρ together with $(1 - a\phi_1)^{-1}a < -1$ could lead to failure of the last condition. This possibility can be eliminated, however, by adding a second WF requirement to rule out a different type of infinite discontinuity. This type pertains to the dynamic response of y_t to the exogenous forcing variable u_t . The response coefficient is $\phi_2 = (1 - a\phi_1 - a\rho)^{-1}$ so to avoid an infinite discontinuity we require that $1 - a\phi_1 - a\rho > 0$ or $1 - a\phi_1 > a\rho$. To see that this condition is sufficient for our purposes, note that with the MSV solution, $1 - a\phi_1 = (1 + d)/2$ is unambiguously positive. Consequently, the WF condition $1 - a\phi_1 > a\rho$ implies that $1 > (1 - a\phi_1)^{-1}a\rho$, which is identical to the E-stability condition under discussion. Thus we have shown that in model (32) with $|\rho| < 1$, the MSV solution is E-stable for all parameter values satisfying our two WF conditions.^{21 22}

What about possible E-stability of the non-MSV solutions? A recent analysis that recognizes not just solutions such as (15) with root (18), but also ones involving “ARMA-type stationary sunspot” phenomena, has recently been conducted by Evans and McGough (2002). Their finding is that such solutions can be E-stable only in regions equivalent to IV and VII.²³ Whether their results are consistent with the position that non-MSV solutions are not E-stable or least-squares learnable in model (32)(8) if its parameters satisfy both of our conditions for being well formulated is unclear. Other relevant results have been provided by

prevail in the current case if an adjustment is made to the model to permit the shock variance to grow along with the y_t values (and u_t is white noise).

²⁰ The last expression is just a rearrangement of (16b).

²¹ A closely related result, more general in some respects but without inclusion of the u_t shock term, has been developed by Gauthier (2003). Also see Wenzelburger (2002), who suggests that some extension to nonlinear models may be possible.

²² A stronger condition than our second WF requirement, process consistency, is considered in the Appendix.

²³ Evans and McGough (2002) do not, however, consider the explosive regions II and V.

Desgranges and Gauthier (2002).

Clearly, the main weakness of the foregoing argument is that the results pertain only to univariate models. It is my conjecture that the results can be extended to rather general multivariate linear formulations, but this extension has not yet been verified.

8. Conclusions

Let us conclude with a brief restatement of the paper's results. After some historical discussion of the RE solution procedures of Walters (1971), Muth (1961), and Lucas (1972b), this paper considers the relevance for actual economies of issues stemming from the existence of multiple RE equilibria. In all linear models, the minimum state variable (MSV) solution—as defined by McCallum (1983, 1999)—is unique by construction. While it might be argued that the MSV solution warrants special status as the (unique) bubble-free solution, the focus in the present paper is on its adaptive, least-squares learnability by individuals not initially endowed with full knowledge of the economy's parameters, as discussed in important recent publications by Evans and Honkapohja (1999, 2001).

Although the MSV solution is learnable and the main alternatives are not learnable in most standard models, Evans and Honkapohja (1992, 1999, 2001) have stressed an example in which the opposite is true. The present paper shows, however, that parameter values yielding that result are such that the model is not well formulated, in a specified sense (one that avoids implausible discontinuities). More generally, analysis of a pair of prominent univariate specifications, featured by Evans and Honkapohja, shows that the MSV solution is invariably learnable in these structures, if they are well formulated.

Appendix

Because of the possibility that $-1 < \rho < 0$, we have ruled out a second type of infinite discontinuity, pertaining to the dynamic response of y_t to the exogenous forcing variable u_t , by requiring that $1 - a\phi_1 - a\rho > 0$. For the MSV solution, $1 - a\phi_1 = (1+d)/2$ so we need $1 + d - 2a\rho > 0$, or $d > 2a\rho - 1$, to avoid the discontinuity. Clearly there is no problem unless $2a\rho > 1$ (so $a < 0$). If it is, the relevant condition may be written (since $d = \sqrt{1 - 4ac}$) as $1 - 4ac > 1 - 4a\rho + 4a^2\rho^2$ or $-4ac > -4a\rho + 4a^2\rho^2$ or, with $a < 0$, $-c < a\rho^2 - \rho$. Now for the latter to hold for all ρ such that $-1 < \rho < 0$, it is necessary and sufficient that $a + c > -1$. That requirement is stronger, however, than the one adopted in this paper.

