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TO EACH ACCORDING TO...? MARKETS, TOURNAMENTS, AND THE MATCHING PROBLEM WITH BORROWING CONSTRAINTS

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ABSTRACT

We compare the performance of markets and tournaments as allocative mechanisms in an economy with borrowing constraints. The model consists of a continuum of individuals who differ in their initial wealth and ability level (e.g. students) and that are to be assigned to a continuum of investment opportunities or inputs of different productivity (e.g. schools of different qualities). With perfect capital markets both mechanisms achieve the efficient allocation, though markets generate higher aggregate consumption because of the waste associated with the production of signals under tournaments. When borrowing constraints are present, however, tournaments dominate markets in terms of aggregate output and, for sufficiently powerful signaling technologies, also in terms of aggregate consumption.

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1. Introduction

Economists have long noted that borrowing constraints can prevent market economies from attaining efficient allocations. In their presence, the principle of allocating the "best" resources to those individuals who can use them more efficiently (and are, thus, willing to pay more for them) is constrained by an affordability requirement: the best resources are allocated to those who, *being able to afford them*, can use them more efficiently. Instruments such as credit subsidies and taxes have usually been the main focus of the literature that examines how efficiency can be improved when borrowing constraints are present. Yet, most of the literature seems to ignore the fact that the choice of an allocative mechanism is itself an important dimension of policy intervention. In this paper we investigate the properties of one such alternative mechanism–a *tournament.*[1]

We place our comparison of markets versus tournaments in the context of a matching problem à la Becker (1973). The assignment problem consists of matching investment projects of varying quality or potential profitability with agents that differ in their ability to exploit these projects. Relevant problems that may fall into this category include, among others, the allocation of land to farmers, of advisors to students, the privatization of firms, or the assignment of rank within a bureaucracy. For the sake of concreteness we identify these projects as schools and our problem is one of matching individuals of different ability to schools of different quality.

Our model consists of a continuum of agents that differ in terms of their initial wealth and ability. There is also a continuum of schools, each characterized by a given quality. The ability of the individual and the quality of the school attended jointly determine the individual's future income. For any given ability, better quality schooling implies greater income. Income is also greater the higher the individual's ability, given the quality of schooling. We furthermore assume that school quality and individual's ability are complements, so that the marginal return to school quality is increasing in ability.

The complementarity assumption implies, as in Becker's matching problem,

[1] A tournament has the property that the allocation of "prizes" among individuals depends solely on each agent's performance relative to that of others'. Our motivation for considering a tournament-based mechanism as an alternative to market prices comes in part from its applicability to reality. Tournaments are commonly used in many areas of economic life: they characterize not only the awarding of prizes in contests, but they also play an important role in the allocation of positions within hierarchical organizations, school and university admissions, recruitment, etc.

that the efficient allocation is characterized by the matching of the ability rank of the student with the quality rank of the school, i.e., by *positively assortative matching.* In the absence of capital market imperfections, the price mechanism, (i.e. a price for each school such that demand equals supply) achieves the efficient allocation; higher-ability individuals are willing to pay higher prices for higher quality schools and the possibility of borrowing against their future income ensures that the final allocation is independent of the initial wealth distribution.

A tournament, in contrast to prices, allocates individuals according to rank order in some contest. The individual's performance in the contest-her score or signal-is assumed to be an increasing function both of her ability and the amount of expenditures she undertakes (think, for example, of the contest as a placement exam such as the SAT in which an individual's score depends not only on her own ability, but also on the quality of high school attended, tutors hired, etc.). We assume that these expenditures are non-productive (i.e. they do not, in themselves, contribute to output nor yield utility) and that the marginal cost of increasing any given score is assumed to be non-increasing in ability. We show that with perfect capital markets a tournament generates the same allocation as does the market mechanism and, accordingly, the same level of aggregate output. Yet, aggregate consumption is lower under tournaments since resources are wasted in the production of signals.

Next we contrast the performance of markets and tournaments when there are borrowing constraints. We show that tournaments always deliver greater output than markets. The intuition for this result can be summarized as follows. With a market mechanism prices do not discriminate among individuals except with respect to their willingness and ability to pay; accordingly, individuals with the same level of schooling expenditures attend the same school, regardless of their abilities. In a tournament, on the other hand, identical expenditures by agents with different abilities do not lead to identical outcomes. In particular, by spending the same amount higher-ability individuals produce higher scores/signals than lower-ability individuals; that makes higher-ability individuals effectively "less credit constrained" than those with lower ability and identical wealth, thus allowing the former to afford a better school. The previous mechanism thus tends to enhance the efficiency of the allocation under tournaments since, under our assumptions, a social planner would wish to assign higher-quality students to higher-quality schools.[2] Whether aggregate consumption is also greater under

[2]Freeman (1996), in a framework with identical agents and borrowing constraints, also obtains the results that exams can improve efficiency. This result, however, is driven by the

tournaments is, however, not guaranteed, since a fraction of available resources are wasted in the competitive production of signals. We discuss the conditions on the tournament technology under which aggregate consumption is also greater under tournaments than under markets.

Our paper is related to several branches of the literature. There is a small literature that examines the performance of tournament-based compensation schemes relative to individualistic reward schemes (e.g., Lazear and Rosen (1981), Green and Stokey (1983), Nalebuff and Stiglitz (1983)). These papers, unlike ours, are concerned with the relative efficiency of tournaments in environments with moral hazard. Their focus is mostly on situations in which agents are homogeneous and the question is how to extract the suitable amount of effort from them given that effort is unobservable.

Second, there is also a relatively small literature that studies matching problems, i.e., problems in which a set of heterogeneous individuals is mapped into a set of heterogeneous objects (which may also happen to be individuals, as in the marriage case) with the payoff from each match depending on some characteristic(s) of both sides of the match (e.g., Becker (1973), Cole, Mailath, and Postlewaite (1992), Kremer (1993)), as well as papers that have introduced search frictions into the matching problem (e.g., Sattinger (1995), Acemoglu (1995), Shimer and Smith (1996), Burdett and Coles (1996)). Our contribution to this literature consists in the analysis of the implications of borrowing constraints, and their interaction with alternative allocation mechanisms, for the outcome of an otherwise standard matching problem. The introduction of borrowing constraints complicates the standard matching problem considerably since it introduces an additional source of heterogeneity–wealth–to the usual unidimensional matching problem.

Lastly, our paper is related to a recent literature on income distribution and borrowing constraints (e.g., Loury (1981), Banerjee and Newman (1993), Galor and Zeira (1993), Fernandez and Rogerson (1994), Benabou (1996)). The main concern of that literature is the study of the steady-state income distributions that result from the presence of borrowing constraints. In contrast, the focus of the present paper is on the relative efficiency of alternative allocation mechanisms when borrowing constraints are present.

The structure of the paper is as follows. Section 2 describes the matching problem and discusses the efficient allocation associated with that problem. Section

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assumption that there are increasing returns in production so that even an exam that is a pure lottery increases efficiency, by allowing agents to specialize.

3 analyzes equilibrium allocations under market prices and tournaments, when perfect capital markets are present. Section 4 characterizes the effects on those allocations of introducing borrowing constraints, and compares the relative efficiency of tournaments and markets in such an environment. Section 5 summarizes our main findings and concludes.

2. The Model

The economy consists of a continuum of agents each of whom is characterized by an endowment of "ability", a, and initial wealth w.[3] To simplify exposition, we assume that these agents are distributed uniformly on the unit square $I^2 \equiv [0,1] \times [0,1]$, and hence that these attributes are uncorrelated across agents.[4]

There exists an exogenous endowment of investment opportunities which vary in their "quality", reflected in the output they generate when combined with a given ability. These can be thought of as, for example, schools of different quality, plots of land of varying fertility, or oil sites or nature parks of different potential profitability. For the sake of concreteness we focus on the example of schools. The quality of a school is given by its index $s \in [0,1] \equiv I$. Again, for expository simplicity, quality is assumed to be distributed uniformly on the unit interval.

An agent with ability a who is allocated to a school of quality s generates an output (income) level $X(a,s)$, where $X : I^2 \rightarrow \Re_+$ can be thought of as a production function. We assume X is twice continuously differentiable, bounded, with $X_a > 0$, $X_s > 0$ (i.e., output is increasing both in ability and school quality), $X_{ss} \leq 0$ (i.e., there are decreasing returns to school quality), and $X_{as} > 0$, i.e., ability and school quality are complements in production. This last assumption plays a key role in the characterization of efficient allocations. We make two additional assumptions that facilitate exposition: (i) $X(0,s) = 0$, $\forall s$ (i.e., the lowest ability agent obtains a payoff of zero independently of her school assignment),[5]

[3]By ability we mean the efficiency/productivity with which a given agent can use a productive resource of a certain quality.

[4]Neither uniformity nor zero correlation is central to our results, but they permit easier diagrammatic and algebraic exposition.

[5]As will be clear further on, the last assumption, while unessential, allows us to get rid of price equilibria under perfect capital markets that differ from each other solely by a constant. This follows solely from $X(0,0) = 0$. The assumption that this is true for all s ensures that, independently of the mechanism used, individuals with zero ability will be unwilling to make strictly positive expenditures. This simplifies the treatment of equilibrium without affecting any of the main results.

and (ii) $X_s(a,s) < 1, \forall (a,s) \in I^2$.[6]

Given the allocation mechanism and presence (or absence) of capital markets, agents' actions take place within the following two-period structure: in the first period, agents incur their desired level of expenditures (possibly borrowing if capital markets are operative), and are allocated to a school; in the second period they generate output, repay their debt (if any), and consume. An agent chooses actions so as to maximize her utility from consumption. The latter equals income X, plus initial wealth, minus any expenditures realized to obtain this allocation, plus a possible lump-sum transfer. The utility function is assumed to be increasing in consumption. We normalize the payoff obtained from not attending any school to equal zero. [7] For the allocation mechanisms that we are examining it does not matter whether ability and wealth are observable. Nonetheless, to make the question of what allocation mechanism should be used in this environment interesting, it is probably best to think of both of these as unobservable.

Our assignment problem is to match each agent with a school. More formally, an allocation is defined by a function $S : I^2 \to I$ that assigns a school of quality $S(a,w)$ to agent (a,w), for all $(a,w) \in I^2$. Let $\Phi(a,s) \equiv \int_0^a \int_0^1 \mathbf{1}[s - S(z,w)]\, dw\, dz$ denote the joint distribution for (a,s) induced by a given allocation S, where $\mathbf{1}$ is an indicator function such that $\mathbf{1}[z] = 1$ if $z \geq 0$ and $\mathbf{1}[z] = 0$ otherwise, for all $z \in \Re$. We say that an allocation S is *feasible* if $\Phi(1,s) = s$, for all $s \in I$, i.e., the measure of individuals that are allocated to schools of quality no greater than s exactly matches the measure of schools of that quality range.

Before proceeding to characterize the equilibrium under perfect capital markets, it is useful to describe the solution to the social planner's problem.

2.1. The Efficient Allocation

The social planner's problem here corresponds to that of Becker's (1973) matching problem. In particular, the efficient allocation is given by a monotonically increasing mapping from agents' abilities to school qualities, i.e., by positively assortative matching. Not surprisingly, that efficient allocation is independent of the wealth distribution.

The efficiency of positively assortative matching is usually demonstrated by showing that, for any other allocation in which some (positive measure) of agents

[6]The role of this last assumption is to rule out "corner" solutions, as will be seen in section 4.

[7]Note that schools are completely passive in the matching problem.

of ability a are matched with schools of quality $s \neq a$, switching the assignment of mismatched agents so that each ability level a is matched with school quality $s = a$ leads to an increase in output. Here we instead present an alternative proof which, though less transparent, relies on a mathematical result (a first-order dominance theorem for a bivariate distribution originally due to Levy and Paroush (1973)) that will play a key role in the derivation of some of the main results of Section 4.[8]

Consider two absolutely continuous cumulative distribution functions $F^*(x_1, x_2)$ and $F(x_1, x_2)$. Let f^* and f be the associated density functions, respectively, with ranges given by $[0, b_i]$, for $i = 1, 2$. Let $U(x_1, x_2)$ be a continuous function with $U_{12} \geq 0$. Furthermore, let the marginal distributions of both functions be identical with $F_1(x_1)$ given by $\int_0^{x_1} \int_0^{b_2} f(t, x_2)\ dx_2\ dt$, and an analogous definition for $F_2(x_2)$. Define

$$\Delta W = \int_0^{b_1} \int_0^{b_2} U(x_1, x_2)\ \Delta f(x_1, x_2)\ dx_2\ dx_1 \qquad (2.1)$$

where Δf denotes $f^* - f$ (and $\Delta F = F^* - F$). Note that ΔW has an interpretation as the difference in expected utility under the f^* distribution relative to the f distribution.

