

NBER WORKING PAPER SERIES

BEQUESTS AS SIGNALS: AN EXPLANATION
FOR THE EQUAL DIVISION PUZZLE

B. Douglas Bernheim
Sergei Severinov

Working Paper 7791
<http://www.nber.org/papers/w7791>

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
July 2000

Seminar participants at the National Bureau of Economic Research, Stanford University, the University of Wisconsin-Madison, Washington University, St. Louis, and Penn State University, provided helpful comments. We thank Katherine Carman for able research assistance, and the NSF for financial support through grant number SBR9511321. The views expressed here are the authors' and do not necessarily reflect those of the National Bureau of Economic Research.

© 2000 by B. Douglas Bernheim and Sergei Severinov. All rights reserved. Short sections of text, not to exceed two paragraphs, may be quoted without explicit permission provided that full credit, including © notice, is given to the source.

Bequests as Signals: An Explanation for the Equal Division Puzzle
B. Douglas Bernheim and Sergei Severinov
NBER Working Paper No. 7791
July 2000
JEL No. D10, H31

ABSTRACT

In the United States, more than two-thirds of decedents with multichild families divide their estates exactly equally among their children. In contrast, *intra vivos* gifts are usually unequal. These findings challenge the validity of existing theories regarding the determination of intergenerational transfers. In this paper, we develop a theory that accounts for this puzzle, based on the notion that the division of bequests provides a signal about a parent's altruistic preferences. The theory can also explain the norm of unigeniture, which prevails in other societies.

B. Douglas Bernheim
Department of Economics
Stanford University
Ralph Landau Building
Stanford, CA 94305
and NBER
bernheim@leland.stanford.edu

Sergei Severinov
Department of Economics
University of Wisconsin
1180 Observatory Dr.
Madison, WI 53706
sseverin@ssc.wisc.edu

1 Introduction

In the United States, more than two-thirds of testate decedents with multichild families divide their estates exactly equally among their children.¹ This finding challenges the validity of existing theories regarding the determination of intergenerational transfers. If testators select the bequest to each heir by balancing marginal benefits against marginal costs, then it is natural to think of equal division as a knife-edge, “measure zero” event. If, for example, bequests reflect altruism (as in Becker [8] or Barro [7]) or intrafamily exchange (as in Bernheim, Shleifer, and Summers [15] or Kotlikoff and Spivak [27]), one would ordinarily expect the optimal bequest to vary with the characteristics of the child, so that children with different characteristics would receive different bequests. These observations give rise to the “equal division puzzle.”

Though economic theory has been almost completely silent on this subject,² two simple possibilities merit consideration. The first is that most individuals are essentially indifferent about the division of their estates. One might, for example, attempt to explain the prevalence of equal division by appealing to the “accidental” theory of intergenerational transfers, in which bequests occur only because individuals are uncertain about the timing of their deaths, and are unable to purchase private annuities due to a failure of private insurance markets.³ The second possibility is that parents feel constrained by a notion of “fairness,” the origins of which are unclear, but presumably external to the process governing the choice of bequests.⁴

¹According to Wilhelm [35], 68.6 percent of all decedents with multichild families divide their estates exactly equally between their children, and 76.6 percent divide their estates so that each child receives within 2 percent of the average inheritance across all children. Two studies by Menchik [29], [30], place the frequency of exact equal division at, respectively, 62.5 percent and 84.3 percent. Tomes [34] obtains a significantly lower figure (21.1 percent), though he also finds that children received within \$500 of the average inheritance in 50.4 percent of all cases. Menchik [30] argues that the lower frequency of equal division in Tomes’ sample reflects data problems.

²Bernheim, Shleifer, and Summers [15] argue that the strategic exchange motive has fewer difficulties with the prevalence of equal division than other theories of bequests, but they do not provide a theoretical framework that yields equal division as a robust prediction.

³There is considerable evidence against the accidental bequest hypothesis, including the fact that many individuals appear to resist annuitization. See, for example, Bernheim Shleifer and Summers [15], Bernheim [10], or Gale and Scholz [22]. See also Hurd [25], [26], for an opposing perspective.

⁴Readers and seminar participants have suggested several other possibilities, but in our view none survives

The “accidental bequest” theory has difficulty accounting for the prevalence of equal division among testate decedents: if an individual is truly indifferent as to the division of his or her bequest, why bother specifying this division in a will? Moreover, both the “accidental bequest” theory and the “fair division” theory run afoul of another well-documented pattern: though bequests are usually equal across children, *intra vivos* gifts are not (see e.g. McGarry [28] and Dunn and Phillips [21]). It is hard to imagine why parents would be indifferent about the division of their estates if they care about the distribution of gifts. It is also difficult to understand why an external notion of equal division as a fair outcome would apply to bequests but not to gifts. Thus, the unequal division of gifts deepens the equal division puzzle.

The absence of a coherent theoretical explanation for the equal division puzzle represents a serious gap in the literature. Bequest motives feature prominently in theoretical and empirical discussions of capital accumulation, fiscal policy, income distribution, and other issues. Moreover, the altruistic model of bequests is often invoked as a justification for widespread practice of studying models with infinite-lived agents (“dynastic” families). As long as one of the most notable empirical regularities concerning bequests remains unexplained, economists must carefully qualify all conclusions that are linked to assumptions about the motivations for bequests.

The primary objective of this paper is to propose a theory of intergenerational transfers that accounts for the equal division puzzle, including the unequal division of gifts. We consider a model in which an altruistic, utility-maximizing parent divides his or her estate between potentially heterogeneous children. To this relatively standard framework we add a close scrutiny. For example, a few of our colleagues have suggested that equal division of bequests will always be *ex post* optimal if parents correctly anticipate this division and compensate appropriately for differences in affection and/or endowments through *intra vivos* gifts. Notably, this theory is equally consistent with any other division of bequests (e.g. 25%-75%) – the parent can always choose gifts that render any given rule for bequest division *ex post* optimal. More importantly, the available evidence indicates that gifts are only partially compensatory. If the theory were correct, then, prior to death, the typical parent would constantly adjust gifts on the margin in response to changes in preferences, resources, and/or other conditions to preserve the optimality of equal bequest division. In practice, neither the frequency nor the volume of gifts seems sufficient for this purpose.

new twist: each child's perception of parental affection directly affects his or her subjective well-being. This assumption is grounded in psychological evidence (see e.g. Coopersmith [19], or Bednar and Peterson [9]). More particularly, patterns of sibling rivalry suggest that children care about the extent to which they are loved and/or valued by a parent, relative to brothers and sisters (see e.g. Baik and Kahn [4]).

By itself, this formulation of children's preferences would not produce a norm of equal division.⁵ Our theory requires one additional plausible assumption: children cannot directly observe the parent's preferences (in particular, the parameter governing the relative strength of altruism for each child). Instead, children may attempt to infer the parent's preferences from his or her actions, including the division of an estate. Being altruistic, the parent must then consider the possibility that certain actions might hurt a child's feelings. In particular, an unequal division of bequests suggests that the parent loves one child less than another.

In this setting, bequests serve as a signal of parental affection. Moreover, since the bequest to each child is observable to all children, the signal is common to all affected audiences. Under certain conditions identified in the text, no separating equilibrium exists, but there is an equilibrium in which a positive fraction of the population adheres to a norm of equal division. Depending on the parameters of the model, this fraction can be arbitrarily large. Though other pooling equilibria may also exist, we justify the selection of the equal division equilibrium formally.

The intuition for this result is straightforward. Under certain conditions, parents wish children to believe that they love their children more equally than they actually do. Thus, parents shade their choices away from their preferred child to differentiate themselves from those who love their children even less equally. But those who actually love their children equally have nowhere to go: they cannot shade their choice to one child or to the other without encouraging imitation by some types of parents who love their children unequally. Consequently, the equilibrium tends to produce a pool at the center of the parental type

⁵If the parent's preferences were public information, equal division would occur in our model only on a set of measure zero.

space.

The discussion in the preceding paragraph suggests that there is a formal relation between the current paper and Bernheim's [11] theory of conformity. We elaborate on this relation in section 2.4. For the time being, we simply note that our results do not follow from Bernheim's. Both models are examples of "multidirectional signaling" (Bernheim and Severinov [14]), but neither is a special case of the other. The analysis in this paper is considerably more complex, and there are some important substantive differences with respect to the structure of equilibria. For example, the equilibria of the current model involve pure communication (sometimes known as "cheap talk"), as well as the use of dissipative signals. In particular, among those adhering to the norm of equal division, subgroups may differentiate themselves through credible verbal statements.

Our analysis also accounts for the unequal division of *intra vivos* gifts. Specifically, we argue that the key difference between gifts and bequests relates to observability: the division of bequests is perfectly observable by all concerned parties, whereas the division of gifts need not be. A parent can give gifts to a favored child without revealing this to another child; however, a parent cannot make bequests to any child without the knowledge of all children. As long as "secret" gifts are feasible, gifts cannot serve as a common signal of the parent's preferences. When neither child is in a position to verify that the parent's resources have been divided equally, then the equilibrium inference function cannot systematically link the children's beliefs about the parent's preferences to the equality of transfers. Without such a link, an equal division norm cannot survive.

There are also conditions under which a parent wishes his or her children to believe that the parent's affection is less equal than it actually is. In such cases, parents shade their choices toward their preferred child to differentiate themselves from those who love their children more equally. But those who prefer to give everything to one child cannot shade their choices in this direction, since they face binding non-negativity constraints on bequests. This leads to pooling at the extreme outcomes where one child or the other receives

the parent's entire estate. Thus, under appropriate parametric assumptions, our model can also account for the pattern known as unigeniture. This is of interest, since unigeniture is a common norm in many societies (see e.g. Chu [18], or Guinnane [23]).

Since our theory provides a possible explanation for the equal division puzzle, it is important to explore ancillary implications for public policy. In a companion piece (Bernheim and Severinov [13]), we examine the effects of government fiscal policies under the hypothesis that bequests signal parental affection. Under conditions normally thought to produce Ricardian equivalence (see Barro [7]), we demonstrate that outcomes can be distinctly non-Ricardian. Indeed, we provide an example in which, with even the tiniest degree of concern about parental affection, an exogenous transfer between parents and children has exactly the same effect as in a model without any intergenerational linkages, even though essentially all families are internally linked by operative, altruistic, intergenerational transfers.

The paper is organized as follows. We describe the model in section 2. Section 3 identifies the conditions under which a norm of equal division emerges. Section 4 investigates the conditions that give rise to unigeniture. Section 5 concludes.

2 The Model

2.1 Participants and preferences

We consider interactions among three parties: a parent (P) and two children ($i = 1, 2$). The parent is endowed with wealth $w_P > 0$, which it divides between the children by making non-negative bequests, $b_i \geq 0$. Throughout this paper, we focus on the division of bequests and abstract from the possibility that the parent might consume some portion of its resources. Thus, the parent chooses bequests to satisfy the constraint $b_1 + b_2 = w_P$. Each child i is endowed with wealth $w_i > 0$, and consumes $c_i = w_i + b_i$. For simplicity, we assume that all parties can observe each others' endowments; we examine the role of this assumption in section 3.3.1.

It is perhaps most natural to interpret the model as follows. A parent nears the end of his

or her life with bequeathable resources w_P , and must decide how to divide these resources between two adult children. Rapid consumption is not an attractive option. Notably, the model does not attempt to explain why so many individuals reach the end of life with positive bequeathable assets; it simply takes this empirical pattern as given. This may appear to be somewhat problematic, since (as we will see) the theory implies that there are incentives for parents to make intergenerational transfers as gifts prior to death, instead of as bequests. This conceptual difficulty is, however, easily resolved, as one can explain the retention of bequeathable assets by appealing to factors outside of the model.⁶ Our theory presupposes the existence of such factors, and attempts to explain the division of whatever bequeathable assets are retained until death.

It will be convenient to think of the parent as dividing the family's total resources, $W \equiv w_P + w_1 + w_2$. Specifically, the parent picks $x \in [\underline{x}, 1 - \bar{x}]$ where $\underline{x} \equiv w_1/W$ and $\bar{x} \equiv w_2/W$, child 1 consumes xW , and child 2 consumes $(1 - x)W$. With this change of variables, it is important to keep in mind that the phrase "equal division" is generally used to refer to the parent's endowment, rather than to the family's resources. That is, equal division occurs when $b_1 = b_2$, or equivalently when

$$x = x^E \equiv \left[1 + \frac{w_1}{W} - \frac{w_2}{W} \right] / 2,$$

and not when $x = 1/2$.

We use U_P and U_i to denote the utilities of the parent and children, respectively. We assume that the parent is an altruist. Since the parent does not consume anything directly, its utility depends only on the outcomes for the children:

$$U_P = tU_1 + (1 - t)U_2,$$

where $t \in [0, 1]$. Parents differ according to the relative weight t that they attach to the first child's utility. We assume that t is known to the parent but not to the children. Children's

⁶The retention of bequeathable assets until the end of life may result from uncertainty concerning the length of life combined with imperfections in annuity markets. Individuals may also derive feelings of security, control, and/or satisfaction from asset ownership.

prior beliefs about t are given by some atomless cumulative distribution function F , and the support of F is the interval $[0, 1]$. We use f to denote the density function associated with F , and we assume that f is symmetric around $1/2$ (that is, $f(t) = f(1 - t)$).

Our assumptions about F merit further discussion. Presumably, every adult has had many opportunities to learn about his or her parent's preference parameter, t . Nevertheless, we believe it is reasonable to assume that some significant degree of uncertainty remains. It is perhaps less reasonable to believe that, after numerous family experiences, the subjective distribution of t is symmetric around $\frac{1}{2}$ (indicating equal probability that the parent prefers each child). We make the symmetry assumption because it allows us to simplify some of the formal proofs. Conceptually, cases in which t is distributed asymmetrically are similar to cases in which the children have unequal endowments. Were we to allow for asymmetric distributions, our central conclusions would survive.⁷

We assume that each child cares about his or her own consumption, c_i , as well as about t . That is, each child's sense of well-being is affected by the extent to which he or she feels "loved" relative to a sibling. Though children cannot observe t directly, they may attempt to infer it from aspects of the parent's behavior, including the choice of x . When the children believe that $t = \hat{t}$, their utilities are given by

$$U_1 = u(c_1) + \beta v(\hat{t})$$

and

$$U_2 = u(c_2) + \beta v(1 - \hat{t}),$$

where u is defined over $[0, +\infty)$, and v is defined over $[0, 1]$. The parent's utility is therefore

$$U_P = [tu(xW) + (1 - t)u((1 - x)W)] + \beta [tv(\hat{t}) + (1 - t)v(1 - \hat{t})] \quad (1)$$

⁷Formally, one must make a different set of technical arguments to accommodate cases in which t is distributed asymmetrically. Analogously to theorem 4 in section 3.2.2, one can show that (generically) an equilibrium with a norm of equal division exists provided that neither the division of children's endowments nor the distribution of t is too asymmetric. Bernheim and Severinov [14] establish some general results along these lines.

$$\equiv U(x, t) + \beta V(\hat{t}, t)$$

We will invoke the following assumption throughout.

Assumption 1: u and v are strictly increasing, strictly concave, twice continuously differentiable on (respectively) $(0, W]$ and $[0, 1]$, $\lim_{c \rightarrow 0} cu'(c) = +\infty$, $v'(0)$ is finite, and $v'(1) = 0$.

Most of assumption 1 is reasonably standard. Weaker conditions would suffice for most of our results. The derivative restrictions are particularly expendable,⁸ but relaxing them complicates some of the proofs. Moreover, they strike us as plausible. Since it is possible to live without parental affection but not without consumption, it seems reasonable to assume that the derivative of v is finite at $t = 0$, while the derivative of u is infinite at $c = 0$. It also strikes us as natural to assume that a child is satiated when he or she has all of the parent's affection.

From assumption 1, it follows that $U(x, t)$ is twice continuously differentiable on $[\underline{x}, 1 - \bar{x}] \times [0, 1]$, $V(\hat{t}, t)$ is twice continuously differentiable on $[0, 1]^2$, $U_{11}(\cdot) < 0$, $V_{11}(\cdot) < 0$, $U_{12}(\cdot) > 0$, and $V_{12}(\cdot) > 0$.

2.2 The timing of decisions

The structure of the game is simple. After observing t , the parent takes actions. In addition to selecting x , the parent may also send a message, $m \in [0, 1]$. This message is intended to represent "pure" communication about the parent's type, in the sense that the value of m does not directly enter the utility function of any party (in the pertinent literature, this is usually referred to as "cheap talk"). We will elaborate further on the role of pure

⁸For example, since c_i is bounded away from zero ($c_i \geq w_i > 0$), the properties of u near zero are inconsequential. We assume that the derivative of u becomes arbitrarily large as c goes to zero only to simplify a construction used in the proof of theorem 2. If u did not satisfy this property, one could simply construct and work with another function \tilde{u} satisfying assumption 1 with $\tilde{u}(c) = u(c)$ on $[\underline{x}, \bar{x}]$.

communication later in this section. Children then observe x and m and draw inferences about the parent's preference parameter, t . The preceding expressions for U_1 , U_2 , and U_P describe the resulting payoffs.⁹

In this setting, the parent's choices of x and m can signal the parent's type, t . Indeed, the model is recognizable formally as a "signaling game," in the sense of Banks and Sobel [6] or Cho and Kreps [16]. Specifically, the parent acts as the "sender," the children act as the "receivers," (x, m) serves as the sender's "message," and \hat{t} serves as both the receivers' inference and the receivers' "response." While it is somewhat unconventional to identify the receivers' inference with the receivers' response, this is easily reconciled with the standard game-theoretic approach.¹⁰

Despite these observations, it is important to emphasize that our model does not give rise to a standard signaling problem. In particular, the Spence-Mirrlees "single-crossing" property is not satisfied. The non-standard aspects of the model lead to atypical results, which we describe in subsequent sections.

2.3 Parental bliss points

Ignoring for the moment the possibility that children may infer \hat{t} from x , we can optimize U_P over x and \hat{t} to find the parent's "bliss point." Obviously, the parent's problem is separable in x and \hat{t} , so the bliss point for x (the "action bliss point") does not depend upon \hat{t} , and the bliss point for \hat{t} (the "perception bliss point") does not depend upon x .

To find the action bliss point for x , we maximize $U(x, t)$ over $x \in [0, 1]$ (notice that, for this purpose, we consider all possible distributions of the family's resources and ignore

⁹Naturally, the children do not receive bequests and draw inferences about the parent's preferences until after the parent dies. In effect, we are assuming that the parent correctly anticipates the inferences that children will make after the parent's death, and that the children attempt to make the best inferences possible. We do not explore the interesting possibility that children might have incentives to engage in self-deception, intentionally forming incorrect inferences. The phenomenon of self-deception merits further consideration in future research.

¹⁰Instead of assuming that the parent cares directly about a child's inference, assume that the parent cares about the child's reaction to his or her inference. One can then renormalize the set of possible reactions to conform with the set of possible inferences. In other words, one can use \hat{t} to denote a child's reaction to the inference that the value of the parent's altruism parameter is \hat{t} .

the requirement that $b_i \geq 0$). Let $X(t)$ denote the mapping from preferences to action bliss points. Because $u(xW)$ is strictly concave, $X(t)$ can be found as the solution of the following first-order condition:¹¹

$$tu'(X(t)W) = (1-t)u'((1-X(t))W) \quad (2)$$

From assumption 1, it follows that $X(t)$ is well-defined, single-valued, strictly increasing, and continuous with $X(0) = 0$, $X(1) = 1$, and $X(1/2) = 1/2$.

Example 1: $u(c) = c^\gamma/\gamma$, with $\gamma < 0$ (so that $\lim_{c \rightarrow 0} cu'(c) = +\infty$, as required in assumption 1). Then $X(t) = (1-t)^\eta [t^\eta + (1-t)^\eta]^{-1}$, where $\eta = 1/(\gamma - 1)$.

To find the perception bliss point for \hat{t} , we maximize $V(\hat{t}, t)$ over $\hat{t} \in [0, 1]$. Let $p(t)$ denote the mapping from preferences to perception bliss points. By strict concavity of $v(\cdot)$, $p(t)$ can be found as the solution to the following first-order condition:¹²

$$tv'(p(t)) = (1-t)v'(1-p(t)) \quad (3)$$

From assumption 1, it follows that $p(t)$ is well-defined, single-valued, strictly increasing, and continuous with $p(0) = 0$, $p(1) = 1$, and $p(1/2) = 1/2$.

Example 2: $v(\hat{t}) = -(1-\hat{t})^2$. Then $p(t) = t$.

Note that $U_1(x, t) > 0$ when $x < X(t)$, and $U_1(x, t) < 0$ when $x > X(t)$. In words, $U(x, t)$ is single-peaked in x , with a maximum at $x = X(t)$. Similarly, $V_1(\hat{t}, t) > 0$ when $\hat{t} < p(t)$, and $V_1(\hat{t}, t) < 0$ when $\hat{t} > p(t)$. In words, $V(\hat{t}, t)$ is single-peaked in \hat{t} , with a maximum at $\hat{t} = p(t)$.

In figure 1, we exhibit indifference contours for two types of parents, t and t' . The indifference contours for type t are ellipses centered at the point $(p(t), X(t))$; similarly, the

¹¹Since $\lim_{c \rightarrow 0} u'(c) = +\infty$, we know that the solution to the first order condition is interior for all $t \in (0, 1)$. Plainly, $X(0) = 0$ and $X(1) = 1$.

¹²Since $u'(0)$ is finite and $u'(1) = 0$, we know that the solution to the first order condition is interior for all $t \in (0, 1)$. Plainly, $p(0) = 0$ and $p(1) = 1$.

indifference contours for type t' are ellipses centered at the point $(p(t'), X(t'))$. Notice that these contours generally cross twice or not at all. This illustrates the failure of the single-crossing property noted at the end of the previous section.

2.4 The direction of imitation

The perception bliss point function, $p(t)$, plays a critical role in our analysis. Its relation to t is particularly important. In this paper, we will largely concern ourselves with two special cases.

Condition 1 (*imitation towards the center*): $p(t) > t$ for $t \in (0, \frac{1}{2})$, $p(t) < t$ for $t \in (\frac{1}{2}, 1)$, and $p'(\frac{1}{2}) < 1$.

We depict a perception bliss point function satisfying condition 1 in figure 2. Note that all types wish to be perceived as loving their children more equally than they actually do. Consequently, imitation tends to occur towards the center of the type space.

Condition 2 (*imitation towards the extremes*): $p(t) < t$ for $t \in (0, \frac{1}{2})$, $p(t) > t$ for $t \in (\frac{1}{2}, 1)$, and $p'(\frac{1}{2}) > 1$.

