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ABSTRACT

Our purpose in this paper is to present a class of convex endogenous growth models, and to analyze their performance in terms of both growth and business cycle criteria. The models we study have close analogs in the real business cycle literature. In fact, we interpret the *exogenous* growth rate of productivity as an *endogenous* growth rate of human capital. This perspective allows us to compare the strengths of both classes of models.

In order to highlight the mechanism that gives endogenous growth models the ability to improve upon their exogenous growth relatives, we study models that are symmetric in terms of human and physical capital formation -- our two engines of growth. More precisely, we analyze models in which the technology used to produce human capital is identical to the technologies used to produce consumption and investment goods, and in which the technology shocks in the two sectors are perfectly correlated.

We find that endogenous growth models can generate levels of labor volatility close to those observed in the data, as well as positively correlated growth rates of output. We also find that these models outperform a related exogenous growth version in most dimensions.

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1. Introduction

The effects of aggregate shocks on economic performance is a topic that has been studied intensively in the real business cycle literature. Even though the real business cycle program has been quite successful in accounting for the cyclical properties of post-war aggregate data (see Cooley (1995) for an excellent review), it has some shortcomings. Those that we find particularly important are: (1) the "trend" in output and its components -- and, hence, the methods used to remove the trend -- have been taken as exogenous, and independent of the sources of fluctuations; (2) the models predict substantially lower variability of labor supply than that observed in the data unless utility is linear in leisure (or, equivalently, there is an indivisibility in labor supply and lotteries are introduced); (3) the models must resort to unobservable, or at least difficult to measure, costs of adjustment, margins of variation, or asymmetries to mimic the persistence of growth rates (see Cogley and Nason (1995) for a discussion); and (4) the properties of the time series implied by the models are not particularly sensitive to the specification of the degree of intertemporal substitution, though this a critical piece of information in understanding the economy's response to exogenous shocks.

Our purpose in this paper is to present a class of convex endogenous growth models, and to analyze their performance in terms of both growth and business cycle criteria. The models we study have close analogs in the real business cycle literature, and hence are a natural first step when moving beyond the standard real business cycle model. In fact, we interpret the *exogenous* growth rate of productivity as an *endogenous* growth rate of human capital. This perspective allows us to compare the strengths of both classes of models using a relatively large number of moments of the joint distribution of macroeconomic time series. Moreover, we deviate from the standard calibration exercise in that we report simulation results for a wide range of the parameters of interest -- specifically, the intertemporal elasticity of substitution -- and their effects on the cyclical properties of endogenous variables.

In order to highlight the mechanism that gives endogenous growth models the ability to improve upon their exogenous growth relatives, we study models that are symmetric in terms of human and physical capital formation -- our two engines of growth. More precisely, we analyze models in which the technology used to produce human capital is identical to the technologies used to produce consumption and investment goods. This is a natural first environment to analyze, since it is very difficult to find evidence that gives reliable information about the capital (both physical and human) to labor ratios across sectors, or the differential impact of productivity shocks.

Since all the models that we consider imply *both* that a number of variables of interest are non-stationary and that some appropriate transformations are, we can compare exogenous and endogenous growth models along a variety of statistics, all of which are stationary conditional on the model. Thus, our approach shifts attention from filtered data (typically, but not exclusively, using the Hodrick-Prescott filter) to either growth rates or ratios of specific concepts (e.g. consumption) to output. A major advantage of our approach is that there is no longer any need to separate the "growth" component from the "cyclical" component, as one model explains both. Indeed, from a formal point of view, it would be incorrect to do so.

Our major findings are:

- The introduction of shocks does not have a large impact on the mean values of simulated data -- including the growth rate of output -- derived from the endogenous growth models we study. Thus, our findings agree with those of Jones, Manuelli and Stacchetti (1999) who study the impact of volatility in fundamentals on the distribution of growth rates.

- The endogenous growth model shows far more labor supply variability than the standard exogenous growth, real business cycle model. This finding highlights a key difference between the two classes of models. In the endogenous growth models that we study, human capital services and hours are jointly supplied to the market. Thus, cyclical fluctuations in labor supply are amplified by cyclical changes in the demand for human capital services. In addition, the model has a fair amount of success in explaining the standard deviation of the growth rate of output, the growth rate of labor productivity and the consumption-output ratio, relative to the exogenous growth analog.

- The endogenous growth model outperforms the real business cycle version in terms of its predictions for the serial correlation properties of annual growth rates of output, and labor productivity. Hence, the models that we study contain important internal propagation mechanisms. The exogenous growth model predicts a value for the autocorrelation of growth rates of capital that is closer to the U.S. value.

- The degree of intertemporal substitution is a major determinant of the second moment properties of time series implied by the endogenous growth models. *Small* differences in the intertemporal elasticity of substitution induce *large* changes in the predicted variability of the consumption-output ratio and the coefficient of variation of hours worked. For our specifications, we find that the "best" fit is obtained for elasticities of substitution lower than one (the logarithmic case). In contrast, the degree of intertemporal substitution has a small effect in the exogenous growth model.

Throughout the paper, we consider only the simplest versions of both endogenous and exogenous growth models, and it is clear that more study is warranted. This is true for both types of models. In order to generate labor volatility values close to those of the U.S. economy, the real business cycle model has been generalized to include indivisible labor (see Hansen (1985)), home production (see Benhabib, Rogerson and Wright (1991), and Greenwood and Hercowitz (1991)), cyclical factor utilization (see Burnside and Eichenbaum (1996), and King and Rebelo (1999)) and a separate, unshocked, sector producing human capital (see Einarsson and Marquis (1998)). Variations of the basic setup designed to produce positive autocorrelation in output growth include labor market search (see Merz (1995) and Andolfatto (1996)), cyclical capital utilization (see Burnside and Eichenbaum (1996)), costs of adjustment (see Cogley and Nason (1995)), extreme "time-to-build" restrictions (see Christiano and Todd (1996)), and differences in the technologies used to produce physical goods and human capital, as well the incidence of shocks across sectors (see Perli and Sakellaris (1998)).

Our paper is not the first to study business cycle effects in an endogenous growth setting.[2] Einarsson and Marquis (1997) study the effects of including human capital accumulation in a model with home production. If the home production technology and the market production technology are sufficiently different, and if shocks do not affect home production, they obtain positive correlation between home and market investment, and the share of (inelastically supplied) labor allocated to consumption and market investment activities. Einarsson and Marquis (1999) study an endogenous growth model with two stocks of human capital, whose production is not affected by shocks. In this setting they are able to generate relatively volatile labor supply, and a small correlation between output and labor productivity. Finally, Collard (1999), studies an endogenous growth model with home production, costs of adjustment in physical capital accumulation, human capital which is accumulated through learning-by-doing and an externality in aggregate labor productivity. The model succeeds at matching the autocorrelation of output growth at the cost of implying, counterfactually, that the consumption-output and the investment-output are constant.

All of the models that have attempted to improve the predictions of the standard real business cycle model for both volatility of hours worked and the serial correlation of output growth have resorted to

[2] For analyses that emphasize cross country differences, see Mendoza (1997), Jones, Manuelli and Stacchetti (1999), de Hek (1999), Fatas (2000).

various asymmetries. These include different production functions in the home, human capital, and physical goods production sectors; in particular, they make strong assumptions about the capital-labor ratio and differential elasticities of substitution across sectors that are not backed by evidence. In addition, for the models to produce the desired results it is necessary to assume a particular pattern of incidence for the technology shocks: in most models, the human capital (or the home production) sector is not subject to any shocks, since this facilitates substitution in and out of market work, increasing the volatility of measured hours. Finally, several of the models resort to (difficult to measure) adjustment costs.

Our model contributes to this literature by showing that realistic values of labor supply volatility, autocorrelation in output growth, and a number of other second moments, can be attained without resorting to asymmetries -- in either production technologies or the incidence of productivity shocks -- and costs of adjustment. Unlike the papers described above, we put emphasis on the role of the intertemporal elasticity of substitution, and on matching model and historical data that are rendered stationary in a manner consistent with the theory.

In section 2, we begin by laying out a general formulation of the class of models that we are interested in studying and provide a simple methodological tool for handling the fact that the natural state space is unbounded. In section 3, we take an initial look at the quantitative properties of a simple class of these models with equal depreciation rates for physical and human capital. In section 4, we consider alternative versions of the model with different depreciation rates for the two capital goods and allow for the possibility that investment in human capital is partly omitted from the income and product accounts altogether. Finally, section 5 provides some concluding comments.

2. A General Model

The class of models that we are interested in studying feature investment in both human and physical capital and a time stationary technology that is subjected to random shocks. They are stochastic versions of the convex models described in Jones and Manuelli (1990). A general specification that captures these features is given by:

(2.1) $$\text{Max } E\{\textstyle\sum_t \beta^t u(c_t,\ell_t)\}$$

subject to,

$$c_t + x_{zt} + x_{ht} + x_{kt} \leq F(k_t,z_t,s_t)$$
$$z_t \leq M(n_{zt},h_t,x_{zt})$$

$$k_{t+1} \leq (1-\delta_k)\, k_t + x_{kt}$$
$$h_{t+1} \leq (1-\delta_h)\, h_t + G(n_{ht}, h_t, x_{ht})$$
$$\ell_t + n_{ht} + n_{zt} \leq 1,$$
h_0 and k_0 given.

