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Heterogeneous Information Arrival
and Option Pricing
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ABSTRACT

We model the arrival of heterogeneous information in a financial market as a doubly-stochastic Poisson process (DSPP). A DSPP is a member of the family of Poisson processes in which the mean value of the process itself is governed by a stochastic mechanism. We explore the implications for pricing stock, index and foreign currency options of the assumption that the underlying security evolves as a mixed diffusion DSPP. We derive an intertemporal CAPM and demonstrate that accounting for heterogeneous information arrival may minimize the ubiquitous pricing bias —“smile-effect”—of standard option pricing models. We propose a conceptually simple but numerically intensive maximum likelihood estimator of the parameters of a DSPP. A simulation study verifies the adequacy of the asymptotic approximations in finite samples.

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1 Introduction

This paper is concerned with the nature of information arrival in financial markets, its implications for the pricing of derivative securities and empirical methods for estimating information arrival. Understanding the nature of the information arrival process is important for several reasons. First, several studies have documented that the nature of information flow to the market affects the temporal patterns and moments of trading activity and returns. In particular, intradaily patterns in the return variance, the moments of trading activity measures, and the probability of (non) trading have been related to information arrivals (Berry & Howe 1994, Penman 1987). For instance, Torbensen (1993) demonstrates that the information flow interpretation of stochastic volatility is compatible with formal models of market microstructure in which informational asymmetries and exogenous liquidity needs motivate trade. The model he develops is consistent with the “mixed distribution hypothesis” for daily returns since it is governed by the mixture of distributions characterizing respectively the return innovations associated with the arrival of news and the interdaily flow of information arrivals (Epps and Epps 1976, Tauchen and Pitts 1983).

Second, the nature of the information arrival process is an important consideration in the specification of the stochastic process that underlies movements in asset prices. Different specifications of the stochastic process governing asset prices have important consequences for the pricing of derivative assets. For instance, the celebrated Black–Scholes model with continuous sample paths has been shown to misprice options (Black 1975 and MacBeth and Merville 1979).¹ Furthermore, considerable empirical evidence has shown that asset price dynamics follow a discontinuous sample path that radically departs from the continuous process underpinning the Black and Scholes (1973), and Merton (1973) models (Brown and Dybvig 1986, Jarrow and Rosenfeld 1984).

Given the limitations of geometric Brownian motion, several authors have proposed op-

¹The Black—Scholes model frequently predicts prices that are lower than those actually observed on call options that are deep out of the money and near to expiration.

tion pricing models based on diffusion–jump processes or semimartingales (Merton 1976 and Cox and Ross 1976).² The jumps are postulated to come about due to the arrival of discrete information governed by a Poisson process. Cox and Ross (1976) derive their model under the assumption of a pure jump process by no–arbitrage arguments and the market is assumed complete. However, as was demonstrated by Naik and Lee (1990), in Merton’s (1976) model the market is incomplete due to discrete jumps with random sizes. Contingent claims can no longer be priced by the no–arbitrage condition in the Harrison and Kreps (1979) sense. The jump risk is assumed to be nonsystematic and therefore not priced implying that the market portfolio has no jumps. However, Jarrow and Rosenfeld have shown that the market portfolio contains jumps. Naik and Lee subsequently proposed an option pricing formula for European options on the market portfolio, an index. Ahn (1992) extended the model to price European options on individual stocks which jump when the market jumps. Bates’(1991) lucid analysis, which is also related to that of Naik and Lee (1992), recognizes the importance of systematic jump risk, and presents an interesting model and empirical results.

An important limitation of existing jump–diffusion models is the implicit assumption that the rate of arrival of information is constant (the mean of a Poisson distribution is a constant). A constant mean implies that discrete information arriving in the market is homogeneous. In this paper we explore the implications for pricing options of relaxing this restrictive and unrealistic assumption. We assume that jumps are caused by the arrival of *heterogeneous* information—extraordinary news items (financial, political, company based or industry wide) announcements about shifts in fiscal and monetary policy or even the result of bursting speculative bubbles.

In this paper we propose a more general diffusion–jump process where the jump process is a doubly–stochastic Poisson process (Snyder and Miller 1991).³ We use the diffusion

²The diffusion–jump processes have sample paths that are continuous from the right and have left hand limits.

³Beaglehole (1993) analyzes a similar problem in which he assumes that the stochastic intensity parameter

doubly—stochastic Poisson process to price stock, index and currency European options.⁴ We demonstrate that our model is able to reduce some of the pricing biases, or “smile effects” generated by option pricing models with diffusion–jump processes. Key to obtaining this result is the assumption that the intensity parameter is driven by a random variable that follows a gamma distribution. This assumption yields a negative–binomial jump process. As a result, the equilibrium risk–free interest rate is lower in our model than in the standard diffusion–jump process. To see this, note that the mean of a negative–binomial distribution is larger than that of a Poisson distribution. Which in turn implies that the jump risk–premium of the negative–binomial case is larger than that of the Poisson case. Hence, a lower equilibrium risk–free interest rate is obtained under the negative–binomial case. We also analyze the effects of the jump risk–premium via the effect of the correlation coefficient of the underlying security’s logarithmic jump with the market portfolio’s logarithmic jump, and we compare the results to those of Ahn (1992).

Finally, we turn to an empirical question. How would one go about estimating information arrival? Existing methods of characterizing information flows have several limitations. For instance, to document patterns of information arrival, Berry and Howe (1996) construct an index of public information flow to financial markets based on the number of news releases by Reuter’s News Service per unit of time. This index measures the “quantity” of information (in terms of the intensity of information arrivals). The authors find that this measure of information arrival is nonconstant, displaying seasonalities and distinct intraday patterns. However, this finding is not based on the estimation of information arrivals. Furthermore, because of the aggregate nature of the index, no statements can be made about the “type” of information (in terms of the specific content of information).

The maximum likelihood method that we develop in this paper is able to characterize both the quantity and type of information flow. The use of maximum likelihood methods

is a linear function of stochastic volatility, which in turn is driven by a mean—reverting diffusion process.

⁴The analysis generalizes the work of Naik and Lee (1990) and Ahn (1992). Our work on the capital asset pricing model is closely related to Merton (1971, 1973), Breeden (1979) and Bates (1991), inter alia.

for the estimation of point processes was suggested by Vere–Jones (1975). The consistency and asymptotic normality of the ML estimates was proven by Ogata (1978).

Our treatment differs from these seminal papers in several ways. First, we focus on estimation of a discrete time DSPP. Second, we consider a model with a parameterized mean value with covariates. Third, we propose a method for approximating the log–likelihood function. The procedure developed requires numerical integration. Statistical properties of the likelihood approximation and the estimator are analyzed by asymptotic and simulation methods.

The rest of the paper is organized as follows. In Section 2 we develop the general option pricing model and the CAPM. In Section 3 we discuss the negative–binomial case and the effects of the jump risk on the option price. In Section 4 we propose a maximum likelihood estimator of a DSPP and provide simulation evidence on the performance of the estimator in finite samples. Section 5 concludes. The appendices contain proofs.

2 Pricing Options with Heterogeneous Information Arrival

In this section we assume the jump risk in the underlying process is systematic (or priced), and discrete information is heterogeneous. This implies the jump risk is undiversifiable. Let (Ω, \mathcal{F}, P) be a probability space, where Ω denotes a set of states of nature, \mathcal{F} denotes the σ –algebra of subsets of Ω , and P denotes the probability measure on \mathcal{F} . The filtration, $\{\mathcal{F}_t : t \geq 0\}$, to which the set of random variables is adapted defines the information set available to the agents. The filtration is right–continuous and P –complete as in Back (1991).

In this economy we have continuous trading and agents have identical endowments and preferences. There is a single consumption good. The investor has two types of investment opportunities, a risky financial security and a risk–free security. The returns from the risky security follow a diffusion–jump process. The consumer splits his wealth between consump-

tion and investment in each of the securities, so as to maximize expected utility subject to his budget constraint.

Next, we define a doubly-stochastic Poisson process (Snyder and Miller 1991, page 341).

DEFINITION 1 (*Doubly-Stochastic Poisson Process*): $\{N(t) : t \geq 0\}$ is a doubly-stochastic Poisson process with intensity process $\{\gamma(t, x(t)) : t \geq 0\}$ if for almost every given path of the process $\{x(t) : t \geq 0\}$, $N(\cdot)$ is a Poisson process with intensity function $\{\gamma(t, x(t)) : t \geq 0\}$.

Put differently, a doubly-stochastic process $\{N(t) : t \geq 0\}$ is conditionally a Poisson process with intensity $\{\gamma(t, x(t)) : t \geq 0\}$ given $\{x(t) : t \geq 0\}$. The unconditional distribution for the doubly-stochastic Poisson process is found by unconditioning over the distribution of the information process of the intensity parameter. Then, the unconditional probability that the number of jumps occurring in $[0, t]$ is j , is given by

$$\begin{aligned} \Pr[N(t) = j] &= \mathbb{E}(\Pr[N(t) = j | x(s) : 0 \leq s < t]) \\ &= \int_0^\infty \frac{1}{j!} \left(\int_0^t \gamma(s, x(s)) ds \right)^j \exp \left(- \int_0^t \gamma(s, x(s)) ds \right) dP(x), \end{aligned} \quad (1)$$

where $P(x)$ is the probability distribution function for x and \mathbb{E} denotes expectation. If $\gamma(\cdot)$ is integrable, $\{N(t) : t \geq 0\}$ is referred to as a *self-exciting counting* process with intensity process $\{\lambda(t), t \geq 0\}$ (Snyder and Miller 1991, Theorem 7.2.1, page 348). In this case $\lambda(t) = \mathbb{E}[\gamma(t, x(t)) | N(s) : 0 \leq s < t]$.

