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POST-'87 CRASH FEARS IN S&P 500 FUTURES OPTIONS

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ABSTRACT

This paper shows that post-crash implicit distributions have been strongly negatively skewed, and examines two competing explanations: stochastic volatility models with negative correlations between market levels and volatilities, and negative-mean jump models with time-varying jump frequencies. The two models are nested using a Fourier inversion European option pricing methodology, and fitted to S&P 500 futures options data over 1988-1993 using a nonlinear generalized least squares/Kalman filtration methodology. While volatility and level shocks are substantially negatively correlated, the stochastic volatility model can explain the implicit negative skewness only under extreme parameters (e.g., high volatility of volatility) that are implausible given the time series properties of option prices. By contrast, the stochastic volatility/jump-diffusion model generates substantially more plausible parameter estimates. Evidence is also presented against the hypothesis that volatility follows a diffusion.

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Deviations of stock index option prices from the benchmark Black-Scholes model have been extraordinarily pronounced since the stock market crash on October 19, 1987. Out-of-the-money (OTM) put options that provide explicit portfolio insurance against substantial downward movements in the market have been trading at high prices (as measured by implicit volatilities) relative to at-the-money options. The OTM puts have been even more "overpriced" relative to OTM calls that will pay off only if the market rises substantially. An illustration of the typical post-crash pattern of implicit volatilities across strike prices for a single day is given in Figure 1. The pronounced implicit volatility patterns emerged immediately after the stock market crash, and have been a permanent feature of the S&P 500 futures options market ever since. The stock market crash was a watershed event, with fundamentally different post-crash patterns of implicit volatilities across strike prices relative to pre-crash patterns.

The implication is that the distribution perceived by market participants and incorporated into options prices since the crash of 1987 is substantially *negatively skewed*, in contrast to the essentially symmetric and slightly positively skewed lognormal distribution underlying the Black-Scholes model. Two approaches have been used in attempting to generate option pricing models with negatively skewed implicit distributions. *Stochastic volatility* models attribute the moneyness biases to the well-documented tendency of market volatility to rise as the market falls. A recent prominent example of this approach is the *implied binomial trees* approach of Dupire (1994), Derman and Kani (1994) and Rubinstein (1994), who postulate a flexible but deterministic functional form for instantaneous conditional volatility in terms of the underlying asset price and time. *Jump* models by contrast attribute the biases to fears of a further stock market crash.

The relevance of the stock market crash of 1987 to the emergence of substantial implicit negative skewness can be rationalized under both models. Grossman and Zhou (1996) point out in a general equilibrium model that even if underlying fundamentals follow geometric Brownian motion, the existence of portfolio insurers induces a negative correlation between the level and volatility of the market. From this viewpoint, the crash of 1987 can be viewed either as revealing the substantial number of portfolio insurers, or as an event that increased the demand for (explicit) portfolio insurance. Jump models would, by contrast, interpret the crash as a revelation that jumps can in fact occur -- a viewpoint somewhat validated by the subsequent 5-8% drops on January 11, 1988 and October 13, 1989.

This paper seeks to explore which of these alternate explanations better explains the negative skewness implicit in option prices. Two diagnostics are proposed. First, it is noted that the two hypotheses have alternate implications for the relationship between option maturity and implicit skewness. Stochastic volatility models postulate that the stock market follows a *diffusion*, with the implication that the conditional distribution is instantaneously normal. Such models consequently imply a *direct* relationship between option maturity and the magnitude of implicit skewness, with little implicit skewness for extremely short-maturity options. By contrast, jump models such as Merton (1976) postulate finite-variance shocks that are independent and identically distributed. By the law of large numbers, such models imply an *inverse* relationship between option maturity and the magnitude of implicit skewness, with little implicit skewness for long-dated options. A model fitted to multiple-maturity option prices should therefore be able to distinguish between the hypotheses.

Second, this paper evaluates the *consistency* of the distributions implicit in option prices with the observed time series properties of S&P 500 futures prices and implicit variances.

Section 1 describes the data and conducts preliminary diagnostics. Section 2 describes the postulated stochastic volatility/jump-diffusion process and the option pricing methodology. Section 3 describes the implicit parameter estimation methodology, and presents estimates. Section 4 examines the consistency of the parameters implicit in options prices with those estimated from the S&P 500 futures and implicit variances time series. Section 5 concludes.

1. Post-crash option pricing patterns -- Preliminary diagnostics

1.1 Data

Transactions data were obtained for American S&P 500 futures options from their inception on January 28, 1983 through December 31, 1993 from the Chicago Mercantile Exchange, along with the underlying futures contracts. Only quarterly options maturing in March, June, September and December were initially available. Serial options written on the quarterly futures contracts and maturing the nearest other two months were subsequently introduced in 1987. The quarterly options' last trading day was initially the third Friday of the month, the expiration date of the underlying futures contract, but was changed in the second quarter of 1986 to the day before because of "triple witching hour" problems. Serial options trade up through the third Friday of their terminal month. All options transactions were matched with the nearest preceding futures price of comparable maturity, provided the lapsed time was less than 5 minutes and no trading halt was in effect.[1]

Preliminary diagnostics were run on a subsample of the full 1983-93 data base. All intradaily transactions for 1- to 4-month quarterly options were selected subject to the selection criteria used in Bates (1991): at least 4 call strikes and 4 put strikes traded per day, and at least 20 call transactions and 20 put transactions per day. Data from days not meeting these criteria were not used in the preliminary diagnostics. *Representative* daily option prices were then constructed using the constrained cubic spline methodology of Bates (1991). Cubic splines subject to option-specific no-arbitrage constraints were fitted daily to pooled intradaily option price/futures price ratios, as a

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[1]In principle all options trading halts when a trading halt in the underlying S&P 500 futures is declared. However, that declaration is not instantaneous. Option trades were recorded on October 13, 1989 after the S&P 500 futures had hit its first price limit.

function of the strike price/futures price ratio. Separate splines were of course fitted to call and put data. Representative call and put option prices for regularly spaced "moneynesses" (X/F - 1 = 0%, ±1%, ±2%, ...) were interpolated using the estimated splines, while associated implicit volatilities were computed using 3-month Treasury bill rates and the Barone-Adesi and Whaley (1987) American option pricing formula. Representative put option values for put strike prices $X_{put}/F = 1/(X_{call}/F)$ were also computed for constructing the skewness premium metric of implicit skewness.

A different subset of the data base was used for the estimates of post-crash stochastic volatility/jump-diffusion processes in sections 2 and 3 below. Both quarterly *and* serial options were used, since serial options were available throughout 1988-93. However, only trades on Wednesday mornings (9-12) were considered, yielding a weekly frequency panel data set. Using daily data was ruled out partly because of the resulting extreme demands on computer memory and time, and partly to avoid modeling day-of-the-week volatility effects. The use of morning trades were reflected a tradeoff between shortening the intradaily interval for greater option price synchronization, and lengthening it to get more observations. Options with less than one week to maturity were discarded. The resulting 1988-93 data set consists of 39,607 transactions (42% calls; 58% puts) in up to 4 options maturities per day on 310 Wednesday mornings over January 6, 1988 to December 29, 1993; an average of 128 trades per morning. Linear interpolations of 3- and 6-month Treasury bill yields were used for the corresponding risk-free discount rates.

1.2 Distributional diagnostics

Figure 1 shows the typical post-'87 "volatility smirk" for implicit volatilities across strike prices: high implicit volatilities for out-of-the-money (OTM) put options relative to the at-the-money (ATM) implicit volatilities, which are in turn higher than out-of-the-money call options' implicit volatilities. As shown in Figure 2, this has been the pattern virtually without exception throughout the 1988-93 period. The implicit volatilities from representative 4% OTM put options were on average 2.4% higher than ATM implicit volatilities during 1988-93, while 4% OTM call implicit volatilities were on average 1.6% lower. The magnitudes of the implicit volatility differentials varied over time, with major shocks (the stock market mini-crashes in January 1988 and October 1989, the Kuwait crisis of 1990-91) substantially increasing the differentials.

The persistence and magnitudes of the post-crash implicit volatility patterns are in sharp contrast to those of the pre-crash period. While pre-crash OTM put implicit volatilities were almost invariably higher than those from ATM options, implicit volatilities from OTM calls were sometimes below, sometimes above ATM implicit volatilities; see Figure 2. In essence, an asymmetric "volatility smirk" pattern alternated with a more symmetric "volatility smile" pattern over 1983-87, with patterns persisting anywhere from 3 months to 1½ years. The substantially smaller magnitudes of the pre-crash smirks and smiles relative to the post-crash smirks is evident in Figure 2.

An alternate and substantially equivalent measure of moneyness biases is given by the "skewness premium," or percentage deviation between call and put prices for options comparably out-of-the-money:

$$SK(x) \equiv \frac{C(F;\ T,\ X_{call})}{P(F;\ T,\ X_{put})} - 1 \qquad \textbf{(1)}$$

where $X_{put}/F = (1+x)^{-1}$, $X_{call}/F = (1+x)$, and $x>0$. Intuitively, since out-of-the money call (put) options pay off only upon realizations in the upper (lower) tail of the distribution of the underlying asset, comparing call and put prices is a direct gauge of the relative (risk-neutral) tail distributions, and therefore assesses implicit skewness. As discussed in Bates (1991, 1997), the skewness premium $SK(x) = x$ for most standard and slightly positively skewed distributional hypotheses: Black and Scholes' lognormal model, Merton's (1976) jump-diffusion with mean-zero jumps, and Hull and White's (1987) stochastic volatility model. "Leverage" models such as the constant elasticity of variance model with standard parameterization, Geske's (1979) compound option model, and Rubinstein's (1983) displaced diffusion model imply roughly a $[0, x]$ range for the skewness premium. Values above (below) the $[0, x]$ range require a distribution more positively (negatively) skewed than the standard theoretical models.