Here $\{s_t\}$ is a stochastic process which we assume is Markov with a time stationary transition probability function, c_t is consumption, x_{kt} is investment in physical capital, k_t is the stock of physical capital, x_{ht} is investment in human capital, h_t is stock of human capital, z_t is "effective labor," n_{zt} is hours spent in the market working, n_{ht} is hours spent in augmenting human capital and ℓ_t is leisure. The depreciation rates on physical and human capital are given by δ_k and δ_h, respectively. If F, M and G are concave and bounded below by a homogenous of degree one function, it is possible to show that the competitive equilibrium allocation coincides with the solution to the planner's problem and, for some parameter values, displays income (and consumption) growth.

Thus, this is a fairly standard endogenous growth model in which effective labor is made up of a combination of hours and human capital which is supplied to the market. For specific choices of functional forms, many models in this literature are special cases of this formulation. For example, if $M = n_z h$ and $G = G_0 h n_h$, the model corresponds to that of Lucas (1988) in the absence of externalities. If $M = n_z h$ and $G = x_h$, this corresponds to the two capital goods version discussed in Jones, Manuelli and Rossi (1993).

The actual solution of models in this class does cause some problems, however. The natural choice of the state is the vector (k_t, h_t, s_t). The problem that this poses is that both k_t and h_t are diverging to infinity (at least for versions of the model that exhibit growth on average). To solve this problem, the key property that we exploit is that for models of this type to have a balanced growth path, both preferences and technology must be restricted in a specific way (see King, Plosser and Rebelo (1988), and Alvarez and Stokey (1998)). For our numerical strategy, it suffices that the model satisfies:

<u>Assumption</u>: *Preferences and Technology*

a) The instantaneous utility function satisfies,

$$u(c,\ell) = \begin{cases} v(\ell)\, c^{1-\sigma}/(1-\sigma) & \text{with } \sigma \neq 1, \text{ but } \sigma > 0, \text{ or} \\ \log(c) + v(\ell) & \text{with } \sigma = 1 \end{cases}$$

b) F is concave and homogeneous of degree one in (k,z)

c) M is concave and homogeneous of degree one in (h,x_z)

d) G is concave and homogeneous of degree one in (h,x_h).

These restrictions, in turn, imply that knowledge of the current shock and the current human capital to physical capital ratio (the two relevant pseudo state variables) is sufficient to determine the optimal choices of employment and next period's human to physical capital ratio. Given the state, the current stocks, the productivity shock and the current level of employment it is possible to determine consumption and future capital stocks using static first order conditions.

Indeed, the property that there is a transformation of the problem in which all the (relevant) variables are stationary is a special case of a much more general (and fairly standard) argument. Our assumptions about the technological side of the model imply that holding the vector of labor supplies fixed, a time path of the endogenous variables, z_t (interpreted as the entire state/date contingent plan) is feasible from initial state (h_0, k_0, s_0) if and only if λz_t is feasible from the initial state $(\lambda h_0, \lambda k_0, s_0)$ $(\lambda > 0)$. That is, the feasible set is linearly homogeneous holding the vector of labor supplies fixed. Moreover, utility also has a homogeneity property -- again holding labor supplies fixed, the utility (i.e., the entire expected discounted sum) realized from λz_t is $\lambda^{1-\sigma}$ times the utility of z_t (at the same labor supplies). Formally, consider the maximization problem:

(2.2) $\quad$ Max $U(z, n)$

subject to

$$(z,n) \in \Gamma(h_0, k_0, s_0),$$

where, as noted, (z, n) is interpreted as the entire date/state contingent path of the endogenous variables and vector of labor supplies and U is the resulting expected discounted sum of utilities. Let $V(h_0, k_0, s_0)$ denote the maximized value in this problem (assuming that it exists) and let $(z^*(h_0, k_0, s_0), n^*(h_0, k_0, s_0))$ denote the optimal plan.

Proposition 1: Assume that the utility function in (2.2) is homogeneous of degree $1-\sigma$ in z (holding n fixed) and that the feasible set, Γ, is linearly homogeneous in (h, k) (holding n and s fixed) and that a solution exists for all (h, k, s). Then, the value function, V, for the problem (P.2) satisfies $V(\lambda k, \lambda h, s) = \lambda^{(1-\sigma)} V(k,h,s)$, for all $\lambda>0$. Moreover, the optimal choice of z is homogeneous of degree one and the optimal choice of n is homogeneous of degree zero -- $(z^*(\lambda k, \lambda h, s), n^*(\lambda k, \lambda h, s)) = (\lambda z^*(h, k, s), n^*(h ,k ,s))$.

Proof: See Appendix A.

3. A Simple Example with Endogenous Growth and Equal Depreciation

In this section, we study the properties of a calibrated version of the model. Our objective is twofold: First, we want to understand how intertemporal substitution affects the implications of the model; second, our intent is to compare this class of endogenous growth models with a more standard real business cycle model (with constant, exogenous growth). To this end, we not only parameterize a version of the model of section 2, but we also analyze a "related" exogenous growth model similar to those studied in the real business cycle literature.

3.1 Calibrating the Model

To specialize the model of section 2, we adopt the restrictions on preferences outlined above, and assume that the production function is given by $F(k, nh, s) = sAk^{\alpha}(nh)^{1-\alpha}$. The laws of motion for physical and human capital are: $k' = (1-\delta_k)\, k + x_k$ and $h' = (1-\delta_h)\, h + x_h$. We also assume that both capital stocks depreciate at the same rate, that is, $\delta_k = \delta_h$. From a formal point of view, our choice of a linear law of motion for capital amounts to an aggregation assumption: the technology used to produce investment in human capital goods (education, training, and health among others) is identical to the technology used to produce general output.[3]

Using the two stochastic Euler equations of the model, it follows that,

$$E_t\, [u_c(t+1)(\alpha F(t+1)/k_{t+1} - (1-\alpha)F(t+1)/h_{t+1})] = 0.$$

Hence, in any interior equilibrium, $h_t/k_t = (1-\alpha)/\alpha$ for all t. This is an important property of the specification of a Cobb-Douglas production function with equal depreciation rates: the human to physical capital ratio is independent of the level of employment and the productivity shock. Given this, and defining $A^* \equiv A(1-\alpha)^{1-\alpha}\alpha^{\alpha}$, it follows that

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[3] This is obviously an extreme assumption. However, we were unable to obtain estimates of the physical capital - labor or physical capital - human capital ratios in specific activities like education and health.

(3.1) $$c_t = k_t\,[s_tA^*\,n_t^{1-\alpha}\,((1-n_t)/n_t)\,((1-\alpha)/\alpha\psi)] \equiv k_t\,g_1(s_t,n_t).$$

Using this, we obtain,

(3.2) $$k_{t+1}=k_t\left[s_t\,A^*\,n_t^{1-\alpha}\left(1-\frac{1-\alpha}{\psi}\frac{1-n_t}{n_t}\right)+1-\delta\right] \equiv k_t g_2(s_t,n_t)$$

Finally, after substitution, the relevant Euler equation becomes:

(3.3) $$[g_1(s_t,n_t)\,(1-n_t)^{\psi}]^{-\sigma}\,(1-n_t)^{\psi} = \beta \int_S\{[g_2(s_t,n_t)g_1(s_{t+1},n_{t+1})(1-n_{t+1})^{\psi}]^{-\sigma} \times(1-n_{t+1})^{\psi}[1-\delta+s_{t+1}A^*(n_{t+1})^{1-\alpha}]P(s_t,ds_{t+1})\}.$$

A solution to this equation is a function $n^*:S \to [0,1]$ with $n_t=n^*(s_t)$. Note that given n^*, the solution to the planner's problem (and the competitive equilibrium for this economy) can be completely described using (3.1) and (3.2) and $h_{t+1} = k_{t+1}(1-\alpha)/\alpha$.

Our model puts restrictions on concepts that -- although clearly identifiable from a theoretical point of view -- are difficult to measure. Prime examples are consumption and investment in human capital. In the data, private expenditures on schooling and health (arguably investments in human capital) are assumed to be part of consumption, and some forms of investment (e.g. training) are likely to remain unmeasured. In the model, those expenditures are more properly viewed as being investments in human capital. The resolution of this problem is not easy. As a first approximation, and we change this in section 4, we assume in our calibration that measured consumption (in the National Income and Product Accounts) corresponds to the sum of consumption and investment in human capital in the model, $c + x_h$. Thus, it is the variable that enters the utility function, along with the level of investment (or spending) in human capital that coincides with measured consumption.

We set capital's share, α, equal to 0.36, and hold β fixed at 0.95. We set $\delta_k = \delta_h = 0.075$. This is a compromise between the relatively high values for physical investment used in the literature and the small values typically estimated for depreciation of human capital. We relax this assumption in section 4. Finally, we choose the remainder of the parameters of the model so as to match the average growth rate of U.S. output over the 1955-1992 period of 1.38% per year and average labor supply equal to 0.17 (see Jones, Manuelli and Rossi, (1993)). These two facts ($\gamma = 1.38\%$ and $n = 0.17$) pin down two of the three remaining parameters of the model, σ, ψ, and A. This leaves one degree of freedom in the choice of these parameters. Since one of our interests is to determine how the degree of intertemporal substitution affects the business cycle properties of the model, we vary σ from 0.9 to 3.0 while simultaneously changing A and ψ so that along the model's non-stochastic balanced growth path, $\gamma = 1.38\%$ and $n = 0.17$.