Lemma 2.1. *(Levy-Paroush) If* $\Delta F(x_1, x_2) \geq 0, \forall (x_1, x_2)$, *then* $\Delta W \geq 0$.

Proof: See Appendix.

We are now set to prove that the efficient allocation in our model assigns all agents with ability a to school $s = a$. Let $\Phi^*(a, s)$ be the cumulative distribution function generated by the allocation $S^*(a, w) = a$, $\forall (a, w) \in I^2$. Clearly, $\Phi^*(a, s) = \min[a, s]$. Let $\Phi(a, s)$ be the cumulative distribution function associated with any alternative feasible allocation S. We are interested in evaluating $\Delta Y = \int_0^1 \int_0^1 X(a, s)\ [d\Phi^*(a, s) - d\Phi(a, s)]\ da\ ds$, i.e., the difference in aggregate output between allocation S^* and allocation S (where S differs non-trivially from S^*, i.e., $\Phi^*(a, s) \neq \Phi(a, s)$ for some (a, s)).

Theorem 2.2. $\Delta Y = \int_0^1 \int_0^1 X(a, s)\ [d\Phi^*(a, s) - d\Phi(a.s)]\ da\ ds > 0$.

Proof: See Appendix.

[8] The version of the Levy-Paroush result presented here (as well as its proof) follows closely the presentation in Atkinson and Bourguignon (1982).

3. The Economy with Perfect Capital Markets

In this section all agents are assumed to have access to an external market for (riskless) loans.[9] Without loss of generality we assume that the interest rate on those loans is zero. Thus, individuals are able to borrow freely in the first period (subject to a solvency requirement), and repay their outstanding loans in the second period.

3.1. Markets

When markets are the allocative mechanism, individuals are confronted with a (common) price schedule P, assigning a price to each school type. An *equilibrium with market prices* is given by a price function $P : I \to \Re_+$ (mapping each school quality s into a price $P(s)$), and a *feasible* allocation S such that, $\forall s \in I, \forall (a, w) \in I^2$, we have:

(m1) $X(a, S(a, w)) - P(S(a, w)) \geq X(a, s) - P(s)$
(m2) $X(a, S(a, w)) - P(S(a, w)) \geq 0$

The first condition states that, given P, no individual should strictly prefer another school to her equilibrium assignment. The second condition is a participation constraint requiring a nonnegative net payoff from the equilibrium school assignment (since, by assumption, an individual can always choose not to attend any school and thereby obtain a payoff of zero). Finally, the requirement that S is feasible is equivalent in our case to a market clearing condition.

Our assumptions on X imply that an equilibrium price schedule P must be strictly increasing (otherwise schools dominated in both cost and quality would be unattended), continuous (otherwise schools of quality greater than but close enough to the point of discontinuity would also be unattended), and bounded in its domain (given the boundedness of X and the participation constraint). It follows from this list of characteristics that P is also (almost everywhere) differentiable, a property that we use extensively in what follows.[10]

[9] Assuming that the loan market is external and riskless allows us to avoid endogenizing the interest rate which is not the focus of our inquiry.

[10] Because we cannot guarantee that P is differentiable everywhere on the unit interval, many of our results below should be qualified as holding "almost everywhere." Since none of the qualitative results or conclusions are affected by that property we will henceforth refrain, for expository convenience, from continuously reminding the reader of that technical qualification.

Thus, taking the price schedule as given, an agent with ability a chooses s to maximize $X(a,s) - P(s)$, yielding the following first and second-order conditions:

$$X_s(a, S(a,w)) - P'(S(a,w)) = 0 \tag{3.1}$$

$$X_{ss}(a, S(a,w)) - P''(S(a,w)) < 0 \tag{3.2}$$

for all $(a,w) \in I^2$. As deduced before, from (3.1) and $X_s > 0$ it follows that $P'(s) > 0$, i.e., the price of a school is strictly increasing in its quality. Furthermore, applying the implicit function theorem to the same condition yields $S(a,w) = S(a,w') \equiv \mathbf{S}(a)$, for all $(a,w),(a,w') \in I^2$, implying that agents with the same ability choose the school of the same quality regardless of their initial wealth.[11]

Next, total differentiation of (3.1) yields $\mathbf{S}'(a) = \frac{X_{as}}{P''-X_{ss}} > 0$. Combined with the feasibility/market clearing requirement, strict monotonicity of $\mathbf{S}$ implies the allocation rule

$$\mathbf{S}(a) = a, \qquad \forall a \in I \tag{3.3}$$

i.e., positively assortative matching obtains in equilibrium. Finally, substitution of (3.3) into (3.1), and forward integration yields the equilibrium price schedule

$$P(s) = \int_0^s X_s(z,z)\, dz \tag{3.4}$$

where we have made use of the result that $P(0) = 0$ as implied by the assumption that $X(0,0) = 0$ and the participation constraint.

Under the price mechanism and perfect capital markets, aggregate output (denoted by Y^{m^*} in this environment) is given by

$$Y^{m^*} = \int_0^1 X(a,a)\, da \tag{3.5}$$

Aggregate consumption, C^{m^*}, is given by aggregate wealth plus aggregate output Y^{m^*}.[12] Hence,

$$C^{m^*} = \frac{1}{2} + \int_0^1 X(a,a)\, da \tag{3.6}$$

[11] This, of course, is a consequence of the existence of perfect capital markets.

[12] We are implicitly assuming that the resources used to pay for the schools are simply distributed to some agent (say, a private owner or the government) who either consumes them or rebates them back to the population in a lump-sum fashion. What matters here is that they are considered not to be wasteful, i.e., they do not subtract from utility-yielding aggregate consumption.

where the first term on the right-hand-side of (3.6) corresponds to aggregate wealth $\int_0^1 w \, dw$.

It is useful to note here that, not surprisingly, in the absence of capital market imperfections the allocation achieved by the market equilibrium as expressed in (3.3) is efficient. Next we turn to the derivation of equilibrium when the allocation mechanism is a tournament.

3.2. Tournaments

First we introduce the technology and tournament rules which are common to both capital market environments. A tournament allocates each individual to the school whose quality rank equals the *rank* of her performance in the contest.[13] We shall hereafter refer to this performance as the signal and to the contest technology as the signaling technology, although no agent is trying to make any inferences about the ability of the contest participants from these scores. Given a signaling technology, the score that an agent achieves in a tournament depends on her ability and on the amount of resources she chooses to spend in generating a signal. We assume that the resources expended by the agent in enhancing her signal are not productive, i.e., they do not in themselves contribute to output and, ceteris paribus, they reduce aggregate consumption.[14]

The signaling technology is represented by a mapping $V : I \times \Re_+ \rightarrow \Re$, with $V(a, e)$ measuring the signal or score generated by an agent of ability a who spends resources e. Often it will be useful to work with the associated cost function $e(v, a)$, defined implicitly by $V(a, e(v, a)) = v$, $\forall a \in I$, and $\forall v \in \Re$. We make the following assumptions: V is continuously differentiable; $V_a \geq 0$ and $V_e \geq 0$ (both holding with strict inequality in the interior of I^2) i.e., signals are increasing in

[13]Thus, the reader may wish to think of these schools or investment opportunities as being owned by the state (or a non-profit institution) in the case of tournaments, since the owner does not receive a rent for them.

[14]One way to think of these expenditures is as using resources (e.g. tutors) that have some constant marginal revenue product elsewhere. Thus, diverting these resources from productive work to signal-generating work robs the economy of their product. While it may be thought that some portion of these expenditures can augment "ability" (and are thus productive), this reformulation of the problem simply creates an additional investment opportunity where credit constraints are potentially binding, thus complicating the analysis. Our analysis would not be affected, in any case, by allowing some fraction of expenditures to be non-wasteful and transferred back to the population in a lump-sum fashion. As will be clear in the next section, this way of setting up the problem simply makes it more difficult for tournaments to dominate markets.

both ability and expenditures; V_e is bounded above in the interior of I^2; $V(a, 0) = 0$, $\forall a \in I$, i.e., expenditures are necessary in order to emit a positive signal.[15] Furthermore, we assume $e_{va} \leq 0$ (the marginal cost of signaling is nonincreasing in ability). We let $F : \Re_+ \rightarrow [0, 1]$ represent the cumulative distribution function for the signals generated in the economy, i.e., $F(v) = \int_0^1 \int_0^1 \mathbf{1}[v - V(a, E(a, w))]\, dw\, da$, $\forall v \in \Re_+$, where $E(a, w)$ denotes the resources expended in signal-enhancing activities by agent (a, w).[16]

In order for the signals' cumulative distribution function to constitute an equilibrium under perfect capital markets, each agent's expenditure choice and consequent school assignment must be utility maximizing given F. More formally, an equilibrium with *tournaments and perfect capital markets* is given by a feasible allocation S, an expenditure rule E, and a signal distribution F satisfying, $\forall (a, w) \in I^2, \forall\, s \in I, \forall v \in \Re$:

(t1) $S(a, w) = F(V(a, E(a, w)))$
(t2) $X(a, S(a, w)) - E(a, w) \geq X(a, s) - e(F^{-1}(s), a)$
(t3) $E(a, w) = e(F^{-1}(S(a, w)), a)$
(t4) $F(v) = \int_0^1 \int_0^1 \mathbf{1}[v - V(a, E(a, w))]\, dw\, da$
(t5) $X(a, S(a, w)) - E(a, w) \geq 0$

Condition *(t1)* simply restates the allocation rule; i.e. an agent is matched to a school whose rank in the quality distribution equals her rank in the signaling distribution. Condition *(t2)* ensures that, given F, an agent's expenditure maximizes her utility. *(t3)* shows the expenditure required by each agent to achieve her equilibrium signal. Condition *(t4)* ensures that the signal distribution that every agent takes as given when solving her optimization problem is, indeed, the distribution observed in equilibrium. Condition *(t5)* is the usual participation constraint.

[15] The last assumption guarantees that agents with zero wealth are allocated to the lowest quality school, simplifying the derivation of the equilibrium allocation under credit constraints.

[16] The tournament can be thought of as an exam, whose score–the signal–depends on the agent's ability and on the amount she spends in order to prepare herself for that test. How well the individual does on that exam is affected by the quality of the high school attended, the extent to which she has had access to tutors or other resources to prepare for the exam, etc. It is totally in keeping with the nature of our model to interpret the expenditures on signaling as being performed prior to "taking the exam." Thus, one possible interpretation is that there is a first period in which high schools are chosen and a price is paid for them either explicitly as with a private school, or implicitly, by choosing to live in a neighborhood (and paying the housing price there) with a particular quality of public education. In the second period, the exam is taken, and the last period is the same as before.

Our assumptions on X and V guarantee that the function mapping school quality into the associated equilibrium signals (i.e., F^{-1}) is strictly increasing and continuous in school quality. It is strictly increasing for, otherwise, schools dominated in quality but with a higher associated signal would be unattended. It is continuous since, otherwise, agents generating a signal greater than but close enough to a discountinuity point would be better off by reducing their expenditures by a discrete amount and attending a school of marginally lower quality. The previous properties imply that the cumulative distribution function over signals F is also strictly increasing, and continuous, and so both it and its inverse F^{-1} are (almost everywhere) differentiable.[17]

Choosing s so as to maximize the payoff from attending school, $X(a,s) - e(F^{-1}(s),a)$, yields the following first and second-order conditions:

$$X_s(a, S(a,w)) = e_v(F^{-1}(S(a,w)),a)\ F^{-1\prime}(S(a,w)) \tag{3.7}$$

$$X_{ss}\ - e_{vv}\ (F^{-1\prime})^2 - e_v\ (F^{-1})'' < 0 \tag{3.8}$$

Analogously to the market equilibrium, it follows from (3.7) that $S(a,w) = S(a,w') = \mathbf{S}(a)$, for all $(a,w),(a,w') \in I^2$. Hence, we have $E(a,w) = e(F^{-1}(\mathbf{S}(a)),a) \equiv \mathbf{e}(a)$, $\forall\ (a,w) \in I^2$. Thus, as with the market mechanism, perfect capital markets ensure that with tournaments agents' initial wealth distribution has no impact on the resulting equilibrium allocation.