The 4% OTM skewness premium shown in Figure 3 confirms that the post-crash moneyness biases have been enormous relative to standard distributional hypotheses. Whereas such hypotheses imply that 4% OTM American call options on S&P 500 futures should be roughly 0-4% more expensive than correspondingly OTM put options, these calls have invariably been substantially *cheaper* than the puts -- 35% cheaper on average over 1988-93. To put this in perspective: a 2-month 4% OTM option with a typical implicit volatility of 16% costs roughly 1% of the underlying asset price. With an average S&P 500 futures price of 362 over 1988-93 and an option tick size of

.05, a 35-39% option pricing error is roughly 25 to 28 price ticks. Standard distributional hypotheses, including leverage models, cannot possibly explain the magnitude of the post-'87 moneyness biases; more negatively skewed distributions are required. Dumas, Fleming and Whaley (1996) show that the comparable biases in the S&P 500 index options market are far too large to be attributable to bid-ask spreads.

A further interesting observation from Figure 3 is that implicit skewness as measured by the skewness premium is strongly and directly related to the relative trading activity in calls *versus* puts of all strike prices.[2] Since in-the-money S&P 500 futures options are thinly traded, the relationship indicates that periods of substantial positive (negative) implicit skewness were typically periods in which OTM calls were more (less) heavily traded than OTM puts. Throughout 1988-93, puts have been heavily traded relative to calls, and negative skewness premia have been consistently observed.

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[2]Bates (1996a) finds a similar relationship for DM and yen futures options over 1984-92 and 1986-92, respectively.

2. A proposed stochastic volatility/jump-diffusion model

Given the pronounced and persistent negative skewness implicit in post-'87 S&P 500 futures options, the following model will be used to nest the two major competing explanations.

Assumption A1: The S&P 500 futures price F is assumed to follow a two-factor geometric jump-diffusion of the following form:

$$
\begin{aligned}
dF/F &= (\mu - \lambda_t \bar{k} + c_{v1} V_{1t} + c_{v2} V_{2t})\,dt + \sqrt{V_{1t}}\,dZ_1 + \sqrt{V_{2t}}\,dZ_2 + k\,dq; \\
dV_{it} &= (\alpha_i - \beta_i V_{it})\,dt + \sigma_{vi}\sqrt{V_{it}}\,dZ_{vi}, \quad i = 1, 2; \\
Cov(dZ_i, dZ_{vi}) &= \rho_i\,dt, \quad i = 1, 2; \\
Cov(dZ_1, dZ_2) &= Cov(dZ_{v1}, dZ_{v2}) = 0;
\end{aligned}
\qquad \textbf{(2)}
$$

where

Z_i and Z_{vi}, $i = 1, 2$ are Wiener processes with the correlation structure specified above;

$\lambda_t = \lambda_0 + \lambda_1 V_{1t} + \lambda_2 V_{2t}$ is the instantaneous conditional jump frequency;

k is the random percentage jump conditional on a jump occurring, with time-invariant lognormal distribution $\ln(1+k) \sim N[\ln(1+\bar{k}) - ½\delta^2, \delta^2]$; and

q is a Poisson counter with instantaneous intensity λ_t: $\text{Prob}(dq = 1) = \lambda_t\,dt$.

The postulated process nests both the stochastic volatility and jump explanations of the strongly negatively skewed distributions implicit in observed S&P 500 futures option prices since the 1987 stock market crash. Negative skewness can arise either because of negative correlations between stock index and volatility shocks ($\rho<0$), or because of non-zero average jumps ($\bar{k} < 0$). Similarly, conditional and unconditional excess kurtosis can arise either from volatile volatility, or from a substantial jump component. The two explanations differ in maturity effects. Jumps primarily affect short-maturity options, whereas stochastic volatility primarily affects longer-maturity options.

The postulated process extends the Bates (1996b) model in several directions potentially consistent with observed S&P 500 futures options. First is the use of a multifactor specification. On any given day, stock index options with a broad array of strike prices and up to 4 maturities are trading simultaneously, all of which must be priced by the model. Consequently, it can be important to have sufficient factors to adequately match time-varying distributional patterns across different strike prices and maturities. For instance, evidence from currency options (Taylor and Xu (1994), Bates (1996b)) indicates that one-factor models can do a poor job in capturing the term structures of implicit volatilities over time, and that two-factor models would do better. Whether comparable improvement is apparent for the stock index options examined here will be examined below.

Second, the jump frequency λ_t is time-varying rather than constant. Given that implicit volatilities ranged from 40% to 10% over 1988-93, assuming constant jump risk throughout the period is implausible.

Options are of course priced not off the true process, but off the corresponding "risk-neutral" process that incorporates the appropriate compensation for volatility risk and jump risk:

$$dF/F = -\lambda_t^* \bar{k}^* \, dt + \sqrt{V_{1t}}\, dZ_1^* + \sqrt{V_{2t}}\, dZ_2^* + k^* dq^*;$$

$$dV_{it} = (\alpha_i - \beta_i V_{it} + \Phi_{vi})dt + \sigma_{vi}\sqrt{V_{it}}\, dZ_{vi}^*, \quad i = 1, 2;$$

$$Cov(dZ_i^*, dZ_{vi}^*) = \rho_i \, dt, \quad i = 1, 2; \tag{3}$$

$$Cov(dZ_1^*, dZ_2^*) = Cov(dZ_{v1}^*, dZ_{v2}^*) = 0;$$

$$Prob(dq^* = 1) = \lambda_t^* \, dt$$

The volatility risk premium $\beta^* - \beta$ reflects the degree to which innovations in the underlying volatility factor are correlated with the marginal utility of nominal wealth J_w, and consequently depends upon investors' preferences. Plausible values for its sign and magnitude can be obtained under the assumption of log utility:

$$(\beta^* - \beta)dt = Cov(dV/V, dJ_w/J_w) = Cov(dV/V, -dW/W); \tag{4}$$

see Cox *et al* (1985). Since volatility shocks are negatively correlated with shocks to the S&P 500 index, which represents a substantial fraction of nominal wealth W, $\beta^* - \beta$ is presumably positive.[3] Conversely, an upper bound on the volatility risk premium can be obtained if volatility shocks are assumed to covary more negatively with the equity than with the non-equity return components of

[3]Intuitively, volatility-sensitive investments such as straddles are "negative-beta" investments that typically pay off in adverse states when the marginal utility of wealth is high. They therefore have a lower conditional mean under the actual than under the risk-neutral distribution. This is of course in contrast to the positive conditional mean differential, or equity premium, of "positive-beta" investments such as the S&P 500.

nominal wealth returns. This assumption implies that $\beta^* - \beta$ is less than $-Cov(dV/V,\ dF/F) = -\rho\,\sigma_v$, a small positive number.

The jump risk premia λ_t^*/λ_t and $\bar{k}^* - \bar{k}$ similarly reflect the compensation required for bearing systematic jump risk:

$$\begin{aligned} \lambda_t^* &= \lambda_t\, E\left(1 + \frac{\Delta J_w}{J_w}\right) \\ \bar{k}^* &= \bar{k} + \frac{Cov(k,\ \Delta J_w/J_w)}{E[1 + \Delta J_w/J_w]}\ . \end{aligned} \tag{5}$$

where ΔJ_w is the jump in the marginal utility of nominal wealth conditional upon a jump occurring. Under systematic jump risk the cost λ_t^* per unit time of Arrow-Debreu crash insurance will diverge from the actuarial rate λ_t at which jumps arrive. Assessing this divergence requires an assessment of how stock market jumps affect other investments. If jumps are assumed to occur only in stock markets and log utility is again assumed, then $\Delta \ln J_w = -\Delta \ln W \approx -f\,\Delta \ln F$ and

$$\begin{aligned} \lambda_t^* &\approx \lambda_t\, E\, \exp(-f\,\Delta \ln F) = \lambda_t (1+\bar{k})^{-f} e^{\frac{1}{2}\delta^2(f^2+f)} \\ \ln(1+\bar{k}^*) &\approx \ln(1+\bar{k}) - f\,\delta^2 \end{aligned} \tag{6}$$

where f is the fraction of nominal wealth held in equity. When average jumps are negative, the "risk-neutral" jump frequency and average drop size will tend to exaggerate the downside risk: $\lambda^* > \lambda$, $\bar{k}^* < \bar{k}$. For plausible parameter values such as those estimated below, however, there is little reason to believe that the jump risk premia introduce a substantial wedge between the "risk-neutral" parameters implicit in option prices and the actual parameters.

The above assumptions generate an analytically tractable method of pricing options without sacrificing accuracy or requiring undesirable restrictions (such as $\rho=0$) on parameter values. European call options that can be exercised only at maturity are priced as the expected value of their terminal payoffs under the "risk-neutral" probability measure:

$$\begin{aligned} c &= e^{-rT} E^* \max(F_T - X, 0) \\ &= e^{-rT} \left[\int_X^\infty F_T \, p^*(F_T) \, dF_T \; - \; X \int_X^\infty p^*(F_T) \, dF_T \right] \\ &= e^{-rT} (F P_1 - X P_2) \end{aligned} \tag{7}$$

where

E* is the expectation with respect to the risk-neutral probability measure;

$F = E^*(F_T)$ is the current futures price;

$P_2 = \text{Prob}^*(F_T > X)$ is one minus the risk-neutral distribution function; and

$P_1 = \int_X^\infty (F_T / F) \, p^*(F_T) dF_T$ is also a probability.