We assume that the process, s_t, is given by $s_t = \exp[z_t - \sigma_\epsilon^2/2(1-\rho^2)]$ with $z_t = \rho z_t + \epsilon_{t+1}$, where the ϵ's are i.i.d., normal with mean zero and variance σ_ϵ^2. It follows that $E(s_t)=1$. To choose ρ and σ_ϵ^2, we use the fact that, with $\delta_k = \delta_h$, the ratio h_t/k_t is identically $(1-\alpha)/\alpha$, and hence, output is given by $y_t = A\ s_t\ (\ ((1-\alpha)/\alpha)^{1-\alpha}\ k_t\ (n_t)^{1-\alpha}$. Thus, given data on output, the capital stock and hours, the time series of s_t can be directly identified up to the constant $A\ ((1-\alpha)/\alpha)^{1-\alpha}$. We use the data set from Burnside and Eichenbaum (1996) (renormalized to reflect annual frequencies) to construct the implied time series of s_t given α. Using this series -- which is not obviously the realization of a stationary process -- we estimate ρ and σ_ϵ^2 to be 0.95 and 0.0146, respectively.[4] Table A.1 in Appendix A shows all the combinations of parameters.

The model that we study has a "related" exogenous growth version. More precisely, if h_t -- our human capital variable -- is assumed to grow exogenously at the rate $\hat{\gamma}$, the technology becomes F(k, nh ,s) $= sA_t k^\alpha (n)^{1-\alpha}$, with $A_t = A(h_t)^{1-\alpha} = \hat{A}\hat{\gamma}^t$. Given this specification, s_t is calculated using the same procedure as before. In this case, the estimated parameters are $\rho = 0.95$ and $\sigma^2_\epsilon = 0.0126$.[5]

To solve the model we use the method of parameterized value function iteration as discussed in Siu (1998). The method finds a linear combination of Chebychev polynomials to approximate the value function in the recursive representation of the model. This is done by iterating upon the contraction mapping over a discretization of the state space until the fixed point is found. See Appendix B for details. We used simulations of time series of the endogenous variables of the models with T=5000 periods in order to calculate estimates of key population moments. To facilitate comparisons, throughout the paper, the same realization of $\{s_t\}$ was used for all cases with the same parameters for the stochastic process.

3.2 Some Results

[4] Since our non-stochastic model has the balanced growth property, our procedure forces the shock to explain the productivity slowdown that started in the mid-seventies. This implies that, in our sample, the estimated shocks are decreasing from a peak of 1.07 (7% above average) in 1966. Even though feasible, modeling the productivity slowdown is beyond the scope of this paper. For alternative explanations see Greenwood and Yorukoglu (1997), Caselli (1999) and Manuelli (2000).

[5] The difference in the estimated standard deviation here relative to the endogenous growth specification is due to the fact that the former allows the growth rate to covary with the shock, while the latter fixes it a 1.38%.

As indicated above, our purpose is to assess the performance of the model in terms of its ability to replicate the distribution of observed variables, and to study the differences in propagation mechanisms between the endogenous and the exogenous growth models. We concentrate on four dimensions of that distribution: mean, standard deviation, autocorrelation and cross correlations.

In previous work, Jones, Manuelli and Stacchetti (1999) find that the introduction of technology shocks in a class of endogenous growth models does not have a large effect on the mean values of the endogenous variables, relative to their (non-stochastic) balanced growth values, unless the shock variance is fairly large. We find the same pattern for the specifications in this paper: the introduction of uncertainty results in slightly higher simulated mean growth rates, but quantitatively the effect is small. Moreover, the results are only slightly affected by the magnitude of the intertemporal elasticity of substitution. The average value of growth rates in the exogenous growth model are virtually identical to their balanced growth values, irrespective of the value of σ. However, the average value of labor supply does differ from its calibrated value, particularly when intertemporal substitution is low. In the endogenous growth version the share of physical capital investment in output, when we "allocate" all of the investment in human capital to consumption, is slightly lower than in the data, and the share of "true" consumption in output is low.[6] The results are displayed in Table A.2 in Appendix A.

Table 3.1 presents the results for the standard deviation of the growth rates of output, γ, and labor productivity, $\gamma_{y/n}$, the investment-output ratio, x_k/y,[7] and the coefficient of variation in hours worked, n; this is done for both the endogenous growth model and its exogenous growth counterpart. In the context of the models that we study all of these variables are stationary. The final row of the table gives the corresponding values for the U.S. from the Burnside and Eichenbaum (1996) data.

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[6] This is a product of our simplifying assumptions. Given our calibrated value for the depreciation rate, the share of output needed to maintain human capital along the balanced growth path is large, and the share needed to maintain physical capital is small (and incompatible with investment's share in the data). See section 4 for versions of the model in which both of these assumptions are relaxed.

[7] We report only the standard deviation of the investment-output ratio since, in the model, $y = c + x_h + x_k$, and it follows that $\sigma(x_k/y) = \sigma((c+x_h)/y)$. Hence, by construction, the variability of the consumption-output ratio and the investment-output ratio coincide.

Table 3.1: Volatility - Endogenous Growth and Real Business Cycle Models

Case	σ	$\sigma(\gamma)$	$\sigma(\gamma)$ - R	$\sigma(x_k/y)$	$\sigma(x_k/y)$-R	$\sigma(n)/\bar{n}$	$\sigma(n)/\bar{n}$ -R	$\sigma(\gamma_{y/n})$	$\sigma(\gamma_{y/n})$-R
1	0.90	.0371	.0182	.0163	.0116	.1008	.0124	.0161	.0100
2	1.00	.0286	.0175	.0102	.0106	.0622	.0113	.0130	.0103
3	1.25	.0216	.0165	.0051	.0090	.0301	.0096	.0130	.0108
4	1.50	.0191	.0159	.0033	.0082	.0187	.0086	.0136	.0111
5	1.75	.0178	.0156	.0023	.0077	.0128	.0081	.0140	.0113
6	2.00	.0171	.0153	.0017	.0074	.0093	.0077	.0142	.0114
7	2.50	.0162	.0150	.0010	.0071	.0052	.0073	.0146	.0116
8	3.00	.0157	.0147	.0006	.0070	.0029	.0071	.0148	.0117
U.S.		.0214	.0214	.0119	.0119	.0342	.0342	.0124	.0124

Note: The column labeled $\sigma(z)$ gives the standard deviation of z, where z corresponds to the growth rate of output (γ), the investment/output ratio (x_k/y), the growth rate of labor productivity ($\gamma_{y/n}$), and labor (n). The term $\bar{n}$ denotes mean number of hours. An R indicates that the column corresponds to the values of a "real business cycle" version with exogenous growth as described in the text.

The first important observation is that, unlike its impact on the first moments of the distribution, changes in the intertemporal elasticity of substitution ($1/\sigma$) have a large impact on the second moments of most variables of the endogenous growth model. In stark contrast, it has a relatively small impact in the exogenous growth model.[8] For example, increasing σ from 0.9 to 3.00 decreases the variability in hours worked by a factor of 35 in the endogenous growth model; in the exogenous growth model this experiment decreases the same variability by a factor of 1.75.

The endogenous growth model does quite well at moderate values of σ. For example, when σ = 1.25, the standard deviation of output growth from the model is virtually identical to that in the data, the standard deviation of the investment-output ratio is 43% of the U.S. value, the coefficient of variation of hours worked is 88% of the estimated U.S. value, and the standard deviation of the growth rate of labor productivity exceeds the U. S. value by 5%. Except for the standard deviation of the investment-output ratio (or, equivalently, the consumption-output ratio) the endogenous growth model studied here performs better than its exogenous growth counterpart. This is particularly evident in the volatility of the growth

[8] This may explain why this literature has typically not explored the effects of alternative values of the elasticity of substitution. For a good survey see Cooley (1995).

rates and hours worked.[9] In the next section, we show how natural extensions of the endogenous growth model improve its ability to match the volatility in the measured consumption- and investment-output ratio.

Table 3.2 reports the autocorrelation properties of the endogenous variables from the simulation, for both the endogenous growth model and its exogenous growth counterpart.

Table 3.2: Autocorrelations - Endogenous Growth and Real Business Cycle Models

Case	σ	$\rho_1(\gamma_y)$	$\rho_1(\gamma_y)$- R	$\rho_1(\gamma_{c+xh})$	$\rho_1(\gamma_c)$- R	$\rho_1(n)$	$\rho_1(n)$- R	$\rho_1(\gamma_{y/n})$	$\rho_1(\gamma_{y/n})$- R
1	0.90	.169	-.019	.247	.316	.949	.744	.972	.201
2	1.00	.145	-.014	.199	.239	.949	.767	.763	.156
3	1.25	.114	-.007	.141	.146	.949	.805	.344	.101
4	1.50	.098	-.004	.115	.104	.949	.830	.211	.074
5	1.75	.089	-.003	.100	.081	.949	.849	.154	.059
6	2.00	.082	-.002	.090	.066	.949	.864	.124	.049
7	2.50	.073	-.001	.077	.049	.949	.888	.093	.037
8	3.00	.067	-.000	.069	.038	.949	.906	.077	.029
U.S.		.213	.213	.400	.400	.825	.825	.295	.295

Note: The column labeled $\rho_1(z)$ corresponds to the first order autocorrelation coefficient of z, where z corresponds to the growth rate of output (γ_y), consumption ($c+x_h$), labor productivity ($\gamma_{y/n}$), and the level of the number of hours worked (n). An R indicates that this corresponds to the output of a "real business cycle" version with exogenous growth as described in the text.