To derive the equilibrium allocation of individuals, note that differentiation of (3.7) with respect to a yields $\mathbf{S}'(a) = -\frac{X_{as}\ - e_{va}\ F^{-1\prime}}{X_{ss}\ - e_{vv}\ (F^{-1\prime})^2 - e_v\ (F^{-1})''} > 0$, where the sign of the inequality follows from our assumptions on X and e and (3.8). As in the market case analyzed above, strict monotonicity of $\mathbf{S}$ and feasibility imply:

$$\mathbf{S}(a) = a \quad \forall a \in I \tag{3.9}$$

that is, the allocation rule under tournaments is identical to the one derived under markets and, therefore, it corresponds to the efficient allocation.

It follows from (t1) and (3.9) that $a = F(V(a, \mathbf{e}(a)))$, implying that

$$1 = (V_a + V_e\ \mathbf{e}')\ F' \qquad \forall a \in I \tag{3.10}$$

holds in equilibrium. This allows us to rewrite (3.7), evaluated at equilibrium, as the differential equation:

$$\mathbf{e}'(a) = X_s(a,a) - \frac{V_a(a,\mathbf{e}(a))}{V_e(a,\mathbf{e}(a))} \quad \forall a \in I \tag{3.11}$$

[17] As noted in footnote 10, we will be assuming differentiability everywhere for expository purposes.

Integrating (3.11) and making use of the result that the lowest ability agents attend the lowest quality school to imposing the participation constraint requiring $\mathbf{e}(0) = 0$, we obtain an (implicit) expression for the equilibrium signaling expenditures:

$$\mathbf{e}(a) = \int_0^a [X_s(z,z) - \frac{V_a(z,\mathbf{e}(z))}{V_e(z,\mathbf{e}(z))}]\, dz \tag{3.12}$$

Aggregate output is given by

$$Y^{t^*} = \int_0^1 X(a,a)\, da \tag{3.13}$$

and aggregate consumption (i.e., initial endowment plus gross output minus resources expended in the signaling process) is

$$C^{t^*} = \frac{1}{2} + \int_0^1 [X(a,a) - \mathbf{e}(a)]\, da \tag{3.14}$$

3.3. Markets vs. Tournaments

Next we compare some of the aggregate outcomes under the two alternative allocation mechanisms. First, and as a direct consequence of the equivalence between their equilibrium allocations, we have

$$Y^{t^*} = Y^{m^*} \tag{3.15}$$

i.e., aggregate output is the same under markets and tournaments. Yet, and because of the waste associated with signaling, the same equivalence does not carry over to aggregate consumption, which is higher under markets by the amount of expenditures realized under tournaments, i.e.,

$$C^{m^*} - C^{t^*} = \int_0^1 \mathbf{e}(a)\, da > 0 \tag{3.16}$$

Why do markets and tournaments attain the same equilibrium allocation? In an environment with perfect capital markets, what sorts individuals out among different investment opportunities is their differential willingness to pay or to generate a higher score or signal for these opportunities. Higher-ability individuals are, ceteris paribus, willing to pay more or to produce a higher signal for higher quality schooling than lower-ability individuals. This sorts the former into higher-quality schools and thus reproduces the same allocation under both systems.

Perhaps the above is most easily understood by noting that a single-crossing condition applies in both environments. Consider markets first. There we can write an individual's payoff as $u(s, p; a) = X(a, s) - p$. The slope of an individual's indifference curve in (s, p) space is given by $\frac{dp}{ds}\mid_{u=\bar{u}}= X_s(a, s)$. Note, as illustrated in Figure 1, that the slope of this curve through any (s, p) point is increasing in a, i.e., $\frac{\partial}{\partial a}(\frac{dp}{ds}\mid_{u=\bar{u}}) = X_{as}(a, s) > 0$. That is, for a given increase in s, the amount that higher-ability individuals are able to pay and still keep their utility constant is greater than for lower-ability individuals. But this necessarily implies that in any equilibrium in which the price of schools differ, higher-ability individuals will be allocated to higher-quality schools.

A similar argument applies for tournaments. There, an individual's payoff can be written as $u(s, v; a) = X(a, s) - e(v, a)$. The slope of the individual's indifference curve in (s, v) space is given by $\frac{dv}{ds}\mid_{u=\bar{u}}= \frac{X_s(a,s)}{e_v(v,a)}$. Note that the slope of this curve through any (s, v) point is increasing in a, i.e., $\frac{\partial}{\partial a}(\frac{dv}{ds}\mid_{u=\bar{u}}) = \frac{X_{sa}e_v - e_{va}X_s}{e_v^2} > 0$, indicating, as before, that for a given increase in s, higher-ability individuals are able to increase their signal by a greater amount than lower ability individuals and keep their utility constant. Again, this necessarily implies that with perfect capital markets, higher-ability individuals will be allocated to higher quality schools.

It is interesting to note, however, that just as the same assignment of individuals to schools under both allocation mechanisms does not imply the same aggregate consumption, neither does it imply the same expenditures in equilibrium. To see this note that we can combine (3.4) and (3.12) to obtain

$$P(s) - \mathbf{e}(s) = \int_0^s \frac{V_a(z, \mathbf{e}(z))}{V_e(z, \mathbf{e}(z))}\, dz > 0 \quad \forall s \in (0, 1] \tag{3.17}$$

As indicated by (3.17), the price of a given quality school in the market equilibrium is never below the expenditure needed by an agent with ability a to obtain the same assignment under tournaments. A simple variational argument may help us gain some intuition for this result. First, note that, in equilibrium, the additional expenditure (i.e., the marginal cost) required for an individual in school s with ability $a = s$ to switch to a school of quality $s + \Delta s$ must equal, as $\Delta s \to 0$, the marginal benefit, i.e., $X_s(s, s)$ under both allocative mechanisms. Under market prices that amount is given by $P(s + \Delta s) - P(s)$, or $P'(s)$ when considering an infinitesimal change. Under tournaments the marginal cost is given by $e(F^{-1}(s+\Delta s), s) - e(F^{-1}(s), s)$ or, as $\Delta s \to 0$, $e_v(F^{-1}(s), s)\ F^{-1\prime}(s) = e_v\ (V_a + V_e\ \mathbf{e}') = \frac{V_a}{V_e} + \mathbf{e}'$. Since in both cases marginal costs must be equated to marginal benefits, it follows that $P'(s) = \frac{V_a}{V_e} + e'(s) = X_s(s, s)$, and thus that $e'(s) < P'(s)$. More intuitively, and following the single-crossing logic developed earlier, it requires lower expenditures for a higher-ability agent to "separate herself out" in a tournament than in a market environment because in order to replicate her signal a lower-ability agent must do more than simply match her expenditures (which is all that he would need to do in a market environment)–he must also compensate for his lower ability by spending an additional amount given by $\frac{V_a}{V_e}$.

A few additional interesting properties of the tournament relative to the market equilibrium can be noted here. First, although we have established that prices must be increasing in the quality of school, note that this property does not carry over to expenditures under tournaments. With the latter, it is the magnitude of the signals that must be increasing in the quality of schooling, but this does not imply that expenditures be increasing as well (note that in equation (3.11), e' may be negative). The reason for this is related to the intuition given above. Depending on the sensitivity of the signalling technology to the increment in ability, it may be possible for an agent with higher ability to achieve the signal associated with her school with lower expenditures than for the agent attending a lower quality school. Second, note that in the limit, as $\frac{V_a}{V_e} \to 0$, the solution to the tournament differential equation is identical to that under markets, i.e., expenditures are the same under both systems. On the other hand, as $\frac{V_a}{V_e} \to \infty$, expenditures under the tournament go to zero so that both tournaments and markets not only achieve the same allocation, but they also are equally efficient since there are no longer any wasteful expenditures associated with tournaments.

4. The Economy with Borrowing Constraints

In this section we assume that there are no capital markets that permit the financing of expenditures beyond an agent's wealth, whether on schools directly, as is the case under market prices, or on the generation of signals, as is the case under tournaments. Although we do not model the microfoundations for this imperfection here, it can be thought of as arising from the absence of an enforcement technology that allows contracts that promise future repayment to be honored. Alternatively, if output cannot be observed, and X is now interpreted as the expected value of output, then the usual moral hazard problems may prevent the capital market from functioning.

4.1. Markets

A *market equilibrium with credit constraints* is given by a price schedule $P : I \rightarrow \Re_{+}$ mapping school qualities into prices, and a feasible allocation S , that respects the same participation constraint (m2) as before but that now must maximize utility subject to the constraint that an agent cannot have expenditures that exceed her wealth, i.e. (m1) must be replaced by:

(mc1) $X(a, S(a, w)) - P(S(a, w)) \geq X(a, s) - P(s)$, $\forall s$ such that $P(s) \leq w$ and $P(S(a, w)) \leq w$.

As in the case of perfect capital markets (and for the same reasons), the equilibrium price function $P(s)$ must be strictly increasing, continuous, bounded, and differentiable. Thus, utility maximization implies:

$$X_s(a, S(a, w)) - P'(S(a, w)) \geq 0, \quad \forall (a, w) \in I^2 \tag{4.1}$$

with the borrowing constraint requiring $P(S(a, w)) \leq w$. The first-order condition must be met with strict equality if $P(S(a, w)) < w$.

In order to characterize equilibrium we need to describe the assignment of agents to schools. This is done in a series of lemmas below. Before proceeding with the formal derivation, we first characterize this allocation informally.

The characterization of equilibrium allocations proceeds by showing that associated with each s there will be a lowest level of ability, $\underline{a}(s)$. For any given s and its associated $\underline{a}(s)$, we show that for levels of wealth greater than the price of this school, agents with this ability level will be effectively unconstrained and optimizing given the entire price schedule and hence have $X_s(\underline{a}(s), s) = P'(s)$. Thus, all agents of that same ability level that can afford to attend s will also do so. Who else will attend s? The remaining agents attending s will have higher ability and be exactly constrained, i.e., their spending will exactly equal their wealth. So, we will show that for each ability level there is an optimal s which all agents (of that ability level) that can afford it attend, and all constrained agents attend the highest quality school that they can afford, i.e.:

$$S(\underline{a}(s), w) = \begin{cases} s, & \forall w \in [P(s), 1] \\ P^{-1}(w), & \forall w \in [0, P(s)] \end{cases} \tag{4.2}$$

Furthermore, we prove that $\underline{a}(s)$ is increasing in s. Hence, the allocation of agents to schools s' and s'', $s' > s''$, can be depicted as in Figure 2. This figure provides a useful way to depict equilibrium in the presence of credit constraints.

On the horizontal axis we measure s and $\underline{a}(s)$. On the vertical axis we represent w and $P(s)$. Given credit constraints, we must have $P(s) \leq 1$ and hence this figure can be restricted to the unit square. The schedule $P(s)$ depicts the equilibrium price schedule. Note that $\underline{a}(s)$ lies to the left of s since otherwise, for any $P(s) > 0$, the set of agents with $a \geq \underline{a}(s)$ and $w \geq P(s)$ would be smaller than $1 - s$ which is the measure of capacity of schools of quality greater than s. The set of agents that attend school s'' are those on the southwest boundary of the shaded area in Figure 2, with agents within that area attending a school at least as highly ranked as s''. Similarly (for the smaller rectangle) for s'.

We now turn to the formal derivation of these results. The first two lemmas establish that the price of the lowest quality school equals zero whereas that of the highest quality is strictly smaller than one. This ensures that, for each ability level, some agents are always unconstrained.

Lemma 4.1. $P(0) = 0$.

Lemma 4.2. $P(s) < 1, \forall s \in [0, 1]$.

Proofs: See Appendix.

Next, we define Q as the set of all agents whose wealth is not smaller than $P(1)$, i.e $Q \equiv \{(a, w) : w \geq P(1)\}$. Note that by the previous lemma this set has positive measure. The next lemma shows that unconstrained agents (i.e., the members of Q) are perfectly sorted by ability level into schools, with higher ability individuals attending higher quality schools.

Lemma 4.3. *(i) for all* $(a, w), (a', w') \in Q$, *if* $a > a'$, *then* $S(a, w) > S(a', w')$; *(ii)* $S(a, w'') = S(a, w)$, *for all* $(a, w) \in Q$ *such that* $w'' > P(S(a, w))$.

Proof: See Appendix.