The distribution functions can be evaluated by Fourier inversion of the underlying characteristic functions:

$$Prob^*(F_T > X \mid P_j) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{Imag[F_j(i\Phi) \, e^{-i\Phi x}]}{\Phi} \, d\Phi \; . \tag{8}$$

where $F_1(\Phi)$ and $F_2(\Phi)$ are the associated real-valued moment generating functions. F_1 and F_2 can be solved using the methodology described in Heston (1993) and Bates (1996b), with straightforward extensions for multiple independent factors and time-varying jump risk:

$$\ln F_j(\Phi \mid V_1, V_2, T) \equiv \ln E^*[e^{\Phi \ln(F_T/F_0)} \mid P_j] \qquad (j = 1, 2)$$
$$= \sum_{i=1}^{2} \left[A_{i,j}(T;\Phi) + B_{i,j}(T;\Phi)\, V_i\right] + \lambda_0^* T C_j(\Phi) \tag{9}$$

where

$$A_{i,j}(T;\Phi) = -\frac{\alpha_i T}{\sigma_{vi}^2}(\rho_i \sigma_{vi}\Phi - \beta_{i,j} - \gamma_{i,j}) - \frac{2\alpha_i}{\sigma_{vi}^2}\ln\left[1 + \tfrac{1}{2}(\rho_i\sigma_{vi}\Phi - \beta_{i,j} - \gamma_{i,j})\frac{1 - e^{\gamma_{i,j}T}}{\gamma_{i,j}}\right], \tag{10}$$

$$B_{i,j}(T;\Phi) = -2\frac{\tfrac{1}{2}[\Phi^2 + (3-2j)\Phi] + \lambda_i^* C_j(\Phi)}{\rho_i\sigma_{vi}\Phi - \beta_{i,j} + \gamma_{i,j}\dfrac{1 + e^{\gamma_{i,j}T}}{1 - e^{\gamma_{i,j}T}}}, \tag{11}$$

$$C_j(\Phi) = (1+\bar{k}^*)^{2-j}\left[(1+\bar{k}^*)^{\Phi} e^{\frac{1}{2}\delta^2[\Phi^2 + (3-2j)\Phi]} - 1\right] - \bar{k}^*\Phi \tag{12}$$

$$\gamma_{i,j} = \sqrt{(\rho_i\sigma_{vi}\Phi - \beta_{i,j})^2 - 2\sigma_v^2\{\tfrac{1}{2}[\Phi^2 + (3-2j)\Phi] + \lambda_i^* C_j(\Phi)\}} \tag{13}$$

and $\beta_{i,j} = \beta_i^* + \rho_i\sigma_{vi}(j-2)$.

Similar inversions can be used to evaluate transition densities.

The above procedure gives the price of a *European* call option as a function of state variables and parameters:

$$c(F,\ V,\ T;\ X,\ \theta)\ =\ e^{-r(T+\Delta t_1)}\,[F P_1\ -\ X P_2] \tag{14}$$

where Δt_1 is the one business day lag in settlement upon terminal exercise of an option, and θ is the vector of parameters. Chaudhury and Wei (1994) show that American futures option prices C and P are bounded above by the *future value* of the European option price:

$$\begin{aligned} \max[F - X,\ c] &\le C \le e^{rT}c \\ \max[X - F,\ p] &\le P \le e^{rT}p \end{aligned} \tag{15}$$

This implies that the proportional markup of American over European prices is within the narrow range $[1,\ e^{rT}]$. Consequently, accurately evaluating the early-exercise premium is not a major issue for the 0-6 month options examined here -- especially given that the in-the-money S&P futures options with maximum potential for approximation error are relatively thinly traded. This article therefore follows Bates (1996b) in inserting expected average jump frequencies and expected average variance into the Bates (1991) jump-diffusion early-exercise approximation, and using "smooth-pasting" conditions based upon the correct European option pricing formula (14).

3. Implicit Parameter Estimation

3.1 Methodology

Implicit parameters were estimated on the panel data set of 39,607 call and put prices for all observed strike prices and up to four maturities on Wednesday mornings over January 6, 1988 to December 29, 1993. The option pricing residual was defined as

$$e_{i,t} \equiv \left(\frac{O}{F}\right)_{i,t} - O\left(1,\ V_t,\ T_{i,t};\ \left(\frac{X}{F}\right)_{i,t},\ \theta\right) \tag{16}$$

where

t is an index over 310 Wednesday mornings within the specified period;

i is an index over transactions (calls and puts of assorted strike prices and at most four maturities) on a given Wednesday morning;

$(O/F)_{i,t}$ is the observed call or put option price/futures price ratio for a given transaction;

$O(\cdot)$ is the theoretical American option price/futures price ratio given the contractual terms of the option (call/put, time to maturity $T_{i,t}$, strike price/spot price ratio $(X/F)_{i,t}$) and given that Wednesday morning's factor realizations V_t, interest rate r_t and the time-invariant parameters θ of the model.

For the full two-factor stochastic volatility/jump-diffusion model, θ was the set of jump and stochastic volatility parameters: $< \lambda_0^*,\ \lambda_1^*,\ \lambda_2^*,\ \bar{k}^*,\ \delta,\ \alpha_1,\ \beta_1^*,\ \sigma_{v1},\ \rho_1,\ \alpha_2,\ \beta_2^*,\ \sigma_{v2},\ \rho_2 >$. 1- and 2-factor subcases of the general model were also estimated, to see which features of the generalized model were important in explaining option pricing deviations from benchmark *ad hoc* Black-Scholes prices with day-specific implicit variances. The average Wednesday morning factor realizations V_t were estimated for all Wednesdays in the 1988-93 data set. Intradaily movements in implicit factors were ignored in the estimation procedure.

A standard econometric method of inferring parameters from option prices is nonlinear ordinary least squares, used *inter alia* by Whaley (1986) and Bates (1991). However, the implicit assumption that option pricing residuals are independent and identically distributed is questionable on theoretical grounds. George and Longstaff (1993) find that market makers' bid-ask spreads in the S&P 100 index options market vary by strike price and maturity, suggesting a *heteroskedastic* impact from bid-ask bounce. A similar heteroskedastic impact arises from imperfect data synchronization with the underlying futures price, which affects in-the-money options more than out-of-the-money options. The pooling error introduced by using a common spot variance for all transactions on a given Wednesday morning introduces more complex intradaily serial and cross-correlations in residuals.

The major issue for implicit parameter estimation is, however, specification error. Any parsimonious time series model imposes a structure on option prices that can capture only some of the features of the true data generating process. Specification error implies that option pricing residuals of comparable moneyness and maturity will be contemporaneously correlated, and serially correlated as well if the conditional "risk-neutral" distribution evolves gradually over time in fashions not captured by the model. Furthermore, no-arbitrage constraints on option prices imply contemporaneous correlations across residuals of different strike prices and maturities. For instance, put-call parity for the European component of American option prices implies positively correlated residuals between calls and puts of identical moneyness and maturity in the presence of specification error. Consequently, implicit parameter estimation via nonlinear ordinary least squares (NL-OLS) would yield misleadingly low estimated standard errors. A further problem when transactions data are used is that NL-OLS can place too much weight on the substantially redundant information provided by heavily traded options while virtually ignoring less actively traded options.

Consequently, implicit parameters are estimated using a nonlinear generalized least squares/Kalman filtration methodology that takes into account the heteroskedasticity, contemporaneous correlation, and serial correlation properties of option residuals. Option pricing residuals are sorted by call/put, maturity, and moneyness criteria into 64 groups,[4] and assumed to include both group-specific and idiosyncratic shocks:

$$\begin{cases} e_{i,t} = \varepsilon_{I,t} + \sigma_I \eta_{i,t} & \text{for } i \in G(I,\ t) \\ \varepsilon_{I,t} = \rho_I \varepsilon_{I^*,t-1} + \upsilon_{I,t} & \end{cases} \tag{17}$$

where

$G(I,\ t)$ is the set of observations in group I at date t;

$\nu_{I,t}$ is a mean-zero, normally distributed shock term common to all option prices in group I at time t, with $E_{t-1} \upsilon_t \upsilon_t' = \boldsymbol{Q}$ for positive semidefinite $\boldsymbol{Q}$;

$\eta_{i,t} \sim N(0,\ 1)$ is an idiosyncratic shock to transaction i at time t, uncorrelated with $\nu_{I,t}$; and

I^* identifies lagged option residuals of the same moneyness and *delivery month* for 0-3 month options, and of the same *maturity* ($I^* = I$) for 3-6 month options.[5]

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[4]The criteria were
1) whether the transaction involved a call or a put;
2) whether the maturity was 0-1, 1-2, 2-3, or 3-6 months;
3) whether the option was in-the-money by 0-1%, 1%-2%, or >2%, or out-of-the-money by 0-1%, 1%-2%, 2%-4%, 4%-8%, or >8%.

The asymmetric moneyness criteria reflected the fact that deep in-the-money options were thinly traded.

[5]Simpler dynamics could have been generated by assuming $I^* = I$ throughout. However, time decay in option prices suggests a closer relationship between residuals of 8-week (1-2 month) maturity and the preceding week's 9-week (2-3 month) maturity option residuals than between 8- and

The set of groups represented on any given day was constantly changing, primarily because of the complicated serial/ quarterly option maturity structure. The average number of groups represented per day was 27.7, 43% of the 64 groups possible.

Given the above specification, the loss function for implicit parameter estimation is

$$\max_{\{V_t\},\,\theta} \ \ln L_{options} = -\frac{1}{2}\sum_t \ln|\Omega_{t|t-1}| + (e_t - E_{t-1}e_t)'\,\Omega^{-1}_{t|t-1}\,(e_t - E_{t-1}e_t). \tag{18}$$

$E_{t-1}e_t$ is a Kalman filtration-based forecast of option residuals conditional upon estimated dynamics (17) and lagged option residuals. The conditional covariance matrix $\Omega_{t|t-1}$ is also estimated using Kalman filtration methods.

The log likelihood in (18) is optimized by a two-stage procedure. Conditional upon the Kalman filtration parameters in (17), (18) is optimized via the nonlinear weighted least squares over the parameters $<\{V_t\},\ \theta>$ that directly determine option residuals.[6] Conditional upon the option residuals, optimization of (18) over $<\{\rho_I,\ \sigma_I\}_{I=1}^{64},\ Q>$ involves estimating a high-dimensional *linear* Kalman filtration with day-specific missing information for particular groups. Alternating between

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5-week option residuals.