There are several interesting results. First, the endogenous growth model generates persistence in output growth for all values of σ considered; the degree of first order autocorrelation increases with the intertemporal elasticity of substitution. Quantitatively, the endogenous growth model with equal depreciation rates can account for about 50% of the degree of persistence in growth rates for values of σ near one. In the exogenous growth model, in all cases, the autocorrelation of output growth is negative. This, of course, is just another instance of the well known failure of real business cycle models to display realistic propagation mechanisms (see Cogley and Nason (1995)).

Second, both models explain a relatively small amount of the observed autocorrelation of the growth rate of consumption. If anything, the exogenous growth model performs better for high values of

[9] This is true irrespective of the calibrated non-stochastic balanced growth value of labor supply, and hence, the calibrated value of ψ. In experiments, we have calibrated the model using n = 0.3, a value close to those used in the real business cycle literature. None of the substantive results presented in this paper are sensitive to this choice.

the intertemporal elasticity of substitution. Third, in the endogenous growth model, the autocorrelation of hours worked coincides with that of the shock and, for our specification, overstates the measured autocorrelation by 15%. The exogenous growth model does better in this dimension and, for some specifications, almost perfectly matches the data. Fourth, the endogenous growth model is able to match the autocorrelation in the growth rate of labor productivity for a value of σ between 1.25 and 1.50, while the real business cycle model understates it for all cases considered.

Finally, for both specifications -- endogenous and exogenous growth -- the autocorrelation of growth rates of the endogenous variables depends on the intertemporal elasticity of substitution. Using the autocorrelation as the target dimension, the endogenous growth model still "prefers" a risk aversion coefficient of $\sigma = 1.25$, while the exogenous growth version "prefers" $\sigma = 0.90$. The latter, however, does a poor job matching the autocorrelation of the growth rate of output at all values of σ.

The last statistic we present are the cross correlations --at several leads and lags-- of the growth rates of output and labor productivity, and the levels of hours worked and the investment-output ratio, with the growth rate of output. Since it is cumbersome to report all the results for the different values of intertemporal substitution, we choose the values of σ that better match the data for both models. Again, this criterion "selects" $\sigma = 1.25$ for the endogenous growth version, and $\sigma = 0.9$ for the exogenous growth version. Table 3.3 contains the estimates of the cross correlations generated by both models, as well as the corresponding statistics from the U.S.

Table 3.3: Cross Correlations with Output Growth - Endogenous Growth (bold), Exogenous Growth, and U.S. Data (*italics*). (Endogenous $\sigma = 1.25$ - Exogenous $\sigma = 0.9$)

Lag (j)	-2	-1	0	1	2
$\gamma_{y,t+j}$	**0.100**	**0.114**	**1.000**	**0.114**	**0.100**
	-0.012	-0.019	1.000	-0.019	-0.012
	-0.081	*0.214*	*1.000*	*0.214*	*-0.081*
n_{t+j}	**0.196**	**0.209**	**0.507**	**0.484**	**0.459**
	-0.053	-0.057	0.624	0.471	0.358
	-0.328	*-0.327*	*0.118*	*0.308*	*0.280*
$\gamma_{y/n,t+j}$	**0.143**	**0.158**	**0.962**	**0.245**	**0.226**
	-0.022	-0.029	0.961	0.158	0.121
	0.272	*0.384*	*0.482*	*-0.166*	*0.144*
$(x_k/y)_{t+j}$	**0.195**	**0.209**	**0.506**	**0.483**	**0.458**
	-0.052	-0.057	0.624	0.470	0.357
	-0.315	*-0.139*	*0.509*	*0.454*	*0.159*

Overall, neither model is a complete successes at matching the cross-correlation structure of the data. As indicated above, the endogenous growth model generates more persistence, and this appears in the form of higher autocorrelation values for the growth rate of output. It also captures the pattern of cross-correlations between productivity and output growth. The exogenous growth model does a better job of matching the cross-correlation between hours and growth rates.

For the sake of comparison with the real business cycle literature we followed the standard practice of taking logs and Hodrick-Prescott filtering the data. We then calculated the standard deviation of all relevant variables -- the analog of Table 3.1 -- for all cases, as well as the cross correlation of detrended output, hours, labor productivity and investment with output at different leads and lags for the "best" cases -- the analog of Table 3.3; the results are in Tables A.3 and A.4 in the appendix. Overall, the general flavor of the simulated results is similar to those found using the stationary ratios: The endogenous growth model predicts that small differences in the degree of intertemporal substitution result in substantial differences in the standard deviation of most variables. In addition, it outperforms the real business cycle model in terms of predicting standard deviations for all variables that are closer to observed U.S. values. In terms of cross-correlations, both models are fairly successful at matching the autocorrelation of detrended output. As in the case of unfiltered data, both models feature high contemporaneous correlations among the variables, and these estimates are too high relative to the U.S. data. For both models, hours lead the cycle much more than in the data while labor productivity lags the cycle more than in the data. Overall, the endogenous growth and the exogenous growth versions are fairly comparable using Hodrick-Prescott filtered data according to these measures.

To summarize, we find that the simple endogenous growth model that we study in this section does quite well at matching some statistics of the U.S. time series. In most cases, it outperforms a related exogenous growth version. In particular, the exercise shows that the class of endogenous growth models has the potential to deliver the kind of internal propagation mechanisms that the real business cycle literature has been trying to find. The endogenous growth model displays greater variability in hours worked, despite the absence of an unshocked sector which "competes" with the market sector for labor resources (e.g. home production as in Benhabib, Rogerson and Wright (1991), and human capital production through formal training as in Einarsson and Marquis (1998)), and utility which is linear in leisure (see Hansen (1985)). However, there are two dimensions in which the model performs relatively poorly: the variability of the consumption-output (or investment-output) ratio is too low relative to the data, and, although better than the exogenous growth model, it fails to account for the first order autocorrelation patterns in the data -- in particular, it exhibits low autocorrelation of the growth rates of consumption.

4. Alternative Approaches to Human Capital Formation

The simple model that we analyzed in the previous section shows that endogenous growth models have the potential to generate increased labor supply volatility and display the kind of internal propagation mechanisms that are necessary to match U.S. observations. Specifications with the intertemporal elasticity of substitution somewhere between 1/2 and 1 ($\sigma = 2$ and $\sigma = 1$, respectively) are closest to matching the data. At the same time, we uncovered two weaknesses of the simple model: the predicted volatility of the consumption-output ratio is too low relative to the data, and the predicted autocorrelations do not match the evidence. In this section we extend the model in two dimensions: first, we allow for a fraction of human capital investment to be unmeasured; second, we do not impose equality of the depreciation rates on physical and human capital.

4.1 Calibration and Results

Since the notion of human capital is empirically difficult to measure, it is not surprising that there is a paucity of estimates of the depreciation rate, δ_h. Haley (1976) estimates δ_h to be somewhere between 1% and 4%. Heckman (1976) obtains point estimates that are similar to Haley's, but are not significantly different from zero. Earlier work by Ben-Porath (1967) estimates δ_h close to 9%. The results in Jorgenson and Fraumeni (1989) are consistent with depreciation rates that range between 1% and 3%. In order to cover the range of estimates, we experimented with three values of δ_h: a low value of 0.01, an intermediate value of 0.04, and a high value of 0.07.

The second issue that we have to deal with is what part of investment in human capital is included in measured GNP. There are several categories of investment in human capital that are likely to be omitted from standard GNP accounting. The single largest category is on-the-job-training and on-the-job-learning. In both cases, it is possible that a fraction of these costs is not counted in standard measures of output.[10] In addition, acquisition of human capital at home (e.g., time of both children and parents), and student inputs

[10] From the "income" side, training that is paid for by firms, i.e. not deducted from wages, appears as decreased profits in the period in which those investments are made. A very good discussion that illustrates the key issues appears in Howitt (1997).

at school are not measured. A second difficulty is that, as indicated above, whatever is measured is scattered between government spending and private consumption. As in section 3, we solve this problem by allocating the measured component of investment in human capital to measured consumption. It is difficult to estimate what fraction of "true" investment in human capital is included in GNP. Considering only training (which is likely to be excluded) its contribution has been estimated somewhere between 50% of total investment (Mincer (1962)) to just over 20% (Heckman, Lochner and Taber (1998)). Our approach is to assume that only a certain fraction, η, of investment in human capital is included in GNP. We vary the possible values of η from a low of 0.25 to a high of 1.0.

Since there is (potentially) an unmeasured component of output, measured GNP will differ from "true" output. In particular, measured GNP is $c + x_k + \eta x_h$. Since we continue to assume that all of the model's labor input is accounted for in the data, we interpret the unmeasured part of x_h as on-the-job-training that workers receive while employed. This interpretation requires recalibrating the model. To see this, note that if we continue to denote labor's share of GNP by $(1-\alpha)$, then it must be that case that $(1-\alpha)$ GNP $= w\,(nh) + (1-\eta)\,x_h$. This implies lower values of α than are commonly used in the literature. From now on, we will use the term GNP to refer to measured output, and use simply output to denote the total amount of goods and services produced.