What is the intuition for the allocation described above? Note that an implication of the single-crossing condition obeyed by indifference curves in this model is that if an agent (a', w) prefers s' to s'', $s' > s''$, then so must all agents with $a > a'$. Unlike the complete markets environment, however, this no longer implies perfect sorting since not all wealth levels of a given ability can afford the associated price. Consequently, market clearing dictates that the price must be such that schools get "filled" by including individuals of lower ability. By single-crossing again, as we proceed to schools of higher quality, and consequently higher prices, those individuals who are most willing to pay the increased price will be those with higher ability.

The preceding discussion allows us to define a function $\mathbf{S} : I \rightarrow I$, as $\mathbf{S}(a) \equiv S(a, w)$, $\forall (a, w) \in Q$. Note that any agent (a, w) with $w \geq P(s)$ will be able to afford the school chosen by the agent with the same ability and highest wealth (and who was shown to be unconstrained). Thus these agents will chose the same school, $\mathbf{S}(a)$. Hence $\mathbf{S}(a)$ represents the quality of the school attended by all agents with ability a who are effectively unconstrained. Its properties are reported in the following lemma.

Lemma 4.4. *(i)* $\mathbf{S}$ *is strictly increasing, (ii)* $\mathbf{S}(0) = 0$, *(iii)* $\mathbf{S}(1) = 1$, *(iv)* $\mathbf{S}$ *is continuous, (v)* $\mathbf{S}$ *is (a.e.) differentiable.*

Proof: See Appendix.

Let $\underline{a}$: $I \rightarrow I$ be the inverse function of $\mathbf{S}$. By construction $\underline{a}\ (s)$ represents the ability of the unconstrained agents that attend school s (who are. in turn, the lowest ability agents attending that school). The properties of $\underline{a}$, stated as a corollary of lemma (4.4), follow directly from those of $\mathbf{S}$.

Corollary 4.5. *(i)* $\underline{a}$ *is strictly increasing, (ii)* $\underline{a}\ (0) = 0$, *(iii)* $\underline{a}\ (1) = 1$. *(iv)* $\underline{a}$ *is continuous, (v)* $\underline{a}$ *is (a.e.) differentiable.*

Notice that, differentiating the first-order condition (given by $X_s(\underline{a}\ (s), s) = P'(s)$), we obtain

$$\underline{a}'\ (s) = -\frac{X_{ss} - P''}{X_{as}} > 0 \tag{4.3}$$

where the numerator is the second-order condition of the agent's maximization problem and its strict negativity is guaranteed by the previous corollary.

Corollary 4.6. $P'(s)$ *is continuous and (a.e.) differentiable.*

Proof: See Appendix.

We can also show that the participation constraint will hold with strict inequality for all agents (but for a subset of zero measure attending school $s = 0$). Formally, $\forall (a, w) \in I^2$ such that $S(a, w) > 0$, we have $X(a, S(a, w)) = X(a, 0) + \int_0^{S(a,w)} X_s(a, z)\, dz > \int_0^{S(a,w)} X_s(\underline{a}(z), z)\, dz = \int_0^{S(a,w)} P'(z)\, dz = P(S(a, w))$.

Next we define the sets $R(s) = \{(a, w),\ a = \underline{a}(s),\ w \geq P(s)\}$ and $T(s) = \{(a, w),\ w = P(s),\ a > \underline{a}(s)\}$. The last proposition in this section characterizes the set of agents allocated to any given school s in equilibrium (see Figure 2) as the union of these two sets.

Proposition 4.7. *$S(a, w) = s$ if and only if $(a, w) \in R(s) \cup T(s)$.*

Proof: See Appendix.

An implication of the previous lemma is that we can express the market clearing condition as $(1 - s) = (1 - \underline{a}(s))\,(1 - P(s))$ or, alternatively,

$$\underline{a}(s) = \frac{s - P(s)}{1 - P(s)}, \quad \forall s \in I \tag{4.4}$$

This yields as an immediate conclusion that

$$\underline{a}(s) < s, \quad \forall s \in (0, 1) \tag{4.5}$$

Note that a direct implication of (4.5) is that in the presence of credit constraints the market equilibrium allocation is inefficient: as shown earlier, the efficient allocation requires that only those individuals with $a = s$ attend school s, irrespective of their endowment. Under credit constraints the lowest ability individual attending s has ability strictly lower than s and the highest ability individual in s has $a = 1$.

We are now set to derive equilibrium expenditures for unconstrained agents. Combining (4.1) and (4.4) we obtain the following differential equation in $P(s)$:

$$P'(s) = X_s\left(\frac{s - P(s)}{1 - P(s)}, s\right), \quad \forall s \in I \tag{4.6}$$

and, using the first-order condition and $P(0) = 0$, we can derive the following expression for equilibrium prices:

$$P(s) = \int_0^s X_s(\underline{a}(z), z)\, dz \tag{4.7}$$

Using P^* and $P_\circ$ to refer to the equilibrium price schedule with perfect capital markets and under credit constraints respectively, it follows from $\underline{a}(s) < s$ and $X_{as} > 0$, that

$$P_\circ(s) < P^*(s) \quad \forall s \in (0, 1]. \tag{4.8}$$

i.e., the presence of borrowing constraints unambiguously lowers the price of schooling for all qualities (with the exception of $s = 0$). The intuition for this result is also straightforward. In equilibrium the marginal benefit from an increase in s for an unconstrained agent with ability $\underline{a}$ is $X_s(\underline{a}(s), s)$. This must equal the marginal cost of increasing s, i.e. $P_\circ'(s)$. The same is, of course, true under perfect capital markets, except that in that environment $a = s > \underline{a}(s)$, $\forall s \in (0, 1)$, and since $X_{as} > 0$, it follows that $P_\circ'(s) < P^{*\prime}(s)$ which, when combined with $P(0) = 0$ (in both environments), yields $P_\circ(s) < P^*(s) \quad \forall s \in (0, 1]$. More intuitively, the presence of credit constraints lowers prices since the marginal (i.e. "market clearing") agent in each school is of lower ability than with perfect capital markets.

4.1.1. Some Remarks on the Political Economy of Markets With Borrowing Constraints[18]

The result expressed in (4.8), together with (4.2), allows us to characterize some of the implications of borrowing constraints on equilibrium allocations and agents' welfare when market prices are used as an allocative mechanism.

Consider the set of agents with ability a. Under perfect capital markets each agent in that set is allocated to a school of quality a, independently of her initial wealth. Under borrowing constraints, as we have seen, assignments are not independent of initial wealth. In this case we can partition the set of agents of ability a into three categories.

For agents with wealth in the interval $[P_\circ(\mathbf{S}(a)), 1]$ the borrowing constraint is not binding, and they are all allocated to the same school $\mathbf{S}(a) > a$ in equilibrium. Since $P_\circ(a) < P_\circ(\mathbf{S}(a))$ they can all afford to attend school a, but they choose

[18]Throughout this subsection we ignore the possibility of transfers.

not to. Hence, it must be the case that

$$X(a, \mathbf{S}(a)) - P_\circ(\mathbf{S}(a)) \geq X(a, a) - P_\circ(a) > X(a, a) - P^*(a) \tag{4.9}$$

where the last inequality follows from (4.8). From this we can conclude that agents of ability a and $w \in [P_\circ(\mathbf{S}(a)), 1]$ must be strictly better off under borrowing constraints than with perfect capital markets.

Consider now the subset of agents of ability a for whom borrowing constraints are binding, i.e., those with wealth in the interval $[0,\ P_\circ(\mathbf{S}(a)))$. For agents in that subset payoffs are given by $X(a, P_\circ{}^{-1}(w)) - w$, which is strictly increasing and continuous in w.[19] Given (4.9), there must exist a wealth level $\mathbf{w}(a) \in [0, P_\circ^{-1}(\mathbf{S}(a)))$, such that an agent with $(a, \mathbf{w}(a))$ is indifferent between the two allocation systems, i.e. her payoffs are the same. Then, all agents with ability a and $w > \mathbf{w}(a)$ are strictly better off under borrowing constraints, whereas those with $w < \mathbf{w}(a)$ are strictly worse off. Furthermore, since the agent with ability a and $w = P^\circ(a)$ is strictly better off, it follows that $\mathbf{w}(a) \in [0, P_\circ(a))$.

Combining the previous results, we can represent the set of agents who will be strictly better off in the equilibrium with borrowing constraints as $\{(a, w) \in I^2 : w > \mathbf{w}(a)\}$, as illustrated in Figure 3. Notice that among the agents with ability a who are made better off by the introduction of borrowing constraints there is a subset–specifically, those with wealth in the interval $(\mathbf{w}(a), P_\circ(\mathbf{S}(a)))$– for whom borrowing constraints are effectively binding in equilibrium.[20] Hence, policies that may tend to eliminate borrowing constraints would also be opposed by some agents who are apparently suffering the impact of such constraints.

4.2. Tournaments

An equilibrium with *tournaments and borrowing constraints* is given by a feasible allocation S, an expenditure rule E, and a signal distribution F, satisfying the same requirements *(t1)-(t5)* as under perfect capital markets except that the utility maximization condition *(t3)* must be modified to take into account the absence of credit markets, i.e., $\forall (a, w) \in I^2$

(tc3) $X(a, S(a, w)) - E(a, w) \geq X(a, s) - e(F^{-1}(s), a), \forall s$ such that $e(F^{-1}(s), a) \leq w$ and $E(a, w) \leq w$

[19] Since in equilibrium the first order condition holds with inequality for these agents, i.e., $X_s(a, P^{-1}(w)) > P'(P^{-1}(w))$.

[20] In fact, a subset among the latter–those with $w \in (\mathbf{w}(a), P(a))$–are made better off by the introduction of borrowing constraints even though it forces them to attend a school of lower quality. This is a consequence of the price reduction induced by the borrowing constraints.

As in the case of perfect capital markets, and for the same reasons, the distribution function of signals F must be strictly increasing in equilibrium (since no agent would be willing to emit a signal higher than necessary to attend a given school), continuous, and consequently (a.e.) differentiable. A necessary condition for utility maximization is given by:

$$X_s(a, S(a,w)) - e_v(F^{-1}(S(a,w)), a)\ F^{-1\prime}(S(a,w)) \geq 0 \tag{4.10}$$

with the borrowing constraint requiring $E(a,w) \leq w$, $\forall (a,w) \in I^2$, and where (4.10) holds with equality if $E(a,w) < w$, i.e., if agent (a,w) is unconstrained.

As with the market mechanism, the characterization of equilibrium allocations proceeds by showing that associated with each school s there is a lowest level of ability, $\underline{a}(s)$. Agents with ability $\underline{a}(s)$ attend school s if they can afford it, i.e., as long as their wealth is sufficient to cover the cost of generating the signal needed to attend school s. Agents with the same ability but wealth below the threshold needed to afford school s, attend the highest-quality shool they can afford. Thus we will show:

$$S(\underline{a}(s), w) = \begin{cases} s, & \forall w \in [e(F^{-1}(s), \underline{a}(s)), 1] \\ F(V(\underline{a}(s), w)), & \forall w \in [0, e(F^{-1}(s), \underline{a}(s))) \end{cases} \tag{4.11}$$

Analogously to Figure 2 in the prior section, we can depict the equilibrium with tournaments and borrowing constraints as shown in Figure 4. As before, the horizontal axis depicts a and s, and the vertical axis measures wealth and expenditures. The schedule denoted $\mathbf{e}(a)$ depicts the equilibirum expenditures of unconstrained agents with this ability level. The set of agents that attend school s is represented by the southwest boundary of the shaded area, with agents within that area attending a school at least as highly ranked as s. Note that, unlike the case of markets, agents with ability greater than $\underline{a}(s)$ do not have the same expenditures as the latter. Instead, the shaded area is downward sloping indicating that as a increases, the level of expenditures needed by an agent to generate the signal associated with school s decreases. We now turn to the formal derivation of these results.

The following two lemmas characterize some properties of the expenditure function in equilibrium.

Lemma 4.8. $e(F^{-1}(0), a) = 0$, $\forall a \in I$.

Lemma 4.9. $e(F^{-1}(S(a,w)), a) < 1$, $\forall s \in [0,1]$.

Proofs: See Appendix

The first lemma states that agents that attend school $s = 0$ incur no expenditures (and thus generate a score or signal of zero given our assumptions on V), while the second lemma guarantees that in equilibrium all agents spend less than the maximum endowment. Consequently, the first-order condition (4.10) will hold with equality for some agents, a result that we will use later.