We depict a perception bliss point function satisfying condition 2 in figure 3. Note that all types wish to be perceived as loving their children less equally than they actually do. Consequently, imitation tends to occur towards the extremes of the type space.

Naturally, it is possible to have configurations in which some types wish to be perceived as loving their children more equally than they actually do, while other types wish to be perceived as loving their children less equally. We do not, however, consider these mixed cases in the current paper; see Bernheim and Severinov [14] for further discussion.

Example 3: Suppose that $v(\hat{t}) = h\left(-\left(1 - \hat{t}\right)^2\right)$, and that v otherwise satisfies assumption 1. Then condition 1 is satisfied if h is concave, and condition 2 is satisfied if h is convex. In example 2, we considered the special case where h is linear; $p(t) = t$ corresponds to the boundary between conditions 1 and 2.

At this point, it is useful to place our model within the broader context of signaling problems. Standard signaling models (e.g. Spence [33]) assume that all senders wish to be perceived as the “highest” type, and imitation occurs in only one direction. In our framework, this would correspond to the assumption that $p(t) = 1$ for all t .

Crawford and Sobel [20] consider cases where, as in our model, $p(t)$ is strictly increasing, but they also assume that, as in the standard setting, $p(t) > t$ for all t . Consequently, imitation only occurs in one direction. They focus exclusively on signaling through pure communication (cheap talk), and do not permit senders to take costly (and therefore potentially discriminatory) actions. They demonstrate that a limited degree of separation through cheap talk is usually possible: in some equilibria, senders willingly segment themselves into a finite number of groups.

Austen-Smith and Banks [3] have extended the model of Crawford and Sobel by allowing costly signalling (burning money). They assume that the marginal cost of action is type-independent. In their model, a set of types take costly action, while other types segment into a number of pools through cheap talk.

Bernheim [11] and Banks [5] consider cases where $p(t) = 1/2$ for all t . Their models are related to the problem that emerges here under condition 1, in that imitation occurs towards the center of the type space (it is therefore multidirectional, as in the current setting). However, these papers do not allow for the possibility that $p(t)$ is increasing (as in Crawford and Sobel), and neither examines the potential role of cheap talk. Both Bernheim and Banks exhibit signaling equilibria in which there is a central pool.

One can think of our model as combining features of the settings studied by Crawford-Sobel on the one hand, and Bernheim-Banks on the other. In our model the non-monotonicity of the payoff from action $U(x, t)$ and the payoff from perception $V(t, p)$ generate double-crossings of indifference curves. Increasing perception bliss point implies that there is a role for cheap talk. However, since imitation is multi-directional (the perception bliss point function also crosses the 45 degree line), the model also yields central pooling with respect

to the costly action, as in Bernheim-Banks. In ongoing work, we are studying more general cases of multidirectional signaling that subsume the current model (in particular, we allow for arbitrary increasing perception bliss point functions).

2.5 The structure of equilibria

A signaling equilibrium involves a pair of choice functions, $\mu(t)$ and $\gamma(t)$, mapping the parent's type t to, respectively, decisions concerning the division of bequests, x , and a pure message, m , as well as an inference function $\phi(\hat{t}, x, m)$ mapping choices of x and m into probability distributions over perceived type, \hat{t} .¹³ The choice function must prescribe optimal decisions for all types t given the inference function. Conversely, the inference function must be consistent with the choice function, in the sense that it is derived from the choice function by applying Bayes law for all choices (x, m) occurring with positive likelihood in equilibrium.

For equilibria with perfect separation of types, the inference function always prescribes a degenerate probability distribution (in other words, it indicates that the children believe they know the parent's type with certainty) for each (x, m) in the image of the choice function. The parent's utility is then determined directly from equation (1). Unfortunately, equation (1) does not allow us to evaluate the parent's utility when the inference is given by a non-degenerate probability distribution. Knowing how to evaluate the parent's utility in such circumstances is, of course, essential if one intends to study equilibria with pooling.

In standard signaling models, non-degenerate inferences do not pose any analytic problems. The receiver selects a response that maximizes expected utility given the inference, and the sender's utility is affected by this response. Non-degenerate inferences are somewhat more problematic in the context of the current model because we have not been explicit about the process governing a child's "reaction" to its beliefs about the parent's type. In particular, we have not explicitly assumed that this reaction is governed by maximization of expected utility. Indeed, in some instances this assumption may be unnatural; for example,

¹³Naturally, $\mu(t)$ and $\phi(\hat{t}, x, m)$ also depend on the initial conditions (\underline{x}, \bar{x}) , which parameterize the problem. We omit the dependence on initial conditions for notational brevity.

the reaction may amount to “hurt feelings” that are not consciously chosen to maximize anything.

We finesse this difficulty by making a stylized but relatively unrestrictive assumption concerning the mapping from non-degenerate inferences to payoffs. In particular, we assume that there is a mapping \widehat{B} from beliefs about types (probability distributions) into types, with the following interpretation: if the children’s beliefs about the parent’s type are summarized by the probability distribution ϕ , they react the same as if they knew with certainty that the parent’s type was $\widehat{B}(\phi)$. The parent’s utility is then given by equation (1), where $\widehat{B}(\phi)$ replaces the term \widehat{t} .

Naturally, it is difficult to proceed analytically unless we impose some restrictions on the mapping \widehat{B} . A small number of minimal and relatively unobjectionable restrictions suffice for our purposes. We impose some of these restrictions only for cumulative distribution functions, $\phi(t)$, of the following type:

$$\phi(t) = \begin{cases} (F(t) - F(r)) / (F(s) - F(r)) & \text{for } t \in [r, s) \\ 1 & \text{if } t \geq s \\ 0 & \text{if } t < r \end{cases}$$

where $0 \leq r \leq s \leq 1$. In other words, these are the posterior probability distributions that emerge when the children learn only that the parent’s type lies between some r and some s . For such ϕ , we can write $\widehat{B}(\phi)$ as $B(r, s)$. We assume that the functions \widehat{B} and B have the following properties:

Assumption 2: (i) If, for some t' , $\phi(t') = 1$ and $\phi(t) = 0$ for $t < t'$, then $\widehat{B}(\phi) = t'$. (ii) Consider two cumulative distribution function, ϕ' and ϕ'' , such that $\phi'(t) \leq \phi''(t) \forall t$, with strict inequality for some t . Then $\widehat{B}(\phi') > \widehat{B}(\phi'')$. (iii) $B(r, s)$ is twice continuously differentiable and $B(r, s) = 1 - B(1 - s, 1 - r)$.

Part (i) is essentially a tautology: it states that, if the children are certain the parents type is t' , then they react as if they know with certainty that the parent’s type is t' . Part (ii) requires that, if the children’s beliefs shift towards higher types, then they react as if the

parent is a higher type. Part (iii) includes a technical differentiability condition along with the requirement that B is symmetric around $1/2$. This is in the spirit of other smoothness and symmetry assumptions that are built into our model. From these assumptions, one can derive two additional properties: first, $B(r, s)$ is increasing in r and s (this follows directly from part (ii)); second, for $r < s$, $B(r, s) \in (r, s)$ (this follows from part (i), along with the fact that $B(r, s)$ is increasing and continuous in r and s).

One example of a function B satisfying assumption 2 is the mean value operator, i.e.

$$B(\underline{t}, \bar{t}) = \int_{\underline{t}}^{\bar{t}} t\phi(t)dt.$$

However, we do not limit our analysis to this particular function.

2.6 Equilibrium, pooling, and social norms

Our objective in this paper is to provide a theory that accounts for the stability of and widespread adherence to social norms governing the division of bequests. We take the view that pooling provides a potential explanation for the existence of a norm. In particular, we explain the prevalence of equal division in some societies by identifying conditions that give rise to equilibria in which a non-trivial fraction of the population chooses $x = x^E$. We account for the emergence of this norm by demonstrating that the corresponding pooling equilibria have attractive properties, and we attribute the stability of this norm to the robustness of the equilibria. Similarly, we explain the prevalence of unigeniture in other societies by identifying conditions that give rise to equilibria in which a non-trivial fraction of the population chooses $x \in \{0, 1\}$, and we account for the emergence and stability of this norm in a similar way. Our analysis reveals that a norm of equal division arises under condition 1 (imitation towards the center), while a norm of unigeniture arises under condition 2 (imitation towards the extremes). The next two sections examine these cases in order.

3 Imitation Toward the Center

We begin our investigation by studying the properties of equilibria when condition 1 (imitation towards the center) is satisfied. Our analysis proceeds in two steps. First, we identify circumstances under which complete separation is impossible. Under these circumstances, equilibrium necessarily entails some pooling. Second, we investigate the existence, properties, and robustness of pooling equilibria for which $x = x^E$ is the unique social norm. The section closes with a discussion of the distinction between gifts and bequests, in which we account for the observation that the equal division norm applies to the latter but not to the former.

3.1 The existence and non-existence of separating equilibria

As mentioned in section 2.4, when condition 1 is satisfied, our framework is related to Bernheim's [11] model of social conformity. The central difference is that Bernheim restricted attention to the case where $p(t) = 1/2$ for all t . As it turns out, this restriction simplifies the analysis considerably. In particular, Bernheim proves at the outset of his investigation that equilibrium action functions must be weakly monotonic, in the sense that higher types (those with higher action bliss points) choose weakly higher actions. This result follows directly from a few simple lines of algebra, and provides a useful starting point for much of his analysis. Unfortunately, for the more general class of perception bliss point functions considered here, monotonicity of the equilibrium action function is not generally guaranteed.

Figure 4 exhibits a potential failure of monotonicity. To keep things simple, we have assumed for the purposes of this illustration that there are only two types, s and $1 - s$ ($s < 1/2$), and that each child has endowment w . Consider the following set of actions and inferences: s chooses $1 - w/W$ and $1 - s$ chooses w/W ; the children infer that the parent is type s if they observe $1 - w/W$, that the parent is of type $1 - s$ if they observe w/W , and that s and $1 - s$ are equally likely if they observe anything else. If the indifference curves of types s and $1 - s$ are as depicted in the figure, then this is a separating equilibrium, even though

it violates monotonicity (the lower type chooses a higher action). From an inspection of the figure, it is readily apparent one cannot construct such an example if $p(s) = p(1-s) = 1/2$.

Fortunately, this does not mean that our analysis must allow for non-monotonic separating equilibria. Our first result demonstrates that the preceding example depends on the discreteness of the type space. When there is a continuum of types (as assumed in section 2), separating equilibria must have the property that x is strictly monotonic in the parent's type, t .

Theorem 1 *Suppose that condition 1 is satisfied. Then, in any separating equilibrium with endowments given by some pair (\underline{x}, \bar{x}) , $\mu(t)$ is strictly monotonically increasing and continuous, $\mu(t) > X(t)$ for $t \in (0, \frac{1}{2})$, $\mu(t) < X(t)$ for $t \in (\frac{1}{2}, 1)$, and $\mu(\frac{1}{2}) = \frac{1}{2}$.*

Proof: See Appendix A.

Although theorem 1 is considerably more subtle and difficult to prove than the corresponding result in Bernheim [11], its implications are similar. For parents with $t < \frac{1}{2}$, the direction of imitation is upwards, while for parents with $t > \frac{1}{2}$, the direction of imitation is downwards; in both cases, parents "lean" towards the center. One can therefore proceed by examining the nature of separating functions emanating from the two extreme points, $t \in \{0, 1\}$.

Accordingly, let $\underline{\mu}(t)$ denote the separating function for $t \in [0, 1/2)$ emanating upward from $t = 0$, $\underline{\mu}(0) = \underline{x}$, and let $\bar{\mu}(t)$ denote the separating function for $t \in (1/2, 1]$ emanating downward from $t = 1$, $\bar{\mu}(t) = \bar{x}$. The indifference contours of each type $t \in [0, \frac{1}{2})$ must be tangent to the equilibrium action function, $\underline{\mu}(t)$, at the equilibrium outcome assigned to that type. Thus, $\underline{\mu}(t)$ corresponds to the solution of the differential equation

$$\underline{\mu}'(t) = -\frac{\beta V_1(t, t)}{U_1(\underline{\mu}(t), t)} \quad (4)$$

on the interval $t \in [0, \frac{1}{2})$, with the initial condition $\underline{\mu}(0) = \underline{x}$ (there are analogous expressions that determines $\bar{\mu}(t)$ for $t \in (\frac{1}{2}, 1]$). The existence and uniqueness of $\underline{\mu}(t)$ and of $\bar{\mu}(t)$ follow

from modifications of standard arguments (see the proof of theorem 2 in appendix A). We know from the proof of theorem 1 that $\lim_{t \uparrow \frac{1}{2}} \underline{\mu}(t) \geq \frac{1}{2}$, and that $\lim_{t \downarrow \frac{1}{2}} \bar{\mu}(t) \leq \frac{1}{2}$. We also know from the theorem that there can be no separating equilibrium if either of these expressions holds with strict inequality.¹⁴ Consequently, full separation of types is impossible unless

$$\lim_{t \uparrow \frac{1}{2}} \underline{\mu}(t) = \lim_{t \downarrow \frac{1}{2}} \bar{\mu}(t) = \frac{1}{2} \quad (5)$$

Equation (5) may or may not hold in any given instance. The equation is less likely to hold – and separating equilibria are therefore less likely to exist – when types $t \in (0, \frac{1}{2})$ must lean further to the right (or when types $t \in (\frac{1}{2}, 1)$ must lean further to the left) to discourage imitation. Generally, for larger values of β (the utility weight attached to perceptions) and/or larger child endowments (\underline{x} and \bar{x}), any given $t \in (0, \frac{1}{2})$ must shade its choice further to the right to differentiate itself from lower types, and any given $t \in (\frac{1}{2}, 1)$ must shade its choice further to the left to differentiate itself from higher types. These observations lead to the following result:

Theorem 2 *Suppose that condition 1 is satisfied. For any pair of endowments (\underline{x}, \bar{x}) , there exists $\beta^*(\underline{x}, \bar{x})$ such that a fully separating equilibrium exists if and only if $\beta \leq \beta^*(\underline{x}, \bar{x})$. $\beta^*(\underline{x}, \bar{x})$ is decreasing in $\max\{\underline{x}, \bar{x}\}$, and strictly positive iff $\max\{\underline{x}, \bar{x}\} < \frac{1}{2}$.*

Proof: See Appendix A.

From theorem 2, we know that full separation is impossible when children attach sufficient importance to parental affection ($\beta > \beta^*(\underline{x}, \bar{x})$). It follows that any equilibrium must then involve some pooling. In the next subsection, we investigate the structure of pooling equilibria.

¹⁴Assume that one or both of these conditions holds with strict inequality. It might appear that one could construct an equilibrium with complete separation using $\underline{\mu}$ and $\bar{\mu}$, relying on cheap talk signals, m , to achieve separation in the region of overlap. However, this is not the case. Theorem 1 establishes that, with a continuum of types, $\mu(t)$ must be strictly monotonic even when we allow for cheap talk. Consequently, there can be no region of overlap.

3.2 Equilibria with pooling

3.2.1 Central pooling equilibria

As is usually the case with signaling models, our problem gives rise to a wide variety of pooling equilibria. We will begin by studying a particular class of equilibria with a relatively simple structure. The action functions for these equilibria are characterized by three parameters, t_ℓ , t_h , and x_p , satisfying $0 \leq t_\ell < \frac{1}{2} < t_h \leq 1$, $x_p \in [\underline{x}, \bar{x}]$, and $\underline{\mu}(t_\ell) \leq x_p \leq \bar{\mu}(t_h)$.¹⁵ Specifically,

$$\mu(t) = \begin{cases} \underline{\mu}(t) & \text{for } t < t_\ell \\ \bar{\mu}(t) & \text{for } t > t_h \\ x_p & \text{for } t \in [t_\ell, t_h] \end{cases}$$

Henceforth, we will refer to these as “central pooling” equilibria. Separation occurs for low types ($t < t_\ell$) and for high types ($t > t_h$). However, all intermediate types ($t \in [t_\ell, t_h]$, including $t = \frac{1}{2}$) select the same action, x_p . We refer to x_p as the *norm* of this equilibrium.

To be clear, this does not necessarily mean that all $t \in [t_\ell, t_h]$ find themselves in the same pool. Even if the intermediate types all choose the same action, they may still choose to differentiate themselves through cheap talk (pure communication, m). Indeed, conditional upon choosing the same action, these individuals effectively play a pure communication game similar to the one studied by Crawford and Sobel [20].¹⁶ As it turns out, we cannot guarantee the existence of central pooling equilibria unless we allow for cheap talk. Thus, as a general matter, dissipative signaling and pure communication coexist in the equilibria of our model. When, for example, a norm of equigeniture prevails, parents who divide their bequests equally may nevertheless make informative statements about the extent to which they favor one child or the other.

Our next result establishes that central pooling equilibria always exist when full separation is infeasible.

¹⁵Here and below we omit the dependence of $\underline{\mu}(t)$ and $\bar{\mu}(t)$ on the endowments (\underline{x}, \bar{x}) .

¹⁶This pure communication game differs from Crawford and Sobel’s framework in two respects. First, it is not the case that $p(t) > t$ for all t within the group choosing x_p . Second, the equilibrium of this pure communication game must be consistent with the equilibrium conditions for the overall game.

Theorem 3 *Suppose that condition 1 is satisfied and that $\beta > \beta^*(\underline{x}, \bar{x})$. Then there exists a central pooling equilibrium with some norm x_p , in which parent types choosing x_p separate themselves into a finite number (possibly just one) of subgroups through pure communication (m), and in which each of these subgroups is a connected interval.*

Proof: See Appendix A.

When a central pooling equilibrium prevails, we interpret x_p as a social norm. Those with sufficiently extreme preferences deviate from the norm, but those with intermediate preferences adhere to it. Those adhering to the norm may further differentiate themselves to a limited extent through credible verbal statements.

It is important to emphasize that one cannot guarantee the existence of central pooling equilibria unless parents have the ability to send pure messages (cheap talk). To understand this point, assume that $p(t)$ is very close to t , so that for all t , $V(t, t) - V(B(t, \frac{1}{2}), t) > 0$. In that case, no type t could serve the role of t_ℓ unless the central pool is further segmented by cheap talk. From the point of view of any such type t , the action associated with the central pool is worse than t 's separating action (since $x_p \geq \underline{\mu}(t) \geq X(t)$). Moreover, from t 's perspective, the inference associated with the central pool, $B(t, t_h)$, is worse than the inference $B(t, \frac{1}{2})$ (since $t_h > \frac{1}{2}$ and since, by assumption, $B(t, \frac{1}{2}) > p(t)$), which is in turn worse than the inference associated with separation (t). Thus, no type t could ever be indifferent between separating and joining the central pool, which is a requirement for equilibrium.

3.2.2 Central pooling equilibria with a norm of equal division

Of course, nothing so far guarantees that it is possible to sustain a norm of equal division (the special case of $x_p = x^E$). When the children have identical endowments, the model is completely symmetric, so one would naturally expect to find a central pooling equilibrium with a norm of equal division. This, however, would not account for the robustness of the norm. Exact equality of endowments is a measure zero event, and in practice many

parents are heavily predisposed to divide bequests equally even when children's resources are unequal.

In our next result, we demonstrate that there almost always (generically) exists a central pooling equilibrium with an equal division norm provided that there is not too much inequality between the children's initial endowments. This establishes the robustness of the equal division norm. Formally, we proceed as follows. Let \mathcal{F} be the set of functions B satisfying assumption 2. We endow \mathcal{F} with the topology of uniform C^1 convergence. In this topology, two functions are close if their values and partial derivatives are close throughout the relevant domain.

Theorem 4 *Fix W and $x_k \equiv \underline{x} + \bar{x}$. Suppose that condition 1 is satisfied and that $\beta > \beta^*(\frac{x_k}{2}, \frac{x_k}{2})$. In the set \mathcal{F} there is an open-dense set subset $\tilde{\mathcal{F}}$ with the following property: for all $B \in \tilde{\mathcal{F}}$ there is some $\delta > 0$ such that if $|x_1 - \frac{x_k}{2}| < \delta$, there exists a central pooling equilibrium with a norm of x^E , in which parent types choosing x^E separate themselves into a finite number (possibly just one) of subgroups through pure communication (m), and in which each of these subgroups is a connected interval.*

Proof: See Appendix A.

Though theorem 4 does not tell us directly the fraction of the population that adheres to the equal division norm, it nevertheless illuminates this issue. In practice, the population is characterized by heterogeneity with respect to the division of resources within families (which we have not formally included in our model), as well as by heterogeneity of preferences (which is included). However, since our model describes equilibria conditional on endowments, we could easily introduce a distribution of endowments. To determine the fraction of the population adhering to the equal division norm, one would proceed as follows. First, identify every profile of endowments giving rise to a central pooling equilibrium with equal division; second, for each such profile, identify the types that divide their bequests equally. Under fairly weak conditions, theorem 4 would imply that the resulting set of individuals with the identified endowments and associated types has strictly positive measure.

Knowing that some positive fraction of the population adheres to an equal division norm does not, of course, imply that this fraction is large. In principle, the central pools might be small, and the norm might be sustainable only for small differences in endowments. It is easy to see, however, that the extent of adherence to the norm depends, among other things, on β (which measures the extent to which children care about parental affection). As β increases, $\underline{\mu}$ shifts to the right and $\bar{\mu}$ shifts to the left, increasing the region of overlap. Since the equilibria under consideration are monotonic, the fraction of the population choosing x_p must grow, and indeed one can make this fraction arbitrarily high by choosing β sufficiently large. Likewise, for larger values of β , it is natural to conjecture that one can sustain an equal division norm with greater inequality between children's endowments. If this is the case, then, for large enough β , one can generate environments in which high fractions of parents divide their estates equally.

We illustrate this point by solving for the equilibria of our model computationally. Naturally, this requires us to parameterize the model. We assume that $u(c) = c^\gamma/\gamma$ with $\gamma < 1$ (as in example 1) and $v(\hat{t}) = -(1-\hat{t})^\lambda$ with $\lambda > 2$ (a special case of example 3 wherein condition 1 is satisfied). For all of the calculations presented here, we assume that $\gamma = 0.5$ and $\lambda = 3$. Under these assumptions, we can rewrite equation (1) - which defines the parent's utility - as

$$\frac{U_P}{W^\gamma} = \left[t \frac{x^\gamma}{\gamma} + (1-t) \frac{(1-x)^\gamma}{\gamma} \right] - \left(\frac{\beta}{W^\gamma} \right) [t(1-\hat{t})^\lambda + (1-t)\hat{t}^\lambda],$$

where we have divided through by the constant W^γ . Note that, for this parameterization, changes in total family resources, W , are essentially equivalent to changes in the preference parameter β . To compute equilibria, one need only know the ratio β/W^γ .