Let the intensity parameter or mean of the unconditional distribution (1) be λ . We are assuming that the unconditional distribution can be evaluated. Later we present examples where (1) is easily evaluated. There is an underlying risky security whose price, S , follows a diffusion-jump process of the form

$$\frac{dS}{S} = (\alpha - \lambda\kappa)dt + \sigma dZ(t) + ydN(t), \quad (2)$$

where $\{Z(t) : t \geq 0\}$ is a standard Wiener process, $\{N(t) : t \geq 0\}$ is a doubly-stochastic Poisson process, unconditioned on the intensity function, with mean λ such that $\Pr(dN(t) = 1) = \lambda dt$ and $\Pr(dN(t) = 0) = 1 - \lambda dt$, $dZ(t)$ and $dN(t)$ assumed to independent. The mean

of the rate of occurrence of jumps is λ . It turns out that $\ln(1+y) \sim \Phi(\mu - 1/2\delta^2, \delta^2)$, where $\Phi(\cdot)$ denotes a normal distribution. The expected proportionate jump change in the stock price if the jump occurs is denoted κ . The standard deviation of stock returns conditional on the jump not occurring is σ and α is the instantaneous mean of log returns. The process in (2) is geometric Brownian motion most of the time, but λ times a year the price jumps discretely by a random amount (percentage) y . Following Gihman and Skorokhod (1972) the stock price at time t when that at time s ($t > s$) is known is given by

$$S(t) = S(s) \exp \left(\left(\alpha - \frac{1}{2}\sigma^2 - \lambda\kappa \right) (t-s) + \sigma(Z(t) - Z(s)) + \sum_{i=N_s+1}^{N_t} y_i \right). \quad (3)$$

Next, we consider the pricing of a European call option when the underlying process is a diffusion-jump with doubly-stochastic Poisson jumps. We assume there are no transaction costs and taxes, no penalties for short sales, and the market operates continuously. Merton's (1976) model is not market-complete in Harrison and Kreps (1979) sense as demonstrated by Naik and Lee (1990). To achieve completeness we need to value the option in equilibrium where a representative agent maximizes expected utility subject to some budget constraint. In equilibrium any arbitrage profit has zero value since marginal utility is zero. Stock, currency and index (market) options will be priced.

2.1 Pricing Stock Options

A stock option pricing model with non-diversifiable jump risk was developed by Ahn (1992). In this section we extend the model by allowing heterogeneous discrete information, implying that the distribution of the number of jumps is given by the doubly-stochastic Poisson process given in (1) above. The linchpin to the model is the fact that the jumps in the market are correlated with jumps in the individual stocks.

Consider a simple economy whose information structure is as described above. In this economy there is a risk-averse representative agent who maximizes lifetime expected utility subject to a budget constraint. We assume the representative agent maximizes logarithmic

utility $U(t) = \ln(C(t))$. Then, the agent's objective is to solve the following stochastic optimal control problem

$$J(W(t)) = \max \mathbb{E}_t \left[\int_t^\infty e^{-\beta t} \ln(C(s)) ds \right], \quad (4)$$

subject to

$$dW = [(a(\alpha - r) + r)W(t) - C(t)] dt + aW(t)\sigma dZ(t) + aW(t)y_w dN(t), \quad (5)$$

where J is the indirect utility function, $W(t)$ is wealth at time t , $C(t)$ is the consumption flow at time t , β is the rate of time preference, a is the amount invested in a risky security and y_w is the jump in wealth arising from the jump in the security price.

The Hamilton–Bellman–Jacobi equation is

$$0 = \max \left[e^{-\beta t} U(C(t)) + J_t + J_w(a(\alpha - r) + r)W(t) - C(t) + \frac{1}{2} J_{ww} \sigma^2 a^2 W^2(t) + \lambda \mathbb{E}[J(aW(1 + y_w) + (1 - a)W) - J(W)] \right], \quad (6)$$

where J is the indirect utility function, J_t is the partial derivative of J with respect to t , and J_w is the partial derivative of J with respect to W . The indirect utility function is $J(W(t)) = (1/\beta) \exp(-\beta t) \ln(W(t))$. The first-order conditions for the consumer's optimization problem (6), optimizing with respect to C and “ a ” are

$$e^{-\beta t} U'(C) - J_w = 0, \quad (7)$$

$$J_w W(\alpha - r) + J_{ww} \sigma^2 a W^2 + \lambda \mathbb{E} \Delta J_w = 0, \quad (8)$$

where the term $\Delta J_w = [J_w(W(1 + y_w) - W) - J_w]$ is the jump in the marginal utility of wealth when a jump occurs. Since $J_w = e^{-\beta t} / (\beta W(t))$ and marginal utility, $U'(C) = 1/C$, optimal consumption, from (7) is

$$C^* = \beta W(t). \quad (9)$$

Since, $J_{ww} = -\exp(-\beta t) / \beta W(t)^2$, from (9), the optimal portfolio holding a^* is given by

$$a^* = - \left[\left(\frac{J_w}{J_{ww}} \sigma^2 W^2 \right) (\alpha - r) + \lambda \mathbb{E} \Delta J_w \right]. \quad (10)$$

To obtain an expression for the equilibrium risk-free interest rate we first need to derive the CAPM. We state this as a proposition.

PROPOSITION 1 *The fundamental Capital Asset Pricing Model (CAPM) when stock returns exhibit doubly-stochastic Poisson jumps with systematic risk is given by*

$$\alpha - r = \sigma_{SW} - \mathbb{E}_{dN=1} \left[\left(\frac{\gamma \Delta J_W}{J_W} \right) \left(\frac{\Delta S}{S} \right) \right], \quad (11)$$

where $\sigma_{SW} = \mathbb{E}_{dN=0}[(dS/S)(dW/W)]/dt$ and $\Delta S = S(1+y) - S$ is the jump in the underlying security price.

Proof: The instantaneous return on a risky asset must satisfy the condition

$$\mathbb{E}_t \left[J_W(W(t+dt)) \left(\frac{S(t+dt)}{S(t)} - e^{r dt} \right) \right] = 0. \quad (12)$$

Dividing (11) by $\partial J/\partial W$ and rearranging gives the standard result

$$\mathbb{E}_t \left(\frac{dS}{S} \right) - r dt = -\mathbb{E}_t \left[\left(\frac{dJ_W}{J_W} \right) \left(\frac{dS}{S} \right) \right] + o(dt), \quad (13)$$

where $\lim o(dt)/dt = 0$ as $dt \rightarrow 0$. To obtain an expression for dJ_W we apply Itô's lemma which yields

$$dJ_W = J_t dt + J_W dW_{dN=0} + \frac{1}{2} J_{WW} \sigma_W^2 W^2 dt + \Delta J_W dN(t) + o(dt), \quad (14)$$

where

$$dW_{dN=0} = (\alpha W - C/W)W dt + \sigma_W W dZ_W.$$

Substituting (14) into (13) and ignoring terms of order odt yields the fundamental CAPM in (11). ■

The term σ_{SW} in (11) is the instantaneous covariance per unit time between the stock and market returns when a jump does not occur. Expression (11) is similar to that in Bates' (1991) Proposition A1 except that here λ is the mean of the stochastic intensity parameter as opposed to a fixed parameter. Equation (11) holds in general and therefore will also hold

for the market's expected excess return, $\alpha_w - r(W)$. Therefore, the instantaneous risk-free rate is

$$r(W) = -\mathbb{E} \left[\frac{(dJ_w/J_w)}{dt} \right] = \alpha_w - \sigma_w^2 + \mathbb{E}_{dN=1} \gamma \left[\left[\frac{(J_w(W(1+y_w)) - J_w)}{J_w} \right] y_w \right]. \quad (15)$$

A closed form expression for the CAPM can be easily derived under the assumption of logarithmic utility. We state this as a theorem.

THEOREM 1 *The Capital Asset Pricing Model (CAPM) when stock returns exhibit doubly-stochastic Poisson jumps with systematic jump risk and the representative agent has log utility is*

$$\alpha - r = \sigma_{sw} - \mathbb{E}_{dN=1} \gamma [((1+y_w)^{-1} - 1)y], \quad (16)$$

where

$$r = \alpha_w - \sigma_w^2 + \mathbb{E}_{dN=1} \gamma [(1+y_w)^{-1} - 1]y_w. \quad (17)$$

Proof: The proof follows from (11) and (15) by noticing that with log utility $(J_w(W(1+y_w)) - J_w)/J_w = (1+y_w)^{-1} - 1$. ■

It follows that the risk-free interest rate is given by

$$r = \alpha_w - \sigma_w^2 - \lambda \kappa_m - \lambda \left(\exp(-\mu_m + \delta_m^2) - 1 \right), \quad (18)$$

and the expected rate of return on the stock is given by

$$\alpha = \lambda \kappa + \omega \sigma \sigma_w + \alpha_w - \sigma_w^2 - \lambda \kappa_m - \lambda \left(\exp(-\mu_m + \delta_m^2 - \rho \delta \delta_m) - 1 \right), \quad (19)$$

where $\kappa = \exp(\mu) - 1$ and $\kappa_m = \exp(\mu_m) - 1$.