The next lemma establishes that positively assortative matching obtains among the subset of agents who can afford to attend any school in equilibrium. Some additional notation is necessary at this point. Let b_s be defined implicitly by $e(F^{-1}(s), b_s) = 1$. Let $a_s = \max[0, b_s]$, that is, a_s represents the lowest ability that can afford to attend school s by spending no more than one. Let $Q(s)$ represent the set of agents who *can afford* to generate the signal $F^{-1}(s)$ necessary to go to school s, i.e., $Q(s) = \{(a, w) \in I^2, a \geq a_s, w \geq e(F^{-1}(s), a)\}$.

Lemma 4.10. *For all* $(a, w), (a', w') \in Q(1)$, *(i) if* $a > a'$, *then* $S(a, w) > S(a', w')$; *(ii) if* $a = a'$ *then* $S(a, w) = S(a', w')$.

Proof: See Appendix

Consider next agent $(a, w) = (a_1, 1)$, that is the lowest ability agent who can afford to attend school $s = 1$ (and, thus, *any* other school). By the previous two lemmas this agent is spending less than her wealth, and is attending some school $s_1 = S(a_1, 1) < 1$. We can associate with this school a set $Q(s_1)$ which, according to the definition above, is the set of agents who can afford s_1. Note that the single-crossing property guarantees that all agents $(a, 1)$ in $Q(s_1)$ are effectively unconstrained. That is, although those members of $Q(s_1)$ who have $a < a_1$ cannot afford to attend some $s \in (s_1, 1]$, it follows from the fact that $(a_1, 1)$ is an unconstrained agent yet chose to attend s_1, that no other agent with ability lower than a_1 would choose a school of quality greater than s_1, even if she could afford it. Similarly, we can define a_{s_1} (the lowest ability agent who can afford to attend s_1) by $e(F^{-1}(s_1), a_{s_1}) = 1$, as well as $s_2 = S(a_{s_1}, 1)$ (the school attended by that agent), and the associated set $Q(s_2)$ (i.e., the set of agents who can afford s_2). Using the same logic as before, we know that all agents $(a, 1) \in Q(s_2)$ are effectively unconstrained. Thus, by iteration we can define a sequence of sets $Q(1), Q(s_1), ..., Q(s_j), ...$, such that any agent $(a, 1) \in Q(s_j)$ is effectively unconstrained for all j. This iterative process, illustrated in Figure 5, converges to some school s_k (and associated set $Q(s_k)$) such that $S(0,1) = s_k$.[21] Hence, from this we can conclude that all agents $(a, 1)$, for all $a \in I$, are effectively unconstrained and that, consequently, the following relationship must be observed: $X_s(a, S(a,1)) = e_v(F^{-1}(S(a,1), a)\ F^{-1\prime}(S(a,1)), \forall a \in [0,1]$.

The preceding discussion allows us to define a function $\mathbf{S} : I \rightarrow I$, as $\mathbf{S}(a) \equiv S(a,1), \forall a \in [0,1]$. Note that any agent (a, w) with $w \geq e(F^{-1}(\mathbf{S}(a)), a)$ will be able to afford the school chosen by the agent with the same ability and highest wealth (and who was shown to be unconstrained). Thus the former will also be effectively unconstrained and choose $\mathbf{S}(a)$ as well. Hence, $\mathbf{S}(a)$ represents the quality of the school attended by all agents with ability a who are unconstrained. Its properties are given in the following lemma.

Lemma 4.11. *(i)* $\mathbf{S}$ *is strictly increasing, (ii)* $\mathbf{S}(0) = 0$, *(iii)* $\mathbf{S}(1) = 1$, *(iv)* $\mathbf{S}$ *is continuous, (v)* $\mathbf{S}$ *is (a.e.) differentiable.*

Proof: See Appendix

Let $\underline{a}$: $I \rightarrow I$ be the inverse function of $\mathbf{S}$. Thus, $\underline{a}\ (s)$ represents the ability of the unconstrained agents attending school s (who are, in turn, the lowest ability agents attending that school, since any agent with ability $a' < a$ will prefer a lower quality school, regardless of her wealth). The properties of $\underline{a}$, stated as a corollary of the preceding lemma, follow directly from those of $\mathbf{S}$.

Corollary 4.12. *(i)* $\underline{a}$ *is strictly increasing, (ii)* $\underline{a}\ (0) = 0$, *(iii)* $\underline{a}\ (1) = 1$, *(iv)* $\underline{a}$ *is continuous, (v)* $\underline{a}$ *is (a.e.) differentiable.*

Proof: See Appendix.

Notice that differentiating the first-order condition (given by $X_s(\underline{a}(s), s) = e_v(F^{-1}(s), \underline{a}(s))\ F^{-1\prime}(s))$ we obtain $\underline{a}'\ (s) = -\frac{X_{ss}\ - e_{vv}\ (F^{-1\prime})^2 - e_v\ (F^{-1})''}{X_{as} - e_{va}\ (F^{-1})'} > 0$, where the numerator is the second-order condition of the agent's maximization problem and its strict negativity is guaranteed by the previous corollary.

[21] The assertion that the sequence cannot, by implication, converge to some $a_{s'} > 0$, follows from the fact that this would imply that $\lim_{s \rightarrow s'} \frac{da_s}{ds} = 0$. But, a_s is defined implicitly by $e(F^{-1}(s), a_s) = 1$, yielding $\lim_{s \rightarrow s'} \frac{da_s}{ds} = \frac{e_v(F^{-1}(s'), a_{s'})F^{-1\prime}(s')}{e_a(F^{-1}(s'), a_{s'},)} = \frac{X_s(a_{s'}, s')}{e_a(F^{-1}(s'), a_{s'},)} > 0, \forall a \in (0,1]$.

Corollary 4.13. *$e_v(F^{-1}(s), \underline{a}(s))F^{-1\prime}(s)$ is continuous and (a.e.) differentiable.*

Proof: See Appendix.

We can also show that the participation constraint will hold with strict inequality for all agents (but for the subset of zero measure attending school $s = 0$). Formally, $\forall (a,w) \in I^2$ such that $S(a,w) > 0$, we have $X(a, S(a,w)) = X(a,0) + \int_0^{S(a,w)} X_s(a,z)\,dz > \int_0^{S(a,w)} X_s(\underline{a}(z),z)\,dz = \int_0^{S(a,w)} e_v(F^{-1}(z), \underline{a}(z))\,F^{-1\prime}(z)\,dz = e(F^{-1}(S(a,w)), a)$.

Define the sets $R(s) = \{(a,w),\ a = \underline{a}(s),\ w \geq e(F^{-1}(s), \underline{a}(s))\}$ and $T(s) = \{(a,w),\ w = e(F^{-1}(s), a),\ a > \underline{a}(s)\}$. The union of these two sets is the boundary of the shaded area in Figure 4. The next proposition establishes that the set of agents allocated to any given school s in equilibrium will be given by the union of those sets.

Proposition 4.14. *$S(a,w) = s$ if and only if $(a,w) \in R(s) \cup T(s)$*

Proof: See Appendix.

Notice that feasibility of S (market clearing) requires $1 - s = \int_{\underline{a}(s)}^1 [1 - e(F^{-1}(s), a)]\,da$ or, more compactly,

$$\underline{a}(s) = s - \int_{\underline{a}(s)}^{1} e(F^{-1}(s), a)\ da, \quad \forall s \in I \tag{4.12}$$

It follows from (4.12) and the properties of $\underline{a}(s)$ derived above that $\underline{a}(s) < s$, $\forall s \in (0,1)$.

Define $\mathbf{e}(a) \equiv e(F^{-1}(\mathbf{S}(a)), a)$, i.e., the equilibrium level of expenditures by agents of ability a who are unconstrained. Accordingly, $\mathbf{e}(\underline{a}(s)) = e(F^{-1}(s), \underline{a}(s))$ gives the level of expenditures undertaken by each agent in set $R(s)$ (i.e., the lowest ability agents attending school s). Notice that $\mathbf{S}(0) = 0$ and our assumptions on V imply that $\mathbf{e}(0) = 0$, i.e., agents with zero ability incur zero expenditures.

The first-order condition and the tournament allocation rule can now be rewritten, respectively, as

$$X_s(\underline{a}(s), s) = e_v(F^{-1}(s), \underline{a}(s))\ F^{-1\prime}(s) \tag{4.13}$$

$$s = F(V(\underline{a}(s), \mathbf{e}(\underline{a}(s)))) \tag{4.14}$$

Differentiation of (4.12), combined with (4.13) yields

$$\underline{a}'(s) = \frac{1 - X_s(\underline{a}(s), s) \int_{\underline{a}(s)}^{1} \frac{e_v(F^{-1}(s),a)}{e_v(F^{-1}(s),\underline{a}(s))} \, da}{1 - \mathbf{e}(\underline{a}(s))} > 0 \quad \forall s \in I \tag{4.15}$$

where the sign of the inequality follows from our assumption of $X_s < 1$ and the fact that $\int_{\underline{a}(s)}^{1} \frac{e_v(F^{-1}(s),a)}{e_v(F^{-1}(s),\underline{a}(s))} \, da \leq 1$ (given $e_{va} \leq 0$).

Finally, differentiating (4.14) to obtain $1 = F' \, (V_a + V_e e') \, \underline{a}'$ and using (4.15) together with a change of variables, we obtain the following differential equation for the equilibrium signaling expenditures $\mathbf{e}$:

$$\mathbf{e}'(a) = \frac{1 - \mathbf{e}(a)}{\frac{1}{X_s(a,\mathbf{S}(a))} - \int_a^1 \frac{e_v(F^{-1}(\mathbf{S}(a)),z)}{e_v(F^{-1}(\mathbf{S}(a)),a)} \, dz} - \frac{V_a(a, \mathbf{e}(a))}{V_e(a, \mathbf{e}(a))} \tag{4.16}$$

defined $\forall a \in I$, and with the boundary condition $\mathbf{e}(0) = 0$.[22]

4.3. Markets vs. Tournaments with Borrowing Constraints

In section 3 we showed that, when agents have access to perfect capital markets, both market prices and tournaments deliver the same (efficient) allocations and hence the same level of aggregate output in equilibrium. Aggregate consumption, however, was always lower under tournaments because of the resources wasted in the signaling process. In this section we show that in the presence of borrowing constraints those results no longer hold. Instead, with borrowing constraints the allocation achieved by tournaments always delivers higher aggregate output than does the market system. Furthermore, for signaling technologies that are sufficiently responsive to ability variations (relative to variations in expenditures) in a sense that will be made precise below, aggregate consumption will also be higher under tournaments.

Before we formalize and prove the above claims, we state a key difference between the equilibrium allocations associated with the two mechanisms considered. Let $\underline{a}^m(s)$ and $\underline{a}^t(s)$ denote the lowest ability agents attending school s under markets and tournaments, respectively. We have,

Lemma 4.15. $\underline{a}^m(s) < \underline{a}^t(s)$, $\forall \; s \in (0, 1)$.

[22] Our assumptions and previous results guarantee the continuity of the function given by the right hand side of (4.16) as well as its derivative with respect to e, which in turn guarantees the existence and uniqueness of a solution to the differential equation (see, e.g., Boyce and DiPrima (1992), Theorem 2.11.1).

Proof: See Appendix.

The lemma, therefore, establishes that the lowest ability agent among those allocated to a school of a given quality has always higher ability under tournaments than under markets. In other words, schools are "more diverse" in terms of ability under markets than under tournaments. On the other hand, schools are more diverse in terms of wealth levels under tournaments than under markets. Loosely speaking, both observations follow from the fact that low ability, high wealth individuals find it easier to "outbid" high ability, low wealth individuals under markets (where the same price applies to everyone) than under tournaments (where the effective price is decreasing in ability). Since the efficient allocation involves zero ability variation within schools, the previous result already suggests that, in the presence of borrowing constraints, the equilibrium allocation may be "closer" to the efficient one under tournaments than under markets.

Before we formalize the intuition expressed above, we first establish that the joint distribution of (a, s) generated by the tournament equilibrium always places greater probability mass in the north-east region of its support than the corresponding equilibrium distribution under markets. This, in turn, implies that the value of the cumulative distribution function at any $(a, s) \in I^2$ under tournaments is never below that under markets.

Lemma 4.16. *Let* $\Gamma^i(a,s) = \int_a^1 \int_0^1 \mathbf{1}[S^i(a,w) - s)]\, dw\, da$, *for* $i = m, t$. *Then* $\Gamma^t(a,s) \geq \Gamma^m(a,s)$ *for all* $(a,s) \in I^2$, *with strict inequality* $\forall\ a > \underline{a}^m(s)$.