Fixing values of \underline{x} , \bar{x} , γ , λ , β/W^γ , and x^E , we compute equilibria as follows. First, we numerically approximate the solutions to the differential equations that define the separating functions $\underline{\mu}$ and $\bar{\mu}$. Second, for some candidate value of t_ℓ , we compute the implied segments of the central pool. Specifically, we find the inference \hat{t} that makes t_ℓ indifferent between $(\underline{\mu}(t_\ell), t_\ell)$ and (x^E, \hat{t}) . We then calculate the highest type in the first segment of the central

pool, t_0 , using the expression $B(t_\ell, t_0) = \hat{t}$. Type t_0 in turn must be indifferent between the first and second segments of the central pool. We can therefore calculate the highest type in the second segment, t_1 , using the indifference condition $V(B(t_\ell, t_0), t_0) = V(B(t_0, t_1), t_0)$. We continue this process until it is no longer possible to extend the sequence, or until we have obtained a candidate value for t_h . Third, we check to see whether the requisite indifference condition is satisfied for this t_h . In particular, t_h must be indifferent between $(x^E, B(t_{N-1}, t_h))$ and $(\bar{\mu}(t_h), t_h)$ (where t_{N-1} is the penultimate type in the sequence). If the indifference condition is satisfied, the configuration is an equilibrium. To find all equilibria, we need only search exhaustively over a single dimension ($t_\ell \in [0, \frac{1}{2})$).

To illustrate these calculations, suppose that $\underline{x} = 0.2$ (child 1 has 20 percent of the family's resources), $\bar{x} = 0.45$ (child 2 has 45 percent of the family's resources), and $\beta/X^\gamma = 11$. For these parameter values, our algorithm identifies two central pooling equilibria with equal division ($x^E = 0.375$). In the first, $t_\ell = 0.0893$, and this type chooses $\underline{\mu}(t_\ell) = 0.310$ as a separating action; $t_h = 0.9200$, and this type chooses $\bar{\mu}(t_h) = 0.470$ as a separating action; all types between 0.0893 and 0.9200 choose equal division ($x^E = 0.375$), and separate through cheap talk into three segments, $[0.0893, 0.1459]$, $[0.1459, 0.7517]$, and $[0.7517, 0.9200]$. In the second equilibrium, $t_\ell = 0.0838$, and this type chooses $\underline{\mu}(t_\ell) = 0.298$ as a separating action; $t_h = 0.9270$, and this type chooses $\bar{\mu}(t_h) = 0.481$ as a separating action; all types between 0.0838 and 0.9270 choose equal division ($x^E = 0.375$), and separate through cheap talk into two segments, $[0.0838, 0.5806]$ and $[0.5806, 0.9270]$.

Figures 5 and 6 provide a more comprehensive picture of the parameter values that are consistent with an equal division norm. For figure 5, we set $x_k = 0.66$ (so that the children have roughly two-thirds of the family's resources) and depict computational results for all $\underline{x} \in [0, 0.66]$ and $\beta/W^\gamma \in [0, 25]$. Equal division of children's endowments corresponds to cases in which $\underline{x} = 0.33$; all other cases involve unequal endowments. The lightly shaded area identifies parameter values for which $\bar{\mu}(\frac{1}{2}) \leq x^E \leq \underline{\mu}(\frac{1}{2})$, and where one of the inequalities is strict (a necessary condition for the existence of an equilibrium with an equal division

norm). The dark area identifies parameter values for which a central pooling equilibrium with equal division exists. Note that, as β increases, an equilibrium norm of equal division is consistent with greater inequality of children's endowments. In fact, for $\beta/W^\gamma > 20$, equigeniture emerges in equilibrium essentially irrespective of how children's endowments are divided.¹⁷ Therefore, if one selects the population distribution of β appropriately for any given distribution of endowments, the model is capable of reproducing any frequency of adherence to an equal division norm, including the actual frequency for US households. Note also that an increase in the family's resources (holding fixed proportional endowments) can make equal division more or less feasible, depending on whether γ is, respectively, negative or positive.

Figure 6 consists of three panels. For each, we fix the value of β/W^γ , and depict computational results for all $\underline{w}/w_k \in [0, 1]$ (where $w_k \equiv (w_1 + w_2)/W$, so $w_1/w_k = 0.5$ signifies equality of the children's endowments) and $w_P/W \in [0, 1]$ (w_P/W denotes the parent's endowed share of the family's resources). In other words, for this figure we fix the family's total resources and preference parameters while varying the division of endowments between the family members. The shaded areas (light and dark) are defined as before. In moving through panels (a), (b), and (c), we consider successively smaller values of β/W^γ .

As we reduce β/W^γ , it becomes easier for any given type to discourage imitation by more extreme types, so decisions are less distorted from $p(t)$. This in turn implies that the lightly shaded area must contract. However, as illustrated in panel (b), for any given value of β/W^γ , the boundaries of the lightly shaded set may be non-monotonic. The necessary condition for an equilibrium with equal division is always satisfied for small enough values of w_P/W . This follows from the fact that $\underline{\mu}(\frac{1}{2})$ must exceed x^E (x^E must exceed $\bar{\mu}(\frac{1}{2})$) if $\underline{\mu}(0)$ is close enough to x^E ($\bar{\mu}(1)$ is close enough to x^E). For large values of w_P/W , the division of children's endowments is essentially irrelevant (they own practically nothing), so the necessary condition is either satisfied for all values of \underline{w}/W (as in panels (a) and (b)) or

¹⁷We say "essentially" any division of children's endowments because our computational approach encounters boundary problems as \underline{x} approaches 0 or x_k .

for no values of w/W (as in panel (c)). For intermediate values of w_p/W , sufficient inequality of the children's endowments may drive x^E outside of the interval $[\underline{\mu}(\frac{1}{2}), \bar{\mu}(\frac{1}{2})]$; this produces the non-monotonicity noted in panel (b).

As before, the heavily shaded areas indicate the parameters for which equal division equilibria exist. Note that, for small values w_p/W , an equilibrium norm of equal division is always consistent with substantial inequality of children's endowments.¹⁸ This finding is intuitive. When the parent holds a small fraction of the family's resources, the costs of giving offense to a less-loved child are potentially large, while the potential benefits of compensation are quite limited. It does not follow, however, that an equal division norm is necessarily less consistent with inequality of children's endowments for large values of w_p/W . On the contrary, as illustrated in panel (a), the boundaries of the heavily shaded area may be non-monotonic. This occurs because, for high values of β/W^γ , an equilibrium with an equal division norm exists for $w_1/w_k = \frac{1}{2}$, and the children are endowed with so few resources that the division between them doesn't matter.¹⁹ In contrast, for intermediate values of w_p/W , x^E varies substantially with w_1/w_k ; consequently, x^E may be inconsistent with the equilibrium conditions for central pooling when the value of w_1/w_k is sufficiently extreme. Nevertheless, for low values of β/W^γ , the boundaries of the heavily shaded area are monotonic, which implies that an equal division norm is less consistent with inequality of children's endowments for large values of w_p/W (see panel (c)). This occurs because, for low values of β/W^γ , the necessary condition for the existence of an equal division equilibrium is only satisfied for small values of w_p/W .

We note that this analysis generates at least one potentially testable implication: the likelihood of equal division rises as the parent's endowed share of family resources falls, at least when this share is small. One must, however, interpret the pertinent empirical evidence

¹⁸We suspect (but have not proven) that, for sufficiently small values of w_p/W , any value w_1/w_k is consistent with the existence of an equal division equilibrium. This is not apparent in panels (b) and (c) because we do not use a sufficiently fine grid for w_p/W .

¹⁹For this reason, we suspect (but have not established) that, for sufficiently large values of w_p/W , any value w_1/w_k is consistent with the existence of an equal division equilibrium in panel (b) as well. Panel (b) may obscure this property because we have not used a sufficiently fine grid for w_p/W .

with caution. Plainly, the relationship between the frequency of equal division and w_P/W may be non-monotonic. Moreover, unless one holds W (total family resources) constant, the model is in principle consistent with almost any relation between parental resources, w_P , and the frequency of equal division.

3.2.3 Equilibrium selection

Intuitively, theorem 4 holds because the set of central pooling equilibria is not locally unique. In general, it is possible to sustain central pooling equilibria for all x_p in some nondegenerate interval. When children have equal endowments, this interval is symmetric around $\frac{1}{2}$, and therefore contains the point corresponding to equal division of bequests (for this case, $x^E = \frac{1}{2}$). Generally, changes in children's endowments lead to continuous changes in the end points of this interval, and in x^E . Thus, x^E remains in the interval of sustainable pooling actions provided that w_1 and w_2 stay within some neighborhood of each other.

The non-uniqueness of equilibria raises an important question: why do we choose to focus on the particular equilibrium described in theorem 4? One possible answer avoids the issue of equilibrium selection entirely. In particular, one can read our results as demonstrating simply that a norm of equal division is consistent with optimization and economic equilibrium. This is of interest because there is, to our knowledge, no other model with this implication; hence equal division of bequests is widely regarded as a puzzle.

We do not, however, wish to suggest that all equilibria are equally compelling as a matter of theory. Indeed, we believe that central pooling equilibria with equal division are especially attractive. There are two separate issues to consider in this context: First, why should we focus on central pooling equilibria? Second, why should we pay particular attention to central pooling equilibria with equal division of bequests?

One can, of course, attempt to address these issues by using available techniques to refine the equilibrium set (see e.g. Cho and Kreps [16]). Unfortunately, existing refinements appear to have little power in settings where $p(t)$ is increasing, unless one also imposes some other criterion. Our approach in this section is to focus on equilibria that have a natural

monotonicity property.

Definition: We will say that an equilibrium is *monotonic* if (i) $\mu(t)$ is weakly increasing in t , and (ii) for all $x' > x''$, arbitrary messages m' and m'' , and $t \in [0, 1]$, $\phi(t, x', m') \leq \phi(t, x'', m'')$.

In words, an equilibrium is monotonic the equilibrium action function, $\mu(t)$, is weakly increasing in t , and if higher actions weakly shift inferences towards higher types, in the sense of first order stochastic dominance. We find these equilibria appealing because preferences are monotonic: higher types prefer to take higher actions.

All central pooling equilibria have weakly increasing action functions, but are not necessarily monotonic. If, however, a central pooling equilibrium is not monotonic, then there is an equivalent central pooling equilibrium that is monotonic (one uses the same action function but adjusts out-of-equilibrium beliefs appropriately). Thus, the set of monotonic central pooling equilibria is essentially equivalent to the set of central pooling equilibria. A similar statement holds for separating equilibria.

We will further refine the set of monotonic equilibria through a device known as the D1 criterion (see Cho and Kreps [16]). In effect, this criterion insists that, on observing a deviation (defined as an action not taken with positive probability by any type of agent in the candidate equilibrium), an individual will infer that the deviating party belongs to the class of agents who had an incentive to make the observed deviation for the largest set of beliefs.

As our next result demonstrates, these criteria exactly characterize the set of outcomes associated with separating equilibria and central pooling equilibria (when each exists):

Theorem 5 *All separating equilibria satisfy the D1 criterion, and for any central pooling equilibrium there exists an equivalent monotonic central pooling equilibrium (same action function and message function, possibly different out-of-equilibrium beliefs) that satisfies the D1 criterion. Moreover, all monotonic equilibria that satisfy the D1 criterion are either separating equilibria or central pooling equilibria.*

Proof: See Appendix A.

Having provided a formal justification for preferring central pooling equilibria, we now turn our attention to the second selection issue: why focus attention on the equilibrium with equal division of bequests? Here, our answer is less formal, and is predicated on the focalness of x^E . When many outcomes are consistent with equilibria, “meeting in the middle” is often the most natural rule for coordinating activity (Schelling [32]). Moreover, any equilibrium not involving some form of equal division would require the parties to share a common understanding of the “ordering” of the children (i.e., which child has received x and which has received $1 - x$). If, for example, the norm is a 60-40 split, there may be confusion as to whether a 40-60 split constitutes a deviation from the norm. Families could base the ordering on some objective criterion such as age, but there are many competing criteria (gender, income, and so forth). With a norm of unequal division, the ordering always favors one child over the other, so the children might even take the ordering itself as a signal of t .

Equal division of the family’s resources (as opposed to equal division of bequests) is also appealing as a focal norm. However, unlike equal division of bequests, it is not always feasible (one child may have more than half of the family’s resources), and it requires parents to constantly adjust their wills through time when children accumulate resources at different rates. We also doubt that it would be robust if one added some degree of asymmetric information about w_1 and w_2 ; unlike equal division of bequests, equal division of total family resources is not easily verified.

3.3 The role of observability

In formulating our model, we have assumed that essentially everything is publicly observable, with the exception of the parent’s preference parameter, t . In this section, we note that our results depend only on the observability of transfers; an equal division norm would emerge under analogous conditions even if the parties had private information concerning their endowments and consumption, or if the children had private information about their own preference parameters, or if the parent could send private “cheap talk” messages. The fact

that we cannot dispense with the observability of transfers explains the empirical observation that an equal division norm applies to bequests, but not to *intra vivos* gifts.

3.3.1 The observability of endowments

In practice, parents and children have limited ability to observe each others' resources and expenditures. Though we assume the opposite, this is primarily a matter of analytic and notational convenience. Imagine, for example, that each party has private information concerning his or her own endowment (but not concerning the endowment of any other party), and that expenditures are unobservable. Since bequests are observable, children can always compute the parent's resources at death after the fact, before they make their inferences concerning the parent's preferences. Consequently, we can continue to treat the parent's endowment as if it is publicly observable. Since children's endowments may be uncertain from the perspective of the parent, we must replace our original equation for the parent's utility with the following expression:

$$U_P = [tE_w^1 u(w + xw_p) + (1-t)E_w^2 u(w + (1-x)w_p)] + \beta [tv(\hat{t}) + (1-t)v(1-\hat{t})],$$

where E_w^i is the expectations operator over endowment w for child i . Though it is no longer convenient to think in terms of the parent dividing total family resources W (since this is uncertain from the perspective of the parent), few other changes in our analysis are required.

The analysis of the previous paragraph is essentially unchanged when we assume that expenditures are observable, but endowments are not. In the absence of stochastic returns, one can infer endowments from expenditures and transfers. However, the parent never has the opportunity to observe expenditures that might shed light on children's resources at the time of the parent's death. Though children can make these calculations, this does not affect their inferences about the parent's type, since they know that the parent did not observe these expenditures prior to dividing his or her estate.

The preceding discussion presupposes that no party has private information concerning the endowment of any other party. This is somewhat restrictive. A parent may know

something about the resources of one child that is not known to the other child. Indeed, children could conceivably undertake activities designed to signal the level of their resources to the parent, and only to the parent. The extension of our analysis to such cases is non-trivial, since the parent's bequests might then signal either the value of the parent's preference parameter, or the nature of the parent's information concerning the resources of each child. We have not yet studied models that allow for these possibilities.

3.3.2 The observability of the children's preference parameters

In formulating our model, we assumed that parents know the utility weight β that children attach to parental affection, but that children do not know the utility weight t that the parent assigns to the first child. This sharp asymmetry may strike the reader as unnatural. It is therefore important to emphasize that our results continue to hold when the children have private information about the parameter β . This is because the informativeness of the parent's choices depends upon the parent's beliefs about the parameter β , rather than on the actual value of β . Provided that the parent and the children share common priors about the distribution of β , and provided that the unconditional distribution of β is the same for both children, we can simply replace our original equation for the parent's utility with the following expression:

$$U_P = [tu(xW) + (1-t)u((1-x)W)] + E(\beta) [tv(\hat{t}) + (1-t)v(1-\hat{t})],$$

where E is the expectations operator over the parameter β . This modification alters nothing of substance. Naturally, the parent's beliefs about the distribution of β may differ across children. However, as a formal matter this case is equivalent to a situation in which each child i has a different preference parameter β_i , and in which these parameters are known with certainty to the parent (one simply replaces each β_i with $E_i(\beta)$).

3.3.3 The observability of cheap talk

Throughout, we have assumed that the “cheap talk” signal m is observed by both children. Conceivably, it would be more natural to assume that, in addition to making public pronouncements, the parent can speak privately to either child. With minor a minor modification, the model can accommodate this possibility. Formally, the parent selects triplet of pure messages, (m, m_1, m_2) , where both children receive the message m , and where child i privately observes the message m_i . It is easy to demonstrate that this modification changes nothing of substance. In particular, the private signals m_i are necessarily uninformative. Assume on the contrary that, for a given (x, m) , m'_i leads to a different inference than m''_i . Changing m_i only affects the utility of child i . Moreover, every parent type t has the same preference ranking over child i 's inference. Thus, if m'_i leads to a more favorable inference than m''_i for one parent type, it does so for all parent types. Consequently, there can be no separation of types along the m_i dimension.

3.3.4 Gifts vs. bequests: the observability of transfers

Next, imagine that, contrary to our assumptions, a transfer to one child is not observable by the other child. If everything else is observable (aside from t), each child can compute the magnitude of the transfer to his or her sibling from endowments and expenditures, so our results are unchanged. However, in more realistic cases where endowments and/or expenditures are also imperfectly observable, the preceding analysis is inapplicable.

In our model, a norm of equal division emerges because transfers serve as a signal of parent's preferences that is common to both children. In other words, there is a single signal and two audiences. When transfers are neither directly observable nor perfectly inferable from other public information, they cannot serve as a common signal. Instead, each transfer provides a private signal to each child. In effect, there are two separate signal-audience pairs. The importance of this distinction is profound. If neither child is in a position to verify that the parent's resources have been divided equally, then the equilibrium inference

function cannot systematically link the children's beliefs about the parent's preferences to the equality of transfers. Without such a link, an equal division norm cannot survive.

To understand this point from a more formal perspective, assume that third parties cannot observe endowments, expenditures, or transfers. Imagine a candidate equilibrium in which a positive fraction of parent types divide their resources equally between their children conditional on each realization of the parent's resources. Provided that the parent transfers to each child i some amount b_i (possibly different for each child) such that $2b_i$ lies in the support of the probability distribution for the parent's endowment, the child must assume that he or she is observing an equilibrium choice, rather than a deviation. Consequently, the parent has the ability to deviate from an equilibrium by giving more to one child and less to the other without encountering undesirable inferences based on out-of-equilibrium beliefs.

As we mentioned in section 1, the available evidence suggests that a norm of equal division applies to bequests, but not to *intra vivos* gifts. We also argued that this observation deepens the equal division puzzle. In particular, one cannot simply attribute equal division to deeply rooted notions of fairness, or to indifference about the division of resources, since these considerations would presumably apply with equal force to both gifts and bequests.

In contrast, our analysis suggests that the key difference between gifts and bequests relates to observability: the division of bequests is perfectly observable by all concerned parties, whereas the division of gifts need not be. A parent can give gifts to a favored child without revealing this to another child; however, a parent cannot make bequests to any child without the knowledge of all children. If secret gifts are feasible, gifts cannot serve as a common signal of the parent's preferences, so the analysis of the previous section does not apply. The ideal treatment of these issues requires a dynamic model in which the parent makes unobservable gifts while alive, followed observable transfers at death. We conjecture that our results will carry over under reasonably general conditions, but we leave this as a topic for subsequent work.

4 Imitation Toward the Extremes

We now turn our attention to the properties of equilibria when condition 2 (imitation toward the extremes) is satisfied. To conserve space, we limit ourselves to an intuitive treatment of this case. Formal statements of results and associated proofs are available upon request.

One can gain insight into the structure of signaling equilibria under condition 2 by noting that the problem is similar to one that we've already considered. To appreciate this point, consider once again figures 2 and 3. Note that, if one restricts attention to the interval $[0, \frac{1}{2}]$ in figure 2 and to the interval $[\frac{1}{2}, 1]$ in figure 3, the parent's perception bliss point functions, $p(t)$, have the same shape. This implies that one analyzes the structure of equilibria in an analogous way, except that type $t = \frac{1}{2}$, rather than $t = 0$, is now regarded as the "lowest" type, while $t = 1$, rather than $t = \frac{1}{2}$, is now regarded as the "highest" type. Naturally, one obtains analogs of theorems 1 and 2, which concern the monotonicity and existence of a separating equilibrium. Moreover, analogously to theorem 3, one obtains equilibria with pools at the "top" end of the type space. The primary difference is that the "top" of the type space refers to the extreme, rather than to the center. Thus, we obtain an extremal pool (as in Cho and Sobel [17]), rather than a central pool. Indeed, the analysis of pooling equilibria is much simpler than in the preceding section, since one must make sure that non-imitation constraints are satisfied on both sides of a central pool, but on only one side of an extremal pool. The relationship between the interval $[\frac{1}{2}, 1]$ in figure 2 and the interval $[0, \frac{1}{2}]$ in figure 3 is analogous. Consequently, the model generates separation from the center, with possible pooling at both extreme points.

Despite the similarities noted in the previous paragraph, there are some important technical differences between the problems considered here and in the section 3. Equation (4) still defines a dynamic system that governs the evolution of the separating action function from any initial condition. However, in section 3, the natural initial condition was $\mu(0) = \underline{x}$. The dynamic system generated a separating function when initialized in this way only because $\underline{x} > 0$ was not the action bliss point of type 0. Since $t = \frac{1}{2}$ is now, in effect, the lowest

type for both halves of the type space, $\mu\left(\frac{1}{2}\right) = \frac{1}{2}$ is the natural initial condition. But this means that, in a separating equilibrium, the lowest type receives both its action bliss point and its perception bliss point. It follows that the initial condition is a stationary point of the dynamic system described by (4). The system does not generate a separating function when initialized in this way; it simply stays put.

To resolve this issue, one must study the stability properties of the dynamic system around the stationary point $\left(\frac{1}{2}, \frac{1}{2}\right)$. In appendix B, we demonstrate that the system is saddle-point stable. This implies that the differential equation has two solutions satisfying $\mu\left(\frac{1}{2}\right) = \frac{1}{2}$, one corresponding to the stable arm, and one corresponding to the unstable arm. The unstable arm has the property that $\mu(t)$ is increasing, less than $X(t)$ for all $t \in \left(0, \frac{1}{2}\right)$, and greater than $X(t)$ for all $t \in \left(\frac{1}{2}, 1\right)$. In contrast, the stable arm has the property that $\mu(t)$ is *decreasing* in t , *greater* than $\frac{1}{2}$ for all $t \in \left(0, \frac{1}{2}\right)$, and *less* than $\frac{1}{2}$ for all $t \in \left(\frac{1}{2}, 1\right)$ (see figure 7). It is straightforward to verify that the first order condition suffices to guarantee mutual non-imitation on the unstable arm, but we have not resolved this issue for the stable arm. Thus, there is a possibility that there might exist non-monotonic separating equilibria. The possibility did not arise in section 3 because 0 was the lowest type, and because it was impossible to choose $x < X(0) = 0$. It arises here because, while $\frac{1}{2}$ is the “lowest” type, $X\left(\frac{1}{2}\right) = \frac{1}{2}$ is not a “lower bound” on x .