The equilibrium expected return $O\alpha_o$ on the option written on the security is

$$\alpha_o O = rO + S \left(\frac{\partial O}{\partial S} \right) \sigma_{sw} - \mathbb{E}_{dN=1} \gamma \left[\frac{\Delta J_w}{J_w} \right] [O(S(1+y)) - O]. \quad (20)$$

Also, from Itô's lemma we know that

$$r_o O = \frac{\partial O}{\partial t} + (\alpha - \lambda \kappa) S O \left(\frac{\partial O}{\partial S} \right) + \frac{1}{2} \sigma^2 S^2 \left(\frac{\partial^2 O}{\partial S^2} \right) + \lambda \mathbb{E}(O(S(1+y)) - O). \quad (21)$$

Substituting (16) for α in (21) and then equating (20) to (21) results in the fundamental differential equation for the option price under systematic jump risk. We state this as a proposition.

PROPOSITION 2 *The partial differential equation that the option satisfies under log utility and systematic jump risk from doubly-stochastic Poisson jumps is*

$$\frac{\partial O}{\partial t} + (r - \lambda^* \kappa^*) S \left(\frac{\partial O}{\partial S} \right) + \frac{1}{2} \sigma^2 S^2 \left(\frac{\partial^2 O}{\partial S^2} \right) + \lambda^* \mathbb{E}(O(S(1 + y^*)) - O) - rO = 0, \quad (22)$$

where $\lambda^* = \mathbb{E}\gamma(J_w(W(1+y))/J_w) = \lambda \mathbb{E}(1+y_w)^{-1}$, and $\kappa^* = \mathbb{E}(y^*) = \kappa + \text{cov}[y, \Delta J_w/J_w] / [\mathbb{E}(1 + \Delta J_w/J_w)] = \exp(\mu - \delta_{sw}) = \exp(\mu^*) - 1$, and $\ln(1 + y^*) \sim \Phi(\mu^* - 1/2\delta^2, \delta^2)$.

Proof Equating (20) to (21) and substituting (16) for α yields

$$0 = \frac{\partial O}{\partial t} + \left(r - \lambda \partial \left[\left[\left(\frac{J_w(W(1+y))}{J_w} \right) \right] y \right] \right) S \left(\frac{\partial O}{\partial S} \right) + \frac{1}{2} \sigma^2 S^2 \left(\frac{\partial^2 O}{\partial S^2} \right) + \mathbb{E} \left(\gamma \left[\frac{J_w(W(1+y))}{J_w} \right] [O(S(1+y^*)) - O] \right) - rO. \quad (23)$$

Notice that under log utility $[J_w(W(1+y))/J_w] = (1+y_w)^{-1}$. Equation (23) follows from (22). ■

As is customary, the fundamental differential equation is solved with respect to the boundary conditions of the option $O(S(T), K, r, 0, \sigma^2) = \max\{0, S(T) - K\}$, and $O(0, K, r, 0, \sigma^2) = 0$, where K is the strike. The call option is priced as if the representative investor is risk-neutral and the risk-neutral stochastic differential equation for the stock price is

$$\frac{dS^*}{S^*} = (r - \lambda^* \kappa^*) dt + \sigma dZ^* + y^* dN^*. \quad (24)$$

The general solution to the partial differential equation which yields the option price is

$$O(S(t), \tau) = \mathbb{E}_t \left\{ e^{-\beta\tau} \left(\frac{J_w(T)}{J_w(t)} \right) \max[0, S(T) - K] \right\}, \quad (25)$$

where $\tau = T - t$ is the time to maturity of the option and $J_w(t) = 1/(W(t))$ is the marginal utility of wealth. The dynamics of the wealth process is given by

$$W(t) = W(s) \exp \left(\left(\alpha_w - \frac{1}{2} \sigma_w^2 - \lambda \kappa_m - \beta \right) (t - s) + \sigma_w (Z_w(t) - Z_w(s)) + \sum_{i=q_s+1}^{N_t} y_{mi} \right). \quad (26)$$

Substituting (26) into (25) yields the call option price

$$O(S(t), \tau) = \mathbb{E}_t \left[\exp \left((\alpha_w - \frac{1}{2}\sigma_w^2 - \lambda\kappa_m - \beta)(T-t) + \sigma_w(Z_w(T) - Z_w(t)) + \sum_{i=q_s+1}^{N_t} y_{mi} \right) \right. \\ \left. \max(S(t)) \exp \left((\alpha - \frac{1}{2} - \lambda\kappa)(T-t) + \sigma(Z(T) - Z(t)) + \sum_{i=N_t+1}^{N_t} y_i \right) - K, 0 \right] \quad (27)$$

We use techniques for bivariate distributions employed by Rubinstein (1976) and Ahn (1992) to the option pricing model. We state the model as a theorem.

THEOREM 2 *The European call option pricing model for stock options when underlying stock returns contain systematic doubly-stochastic Poisson jumps is*

$$O(S(t), \tau) = \sum_{j=0}^{\infty} \int_0^{\infty} \frac{1}{j!} \left(\int_0^t \gamma(s, x(s)) ds \right)^j \exp \left(- \int_0^t \gamma(s, x(s)) ds \right) dP(x) \\ \exp(-\lambda\tau(e^{\xi}-1)e^{(\xi j)}) [S\Phi(d_1) - \exp(-r_n\tau)K\Phi(d_2)], \quad (28)$$

where

$$d_1 = \frac{\ln(S/K) + r_n\tau + 1/2(\sigma^2\tau + \delta^2j)}{(\sigma^2\tau + \delta^2j)^{1/2}}, \\ d_2 = d_1 - [\sigma^2\tau + \delta^2j]^{1/2}, \\ \xi = \mu - \mu_m + \delta_m^2 - \rho\delta\delta_m, \\ r_n = r - \lambda(e^{\xi} - \exp(-\mu_m + \delta_m^2)) + j\frac{\mu}{\tau} - \frac{j\delta\delta_m}{\tau}, \\ r = \alpha_w - \sigma_w^2 - \lambda(\exp(-\mu_m + \delta_m^2) + \exp(\mu_m) - 2),$$

$\Phi(\cdot)$ is a standard normal distribution, and $P(x)$ is the probability distribution of the random variable x .

Proof: Appendix A. ■

Merton's (1976) model with non-systematic jumps, can be obtained from (28) by setting the expected jump amplitudes of the underlying stock with the market portfolio and the variance of jumps equal to zero, (that is $\mu_m = \delta_m = 0$).

2.2 Pricing Index and Currency Options

In this subsection we develop a model for valuing index and currency options. Assume that the dividend yield on the index is d . Since the index is taken to be the market portfolio the correlation coefficient is unity, that is $\rho = 1$. Also the dividend yield on the index, d , equals that of the market portfolio, β . Further, the mean and variance of the diffusion and jump components of the index equal those of the underlying security. That is $\alpha = \alpha_w$, $\sigma^2 = \sigma_w^2$, $\mu = \mu_m$, $\delta^2 = \delta_m^2$ and $\xi = 0$. Substituting the restrictions into (28) yields the price of an index option. We state this as a theorem.

THEOREM 3 *The European call option pricing model for index options under systematic doubly-stochastic Poisson jumps is*

$$O(S(t), \tau) = \sum_{j=0}^{\infty} \int_0^{\infty} \frac{1}{j!} \left(\int_0^t \gamma(s, x(s)) ds \right)^j \exp \left(- \int_0^t \gamma(s, x(s)) ds \right) dP(x) \exp(-d\tau) [S\Phi(d_1) - \exp(-r_n\tau)K\Phi(d_2)], \quad (29)$$

where

$$\begin{aligned} d_1 &= \frac{[\ln(S/K) + r_n\tau + 1/2(\sigma^2\tau + \delta^2j)]}{[\sigma^2\tau + \delta^2j]^{1/2}}, \\ d_2 &= d_1 - [\sigma^2\tau + \delta^2j]^{1/2}, \\ \xi &= \mu - \mu_m + \delta_m^2 - \rho\delta\delta_m, \\ r_n &= r - \lambda + \lambda \exp(-\mu + \delta^2) + j \frac{\mu}{\tau} - \frac{j\delta^2}{\tau}, \\ r &= \alpha_w - \sigma_w^2 - \lambda(\exp(-\mu + \delta^2) + \exp(\mu) - 2), \end{aligned}$$

d is the dividend yield and $\Phi(\cdot)$ is a standard normal distribution.

Proof: Follows immediately from the proof of Theorem 2. ■

To find the pricing formula for currency options set the dividend yield equal to the foreign interest rate and replace the underlying security price by the forward price of the currency.

3 Negative–Binomial Jumps

In this section we consider a particular example for the distribution of the stochastic intensity parameter $\gamma(t, x(t))$. Suppose the information process behind the stochastic intensity parameter $\gamma(t, x(t))$ is a random variable $\gamma(t, x) = xv(t)$, where $v(t)$ is a deterministic function of time, t . In this case $\{N(t) : t \geq 0\}$ is a stochastic process with an intensity parameter $v(t)$ that is scaled by the random variable x . If the distribution of x is a Gamma distribution then it is possible to show that the resultant unconditional distribution of jumps is negative–binomial. We state this as a proposition.

PROPOSITION 3 *If there is discrete information arriving in the market and the rate of arrival, $v(t)$ is randomly scaled by a gamma distributed random variable x , with density*

$$f(x) = \begin{cases} \frac{b^\xi}{\Gamma(\xi)} e^{-bx} x^{\xi-1} & 0 \leq x \leq \infty, \quad \xi > 0, \quad b > 0 \\ 0 & \text{elsewhere,} \end{cases} \quad (30)$$

then the unconditional probability distribution of the number of jumps, j , is a negative–binomial distribution

$$f(j) = \frac{(j + \xi - 1)!}{j!(\xi - 1)!} \pi^j (1 - \pi)^\xi \quad j = 0, 1, 2, \dots \quad (31)$$

where $\pi = b / (b + \int_0^t v(s) ds)$.