[6]The Davidon-Fletcher-Powell quadratic hill-climbing algorithm was used (GQOPT subroutine DFP) for this optimization. The score was computed numerically, exploiting specific features of the log likelihood function to increase efficiency. For instance, computing $\partial \ln L / \partial V_t$ required perturbing only date-*t* options and measuring subsequent propagation effects. Nonnegativity constraints were enforced through log transformations of parameters and spot variances, while correlations were constrained via a cumulative normal transformation.

the two optimization steps until joint convergence yields estimates of implicit parameters and factor realizations for a specific model, estimates of the relative importance of idiosyncratic and common shocks, and a full dynamic description of the specification error (as captured by the vector of common shocks). A slightly improved variant of the Shumway and Stoffer (1982) and Watson and Engle (1983) EM algorithm approach to estimating Kalman filtrations is developed in the appendix for this particular application.[7]

Optimization yields the estimates of parameters and state variable realizations that best fit observed option prices. However, such an optimization does not constrain the state variable estimates to evolve consistently with the underlying option pricing model. For instance, the implicit variances estimated under the Black-Scholes submodel are not constrained to be identical, contrary to the assumptions of that model. Consequently, the stochastic volatility and stochastic volatility/jump-diffusion models were also estimated using the likelihood function

$$\ln L(\{V_t\}, \theta, \beta_1, \beta_2) = \ln L_{options} + \ln L_{V1} + \ln L_{V2} \tag{19}$$

where

$\ln L_{options}(\{V_{1t}, V_{2t}\}, \theta)$ is the function of option pricing residuals given above in equation (18), and

$\ln L_{\{V\}} = \Sigma_t \ln p(\ln V_t \mid \alpha, \beta, \sigma_v; V_{t-1})$ is the log likelihood of an estimated $\{V_t\}$ sample path given the *actual* (as opposed to risk-neutral) rate of variance mean reversion β.

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[7]The procedure is superior to that in Bates (1996b) in two regards. First, it copes better with missing observations. Second, the EM algorithm approach to estimating high-dimensional common shock vectors is substantially faster.

The spot variance transition densities are related to the noncentral chi-squared density, with series representation

$$p(\ln V_{t+\Delta t} \mid V_t) = \frac{2}{\kappa} \frac{e^{-\frac{1}{2}(y+\Lambda)} y^{\frac{1}{2}\nu}}{2^{\frac{1}{2}\nu}} \sum_{j=0}^{\infty} \frac{(\frac{1}{4} y \Lambda)^j}{\Gamma(\frac{1}{2}\nu + j)\, j!} \tag{20}$$

where $\kappa = \frac{1}{2}\sigma_v^2(1 - e^{-\beta\Delta t})/\beta$, $y = 2V_{t+\Delta t}/\kappa$, $\nu = 4\alpha/\sigma_v^2$, $\Lambda = 2V_t e^{-\beta\Delta t}/\kappa$, and $\Gamma(\cdot)$ is the gamma function.[8]

The constrained estimates serve three functions. First, since the underlying hypothesis is that the state variables follow a diffusion, the constrained estimates yield *smoothed* state variable sample paths that are useful in assessing major and persistent developments in S&P 500 futures option prices. The appropriate degree of smoothing is determined endogenously, based upon the estimated volatility of volatility σ_v. For instance, optimization of (19) under the Black-Scholes assumption $\alpha = \beta^* = \sigma_v = 0$ would be equivalent to estimating a single implicit variance over the entire 1988-93 period. Second, the constrained parameter estimates are of course more plausible relative to the time series properties of the state variable estimates. Finally, a comparison of the constrained and unconstrained parameter estimates can be used to test the option pricing models.

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[8]The transition densities $p(\ln V_t) = V_t\, p(V_t)$ were used rather than the transition densities of V_t because the former is strictly finite whereas the latter is infinite at $V = 0$ when the reflecting barrier at zero is attainable ($2\alpha < \sigma_v^2$), yielding nonsensical results when estimating implicit factors.

3.2 Results

Model-specific estimates of implicit distributions indicate extremely turbulent conditions in the S&P 500 futures option market over 1988-93, with somewhat quieter conditions following the end of the Gulf war. Assorted shocks hitting the stock market provoked substantial movements in implicit distributions -- not just implicit volatility, but higher moments as well. The evolution of implicit distributions is especially manifest in the full two-factor stochastic volatility/jump-diffusion estimates, which indicate a fundamentally different role for the two factors. V_1 is a "volatility-and-skewness" factor that heavily affects implicit skewness and leptokurtosis both through its contribution to jump risk and through a high implicit negative correlation with market shocks. V_2 by contrast captures parallel shifts in the term structure of implicit volatilities that do not especially influence higher moments. Because of the jump risk channel, V_1 has roughly twice the impact of V_2 on instantaneous and longer-maturity expected average conditional variances.

The smoothed estimates of these factors in Figure 4 indicate the major shocks that affected the options market over 1988-93: an 8% intradaily drop in S&P 500 futures prices on January 8, 1988, the mini-crash of October 13, 1989, and the Kuwait crisis from Iraq's invasion on August 2, 1990 through the conclusion of the Gulf war on March 3, 1991. Smaller shocks also appear, such as the Clinton tax increase announced on February 17, 1993.

The typical option pricing shift accompanying substantial market drops was higher implicit downside risk -- not just through increases in implicit volatilities, but through increases in higher implicit moments as well. By contrast, the substantial run up in the market over 1991-93 was largely accompanied by declining assessments of non-jump volatility and correspondingly lower downside risk. As indicated in Figure 4, however, residual jump fears inferred from the two-factor models persisted substantially without change at a lower level throughout the post-Kuwait period. Judging from option prices, market participants did *not* view the stock market as overvalued and more prone to crash following the run up; quite the contrary. It appears that crashes begat crash fears in the S&P 500 futures options market, while an absence of crashes reduced crash fears to an assessed biannual frequency.[9]

The unconstrained stochastic volatility (SV) and stochastic volatility/jump-diffusion (SVJD) model estimates reported in Table 1 eliminate most of the moneyness- and maturity-related option pricing biases of the *ad hoc* Black-Scholes (BS) model. The major improvement clearly originates in relaxing the conditionally lognormal assumption to capture the "volatility smirk," symptomatic of substantially negative implicit skewness. By contrast, relaxing the assumption of a flat term structure of implicit volatilities (DV1 and DV2 estimates) contributed relatively little to the improved fit, indicating predominantly flat term structures of implicit volatilities across different option maturities throughout the 1988-93 period. And while full-model estimates indicate substantial implicit jump risk, the 1- and 2-factor estimates still attribute much of the negative implicit skewness (at longer horizons) to substantial volatility shocks that are negatively correlated with market shocks. The estimates indicate that jumps alone cannot capture the skewed and leptokurtic

[9]The experience of other countries appears to be quite different from the U.S. experience. Gemmill (1995) found little change in implicit skewness from British stock index options following the British stock market crash in 1987, while Beinert and Trautmann (1994) found that German stock options exhibited increased *positive* skewness (rebound expectations) following the German 1987 crash.

implicit distributions evident at both short and longer-maturity horizons -- basically because of too-rapid convergence towards lognormality at longer maturities. Allowing for time-varying jump risk is quite important in improving the fit over time of models with jumps. The constant jump risk component λ_0^* is of negligible importance.

The 2-factor models' ability to distinguish between implicit volatility shifts that accompany higher implicit moments and those that do not yields substantially improved fits relative to 1-factor models throughout the 1988-93 period. The downside is that essentially twice as many parameters and factor realizations must be estimated. The two-factor SV model appears somewhat overfitted, in that implicit factor realizations become highly volatile relative to the one-factor estimates[10] and frequently hit the nonnegativity constraint. Smoothing the SV estimates using (19) heavily affects inferred factor realizations at relatively little cost in option-specific log likelihood; another indication of overfitting. By contrast, implicit factor realizations from the two-factor SVJD model are more stable, hit nonnegativity constraints far less frequently, and are modified less by smoothing.

Estimates of Kalman filtration parameters indicates severe persistence in option pricing residuals; see Table 1. The problem is most pronounced for residuals from the conditionally lognormal models (BS, DV1, DV2) -- not surprising given that the "volatility smirk" was present throughout 1988-93. The filtration-based serial correlation correction cuts option residuals' standard errors in half for those models, implying an associated R^2 of roughly 75% in "explaining" option

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[10]Estimates of the volatility of volatility σ_v for implicit factor realizations from constrained and unconstrained models are reported in Table 2 below.

residuals. Moving to more complicated models does not especially reduce the autocorrelation estimates, but the serial correlation correction becomes less important in reducing standard errors.[11]

The relative importance of idiosyncratic and common shocks varies by model, and by group classification (call/put, maturity, and strike class). For the two-factor SVJD model, idiosyncratic noise as a percentage of the underlying futures prices ranged from .009% to .067% (0.6 to 4.9 price ticks)[12], with a tendency to increase for longer-maturity and deeper in-the-money options. Common shock magnitudes suggestive of specification error were typically comparable and slightly higher: .011% - .079%, or 0.8 - 5.7 price ticks. However, the SVJD model had difficulty fitting the longest-maturity (3-6 month) options and deepest in-the-money 2-3 month put options, with common shock standard deviations of .113% - .174% (8.2 - 12.6 price ticks). The relative importance of common shocks was of course substantially more pronounced for the more parsimonious and poorer-fitting models.

While the 6-tick standard errors achieved by the SV and SVJD models eliminates most of the 15-16 tick standard errors of the *ad hoc* lognormal models (BS, DV1, DV2), there is room for further improvement. The maturity-specific constrained cubic splines fitted in section 1 to 1-4 month quarterly options have an overall standard error of about 2 price ticks over 1988-93-- a fit also

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[11]Since the estimation procedure uses a nonlinear generalized least squares approach, equally-weighted standard errors premised on homoskedasticity are not equivalent to the maximization criterion. They are reported to provide a broad-based and relatively intuitive measure of model performance for comparison with other empirical work.

[12]Price ticks in the S&P 500 futures market are .05. The average 1988-93 S&P 500 futures price level of 362 was used to converting standard errors into price ticks.

achieved by Bates (1991) on pre-crash S&P 500 futures options using an *ad hoc* 4-parameter jump-diffusion model with day- and maturity-specific implicit parameters. A comparable fit could probably be achieved using implied binomial trees with daily parameter re-estimation. The presence of idiosyncratic noise from bid-ask bounce, synchronization error, and intradaily pooling error makes it unlikely that a fit substantially better than two ticks could be achieved on this data set.

While profligately parameterized, such *ad hoc* methods do impose no-arbitrage constraints on relative option prices, and are therefore consistent with probability distributions generated by some deeper, unspecified data generating process. Furthermore, any freely parameterized model that fits options reasonably well will generate similar implicit distributions at option-specific maturities, and is therefore as valid as any other model in *describing* moneyness biases. The choice of model is less innocuous when extrapolating from monthly/quarterly option maturities to the daily or weekly frequencies used in *testing* option models.