Irrespective of whether one selects the separating function corresponding to the unstable arm or the one corresponding to the stable arm, one obtains equilibria with extremal pools, at least for sufficiently large β . In fact, for the unstable arm (the most natural equilibrium), one *always* obtains extremal pools, even for very small β .²⁰ Equilibria with pools at the extremes are of interest because we can interpret them as supporting social norms in which one child receives the parent’s entire estate. This practice is commonly known as unigeniture,

²⁰To see this, note that the separating function corresponding to the stable arm must have $\mu(t) < X(t)$ for $t \in (0, 1)$ (the argument is essentially identical to one given in the proof of theorem 1). But then, with $\underline{x} > 0$, separation would require $\mu(t) < \underline{x}$ for sufficiently small t , which is impossible. Of course, since $X(t) < \underline{x}$ for sufficiently small t , one also obtains strictly positive mass at the extreme points even in the absence of signaling (i.e. if each type simply chose its action bliss point). The effect of signaling is to increase the sizes of the extremal pools.

and is found in many societies outside of the United States.

In contrast to equigeniture (equal division), there are a number of existing theories that provide explanations for unigeniture (see e.g. Chu [18], or Guinnane [23]). We have opted for a somewhat cursory discussion of condition 2 partly for this reason. It is important to emphasize, however, that our model is completely complementary to other theories of unigeniture. In essence, the factors described in these theories determine the population distribution of the parental action bliss point, $X(t)$. The mechanism outlined in this section then accentuates any preexisting tendency towards unigeniture.

Usually (but not always), unigeniture takes the form of primogeniture, which means that the oldest child typically receives the parent's estate. Our model cannot explain a preference for older children. However, any other consideration that favors transfers to the oldest child would skew the distribution of parental action bliss points to one side of the parent-type space (the side that represents favoritism towards the oldest child). The mechanism considered here would then accentuate the preexisting tendency towards primogeniture.

5 Conclusion

In this paper, we have studied environments in which parental choices affecting the level and division of bequests provide children with information about the parent's preferences, and in which children are directly affected by their perceptions of parental affection. Under conditions identified in the text, the model gives rise to equilibria that support norms of equal division, and these equilibria have attractive properties that argue in favor of their selection. Since these results depend critically on the assumption that transfers are necessarily observable by all children, the theory applies to bequests, but not to *intra vivos* gifts. Consequently, our model not only provides an explanation for the equal division of bequests, but also reconciles this pattern with the unequal division of gifts. Under an alternative set of conditions, the model gives rise to equilibria that support norms of unigeniture. This is of interest because unigeniture is a common pattern outside of the United States.

Appendix A

Proof of theorem 1

Let $\mu(t)$ be some equilibrium mapping from values of t to values of x . Since parents also choose a pure message m , μ may map more than one value of t to the same value of x . However, since the equilibrium generates complete separation, children correctly identify every type t . Thus, the outcome for type t is $(\mu(t), t)$ (signifying $x = \mu(t)$ and $\hat{t} = t$). Since it is costless to imitate pure messages, the mutual non-imitation restrictions are satisfied iff every type t weakly prefers $(\mu(t), t)$ to $(\mu(t'), t')$ for all $t' \in [0, 1]$.

First we prove the following preliminary result.

Lemma 1: $\forall t \in [0, \frac{1}{2}), \mu(t) = X(t)$.

Proof. Note that $\mu(0) \geq \underline{x} > 0$. Thus, if the lemma is false, there must exist some $t \in (0, \frac{1}{2})$ with $\mu(t) \neq X(t)$. We argue that, for some small $\varepsilon > 0$, type $t - \varepsilon$ would imitate type t , which is a contradiction. Plainly, type $t - \varepsilon$ weakly prefers $(X(t - \varepsilon), t - \varepsilon)$ to $(\mu(t - \varepsilon), t - \varepsilon)$ (where the first element in each pair refers to an action, and the second refers to a perception). We complete the argument by showing that, for small ε , type $t - \varepsilon$ strictly prefers $(X(t), t) = (\mu(t), t)$ to $(X(t - \varepsilon), t - \varepsilon)$. In particular, define

$$\psi(\varepsilon) \equiv [U(X(t), t - \varepsilon) + \beta V(t, t - \varepsilon)] - [U(X(t - \varepsilon), t - \varepsilon) + \beta V(t - \varepsilon, t - \varepsilon)].$$

Note that $\psi(0) = 0$. Using the fact that $U_1(X(t), t) = 0$, it is easy to verify that $\psi'(0) = \beta V_1(t, t) > 0$. So for small ε we have $\psi(\varepsilon) > 0$, as desired. Q.E.D.

The remainder of the proof consists of four main steps.

Step 1. $\mu(t) > X(t) \forall t \in [0, \frac{1}{2})$ (a mirror-image argument establishes that $\mu(t) < X(t) \forall t \in (\frac{1}{2}, 1]$).

Define $\underline{t} \equiv X^{-1}(\underline{x})$. Since $\underline{x} > 0$, $\underline{t} > 0$. For $t < \underline{t}$, $X(t) < \underline{x}$, but $\mu(t) > \underline{x}$, so $\mu(t) > X(t)$. Thus, the claim can only be false if $\exists t' \in (\underline{t}, \frac{1}{2})$ s.t. $\mu(t') < X(t')$ (the strict inequality follows from lemma 1). Define $t^s = \sup\{t \in [\underline{t}, t'] \mid \mu(t) > X(t)\}$; note that $\frac{1}{2} > t^s > 0$. By lemma 1, there are two possibilities: $\mu(t^s) < X(t^s)$, and $\mu(t^s) > X(t^s)$.

(i) Assume first that $\mu(t^s) < X(t^s)$. Then there exists an increasing sequence t_k s.t. $\lim_{k \rightarrow \infty} t_k = t^s$ with $\mu(t_k) > X(t_k)$. Without loss of generality, we can assume that $\mu(t_k)$ converges to some limit $\bar{\mu}$ (if necessary, select a convergent subsequence). Plainly, $\bar{\mu} > \mu(t^s)$.

The following no-imitation conditions must hold in a separating equilibrium:

$$\begin{aligned} U(\mu(t_k), t_k) + \beta V(t_k, t_k) &\geq U(\mu(t^s), t_k) + \beta V(t^s, t_k) \\ U(\mu(t^s), t^s) + \beta V(t^s, t^s) &\geq U(\mu(t_k), t^s) + \beta V(t_k, t^s) \end{aligned} \quad (6)$$

Adding up the inequalities and rearranging, we obtain:

$$\beta \int_{t_k}^{t^s} \int_{t_k}^{t^s} V_{12}(r, s) dr ds \geq \int_{t_k}^{t^s} \int_{\mu(t^s)}^{\mu(t_k)} U_{12}(r, s) dr ds \quad (7)$$

Since $U(.,.)$ and $V(.,.)$ are twice continuously differentiable on, respectively, $[\underline{x}, 1 - \bar{x}] \times [0, 1]$ and $[0, 1]^2$, and since $U_{12}(.,.) > 0$ and $V_{12}(.,.) > 0$, it follows that there exists $\underline{u}, \underline{v}$ with $U_{12}(r, s) > \underline{u} > 0$ and $\underline{v} > V_{12}(r, s) > 0$ on the respective domains of integration. Thus (7) implies that

$$\beta(t^s - t_k)^2 \underline{v} \geq (t^s - t_k)(\mu(t_k) - \mu(t^s)) \underline{u}, \quad (8)$$

or equivalently that $\beta(t^s - t_k) \underline{v} \geq (\mu(t_k) - \mu(t^s)) \underline{u}$. Since $t^s - t_k$ converges to zero and $\lim_{k \rightarrow \infty} (\mu(t_k) - \mu(t^s)) = \bar{\mu} - \mu(t^s) > 0$, this inequality fails for large k - a contradiction.

(ii) Now consider the case $\mu(t^s) > X(t^s)$. Then there exists a decreasing sequence t_k s.t. $\lim_{k \rightarrow \infty} t_k = t^s$ with $\mu(t_k) < X(t_k)$. Without loss of generality, we can assume that $\mu(t_k)$ converges to some limit $\bar{\mu}$ (if necessary, select a convergent subsequence). Plainly, $\bar{\mu} < \mu(t^s)$.

Formulate non-imitation constraints as in (6) and rearrange them as in (7). Following the same steps as in the previous case, we conclude that for $\underline{v} > 0$ and $\underline{u} > 0$, the following inequality must be true for all k :

$$\beta(t_k - t^s)^2 \underline{v} \geq (t_k - t^s)(\mu(t^s) - \mu(t_k)) \underline{u},$$

or equivalently that $\beta(t_k - t^s) \underline{v} \geq (\mu(t^s) - \mu(t_k)) \underline{u}$. Since $\lim_{k \rightarrow \infty} (t_k - t^s) = 0$ and $\lim_{k \rightarrow \infty} (\mu(t^s) - \mu(t_k)) = \mu(t^s) - \bar{\mu} > 0$, this inequality fails for large k - a contradiction.

Step 2. $\mu(t)$ is continuous on $[0, \frac{1}{2})$ (a mirror-image argument establishes continuity on $(\frac{1}{2}, 1]$).

Suppose that $\mu(t)$ is discontinuous at some $t^d \in [0, 1/2)$. Then there is some sequence t_k s.t. $\lim_{k \rightarrow \infty} t_k = t^d$, $\mu(t_k)$ converges to some limit $\bar{\mu}$ (if necessary, select a convergent subsequence), and either: (i) $\bar{\mu} < \mu(t^d)$, or (ii) $\bar{\mu} > \mu(t^d)$.

Consider (i). By step 1, $X(t_k) < \mu(t_k)$, so $X(t^d) \leq \bar{\mu} < \mu(t^d)$. Therefore $\exists \varepsilon > 0$ and N_ε s.t. for all $k \geq N_\varepsilon$, $U(\mu(t_k), t^d) - U(\mu(t^d), t^d) > \varepsilon$. Since $V(t_k, t^d) - V(t^d, t^d)$ converges to zero, t^d would imitate some t_k – a contradiction.

In case (ii) $\bar{\mu} > \mu(t^d) > X(t^d)$ (the second inequality follows from step 1). Consequently, $\exists \varepsilon > 0$ s.t. for large k , $U(\mu(t^d), t_k) - U(\mu(t_k), t_k) > \varepsilon$, while $V(t_k, t_k) - V(t^d, t_k)$ converges to zero. Thus, some t^k would imitate t^d – a contradiction.

Step 3. $\mu(t)$ is increasing on $[0, \frac{1}{2})$ (a mirror-image argument establishes that $\mu(t)$ is increasing on $(\frac{1}{2}, 1]$).

Suppose that this claim is false. Then $\exists t \in (0, \frac{1}{2})$ s.t. for some $t_1 \in (t, p(t))$ we have $\mu(t_1) < \mu(t)$. Since $X(t) < X(t_1) < \mu(t_1) < \mu(t)$ (where the middle inequality follows from step 1), $U(\mu(t_1), t) > U(\mu(t), t)$. Also, $V(t, t) < V(t_1, t)$. Therefore, t would imitate t_1 – a contradiction.

Step 4. $\lim_{t \uparrow \frac{1}{2}} \mu(t) = \lim_{t \downarrow \frac{1}{2}} \mu(t) = \mu(\frac{1}{2}) = \frac{1}{2}$.

First we argue that $\lim_{t \uparrow \frac{1}{2}} \mu(t) = \lim_{t \downarrow \frac{1}{2}} \mu(t) = \frac{1}{2}$. From the previous steps we know that

$$\lim_{t \uparrow \frac{1}{2}} \mu(t) \geq \frac{1}{2} \geq \lim_{t \downarrow \frac{1}{2}} \mu(t)$$

Suppose that one of these inequalities is strict; without loss of generality, assume that it is the first one. Then for some $\delta > 0$, there exists a sequence t_k converging to $\frac{1}{2}$ from the left and a sequence t'_k converging to $\frac{1}{2}$ from the right s.t. $\mu(t_k) > \frac{1}{2} + 2\delta$ and $\mu(t'_k) < \frac{1}{2} + \delta$ for all k large enough. Using the non-imitation conditions for t_k and t'_k as in (6) and rearranging them as in (7), we obtain:

$$\beta \int_{t_k}^{t'_k} \int_{t_k}^{t'_k} V_{12}(r, s) dr ds \geq \int_{t_k}^{t'_k} \int_{\mu(t'_k)}^{\mu(t_k)} U_{12}(r, s) dr ds$$

Since $\mu(t_k) - \mu(t'_k) > \delta$ for large k , we can invoke the same argument as in step 1 to show that the inequality must fail – a contradiction.

Finally, we establish that $\mu(\frac{1}{2}) = \frac{1}{2}$. Assume on the contrary that $\mu(\frac{1}{2}) = \frac{1}{2}$. By continuity of U , V , and μ , $U(\mu(t_k), \frac{1}{2}) + \beta V(t_k, \frac{1}{2})$ converges to $U(\frac{1}{2}, \frac{1}{2}) + \beta V(\frac{1}{2}, \frac{1}{2}) > U(\mu(\frac{1}{2}), \frac{1}{2}) + \beta V(\frac{1}{2}, \frac{1}{2})$. Thus, type $\frac{1}{2}$ would imitate type t_k for large k – a contradiction. **QED.**

Proof of theorem 2

The proof consists of 4 steps.

Step 1: Fix the action for type $t = 0$ at $\underline{\mu}(0) = \underline{x}$. We claim that there exists a unique separating function $\underline{\mu}(t)$ on $t \in [0, \frac{1}{2})$. (A similar statement holds for the upper half of the type space).

From theorem 1, we know that, in our search for separating functions, we can confine attention to $\underline{\mu}(t)$ that are strictly increasing and continuous, and that satisfy $\underline{\mu}(t) > X(t)$ for $t \in [0, \frac{1}{2})$. We argue that any $\underline{\mu}(t)$ with these properties is a separating function if and only if it satisfies the following condition:

$$\underline{\mu}'(t) = -\frac{\beta V_1(t, t)}{U_1(\underline{\mu}(t), t)} \quad (9)$$

at all points of differentiability. Necessity follows from the first-order condition governing type t 's optimal decision:

$$U_1(\underline{\mu}(t), t)\underline{\mu}'(t) + \beta V_1(t, t) = 0. \quad (10)$$

To establish sufficiency, consider $t' < t$. Since $\underline{\mu}(t)$ is continuous and increasing, it is also almost everywhere differentiable. We therefore have:

$$\begin{aligned} & U(\underline{\mu}(t), t) + \beta V(t, t) - U(\underline{\mu}(t'), t) - \beta V(t', t) \\ &= \int_{t'}^t U_1(\underline{\mu}(s), t)\underline{\mu}'(s)ds + \beta \int_{t'}^t V_1(s, t)ds \\ &\geq \int_{t'}^t U_1(\underline{\mu}(s), s)\underline{\mu}'(s)ds + \beta \int_{t'}^t V_1(s, s)ds = 0 \end{aligned} \quad (11)$$

where the inequality follows from $U_{12}(\cdot, \cdot) > 0$, $V_{12}(\cdot, \cdot) > 0$, and $\underline{\mu}'(t) \geq 0$ (since $\underline{\mu}(t)$ is increasing), and the second equality follows from (9). Thus, type t does not imitate t' . The proof for the case of $t' > t$ is similar.

Note that (9) is a first-order differential equation for $\underline{\mu}(t)$. Unfortunately, since $U_1(X(t), t) = 0$ for all t , the right-hand side of (9) is not Lipschitz on $(t, x) \in [0, \frac{1}{2}] \times [\underline{x}, 1]$. Consequently, standard results on the existence and uniqueness of a solution do not apply directly. Instead, we proceed as follows.

Consider the following dynamical system with parameter s :

$$\begin{pmatrix} t'(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} -U_1(x(s), t(s)) \\ \beta V_1(t(s), t(s)) \end{pmatrix} \quad (12)$$

Choose some small $\varepsilon > 0$, and define $D \equiv [0, \frac{1}{2}] \times [\underline{x}, 1 - \varepsilon]$. Consider the behavior of the dynamical system (12) on D given the initial condition $(t(0), x(0)) = (0, \underline{x})$. Assumption 1, $\underline{x} > 0$, and $\varepsilon > 0$ guarantee that both $U_1(x, t)$ and $V_1(t, t)$ are Lipschitz functions on the pertinent domains, so the existence and uniqueness of the solution to (9) follows from standard results in the theory of differential equations.

We need to show that the solution generates a separating function with the required properties. The functions $t(s)$ and $x(s)$ are clearly continuous. Define $\widehat{D} \equiv \{(t, x) \in D \mid t < x\}$. As long as $(t(s), x(s)) \in \widehat{D}$, both $t(s)$ and $x(s)$ are strictly increasing. Since $(t(0), x(0)) = (0, \underline{x}) \in \widehat{D}$, $t(s)$ and $x(s)$ are both increasing for small s . The pair can never reach the lower boundary of \widehat{D} (defined by the line $x = t$) since $\frac{x'(s)}{t'(s)}$ approaches infinity as $x(s)$ approaches $t(s)$ for $t(s) < \frac{1}{2}$. Similarly, provided we take ε sufficiently small, $(t(s), x(s))$ can never reach the upper boundary of \widehat{D} . To see this, note that for sufficiently small ε , if $1 - x(s) \leq 2\varepsilon$ and $t(s) \in [0, \frac{1}{2}]$, then $\frac{t'(s)}{x'(s)}(1 - x(s)) > 1$ (this follows from the assumption that $\lim_{c \rightarrow 0} cu'(c) = +\infty$). But then $x(s)$ can never reach $1 - \varepsilon$ while $t(s) < \frac{1}{2}$.

Consequently, there exists $s^* \in \mathfrak{R}_+ \cup \{+\infty\}$ such that $t(s)$ and $x(s)$ are strictly increasing in s with $t(s) < x(s)$ on $[0, s^*)$, and $\lim_{s \rightarrow s^*} t(s) = \frac{1}{2}$. This implies that the inverse function $t^{-1}(t)$ exists, is strictly increasing and continuous, and maps $[0, \frac{1}{2}]$ to $[0, s^*)$. Let $\underline{\mu}(t) \equiv$

$x(t^{-1}(t))$. This function is well-defined, strictly increasing, continuous, and satisfies $\underline{\mu}(t) > t$ for $t \in [0, \frac{1}{2})$. Condition (9) is therefore sufficient to guarantee that $\underline{\mu}(t)$ is a separating function, and $\underline{\mu}(t)$ satisfies this condition by construction.

Finally, we argue that there is no other separating equilibrium for types $t \in [0, \frac{1}{2})$ in which type $t = 0$ chooses $x = \underline{x}$. Note that the right-hand side of (9) is Lipschitz in a neighborhood of every point along the trajectory of the solution to (12), including the initial condition $(0, \underline{x})$. Consequently, the solution to (9) is locally unique when the process is initialized at every such point, and coincides locally with the solution to (12). But then, fixing the initial condition at $(0, \underline{x})$, the unique solution of (12) in D must be the unique solution of (9) in D .

Step 2: For any arbitrary initial condition (endowment) $\underline{x} > 0$, we establish the existence of $\underline{\beta}^*(\underline{x})$ s.t. for $\beta \leq \underline{\beta}^*(\underline{x})$, $\lim_{t \uparrow \frac{1}{2}} \underline{\mu}(t) = \frac{1}{2}$, and for $\beta > \underline{\beta}^*(\underline{x})$, $\lim_{t \uparrow \frac{1}{2}} \underline{\mu}(t) > \frac{1}{2}$. Moreover, we show that $\underline{\beta}^*(\underline{x}) > 0$ iff $\underline{x} < \frac{1}{2}$. Since the problem is completely symmetric (apart from endowments), it also follows that, for $\beta \leq \underline{\beta}^*(\bar{x})$, $\lim_{t \downarrow \frac{1}{2}} \bar{\mu}(t) = \frac{1}{2}$, and for $\beta > \underline{\beta}^*(\bar{x})$, $\lim_{t \downarrow \frac{1}{2}} \bar{\mu}(t) < \frac{1}{2}$.

There are two cases to consider: (i) $\underline{x} \in (0, \frac{1}{2})$, and (ii) $\underline{x} \in [\frac{1}{2}, 1)$.

Case (i): Suppose $\underline{x} \in (0, \frac{1}{2})$. First we show that there exists some $\widehat{\beta}(\underline{x}) > 0$ such that if $\beta = \widehat{\beta}$, then $\lim_{t \uparrow \frac{1}{2}} \underline{\mu}(t) = \frac{1}{2}$. Define $R(t, \theta) \equiv \frac{1}{2}(1 - \theta) + \theta t$ and

$$H(t, \theta) \equiv -\frac{V_1(t, t)}{U_1(R(t, \theta), t)} \quad (13)$$

Since $X'(\frac{1}{2}) > 0$ (this follows from implicit differentiation of (2)) and $\underline{x} < \frac{1}{2}$, we can choose some small $\theta^* > 0$ such that $\forall t \in [0, \frac{1}{2})$, $R(t, \theta^*) > X(t)$, and $R(0, \theta^*) > \underline{x}$. Consequently $H(t, \theta^*) > 0 \forall t \in [0, \frac{1}{2})$. We will show that $\exists K > 0$ s.t. $\forall t \in [0, \frac{1}{2})$, $H(t, \theta^*) < K$.