As in the previous section, we set $\beta = 0.95$, and choose the remainder of the parameters to match long run U.S. observations. These include the average growth rate of output of 1.38% and average labor supply equal to 0.17. In addition, we require payments to capital to be 36% of measured output, and physical capital investment's share of measured output to be 24.4%. This leaves three degrees of freedom in the choice of parameters. We experiment with different values of σ, δ_h and η to see how the model differs from those in section 3. Note that in performing these adjustments, the calibrated values of α, A, δ_k and ψ respond to maintain our identifying assumptions. For our numerical exercise we considered the following configurations:

$$\sigma \in \{1.00, 1.25. 1.50, 2.00\},\ \eta \in \{0.25, 0.50, 1.00\},\text{ and } \delta_h \in \{0.01, 0.04, 0.07\}$$

In all, we computed 36 different cases. A complete description of the parameters is in Table A.5.[11]

For the stochastic shock process, we used the same parameters as in the previous section. (Indeed, we used the same realization for the simulations.) This allows us to do direct comparisons of the results

[11] The table includes a 37^{th} case that will be discussed later.

with those of the previous section.[12] Finally, since there is no "obvious" exogenous growth analog of the model in this section, we continue using the results from section 3 for the purposes of comparison.

As in section 3, the model's implications for the *means* of the variables of interest result in small deviations from their calibrated values, and we do not report the results here. The implications of the model for the *standard deviations* of the growth rates of GNP, γ, and labor productivity, $\gamma_{y/n}$, the ratio of measured consumption to GNP, $(c + \eta x_h)$/GNP, and the coefficient of variation of hours worked, $\sigma(n)/\bar{n}$, are in Table A.6. There are several results of interest:

- The volatility levels for the growth rate of GNP, the share of measured consumption in GNP, hours and labor productivity can be matched simultaneously with reasonable values of σ, δ_h and η. One set of parameters that works fairly well is $\sigma = 1.25$, $\delta_h = .04$ and $\eta = 1.0$. We discuss a case similar to this one in detail below.

- In almost all cases, logarithmic utility generates volatility levels for all of the variables that are higher than in the data, while they are typically too low when $\sigma = 2.0$. Again, this is in contrast to the exogenous growth model, where these volatilities are uniformly too low.

- For the most part, all standard deviations decrease in δ_h. The magnitude of this decrease is quite sensitive to the size of the depreciation rate of human capital.

- The effects of η -- the share of "true" investment in human capital that is measured in GNP -- are small for all variables except for the growth rate of labor productivity, $\gamma_{y/n}$. For this variable, the model predicts too much variability unless all of x_h is measured ($\eta = 1$).

Thus, one lesson learned from these experiments is that the model's predictions for the standard deviations of the endogenous variables are quite sensitive to the choice of parameters. We experimented with our choices of (σ, η, δ_h) seeking to match the values of $\sigma(\gamma)$, $\sigma[(c+\eta x_h)/\text{GNP}]$ and $\sigma(n)/\bar{n}$, found in the U.S. data. The results, along with those of the "best" exogenous growth model from section 3, are in Table 4.1.

[12] Formally the identification assumption that allowed us to estimate the shock, $\delta_h = \delta_k$, does not hold any longer. However, the estimation in the exogenous growth version yielded a very similar process for s_t. Moreover, we experimented using an actual series for h_t from the work of Kendrick (1976), Eisner (1989), and Jorgensen and Fraumeni (1989) and, in each case, we obtained ranges for ρ and σ_ε that include those used in section 2. Given the large amount of uncertainty associated with these estimates, we did not see any compelling reason to change the estimates used in section 3.

Table 4.1: Extended Model - Volatility

Case	σ, η, δ_h	$\sigma(\gamma)$	$\sigma[(c+\eta x_h)/GNP]$	$\sigma[x_k/GNP]$	$\sigma(n)/\bar{n}$	$\sigma(\gamma_{y/n})$
37	1.273, 1.0, 0.0445	0.0219	0.0118	0.0118	0.0341	0.0126
RBC	0.90, 1.0, 0.075	0.0182	0.0116	0.0116	0.0124	0.0100
U.S.		0.0214	0.0119	0.0119	0.0342	0.0124

For our preferred case -- case 37 -- the match between model and data is almost perfect. The estimates are, at most, within 2% of the observed valuesOn the other hand, the exogenous growth version falls moderately short in the standard deviations of the growth rates of GNP and labor productivity (about 80% of the U.S. values), and only reproduces approximately 1/3 of the U.S. coefficient of variation for hours worked.

Why is it that relaxing the assumption of equal depreciation rates generates such a big difference in the volatility of measured consumption? The reason is simple: If δ_h is smaller than δ_k the balanced growth value of h/k increases. This has two effects: on the one hand, a higher value of the stock of human capital requires more investment in it. On the other hand, the lower depreciation rate implies that less investment is required. In the examples we have looked at, the second effect dominates. This is important because our definition of measured consumption is $c + \eta x_h$, and it was the divergent behavior of the two components that resulted in the low estimates of its variability in section 3.[13]

In section 3 we pointed out that the model failed to account for the autocorrelation of consumption growth. The extended model is a substantial improvement. Table A.7 in the appendix contains the results for all 37 cases. There are a few interesting regularities:

- If human capital depreciation is small and the share of measured human capital investment is close to-one, it is relatively easy for the model to produce persistence in output growth resembling values found in the U.S. data (around 0.2). The first order autocorrelation of output growth is decreasing in δ_h when $\eta = 1$, whereas it is increasing in δ_h when $\eta = 0.5$ and 0.25. Hence, the autocorrelation of output growth seems to be more sensitive to η than to δ_h.

[13] We also varied the share of human capital investment included in measured GNP, η. Our preferred specification has η equal to one. Moreover, the volatility of the measured consumption-output ratio is fairly insensitive to η and, hence, we do not discuss the effects of changing this parameter.

- On the other hand, the autocorrelation of measured consumption growth is extremely sensitive to the value of δ_h. In fact, when δ_h equals 0.01 the model produces negative autocorrelation.

As before, we find it encouraging that small variations in the parameters result in relatively large changes in the predicted values. It is of interest to evaluate how well our preferred specification -- chosen to match three measures of volatility in the data -- does in terms of accounting for the autocorrelation of endogenous variables. The results, along with those for the exogenous growth model, are reported in Table 4.2.

Table 4.2: Extended Model - Autocorrelations

Case	σ, η, δ_h	$\rho_1(\gamma_y)$	$\rho_1(\gamma_k)$	$\rho_1(\gamma_{c+\eta x h})$	$\rho_1(n)$	$\rho_1(\gamma_{y/n})$
37	1.273, 1.0, 0.0445	0.148	0.484	0.358	0.956	0.342
RBC	0.9, 1.0, 0.075	-0.019	0.740	0.316	0.744	0.201
U.S.		0.213	0.774	0.400	0.852	0.295

Even though the fit is not perfect, our endogenous growth model displays distinct propagation mechanisms. It accounts for 70% of the first order serial correlation in the annual growth rate of output, and it overestimates the first order serial correlation of productivity growth by approximately 15%. The autocorrelations of consumption growth and hours worked are within 10% of the U.S. values. The one significant deviation is the autocorrelation in the growth rate of capital: the endogenous growth model's prediction is close to half the U.S. value, while the exogenous growth model implies a much closer fit. Using the first order autocorrelation as a metric, the endogenous growth model outperforms its exogenous growth counterpart if the growth rate of capital is ignored. Again, the important difference lies in the two models' abilities to account for the first order serial correlation properties of the growth rate of output. The exogenous growth model predicts a negative value, while the endogenous growth version predicts a significantly positive autocorrelation.

Finally, we computed the cross correlations with GNP growth for our preferred specification. The results are reported in Table A.8, and show no substantial improvement over our findings in section 3. The model still generates an excessively high contemporaneous correlation between GNP growth and labor productivity. Moreover, the contemporaneous correlation between output growth and the physical capital investment-output ratio represents a distinct deterioration over the equal depreciation rate version analyzed in section 3. Hours worked now lags the cycle, as it does in the data, though this correlation pattern is marginal at best.[14]

4.2 The Dynamics of a Response to Shock

To understand how the extended model improves the serial correlation properties of measured consumption, it is necessary to explore how a shock affects investment in both physical and human capital. For the class of models in which $\delta_h < \delta_k$, a positive technology shock results not only in an increase in investment in physical capital, x_k, but also in an increase in x_k *relative* to x_h.