Lemma 4.17. *Let* $\Phi^i(a,s) = \int_0^a \int_0^1 \mathbf{1}[s - S^i(z,w)]\ dw\ dz$, *for* $i = m, t$. *Then* $\Phi^t(a,s) \geq \Phi^m(a,s)$ *for all* $(a,s) \in I^2$, *with strict inequality* $\forall\ a > \underline{a}^m(s)$.

Proofs: See Appendix.

We are now in a position to state the main result of the paper. Let $Y_\circ^m = \int_0^1 \int_0^1 X(a, S^m(a,w))\ da\ dw$ and $Y_\circ^t = \int_0^1 \int_0^1 X(a, S^t(a,w))\ da\ dw$ denote aggregate output under markets and tournaments, respectively, in the presence of borrowing constraints.

Theorem 4.18. $Y_\circ^t > Y_\circ^m$

Proof: See Appendix.

Thus, the level of aggregate output generated in the tournament equilibrium is always strictly greater than the corresponding output level under markets. The intuition for this result is as follows: with a market mechanism, identical expenditures by different agents generate the same outcome–that is, individuals are allocated to the same school since prices do not discriminate among individuals except with respect to their willingness and ability to pay. In a tournament, on the other hand, identical expenditures by non-identical agents do not lead to identical outcomes. In particular, by spending the same amount higher-ability individuals produce higher scores/signals than lower-ability individuals. This implies that, ceteris paribus, higher-ability individuals are, effectively, "less credit constrained" than lower-ability agents. This serves to enhance allocative efficiency since a social planner would wish to allocate the higher-ability individual to the higher-quality school.

The dominance of tournaments over markets with respect to output, however, does not necessarily carry over to aggregate consumption, since a fraction of aggregate output in the tournament equilibrium is wasted in signaling activities. Next we analyze the conditions under which the equilibrium allocation under tournaments will also be associated with higher aggregate consumption.

Given an arbitrary signaling technology represented by V, define $\pi(a,e) \equiv \frac{V_a(a,e)}{V_e(a,e)}$, and consider the family of signaling technologies $\{V^\theta : I \times \Re_+,\ s.t.\ \frac{V_a^\theta(a,e)}{V_e^\theta(a,e)} = \theta\ \pi(a,e), \theta \in \Re_+\}$. Let $\mathbf{e}^\theta(a)$ denote the expenditure of unconstrained agents with ability a in the tournament equilibrium under borrowing constraints when the signaling technology is V^θ.

Lemma 4.19. $\lim_{\theta \to +\infty} \int_0^1 \mathbf{e}^\theta(a)\ da = 0.$

Proof: See Appendix.

The preceding lemma states that by increasing the sensitivity of the signaling technology to variations in ability (relative to variations in expenditures), aggregate signal expenditures become arbitrarily close to zero. The next theorem establishes that increased "power" of the signaling technology brings the equilibrium allocation under tournaments arbitrarily close to the efficient allocation (i.e., the one which maximizes aggregate output).

Theorem 4.20. *Given any $\epsilon > 0$, there exists a $\theta^* \in \Re_+$ such that, if $\theta > \theta^*$, then $s - \underline{a}^\theta(s) \leq \epsilon$ almost everywhere (a.e.) in the unit interval .*

Proof: See Appendix.

Thus, by increasing θ sufficiently, the equilibrium allocation under tournaments is arbitrarily close to having the *perfect positively assortative matching* property–given by $S(a, w) = a, \forall a \in I$–that characterizes the efficient allocation. Combining the results of the previous lemma and theorem we can establish the existence of a threshold value for θ, beyond which tournaments will dominate markets in terms of aggregate consumption. Let $C_{\circ}^{t_\theta}$ and $Y_{\circ}^{t_\theta}$ denote, respectively, aggregate consumption and aggregate output in an equilibrium with tournaments and borrowing constraints when the signaling technology is V^{θ}. As before $C_{\circ}^{m}$, and $Y_{\circ}^{m}$ are the corresponding counterparts under markets and borrowing constraints.

Corollary 4.21. *There exists a* $\theta^* \in \Re_+$ *such that* $C_{\circ}^{t_\theta} > C_{\circ}^{m}, \forall \theta > \theta^*$.

Proof: See Appendix.

5. Summary and Conclusions

In this paper we have studied a version of the matching problem, focusing on the impact of borrowing constraints on equilibrium allocations under two alternative allocative mechanisms, markets and tournaments. With perfect capital markets both mechanisms achieve the efficient allocation, characterized by positively assortative matching, though markets generate higher aggregate consumption because of the waste associated with the signaling process under tournaments. When borrowing constraints are present, tournaments are shown to dominate markets in terms of aggregate output and, for sufficiently powerful signaling technologies, also in terms of aggregate consumption. In the latter case, the consumption losses resulting from signaling waste are more than offset by the efficiency gains that arise from the fact that the allocation under tournaments is "closer" to the efficient one.

We have not attempted to find the optimal mechanism in this environment. It would be of interest to characterize it and to require that it be robust, as both the price mechanism and tournament are, to allowing ignorance on the part of the "social planner" of such things as the exact utility functions, income distribution, and ability distribution in the economy. Another direction for future work would be to compare how a tournament does against more traditional interventions in the credit market.

There are many other interesting questions that one could address in the framework proposed in this paper. What are the implications for the income distribution under both systems? Which agents would prefer which system? Perhaps not surprisingly, our model raises a number of issues that are interesting from a political economy perspective. In particular, the heterogeneity underlying our economy implies that reforms that would be necessary to achieve a more efficient allocation of resources may be difficult to implement politically (say, under a majority voting scheme) in the absence of suitable compensating transfers. Such reforms may include policies that tend to alleviate or eliminate the effects of borrowing constraints, the substitution of tournaments for markets as an allocative mechanism, as well as the choice of a more powerful signaling technology (when several are available).

6. Appendix

6.1. Proofs for Section 2.1

Proof of Lemma 2.1: Integration by parts of (2.1) with respect to x_2 yields $\Delta W = \int_0^{b_1}[U(x_1,x_2)\ \int_0^{b_2} \Delta f(x_1,x_2)\ dx_2 - \int_0^{b_2} U_2(x_1,x_2) \int_0^{x_2} \Delta f(x_1,t)\ dtdx_2]\ dx_1$. Integrating again by parts, this time with respect to x_1 yields, $\Delta W = -\int_0^{b_1} U_1(x_1,b_2)\ \Delta F_1(x_1)\ dx_1$ $- \int_0^{b_2} U_2(b_1,x_2)\ \Delta F_2(x_2)\ dx_2\ + \int_0^{b_1}\int_0^{b_2} U_{12}(x_1,x_2)\ \Delta F(x_1,x_2)\ dx_2\ dx_1$

Recalling that the two distributions have identical marginals $\Delta F_1(x_1) = \Delta F_2(x_2) = 0$, the above expression reduces to:

$$\Delta W = \int_0^{b_1}\int_0^{b_2} U_{12}(x_1,x_2)\ \Delta F(x_1,x_2)\ dx_2 dx_1 \geq 0 \qquad (6.1)$$

Proof of Theorem 2.2: From (6.1) we have $\int_0^1\int_0^1 X(a,s)\ [d\Phi^*(a,s) - d\Phi(a,s)]\ da\ ds = \int_0^1\int_0^1 X_{as}(a,s)\ [\Phi^*(a,s) - \Phi(a,s)]\ dads$. Since $X_{as} > 0$, in order to show that output is higher under S^* than under S we simply need to show that $\Phi^*(a,s) - \Phi(a,s) \geq 0, \forall(a,s)$. But, $\Phi(a,s) \leq \Phi(1,s) = s$ and $\Phi(a,s) \leq \Phi(a,1) = a$, hence $\Phi(a,s) \leq \min[a,s] = \Phi^*(a,s)$, $\forall(a,s)$ and since $\Phi^*(a,s) \neq \Phi(a,s)$ for some (a,s), then we obtain a strict inequality.

6.2. Proofs for Section 4.1

Proof of Lemma 4.1: Suppose $P(0) = \epsilon > 0$. Then there would be a subset of agents $\{(a,w) \in I^2, 0 \leq w < \epsilon\}$ with positive measure ϵ whom would not be able to afford any school. But that implies that the measure of agents allocated to schools in equilibrium would be strictly less than the measure of total school capacity, which is inconsistent with market clearing.

Proof of Lemma 4.2: Continuity of P and market clearing trivially rule out $P(s) > 1$. Suppose that $P(s) = 1$ for some $s \in [0,1]$. Let (a,w) be such that $S(a,w) = s$. Revealed preference and the fact that $P(0) = 0$ then imply $X(a,s) - 1 \geq X(a,0)$. But $X(a,s) - 1 = X(a,0) + \int_0^s X_s(a,z)\ dz - 1 < X(a,0) - s \leq X(a,0)$, where the strict inequality follows from the assumption $X_s(a,s) < 1$, $\forall(a,s) \in I^2$. The resulting contradiction implies that $P(s) = 1$ cannot hold either.

Proof of Lemma 4.3: (i) Suppose that there did exist some $(a, w), (a', w') \in Q$, with $a > a'$ such that $s = S(a, w) \leq S(a', w') = s'$. By revealed preference $X(a, s) - P(s) \geq X(a, s') - P(s')$ and $X(a', s') - P(s') \geq X(a', s) - P(s)$. Combining both inequalities we obtain $X(a', s') - X(a', s) \geq X(a, s') - X(a, s)$, which rules out $s < s'$ since the latter would make the expression inconsistent with the assumption $X_{as} > 0$ (note that this is an algebraic restatement of the single-crossing logic developed earlier). What if $s = s'$? By construction both agents are unconstrained so their FOC must be satisfied at s and s', i.e., $X_s(a, s) = P'(s)$ and $X_s(a', s) = P'(s)$. But that implies $X_s(a, s) = X_s(a', s)$ which can hold only if $a = a'$.

(ii) Suppose first that $\overline{s} = S(a, w) \neq S(a, w'') = \underline{s}$ for two agents $(a, w), (a, w'') \in Q$, and $\overline{s} > \underline{s}$. Then both must be indifferent between $\overline{s}$ and $\underline{s}$ and, since unconstrained, the FOC must hold for both of them. Note that no other $(a', w') \in Q$, $a' \neq a$, can attend any $s \in [\underline{s}, \overline{s}]$ since this would contradict part (i). Furthermore, any (a', w') such that $w' \in (P(\overline{s}), P(1))$ and $a' \neq a$ also does not attend any $s \in (\underline{s}, \overline{s})$ since, given that a is indifferent between $\overline{s}$ and $\underline{s}$, the single-crossing condition implies that $a' > a$ strictly prefers $\overline{s}$ to any $s \in [\underline{s}, \overline{s})$ and that $a' < a$ strictly prefers $\underline{s}$ to any $s \in (\underline{s}, \overline{s}]$. Who then attends the subset of schools $(\underline{s}, \overline{s})$? The above reasoning and market clearing imply that, aside from agents with ability a (who have no positive measure), it must be a subset of those agents z with $w \in [P(\underline{s}), P(\overline{s})]$ who attend $s \in [\underline{s}, \overline{s}]$. Thus market clearing would require $\overline{s} - \underline{s} \leq P(\overline{s}) - P(\underline{s})$. But the indifference of (a, w) between $\overline{s}$ and $\underline{s}$ implies that $P(\overline{s}) - P(\underline{s}) = X(a, \overline{s}) - X(a, \underline{s}) = \int_{\underline{s}}^{\overline{s}} X_s(a, s)\, ds < \overline{s} - \underline{s}$, given $X_s(a, s) < 1$, $\forall (a, s) \in I^2$. Thus, $\overline{s} - \underline{s} \leq P(\overline{s}) - P(\underline{s}) < \overline{s} - \underline{s}$, which clearly cannot be satisfied. Hence, it must be the case that $S(a, w) = S(a, w'')$ for any two agents $(a, w), (a, w'') \in Q$. Note that the result extends to any agent (a, w'') with $P(S(a, w)) \leq w'' \leq P(1)$, since she too can afford $S(a, w)$.