Suppose not. Then there exists a convergent sequence $t_n \in [0, \frac{1}{2})$, with $\lim_{n \rightarrow \infty} t_n \equiv \widehat{t}$ such that $\lim_{n \rightarrow \infty} H(t_n, \theta^*) = \infty$. Since $H(t, \theta^*)$ is continuous in t and finite for $t \in [0, \frac{1}{2})$, we must have $\widehat{t} = \frac{1}{2}$. Since $U(., .)$ and $V(., .)$ are twice continuously differentiable on the pertinent domains, we can compute the limit of $H(t_n, \theta^*)$ as $t_n \rightarrow \frac{1}{2}$ by applying L'Hospital's

rule:

$$\lim_{n \rightarrow \infty} H(t_n, \theta^*) = -\frac{V_{11}(\frac{1}{2}, \frac{1}{2}) + V_{12}(\frac{1}{2}, \frac{1}{2})}{U_{11}(\frac{1}{2}, \frac{1}{2})\theta^* + U_{12}(\frac{1}{2}, \frac{1}{2})} \quad (14)$$

Because $U_1(X(t), t) = 0 \forall t$, it follows that $U_{11}(X(t), t)X'(t) + U_{12}(t, t) = 0$. Since $\theta^* < X'(\frac{1}{2})$ and $U_{11}(\frac{1}{2}, \frac{1}{2}) < 0$, we know that $U_{11}(\frac{1}{2}, \frac{1}{2})\theta^* + U_{12}(\frac{1}{2}, \frac{1}{2}) > 0$, so the denominator is strictly positive. Since $V_{11}(\frac{1}{2}, \frac{1}{2})$ and $V_{12}(\frac{1}{2}, \frac{1}{2})$ are finite, the numerator is finite. Hence the ratio is finite – a contradiction.

Now choose $\hat{\beta}(\underline{x}) > 0$ such that $\hat{\beta}(\underline{x})K < \theta^*$. Assuming that $\beta = \hat{\beta}(\underline{x})$, we claim that $\underline{\mu}(t) < R(t, \theta^*) \forall t \in [0, \frac{1}{2})$. By construction, $R(0, \theta^*) > \underline{x} = \underline{\mu}(0)$, so this is obviously true for small enough t . Suppose now that $\exists t'' \in [0, \frac{1}{2})$ s.t. $\underline{\mu}(t'') \geq R(t'', \theta^*)$. Then $\exists t' \leq t''$ s.t. $\underline{\mu}(t') = R(t', \theta^*)$ and $\underline{\mu}'(t') \geq R_1(t', \theta^*)$. But $\underline{\mu}'(t') \leq \hat{\beta}(\underline{x})K < \theta^* = R_1(t', \theta^*)$. This contradiction establishes the claim. We therefore know that $\underline{\mu}(t) \in (X(t), R(t, \theta^*))$ and $R(t, \theta^*) < \frac{1}{2}$ for $t \in [0, \frac{1}{2})$. It follows that $\lim_{t \uparrow \frac{1}{2}} \underline{\mu}(t) = \frac{1}{2}$, as desired.

Next, we establish that $\forall t \in [0, \frac{1}{2})$, $\underline{\mu}(t)$ is increasing in β . Choose any $\beta_1 < \beta_2$ and denote the corresponding separating action functions by $\underline{\mu}_1(t)$ and $\underline{\mu}_2(t)$. From (9) it follows that $\underline{\mu}'_1(0) = \underline{\mu}'_2(0) = 0$. But with $\underline{\mu}_1(0) = \underline{\mu}_2(0) = \underline{x} > 0$, it is easy to verify (by differentiating (9)) that $\underline{\mu}''_2(0) > \underline{\mu}''_1(0) > 0$. Thus, for t sufficiently small, $\underline{\mu}_2(t) > \underline{\mu}_1(t)$. If for some $t \in (0, \frac{1}{2})$, $\underline{\mu}_1(t) \geq \underline{\mu}_2(t)$, then $\exists s \leq t$ s.t. $\underline{\mu}_1(s) = \underline{\mu}_2(s)$ and $\underline{\mu}'_1(s) \geq \underline{\mu}'_2(s)$. But from (9) it follows that if $\underline{\mu}_1(s) = \underline{\mu}_2(s)$, then $\underline{\mu}'_1(s) < \underline{\mu}'_2(s)$, which is a contradiction.

From the monotonicity of $\underline{\mu}(t)$ in the value of β it follows that, for $\beta_1 < \beta_2$, if $\lim_{t \uparrow \frac{1}{2}} \underline{\mu}_2(t) = \frac{1}{2}$, then $\lim_{t \uparrow \frac{1}{2}} \underline{\mu}_1(t) = \frac{1}{2}$, and if $\lim_{t \uparrow \frac{1}{2}} \underline{\mu}_1(t) > \frac{1}{2}$, then $\lim_{t \uparrow \frac{1}{2}} \underline{\mu}_2(t) > \frac{1}{2}$. Thus, there exists $\underline{\beta}^*(\underline{x}) \geq \hat{\beta}(\underline{x})$ such that $\lim_{t \uparrow \frac{1}{2}} \underline{\mu}(t) = \frac{1}{2}$ for $\beta < \underline{\beta}^*(\underline{x})$ and $\lim_{t \uparrow \frac{1}{2}} \underline{\mu}(t) > \frac{1}{2}$ for $\beta > \underline{\beta}^*(\underline{x})$. Moreover, if $\underline{\mu}(t)$ is continuous in β , then it follows that $\lim_{t \uparrow \frac{1}{2}} \underline{\mu}(t) = \frac{1}{2}$ for $\beta = \underline{\beta}^*(\underline{x})$.

To establish the continuity of $\underline{\mu}(t)$ in β , again select $\beta_1 < \beta_2$. We can use (i) the monotonicity of $\underline{\mu}(t)$ in β (proven above), (ii) $\underline{\mu}(t) > X(t)$, and (iii) $U_{11}(x, t) < 0$ to establish that $U_1(\underline{\mu}_2(t), t) < U_1(\underline{\mu}_1(t), t) < 0$. Combining this observation with equation (9) and using the fact that $\underline{\mu}(t) = \underline{x} + \int_0^t \underline{\mu}'(s)ds$, it is easy to check that $\underline{\mu}_1(t) < \underline{\mu}_2(t) < \underline{x} + \frac{\beta_2}{\beta_1}(\underline{\mu}_1(t) - \underline{x})$. Thus, when $\beta_1 - \beta_2$ converges to zero, $\underline{\mu}_2(t) - \underline{\mu}_1(t)$ also does so.

Case (ii): $\underline{x} \in [\frac{1}{2}, 1)$. Since $\underline{\mu}(t)$ is strictly increasing and since $\underline{\mu}(0) = \underline{x} \geq \frac{1}{2}$, $\lim_{t \uparrow \frac{1}{2}} \underline{\mu}(t) > \frac{1}{2}$ for all $\beta > 0$. Thus, $\underline{\beta}^*(\underline{x}) = 0$, as desired.

Step 3. Define $\beta^*(\underline{x}, \bar{x}) \equiv \min\{\underline{\beta}^*(\underline{x}), \underline{\beta}^*(\bar{x})\}$. $\beta^*(\underline{x}, \bar{x})$ is decreasing in $\max\{\underline{x}, \bar{x}\}$.

First we argue that $\underline{\beta}^*(\underline{x})$ is decreasing in the initial condition \underline{x} . Select two endowments levels $\underline{x}_1 < \underline{x}_2$, and consider the associated trajectories $\underline{\mu}_1(t)$ and $\underline{\mu}_2(t)$ that solve (9). Since the right-hand side of (9) is Lipschitz in a neighborhood of every point along each trajectory, these trajectories cannot cross for $t \in (0, \frac{1}{2})$. Since $\underline{\mu}_1(0) = \underline{x}_1 < \underline{x}_2 = \underline{\mu}_2(0)$, it follows that $\underline{\mu}_1(t) < \underline{\mu}_2(t) \forall t \in (0, \frac{1}{2})$. Thus, if $\lim_{t \uparrow \frac{1}{2}} \underline{\mu}_2(t) = \frac{1}{2}$ (equivalently, $\beta \leq \underline{\beta}^*(\underline{x}_2)$), then $\lim_{t \uparrow \frac{1}{2}} \underline{\mu}_1(t) = \frac{1}{2}$ (equivalently, $\beta \leq \underline{\beta}^*(\underline{x}_1)$). This implies $\underline{\beta}^*(\underline{x}_1) \geq \underline{\beta}^*(\underline{x}_2)$, as desired.

Since $\underline{\beta}^*(x)$ is decreasing in x , it follows that $\beta^*(\underline{x}, \bar{x}) = \min\{\underline{\beta}^*(\underline{x}), \underline{\beta}^*(\bar{x})\} = \underline{\beta}^*(\max\{\underline{x}, \bar{x}\})$, from which it follows that $\beta^*(\underline{x}, \bar{x})$ is a decreasing function of $\max\{\underline{x}, \bar{x}\}$.

Step 4. A separating equilibrium exists iff $\beta \leq \beta^*(\underline{x}, \bar{x})$.

First suppose that $\beta > \beta^*(\underline{x}, \bar{x})$. Any separating equilibrium must satisfy $\mu(0) \geq \underline{x}$ and $\mu(1) \leq 1 - \bar{x}$. Since, as argued in step 3, the value of the separating function is strictly increasing in the initial condition (decreasing for the top half of the type space), $\lim_{t \uparrow \frac{1}{2}} \mu(t) \geq \lim_{t \uparrow \frac{1}{2}} \underline{\mu}(t)$ (where $\underline{\mu}(t)$ is the separating function for the lower half of the type space emanating upward from \underline{x}), and $\lim_{t \downarrow \frac{1}{2}} \mu(t) \leq \lim_{t \downarrow \frac{1}{2}} \bar{\mu}(t)$ (where $\bar{\mu}(t)$ is the separating function for the upper half of the type space emanating downward from $1 - \bar{x}$). Moreover, by step 2, $\lim_{t \uparrow \frac{1}{2}} \underline{\mu}(t) \geq \frac{1}{2}$ and $\lim_{t \downarrow \frac{1}{2}} \bar{\mu}(t) \leq \frac{1}{2}$, and since $\beta > \beta^*(\underline{x}, \bar{x})$, at least one of these inequalities is strict. But this implies that $\lim_{t \uparrow \frac{1}{2}} \mu(t) > \lim_{t \downarrow \frac{1}{2}} \mu(t)$, which in turn implies that the separating function is non-monotonic, contrary to theorem 1.

Now suppose that $\beta \leq \beta^*(\underline{x}, \bar{x})$. Let

$$\mu(t) = \begin{cases} \underline{\mu}(t) & \text{for } t \in [0, \frac{1}{2}) \\ \frac{1}{2} & \text{for } t = \frac{1}{2} \\ \bar{\mu}(t) & \text{for } t \in (\frac{1}{2}, 1] \end{cases}$$

We argue that this is a separating equilibrium. Note first that type $t = \frac{1}{2}$ receives its bliss point, and therefore has no incentive to imitate any other type. Now consider some type

$t' \in [0, \frac{1}{2})$. From the construction of $\underline{\mu}(t)$, t' has no incentive to imitate any other $t \in [0, \frac{1}{2})$. Since $\lim_{t \uparrow \frac{1}{2}} \underline{\mu}(t) = \frac{1}{2}$, t' has no incentive to imitate $t = \frac{1}{2}$ (if it did, it would also have an incentive to imitate $t = \frac{1}{2} - \varepsilon$ for some small $\varepsilon > 0$, contrary to the construction of $\underline{\mu}(t)$). For any $t \in (\frac{1}{2}, 1]$, $\underline{\mu}(t) > \frac{1}{2} > X(t')$ and $t > \frac{1}{2} > p(t')$, so type t' prefers $(\frac{1}{2}, \frac{1}{2})$ to $(t, \underline{\mu}(t))$ (where the first element in each pair denotes a perceived type and the second denotes an action). Since t' has no incentive to imitate $t = \frac{1}{2}$, it therefore has no incentive to imitate type $t \in (\frac{1}{2}, 1]$. A symmetric argument establishes that no type $t' \in (\frac{1}{2}, 1]$ has an incentive to imitate any other type. **QED.**

Proof of theorem 3:

Fix a pair of endowments \underline{x} and \bar{x} . Let $\underline{\mu}(t)$ and $\bar{\mu}(t)$ be the separating functions corresponding to these endowment constructed as in the proof of theorem 2, extended continuously to the domains $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ respectively (that is, $\underline{\mu}(\frac{1}{2}) = \lim_{t \uparrow \frac{1}{2}} \underline{\mu}(t)$, and $\bar{\mu}(\frac{1}{2}) = \lim_{t \downarrow \frac{1}{2}} \bar{\mu}(t)$). Since $\beta > \beta^*(\underline{x}, \bar{x})$ by assumption, we must have $\bar{\mu}(\frac{1}{2}) < \underline{\mu}(\frac{1}{2})$.

Consider the expression $\underline{\mu}(t) - \bar{\mu}(1-t)$. We know that it is continuous, strictly increasing in t , strictly negative for $t = 0$, and strictly positive for $t = \frac{1}{2}$. Consequently, there exists a unique $t^* \in (0, \frac{1}{2})$ such that $\underline{\mu}(t^*) = \bar{\mu}(1-t^*)$. We will look for central pooling equilibria with $t_\ell < t^*$ and $t_h > 1-t^*$.

Since $\underline{\mu}(t) < \bar{\mu}(1-t) \forall t \in [0, t^*]$, we can define $x_p(t) \in [\underline{\mu}(t), \bar{\mu}(1-t)]$ to be the value of x that solves:

$$U(\underline{\mu}(t), t) - U(x, t) = U(\bar{\mu}(1-t), 1-t) - U(x, 1-t) \quad (15)$$

The existence and uniqueness of $x_p(t)$ follows from the fact that the left-hand side of (15) is zero for $x = \underline{\mu}(t)$ and, since $\underline{\mu}(t) > X(t)$, is strictly increasing in x on $[\underline{\mu}(t), \bar{\mu}(1-t)]$, while the right-hand side is strictly positive for $x = \underline{\mu}(t)$ (since $\bar{\mu}(1-t) < X(1-t)$) and strictly decreasing on this interval, reaching zero at $x = \bar{\mu}(1-t)$. Note that $x_p(t^*) = \underline{\mu}(t^*) = \bar{\mu}(1-t^*)$, and that $\underline{\mu}(t) < x_p(t) < \bar{\mu}(1-t)$ for $t < t^*$. From (15) it follows that the $x_p(t)$ is continuous in t . We will look for central pooling equilibria with $x_p = x_p(t_\ell)$ and $t_h = 1-t_\ell$.

Now consider the expression

$$\Gamma(t) \equiv U(\underline{\mu}(t), t) + \beta V(t, t) - U(x_p(t), t) - \beta V(p(t), t) \quad (16)$$

Since $t^* > 0$, $\underline{\mu}(0) < x_p(0)$, and $\Gamma(0) > 0$. Moreover, since $\Gamma(t^*) < 0$ and Γ is continuous, there exists some $\tilde{t} \in (0, t^*)$ with $\Gamma(\tilde{t}) = 0$. It is easy to check that \tilde{t} is unique. We will look for central pooling equilibria with $t_\ell > \tilde{t}$.

A central pooling equilibrium may also involve informative messages m among those choosing x_p . In particular, these messages will partition those types selecting x_p into a finite number of segments. The following discussion facilitates the construction of these segments.

To begin with, type t_ℓ must be indifferent between separating, and being a member of the lowest differentiated segment selecting x_p . This indifference condition places a restriction on the identify of the highest type in this segment. We formulate this restriction as follows.

For $t \in [\tilde{t}, t^*]$, let $t^p(t) \in (p(t), 1 - t)$ be defined implicitly by the equation $B(t, t^p(t)) = p(t)$. The existence and uniqueness of $t^p(t)$ is assured because $B(t, t) = t < p(t)$, $B(t, 1 - t) = \frac{1}{2} > p(t)$, and by assumption $B(t, s)$ is strictly increasing and continuous in s .

Now consider the function

$$\psi(t, s) = U(\underline{\mu}(t), t) + \beta V(t, t) - U(x_p(t), t) - \beta V(B(t, s), t)$$

By the definition of \tilde{t} , for $t \in [\tilde{t}, t^*]$, $\psi(t, t^p(t)) \leq 0$. Furthermore, $\psi(t, t) \geq 0$. Since the pertinent functions are continuous, it follows that, for $t \in [\tilde{t}, t^*]$, there exists $s_1^L(t) \in [t, t^p(t)]$ such that $\psi(t, s_1^L(t)) = 0$. Moreover, this solution is unique because $\psi(t, s)$ is strictly decreasing in s on this interval. Note that $s_1^L(\tilde{t}) = t^p(\tilde{t})$, $s_1^L(t^*) = t^*$, and $s_1^L(t)$ is continuous.

Similarly, for $t \in [\tilde{t}, t^*]$, define the function $s_1^U(t)$ as follows. If $\psi(t, 1) \geq 0$, $s_1^U(t)$ is the solution to the equation $\psi(t, s_1^U(t)) = 0$ on $[t^p(t), 1]$ (existence and uniqueness follow because $\psi(t, s)$ is continuous and strictly increasing in s on this interval). If $\psi(t, 1) < 0$, let $s_1^U(t) = 1$. Note that $s_1^U(\tilde{t}) = t^p(\tilde{t})$. Also note that $s_1^U(t)$ is continuous.

Intuitively, the interpretation of the functions $s_1^L(t)$ and $s_1^U(t)$ is as follows: for type t_ℓ to be indifferent between separating and being a member of the lowest differentiated segment

selecting x_p , the highest type in this segment must be either $s_1^L(t_\ell)$ or $s_1^U(t_\ell)$.

Next, any type on the boundary between two differentiated segments within the central pool must be indifferent between belonging to either segment. This indifference condition places a restriction on the identity of the highest type in the next segment. In particular, for $t \leq s$, define the following function:

$$\Omega(t, s) = \begin{cases} 1 & \text{if } V(B(t, s), s) - V(B(s, r), s) < 0 \forall r \in (s, 1) \\ s & \text{if } V(B(t, s), s) - V(B(s, r), s) > 0 \forall r \in (s, 1) \\ r \in (s, 1) \text{ such that } V(B(t, s), s) = V(B(s, r), s) & \text{otherwise} \end{cases} \quad (17)$$

Note that $\Omega(t, s) > p(s) > s$ when $s < \frac{1}{2}$. In particular, one subtlety of this definition is that $\Omega(s, s) > s$ for $s < \frac{1}{2}$, but $\Omega(s, s) = s$ for $s \geq \frac{1}{2}$. One can check that $\Omega(t, s)$ is continuous (continuity of interior solutions is trivial; one must check that the function cannot jump from one extreme solution to the other).

For $i \geq 2$, we define the following two sequences of functions over the domain $[\tilde{t}, t^*]$: $s_i^L(t) = \Omega(s_{i-2}^L(t), s_{i-1}^L(t))$, and $s_i^U(t) = \Omega(s_{i-2}^U(t), s_{i-1}^U(t))$ (where $s_1^L(t)$ and $s_1^U(t)$ are defined above, and where $s_0^L(t) \equiv s_0^U(t) \equiv t$). Continuity of $s_i^L(t)$ and $s_i^U(t)$ follows by induction on i . Note that $s_i^L(\tilde{t}) = s_i^U(\tilde{t})$.

Intuitively, the interpretation of the functions $s_i^L(t)$ is as follows: for type $s_{i-1}^L(t_\ell)$ to choose x_p and be indifferent between the two contiguous segments $[s_{i-2}^L(t_\ell), s_{i-1}^L(t_\ell)]$ and $[s_{i-1}^L(t_\ell), s]$, it must be the case the s (the highest type in the next segment) equals $s_i^L(t_\ell)$. A similar statement holds for $s_{i-1}^U(t_\ell)$, except that the first segment in the series is defined differently.

We now prove a result that allows us to focus on equilibria with a finite number of differentiated segments within the central pool.

Lemma 2: Suppose that assumption 2 is satisfied. Then there exists $N \geq 1$ such that $s_{N-1}^U(t^*) < \frac{1}{2}$ and $s_N^U(t^*) \geq \frac{1}{2}$.

Proof: For notational simplicity, define $t_n \equiv s_n^U(t^*)$. The proof is by contradiction. Thus, suppose that $\forall n, t_n < \frac{1}{2}$. The sequence t_n is strictly increasing and, since $t_{n-1} < p(t_{n-1}) < t_n$, the limit point \bar{t} must have the property that $p(\bar{t}) = \bar{t}$, so $\bar{t} = \frac{1}{2}$.

Step 1. Since $B(x, y) = 1 - B(1 - y, 1 - x)$, we know that $B(x, 1 - x) = \frac{1}{2}$, from which it follows that $B_1(\frac{1}{2}, \frac{1}{2}) = B_2(\frac{1}{2}, \frac{1}{2})$. From $B(x, x) = x$, we have $B_1(\frac{1}{2}, \frac{1}{2}) + B_2(\frac{1}{2}, \frac{1}{2}) = 1$. Combining these observations yields $B_1(\frac{1}{2}, \frac{1}{2}) = B_2(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$. Clearly, $B(t_{n-1}, t_n) = t_{n-1} + \int_{t_{n-1}}^{t_n} B_2(t_{n-1}, z) dz$. Since $B(\cdot, \cdot)$ is continuously differentiable and since t_n converges to $\frac{1}{2}$, $\forall \zeta > 0 \exists N_1(\zeta)$ s.t. for $n > N_1(\zeta)$,

$$\frac{t_{n-1} + t_n}{2} - \zeta(t_n - t_{n-1}) < B(t_{n-1}, t_n) < \frac{t_{n-1} + t_n}{2} + \zeta(t_n - t_{n-1}) \quad (18)$$

Step 2. Consider the equation $V(a, s) = V(b, s)$ for $a < s < p(s) < b$. We can rewrite this as $V(p(s), s) - \int_a^{p(s)} V_1(z, s) dz = V(p(s), s) + \int_{p(s)}^b V_1(z, s) dz$, from which it follows that

$$\int_a^{p(s)} \int_z^{p(s)} V_{11}(r, s) dr dz = \int_{p(s)}^b \int_{p(s)}^z V_{11}(r, s) dr dz$$

Since $V(\cdot)$ is twice continuously differentiable, this implies that, $\forall \delta > 0, \exists \varepsilon > 0$ such that if $b - a < \varepsilon$, then $(1 - \delta)(p(s) - a) < (b - p(s))$. From this it follows that $\exists N_2(\delta)$ s.t. if $n \geq N_2(\delta)$

$$(1 - \delta)(p(t_n) - B(t_{n-1}, t_n)) < (B(t_n, t_{n+1}) - p(t_n)) \quad (19)$$

Step 3. Combining (18) and (19) we obtain that $\forall \zeta > 0$ and $\forall \delta > 0$, if $n \geq N(\zeta, \delta) \equiv \max\{N_1(\zeta), N_2(\delta)\}$, then

$$(1 - \delta) \left(p(t_n) - \frac{t_{n-1} + t_n}{2} - \zeta(t_n - t_{n-1}) \right) < \left(\frac{t_n + t_{n+1}}{2} + \zeta(t_{n+1} - t_n) - p(t_n) \right) \quad (20)$$

Since condition 1 is assumed to hold, $p'(\frac{1}{2}) < 1$. Therefore $\exists k < 1$ s.t. $p'(t) < k$ for all t in some neighborhood of $\frac{1}{2}$. It follows that $\exists M$ s.t. for $n > M$,

$$p(t_{n+1}) - p(t_n) < k(t_{n+1} - t_n)$$

Since $t_{n+1} < p(t_{n+1}) < \frac{1}{2}$, the previous inequality can hold only if:

$$p(t_n) > (1 - k)(t_{n+1} - t_n) + t_n \quad (21)$$

Using (20) in (21) we obtain that $\forall n \geq \max(M, N(\zeta, \delta))$ the following inequality is satisfied:

$$\begin{aligned} & (1 - \delta) \left(t_n + (1 - k)(t_{n+1} - t_n) - \frac{t_{n-1} + t_n}{2} - \zeta(t_n - t_{n-1}) \right) \\ < & \left(\frac{t_n + t_{n+1}}{2} + \zeta(t_{n+1} - t_n) - (1 - k)(t_{n+1} - t_n) - t_n \right) \end{aligned} \quad (22)$$

which after simplification becomes:

$$(1 - \delta) \left(\frac{1}{2} - \zeta \right) (t_n - t_{n-1}) < \left(\frac{1}{2} + \zeta - (1 - k)(2 - \delta) \right) (t_{n+1} - t_n) \quad (23)$$

When $k < \frac{3}{4}$, for small ζ and δ the right-hand side of (23) is negative while the left-hand side is positive. Thus, the inequality cannot hold. This contradicts the assumption that $t_{n+1} < p(t_{n+1}) < \frac{1}{2}$ (which we used to construct (21)). When $1 > k \geq \frac{3}{4}$, the right-hand side is strictly positive. One can then choose δ and ζ sufficiently small so that $(1 - \delta) \left(\frac{1}{2} - \zeta \right) > \left(\frac{1}{2} + \zeta - (1 - k)(2 - \delta) \right)$. Then for $n \geq \max(M, N(\zeta, \delta))$ we must have

$$t_n - t_{n-1} < t_{n+1} - t_n$$

contradicting $t_n < \frac{1}{2} \forall n$. **QED.**

Next we prove a result that identifies t_ℓ for the central pooling equilibrium that we subsequently construct.