4. Dynamic Implications

The implicit parameter estimates and factor realizations above essentially describe distributions substantially more consistent with post-1987 S&P 500 futures option prices than the lognormal distribution underlying Black-Scholes, and how those implicit distributions have varied over time. While an ability to reduce or eliminate systematic option pricing errors is an important attribute of any option pricing model, such models are also important for their purported ability to predict the future evolution of asset and option prices. Indeed, standard implicit parameter-based tests of option pricing models can be categorized by whether the implicit parameter estimates are tested for consistency with the evolution of the underlying asset price, with the evolution of option prices, or with the *joint* option price/asset price evolution. Examples of the first include the tests of whether implicit volatilities are unbiased and informationally efficient predictors of the subsequent realized volatility of the underlying asset.[13] The studies of whether the term structure of implicit volatilities predicts future implicit volatilities fall within the second category.[14] As noted by Dumas, Fleming, and Whaley (1996), "market efficiency" tests of no-arbitrage models such as Black-Scholes and implied binomial trees are equivalent to comparing first-differenced option prices or option returns with those predicted by the model *conditional* on the realized change in the underlying asset price.[15,16]

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[13]See Day and Lewis (1992), Canina and Figlewski (1993), and Fleming (1993) for such tests of S&P 100 index options.

[14] See Stein (1989) and Diz and Finucane (1993).

[15]See Whaley (1986) and Dumas, Fleming, and Whaley (1996).

[16] Market efficiency tests typically suffer from several problems. First is a severe selection bias: the typical approach of selling "overvalued" options, buying "undervalued" options and delta-hedging tends to select those options most severely out-of-sync with the underlying asset price.

4.1 Tests of the stochastic evolution of option prices

While there is no presumption under jump or stochastic volatility models that a delta-hedged option position will be riskfree, the models do specify the distribution from which option price changes should be drawn. For the postulated stochastic volatility/jump-diffusion process, the stochastic component of call or put option price changes can be roughly decomposed into "moneyness" and "implicit factor" effects:

$$\Delta C = [C(F+\Delta F, V, t+\Delta t) - C(F, V, t)] + C_{V'}\Delta V + O(\Delta T) \qquad \textbf{(21)}$$

where $V' = (V_1, V_2)$ and $O(\Delta T)$ captures deterministic terms of order ΔT. The "moneyness" effect reflects the option pricing impact from the option moving deeper in- or out-of-the-money as the underlying asset price changes. The "implicit factor" effect captures how option prices of a standardized moneyness and maturity evolve. Models with stationary return distributions such as Black and Scholes (1973) and Merton (1976) attribute all stochastic option price variation to the moneyness effect.[17]

Second is a focus on *average* option returns. The fact that the *variance* of returns to delta-hedged options positions should be close to zero under Black-Scholes assumptions has been less thoroughly tested -- perhaps because it is so obviously rejected by random fluctuations in implicit volatilities. Third, the evidence above of severe persistence in Black-Scholes option residuals reduces the power of tests based on first-differenced option prices, and may explain Dumas, Fleming and Whaley's conclusion that implied binomial trees models do no better than Black-Scholes from a hedging criterion.

[17]The constant elasticity of variance and binomial trees models also attribute all stochastic option price changes to movements in the underlying asset price. These models are not homogeneous in the asset price and strike price, precluding meaningful use of the strike price/futures price ratio as the measure of moneyness. They also predict nonstationary implicit volatilities, contrary to fact.

The unconstrained stochastic volatility (SV) and stochastic volatility/jump-diffusion (SVJD) models examined above fit option prices somewhat comparably. Equivalently, the two models yield similar predictions regarding the risk-neutral distributions of underlying asset prices and option prices at option maturation. However, the models yield quite different predictions regarding option price evolution -- how we get there from here. The SV models attribute the substantial negative skewness implicit in S&P 500 futures options to highly volatile stochastic volatility factors that typically rise as the market falls. The SVJD models by contrast assigns less weight to implicit factor movements and more weight to the moneyness impact of occasional large and predominantly downward changes in the market. Both models predict a substantial correlation between moneyness and implicit factor shocks.

In contrasting the two models, therefore, it is useful to single out the second component and examine the models' predictions for the stochastic evolution of option prices of a standardized moneyness and maturity. That evolution is conveniently summarized across options of *all* moneynesses and maturities by the estimates of the stochastic factors V_1 and V_2. Whether those estimates evolve consistently with the postulated square-root diffusions and with the parameters inferred from option prices can be tested. The test is analogous to testing the Black-Scholes model based upon its prediction that implicit volatilities inferred daily from pooled option prices of all strike prices and maturities should not change over time.

Maximum likelihood estimates of implicit factor processes under the postulated square root process strongly reject the hypothesis that standardized option prices evolve as predicted by implicit parameter estimates; see Table 2. Within that specification, the major disagreement is over the volatility of volatility parameter σ_v. Stochastic volatility models require high values of σ_v to generate substantial implicit skewness and leptokurtosis; and those values are not justified by the observed volatility of factor realizations and standardized options prices. Models with jumps yield smaller values of σ_v that are nonetheless implausible. Constrained estimation using (19) to impose time series plausibility upon implicit parameter estimates cannot reconcile a model-specific incompatibility between how options are priced and how option prices evolve.

A useful diagnostic of the misspecification of the factor process under parameters inferred from option prices is generated by "normalizing" factor transitions using the monotonic transformation

$$z_{t+1} \equiv N^{-1}\left[F(V_{t+1} \mid V_t;\ \hat{\alpha},\ \hat{\beta},\ \hat{\sigma}_v\ )\right],\ t = 1,\ \ldots,\ 309, \qquad \textbf{(22)}$$

where $F(\ln V_{t+1} \mid V_t;\ \bullet)$ is the conditional distribution function and $N^{-1}(\bullet)$ is the inverse of the cumulative normal. If the conditional distribution function is correctly specified with correct parameters, then the z's should be independent and identically distributed draws from a normal $N(0,\ 1)$ density -- a testable hypothesis. Conversely, if the conditional distribution is *not* correctly specified, analysis of the z's usefully summarizes the overall misspecification of conditional distributions.[18]

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[18]I am indebted to Charles Thomas for drawing my attention to "calibration" approach used by Fackler and King (1990) and Silva and Kahl (1993), which inspired the above transformation. Those papers work with uniform distributions. The additional transformation into normally distributed residuals appears preferable for highlighting outliers and for permitting use of standard normality tests such as Shapiro-Wilks. If V_{t+1} were drawn from a conditionally normal distribution,

The histograms in Figure 5 illustrates the extent of the misspecification of the 1-factor constrained stochastic volatility (SVC1) model. Factor realizations are substantially less volatile than predicted by implicit σ_v values, yielding a concentration of probability mass of normalized residuals well within the standard normal curve. While most pronounced for the SVC1 model, the standard deviations of normalized residuals in Table 3 indicate comparable problems for other specifications.

Normalizing implicit factor transitions using time series-based parameter estimates reveals that the postulated square root process is fundamentally misspecified. While maximum likelihood estimation successfully matches the first two moments, there are far too many outliers. The high improbability of those outliers relative to the diffusion-based assumptions indicates true conditional transition distributions are far more leptokurtic than hypothesized, and suggests that the underlying volatility processes follow a jump process. Outliers are almost all positive for the one-factor models, and correspond to sharp increases in implicit volatilities accompanying events such as Kuwait-related shocks and the mini-crashes of January 8, 1988 and October 13, 1989.[19] Two-factor models' outliers are more complicated, and include substantial shifts in the relative importance of the two factors at various times.

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the transformation (22) would be equivalent to the standard normalization $\tilde{z}_{t+1} = [V_{t+1} - E_t(V_{t+1})] / \sqrt{Var_t(V_{t+1})}$.

[19]The one large negative outlier in Figure 4 occurred on January 16-23, 1989, following the start of the Gulf war, and reversed the previous week's increase.

4.3 Tests of consistency with the time series properties of futures prices

The central empirical question is, of course, whether the substantial negative skewness implicit in options on S&P 500 stock index futures throughout 1988-1993 was in fact validated by subsequent developments in S&P 500 futures prices. For instance, implicit risk-neutral jump frequencies from the 1- and 2-factor models with jumps predict a total of 4 and 6 jumps over 1988-93, respectively. Those jumps are predicted to be drawn from a distribution with substantial negative mean (-9.5% and -5.7%, respectively) and standard deviation (10-11%). And while there were four large daily moves over 1988-93 of 4-6% in magnitude that might be interpreted as jumps, implicit and observed jump magnitudes are not especially compatible.[20] Indeed, the impact of the outliers cannot be discerned at *weekly* frequencies; see Table 4. Ex post, all models clearly exaggerate stock market risk. For instance, no weekly move greater than 10% in magnitude was observed over 1988-93, despite a 97-99% probability of such a move according to risk-neutral implicit distributions.

To examine the informational content of implicit distributions, the short-maturity (0-3 month) futures price process was modeled as in (2). The conditional mean μ was specified as

$$E(dF/F)/dt = c_0 + c_1 r_t + c_2 y_t + cv_1 V_{1t} + cv_2 V_{2t} \qquad \mathbf{(35)}$$

where r_t is the preceding day's 3-month Treasury bill yield and y_t is the previous day's implicit dividend yield from synchronous futures prices of different maturities.[21] The last two terms generate

[20]On a noon-to-noon basis, the four largest outliers were January 8-11, 1988 (-5.8%), October 13-16, 1989 (-4.8%), August 24-27, 1990 (+5.0%), and September 28 - October 1, 1990 (+4.4%).

[21]The average cost-of-carry was computed as $COC = \frac{1}{N}\sum_{n=1}^{N} \ln[F_n^{MT}/\overline{F}_n^{ST}]$, where $\overline{F}_n^{ST}$ was the average of all short-term futures prices observed within a ±20 second window around the corresponding medium-term futures price F_n^{MT}. The shortest two futures maturities available were

instantaneous "GARCH-in-mean" effects, although higher moments are also affected in discrete time.

The futures data were short-maturity (typically 0-3 month) noon quotes on Wednesdays for which there were options data available. The typical time interval was therefore one week, although holidays occasionally induced a longer time interval. To avoid maturity shifts, the futures contract maturity was the shortest maturity such that futures contracts with identical delivery date existed at the next available Wednesday.