Proof: Appendix B. ■

The associated process $\{N(t) : t \geq 0\}$ for the negative–binomial distribution is known as an *inhomogeneous Polya Process*, Snyder and Miller (1991). An option pricing formula can be easily derived using the approach developed in section 2 and Theorem 3. We state this as a theorem.

THEOREM 4 *The European call option pricing model for stock options when underlying stock returns contain systematic negative–binomial jumps is*

$$O(S(t), K, r, \tau, \sigma^2) = \sum_{j=0}^{\infty} \frac{(\xi + j - 1)!}{j!(\xi - 1)!} \pi^j (1 - \pi)^\xi \exp(-\lambda\tau(e^\xi - 1)) e^{\xi j} [S\Phi(d_1) - \exp(-r_n\tau)K\Phi(d_2)], \quad (32)$$

where

$$\begin{aligned}
d_1 &= \frac{[\ln(S/K) + r_n\tau + 1/2(\sigma^2\tau + \delta^2j)]}{[\sigma^2\tau + \delta^2j]^{1/2}}, \\
d_2 &= d_1 - [\sigma^2\tau + \delta^2j]^{1/2}, \\
\xi &= \mu - \mu_m + \delta_m^2 - \rho\delta\delta_m, \\
\lambda &= \frac{(\xi + j)\pi}{(1 - \pi)} \\
r_n &= r - \lambda(e^\xi - \exp(-\mu_m + \delta_m^2)) + j\frac{\mu}{\tau} - \frac{j\delta\delta_m}{\tau}, \\
r &= \alpha_w - \sigma_w^2 - \lambda(\exp(-\mu_m + \delta_m^2) + \exp(\mu_m) - 2),
\end{aligned}$$

$\Phi(\cdot)$ is a standard normal distribution.

Proof: Follows immediately from the proof of Theorem 2. ■

Note that λ is now the mean of a negative-binomial distribution. To value options on an index we let $\alpha = \alpha_w$, $\sigma^2 = \sigma_w^2$, $\mu = \mu_m$, $\delta^2 = \delta_m^2$, $\xi = 0$, and the dividend yield is $d = \beta$. We state this as a theorem.

THEOREM 5 *The European call option pricing model for indices under systematic negative-binomial jumps is*

$$O(S(t), \tau) = \sum_{j=0}^{\infty} \frac{(\xi + j - 1)!}{j!(\xi - 1)!} \pi^j (1 - \pi)^\xi [\exp(-d\tau)S\Phi(d_1) - \exp(-r_n\tau)K\Phi(d_2)], \quad (33)$$

where

$$\begin{aligned}
d_1 &= \frac{[\ln(S/K) + r_n\tau + 1/2(\sigma^2\tau + \delta^2j)]}{[\sigma^2\tau + \delta^2j]^{1/2}}, \\
d_2 &= d_1 - [\sigma^2\tau + \delta^2j]^{1/2}, \\
\lambda &= \frac{(\xi + j)\pi}{(1 - \pi)} \\
r_n &= r - \lambda + \lambda\exp(-\mu + \sigma^2) + j\frac{\mu}{\tau} - \frac{j\delta^2}{\tau}, \\
r &= \alpha_w - \sigma_w^2 - \lambda(\exp(-\mu + \delta^2) + \exp(\mu) - 2),
\end{aligned}$$

where d is the dividend yield and $\Phi(\cdot)$ is a standard normal distribution.

Proof: Follows immediately from the proof of Theorem 2. ■

To obtain the formula for pricing currency options set the dividend yield equal to the foreign interest rate $d = r_f$ and replace the underlying security price with the forward price of the currency. Note that if the intensity parameter is fixed then we are back to the standard Poisson jump process and it is straightforward to show that the option pricing models of Naik and Lee and Ahn can be obtained.

3.1 *The Effect of Systematic Jump Risk*

In this subsection we analyze the effect of systematic jump risk on the price of derivative securities. We compare our results to those of Ahn (1992) which we use as a benchmark. To understand the effect of a random intensity parameter implied by negative binomial jumps we first compare the mean of a Poisson distribution with that of the negative-binomial distribution. We state the relationship between the two means in Lemma 1.

LEMMA 1 *The mean, λ , of a negative-binomial random variable is at least as large as that of a Poisson distributed random variable.*

Proof: Let the mean of a Poisson distributed random variable be λ_p and that of the negative-binomial be λ as indicated above. It is well-known that $\lambda = (\xi + j)\pi / (1 - \pi)$ and $\lambda_p = (\xi + j)\pi$. Since $1 - \pi < 1$ for any $\pi > 0$, it follows that $\lambda > \lambda_p$. ■

Next, we consider the effect of the negative-binomial jump risk. To simplify the argument assume the instantaneous dividend yield is zero ($d = 0$) and the expected jump amplitudes of the market portfolio and the underlying security are zero ($\mu = \mu_m = 0$). Then, the value of the call option in (32) is given by

$$O(S(t), K, r, \tau, \sigma^2) = \sum_{j=0}^{\infty} \frac{(\xi + j - 1)!}{j!(\xi - 1)!} \pi^j (1 - \pi)^\xi \exp(-\lambda\tau(e^\xi - 1)e^{\xi j}) [S\Phi(d_1) - \exp(-r_n\tau)K\Phi(d_2)], \quad (34)$$

where

$$\begin{aligned}
d_1 &= \frac{[\ln(S/K) + r_n\tau + 1/2(\sigma^2\tau + \delta^2j)]}{[\sigma^2\tau + \delta^2j]^{1/2}}, \\
d_2 &= d_1 - [\sigma^2\tau + \delta^2j]^{1/2}, \\
\xi &= \delta_m^2 - \rho\delta\delta_m \\
\lambda &= \frac{(\xi + j)\pi}{(1 - \pi)} \\
r_n &= r - \lambda(e^\xi - \exp(-\delta_m^2)) - \frac{j\rho\delta\delta_m}{\tau}, \\
r &= \alpha_w - \sigma_w^2 - \lambda(\exp(\delta_m^2) - 1).
\end{aligned}$$

Notice that the systematic jump risk affects the equilibrium risk-free rate. In particular, as the jump variance (δ_m^2) increases the interest rate falls. Since the mean of the negative-binomial (λ) is larger than that of the Poisson distribution (λ_p), this implies that the risk-free rate with DSPP is lower than that in the Poisson jump case of Ahn (1992) when all other parameters are fixed. Furthermore, it is well known that as the risk-free rate falls the value of a call option also falls. Hence, our model will yield lower option values than that of Ahn. Our model reduces the ubiquitous “smile effect”, where standard models tend to overprice options that are deep in or out of the money when all other parameters remain fixed.

Next, we consider the precise effect of the jump risk. We have alluded to the fact that the risk-free rate falls as the jump variance of the market portfolio increases. The expectation from this effect is a rise in hedging demand for the risk-free bonds as the uncertainty from jumps in the market portfolio increases. To analyze the effect of jump risk we differentiate (34) with respect to ρ , the correlation coefficient of the underlying stock’s logarithmic jump with the market portfolio’s logarithmic jump. The derivative is

$$\frac{\partial O_t}{\partial \rho} = S\delta_m\delta\text{cov}[j, \Phi(d_1(j))] \tag{35}$$

$$= S\delta_m\delta\lambda\tau \left[\frac{(\xi + j - 1)!}{(\xi - 1)!j!} \pi^j (1 - \pi)^\xi (\Phi(d_1(j)) - \Phi(d_1(j + 1))) \right]. \tag{36}$$

As j increases the kurtosis of the normal cumulative distribution also increases for out of

the money options but decreases for in the money options. The increase in the kurtosis is larger for the negative-binomial jumps case than for the Poisson jump case, since the mean of jumps is larger for the negative-binomial case. Then, the covariance, $\text{cov}[j, \Phi(d1)] > 0$ for out of the money options but for in the money options $\text{cov}[j, \Phi(d1)] < 0$. For $S < K$, $\partial O_t / \partial \rho < 0$, and for $S > K$, $\partial O_t / \partial \rho > 0$.

The above discussion implies that when the option is deep out of the money, as the correlation increases, the option becomes less valuable. When the option is deep in the money, as the correlation increases, the option becomes more valuable. The effect is greater in either case under the negative-binomial case than under the Poisson case.

It turns out that the effect of changes in ρ operate in two ways. First, as ρ increases, the risk-adjusted mean of jumps $\lambda^* = \lambda e^{\xi}$ falls. The corollary is a fall in the option price. Second, as ρ increases the expected return on the stock increases, due to the increase in the risk-premium. The terminal stock price has a higher probability of being quite high, increasing the chances of the option finishing deep in the money. This effect is captured by the risk-adjusted mean of jumps in the expression for r_n above. Since the two effects work in opposite directions their relative importance is what matters. For deep out of the money options, an increase in expected stock returns may not significantly increase the probability of being in the money, and so the second factor is dominated by the first one which does not depend on the moneyness of the option. Thus the option falls as ρ increases. For deep in the money options factor two dominates factor one leading to a rise in the option value.