Lemma 3: $\exists t_\ell \in [\tilde{t}, t^*]$ and $N \geq 1$ such that either $s_N^L(t_\ell) = \frac{1}{2}$ or $s_N^U(t_\ell) = \frac{1}{2}$.

Proof: We begin by noting that, since $x_p(t^*) = \underline{\mu}(t^*)$ and since $s_1^L(t^*) = s_0^U(t^*) = t^*$, we must also have $s_2^L(t^*) = s_1^U(t^*)$. Then it follows inductively that $s_i^L(t^*) = s_{i-1}^U(t^*)$ for all $i \geq 1$, including $i = N$. Consequently, $s_N^L(t^*) < \frac{1}{2} \leq s_N^U(t^*)$.

We now introduce a family of $N + 1$ mappings, $\sigma_0(\tau) : [0, 1] \mapsto [\tilde{t}, t^*]$ and, for $i > 1$, $\sigma_i(\tau) : [0, 1] \rightarrow [0, 1]$, with the following characteristics. $\sigma_0(\tau)$ is continuous, strictly

decreasing on $[0, \frac{1}{2}]$, strictly increasing on $(\frac{1}{2}, 1]$, and takes the following values: $\sigma_0(0) = \sigma_0(1) = t^*$ and $\sigma_0(\frac{1}{2}) = \tilde{t}$. Thus $\sigma_0(\tau)$ maps twice into all $t \in (\tilde{t}, t^*]$. Next, define $\sigma_i(\tau)$ as follows: if $\tau \in [0, \frac{1}{2}]$, then $\sigma_i(\tau) = s_i^L(\sigma_0(\tau))$; if $\tau \in [\frac{1}{2}, 1]$, then $\sigma_i(\tau) = s_i^U(\sigma_0(\tau))$. In other words, if $\tau \in [0, \frac{1}{2}]$, then $\sigma_i(\tau)$ parameterizes the sequence $(t, \{s_i^L(t)\}_{i=1}^N)$ where $t = \sigma_0(\tau)$, while if $\tau \in [\frac{1}{2}, 1]$, then $\sigma_i(\tau)$ parameterizes the sequence $(t, \{s_i^U(t)\}_{i=1}^N)$ where $t = \sigma_0(\tau)$. Note that $\sigma_i(0) = s_i^L(t^*)$, and $\sigma_i(1) = s_i^U(t^*)$. Since $\forall i$, $\sigma_0(\cdot)$, $s_i^L(\cdot)$, and $s_i^U(\cdot)$ are continuous, and since $s_i^L(\sigma_0(\frac{1}{2})) = s_i^U(\sigma_0(\frac{1}{2}))$ (which follows from $\sigma_0(\frac{1}{2}) = \tilde{t}$), $\sigma_i(\tau)$ is continuous, and thus the collection $\{\sigma_i(\tau)\}_{i=0}^N$ defines a continuous transformation (homotopy) between sequences $(t^*, \{s_i^L(t^*)\}_{i=1}^N)$ and $(t^*, \{s_i^U(t^*)\}_{i=1}^N)$. Since $\sigma_N(0) = s_N^L(t^*) < \frac{1}{2}$ and $\sigma_N(1) = s_N^U(t^*) \geq \frac{1}{2}$, there exists some $\tau^* \in [0, 1]$ such that $\sigma_N(\tau^*) = \frac{1}{2}$ (non-uniqueness is unimportant, since any such τ^* will suffice). **Q.E.D.**

Now we construct the central pooling equilibrium. Choose $t_i^* = \sigma_i(\tau^*)$ for $i = 0, \dots, N$, and let $t_\ell = t_0^* (\equiv \sigma_0(\tau^*))$. Consider $t \in [0, \frac{1}{2}]$. If $t \in [0, t_\ell]$, then t separates by taking the action $\underline{\mu}(t)$. If $t \in (t_i^*, t_{i+1}^*)$ for any $i \in \{0, 1, \dots, N-1\}$, then t takes the action $x_p(t_\ell)$ and sends a 'cheap talk' message $m_i^* \in [t_i^*, t_{i+1}^*]$. The strategies for $t \in [\frac{1}{2}, 1]$ are constructed symmetrically. If $t \in [1 - t_\ell, 1]$ then t separates by taking actions $\overline{\mu}(t)$, while if $t \in [1 - t_{i+1}^*, 1 - t_i^*]$, then t takes the action $x_p(t_0)$ and sends a message $1 - m_i^* \in [1 - t_{i+1}^*, 1 - t_i^*]$. For off-equilibrium actions in $(\underline{\mu}(t_\ell), x_p(t_\ell))$, we set the inferences equal to t_ℓ . Similarly, for off-equilibrium actions $(x_p(t_\ell), \overline{\mu}(1 - t_\ell))$, we set inferences equal to $1 - t_\ell$.

It is easy to verify that types $t \leq t_\ell$ and $t \geq 1 - t_\ell$ have no incentive to imitate any other type. Now consider $t \in (t_\ell, 1 - t_\ell)$. Since the 'cheap talk' segments are symmetric around $\frac{1}{2}$, type $\frac{1}{2}$ is indifferent between the two segments of the central pool for which it forms the boundary, and strictly prefers these to all other segments, as well as to separating alternatives. Also, by construction, all other 'boundary' types t_i within the central pool are indifferent between contiguous segments (on the lower half of the type space, $[t_{i-1}, t_i]$ and $[t_i, t_{i+1}]$), and strictly prefer these to all other segments, as well as to separating alternatives. Any type that is interior to a segment within the central pool strictly prefers that segment

to all others, as well as to separating alternatives. **Q.E.D.**

Proof of theorem 4: Since we will be considering equilibria for ranges of endowments, it is important to alter our notation so as to reflect dependence on endowments. Accordingly, let $\mu(t, z)$ denote the separating function on $t \in [0, 1/2]$ initialized by the child's endowed share of family resources, z ($\mu(0, z) = z$).

We will be concerned with equilibria that involve equal division of the parent's resources. Throughout, total family resources are fixed at some level W , and the children's total share of the resources is fixed at x_k . The equal division choice can be rewritten as $x^E(z) = \frac{1}{2} + (z - \frac{x_k}{2})$. For $x^E(z) \in [z, \mu(\frac{1}{2}, z)]$, $t^*(z)$ solves $\mu(t^*, z) = x^E(z)$. Note that if $\beta > \beta^*(\frac{x_k}{2}, \frac{x_k}{2})$, then $t^*(z)$ exists and is less than $\frac{1}{2}$ for all z in some neighborhood of $\frac{x_k}{2}$. Also define $\tilde{t}(z)$ as the unique solution in $[0, t^*(z)]$ to $U(\mu(\tilde{t}, z), \tilde{t}) + \beta V(\tilde{t}, \tilde{t}) = U(x^E(z), \tilde{t}) + \beta V(p(\tilde{t}), \tilde{t})$.

As in the proof of theorem 3, we will parameterize potential partitions of the central pool. However, this requires some preliminaries.

For $t \in [\tilde{t}(z), t^*(z)]$, there are two solutions for the equation $\psi(t, s, z) \equiv U(\mu(t, z), t) + \beta V(t, t) - U(x^E(z), t) - \beta V(B(t, s), t) = 0$. Let $s_1^L(t, z)$ be the solution on $[t, p(t)]$, and let $s_1^U(t, z)$ be the solution on $[p(t), 1]$ (if $\psi(t, 1, z) < 0$, let $s_1^U(t, z) = 1$). For integers $i > 1$, define the following two sequences of functions (indexed $m = L, U$) over the domain $[\tilde{t}(z), t^*(z)]$: $s_i^m(t, z) \equiv \Omega(s_{i-2}^t(t, z), s_{i-1}^m(t, z))$ where $s_0^m(t, z) \equiv t$, $s_1^m(t, z)$ is as defined above, and Ω is as defined in the proof of theorem 3.

Note that

$$\frac{dB(t, s_1^L(t, z))}{dt} = \frac{- \left[\int_{\mu(t, z)}^{x^E(z)} U_{12}(x, t) dt + \int_t^{B(t, s_1^L(t, z))} V_{12}(y, t) dy \right]}{\beta V_1(B(t, s_1^L(t, z)), t)}$$

We know that $U_{12} > 0$ and $V_{12} > 0$; also, for $t \in (\tilde{t}(z), t^*(z))$, $B(t, s_1^L(t, z)) < p(t)$, so $V_1(B(t, s_1^L(t, z)), t) > 0$. Thus, for $t \in (\tilde{t}(z), t^*(z))$, $dB(t, s_1^L(t, z))/dt < 0$; over this domain, $B(t, s_1^L(t, z))$ decreases (strictly) monotonically from $p(\tilde{t}(z))$ to $t^*(z)$. It also follows that $dB(t^*(z), s_1^L(t^*(z), z))/dt = 0$ and $\lim_{t \downarrow \tilde{t}(z)} dB(t, s_1^L(t, z))/dt = -\infty$ (where the second statement follows from $V_1(B(\tilde{t}(z), s_1^L(\tilde{t}(z), z)), \tilde{t}(z)) = 0$). We conclude that $B(t, s_1^L(t, z))$ is

invertible on $[t^*(z), p(\tilde{t}(z))]$. Define $T(b, z)$ on this interval as the inverse function; it maps $b \in [t^*(z), p(\tilde{t}(z))]$ to $[\tilde{t}(z), t^*(z)]$, it is (strictly) monotonically decreasing with $T(t^*(z), z) = t^*(z)$ and $T(p(\tilde{t}(z)), z) = \tilde{t}(z)$, and it is continuously differentiable on $(t^*(z), p(\tilde{t}(z)))$. Moreover, $\lim_{b \downarrow t^*(z)} T_1(b, z) = -\infty$ and $\lim_{b \uparrow p(\tilde{t}(z))} T_1(b, z) = 0$.

Through similar arguments, one can show that $B(t, s_1^U(t, z))$ increases (strictly) monotonically from $p(\tilde{t}(z))$ to $\bar{B}(z) \equiv B(t^*(z), s_1^U(t^*(z), z))$ as t increases from $\tilde{t}(z)$ to $t^*(z)$. So $B(t, s_1^U(t, z))$ is invertible on $[p(\tilde{t}(z)), \bar{B}(z)]$. Define $T(b, z)$ on this interval as the inverse function; it maps $b \in [p(\tilde{t}(z)), \bar{B}(z)]$ to $[\tilde{t}(z), t^*(z)]$, it is (strictly) monotonically increasing with $T(p(\tilde{t}(z)), z) = \tilde{t}(z)$ and $T(\bar{B}(z), z) = t^*(z)$, and is continuously differentiable on $(p(\tilde{t}(z)), \bar{B}(z))$. Moreover, one can verify that $\lim_{b \downarrow p(\tilde{t}(z))} T_1(b, z) = 0$ and $\lim_{b \uparrow \bar{B}(z)} T_1(b, z)$ is finite.

Now we parameterize the appropriate partitions. For $\tau \in [0, 1]$, let $\sigma(\tau, z) = T(t^*(z) + \tau[\bar{B}(z) - t^*(z)], z)$. Also, let $\tau^s(z)$ be the unique solution of $\sigma(\tau^s(z), z) = \tilde{t}(z)$. For any strictly positive integer K , if $\tau \in [K, K + \tau^s(z)]$, let $\sigma(\tau, z) = s_K^L(\sigma(\tau - K, z), z)$, and if $\tau \in [K + \tau^s(z), K + 1]$, let $\sigma(\tau, z) = s_K^U(\sigma(\tau - K, z), z)$. It is easy to check that $\sigma(\cdot)$ is continuous. Let $N(z)$ denote the smallest integer such that $\sigma(N(z), z) \leq \frac{1}{2} < \sigma(N(z) + 1, z)$ (lemma 2 assures the existence of $N(z)$, and we know that $N(z) > 1$). $\sigma(\cdot)$ is continuously differentiable on $\tau \in (K, K + 1)$ for any integer $K \leq N(z)$ (since $\sigma(\tau, z) > \sigma(\tau - 1, z)$ for such τ).

To simplify notation, we will make use of some additional definitions. Let $\sigma_e(\tau) \equiv \sigma(\tau, \frac{x_k}{2})$, $N_e \equiv N(\frac{x_k}{2})$, $\tau_e \equiv \min\{\tau \mid \sigma_e(\tau) = \frac{1}{2}\}$ (note $\tau_e \in [N_e, N_e + 1]$), $\bar{\tau}_e \equiv \max_{\tau \leq \tau_e - 1} \sigma_e(\tau)$ (note $\bar{\tau}_e < \frac{1}{2}$), $G(\tau, z) \equiv B(\sigma(\tau - 1, z), \sigma(\tau, z))$, and $G_e(\tau) \equiv G(\tau, \frac{x_k}{2})$.

At various points in the proof, we require a measure of the distance between two function $B', B'' \in \mathcal{F}$. We use the following distance metric, which generates the topology of uniform C^1 convergence: $d(B', B'') = \sup_{(t_1, t_2) \in [0, 1]^2} |B'(t_1, t_2) - B''(t_1, t_2)| + \sup_{i, (t_1, t_2) \in [0, 1]^2} |B'_i(t_1, t_2) - B''_i(t_1, t_2)|$.

With these preliminaries out of the way, the proof proceeds in four steps.

Step 1: Let $B \in \mathcal{F}'$ iff τ_e is not an integer and $\sigma'_e(\tau_e) > 0$. \mathcal{F}' is open-dense in \mathcal{F} .

First, we establish denseness. Consider $B \notin \mathcal{F}'$. By the definition of τ_e , there exists $\underline{\tau} < \tau_e$ such that $[\underline{\tau}, \tau_e)$ contains no integer and $\sigma'_e(\tau) > 0$ for $\tau \in [\underline{\tau}, \tau_e)$. For any $\varepsilon > 0$, we can select some $\tilde{B} \in \mathcal{F}$ with $d(\tilde{B}, B) < \varepsilon$ and the following properties:

(i) $\tilde{B}(t_1, t_2) = B(t_1, t_2)$ for $t_2 \leq \bar{t}_e$ (and, symmetrically, for $t_1 \geq 1 - \bar{t}_e$).

(ii) There exists some $\tau' \in [\underline{\tau}, \tau_e)$ such that (a) $\tilde{B}(\sigma_e(\tau' - 1), \frac{1}{2}) = B(\sigma_e(\tau' - 1), \sigma_e(\tau'))$, and (b) $\tilde{B}_k(\sigma_e(\tau' - 1), \frac{1}{2}) = B_k(\sigma_e(\tau' - 1), \sigma_e(\tau'))$, $k = 1, 2$ (with, of course, a symmetric perturbation at the point $(\frac{1}{2}, 1 - \sigma_e(\tau' - 1))$).

By property (i), $\tilde{\sigma}_e(\tau) = \sigma_e(\tau)$ for $\tau \leq \tau_e - 1$. Thus, from the indifference conditions for type $\tilde{\sigma}_e(\tau' - 1) = \sigma_e(\tau' - 1)$ that define $\tilde{\sigma}_e(\tau')$ and $\sigma_e(\tau')$, it follows that $V(\tilde{B}(\sigma_e(\tau' - 1), \tilde{\sigma}_e(\tau')), \sigma_e(\tau' - 1)) = V(B(\sigma_e(\tau' - 1), \sigma_e(\tau')), \sigma_e(\tau' - 1))$. But from (ii) (a), this implies that $\tilde{\sigma}_e(\tau') = \frac{1}{2}$ (equivalently, $\tau' = \bar{\tau}_e$). Moreover, if one implicitly differentiates the indifference conditions defining $\tilde{\sigma}_e(\tau')$ and $\sigma_e(\tau')$ with respect to τ , (ii) (b) implies that $\tilde{\sigma}'_e(\tau') = \sigma'_e(\tau')$, which is strictly positive by construction. Thus, $\tilde{B} \in \mathcal{F}$.

Now, we establish openness. Choose some $B \in \mathcal{F}'$. Choose an interval $[\tau_1, \tau_2]$ such that $N_e - 1 < \tau_1 < \tau_e < \tau_2 < N_e$. Given that B and V are continuously differentiable, one can show that, as $\tilde{B} \rightarrow B$, $\tilde{\sigma}_e(\tau) \rightarrow \sigma_e(\tau)$ uniformly on $[0, \tau_2]$, $\tilde{\sigma}'_e(\tau) \rightarrow \sigma'_e(\tau)$ uniformly on $[\tau_1, \tau_2]$, and, consequently, $\tilde{\tau}_e \rightarrow \tau_e$ and $\tilde{\sigma}'_e(\tilde{\tau}_e) \rightarrow \sigma'_e(\tau_e) > 0$. Thus, for \tilde{B} sufficiently close to B , $\tilde{B} \in \mathcal{F}'$.

Step 2. Let $B \in \mathcal{F}''$ iff $B \in \mathcal{F}'$ and $G'_e(\tau_e) = 0$. \mathcal{F}'' is open-dense in \mathcal{F} .

We will show that \mathcal{F}'' is open-dense in \mathcal{F}' ; the conclusion then follows from step 1.

First we establish denseness. Consider $B \in \mathcal{F}' - \mathcal{F}''$. Recall that $\tau_e > 1$. By construction, $G_e(\tau) = t^*(z) + (\tau - 1)(\bar{B}(z) - t^*(z))$, so $G'_e(\tau) > 0$ for $\tau \in [1, 2]$. Thus, $G'_e(\tau_e) = 0$ implies $\tau_e > 2$.

Notice that, given $\sigma_e(\tau - 2)$ and $\sigma_e(\tau - 1)$, $G_e(\tau)$ is determined (in a neighborhood of τ_e) as the solution on $[\sigma_e(\tau - 1), 1]$ of the equation

$$V(B(\sigma_e(\tau - 2), \sigma_e(\tau - 1)), \sigma_e(\tau - 1)) = V(G_e(\tau), \sigma_e(\tau - 1))$$

and $\sigma_e(\tau)$ is then determined as the solution to

$$G_e(\tau) = B(\sigma_e(\tau - 1), \sigma_e(\tau))$$

Differentiating these expressions at $\tau = \tau_e$, we obtain

$$G'_e(\tau_e) = \frac{V_1(B_{e-1}, t_{e-1})G'_e(\tau_e - 1)}{V_1(B_e, t_e)} + \sigma'_e(\tau_e - 1) \left[\frac{V_2(B_{e-1}, t_{e-1}) - V_2(B_e, t_{e-1})}{V_1(B_e, t_e)} \right]$$

and

$$\sigma'_e(\tau_e - 1) = \frac{G'_e(\tau_e - 1) + B_1(t_{e-2}, t_{e-1})\sigma'_e(\tau_e - 2)}{B_2(t_{e-2}, t_{e-1})}$$

where we have used the following shorthand notation: $t_{e-n} = \sigma_e(\tau_e - n)$, and $B_{e-n} = B(t_{e-n-1}, t_{e-n})$.

Since $B \in \mathcal{F}' - \mathcal{F}''$, $\sigma'_e(\tau_e) > 0$ and $G'_e(\tau_e) = 0$, which can only be true if $\sigma'_e(\tau_e - 1) < 0$. Thus, we know that the numerator in the expression for $\sigma'_e(\tau_e - 1)$ is non-zero. We also claim that the final bracketed term in the expression for $G'_e(\tau_e)$ is non-zero. This follows from the fact that $V_1(B_e, t_e) > 0$ and

$$V_2(B_{e-1}, t_{e-1}) - V_2(B_e, t_{e-1}) = - \int_{B_{e-1}}^{B_e} V_{12}(t, t_{e-1}) dt < 0$$

For any $\varepsilon > 0$, we can find a \tilde{B} with $d(\tilde{B}, B) < \varepsilon$ that differs from B only in some small neighborhood of (t_{e-2}, t_{e-1}) (one that excludes all other points (t_{e-n-1}, t_{e-n}) for $n = 1$), and that satisfies $\tilde{B}(t_{e-2}, t_{e-1}) = B(t_{e-2}, t_{e-1})$ and $\tilde{B}_1(t_{e-2}, t_{e-1}) = B_1(t_{e-2}, t_{e-1})$, but for which $\tilde{B}_2(t_{e-2}, t_{e-1}) = B_2(t_{e-2}, t_{e-1})$. This perturbation alters the value of $B_2(t_{e-2}, t_{e-1})$ but leaves all of the other terms in the equations for $G'_e(\tau_e)$ and $\sigma'_e(\tau_e - 1)$ unaffected. Based on the observations in the preceding paragraph, for such a perturbation, $\tilde{G}'_e(\tau_e) = G'_e(\tau_e) = 0$, so $\tilde{B} \in \mathcal{F}''$.