To understand the effects involved assume that, initially, the economy is operating along its balanced growth path with $s_t = 1$. From the Euler equation it follows that,

$$F(t)/h_t(\alpha h_t/k_t - (1-\alpha)) = \delta_k - \delta_h > 0.$$

Next, the no arbitrage condition is just,

$$E_t[u_c(t+1)\,(\delta_h - \delta_k + s_{t+1}F(t+1)/h_{t+1})(\alpha h_{t+1}/k_{t+1} - (1-\alpha))] = 0,$$

where, following a positive shock to s_t, s_{t+1} is likely to be greater than one. If the solution kept $h_{t+1}/k_{t+1} = h_t/k_t$, then it follows that $(\delta_h - \delta_k + s_{t+1}F(t+1)/h_{t+1})(\alpha h_{t+1}/k_{t+1} - (1-\alpha))$ would be positive, violating the no-arbitrage condition. Thus, the optimal policy is such that $h_{t+1}/k_{t+1} < h_t/k_t$, and this, in turn, results in relatively low investment in human capital in the period of a positive shock.[15] One interpretation for this is

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[14]The intuition behind these cross correlation results becomes apparent when we consider the impulse response behavior of the model. This is done in the next subsection.

[15] Of course the explanation is only approximate since the theory only restricts the integral, and not each term, to be zero. However, it captures the right effects. In addition, the reader can check that our arguments go through for any homogeneous of degree one function, and not just the Cobb-Douglas case.

that in good times there is a (relative) increase in physical capital investment and a (relative) decrease in human capital investment, while individuals increase their participation in the labor market. It is important to note that this is not driven by competing uses of time. In our extended model, physical and human capital are produced using the same technology and, hence, are subject to the same stochastic shock. The true reason lies in the durability of the two forms of capital: since human capital depreciates slowly, it is optimal to postpone investing in it until the technology shock is lower. In terms of our aggregate model, one interpretation of this result is not only that individuals invest less in human capital, but also that firms postpone training in good times, even though they go ahead with other investment plans. This is consistent with anecdotal evidence.

In this model, "measured" consumption is $c + \eta x_h$. Thus, the response of measured consumption to a shock is the sum of two individual effects, "pure" c and "pure" x_h, that, to some extent, reinforce each other, instead of moving in opposite directions, as in the case where $\delta_k = \delta_h$. It follows that both "permanent income" and "equality of rate of return" type of arguments work in the same direction and result in long lasting impacts on measured consumption as a result of a technology shock.

In Figure 1, we show the response of the two forms of investment to a positive one standard deviation shock to productivity. The values are computed from our preferred specification (case 37 above) and are such that the shock "hits" the economy in period 3. The data displayed in Figure 1 correspond to percentage deviations from the balanced growth path in the absence of shocks. Figure 1 illustrates the arguments sketched above. During the period of the shock there is a large response, over 6% above trend, of investment in physical capital. Since there are no adjustment frictions, this increase in investment is short lived, and it is only 0.5% above trend in the period after the shock. From then on, it decreases slowly but, even after 20 periods, it remains slightly above trend. The response of x_h is quite different. During the period of the shock it increases only 0.2% over its trend value. However, after one period, it rises more than 3.5% over trend and, subsequently, decreases to close to its unshocked value.[16]

[16] It is possible with small values of δ_h for the model to generate <u>decreases</u> in human capital investment at the time of the shock, and larger increases afterwards.

Figure 1: Impulse Response Functions - Investment

(Percentage Deviation from Balanced Growth Path)

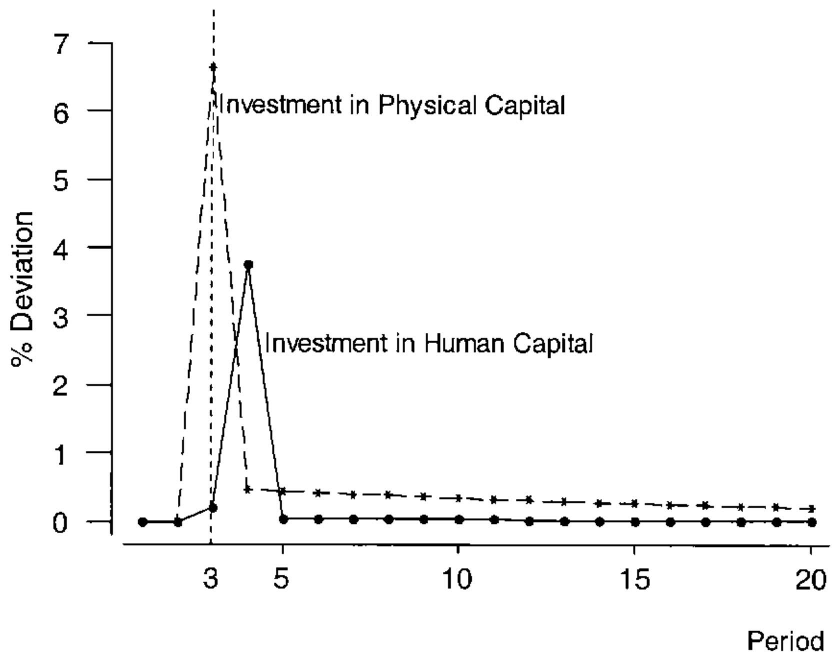
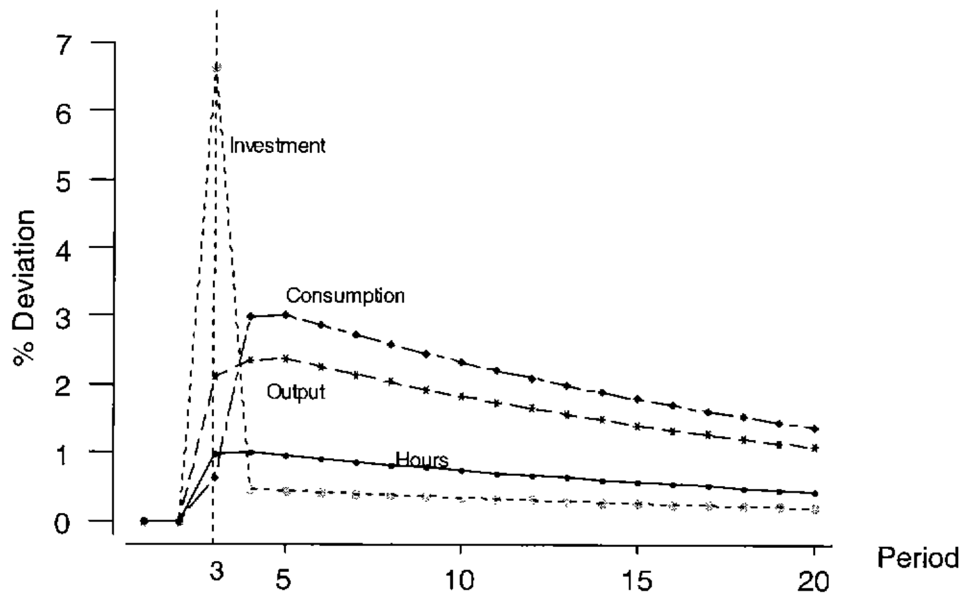


Figure 2 displays the impulse response functions for all the measured variables: output, consumption (defined as $c+x_h$), hours and investment (in physical capital). The two most interesting features are the delayed response of measured consumption to a shock, and the relatively long lasting increase in hours. As indicated above, the behavior of consumption is driven not only by intertemporal substitution effects, as in the standard exogenous growth model, but also by the delayed response of investment in human capital (included in our measure of consumption) to a technology shock. Similarly, the response of hours worked is not highest during the period of the shock: For this specification, hours are slightly higher in the period after the shock. Moreover, the effects of the positive impact are relatively long lasting: after 10 periods, hours are more than 0.7% above normal, and after 20 periods they are 0.4% above trend.

Figure 2: Impulse Response Functions - Several Categories

(Percentage Deviation from Balanced Growth Path)



5. Conclusion

In this paper we have taken a preliminary look at a class of stochastic endogenous growth models. We find our results encouraging. Our artificial economies show an improvement -- over the exogenous growth models -- in accounting for the first order serial correlation of the growth rate of output and the measured variability of hours per worker. This is true despite the fact that the models that we study do not rely on asymmetries in either the technologies of production across sectors, or the incidence of shocks.

One important finding is that, in contrast to exogenous growth models, the economies that we study imply that the intertemporal elasticity of substitution plays a major role in determining the second moment properties of macroeconomic time series. On the basis of our specification, we find that an intertemporal elasticity of substitution of approximately 0.8 does best at replicating the moments in the data. Moreover, from the perspective of the model, there is a substantial difference between 0.8 and 1.0 (logarithmic preferences), which is the most common specification used in the real business cycle literature.

There are several dimensions in which the model is found lacking. In particular, the contemporaneous correlation between the growth rate of output and both labor supply and the growth rate of labor productivity are higher than the U.S. values; this is also true of the real business cycle model. In addition, the first order autocorrelation of the growth rate of capital is lower than in the data, and represents a deterioration relative to the exogenous growth version.

It seems to us that the next step is to carefully explore the effects of generalizing the model. This includes generalizing both the details of the market production technologies, and the human capital production technologies. Some of this work has been done, and we plan to use more evidence to recover the differences in factor intensities across sectors.