4 Maximum Likelihood Estimation

In this section we use likelihood theory to derive estimators of the parameters of a DSPP. Let $N_t : t = 1, 2, \dots$ be a doubly stochastic Poisson process with a mean value process $\lambda_t : t = 1, 2, \dots$. Suppose that for each T the mean value process has a density with respect to a σ -finite product measure μ^T , $f(\lambda_1, \dots, \lambda_T, \varepsilon)$ which depends on a parameter $\varepsilon \in \mathbb{R}^q$.

Then the marginal probability function of $\{N_t\}$ is given by

$$f(n_1, \dots, n_T; \varepsilon) = \int \prod_{t=1}^T \frac{e^{-\lambda_t} \lambda_t^{n_t}}{n_t!} f(\lambda_1, \dots, \lambda_T; \varepsilon) d\varphi^T(\bar{\lambda}). \quad (37)$$

Suppose that the mean value process $\{\lambda_t\}$ has the following form

$$\lambda_t = \lambda_t(u_t, \theta),$$

where θ is a parameter in \mathbb{R}^d and $\{u_t\}$ is a sequence of iid random variables having density $f(u; \vartheta)$ with respect to Lebesgue measure. The mean value may also depend on some covariate x_t . Then (37) becomes

$$f(n_1, \dots, n_T; \theta, \vartheta) = \prod_{t=1}^T \int \frac{e^{-\lambda_t(u, \theta)} \lambda_t(u, \theta)^{n_t}}{n_t!} f(u; \vartheta) du. \quad (38)$$

This is the probability density function of a Poisson mixture with a mixing density $f(u; \vartheta)$.

The log likelihood function is

$$\mathcal{L}_T(\theta) = \sum_{t=1}^T \log \left[\int \frac{e^{-\lambda_t(u, \theta)} \lambda_t(u, \theta)^{n_t}}{n_t!} f(u; \vartheta) du \right]. \quad (39)$$

As usual the maximum likelihood estimator $\hat{\theta}$ is the root of the likelihood equation

$$\frac{\partial}{\partial \theta} \mathcal{L}_T(\hat{\theta}) = \sum_{t=1}^T l_t(\hat{\theta}) = 0, \quad (40)$$

where

$$l_t(\theta) = \frac{\partial}{\partial \theta} \log f(N_t; \theta, \vartheta) \quad (41)$$

$$= \frac{1}{f(n_t; \theta, \vartheta)} \int [n_t - \lambda_t(u, \theta)] \frac{\partial}{\partial \theta} \log \lambda_t(u, \theta) \frac{e^{-\lambda_t(u, \theta)} \lambda_t(u, \theta)^{n_t}}{n_t!} f(u; \vartheta) du, \quad (42)$$

provided that $\lambda_t(u, \theta)$ has continuous partial derivatives such that

$$\int \left| \frac{\partial}{\partial \theta} \lambda_t(u, \theta) f(u; \vartheta) \right| du < \infty,$$

$$\int \left| \frac{\partial}{\partial \theta} \log \lambda_t(u, \theta) f(u; \vartheta) \right| du < \infty,$$

so that differentiation can be done under integration. Furthermore, if

$$\frac{\partial}{\partial \theta} \log \lambda_t(u, \theta) = h(\theta),$$

then we have the following relationship;

$$\frac{\partial}{\partial \theta} \log f(n_t; \theta, \vartheta) = h(\theta) \left[n_t - (n_t + 1) \frac{f(n_t + 1; \theta, \vartheta)}{f(n_t; \theta, \vartheta)} \right]. \quad (43)$$

If higher order partial derivatives of $\log \lambda_t(u, \theta)$ exist then all the higher order partial derivatives of the log likelihood can be written in terms of the probability functions themselves. Identifiability of θ is guaranteed by the condition on the functional form of λ_t , specifically

$$\lambda_t = (u, \theta_1) = \lambda_t = (u, \theta_2) \quad \text{for all } u \text{ implies } \theta_1 = \theta_2 .$$

The existence and uniqueness of the MLE depends on the mixing density and the form of $\lambda_t(u, \theta)$. We present this point as a theorem below.

THEOREM 6 *If: (i) $\frac{\partial}{\partial \theta} \log \lambda_t(u, \theta)$ exists and as a function of Θ is continuous on a convex set Θ for each $u \in \mathbb{R}$, (ii) $\log \lambda_t(u, \theta)$ is concave and $\lambda_t(u, \theta)$ is convex on Θ , then the roots of the likelihood equation correspond to the maxima of the log likelihood function. Moreover if at least one of $n_t \log \lambda_t(u, \theta) - \lambda_t(u, \theta)$ is strictly concave, then the solution is unique.*

Proof: Let

$$q(\theta, u) = e^{\lambda_t(u, \theta)} \lambda_t(u, \theta)^{-n_t},$$

which is log-convex in θ on Θ under the conditions stated above. So for $0 \leq \alpha \leq 1$ and $\theta_1, \theta_2 \in \Theta$

$$q(\alpha\theta_1 + (1 - \alpha)\theta_2, u) \leq q(\theta_1, u)^\alpha q(\theta_2, u)^{(1-\alpha)}.$$

Then by Hölders inequality

$$\mathbb{E}_u q(\alpha\theta_1 + (1 - \alpha)\theta_2, u) \leq [\mathbb{E}_u q(\theta_1, u)^\alpha] [\mathbb{E}_u q(\theta_2, u)^{(1-\alpha)}],$$

where \mathbb{E}_u is the expectation with respect to the mixing density. Hence $\mathbb{E}_u q(\theta, u)$ is log convex in θ . The result then follows from the differentiability of convex functions. \blacksquare

The following two Lemma are needed to establish the asymptotic properties of the MLE.

Lemma 2 provides a suitable law of large numbers.

LEMMA 2 *If $\{X_t\}$ are independent random variables with $\mathbb{E}X_t = 0$ and $\mathbb{E}X_t^2 < \infty$ and $\sum_t^T X_t^2/T^2 \rightarrow 0$, then $1/T \sum_{t=1}^T X_t \rightarrow 0$ in probability.*

Proof: See Rao (1973). ■

Lemma 3 provides a Lyapunov form of the central limit theorem.

LEMMA 3 *Let $\{X_t\}$ be independent d -dimensional random vectors such that $\mathbb{E}X_t = 0$, $\text{cov}(X_t = I_t)$. If:*

(a) $I_t \rightarrow I$ for a positive definite matrix I

(b) For some $\alpha > 2 \sum_{t=1}^T \mathbb{E}|vX_t|^\alpha/T^{\alpha/2} \rightarrow 0$ for any row vector $v \in \mathbb{R}^d$. Then $T^{-1/2} \sum_{t=1}^T X_t \rightarrow N(0, I)$ in distribution.

Proof: See Hoadley (1971). ■

The next theorem establishes conditions under which a consistent root of the likelihood equations exists

THEOREM 7 *If:*

(i) $\lambda_t(u, \theta) > 0$ for all θ and u and $\lambda_t(u, \theta)$ is continuous in θ .

(ii) up to third order partial derivatives of $\lambda_t(u, \theta)$ exist and are continuous for all $\theta \in \Theta$.

(iii)

$$\begin{aligned} \int \left| \frac{\partial}{\partial \theta_i} \lambda_t(u, \theta) \right|^2 f(u; \vartheta) du < \infty, & \quad \int \left| \frac{\partial}{\partial \theta_i} \log \lambda_t(u, \theta) \right|^2 f(u; \vartheta) du < \infty, \\ \int \left| \frac{\partial^2}{\partial \theta_i \partial \theta_j} \lambda_t(u, \theta) \right|^2 f(u; \vartheta) du < \infty, & \quad \int \left| \frac{\partial^2}{\partial \theta_i \partial \theta_j} \log \lambda_t(u, \theta) \right|^2 f(u; \vartheta) du < \infty, \\ \int \left| \frac{\partial^3}{\partial \theta_i \partial \theta_j \partial \theta_k} \lambda_t(u, \theta) \right|^2 f(u; \vartheta) du < \infty, & \quad \int \left| \frac{\partial^3}{\partial \theta_i \partial \theta_j \partial \theta_k} \log \lambda_t(u, \theta) \right|^2 f(u; \vartheta) du < \infty. \end{aligned}$$

(iv) The information matrix $I_T(\theta) = \mathbb{E}[\dot{l}_t(\theta)/\dot{l}_t'(\theta)]$ is well defined and there exists a positive definite matrix $\mathbf{I}(\theta)$ such that $1/T \sum_{t=1}^T I_T(\theta) \rightarrow \mathbf{I}(\theta)$ in probability, where

$$\dot{l}(n_t; \theta) = \begin{bmatrix} \frac{\partial}{\partial \theta_1} \log f(n_t; \theta) \\ \dots \\ \frac{\partial}{\partial \theta_d} \log f(n_t; \theta) \end{bmatrix}.$$

(v)

$$\frac{\sum_{t=1}^T \mathbb{E} \left(\frac{\partial}{\partial \theta} \log f(n_t; \theta) \right)^2}{T^2} \rightarrow 0.$$

(vi) For some $\alpha > 2$ and for all row vectors $v \in \mathbb{R}^d$

$$\frac{\sum_{t=1}^T \mathbb{E} |v \dot{l}(n_t; \theta)|^\alpha}{T^{\alpha/2}} \rightarrow 0.$$

Then there exists a consistent root, $\hat{\theta}$ of the likelihood equations, such that

$$T^{1/2}(\hat{\theta} - \theta_o) \rightarrow N(0, \mathbf{I}^{-1}(\theta_o)),$$

in distribution.