Proof of Lemma 4.4: i) is an immediate consequence of Lemma 4.3. (ii) $\mathbf{S}(0) = 0$ follows from the participation constraint, given that $P(s) > X(0, s) = 0, \forall s \in (0, 1]$. (iii) follows directly from the strict monotonicity of $\mathbf{S}$ and market clearing. (iv) Suppose that $\underline{s} = \lim_{z \to a_-} \mathbf{S}(z) < \lim_{z \to a_+} \mathbf{S}(z) = \overline{s}$; monotonicity of $\mathbf{S}$ implies that no agent in Q will be attending schools in the interval $(\underline{s}, \overline{s})$ and from the previous lemma the same is true for any agent with wealth $w \in [P(\overline{s}), P(1)]$. Market clearing thus implies $\overline{s} - \underline{s} \leq P(\overline{s}) - P(\underline{s})$, and the same reasoning as in Lemma 4.3 leads to a contradiction. Thus, $\lim_{z \to a_-} \mathbf{S}(z) = \lim_{z \to a_+} \mathbf{S}(z)$, which, combined with monotonicity, implies that $\mathbf{S}$ is continuous. (v) follows from (i) and the boundedness of the range of **S**.

Proof of Corollary 4.6: Follows from continuity and differentiability of $\underline{a}\ (s)$ and from the fact that $P'(s) = X_s(\underline{a}\ (s), s)$ holds $\forall(a, w) \in Q$.

Proof of Proposition 4.7: Above we showed that all agents in $R(s)$ are assigned to school s. Who else attends s ? Note that any $(a, w) \in Q$ does not attend s unless $(a, w) \in R(s)$. Also, trivially, agents with $w \in [0, P(s))$ are ruled out, because they cannot afford it. Consider agents with $w \in (P(s), P(1))$, and $a > \underline{a}\ (s)$. We know that $X_s(\underline{a}\ (s), s) = P'(s)$, so it must be the case that $X_s(a, s) > P'(s)$ for those agents. Consequently, they would always find it optimal to switch to a school $s' > s$, since for s' sufficiently close to s the school would be affordable and the participation constraint would remain satisfied. So these agents do not attend s. Consider next agents with $w \in (P(s), P(1))$, and $a < \underline{a}\ (s)$. Those agents cannot find it optimal to attend s since in their case $X_s(a, s) < P'(s)$, implying that they would be better off at a lower ranked school. Consider, finally, agents in $T(s)$. They cannot afford a school of quality greater than s, so they will choose to attend some $s' \in [0, s]$. The same single crossing logic used in Lemma 4.3 implies that if agent $(\underline{a}\ (s), 1)$ prefers s to any $s \in [0, s]$ then so must all $a > \underline{a}\ (s)$. Consequently, we can conclude that $S(a, w) = s$ if and only if $(a, w) \in R(s) \cup T(s)$.

6.3. Proofs for Section 4.2

Proof of Lemma 4.8: Suppose $e(F^{-1}(0), \widehat{a}) = \epsilon > 0$ for some $\widehat{a} \in I$. Then there would be a subset of agents $\{(a, w) \in I^2 : 0 \leq w < e(F^{-1}(0), a)\}$ with positive measure whom would not be able to afford any school. But that implies that the measure of agents allocated to schools in equilibrium would be strictly less than the measure of school capacity, which is inconsistent with the definition of a feasible allocation.

Proof of Lemma 4.9: Suppose that $e(F^{-1}(S(a, w)), a) \geq 1$ for some $s \in [0, 1]$. Take (a, w) such that $S(a, w) = s$. Revealed preference then implies $X(a, s) - e(F^{-1}(s), a) \geq X(a, 0)$. But $X(a, s) - e(F^{-1}(s), a) = X(a, 0) + \int_0^s X_s(a, z)\ dz - e(F^{-1}(s), a) \leq X(a, 0) + \int_0^s X_s(a, z)\ dz - 1 < X(a, 0) - 1 + s \leq$

$X(a,0)$, a contradiction (where the strict inequality follows from our assumption $X_s(a,s) < 1, \forall (a,s) \in I^2$).

Proof of Lemma 4.10: (i) The proof of this follows the same single-crossing logic of the equivalent lemma in the markets section, with $e(F^{-1}(s),a)$ replacing $P(s)$. Hence, we omit it here. (ii) Suppose instead that $\overline{s} = S(a,w) > S(a,w') = \underline{s}$ for some a. Then both $(a,w),(a,w')$ must be indifferent between both schools and, since unconstrained, the first-order condition must hold in both cases. Note that no other agent $(a'',w'') \in Q(1)$, $a'' \neq a$, attends any $s \in [\underline{s},\overline{s}]$ since this would contradict part (i). Furthermore, any (a'',w'') such that $w'' \in (e(F^{-1}(\overline{s}),a''), e(F^{-1}(1),a''))$ and $a'' \neq a$ also does not attend any $s \in (\underline{s},\overline{s})$ since, given that (a,w) is indifferent between $\overline{s}$ and $\underline{s}$, the single-crossing condition implies that if $a'' > a$ our agent (a'',w'') would strictly prefer $\overline{s}$ (which she can afford) to any $s \in [\underline{s},\overline{s})$, and that if $a'' < a$ she would strictly prefer $\underline{s}$ to any $s \in (\underline{s},\overline{s}]$. Then, who could possibly attend schools in the interval $[\underline{s},\overline{s}]$? The above reasoning and market clearing imply that aside from agents with ability a (who have zero measure), those schools must be filled by a subset of $\{(a'',w'') \in I^2,\ w \in [e(F^{-1}(\underline{s}),a''), e(F^{-1}(\overline{s}),a'')]\}$. So, it follows from market clearing that $\overline{s} - \underline{s} \leq \int_0^1 [e(F^{-1}(\overline{s}),z) - e(F^{-1}(\underline{s}),z)]\ dz = \int_0^1 \int_{\underline{s}}^{\overline{s}} e_v(F^{-1}(s),z)\ F^{-1\prime}(s)\ ds\ dz \leq \int_{\underline{s}}^{\overline{s}} X_s(1,s)\ ds < (\overline{s} - \underline{s})$, a contradiction (where, in obtaining the last two inequalities we made use of (4.10), $X_{as} > 0$ and $X_s < 1$).

Proof of Lemma 4.11: The proof is similar to its equivalent in the market case and hence we omit it.

Proof of Corollary 4.12: Ibid.

Proof of Corollary 4.13: Ibid.

Proof of Proposition 4.14: Above we showed that all agents in $R(s)$ are assigned to school s. Who else attends s? Note that any $(a,w) \in Q(1)$ (and thus with $w \geq e(F^{-1}(1),a))$ does not attend s unless $(a,w) \in R(s)$. Trivially, any agent (a,w) with $w \in [0, e(F^{-1}(s),a))$ is ruled out, because she cannot afford it. Consider an agent (a,w) with $w \in (e(F^{-1}(s),a), e(F^{-1}(1),a))$, and $a > \underline{a}\ (s)$. We know that $X_s(\underline{a}\ (s),s) = e_v(F^{-1}(s),\underline{a}(s))\ F^{-1\prime}(s)$, so the strict negativity of the second-order condition guarantees that $X_s(a,s) > e_v(F^{-1}(s),a)\ F^{-1\prime}(s)$ for those agents. Consequently, they would always find it optimal to switch to a school $s' > s$, since for s' sufficiently close to s the school would be affordable and the participation constraint would remain satisfied. Thus these agents do not attend s. Consider next agents with $w \in [e(F^{-1}(s), a), e(F^{-1}(1), a))$, and $a < \underline{a}\ (s)$. Those agents cannot find it optimal to attend s since in their case $X_s(a, s) < e_v(F^{-1}(s), a)\ F^{-1\prime}(s)$, implying that they would be better off at a lower ranked school. Consider, finally, agents in $T(s)$. They cannot afford a school of quality greater than s, so they will choose to attend some $s' \in [0, s]$. But the same single-crossing logic used in Lemma 4.10 implies that if agent $(\underline{a}\ (s), 1)$ prefers s to any $s \in [0, s]$ then so must all $a > \underline{a}\ (s)$. Consequently, we can conclude that $S(a, w) = s$ if and only if $(a, w) \in R(s) \cup T(s)$.

6.4. Proofs for Section 4.3

Proof of Lemma 4.15: Totally differentiating (4.4) yields

$$\underline{a}^{m\prime}(s) = \frac{1 - P'(s)\ (1 - \underline{a}^m(s))}{1 - P(s)} \tag{6.2}$$

which evaluated at $s = 0$ yields $\underline{a}^{m\prime}(0) = 1 - P'(0)$.

Evaluating (4.15) at $s = 0$, we obtain:

$$\begin{aligned} \underline{a}^{t\prime}(0) &= 1 - X_s(0,0) \int_0^1 \frac{e_v(F^{-1}(0), z)}{e_v(F^{-1}(0), 0)}\, dz \\ &\geq 1 - X_s(0,0) = 1 - P'(0) = \underline{a}^{m\prime}(0) \end{aligned}$$

where the inequality follows from our assumption that $e_{va} \leq 0$ and hence that $\frac{e_v(F^{-1}(0),z)}{e_v(F^{-1}(0),0)} \leq 1$.

Suppose next that, contrary to the statement of the Lemma, $\underline{a}^m(s') > \underline{a}^t(s')$ held for some $s' \in (0,1)$. Given $\underline{a}^t(0) = \underline{a}^m(0) = 0$ and the result that $\underline{a}^{t\prime}(0) > \underline{a}^{m\prime}(0)$, continuity implies that there must then exist some $s^* \in (0, s')$ such that $\underline{a}^m(s^*) = \underline{a}^t(s^*)$ and $\underline{a}^{m\prime}(s^*) \geq \underline{a}^{t\prime}(s^*)$. Furthermore, it must be the case that $\mathbf{e}(\underline{a}^t(s^*)) > P(s^*)$, since otherwise, given $V_a > 0$ the mass of agents in schools ranked s and greater would be larger under tournaments than under markets. Thus,

$$\begin{aligned} \underline{a}^{t\prime}(s^*) &= \frac{1 - X_s(\underline{a}^t(s^*), s^*) \int_{\underline{a}^t(s^*)}^1 \frac{e_v(F^{-1}(s^*), z)}{e_v(F^{-1}(s^*), \underline{a}^t(s^*))}\, dz}{1 - \mathbf{e}(\underline{a}^t(s^*))} \\ &> \frac{1 - P'(s^*)\ (1 - \underline{a}^m(s^*))}{1 - P(s^*)} = \underline{a}^{m\prime}(s^*) \end{aligned}$$

but this contradicts the previous requirement that $\underline{a}^{m\prime}(s^*) \geq \underline{a}^{t\prime}(s^*)$.

Proof of Lemma 4.16: Consider first the case of $a \leq \underline{a}^m(s)$. From Lemma 4.15 we have $a \leq \underline{a}^t(s)$. Thus, clearly $\Gamma^m(a,s) = \Gamma^t(a,s) = 1-s$ in that case. Next, suppose that $\underline{a}^m(s) < a < \underline{a}^t(s)$. Then $\Gamma^m(a,s) < 1-s = \Gamma^t(a,s)$. Finally, let $a > \underline{a}^t(s)$. In that case we have $\Gamma^t(a,s) - \Gamma^m(a,s) = \int_a^1 [P(s) - e(F^{-1}(s),a)]\ da > 0$, where the inequality follows from the fact that $\int_{\underline{a}^t(s)}^1 [P(s) - e(F^{-1}(s),a)]\ da = \Gamma^t(\underline{a}^t(s),s) - \Gamma^m(\underline{a}^t(s),s) > 0$, combined with the observation that $P(s) - e(F^{-1}(s),a)$ is strictly increasing in a for $a \geq \underline{a}^t(s)$. Note that since $\underline{a}^m(s) < 1, \forall\ s \in [0,1)$, this implies that the set $\{a \in I : \underline{a}^m(s) < a \leq 1\}$ has a positive measure for all $s \in [0,1)$.

Proof of Lemma 4.17: $\Gamma^i(a,s) = 1 - \Phi^i(a,1) - \Phi^i(1,s) + \Phi^i(a,s)$, for $i = m,t$. Furthermore, feasibility of S (market clearing) implies identical marginal distributions for markets and tournaments, i.e., $\Phi^t(a,1) = \Phi^m(a,1) = a$, and $\Phi^t(1,s) = \Phi^m(1,s) = s$. It then follows from lemma 4.16 that $\Phi^t(a,s) - \Phi^m(a,s) = \Gamma^t(a,s) - \Gamma^m(a,s) \geq 0$, for all $(a,s) \in I^2$, with strict inequality if $a > \underline{a}^m(s)$.