To examine whether implicit variances are biased forecasts of future variance, the instantaneous conditional variance was modeled as a linear transform of the factor realizations inferred from option prices:

$$Var_t(dF/F) = V_0 + d_1 V_{1t} + d_2 V_{2t} \qquad \textbf{(36)}$$

where V_0, d_1, and d_2 are constants. Similarly, the actual jump frequency was modeled as a linearly transformed version of the implicit jump frequency:

$$\lambda_t = l_0 + l_1 \lambda_t^* = (l_0 + \lambda_0^*) + (l_1 \lambda_1^*) V_{1t} + (l_2 \lambda_2^*) V_{2t}. \qquad \textbf{(37)}$$

The resulting log-likelihood function

$$\ln L_{\{F\}} = \sum_n \ln p\left[\ln\left(\frac{F_n}{F_{n-1}}\right) \mid V_{n-1}\right]. \qquad \textbf{(39)}$$

for weekly log-differenced futures prices *conditional* upon observed factor realization can be calculated via Fourier inversion of a slightly modified variant of (10)-(13), and was optimized over the parameter space $<l_0, l_1, \bar{k}, \delta, c_0, c_1, c_2, c_{v1}, c_{v2}, V_0, d_1, d_2>$. The stochastic variance parameters

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used. The implicit dividend yield is $r_t - COC_t$.

$<\alpha_i, \beta_i, \sigma_{vi}, \rho_i>$ were set equal to those inferred from option prices, ignoring the possible divergence between β and β^*.

Maximum-likelihood estimates of the parameters are presented in Table 5. Variables affecting the conditional mean exhibited no significant ability to forecast futures prices, with R^2 typically around 1%. By contrast, implicit factor realizations were informative but biased forecasts of subsequent S&P 500 volatility, with implicit variances typically overstating realized variance. The biases were most pronounced for implicit variances from the stochastic volatility models, and are roughly comparable to those in Day and Lewis (1992) and Fleming (1993). Factor realizations inferred under jump models are less biased in describing realized distributions, suggesting that fears of "rare events" may be contributing to the implicit volatility forecasting biases typically found for stock and stock index options.[22] Unconstrained estimates of jump parameters from weekly S&P 500 futures returns pick up an infrequent *positive* jump component that is significantly different from zero but not significantly different from implicit parameter estimates.

4.4 Correlation tests

While it is possible using Fourier inversion techniques to evaluate and estimate *joint* transition densities of S&P 500 futures prices and factor realizations, the inconsistencies evident above between implicit and observed *marginal* densities suggests little to be gained from the exercise. However, both the stochastic volatility and stochastic volatility/jump-diffusion models attribute some of the negative implicit skewness to negative correlations between market and volatility shocks. That there

[22]See Bates (1996) for a survey of the empirical literature.

exists such a correlation between stock market returns and *actual* conditional volatility changes is of course fundamental to the EGARCH approach, while corresponding negative correlations between returns and *implicit* volatility changes have been found for individual stocks (Schmalensee and Trippi (1978)) and for the British stock market (Franks and Schwartz (1991)). Some simple correlation computations on weekly data reported in Table 6 confirm that the underlying assumption of substantial negative correlations between S&P 500 futures returns and implicit factor changes are in fact observed. In the 2-factor models, the correlation is most pronounced for innovations in the factor (V_1) that most influences higher moments, as predicted by implicit parameter estimates.

The two-factor model was premised upon the assumption of independent factors, whereas innovations in the two unconstrained implicit factor estimates are in fact substantially negatively correlated. This is symptomatic of overfitting. The correlations between factors are less pronounced for the constrained (smoothed) estimates, in which the second factor plays more of a "parameter drift" role.

5. Summary and conclusions

This article has presented evidence that post-'87 distributions implicit in S&P 500 futures options are strongly negatively skewed, and has examined two competing hypotheses: a stochastic volatility model with negative correlations between index and volatility shocks, and a stochastic volatility jump-diffusion model with time-varying jump risk. The fundamental premise underlying the stochastic volatility model is confirmed: index and implicit volatility shocks are in fact strongly negatively correlated. However, this negative correlation is not sufficient of itself to generate sufficiently negative implicit skewness. An extremely high volatility of volatility is also necessary -- implausibly high when judged against the time series properties of option prices. By contrast, the stochastic volatility/jump-diffusion model explanation is more compatible with plausible stochastic volatility parameter values. The crash fears explanations is also somewhat more compatible with observed S&P 500 futures returns over 1988-93 than the stochastic volatility specification, and offers a partial explanation for previously reported biases in stock and stock index implicit volatility forecasts of future volatility. All models examined here clearly exaggerated the stock market risk ex post, given predicted large movements that did not in fact occur over 1988-93. And while the difference between risk-neutral and actual distributions could conceivably explain the divergence, enormous volatility and/or jump risk premia would probably be required.

This article has also presented strong evidence against the hypothesized square root diffusion processes driving instantaneous volatility and jump risk. Such processes possess many desirable features (nonnegativity, leverage effects, analytic tractability), but cannot account for the substantial and typically positive implicit volatility shocks observed in the S&P 500 futures options market. A volatility-jump model is clearly necessary, and will be explored in future work.

Finally, neither of the two models quite captures the profile of implicit skewness across different option maturities. Whereas the jumps hypothesis predicts a strong inverse relationship between skewness and maturity and the stochastic volatility model predicts a direct relationship at short horizons, the truth appears to lie somewhere in between: a flat to declining relationship that declines slower than predicted by the jumps explanation. Within the framework of this article's model, the problems are most evident in the difficulties in pricing the longest-maturity (3-6 month) options in the data set. While it is possible that an alternate volatility-jumps model might do better, I suspect that the problem originates with the independent and finite-variance shocks assumption, with the resulting rapid convergence towards lognormal distributions at longer maturities. Equivalently, the "volatility smirk" remains too pronounced at longer maturities. Alternate infinite-variance option pricing models such as the stable Paretian model are consequently worth exploring.

Appendix

Conditional upon particular estimates $<\{V_t\}, \theta>$ of the volatility state realization and implicit parameters, option pricing residuals are assumed to satisfy

$$\begin{cases} e_{i,t} = \varepsilon_{I,t} + \sigma_I \eta_{i,t} & for\ i \in G(I,t) \\ \varepsilon_t = F_{t-1}\varepsilon_{t-1} + \upsilon_t \end{cases} \tag{A.1}$$

where

$\eta_{i,t} \sim N(0, 1)$ is an idiosyncratic shock to transaction I at time t, uncorrelated with all other shocks;

$G(I, t)$ is the set of residuals in group I at time t;

ε_t is the N-dimensional vector of common shocks, with I-th entry $\varepsilon_{I,t}$;

$F_t = D_\rho A_t$ is the product of an $N \times N$ diagonal matrix D_ρ with serial correlations $\rho = \{\rho_I\}$ along the diagonal, and a permutation matrix A_t that captures the assumed maturity-related serial dependency of common shocks; and

υ_t is a mean-zero, normally distributed vector with $E_{t-1}\upsilon_t\upsilon_t' = Q$ for positive semidefinite Q.

It is useful to orthogonalize option residuals by dividing into group-average and deviation from group-average components:

$$e_{i,t} = \bar{e}_{I,t} + u_{i,t} \tag{A.2}$$

where $\bar{e}_{I,t} = \frac{1}{N_{I,t}} \sum_{i \in G(I,\ t)} e_{i,t} \sim N\left(\varepsilon_{I,t},\ \sigma_I^2/N_{I,t}\right)$ is a reduced-noise signal that summarizes all relevant date t, group I information about the level of the underlying common shocks. The precision of the signal varies observably with the group-specific number of observations $N_{I,t}$; frequently no information is available for particular groups. The deviations from group-average $u_{i,t}$ collected into the vector $\boldsymbol{u}_t$ are useful in identifying the magnitudes of idiosyncratic noise σ_I but are otherwise orthogonal to the Kalman filtration. The covariance matrix $\boldsymbol{S}_t = E\,\boldsymbol{u}_t\boldsymbol{u}_t'$ is block-diagonal, and depends only upon $\{\sigma_I,\ N_{I,t}\}$.

Let $\boldsymbol{x}_t$ represent the n_t-dimensional subset of $\bar{\boldsymbol{e}}_t$ observed on date t, where the number $n_t \leq N$ of groups represented changes constantly over time. Estimating the parameters $<\{\rho_I,\ \sigma_I\}_{I=1}^{64},\ \boldsymbol{Q}>$ of (A.1) is a standard Kalman filtration problem with missing observations. Let $\hat{\boldsymbol{\varepsilon}}_{t|s}$ and $\boldsymbol{P}_{t|s}$ be the mean and variance of the unobserved vector $\boldsymbol{\varepsilon}_t$ conditional on information through time s. By linear projection, the observed $\boldsymbol{x}_t$ can be used to update the conditional distribution of $\boldsymbol{\varepsilon}_t$:

$$\begin{aligned} \hat{\boldsymbol{\varepsilon}}_{t|t} &= \hat{\boldsymbol{\varepsilon}}_{t|t-1} + \boldsymbol{P}^{\varepsilon x}_{t|t-1}\,(\boldsymbol{P}^{xx}_{t|t-1} + R^{xx}_t)^{-1}(\boldsymbol{x}_t - \hat{\boldsymbol{x}}_{t|t-1}) \\ \boldsymbol{P}_{t|t} &= \boldsymbol{P}_{t|t-1} - \boldsymbol{P}^{\varepsilon x}_{t|t-1}\,(\boldsymbol{P}^{xx}_{t|t-1} + R^{xx}_t)^{-1}\boldsymbol{P}^{x\varepsilon}_{t|t-1} \end{aligned} \qquad \textbf{(A.3)}$$

where

$\boldsymbol{P}^{\varepsilon x}_{t|t-1} = Cov_{t-1}(\boldsymbol{\varepsilon}_t,\ \boldsymbol{x}_t)$ is an $N \times n_t$ matrix consisting of columns of $\boldsymbol{P}_{t|t-1}$ corresponding to observed $\boldsymbol{x}_t$;

$\boldsymbol{P}^{x\varepsilon}_{t|t-1}$ is its transpose;

$\boldsymbol{P}^{xx}_{t|t-1}$ is an $n_t \times n_t$ submatrix of $\boldsymbol{P}_{t|t-1}$ based on selecting rows and columns corresponding to observed $\boldsymbol{x}_t$;

$\boldsymbol{R}^{xx}_t$ is an $n_t \times n_t$ diagonal matrix with $\boldsymbol{x}$-specific diagonal entries $\sigma_I^2 / N_{I,t}$; and

$\hat{\boldsymbol{x}}_{t|t-1} = \hat{\boldsymbol{\varepsilon}}^{x}_{t|t-1}$ is the $n_t \times 1$ $\boldsymbol{x}$-specific subvector of $\boldsymbol{\varepsilon}_{t|t-1}$.