The argument for openness parallels that given in step 1.

Step 3. There exists an open-dense set $\tilde{\mathcal{F}} \subset \mathcal{F}$ such that if $B \in \tilde{\mathcal{F}}$, then $\exists \tau_1, \tau_2$ such that $N_e - 1 < \tau_1 < \tau_e < \tau_2 < N_e$ satisfying (i) $\sigma'_e(\tau) > 0$ for $\tau \in [\tau_1, \tau_2]$, (ii) $G'_e(\tau) = 0$ for $\tau \in [\tau_1, \tau_2]$, (iii) $\sigma_e(\tau_1) = 1 - \sigma_e(\tau_2)$, and (iv) $V(G_e(\tau_1), \sigma_e(\tau_1)) = V(G_e(\tau_2), \sigma_e(\tau_2))$.

For $B \in \mathcal{F}''$, properties (i) and (ii) are plainly satisfied in a neighborhood of τ_e (since the relevant functions are continuously differentiable). Moreover, since $\sigma_e(\tau_e) = \frac{1}{2}$, it is plainly possible to pick τ_1 and τ_2 to satisfy (iii). Thus, we need only demonstrate that property (iv) holds on an open-dense subset of \mathcal{F}'' .

First, we establish denseness. Consider $B \in \mathcal{F}''$ for which property (iv) does not hold for any τ_1, τ_2 satisfying properties (i)-(iii). Choose some interval $[\tau_1, \tau_2]$ over which (i)-(iii) hold, such that $1 - \sigma_e(\tau_2) > \max_{\tau \leq \tau_2 - 1} \sigma_e(\tau)$ (since $\bar{t}_e < \frac{1}{2}$, one can always do this by taking τ_2 sufficiently close to τ_e). By step 1, we know that there is some $\delta > 0$ such that, for any \tilde{B} with $d(\tilde{B}, B) < \delta$, $\tilde{\sigma}'_e(\tau) > 0$ for all $\tau \in [\tau_1, \tau_2]$. For any $\varepsilon > 0$, choose \tilde{B} with $d(\tilde{B}, B) < \max\{\varepsilon, \delta\}$ to have the following properties: (a) $\tilde{B}(t_1, t_2) = B(t_1, t_2)$ outside of η -neighborhoods of the points $(\sigma_e(\tau_2 - 1), \sigma_e(\tau_2))$ and $(1 - \sigma_e(\tau_2), 1 - \sigma_e(\tau_2 - 1))$, and (b) $\tilde{B}(\sigma_e(\tau_2 - 1), \sigma_e(\tau_2)) < B(\sigma_e(\tau_2 - 1), \sigma_e(\tau_2))$ (which of course implies a symmetric perturbation at $(1 - \sigma_e(\tau_2), 1 - \sigma_e(\tau_2 - 1))$).

Property (a) implies that, provided we take η sufficiently small, $\tilde{G}_e(\tau) = G_e(\tau)$ for $\tau \leq \tau_2$ and $\tilde{\sigma}_e(\tau) = \sigma_e(\tau)$ for $\tau \leq \max\{\tau_1, \tau_2 - 1\}$ (note that we can choose this η independently of ε). Combined with the indifference conditions for type $\sigma_e(\tau_2 - 1)$, this implies that $\tilde{B}(\sigma_e(\tau_2 - 1), \tilde{\sigma}_e(\tau_2)) = B(\sigma_e(\tau_2 - 1), \sigma_e(\tau_2))$. From property (b), it then follows that $\tilde{\sigma}_e(\tau_2) > \sigma_e(\tau_2)$. Consequently, there is some $\tilde{\tau}_2 \in (\tau_1, \tau_2)$ such that $\tilde{\sigma}_e(\tilde{\tau}_2) = \sigma_e(\tau_2)$.

Now consider the interval $[\tau_1, \tilde{\tau}_2]$. For small ε , we have $\tilde{N}_e - 1 < \tau_1 < \tilde{\tau}_e < \tilde{\tau}_2 < \tilde{N}_e$. By construction, $\tilde{\sigma}'_e(\tau) > 0$ and $\tilde{G}_e(\tau) = G_e(\tau)$ for $\tau \in [\tau_1, \tau_2] \supset [\tau_1, \tilde{\tau}_2]$, so properties (i) and (ii) are satisfied. Note that $\tilde{\sigma}_e(\tau_1) = \sigma_e(\tau_1) = 1 - \sigma_e(\tau_2) = 1 - \tilde{\sigma}_e(\tilde{\tau}_2)$, so property (iii) is satisfied. Finally,

$$\begin{aligned} V(\tilde{G}_e(\tau_1), \tilde{\sigma}_e(\tau_1)) &= V(G_e(\tau_1), \sigma_e(\tau_1)) = V(G_e(\tau_2), \sigma_e(\tau_2)) \\ &= V(G_e(\tilde{\tau}_2), \sigma_e(\tau_2)) = V(\tilde{G}_e(\tilde{\tau}_2), \tilde{\sigma}_e(\tilde{\tau}_2)) \end{aligned}$$

(where the inequality follows from the fact that $G_e(\tau)$ is strictly monotonic for $\tau \in [\tau_1, \tau_2]$), so property (iv) is satisfied.

The argument for openness parallels that given in step 1.

Step 4. For any $B \in \tilde{\mathcal{F}}$, there exists $\varepsilon > 0$ such that if $|\underline{x} - \frac{x_k}{2}| < \varepsilon$, there exists a central pooling equilibrium with a norm of equal division.

Let $[\tau_1, \tau_2]$ be the interval satisfying properties (i)-(iv) of step 3. Then for sufficiently small ε , $\sigma(\tau, \underline{x})$ and $\sigma(\tau, x_k - \underline{x})$ are both strictly monotonic – and therefore invertible – on $[\tau_1, \tau_2]$. Let $t_L \equiv \max\{\sigma(\tau_1, \underline{x}), 1 - \sigma(\tau_2, x_k - \underline{x})\}$ and $t_R \equiv \min\{\sigma(\tau_2, \underline{x}), 1 - \sigma(\tau_1, x_k - \underline{x})\}$. Let $\pi_1(t)$ denote the inverse of $\sigma(\tau, \underline{x})$ and let $\pi_2(t)$ denote the inverse of $1 - \sigma(\tau, x_k - \underline{x})$; both are well-defined and monotonic on $[t_L, t_R]$. As $\varepsilon \rightarrow 0$, $t_L \rightarrow \sigma_e(\tau_1)$ and $t_R \rightarrow \sigma_e(\tau_2)$. Thus, for small ε , given property (iv), either $V(G(\pi_1(t_L), \underline{x}), t_L) > V(G(\pi_2(t_L), x_k - \underline{x}), 1 - t_L)$ and $V(G(\pi_1(t_R), \underline{x}), t_R) < V(G(\pi_2(t_R), x_k - \underline{x}), 1 - t_R)$, or both inequalities are reversed. In either case, by continuity, there exists some $\hat{t} \in [t_L, t_R]$ such that $V(G(\pi_1(\hat{t}), \underline{x}), \hat{t}) = V(G(\pi_2(\hat{t}), x_k - \underline{x}), 1 - \hat{t})$. But this implies the existence of an equilibrium with cheap talk partitions at $\sigma(\pi_1(\hat{t}) - K, \underline{x})$ for non-negative integers $K \leq \pi_1(\hat{t})$, and at $1 - \sigma(\pi_2(\hat{t}) - K, x_k - \underline{x})$ for non-negative integers $K \leq \pi_2(\hat{t})$. **QED**

Proof of theorem 5: In the proof of this theorem, we use the phrase ‘action pool’ to indicate a set of types that take the same action x , and we use the term ‘segment’ to indicate a set of types that select the same action x and message m .

Step 1: Any monotonic equilibrium with an action pool that does not include type $t = \frac{1}{2}$ does not satisfy the D1 criterion.

Consider a monotonic equilibrium $(\mu(t), \gamma(t), \phi(t, x, m))$. Since $\mu(t)$ is weakly increasing, any action pool must consist of a non-degenerate interval $[t_l, t_h]$. Moreover, since $V_{12}(\cdot) > 0$, every segment within $[t_l, t_h]$ must also consist of an interval (if t' and t'' both weakly prefer some segment to all other segments, then $t \in (t', t'')$ must strictly prefer this segment). Suppose that the equilibrium contains an action pool such that $\frac{1}{2} \notin [t_l, t_h]$. Without loss of generality, assume that $t_h < \frac{1}{2}$. Let μ_p denote the associated pooling action. Let $[t_h^-, t_h]$ denote the segment containing t_h . We know that $t_h^- < t_h$ (otherwise some $t \in [t_l, t_h]$ s.t. $t_h \in (t, p(t))$ would imitate t_h).

Let $\hat{b}(x, m) \equiv \hat{B}(\phi(\cdot, x, m))$, and for the equilibrium action and message functions $\mu(t)$

and $\gamma(t)$, let $b(t) \equiv \widehat{b}(\mu(t), \gamma(t))$. The function $b(t)$ represents the type-equivalent of the inference associated with the equilibrium action and message chosen by type t (note in a separating equilibrium one always has $b(t) = t$). Monotonicity along with assumption 2 (part (ii)) imply that for $x' > x''$ and any messages m' and m'' , $\widehat{b}(x', m') > \widehat{b}(x'', m'')$, and that $b(t)$ is weakly increasing in t . Since $t_h^- < t_h$, it follows that $b_p \equiv b(t_h) < t_h$.

Let $\mu' = \lim_{t \downarrow t_h} \mu(t)$ and let $b' = \lim_{t \downarrow t_h} b(t)$. Since $\mu(t)$ and $b(t)$ are weakly increasing, both limits are well-defined. Note that $\mu' \geq \mu_p$ and $b' \geq t_h > b_p$ (where $b' \geq t_h$ follows from the fact that $\mu(t') < \mu(t)$ when $t' \leq t_h < t$, which implies $\inf\{\text{supp}(\phi(\cdot, \mu(t), \gamma(t)))\} > t_h$ for all $t > t_h$, and from the fact that $B(r, s) \geq r$). Define W as follows:

$$W \equiv U(\mu_p, t_h) + \beta V(b_p, t_h) - U(\mu', t_h) - \beta V(b', t_h) \quad (24)$$

Then in equilibrium $W = 0$. Clearly, $W \geq 0$, because μ_p is an equilibrium action for t_h . Moreover, if $W > 0$, then for some small $\varepsilon > 0$, type $t_h + \varepsilon$ would prefer to imitate type t_h .

Next, we claim that $\mu' > \mu_p$. There are two cases to consider: $b' > t_h$, and $b' = t_h$. First suppose $b' > t_h$. Since $b(t) \geq b' \forall t > t_h$, all $t \in (t_h, b']$ lie in the same segment. This implies $\mu(t) = \mu(b')$, from which it follows that $\mu' = \mu(b')$. We cannot have $\mu(b') = \mu_p$ since t_h is defined as the highest type in the action pool at μ_p . Therefore, $\mu' = \mu(b') > \mu_p$, as desired. Next suppose $b' = t_h$. Suppose contrary to our claim that $\mu' = \mu_p$. Since $b_p < t_h$, type t_h strictly prefers (μ', b') to (μ_p, b_p) . But then type t_h would imitate some type slightly greater than t_h .

Thus, there exists a non-empty interval of off-equilibrium actions (μ_p, μ') . Fix any off-equilibrium action $\mu_d \in (\mu_p, \mu')$ and let $b_d = \widehat{b}(\mu_d, m)$ for some m . Monotonicity requires that $b' \geq b_d \geq b_p$.

We claim that, if the D1 criterion is satisfied, then $b_d = t_h$. Once we have shown this, step 1 follows immediately: since t_h strictly prefers (μ_p, t_h) to (μ_p, b_p) , t_h would have an incentive to choose an action $\mu_p + \varepsilon$ for some sufficiently small $\varepsilon > 0$, which contradicts the assumption that $(\mu(t), \gamma(t), \phi(t, x, m))$ is an equilibrium.

To show that the D1 criterion implies $b_d = t_h$, we need to demonstrate that, for all

$b \in [b_p, b']$, $t = t_h$,

$$U(\mu_d, t) + \beta V(b, t) \geq U(\mu(t), t) + \beta V(b(t), t) \quad (25)$$

implies

$$U(\mu_d, t_h) + \beta V(b, t_h) > U(\mu_p, t_h) + \beta V(b_p, t_h) \quad (26)$$

In words, this condition is satisfied if t_h is willing to deviate to μ_d for a strictly larger set of allowable inferences than any other t . There are two cases to consider: $t < t_h$, and $t > t_h$.

First consider $t < t_h$. Notice that

$$U(\mu(t), t) + \beta V(b(t), t) \geq U(\mu_p, t) + \beta V(b_p, t) \quad (27)$$

Therefore, (25) implies:

$$U(\mu_d, t) + \beta V(b, t) \geq U(\mu_p, t) + \beta V(b_p, t) \quad (28)$$

Thus (26) follows from (25) if:

$$\begin{aligned} & U(\mu_d, t_h) - U(\mu_p, t_h) + \beta V(b, t_h) - \beta V(b_p, t_h) \\ & > U(\mu_d, t) - U(\mu_p, t) + \beta V(b, t) - \beta V(b_p, t) \end{aligned} \quad (29)$$

We can rewrite the preceding inequality as:

$$\int_{\mu_p}^{\mu_d} \int_t^{t_h} U_{12}(r, s) ds dr + \beta \int_{b_p}^b \int_t^{t_h} V_{12}(r, s) ds dr > 0 \quad (30)$$

This inequality holds because $V_{12}(\cdot, \cdot) > 0$ and $U_{12}(\cdot, \cdot) > 0$.

Next consider $t > t_h$. Earlier in this proof, we established that

$$U(\mu_p, t_h) + \beta V(b_p, t_h) = U(\mu', t_h) + \beta V(b', t_h) \quad (31)$$

We also know that

$$U(\mu(t), t) + \beta V(b(t), t) \geq U(\mu', t) + \beta V(b', t) \quad (32)$$

Therefore to establish that (26) follows from (25), it suffices to show that

$$U(\mu_d, t) + \beta V(b, t) \geq U(\mu', t) + \beta V(b', t)$$

implies

$$U(\mu_d, t_h) + \beta V(b, t_h) > U(\mu', t_h) + \beta V(b', t_h)$$

The above statement holds if the following inequality is satisfied:

$$\begin{aligned} & U(\mu_d, t_h) + \beta V(b, t_h) - U(\mu', t_h) - \beta V(b', t_h) \\ & > U(\mu_d, t) + \beta V(b, t) - U(\mu', t) - \beta V(b', t) \end{aligned} \quad (33)$$

This inequality can be rewritten as:

$$\int_{\mu_d}^{\mu'} \int_{t_h}^t U_{12}(r, s) ds dr + \beta \int_b^{b'} \int_{t_h}^t V_{12}(r, s) dr ds > 0 \quad (34)$$

This inequality holds because $U_{12}(\cdot, \cdot) > 0$ and $V_{12}(\cdot, \cdot) > 0$.

Step 2. Separating equilibria are obviously monotonic and trivially satisfy the D1 criterion, because there are no out-of-equilibrium actions. We complete the proof by demonstrating that, for any central pooling equilibrium there exists a equivalent monotonic central pooling equilibrium (same action function and message function, possibly different out-of-equilibrium beliefs) that satisfies the D1 criterion.

Consider any central pooling equilibrium characterized by the parameters t_ℓ , t_h , and x_p . The set of out-of-equilibrium actions is $(\underline{\mu}(t_\ell), x_p) \cup (x_p, \bar{\mu}(t_h))$. Construct out-of-equilibrium beliefs as follows: for $x \in (\underline{\mu}(t_\ell), x_p)$ let $\phi(t_\ell, x, m) = 1$ and $\phi(t, x, m) = 0$ for $t < t_\ell$, and for $x \in (x_p, \bar{\mu}(t_h))$ let $\phi(t_h, x, m) = 1$ and $\phi(t, x, m) = 0$ for $t < t_h$. It is easy to check that this is an equilibrium.

To show that it satisfies the D1 criterion, we argue as follows. For all $x \in (x_p, \bar{\mu}(t_h))$, define $t^I(x) \in (b_p, t_h)$ (where $b_p \equiv B(t_\ell, t_h)$) as the solution to the following equation:

$$U(x, t_h) + \beta V(t^I, t_h) = U(x_p, t_h) + \beta V(b_p, t_h) \quad (35)$$

Existence is guaranteed because, for $t^I = b_p$, the left-hand side exceeds the right-hand side, while for $t^I = t_h$, the opposite is true (this follows from the fact that type t_h is indifferent

between (x_p, b_p) and $(\bar{\mu}(t_h), t_h)$, and from the fact that $x < \bar{\mu}(t_h) < X(t_h)$. From an argument analogous to one used in the proof of step 1, it follows that, for all $t = t_h$,

$$U(x, t) + \beta V(t^I(x), t) < U(\mu(t), t) + \beta V(b(t), t) \quad (36)$$

But then, for all such x , D1 never rules out beliefs that place positive probability on type t_h , and in particular it cannot rule out beliefs that place all probability on type t_h . In words, expressions (35) and (36) imply that there is no type t that would be willing to deviate to the action x for a strictly larger set of allowable inference than type t_h ; in particular, only t_h would be willing to deviate to x for the inference $t^I(x)$. Naturally, one can prove a symmetric property for the lower half of the type space, from which it follows that for all $x \in (\underline{\mu}(t_\ell), x_p)$, D1 never rules out beliefs that place positive probability on type t_ℓ , and in particular it cannot rule out beliefs that place all probability on type t_ℓ . Thus, the proposed equilibrium satisfies the D1 criterion. **QED.**

Appendix B

Recall that we can represent equation (9) as the dynamical system (12). The initial condition $(\frac{1}{2}, \frac{1}{2})$ is a fixed point of this system. To study its behavior near $(\frac{1}{2}, \frac{1}{2})$, we linearize the system at $(\frac{1}{2}, \frac{1}{2})$ to obtain:

$$\begin{pmatrix} \mu'(s) \\ t'(s) \end{pmatrix} = \begin{pmatrix} 0 & \beta(V_{11} + V_{12}p'(\frac{1}{2})) \\ -U_{11} & -U_{12} \end{pmatrix} \begin{pmatrix} \mu(s) \\ t(s) \end{pmatrix}$$

where the functions $U_{11}(\cdot, \cdot)$, $U_{12}(\cdot, \cdot)$, $V_{11}(\cdot, \cdot)$, $V_{12}(\cdot, \cdot)$ are evaluated at $(\frac{1}{2}, \frac{1}{2})$. Solving the characteristic equation, we can compute the eigenvalues $\lambda_{1,2}$:

$$\lambda_{1,2} = -\frac{1}{2}U_{12} \pm \frac{1}{2}\sqrt{U_{12}^2 - 4\beta U_{11} [V_{11} + V_{12}p'(\frac{1}{2})]}$$

We know that $V_1(t, t) = 0$ for all t , so $V_{11}(t, t) + V_{12}(t, t) = 0$. Since $V_{12}(t, t) > 0$ and since $p'(\frac{1}{2}) > 1$ (condition 2 holds), $V_{11}(\frac{1}{2}, \frac{1}{2}) + V_{12}(\frac{1}{2}, \frac{1}{2})p'(\frac{1}{2}) > V_{11}(\frac{1}{2}, \frac{1}{2}) + V_{12}(\frac{1}{2}, \frac{1}{2}) = 0$. Therefore, $\lambda_1 > 0 > \lambda_2$, and it follows that $(\frac{1}{2}, \frac{1}{2})$ is a saddle point of the linearized system. Then the behavior of the linearized system around $(\frac{1}{2}, \frac{1}{2})$ is described by the following differential equation:

$$\begin{pmatrix} \mu(s) - \frac{1}{2} \\ t(s) - \frac{1}{2} \end{pmatrix} = c_1 \vec{x}_1 e^{\lambda_1 s} + c_2 \vec{x}_2 e^{\lambda_2 s}$$

where c_1 and c_2 are arbitrary constants, and \vec{x}_1 and \vec{x}_2 are eigenvectors associated with eigenvalues λ_1 and λ_2 respectively.

The trajectory along the unstable arm of the linearized dynamical system is determined by the eigenvector $\vec{x}_1 = (x_1^1, x_1^2)'$ ($c_2 = 0$). By Hartman-Grobman theorem (theorem 5.3.5, p.153, Robinson (1995)), the original non-linearized system also has a unique trajectory emanating from $(\frac{1}{2}, \frac{1}{2})$ with the slope equal to $\frac{x_1^1}{x_1^2}$, and a unique trajectory converging to $(\frac{1}{2}, \frac{1}{2})$ with slope $\frac{x_2^1}{x_2^2}$. The eigenvector \vec{x}_i satisfies:

$$\begin{pmatrix} 0 & \beta(V_{11}(\frac{1}{2}, \frac{1}{2}) + V_{12}(\frac{1}{2}, \frac{1}{2})p'(\frac{1}{2})) \\ -U_{11}(\frac{1}{2}, \frac{1}{2}) & -U_{12}(\frac{1}{2}, \frac{1}{2}) \end{pmatrix} \begin{pmatrix} x_i^1 \\ x_i^2 \end{pmatrix} = \lambda_i \begin{pmatrix} x_i^1 \\ x_i^2 \end{pmatrix}$$

Since $\forall t U_1(t, t) = 0$, we have $U_{12}(\frac{1}{2}, \frac{1}{2}) = -U_{11}(\frac{1}{2}, \frac{1}{2}) > 0$. But this implies that

$$U_{12} \left(\frac{1}{2}, \frac{1}{2} \right) (x_1^1 - x_1^2) = \lambda_1 x_1^2,$$

which in turn implies that

$$\frac{x_1^1}{x_1^2} = 1 + \frac{\lambda_1}{U_{12} \left(\frac{1}{2}, \frac{1}{2} \right)} > 1,$$

where the inequality follows from the fact that $\lambda_1 > 0$ and $U_{12} \left(\frac{1}{2}, \frac{1}{2} \right) > 0$. This implies that, locally along the unstable arm, $\mu(t)$ is increasing in t , $\mu(t) < t$ for $t < \frac{1}{2}$, and $\mu(t) > t$ for $t > \frac{1}{2}$. An argument similar to that provided in the proof of theorem 1 allows us to extend this conclusion to the entire interval $[0, 1]$.