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Appendix A

1) **Proof of Proposition 1**: Fix an arbitrary initial state, (h, k, s) and let (z*(h, k, s), n* (h, k, s)) denote the solution to (P.2) from this state. Now consider (P.2) when the initial state is (λk,λh,s). It follows immediately from the linear homogeneity of Γ that (λz*(h, k, s), n* (h, k, s)) is feasible for the problem with initial state (λk,λh,s). Contrary to the conclusion of the proposition, assume that (λz*(h, k, s), n* (h, k, s)) is not optimal. Then, take some alternative plan, (z, n) that is feasible and generates higher utility--

$$\text{(A.1)} \qquad U(z, n) > U(\lambda z^*(h, k, s), n^*(h, k, s)).$$

Since (z, n) is feasible given initial state (λk,λh,s), it follows from the the linear homogeneity of Γ that (z/λ, n) is feasible when the initial state is (λk/λ,λh/λ,s) = (h, k, s). Moreover, the utility of (z/λ, n) is given by $U(z/\lambda, n) = U(z, n)/\lambda^{1-\sigma}$. Using this and (A.1) we have that

$$U(z/\lambda, n) = U(z, n)/\lambda^{1-\sigma} > U(\lambda z^*, n^*)/\lambda^{1-\sigma} = \lambda^{1-\sigma}U(z^*, n^*)/\lambda^{1-\sigma} = U(z^*, n^*).$$

That is, (z/λ, n) is feasible when the initial state is (h, k, s) and it gives higher utility than (z*, n*), a contradiction.

That the value function is homogeneous of degree 1-σ in z (holding n fixed) follows immediately from the fact that the policy rules have the property that they do. ■

2) **Additional Tables**

Table A.1: Basic Model - Parameters

Case	σ	ρ	σ^2_ϵ	ψ	A	δ
1	0.90	0.95	0.0146	8.47	.841	.075
2	1.00	0.95	0.0146	8.32	.849	.075
3	1.25	0.95	0.0146	7.99	.871	.075
4	1.50	0.95	0.0146	7.70	.893	.075
5	1.75	0.95	0.0146	7.43	.915	.075
6	2.00	0.95	0.0146	7.20	.937	.075
7	2.50	0.95	0.0146	6.80	.982	.075
8	3.00	0.95	0.0146	6.47	1.026	.075

Table A.2: Basic Model - Average Values

Case	σ	$E(\gamma)$	$E(c/y)$	$E[(c+x_h)/y]$	$E[x_k/y]$	$E(n)$
1	0.90	1.49	.372	.774	.226	.170
2	1.00	1.42	.377	.776	.224	.170
3	1.25	1.40	.392	.781	.219	.170
4	1.50	1.40	.407	.786	.214	.170
5	1.75	1.40	.421	.791	.209	.170
6	2.00	1.40	.434	.796	.204	.170
7	2.50	1.41	.459	.805	.195	.170
8	3.00	1.41	.482	.814	.186	.170
U.S.		1.38	*	.756	.244	*

Table A.3: Basic Model - Volatility - Endogenous Growth and Real Business Cycle Models Hodrick-Prescott Filtered Data

Case	σ	σ(y)	σ(y) - R	σ(x_k)	σ(x_k)-R	σ(n)	σ(n) -R	σ(y/n)	σ(y/n)-R	σ(c+x_h)	σ(c)-R
1	0.90	3.911	1.958	6.390	5.506	3.539	0.895	1.473	1.236	3.239	1.120
2	1.00	3.029	1.886	4.584	5.042	2.189	0.784	1.243	1.231	2.600	1.118
3	1.25	2.295	1.776	3.092	4.365	1.060	0.619	1.331	1.232	2.077	1.130
4	1.50	2.035	1.713	2.560	3.995	0.658	0.527	1.419	1.236	1.894	1.143
5	1.75	1.901	1.671	2.283	3.762	0.451	0.466	1.474	1.241	1.802	1.154
6	2.00	1.821	1.640	2.112	3.603	0.326	0.423	1.510	1.245	1.747	1.164
7	2.50	1.728	1.599	1.907	3.401	0.182	0.365	1.554	1.252	1.684	1.179
8	3.00	1.676	1.571	1.786	3.280	0.102	0.327	1.579	1.256	1.651	1.191
U.S.		2.592	2.592	6.046	6.046	2.006	2.006	1.235	1.235	2.095	2.095

Note: The column labeled σ(z) gives the standard deviation of z, where z corresponds to output, y, investment in physical capital, x_k, hours worked, n, labor productivity, y/n, and measured consumption, c+x_h in the endogenous growth model and c in the real business cycle model. An R indicates that the column corresponds to the values of a "real business cycle" version with exogenous growth as described in the text.

Table A.4: Cross Correlations with Output - Endogenous Growth (bold), Exogenous Growth, and U.S. Data (*italics*). Logged and H-P Filtered Data. (Endogenous σ = 1.25 , Exogenous σ = 0.9)

Lag (j)	-2	-1	0	1	2
y_{t+j}	**0.373**	**0.653**	**1.000**	**0.653**	**0.373**
	0.316	0.615	1.000	0.615	0.316
	0.292	*0.693*	*1.000*	*0.693*	*0.292*
n_{t+j}	**0.465**	**0.687**	**0.949**	**0.494**	**0.159**
	0.425	0.635	0.886	0.302	-0.065
	0.051	*0.474*	*0.886*	*0.682*	*0.184*
$(y/n)_{t+j}$	**0.274**	**0.579**	**0.968**	**0.733**	**0.517**
	0.192	0.515	0.942	0.756	0.547
	0.538	*0.705*	*0.659*	*0.339*	*0.322*
x_{kt+j}	**0.401**	**0.669**	**0.996**	**0.618**	**0.320**
	0.398	0.644	0.950	0.422	0.069
	0.308	*0.665*	*0.871*	*0.373*	*-0.116*

Table A.5: Extended Model - Parameters

Case	σ	η	δ_h	A	α	δ_k	ψ
1	1.0	1.0	.01	0.607	0.360	0.098	5.60
2	1.0	1.0	.04	0.749	0.360	0.098	7.20
3	1.0	1.0	.07	0.878	0.360	0.098	8.58
4	1.0	.50	.01	0.615	0.322	0.098	5.78
5	1.0	.50	.04	0.779	0.296	0.098	7.70
6	1.0	.50	.07	0.936	0.281	0.098	9.48
7	1.0	.25	.01	0.618	0.302	0.098	5.88
8	1.0	.25	.04	0.790	0.260	0.098	7.99
9	1.0	.25	.07	0.960	0.234	0.098	10.02
10	1.25	1.0	.01	0.640	0.360	0.106	5.51
11	1.25	1.0	.04	0.784	0.360	0.106	7.02
12	1.25	1.0	.07	0.913	0.360	0.106	8.33
13	1.25	.50	.01	0.648	0.324	0.106	5.68
14	1.25	.50	.04	0.813	0.299	0.106	7.49
15	1.25	.50	.07	0.971	0.283	0.106	9.17
16	1.25	.25	.01	0.651	0.305	0.106	5.77
17	1.25	.25	.04	0.823	0.264	0.106	7.75
18	1.25	.25	.07	0.993	0.238	0.106	9.66
19	1.5	1.0	.01	0.674	0.360	0.114	5.43
20	1.5	1.0	.04	0.819	0.360	0.114	6.87
21	1.5	1.0	.07	0.951	0.360	0.114	8.13
22	1.5	.50	.01	0.681	0.326	0.114	5.59
23	1.5	.50	.04	0.846	0.301	0.114	7.30
24	1.5	.50	.07	1.005	0.285	0.114	8.90
25	1.5	.25	.01	0.684	0.307	0.114	5.67
26	1.5	.25	.04	0.856	0.267	0.114	7.54
27	1.5	.25	.07	1.027	0.241	0.114	9.34
28	2.0	1.0	.01	0.741	0.360	0.129	5.29
29	2.0	1.0	.04	0.888	0.360	0.129	6.60
30	2.0	1.0	.07	1.022	0.360	0.129	7.76
31	2.0	.50	.01	0.748	0.329	0.129	5.43
32	2.0	.50	.04	0.914	0.305	0.129	6.98
33	2.0	.50	.07	1.074	0.289	0.129	8.42
34	2.0	.25	.01	0.750	0.312	0.129	5.50
35	2.0	.25	.04	0.923	0.273	0.129	7.18
36	2.0	.25	.07	1.094	0.248	0.129	8.80
37	1.273	1.0	0.0445	0.807	0.360	0.107	7.21