For the stronger condition, an alternative and more general argument can be based on the concept of “process consistency,” discussed by Flood and Garber (1980), McCallum (1983, pp. 159-160), and Evans and Honkapohja (1992, pp. 10-12). A model fails to be process consistent when solving out expectational variables, by iteration into the infinite future, is illegitimate because the implied infinite series does not converge.²⁴ For model (32)(8) to be process consistent, then, it must be the case that at least one of the roots to (16b) exceeds 1.0 in absolute value. Thus process consistency obtains in region V of Figure 2, but not in region VII, according to the root properties reported by E&H (2001, p. 203). Requiring process consistency is therefore consistent with our main result but rules out some MSV solutions that are E-stable and permitted by the weaker condition adopted in Section 7.

²⁴ An extensive discussion of related issues is given by Sargent (1987, pp. 176-204 and 305-308).

References

- Alstadheim, Ragna, and Dale W. Henderson, "Price-Level Determinacy, the Zero Lower Bound on the Interest Rate, and the Liquidity Trap," Working Paper, May 2002.
- Benhabib, Jess, Stephanie Schmitt-Grohe, and Martin Uribe. "The Perils of Taylor Rules," Journal of Economic Theory 96 (January/February 2001), 40-69.
- Bernanke, Ben S., and Michael Woodford, "Inflation Forecasts and Monetary Policy," Journal of Money, Credit, & Banking 24 (November 1997), 653-684.
- Brayton, Flint, Andrew Levin, Ralph W. Tryon, and John C. Williams, "The Evolution of Macro Models at the Federal Reserve Board," Carnegie-Rochester Conference Series in Public Policy 47 (December 1997), 115-167.
- Bullard, James, and Kaushik Mitra, "Learning About Monetary Policy Rules," W.P. 2000-001B (July 2000), Federal Reserve Bank of St. Louis.
- Cagan, Phillip, "The Monetary Dynamics of Hyperinflation," in Studies in the Quantity Theory of Money, M. Friedman. ed. University of Chicago Press, 1956.
- Clarida, Richard, Jordi Gali, and Mark Gertler, "The Science of Monetary Policy: A New Keynesian Perspective," Journal of Economic Literature 37 (Dec. 1999), 1661-1707.
- _____, _____, and _____, "Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory," NBER W.P. 6442, 1997.
- Cochrane, John H., "A Frictionless View of U.S. Inflation," NBER Macroeconomics Annual 1998. MIT Press, 1998.
- Evans, George W., "Expectational Stability and the Multiple Equilibrium Problem in Linear Rational Expectations Models," Quarterly Journal of Economics 100 (November 1985), 1217-1233.