If there were no idiosyncratic noise ($\boldsymbol{R}^{xx}_t = 0$), the $\boldsymbol{x}$-specific components of $\hat{\boldsymbol{\varepsilon}}_{t|t}$ would be known exactly, and the corresponding rows and columns of $\boldsymbol{P}_{t|t}$ would be zero. However, it would still be necessary to estimate unobserved values of $\boldsymbol{\varepsilon}_t$ based on the conditional covariance structure in (A.3).

The conditional distribution of next period's ε_{t+1} is given by

$$\begin{aligned} \hat{\varepsilon}_{t+1|t} &= F_t \hat{\varepsilon}_{t|t} \\ P_{t+1|t} &= F_t P_{t|t} F_t' \end{aligned} \qquad \textbf{(A.4)}$$

while next period's observed group-average residuals x_{t+1} are conditionally distributed $N[\hat{\varepsilon}^x_{t+1|t}, P^{xx}_{t+1|t} + R^{xx}_{t+1}]$. The log-likelihood of observed option pricing residuals is consequently

$$\begin{aligned} \ln L_{options} = -\tfrac{1}{2}\sum_t \Big[& \ln|P^{xx}_{t|t-1} + R^{xx}_t| + (x_t - \hat{x}_{t|t-1})'(P^{xx}_{t|t-1} + R^{xx}_t)^{-1}(x_t - \hat{x}_{t|t-1}) \\ & + \ln|S_t| + u_t' S_t^{-1} u_t \Big] \end{aligned} \qquad \textbf{(A.6)}$$

where $P_{1|0}$, the unconditional covariance matrix of ε_1, depends upon ρ and Q,[23] and $\hat{\varepsilon}_{1|0} = \mathbf{0}$.

The log likelihood function could in principle be optimized with regard to the option pricing parameters $<\{V_t\}, \theta>$ and the parameters $<\{\rho_I, \sigma_I\}_{I=1}^{64}, Q>$ governing the volatility and dynamics of option pricing residuals. As discussed in Watson and Engle (1983) in a strictly linear framework, sequential optimization over subsets of the parameters is convenient and reasonably efficient. Conditional upon the Kalman filtration parameters, optimization of (A.5) over $<\{V_t\}, \theta>$ involves nonlinear weighted least squares, and can be achieved by quadratic hill-climbing. Conditional on option parameters and the resulting option residuals $\{x_t, u_t\}_{t=1}^{T}$, optimization of (A.5) over $<\{\rho_I, \sigma_I\}_{I=1}^{64}, Q>$ could in principle also be optimized by quadratic hill-climbing.

[23]Given that serial persistence of common shocks is assumed to depend on delivery month whereas groups are categorized by maturity (0-1, 1-2, 2-3 and 3-6 months), computing $P_{1|0}$ is slightly tricky. The longest-maturity (3-6 month) groups have unconditional covariances $q_{IJ}/(1 - \rho_I \rho_J)$, where q_{IJ} is the (I, J)-th entry of Q. Unconditional covariances involving shorter-maturity groups are computed recursively off longer-maturity covariances based on an assumption of maturity shifts every 4 weeks, with the first shift occurring (for this data set) 2 weeks prior to January 4, 1988.

In practice, the high dimensionality of ε makes direct optimization of (A.5) with regard to filtration parameters quite slow. The major problem is estimating Q, which has $\frac{1}{2}(N^2 + N) = 2080$ free parameters in this case. Furthermore, nonlinear parameter transformations (Cholesky factorization) must be applied to ensure positive semidefinite Q, further slowing direct parameter optimization when the likelihood gradient is evaluated numerically. It is far more efficient in this case to estimate filtration parameters via an EM algorithm approach.

The EM algorithm of Dempster, Laird and Rubin (1977) proceeds in two alternating steps. First, the expectation of the *joint* log likelihood of option residuals and of the (unobserved) ε_t's is computed *conditional* upon observed residuals and an initial guess of the filtration parameters. Second, the expected log likelihood is maximized -- or at least increased -- with regard to its direct dependence upon filtration parameters, yielding new parameter values to be used in the first step.[24] The algorithm always increases the true log likelihood of option residuals (A.5), and a (local) optimum is attained when parameter estimates are no longer revised at the maximization step. For exponential distributions such as the one considered here, the expectation step is quite tractable and the steps that increase expected log likelihood can be computed analytically.

In this problem, the joint log likelihood of observed option residuals and common shocks is[25]

---

[24]Ruud (1984) notes that it is sufficient for the new parameter estimates to increase the expected log likelihood. Maximization is not necessary, and can slow the algorithm.

[25]It is not necessary to include missing option residuals in the log likelihood, since the relevant function is the joint log likelihood of *observed* data and underlying common shocks. Shumway and Stoffer's (1982) procedure of including missing observations and zeroing out relevant entries of vectors and matrices is equivalent to not including those missing data in the first place.

$$\begin{aligned}\ln L \;=\; & -\tfrac{1}{2}\ln|\boldsymbol{P}_{0|0}| \;-\; \tfrac{1}{2}\boldsymbol{\varepsilon}_0'\boldsymbol{P}_{0|0}^{-1}\boldsymbol{\varepsilon}_0 \\ & -\frac{T}{2}\ln|\boldsymbol{Q}| \;-\; \tfrac{1}{2}\sum_{t=1}^{T}(\boldsymbol{\varepsilon}_t - \boldsymbol{F}_{t-1}\boldsymbol{\varepsilon}_{t-1})'\boldsymbol{Q}^{-1}(\boldsymbol{\varepsilon}_t - \boldsymbol{F}_{t-1}\boldsymbol{\varepsilon}_{t-1}) \\ & -\frac{1}{2}\sum_{t=1}^{T}\sum_{I=1}^{64}\sum_{i\in G(I,t)}\left[\ln\sigma_I^2 + \frac{(e_{i,t} - \varepsilon_{I,t})^2}{\sigma_I^2}\right]\end{aligned} \qquad \textbf{(A.6)}$$

where $\boldsymbol{P}_{0|0}$ is the unconditional covariance matrix of the initial common shocks $\boldsymbol{\varepsilon}_0$.[26] The expectation of this conditional upon observed option residuals can be computed using a Kalman smoother:

$$\begin{aligned}E_T \ln L \;=\; & -\tfrac{1}{2}\ln|\boldsymbol{P}_{0|0}| \;-\; \tfrac{1}{2}\,trace\left[\boldsymbol{P}_{0|0}^{-1}\left(\hat{\boldsymbol{\varepsilon}}_{0|T}\hat{\boldsymbol{\varepsilon}}_{0|T}' + \boldsymbol{P}_{0|T}\right)\right] \\ & -\frac{T}{2}\ln|\boldsymbol{Q}| \;-\; \tfrac{1}{2}\sum_{t=1}^{T} trace\left[\boldsymbol{Q}^{-1}E_T(\boldsymbol{\varepsilon}_t - \boldsymbol{F}_{t-1}\boldsymbol{\varepsilon}_{t-1})(\boldsymbol{\varepsilon}_t - \boldsymbol{F}_{t-1}\boldsymbol{\varepsilon}_{t-1})'\right] \\ & -\frac{1}{2}\sum_{t=1}^{T}\sum_{I=1}^{64}\sum_{i\in G(I,t)}\left[\ln\sigma_I^2 + \frac{(e_{i,t} - \hat{\varepsilon}_{I,t|T})^2 + \boldsymbol{P}_{t|T}(I,I)}{\sigma_I^2}\right]\end{aligned} \qquad \textbf{(A.7)}$$

where

$$\begin{aligned}& E_T(\boldsymbol{\varepsilon}_t - \boldsymbol{F}_{t-1}\boldsymbol{\varepsilon}_{t-1})(\boldsymbol{\varepsilon}_t - \boldsymbol{F}_{t-1}\boldsymbol{\varepsilon}_{t-1})' \\ & \quad = (\hat{\boldsymbol{\varepsilon}}_{t|T} - \boldsymbol{F}_{t-1}\hat{\boldsymbol{\varepsilon}}_{t-1|T})(\hat{\boldsymbol{\varepsilon}}_{t|T} - \boldsymbol{F}_{t-1}\hat{\boldsymbol{\varepsilon}}_{t-1|T})' + \boldsymbol{P}_{t|T} \\ & \quad\quad - Cov_T(\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_{t-1})\boldsymbol{F}_{t-1}' - \boldsymbol{F}_{t-1}Cov_T(\boldsymbol{\varepsilon}_{t-1}, \boldsymbol{\varepsilon}_t) + \boldsymbol{F}_{t-1}\boldsymbol{P}_{t-1|T}\boldsymbol{F}_{t-1}'\end{aligned} \qquad \textbf{(A.8)}$$

and $\boldsymbol{P}_{t|T}(I,I)$ is the Ith diagonal term of $\boldsymbol{P}_{t|T}$. Smoothed conditional means and variances are computed by updating filtration-based estimates recursively backwards from the terminal values $\hat{\boldsymbol{\varepsilon}}_{T|T}$ and $\boldsymbol{P}_{T|T}$:

---

[26] $\boldsymbol{P}_{0|0}$ and $\boldsymbol{P}_{1|0}$ are related ($\boldsymbol{P}_{1|0} = \boldsymbol{F}_0\boldsymbol{P}_{0|0}\boldsymbol{F}_0' + \boldsymbol{Q}$) but not identical given that the unconditional covariance matrices have intramonthly seasonals determined by the timing of maturity shifts.

$$
\begin{aligned}
\hat{\boldsymbol{\varepsilon}}_{t|T} &= \hat{\boldsymbol{\varepsilon}}_{t|t} + \boldsymbol{J}_t(\hat{\boldsymbol{\varepsilon}}_{t+1|T} - \hat{\boldsymbol{\varepsilon}}_{t+1|t}) \\
\boldsymbol{P}_{t|T} &= \boldsymbol{P}_{t|t} + \boldsymbol{J}_t(\boldsymbol{P}_{t+1|T} - \boldsymbol{P}_{t+1|t})\boldsymbol{J}_t'
\end{aligned}
\qquad \textbf{(A.9)}
$$

where $\boldsymbol{J}_t = \boldsymbol{P}_{t|t}\boldsymbol{F}_t'\boldsymbol{P}_{t+1|t}^{-1}$.