Proof: Conditions (i) (ii) (iii) insure that a Taylor series expansion of the likelihood function exists and that the operations of integration and differentiation can be interchanged. A Taylor series expansion yields

$$\mathcal{H}_i(\theta) = \mathcal{H}_i(\theta_o) + \sum_{j=1}^d \delta_j \mathcal{H}_{ij}(\theta_o) + \frac{1}{2} \sum_{j,k=1}^d \delta_j \delta_k \mathcal{H}_{ijk},$$

where

$$\begin{aligned} \delta_j &= \theta_j - \theta_{0,j}, \\ \mathcal{H}_i(\theta) &= \frac{1}{T} \sum_{t=1}^T \frac{\partial}{\partial \theta_i} \log f(n_t; \theta) \\ \mathcal{H}_{ij}(\theta) &= -\frac{1}{T} \sum_{t=1}^T \frac{\partial^2}{\partial \theta_i \partial \theta_j} \log f(n_t; \theta) \\ \mathcal{H}_{ijk}(\theta) &= -\frac{1}{T} \sum_{t=1}^T \frac{\partial^3}{\partial \theta_i \partial \theta_j \partial \theta_k} \log f(n_t; \theta) \end{aligned}$$

and θ' is a point on the line segment connecting θ and θ_o .

The result follows by applying Lemma 1 and 2 to

$$\sum \frac{\partial}{\partial \theta} \log f(n_t; \theta).$$

With condition (v),

$$\frac{\sum \frac{\partial}{\partial \theta} \log f(n_t; \theta)}{T} \longrightarrow 0$$

in probability. Condition (iv) and the Lyapunov condition (vi) give $\sum l(n_t; \theta_o)/T^{1/2} \rightarrow N(0, \mathbf{I}(\theta_o))$ in distribution. \blacksquare

4.1 *Approximate Likelihood*

In practice, the integral in the likelihood cannot be evaluated numerically so some kind of numerical integration procedure such as Gaussian quadrature has to be employed. We therefore turn to consider the approximate likelihood.

Suppose that an approximation for the probability element is available in the following form:

$$f(u)du = \sum_{j=1}^M \pi_j \Delta_{u-u_j},$$

where

$$\Delta_u = \begin{cases} 1 & \text{if } u = 0 \\ 0 & \text{if } u \neq 0. \end{cases}$$

The accuracy of the approximation depends on the nodes u_j , the weights π_j , the number of nodes M and the form of the function f . With this approximation, we have as an approximation for the marginal probability function,

$$\tilde{f}(n_t; \theta, \vartheta) = \sum_{j=1}^M f(n_t|u_j; \theta) \pi_j, \quad (44)$$

leading to

$$\begin{aligned} \tilde{l}_{t,M}(\theta) &= \frac{\partial}{\partial \theta} \log \tilde{f}(n_t; \theta, \vartheta) \\ &= \frac{\sum_{j=1}^M [n_t - \lambda_t(u_j; \theta)] \frac{\partial}{\partial \theta} \log \lambda_t(u_j; \theta) f(n_t|u_j; \theta) \pi_j}{\tilde{f}(n_t; \theta, \vartheta)}. \end{aligned} \quad (45)$$

In the DSPP case

$$f(n_t|u_j; \theta) = \frac{e^{-\lambda_t(u_j; \theta)} \lambda_t(u_j; \theta)^{n_t}}{n_t},$$

Note that (44) can be seen as a probability function of a finite mixture with mass points $\{\pi_j\}$ when $\sum_{j=1}^M \pi_j = 1$ and $\pi_j \geq 0$. The approximate likelihood equations for θ have the following simple form

$$\sum_{t=1}^T \sum_{j=1}^M [n_t - \lambda_t(u_j; \hat{\theta})] \hat{q}_{tj} = 0, \quad (46)$$

where

$$\hat{q}_{tj} = \frac{\frac{\partial}{\partial \theta} \log \lambda_t(u_j; \hat{\theta}) f(n_t|u_j; \hat{\theta}) \pi_j}{\tilde{f}(n_t; \theta, \vartheta)}.$$

Equation (46) can now be seen to be equivalent to weighted least squares normal equations.

An important problem that arises with numerical methods is truncation error due to using the discrete sums instead of the desired integrals. Let \mathcal{R} be the truncation error in the numerical integration such that

$$f(n_t; \theta, \vartheta) = \tilde{f}(n_t; \theta, \vartheta) + \mathcal{R}.$$

The error term depends on the nodes, the weights, the number of nodes and the integrand. In the Gaussian quadrature formulas it involves the $2M$ 'th derivative of the Poisson term with respect to the mixing variable.

The asymptotic behavior of the approximate MLE has to be considered in terms of M as well as T .⁵ To consider the convergence of the approximate likelihood we make the following three assumptions: (A) Θ is compact, (B) $\lambda(u, \theta)$ for all u is uniformly continuous on Θ , and (C) for all n_t with probability one

$$\sup_{\Theta} \left| \frac{\partial}{\partial \theta} \log f(n_t; \theta) - \frac{\partial}{\partial \theta} \log \tilde{f}(n_t; \theta) \right| \longrightarrow 0 \text{ as } M \longrightarrow \infty,$$

for some $\theta \in \Theta$. And

$$\sup_{\Theta} \left| \frac{\partial^2}{\partial \theta_i \partial \theta_j} \log f(n_t; \theta) - \frac{\partial^2}{\partial \theta_i \partial \theta_j} \log \tilde{f}(n_t; \theta) \right| \longrightarrow 0 \text{ as } M \longrightarrow \infty.$$

⁵We are grateful to a referee for pointing this out.

Then with probability tending to one

$$\tilde{l}_{t,M}(\theta) \longrightarrow l_t(\theta) \text{ as } M \longrightarrow \infty, \text{ uniformly in } \theta,$$

$$\frac{\partial}{\partial \theta} \tilde{l}_{t,M}(\theta) \longrightarrow \frac{\partial}{\partial \theta} l_t(\theta) \text{ as } M \longrightarrow \infty, \text{ uniformly in } \theta,$$

since under the conditions given above if a sequence of derivatives of functions converges uniformly, then so does a sequence of functions themselves, Rudin (1976).

Furthermore, note that if

$$\frac{\partial}{\partial \theta} \log \lambda_t(u, \theta) = h(\theta),$$

that is there is no dependence on u then since all the partials can be expressed in terms of the probability functions as suggested in (43) we only need convergence of $\tilde{f}(n_t; \theta)$ in θ for each n_t .

Let

$$\sup_y \sup_{\theta} |f(y; \theta) - \tilde{f}(y; \theta)| = r_{M_T}.$$

Here r_{M_T} depends on the particular numerical integration used. Then we can state the following theorem.

THEOREM 8 *Under the conditions of Theorem 7, conditions (A), (B), (C) and if*

$$\left| \frac{\partial}{\partial \theta} \lambda_t(u, \theta) \right| < C_1,$$

and

$$\left| \frac{\partial}{\partial \theta} \lambda_t(u, \theta) \right| < C_2,$$

for all u and θ where C_1 and C_2 are constants and if $r_{M_T} \rightarrow 0$ as $T \rightarrow \infty$.

Then the approximate MLE $\tilde{\theta}$ is consistent as $T \rightarrow \infty$ and

$$\sqrt{T}(\tilde{\theta} - \theta_o) \rightarrow N(0, \mathbf{I}^{-1}(\theta_o)),$$

in distribution as $T \rightarrow \infty$.

Proof: Let

$$\begin{aligned}\mathcal{R} &= f(n_t; \theta) - \tilde{f}(n_t; \theta), \\ \mathcal{R}' &= \frac{\partial}{\partial \theta} f(n_t; \theta) - \frac{\partial}{\partial \theta} \tilde{f}(n_t; \theta),\end{aligned}$$

and

$$\tilde{\mathcal{H}}_T(\theta) = \frac{1}{T} \sum_{t=1}^T \frac{\partial}{\partial \theta} \log \tilde{f}(n_t; \theta).$$

First, note that

$$\begin{aligned}\left| \frac{\partial}{\partial \theta} f(n_t; \theta) - \frac{\partial}{\partial \theta} \tilde{f}(n_t; \theta) \right| &= \left| \int [n_t - \lambda_t(u_j; \theta)] \frac{\partial}{\partial \theta} \log \lambda_t(u_j; \theta) f(n_t|u_j; \theta) f(u) du - \right. \\ &\quad \left. \sum_{j=1}^M [n_t - \lambda_t(u_j; \theta)] \frac{\partial}{\partial \theta} \log \lambda_t(u_j; \theta) f(n_t|u_j; \theta) \pi_j \right| \\ &\leq n_t C_2 |\mathcal{R}| + C_1 |\mathcal{R}'|,\end{aligned}\tag{47}$$

so that $|\mathcal{R}'| \leq 2C|\mathcal{R}|$ where $C = \max[C_1, n_t C_2]$ and

$$\begin{aligned}\frac{1}{T} \sum_{t=1}^T [\tilde{l}_{t,M}(\theta_o) - l_t(u_t, \theta_o)] &= \frac{1}{T} \sum_{t=1}^T \left[\frac{\frac{\partial}{\partial \theta} \tilde{f}(n_t|u_j; \theta_o)}{\tilde{f}(n_t|u_j; \theta_o)} - \frac{\frac{\partial}{\partial \theta} f(n_t|u_j; \theta_o)}{f(n_t|u_j; \theta_o)} \right] \\ &\leq \frac{K}{T} \sum_{t=1}^T [l_t(\theta_o) \sup |\mathcal{R}| + \sup |\mathcal{R}'|] \leq K' r_{M_T},\end{aligned}\tag{48}$$

where K and K' are constants as $T \rightarrow \infty$. If $r_{M_T} \rightarrow 0$ as $T \rightarrow \infty$, then by the law of large numbers following condition (v) of Theorem 7 as $T \rightarrow \infty$,

$$\tilde{\mathcal{H}}_T(\theta_o) \rightarrow 0,\tag{49}$$

in probability.