Table A.6: Extended Model -Volatilities

Case	σ, η, δ_h	$\sigma(\gamma)$	$\sigma[(c+\eta\ x_h)/GNP]$	$\sigma[x_k/GNP]$	$\sigma(n)/\bar{n}$	$\sigma(\gamma_{y/n})$
1	1.0, 1.0, .01	.0302	.0344	.0344	.0830	.0108
2	1.0, 1.0, .04	.0290	.0187	.0187	.0694	.0121
3	1.0, 1.0, .07	.0287	.0106	.0106	.0626	.0134
4	1.0, .5, .01	.0421	.0352	.0352	.0877	.0253
5	1.0, .5, .04	.0352	.0221	.0221	.0764	.0169
6	1.0, .5, .07	.0310	.0149	.0149	.0702	.0143
7	1.0, .25, .01	.0512	.0370	.0370	.0904	.0363
8	1.0, .25, .04	.0390	.0258	.0258	.0810	.0213
9	1.0, .25, .07	.0313	.0193	.0193	.0756	.0151
10	1.25, 1.0, .01	.0241	.0249	.0249	.0501	.0115
11	1.25, 1.0, .04	.0224	.0132	.0132	.0370	.0124
12	1.25, 1.0, .07	.0214	.0066	.0066	.0295	.0133
13	1.25, .50, .01	.0332	.0241	.0241	.0516	.0221
14	1.25, .50, .04	.0271	.0139	.0139	.0388	.0168
15	1.25, .50, .07	.0231	.0077	.0077	.0308	.0145
16	1.25, .25, .01	.0395	.0243	.0243	.0524	.0294
17	1.25, .25, .04	.0299	.0148	.0148	.0399	.0198
18	1.25, .25, .07	.0237	.0090	.0090	.0317	.0149
19	1.5, 1.0, .01	.0215	.0207	.0207	.0366	.0122
20	1.5, 1.0, .04	.0198	.0115	.0115	.0248	.0131
21	1.5, 1.0, .07	.0186	.0056	.0056	.0177	.0139
22	1.5, .50, .01	.0296	.0199	.0199	.0376	.0214
23	1.5, .50, .04	.0242	.0111	.0111	.0256	.0173
24	1.5, .50, .07	.0207	.0058	.0058	.0182	.0155
25	1.5, .25, .01	.0347	.0193	.0193	.0379	.0271
26	1.5, .25, .04	.0268	.0114	.0114	.0262	.0200
27	1.5, .25, .07	.0216	.0064	.0064	.0186	.0163
28	2.0, 1.0, .01	.0192	.0167	.0167	.0246	.0132
29	2.0, 1.0, .04	.0177	.0097	.0097	.0142	.0140
30	2.0, 1.0, .07	.0167	.0052	.0052	.0077	.0146
31	2.0, .50, .01	.0258	.0153	.0153	.0251	.0204
32	2.0, .50, .04	.0216	.0089	.0089	.0147	.0179
33	2.0, .50, .07	.0189	.0050	.0050	.0082	.0166
34	2.0, .25, .01	.0300	.0147	.0147	.0253	.0250
35	2.0, .25, .04	.0241	.0088	.0088	.0151	.0203
36	2.0, .25, .07	.0202	.0049	.0049	.0084	.0178
37	1.273, 1.0, 0.044	.0219	.0118	.0118	.0341	.0126
U.S.		.0214	.0119	.0119	.0342	.0124

Table A.7: Extended Model - Autocorrelations

Case	σ, η, δ_h	$\rho_1(\gamma_y)$	$\rho_1(\gamma_k)$	$\rho_1(\gamma_{c+\eta xh})$	$\rho_1(\mathbf{n})$	$\rho_1(\gamma_{y/n})$
1	1.0, 1.0, .01	.264	.192	-.264	.964	.725
2	1.0, 1.0, .04	.180	.512	.221	.956	.748
3	1.0, 1.0, .07	.167	.819	.591	.950	.774
4	1.0, .5, .01	-.270	.180	-.087	.964	-.373
5	1.0, .5, .04	-.133	.481	.528	.955	.047
6	1.0, .5, .07	.051	.800	.600	.950	.680
7	1.0, .25, .01	-.403	.175	.213	.963	-.425
8	1.0, .25, .04	-.267	.464	.728	.955	-.132
9	1.0, .25, .07	-.010	.789	.676	.950	.682
10	1.25, 1.0, .01	.227	.156	-.216	.964	.389
11	1.25, 1.0, .04	.154	.439	.267	.957	.360
12	1.25, 1.0, .07	.141	.759	.569	.951	.363
13	1.25, .50, .01	-.258	.146	.060	.963	-.403
14	1.25, .50, .04	-.150	.407	.565	.956	-.177
15	1.25, .50, .07	-.006	.736	.489	.951	.128
16	1.25, .25, .01	-.382	.142	.444	.963	-.472
17	1.25, .25, .04	-.268	.393	.608	.955	-.325
18	1.25, .25, .07	-.083	.721	.436	.951	.018
19	1.5, 1.0, .01	.213	.137	-.170	.965	.276
20	1.5, 1.0, .04	.147	.381	.275	.958	.239
21	1.5, 1.0, .07	.132	.702	.564	.953	.232
22	1.5, .50, .01	-.247	.123	.157	.964	-.382
23	1.5, .50, .04	-.151	.360	.564	.957	-.197
24	1.5, .50, .07	-.032	.678	.434	.952	-.010
25	1.5, .25, .01	-.365	.099	.511	.964	-.471
26	1.5, .25, .04	-.264	.334	.504	.956	-.340
27	1.5, .25, .07	-.113	.659	.336	.952	-.115
28	2.0, 1.0, .01	.207	.089	-.129	.967	.179
29	2.0, 1.0, .04	.146	.305	.294	.963	.141
30	2.0, 1.0, .07	.129	.601	.547	.958	.130
31	2.0, .50, .01	-.218	.080	.288	.967	-.347
32	2.0, .50, .04	-.144	.284	.531	.961	-.208
33	2.0, .50, .07	-.055	.572	.407	.956	-.079
34	2.0, .25, .01	-.336	.078	.525	.966	-.439
35	2.0, .25, .04	-.251	.268	.408	.960	-.3193
36	2.0, .25, .07	-.137	.554	.273	.955	-.169
37	1.273, 1.0, 0.0445	0.141	0.484	0.358	0.956	0.342
U.S.		.213	.774	.3997	.852	.295

Table A.8: Cross Correlations with Output Growth - Endogenous Growth (bold), and U.S. Data (*italics*).

$\sigma = 1.273$, $\eta = 1.0$, $\delta_h = 0.00445$

Lag (j)	-2	-1	0	1	2
$\gamma_{y,t+j}$	**0.080**	**0.148**	**1.000**	**0.148**	**0.080**
	-0.081	*0.214*	*1.000*	*0.214*	*-0.081*
n_{t+j}	**0.156**	**0.186**	**0.467**	**0.468**	**0.445**
	-0.328	*-0.327*	*0.118*	*0.308*	*0.280*
$\gamma_{y/n,t+j}$	**0.114**	**0.178**	**0.962**	**0.252**	**0.203**
	0.272	*0.384*	*0.482*	*-0.166*	*0.144*
$(x_k/y)_{t+j}$	**0.056**	**0.125**	**0.978**	**0.022**	**0.014**
	-0.315	*-0.139*	*0.509*	*0.454*	*0.159*

Appendix B: Numerical Methods used for Solving the Models

This appendix outlines the numerical method used to solve the endogenous growth models studied in this paper. The method is an extension of the general class of projection methods developed in Judd (1992). For further details, as well as the description of the method's use in solving the standard real business cycle model, see Siu (1998).

Let $\kappa = h/k$, $\hat{c} = c/k$, and $\eta = k'/k$. The method begins by specifying the approximation to the value function to be of the form:

$$\hat{v}(\kappa,z) = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} a_{ij} T_i(\phi(\kappa))\, T_j(\phi(z))$$

where, T_i is the i-th order Chebychev polynomial, and $\varphi: \mathbb{R}^1 \rightarrow [-1,1]$ is a linear transformation mapping the bounded, ergodic capture regions of κ and z into the Chebychev polynomial's domain of definition.

The $N^2 \times 1$ coefficient vector, $\boldsymbol{a} = \{a_{ij}\}$, characterizing the approximation $\hat{v}$ is chosen to solve the following system of N^2 equations:

$$\int\int R(\kappa,z;\hat{v}) w_{ij}\, \partial\phi(z)\partial\phi(\kappa) = 0, \quad i,j = 1,\ldots,N,$$

where

$$R(\kappa,z;\hat{v}) \equiv \hat{v}(\kappa,z) - \max_{\eta,\kappa'\in\Gamma(\kappa,z)}\left(\frac{[\hat{c}(1-n)^{\psi}]^{1-\sigma}}{1-\sigma} + \beta\eta^{1-\sigma}\int\hat{v}(\kappa',z')P(z,dz')\right),$$

and

$$\Gamma(\kappa,z;\hat{v}) = \{\eta,\kappa' : 0 \le \hat{c} \le \mu A\kappa^{1-\alpha}n^{1-\alpha} + (1-\delta)(1+\kappa) - \eta(1+\kappa'),\ 0 \le n \le 1\}.$$

We select the N^2 functions $\{w_{ij}\}$ to be:

$$w_{ij} = \frac{T_{i-1}(\varphi(\kappa))}{\sqrt{1-\varphi(\kappa)^2}}\frac{T_{j-1}(\varphi(z))}{\sqrt{1-\varphi(z)^2}}.$$

To compute the double integral above, we use M^2-point Gauss-Chebychev quadrature integration (with $M \ge N$). Note that in the special case where $\delta_h = \delta_k$, $\kappa = (1-\alpha)/\alpha$ in every period, so that the dimension of the state space is reduced in half and the dimension of the unknown coefficient vector $\boldsymbol{a}$ is reduced by a factor of N.

Due to the orthogonality conditions possessed by Chebychev polynomials, we are able to rewrite the system of equations as the following matrix expression:

$$X'(X \cdot \boldsymbol{a} - Y_a) = 0$$

where X is an appropriately defined matrix of Chebychev polynomials and Y_a is a vector consisting of nonlinear functions of the unknown coefficients $\boldsymbol{a}$.

To solve this matrix problem easily, we implement the following iterative procedure. Given an initial guess for the coefficient vector, a_0, the vector Y_{a0} is computed. A new guess, a_1, is computed as:

$$a_1 = (X'X)^{-1}X'Y_{a0}.$$

This procedure is repeated until we converge upon the limit point, a^*.