Since $\lambda_2 < 0$ and since $\beta \left(V_{11} \left(\frac{1}{2}, \frac{1}{2} \right) + V_{12} \left(\frac{1}{2}, \frac{1}{2} \right) p' \left(\frac{1}{2} \right) \right) > 0$, x_2^1 and x_2^2 have opposite signs. This implies that, locally along the stable arm, $\mu(t)$ is decreasing in t , $\mu(t) > \frac{1}{2}$ for $t < \frac{1}{2}$, and $\mu(t) < \frac{1}{2}$ for $t > \frac{1}{2}$. Once again, it is possible to extend this conclusion to the entire interval $[0, 1]$.

References

- [1] Abel, Andrew, and B. Douglas Bernheim, "Fiscal Policy with Impure Intergenerational Altruism," *Econometrica* 59 (6), November 1991, 1687-1712.
- [2] Andreoni, James, "Giving with Impure Altruism: Applications to Charity and Ricardian Equivalence," *Journal of Political Economy* 97, 1989, 1447-1458.
- [3] Austen-Smith, David, and Jeffrey S. Banks, "Cheap Talk and Burned Money," mimeo, University of Rochester, 1995.
- [4] Baik, Stephen P., and Michael D. Kahn, *The Sibling Bond*, Harper Collins Publishers Inc.: N.Y., 1997.
- [5] Banks, Jeffrey S., "A Model of Electoral Competition with Incomplete Information," *Journal of Economic Theory* 50, April 1990, 309-325.
- [6] Banks, Jeffrey S., and Joel Sobel, "Equilibrium Selection in Signaling Games," *Econometrica* 55(3), May 1987, 647-661.
- [7] Barro, Robert J., "Are Government Bonds Net Wealth?" *Journal of Political Economy* 82, November/December 1974, 1095-1117.
- [8] Becker, Gary S., "A Theory of Social Interactions," *Journal of Political Economy* 82, November/December 1974, 1063-1093.
- [9] Bednar, Richard L., and Scott R. Peterson, *Self-Esteem: Paradoxes and Innovation in Clinical Theory and Practice*, 2nd edition, American Psychological Association: Washington, D.C., 1996.

- [10] Bernheim, B. Douglas, "How Strong are Bequest Motives? Evidence Based on Estimates of the Demand for Life Insurance and Annuities," *Journal of Political Economy* 99 (5), October 1991, 899-927.
- [11] Bernheim, B. Douglas, "A Theory of Conformity," *Journal of Political Economy* 102, 1994, 841-877.
- [12] Bernheim, B. Douglas, and Kyle Bagwell, "Is Everything Neutral?" *Journal of Political Economy* 96(2), 1988, 308-338.
- [13] Bernheim, B. Douglas, and Sergei Severinov, "Bequests as Signals: Implications for Fiscal Policy," mimeo, Stanford University, 1999.
- [14] Bernheim, B. Douglas, and Sergei Severinov, "Multidirectional Signaling," 1999 (in progress).
- [15] Bernheim, B. Douglas, Andrei Shleifer, and Lawrence Summers, "The Strategic Bequest Motive," *Journal of Political Economy* 93 (6), December 1985, 1045-1076.
- [16] Cho, In-Koo, and David M. Kreps, "Signaling Games and Stable Equilibria," *Quarterly Journal of Economics* 102(2), May 1987, 179-221.
- [17] Cho, In-Koo, and Joel Sobel, "Strategic Stability and Uniqueness in Signaling Games," *Journal of Economic Theory* 50, April 1990, 381-413.
- [18] Chu, C. Y. Cyrus, "Primogeniture," *Journal of Political Economy* 99(1), 1991, 78-99.
- [19] Coopersmith, Stanley, *The Antecedents of Self-Esteem*, San Francisco, CA: Freeman, 1967.
- [20] Crawford, V. and J. Sobel, "Strategic Information Transmission," *Econometrica* 50, 1982, 1431-1451.

- [21] Dunn, Thomas, and John Phillips, "Do Parents Divide Resources Equally Among Children? Evidence from the AHEAD Survey," Aging Studies Program Paper No. 5, Syracuse University, 1997.
- [22] Gale, William G., and John Karl Scholz, "Intergenerational Transfers and the Accumulation of Wealth," *Journal of Economic Perspectives* 8, 1994, 145-160.
- [23] Guinnane, Timothy W., "Intergenerational Transfers, Emigration, and the Rural Irish Household System," *Explorations in Economic History* 29, 1992, 456-476.
- [24] Hirsch, M. W.. and S. Smale, *Differential Equations, Dynamical Systems and Linear Algebra*, Academic Press, 1974.
- [25] Hurd, Michael, "Saving of the Elderly and Desired Bequests," *American Economic Review* 77, 1987, 298-312.
- [26] Hurd, Michael, "Mortality Risk and Bequests," *Econometrica* 57, 1989, 779-814.
- [27] Kotlikoff, Laurence J., and Avia Spivak, "The Family as an Incomplete Annuities Market," *Journal of Political Economy* 89, April 1981, 372-391.
- [28] McGarry, Kathleen, "Inter vivos Transfers and Intended Bequests," mimeo, Department of Economics, UCLA, 1998.
- [29] Menchik, Paul L., "Primogeniture, Equal Sharing and the U.S. Distribution of Wealth," *Quarterly Journal of Economics* 94(2), March 1980, 299-316.
- [30] Menchik, Paul L., "Unequal Estate Division: Is it Altruism, Reverse Bequests, or Simply Noise?" *Modelling the Accumulation and the Distribution of Wealth*, New York: Oxford University Press, 1988, 105-116.
- [31] Robinson, C., *Dynamical Systems*, 1995, CRC Press.

- [32] Schelling, Thomas, *The Strategy of Conflict*, Harvard University Press: Cambridge, Mass., 1960.
- [33] Spence, A. Michael, *Market Signaling*, Harvard University Press: Cambridge, Mass., 1974.
- [34] Tomes, Nigel, "The Family, Inheritance and the Intergenerational Transmission of Inequality," *Journal of Political Economy* 89(5), October 1981, 928-958.
- [35] Wilhelm, Mark O., "Bequest Behavior and the Effect of Heirs' Earnings: Testing the Altruistic Model of Bequests," *American Economic Review* 86(4), September 1996, 874-892.

Figure 1 : Illustration of Indifference Contours

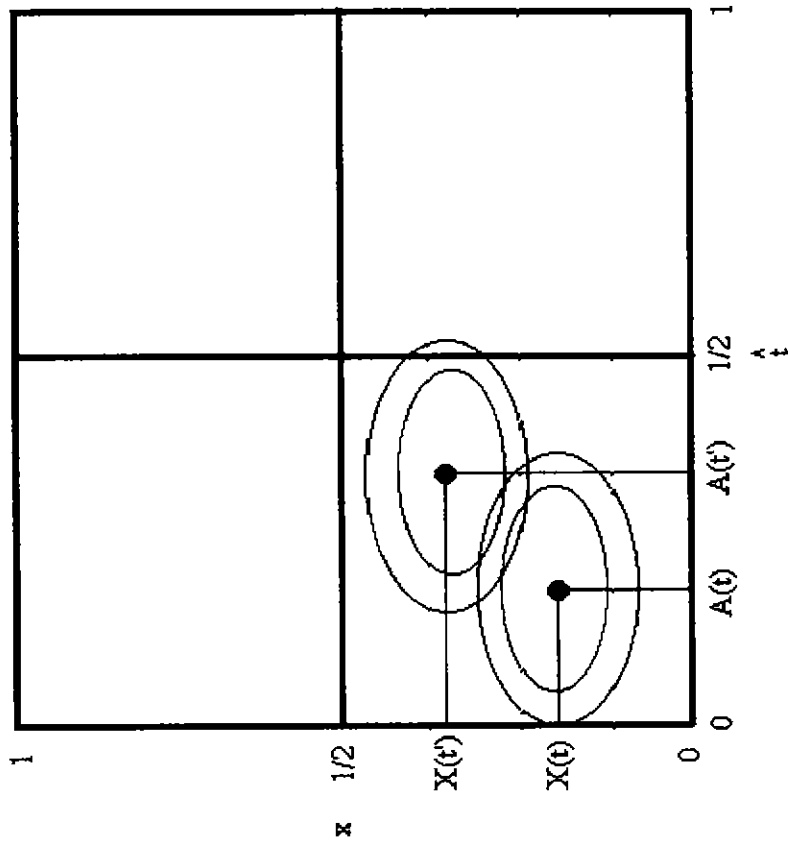


Figure 2: Illustration of Condition 1

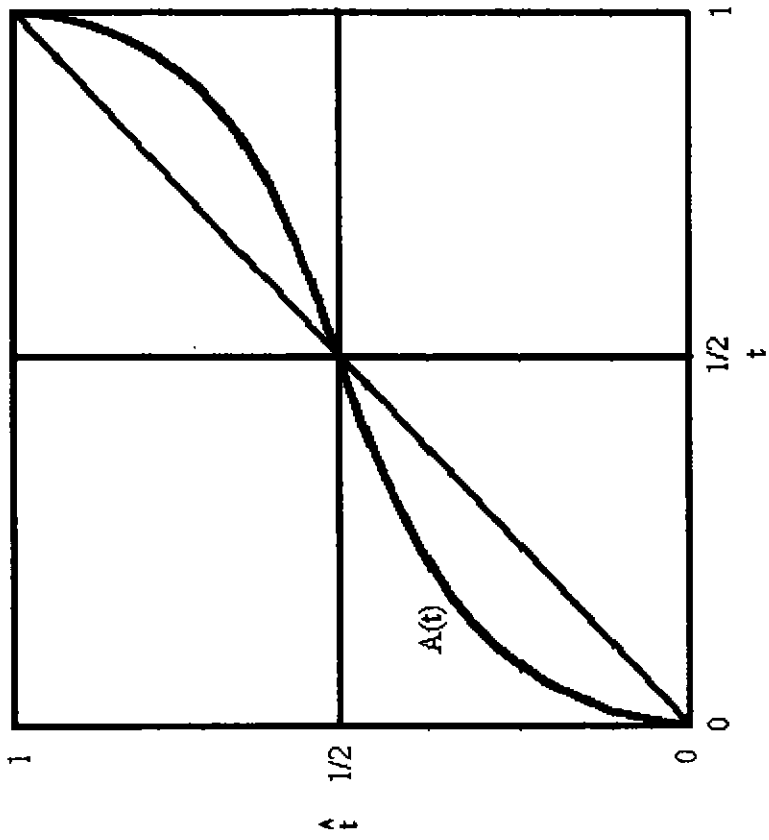


Figure 3 : Illustration of Condition 2

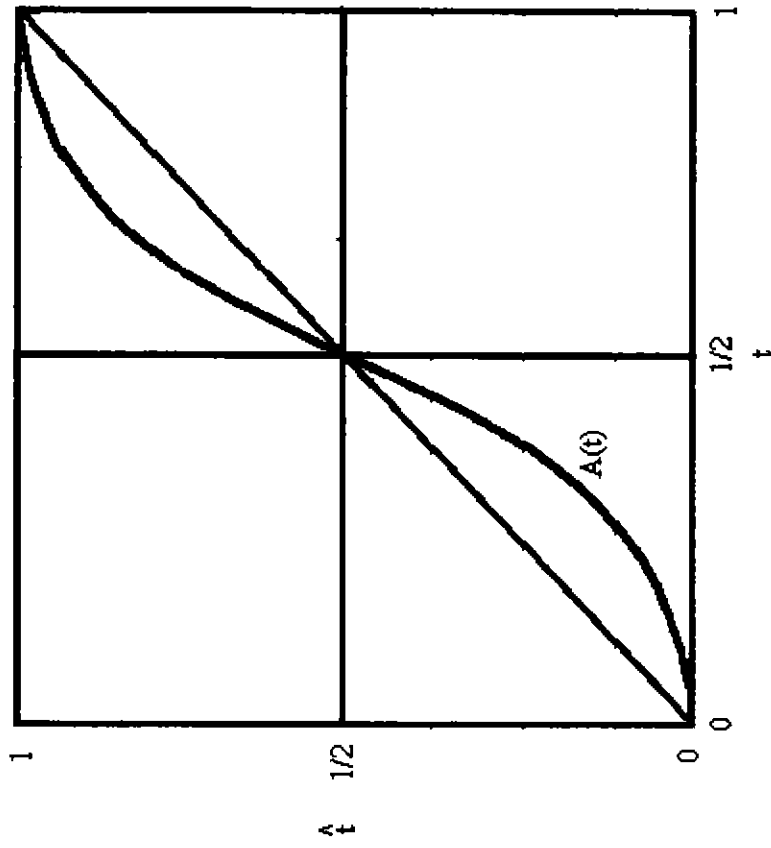


Figure 4: An Example of Non-monotonicity

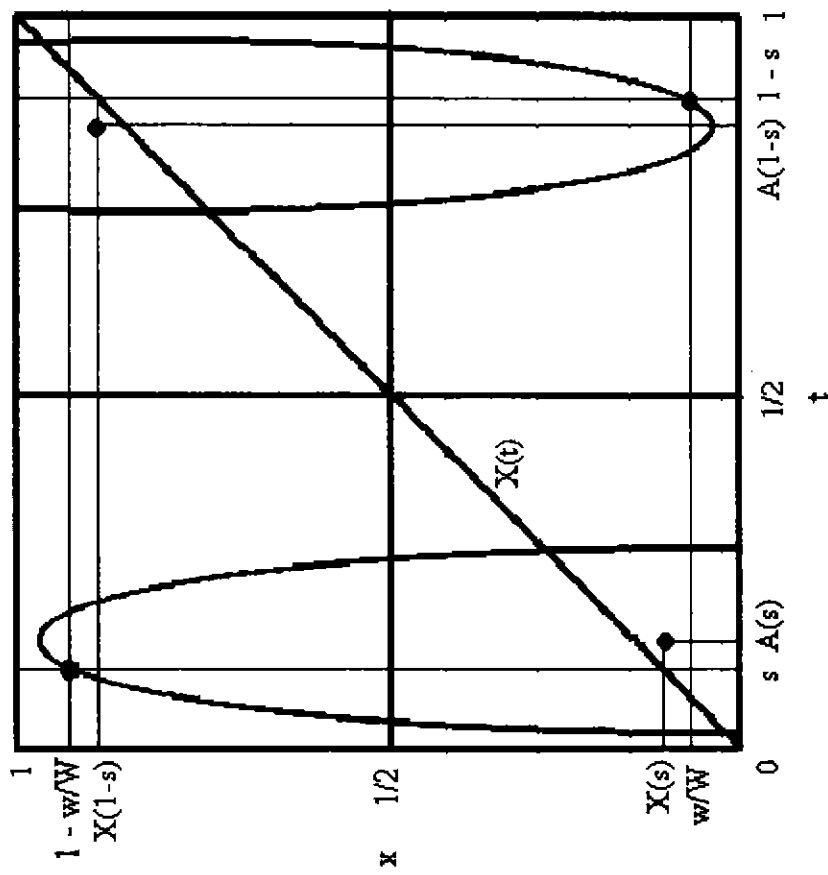


Figure 5: Existence of Equilibria with an Equal Division Norm: The Effects of β and W .

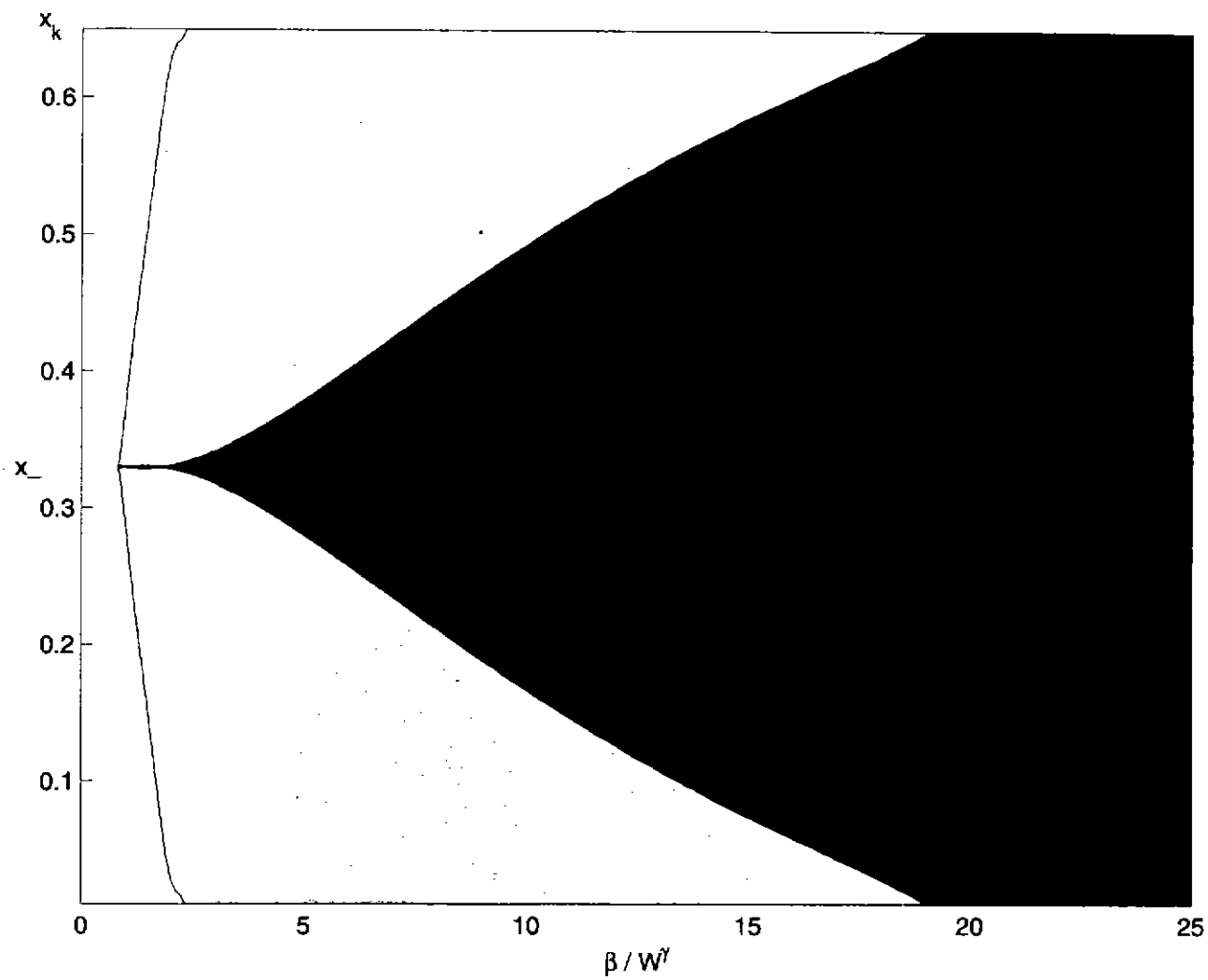


Figure 6: Existence of Equilibria with an Equal Division Norm: The Effects of w_p/W .

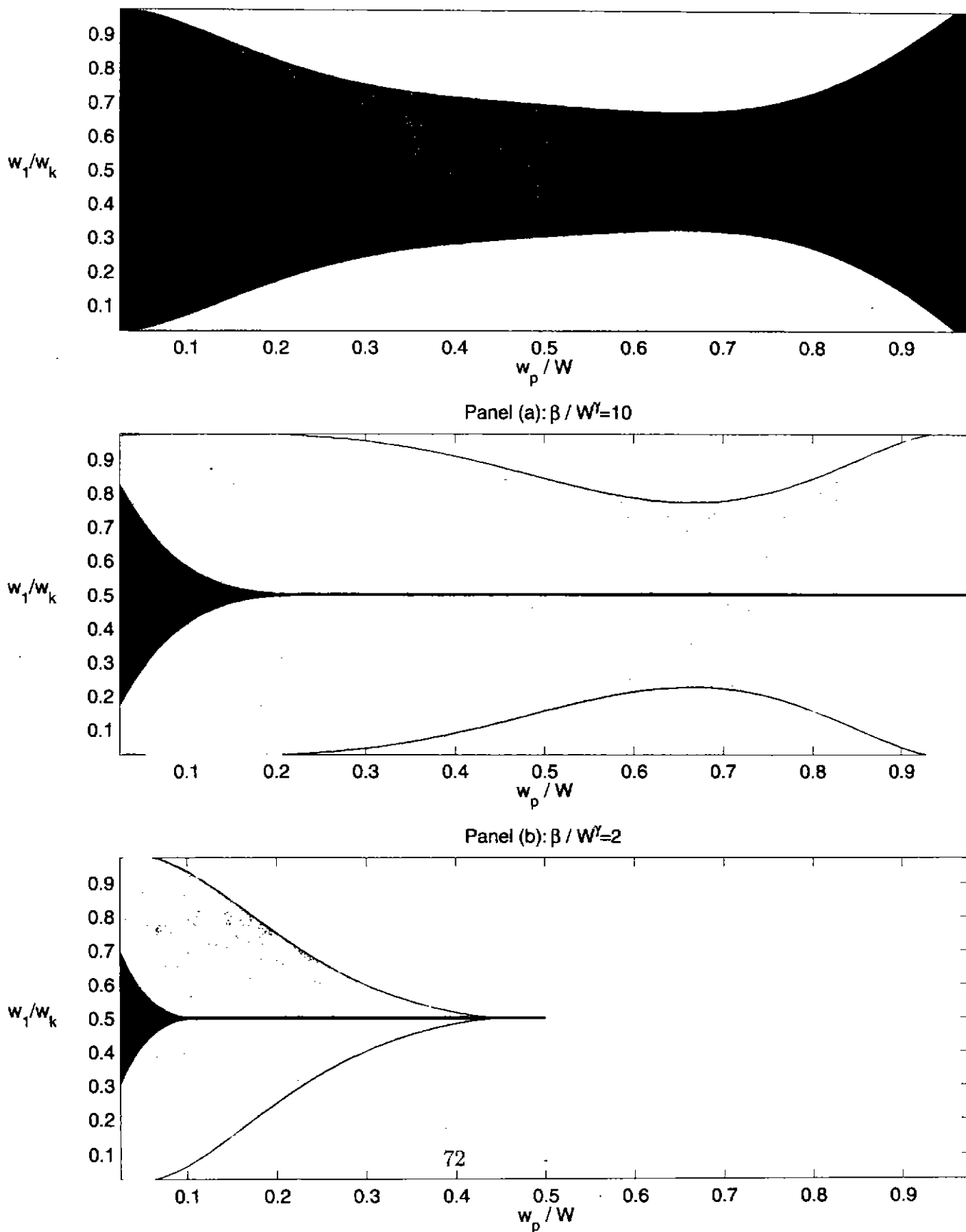


Figure 7: Local Dynamics Under Condition 2