- _____, "Selection Criteria for Models with Non-Uniqueness," Journal of Monetary Economics 18 (September 1986), 147-157.
- _____, "The Fragility of Sunspots and Bubbles," Journal of Monetary Economics 23 (1989), 297-317.
- Evans, George W., and Seppo Honkapojha, "On the Robustness of Bubbles in Linear RE Models," International Economic Review 33 (February 1992), 1-14.
- _____ and _____, "Learning Dynamics," in Handbook of Macroeconomics, J.B. Taylor and M. Woodford, eds. North-Holland, 1999.
- _____, and _____, Learning and Expectations in Macroeconomics, Princeton Univ. Press, 2001.
- Evans, George W., and Bruce McGough, "Stable Sunspot Solutions in Models with Predetermined Variables," Working Paper, University of Oregon, April 2002.
- Flood, Robert P., and Peter M. Garber, "An Economic Theory of Monetary Reform," Journal of Political Economy 88 (February 1980), 24-58.
- Gauthier, Stéphane, "On the Dynamic Equivalence Principle in Linear Rational Expectations Models," Macroeconomic Dynamics 7 (Feb. 2003), forthcoming.
- Gurley, John G., and Edward S. Shaw, Money in a Theory of Finance. Brookings Institution, 1960.
- Isard, Peter, Douglas Laxton, and Ann-Charlotte Eliasson, "Simple Monetary Policy Rules Under Model Uncertainty," International Tax and Public Finance 6 (November 1999), 537-577. Also in International Finance and Financial Crises: Essays in Honor of Robert P. Flood, Jr., P. Isard, A. Razin, and A.K. Rose, eds. Kluwer Academic Publishing, 1999.

- Johnson, Harry G., "Monetary Theory and Policy," American Economic Review 52 (June 1962), 325-384.
- Kocherlakota, Narayana, and Christopher Phelan, "Explaining the Fiscal Theory of the Price Level," Federal Reserve Bank of Minnesota Quarterly Review 23 (Fall 1999), 14-23.
- King, Robert G., "The New IS-LM Model: Language, Logic, and Limits," Federal Reserve Bank of Richmond Economic Quarterly 86 (Summer 2000), 45-103.
- Krugman, Paul, Peddling Prosperity. W.W. Norton, 1994.
- Lange, Oscar, "Say's Law: A Restatement and Criticism," in Studies in Mathematical Economics and Econometrics, O. Lange, F. McIntyre, and T.O. Yntema, eds. Chicago, 1942.
- Lucas, Robert E., Jr., "Expectations and the Neutrality of Money," Journal of Economic Theory 4 (April 1972a),
 _____, "Econometric Testing of the Natural Rate Hypothesis," in Econometrics of Price Determination O. Eckstein, ed. Board of Governors of the Federal Reserve System, 1972b.
- Marcet, Albert, and Thomas J. Sargent, "Convergence of Least Squares Learning Mechanisms in Self-Referential Linear Stochastic Models," Journal of Economic Theory 48 (April 1989), 337-368.
- Matthews, Kent, "Introduction," in The Economics and Politics of Money: The Selected Essays of Alan Walters, K. Matthews, ed. Cheltenham: Edward Elgar Publishing, 1998.
- McCallum, Bennett T., "Price Level Determinacy with an Interest Rate Policy Rule and Rational Expectations," Journal of Monetary Economics 8 (Nov. 1981), 319-329.

- _____, "On Non-Uniqueness in Rational Expectations Models: An Attempt at Perspective," Journal of Monetary Economics 11 (March 1983), 139-168.
- _____, "Role of the Minimal State Variable Criterion in Rational Expectations Models," in International Tax and Public Finance 6 (November 1999), 621-639. Also in International Finance and Financial Crises: Essays in Honor of Robert P. Flood, Jr., P. Isard, A. Razin, and A.K. Rose, eds. Kluwer Academic Publishing, 1999.
- _____, "Indeterminacy, Bubbles, and the Fiscal Theory of the Price Level," Journal of Monetary Economics 47 (February 2001), 19-30.
- _____, "Inflation Targeting and the Liquidity Trap," in Inflation Targeting: Design, Performance, Challenges, N. Loyaza and R. Soto, eds. Central Bank of Chile, 2002.
- Muth, John H., "Rational Expectations and the Theory of Price Movements," Econometrica 29 (June 1961), 315-335.
- Patinkin, Don, "The Indeterminacy of Absolute Prices in Classical Economic Theory," Econometrica 17 (January 1949), 1-27.
- _____, "Financial Intermediaries and the Logical Structure of Monetary Theory: A Review Article," American Economic Review 51 (March 1961), 95-116.
- _____, Money, Interest, and Prices, 2nd ed. Harper and Row, 1965.
- Sargent, Thomas J., "A Note on the Accelerationist Controversy," Journal of Money, Credit, and Banking 3 (1971 August), 721-725.
- _____, Macroeconomic Theory, 2nd ed. Academic Press, 1987.