Proof of Theorem 4.18: Using the joint c.d.f. for (a,s) introduced above we can write $Y^i = \int_0^1 \int_0^1 X(a,s)\ d\Phi^i(a,s)\ da\ ds$, for $i = m,t$. Accordingly,

$$Y^t - Y^m = \int_0^1 \int_0^1 X(a,s)\ [d\Phi^t(a,s) - d\Phi^m(a,s)]\ da\ ds$$

Since X and Φ^i satisfy the assumptions needed to apply the Levy-Paroush result, we have:

$$Y^t - Y^m = \int_0^1 \int_0^1 X_{as}(a,s)\ [\Phi^t(a,s) - \Phi^m(a,s)]\ da\ ds > 0$$

(where the inequality follows from our assumption $X_{as} > 0$ and corollary 4.17).

Proof of Lemma 4.19: The proof involves two steps. First we derive an upper bound for $\int_0^1 \mathbf{e}^\theta(a)\ da$, given θ. Then we show that this upper bound becomes arbitrarily small as $\theta \to +\infty$.

Define the set $D^\theta = \{a \in I : \mathbf{e}^\theta(a) \geq \epsilon\}$. Notice that $D^\theta \subset (0,1]$, since $\mathbf{e}^\theta(0) = 0$. Notice also that, $\forall\theta$, continuity of $\mathbf{e}^\theta(a)$ implies the existence of a partition $\{D_1^\theta, D_2^\theta, ..., D_{N(\theta)}^\theta\}$, where $D_i^\theta = [\underline{d}_i^\theta, \overline{d}_i^\theta]$, with measure given by $\mu(D_i^\theta) = \overline{d}_i^\theta - \underline{d}_i^\theta$, and such that $D^\theta = \cup_i D_i^\theta$ and $D_i^\theta \cap D_j^\theta = \emptyset$, if $j \neq i$. Let $\alpha \equiv \frac{\overline{X}_s}{1-\overline{X}_s}$, where

$\overline{X}_s = \sup_{(a,s)\in I^2} X_s(a,s) < 1$. Note that we can write

$$\begin{aligned}
\int_{D_i^\theta} \mathbf{e}^\theta(a)\ da &= \int_{D_i^\theta} (\epsilon + \int_{\underline{d}_i^\theta}^{a} \mathbf{e}^{\theta\prime}(z)\ dz)\ da \\
&= \epsilon\ \mu(D_i^\theta) + \int_{D_i^\theta}\int_{\underline{d}_i^\theta}^{a}[\frac{1-\mathbf{e}^\theta(z)}{\frac{1}{X_s(z,\mathbf{S}^\theta(z))} - \int_z^1 \frac{e_v(v,u)}{e_v(v,z)}du} - \theta\ \pi(z,\mathbf{e}^\theta(z))]\ dz\ da \\
&< \epsilon\ \mu(D_i^\theta) + \int_{D_i^\theta}\int_{\underline{d}_i^\theta}^{a}(\alpha - \theta\ \pi(z,\mathbf{e}^\theta(z)))\ dz\ da \\
&< (\epsilon+\alpha)\ \mu(D_i^\theta) - \theta\ Q_i^\theta \\
&< (\epsilon+\alpha)\ - \theta\ Q_i^\theta
\end{aligned}$$

where $Q_i^\theta \equiv \int_{D_i^\theta}\int_{\underline{d}_i^\theta}^{a} \pi(z,\mathbf{e}^\theta(z))\ dz\ da$. By construction, both arguments of π in the previous integral take values that are bounded away from zero, thus implying that $Q_i^\theta \geq 0$, with equality holding if and only if $\mu(D_i^\theta) = 0$. Combining the fact that $\int_0^1 \mathbf{e}^\theta(a)\ da < \sum_{i=1}^{N(\theta)} \int_{D_i^\theta} \mathbf{e}^\theta(a)\ da + \epsilon\ (1-\mu(D^\theta))$ with the previous result we have

$$\int_0^1 \mathbf{e}^\theta(a)\ da < \epsilon + \alpha\ - \theta\ \sum_{i=1}^{N(\theta)} Q_i^\theta$$

Since $\mathbf{e}^\theta(a)$ is non-negative for all θ, it follows that

$$\theta\ \sum_{i=1}^{N(\theta)} Q_i^\theta < \epsilon + \alpha$$

Notice that the previous inequality requires that $\lim_{\theta\to+\infty} \sum_{i=1}^{N(\theta)} Q_i^\theta = 0$, which in turn implies $\lim_{\theta\to+\infty} \mu(D^\theta) = 0$. Given our definition of D^θ it follows that $\lim_{\theta\to+\infty} \int_0^1 \mathbf{e}^\theta(a)\ da < \epsilon$. Since the initial choice of $\epsilon > 0$ was arbitrary, it must be the case that $\lim_{\theta\to+\infty} \int_0^1 \mathbf{e}^\theta(a)\ da = 0$.

Proof of Theorem 4.20: Suppose not and define the set $R^\theta = \{s \in I : s - \underline{a}^\theta(s) > \epsilon\ \}$. Then there exists a sequence of $\{\theta_j\}_{j=1}^{\infty}$ where $\theta_j > 0$, $\forall j$, and $\theta_j \to +\infty$, as $j \to +\infty$, such that $\forall\theta$ in that sequence, $\int_0^1 [s - \underline{a}^\theta(s)]\ ds > \epsilon\ \mu(R^\theta)$, where $\mu(R^\theta)$ is the measure of R^θ. Note that by market clearing (i.e. equation (4.12)), $\mathbf{e}^\theta(s) \geq \frac{s-\underline{a}^\theta(s)}{1-\underline{a}^\theta(s)} > \epsilon$, $\forall s \in R^\theta$. Hence, $\int_0^1 \mathbf{e}^\theta(a)\ da > \int_0^1 [s - \underline{a}^{\theta_j}(s)]\ ds > \epsilon\ \mu(R^\theta)$. But this contradicts the previous lemma for any $\mu(R^\theta) > 0$.

Proof of Corollary 4.21 : From the previous theorem and lemma, as $\theta \to \infty$ the allocation under tournaments approaches the efficient allocation (and hence maximum output) and signaling waste converges to zero. Consequently, aggregate consumption is also approaching its maximum level. Output and consumption under markets, on the other hand, are always below the efficient levels. Hence for θ sufficiently large, $C_{\circ}^{t_\theta} > C_{\circ}^{m}$, $\forall\theta > \theta^{*}$.

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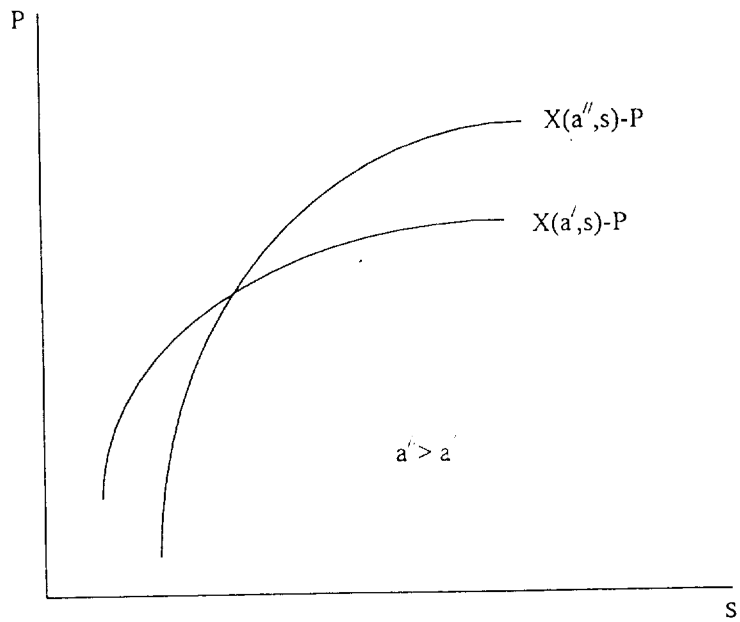

Figure 1

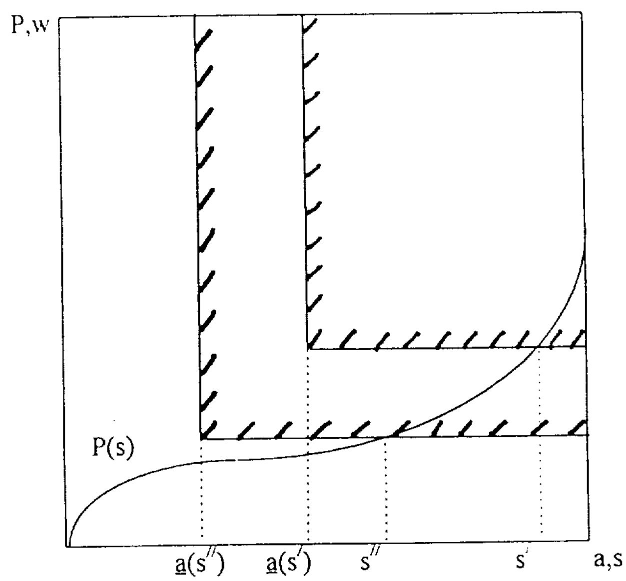

Figure 2

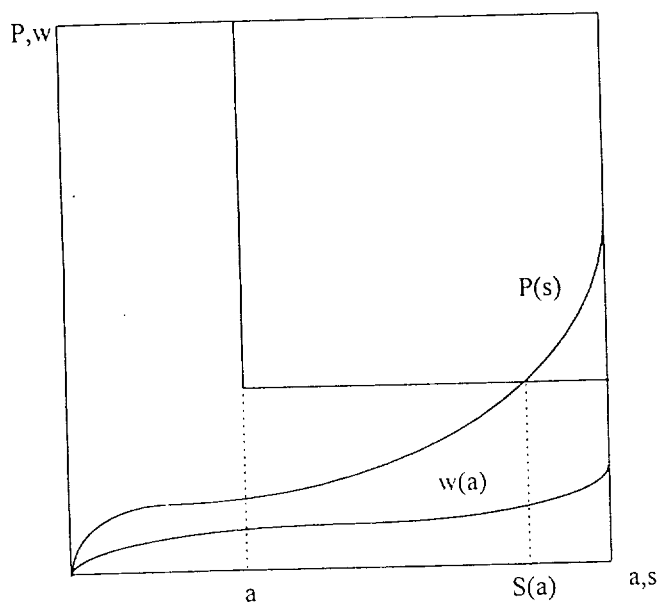

Figure 3

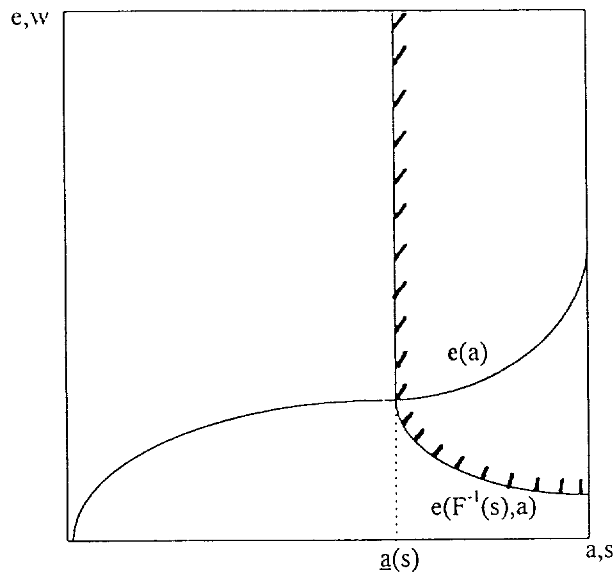

Figure 4

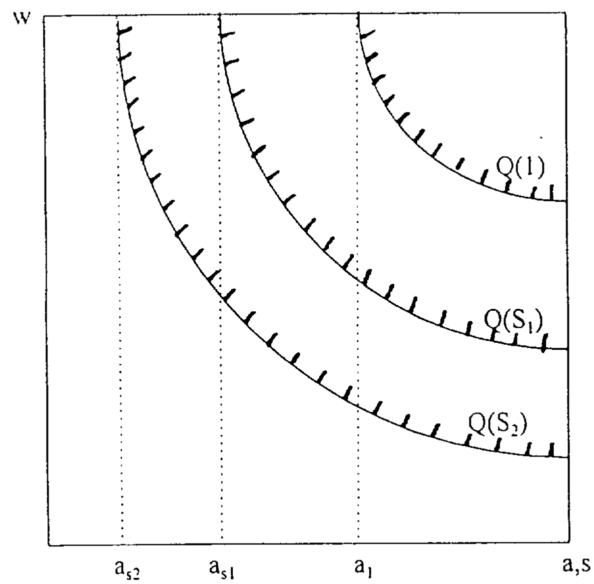

Figure 5