Taking a neighborhood, U_δ , of Θ containing θ_o of radius $\delta > 0$, from (49) we see that $0 \in \tilde{\mathcal{H}}_T(U_\delta)$ with probability tending to one as $T \rightarrow \infty$. With the assumption of uniform continuity (B), uniform convergence (C), condition (iv) of Theorem 7 and by the inverse function theorem; $\tilde{\mathcal{H}}_T(\theta_o)$ is one-to-one on U_δ , and the inverse function $\mathcal{H}_T^{-1} : \mathcal{H}_T(U_\delta) \rightarrow U_\delta$ exists for sufficiently large T .

Since δ is arbitrary, letting δ tend to 0, $\tilde{\theta} = \mathcal{H}_T^{-1}(0)$ converges to θ_o in probability as $T \rightarrow \infty$.

For asymptotic normality, observe that

$$\left| \tilde{\mathcal{H}}_T(\theta_o) - \mathcal{H}_T(\theta_o) \right| \rightarrow 0,$$

in probability.

The result follows from Slutsky's Theorem, the Lyapunov condition of Theorem 7 and the asymptotic behavior of the score functions, $\mathcal{H}_T(\theta_o)$, Rao (1973). ■

Suppose $\{u_t\}$ are iid normal with mean zero and variance σ^2 so that

$$f(n_t|\theta) = \int_0^\infty \frac{e^{-\lambda_t(u,\theta)} \lambda_t(u,\theta)^{n_t}}{n_t!} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}u^2} du.$$

Then a suitable numerical integration procedure is given by the Gauss–Hermite formula, Davis and Rabinowitz (1984)

$$\int_{-\infty}^{\infty} e^{-u^2} g_t(u) du = \sum_{j=1}^M \pi_j g_t(u_j) + \frac{M! \sqrt{\pi}}{2^M (2M)!} g_t^{(2M)}(\vartheta) \quad -\infty < \vartheta < \infty,$$

with

$$g_t(u) = \frac{1}{\sqrt{\pi n_t!}} e^{-\lambda_t(\sqrt{2\sigma u}, \theta)} \hat{\lambda}_t^{n_t}(\sqrt{2\sigma u}, \theta),$$

and with $g_t^{(2M)}(u)$ denoting the $2M$ 'th derivative of $g_t(u)$.

In this case since $g_t(u)$ is analytic (provided that $\lambda(u, \theta)$ is analytic in u) all the derivatives of any order are bounded above so that there exists some constant K such that $\sup_{\theta} |g_t^{(2M)}(\vartheta)| < K$. Using Stirling's formula for factorials we have

$$|\mathcal{R}| \leq \frac{\pi}{2^{2M-1}} K,$$

if $M \rightarrow \infty$ as $T \rightarrow \infty$ the $r_{M_T} \rightarrow 0$ as $T \rightarrow \infty$. As can be seen from the form of the error term, the approximation error is small for a large number of nodes. Also we see that the derivative in the error term involves powers of σ , so it is small for σ but otherwise the error term persists.

5 *Simulation Results*

In this section we present some Monte Carlo simulation results for a simple parameterized mean function

$$\lambda(\phi_1, \phi_2, H) = \exp \{ \phi_1 M[t-1, t] + \phi_2 M[t-2, t-1] - H \}, \quad (50)$$

where H is the history of the process and M is generated as a doubly stochastic Poisson process. For generating the counting process we use the fact that conditional on M , λ_t is piecewise constant and the counting process behaves as a Poisson process on the intervals of constancy of $\lambda(\cdot)$.⁶

Each simulation cycle consists of picking values of ϕ_1, ϕ_2, H and generating 1000 samples, each of a given length T and then estimating ϕ_1, ϕ_2, H by maximum likelihood. The method used for finding the solution to the approximate MLE is described above.

The results in Tables 1, 2, 3 indicate an apparent bias in the maximum likelihood estimates. This bias seems to be of order $O(1/T)$, approximately equal to $(7, 4, 16)/T$ even for the relatively small interval length used for Table 3. However the bias is small when compared with the random variations: about one half of a standard deviation.

The variance covariance estimates also appear as being somewhat biased. They consistently tend to underestimate the variability of the MLEs, but this underestimation error never goes beyond 10 % of the true value. Their variability, however, is somewhat large especially for the small interval lengths used for the simulations in Table 3. It is likely that this variability can be reduced by using more exact approximations.

Panel D in each table, shows the cumulative distribution function (cdf) of two variates

⁶For processes not having piecewise constant intensity functions, the more general method of simulation by thinning can be used, Lewis & Shedler (1979).

that can be used for setting up confidence limits. These two variates are approximately χ squared distributed with three degrees of freedom.

For various cut-off values, Panel D shows the theoretical percentile (based on the chi-square distribution) and the empirical percentile obtained from 1000 simulations. In Table 1, the empirical cdf's follow their theoretical values well within the expected random variations. This is not the case for Table 2 and 3 where probably due to the bias of the MLE the two variates seem to have stochastically higher values than those predicted by the asymptotic approximations. The disagreement although statistically significant is not as big as to invalidate most practical inference used of the confidence regions that could be built using the asymptotic distributions. We should also notice the fact that variate 1 seems to show a distribution closer to the chi-squared approximation that displayed by variate 2. Table 1, 2, 3 show the effects of changes in the length of the observation interval. Table 4, in turn, corresponds to $T=200$, as does Table 1, but the parameter values are now doubled.

The general qualitative conclusions we draw from Tables 1, 2, 3 are still valid for Table 4. We notice higher values of bias and standard deviation than those observed in Table 1 but these do no more than keep their magnitudes in proportion with the true parameter values. Variate 1 still shows a reasonable agreement with its true parameter values. Variate 1 still shows a reasonable agreement with its true asymptotic distribution, specially in the higher percentage points (those most used for setting confidence bounds). The behavior of variate 2 is more erratic.

Taken together these results are encouraging because they show that the asymptotic distributions yield reasonable approximations even for relatively small observation intervals (notice that Table 3 refers to simulations having on average on 16 information arrivals) . Of course these results are incomplete as they refer only to given model and to a very restricted range of parameter values for that model.

6 Conclusion

This paper makes two contributions. First, we develop an equilibrium formulation of option pricing based on mixed diffusion–doubly stochastic Poisson processes. Our formulation enables us to explore the consequences of heterogeneous information arrival on the value of several derivative securities. This model has potential for application in a broad variety of economic settings where jump risk exists and where the rate of occurrence of jumps may be random.

Second, we present a maximum likelihood estimator of the parameters of a DSPP and derive its asymptotic properties. The MLE enables us to empirically evaluate the assumption that information arrival is well characterized by a DSPP and in a more general study can be used to evaluate the importance of different covariates on the information flow process. A simulation study verifies the adequacy of the asymptotic approximations in finite samples. A companion paper, Asea & Ncube (1996a) models the information arrival process as a Markov–modulated Poisson process. The Markov–modulated Poisson process (MMPP) is a doubly stochastic Poisson process in which the arrival rate varies according to a finite state irreducible Markov process.⁷ Further work along these lines will improve our understanding of the relationship between information arrival and asset pricing.

Appendix A

PROOF OF THEOREM 3: Following Rubinstein (1976) and Ahn (1992), we know that if x and y are two random variables which are bivariate normal, then

$$\mathbb{E}_t(e^y|x \geq a) = \exp \left[\mu_y + \frac{1}{2} \sigma_y^2 \right] \Phi \left[\frac{-a + \mu_x + \text{cov}(x, y)}{\sigma_x} \right], \quad (\text{A. 1})$$

⁷See also Asea and Ncube (1996b, 1996c) for additional implications of heterogeneous information arrival for pricing derivative securities.

and

$$\mathbb{E}_t(e^{x+y}|x \geq a) = \exp\left[\mu_x + \mu_y + \frac{1}{2}\text{var}(x+y)\right] \Phi\left[\frac{-a + \mu_x + \sigma_x^2 + \text{cov}(x,y)}{\sigma_x}\right], \quad (\text{A. 2})$$

where a is a constant, $\Phi(\cdot)$ is the standard normal cumulative distribution, μ_x and μ_y are means of x and y , respectively. Let $Z_x = \sigma(Z(T) - Z(t))$ and $Z_y = \sigma_w(Z(T) - Z(t))$. Then, $\mathbb{E}(Z_y(j)) = j(\mu_m - 1/2\sigma_m^2)$, $\text{var}(Z_y(j)) = \sigma_w^2\tau + j\delta_m^2$, $\mathbb{E}(Z_x(j)) = j(\mu - 1/2\sigma^2)$, $\text{var}(Z_x(j)) = \sigma^2\tau + j\delta^2$, and $\text{cov}(Z_x(j), Z_y(j)) = \sigma\sigma_w\tau + j\delta\delta_m$.