Schmitt-Grohe, Stephanie, and Martin Uribe, "Liquidity Traps with Global Taylor Rules,"
Working Paper, July 2000.

Sims, Christopher A., "A Simple Model for the Study of the Determination of the Price
Level and the Interaction of Monetary and Fiscal Policy," Economic Theory 4 (1994),
381-399.

Taylor, John B., "Discretion Versus Policy Rules in Practice," Carnegie-Rochester
Conference Series on Public Policy 39 (December 1993), 195-214.

Walters, Alan A., "Consistent Expectations, Distributed Lags, and the Quantity Theory,"
Economic Journal 81 (June 1971), 273-281.

_____, "Monetary and Fiscal Policy in the UK," in Keynes and Economic Policy,
W. Eltis and P. Sinclair, eds. London: Macmillan, 1988.

Wenzelburger, Jan, "Learning in Linear Models with Expectational Leads," Working Paper,
University of Bielefeld, January 2002.

Woodford, Michael, "Nonstandard Indicators for Monetary Policy: Can Their Usefulness
Be Judged From Forecasting Regressions?" In Monetary Policy, G.N. Mankiw, ed.
University of Chicago Press for NBER, 1994.

_____, "Price-Level Determinacy Without Control of a Monetary Aggregate,"
Carnegie-Rochester Conference Series on Public Policy 43 (December 1995), 1-46

_____, "Price Level Determination Under Interest Rate Rules," Ch. 2 of Interest and
Prices, Manuscript, 2001.

Figure 1

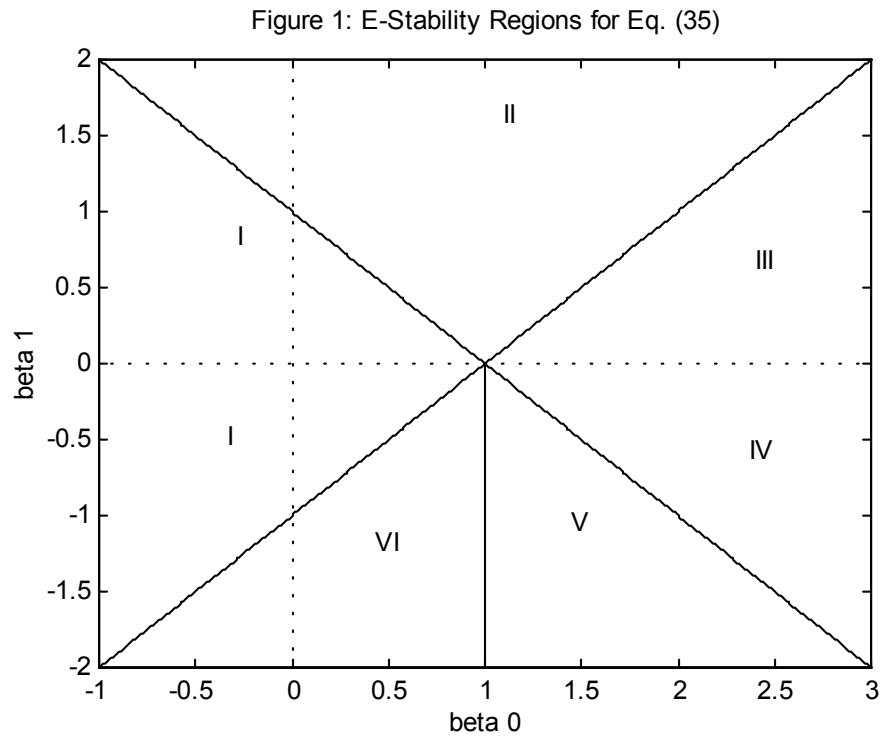


Figure 2