Then using (A1) and (A2), the value of the option is an expectation conditional on $(N(T) - N(t)) = j$. From (27) the value of the option is

$$O(S(t), \tau) = \mathbb{E}_t \exp[(\alpha - \bar{\lambda}\kappa_m - \alpha_w + \sigma_w^2 + \iota\kappa_m - \omega\sigma\sigma_w)\tau + j(\mu - \mu_m + \delta_m^2 - \rho\delta\delta_m)] K \exp[-(\alpha - \sigma_w^2 - \iota\kappa_m)\tau - j(\mu_m - \delta_m^2)] \Phi(d_2(j)), \quad (\text{A. 3})$$

where

$$d_1(j) = \frac{\ln(S(t)/K) + (\alpha - \iota\kappa - \omega\sigma\sigma_w)\tau}{\sqrt{\sigma^2\tau + j\delta^2}},$$

$$d_2(j) = d_1 - \sqrt{\sigma^2\tau + j\delta^2}.$$

Using (18) and (19) we can rewrite (A3) as

$$O(S(t), \tau) = \exp[(-\iota\tau(\exp(\mu - \mu_m + \sigma_m^2 - \rho\delta\delta_m) - 1) + j(\mu - \mu_m + \delta_m^2 - \rho\delta\delta_m))] S(t)\Phi(d_1(j)) - \exp[(-\iota\tau(\exp(\mu - \mu_m + \sigma_m^2 - \rho\delta\delta_m) - 1) + j(\mu - \mu_m + \delta_m^2 - \rho\delta\delta_m))] K \exp[-(r_n\tau)\Phi(d_2(j)), \quad (\text{A. 4})$$

where

$$d_1(j) = \frac{\ln(S(t)/K) + r_n + \frac{1}{2}(\sigma^2\tau + j\delta^2)}{\sqrt{\sigma^2\tau + j\delta^2}},$$

$$d_2(j) = d_1 - \sqrt{\sigma^2\tau + j\delta^2},$$

$$r_n = r - \lambda(e^\xi - \exp(-\mu_m + \sigma^2) + j\delta^2) + \frac{j\mu}{\tau} - \frac{j\rho\delta\delta_m}{\tau},$$

$$\xi = \mu - \mu_m + \delta^2 - \rho\delta\delta_m.$$

Then, unconditioning on the negative-binomial distribution of jumps we obtain expression (28) where d_1 , d_2 , r , r_n , and ξ are defined above. ■

Appendix B

PROOF OF PROPOSITION 3:

The conditional Poisson distribution of the number jumps, j , is

$$f(j|\gamma(t, x(t))) = \frac{e^{-\gamma(t, x(t))} \gamma^j}{j!} \quad j = 1, 2, \dots, \gamma(t, x(t)) > 0. \quad (\text{B. 1})$$

Given that $\gamma(t, x(t)) = xv(t)$, where x follows a gamma distribution given in (1), from Bayes Theorem, the unconditional distribution of j jumps in the underlying asset price, is

$$\begin{aligned} f(j) &= \int_0^\infty f(j|\gamma(t, x(t))) f(\gamma(t, x(t))) d\gamma(t, x(t)) \\ &= \frac{1}{j!} \int_0^\infty x^j \exp\left(-x \int_0^t v(s) ds\right) \frac{b^\xi}{\Gamma(\xi)} e^{-bx} x^{\xi-1} dx, \end{aligned}$$

Then by noticing that

$$\Gamma(\xi) = \int_0^\infty e^{-bx} x^{\xi-1} dx = (\xi - 1)!,$$

letting

$$\pi = \frac{b}{(b + \int_0^t v(s) ds)},$$

and manipulating terms we obtain the negative-binomial distribution given by expression (31). ■

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TABLE 1
SIMULATION RESULTS ^a

A: Summary Statistics			
	Mean	S.D. of mean	S.D. of estimates
$\hat{\phi}_1$	2.03	.005	.152
$\hat{\phi}_2$	1.01	.007	.224
\hat{H}	4.07	.012	.371
B. Empirical Var-Cov Matrix of Estimates			
	.023	.011	.050
	.011	.050	.044
	.050	.044	.138
C. Average (S.D.) of Var-Cov Estimates			
	.022 (.005)	.009 (.005)	.049 (.011)
	.009 (.005)	.049 (.010)	.040 (.012)
	.049 (.011)	.040 (.012)	.130 (.030)
D. Chi-squared variates			
Theoretical Percentile	Empirical Percentile		
	Variate 1	Variate 2	
0.5%	0.6%	0.6%	
1.0%	1.1%	1.1%	
2.5%	2.3%	2.3%	
5.0%	4.3%	4.5%	
50.0%	47.5%	48.8%	
95.0%	95.0%	94.7%	
97.5%	98.1%	97.5%	
99.0%	99.2%	98.6%	
99.5%	99.6%	99.7%	

^aAs detailed in Section 5.1 the following was done: 1000 simulations each of length 200 was generated where the input process is Poisson with unit intensity and the output process is generated according to equation (50), with parameters $\phi_1 = 2$, $\phi_2 = 1$ and $H = 4$. The average input intensity of the generated process is 1.000 (200 points/simulation). The average output intensity of the generated process is .325 (65 points/simulation). The figures reported in this table are from the 1000 estimates obtained.

TABLE 2
SIMULATION RESULTS ^a

A: Summary Statistics			
	Mean	S.D. of mean	S.D. of estimates
$\hat{\phi}_1$	2.07	0.007	0.226
$\hat{\phi}_2$	1.05	0.011	0.341
\hat{H}	4.16	0.017	0.542
B. Empirical Var-Cov Matrix of Estimates			
	.051	.022	.109
	.022	.116	.097
	.109	.097	.294
C. Average (S.D.) of Var-Cov Estimates			
	.050 (.017)	.020 (.016)	.107 (.039)
	.020 (.016)	.106 (.037)	.087 (.042)
	.107 (.039)	.087 (.042)	.285 (.103)
D. Chi-squared variates			
Theoretical Percentile	Empirical Percentile		
	Variate 1	Variate 2	
0.5%	.2%	.3%	
1.0%	.7%	.8%	
2.5%	2.6%	2.8%	
5.0%	4.3%	4.6%	
50.0%	49.0%	48.6%	
95.0%	94.6%	93.9%	
97.5%	97.1%	96.0%	
99.0%	98.2%	97.9%	
99.5%	99.0%	98.4%	

^aAs detailed in Section 5 the following was done: 1000 simulations each of length 200 was generated where the input process is Poisson with unit intensity and the output process is generated according to equation (50), with parameters $\phi_1 = 2$, $\phi_2 = 1$ and $H = 4$. The average input intensity of the generated process is .998 (100 points/simulation). The average output intensity of the generated process is .322 (32 points/simulation). The figures reported in this table are from the 1000 estimates obtained.

TABLE 3
SIMULATION RESULTS ^a

A: Summary Statistics			
	Mean	S.D. of mean	S.D. of estimates
$\hat{\phi}_1$	2.15	.011	.363
$\hat{\phi}_2$	1.11	.016	.505
\hat{H}	4.33	.027	.862
B. Empirical Var-Cov Matrix of Estimates			
	.132	.056	.284
	.056	.255	.219
	.284	.219	.743
C. Average (S.D.) of Var-Cov Estimates			
	.124 (.081)	.052 (.068)	.271 (.186)
	.052 (.068)	.252 (.133)	.211 (.179)
	.271 (.186)	.211 (.179)	.708 (.489)
D. Chi-squared variates			
Theoretical Percentile	Empirical Percentile		
	Variate 1	Variate 2	
0.5%	.2%	.4%	
1.0%	.7%	.8%	
2.5%	1.5%	1.7%	
5.0%	4.3%	4.5%	
50.0%	52.1%	51.3%	
95.0%	94.4%	92.1%	
97.5%	96.7%	94.3%	
99.0%	98.5%	96.8%	
99.5%	99.5%	98.0%	

^aAs detailed in Section 5 the following was done: 1000 simulations each of length 50 as generated where the input process is Poisson with unit intensity and the output process is generated according to equation (50), with parameters $\phi_1 = 2$, $\phi_2 = 1$ and $H = 4$. The average input intensity of the generated process is .997 (50 points/simulation). The average output intensity of the generated process is .322 (16 points/simulation). The figures reported in this table are from the 1000 estimates obtained.

TABLE 4
SIMULATION RESULTS ^a

A: Summary Statistics			
	Mean	S.D. of mean	S.D. of estimates
$\hat{\phi}_1$	4.06	.008	.263
$\hat{\phi}_2$	2.05	.009	.270
\hat{H}	8.15	.021	.667
B. Empirical Var-Cov Matrix of Estimates			
	.069	.034	.169
	.034	.073	.109
	.169	.109	.445
C. Average (S.D.) of Var-Cov Estimates			
	.065 (.016)	.032 (.011)	.160 (.040)
	.032 (.011)	.072 (.016)	.104 (.032)
	.160 (.040)	.104 (.032)	.422 (.105)
D. Chi-squared variates			
Theoretical Percentile	Empirical Percentile		
	Variate 1	Variate 2	
0.5%	.4%	.4%	
1.0%	.6%	.6%	
2.5%	1.7%	1.6%	
5.0%	3.8%	3.9%	
50.0%	46.3%	46.5%	
95.0%	94.5%	93.9%	
97.5%	97.1%	96.0%	
99.0%	98.8%	98.6%	
99.5%	99.4%	98.8%	

^aAs detailed in Section 5 the following was done: 1000 simulations each of length 200 was generated where the input process is Poisson with unit intensity and the output process is generated according to equation (50), with parameters $\phi_1 = 4$, $\phi_2 = 2$ and $H = 8$. The average input intensity of the generated process is .998 (200 points/simulation). The average output intensity of the generated process is .322 (57 points/simulation). The figures reported in this table are from the 1000 estimates obtained.