Shumway and Stoffer (1982, p. 263) give a recursion for evaluating the autocovariances in (A.8), while Watson and Engle (1983) similarly advocate augmenting the state vector to include lagged variables. However, a simpler expression can be derived. Hamilton (1994, p. 395) shows that if next period's vector $\boldsymbol{\varepsilon}_{t+1}$ were observed, the conditional expectation $E_T[\boldsymbol{\varepsilon}_t \mid \boldsymbol{\varepsilon}_{t+1}]$ = $\boldsymbol{\varepsilon}_{t|t} + \boldsymbol{J}_t(\boldsymbol{\varepsilon}_{t+1} - \hat{\boldsymbol{\varepsilon}}_{t+1|t})$. Consequently,

$$
\begin{aligned}
E_T[\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_{t+1}'] &= E_T[E_T(\boldsymbol{\varepsilon}_t|\boldsymbol{\varepsilon}_{t+1})\boldsymbol{\varepsilon}_{t+1}'] \\
&= E_T\{[\hat{\boldsymbol{\varepsilon}}_{t|t} + \boldsymbol{J}_t(\boldsymbol{\varepsilon}_{t+1} - \hat{\boldsymbol{\varepsilon}}_{t+1|t})]\boldsymbol{\varepsilon}_{t+1}'\}
\end{aligned}
\qquad \textbf{(A.10)}
$$

Similarly from (A.9) above,

$$
\hat{\boldsymbol{\varepsilon}}_{t|T}\hat{\boldsymbol{\varepsilon}}_{t+1|T}' = [\hat{\boldsymbol{\varepsilon}}_{t|t} + \boldsymbol{J}_t(\hat{\boldsymbol{\varepsilon}}_{t+1|T} - \hat{\boldsymbol{\varepsilon}}_{t+1|t})]\hat{\boldsymbol{\varepsilon}}_{t+1|T}' \qquad \textbf{(A.11)}
$$

Subtracting (A.11) from (A.10) yields

$$
\begin{aligned}
Cov_T(\boldsymbol{\varepsilon}_t, \hat{\boldsymbol{\varepsilon}}_{t+1}) &\equiv E_T[\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_{t+1}'] - \hat{\boldsymbol{\varepsilon}}_{t|T}\hat{\boldsymbol{\varepsilon}}_{t+1|T}' \\
&= \boldsymbol{J}_t[E_T(\boldsymbol{\varepsilon}_{t+1}\boldsymbol{\varepsilon}_{t+1}') - \hat{\boldsymbol{\varepsilon}}_{t+1|T}\hat{\boldsymbol{\varepsilon}}_{t+1|T}'] \\
&= \boldsymbol{J}_t\boldsymbol{P}_{t+1|T}.
\end{aligned}
\qquad \textbf{(A.12)}
$$

Direct substitution confirms that this solution satisfies Shumway and Stoffer's recursion.

Apart from the nonlinear dependency of the initial unconditional covariance matrix $\boldsymbol{P}_{0|0}$ upon parameter values, optimizing (A.7) with regard to $<\{\rho_l, \sigma_l\}_{l=1}^{64}, \boldsymbol{Q}>$ is a relatively straightforward linear exercise in estimating a constrained vector autoregression (VAR) using modified moments. Improved estimates of idiosyncratic noise are generated by

$$\hat{\sigma}_I^2 = \frac{1}{N_I} \sum_{t=1}^{T} \sum_{i \in G(I,t)} \left[(e_{i,t} - \hat{\varepsilon}_{I,t|T})^2 + P_{t,T}(I,I) \right], \tag{A.13}$$

where $N_I = \sum_t N_{I,t}$. Conditional on earlier Q estimates, improved $\{\rho_I\}$ estimates (which determine $F_t = D_\rho A_t$) are given by the constrained VAR moment conditions in Hamilton (1994, p. 318):

$$\hat{\rho} = \begin{pmatrix} q^{11}\sum_t E_T(x_{1t}x_{1t}) & \cdots & q^{1n}\sum_t E_T(x_{1t}x_{nt}) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ q^{n1}\sum_t E_T(x_{nt}x_{1t}) & \cdots & q^{nn}\sum_t E_T(x_{nt}x_{nt}) \end{pmatrix}^{-1} \times \begin{pmatrix} \sum_t \sum_j q^{1j} E_T(x_{1t}y_{jt}) \\ \cdot \\ \cdot \\ \sum_t \sum_j q^{nj} E_T(x_{nt}y_{jt}) \end{pmatrix} \tag{A.14}$$

where

q^{ij} is the (i, j)-th element of Q^{-1};

$E_T(x_{it}x_{jt})$ is the (i, j)-th element of $A_{t-1}(\hat{\varepsilon}_{t-1|T}\hat{\varepsilon}_{t-1|T}' + P_{t-1|T})A_{t-1}'$; and

$E_T(x_{it}y_{jt})$ is the (i, j)-th element of $A_{t-1}(\hat{\varepsilon}_{t-1|T}\hat{\varepsilon}_{t|T}' + J_{t-1}P_{t|T})$.

Conditional upon the $\{\rho_I\}$ estimates, improved Q estimates are given by

$$\hat{Q} = \frac{1}{T}\sum_{t=1}^{T}\Big[(\hat{\varepsilon}_{t|T} - F_{t-1}\hat{\varepsilon}_{t-1|T})(\hat{\varepsilon}_{t|T} - F_{t-1}\hat{\varepsilon}_{t-1|T})' + P_{t|T} + P_{t|T}J_{t-1}'F_{t-1}' + F_{t-1}J_{t-1}P_{t|T} + F_{t-1}P_{t-1|T}F_{t-1}\Big] \tag{A.15}$$

Each step in the EM algorithm estimation of $<\{\rho_I, \sigma_I\}_{I=1}^{64}, Q>$ therefore consists of the following steps:

1. applying a Kalman smoother to estimate $\{\hat{\varepsilon}_{t|t}, \hat{\varepsilon}_{t|T}, P_{t|t}, P_{t|T}, J_t\}$ and other relevant moments conditional upon particular parameter values $<\{\rho_I, \sigma_I\}, Q>$;

2. Revising estimates of $\{\rho_I, \sigma_I\}_{I=1}^{64}$ based upon estimated summary statistics from the first step;

3. Revising estimates of Q based upon estimated summary statistics and upon revised estimates of $\{\rho_I\}$.

The estimation procedure ensures positive definite estimates for Q, while the final parameter estimates $<\{\rho_I, \sigma_I\}_{I=1}^{64}, Q>$ from repeated applications of the algorithm approximately optimize (A.5).[27]

Conditional upon fixed-point estimates of $<\{\rho_I, \sigma_I\}_{I=1}^{64}, Q>$ from the EM algorithm, option-specific parameters $<\{V_t\}, \theta>$ were estimated by optimizing the nonlinear weighted least squares function (A.5) via the Davidon-Fletcher-Powell (DFP) algorithm, using a numerically computed gradient. The DFP and EM optimizations were alternated until joint convergence. The bulk of the computer time was taken up in the DFP stage.

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[27]The maximization step ignores the direct dependency of $P_{0|0}$ upon ρ and Q, so that the EM algorithm approach is not exactly equivalent to optimizing (A.5). This difference arises from the difference between maximum likelihood- and regression-based estimation of VAR's, and is not an issue in large samples.

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Table 1. S&P 500 futures options, 1988-93: implicit parameter estimates.

	Stochastic volatility parameters					autocorrelations				
Model	α	β^*	β	σ_v	ρ	range	med.	SE1[a] $(\times 10^2)$	SE2[a] $(\times 10^2)$	**ln $L_{options}$**
One-factor models										
BS	0	0		0	0	[.35, .79]	.55	.221	.113	249,793.27
DV1	.032	.55		0	0	[.36, .81]	.59	.218	.106	250,004.64
SV1	.100	1.49		.742	-.571	[.31, .84]	.58	.110	.078	255,335.03
SVJD1	.049	2.45		.378	-.545	[.24, .78]	.54	.098	.077	256,483.87
Two-factor models										
DV2	.112	7.14		0	0	[.26, .80]	.56	.209	.102	250,435.88
	.010	.00		0	0					
SV2	.028	.00		1.029	-.770	[.20, .76]	.52	.087	.066	256,995.76
	.130	5.67		.669	-.385					
SVJD2	.003	1.07		.560	-.851	[.15, .77]	.50	.079	.069	258,064.59
	.033	.02		.333	-.412					
Constrained estimates										
SVC1	.089	1.21	3.45	.693	-.584	[.30, .83]	.56	.110	.078	255,137.47
SVJDC1	.043	2.48	2.25	.322	-.599	[.23, .78]	.54	.099	.077	256,475.28
SVC2	.064	.00	4.31	.945	-1.000	[.20, .76]	.52	.087	.066	256,542.57
	.007	.01	1.10	.152	.279					
SVJDC2	.018	1.65	3.21	.485	-1.000	[.11, .75]	.49	.078	.068	257,713.02
	.011	.01	.85	.192	-.324					

Jump parameters

SVJD1:	$\lambda_t^* = .0000 + 27.19\, V_t$,	$\bar{k}^* = -.095,\ \delta = .109.$
SVJDC1:	$\lambda_t^* = .0000 + 31.62\, V_t$,	$\bar{k}^* = -.085,\ \delta = .113.$
SVJD2:	$\lambda_t^* = .0002 + 87.64\, V_{1t} + .06\, V_{2t}$,	$\bar{k}^* = -.057,\ \delta = .102.$
SVJDC2:	$\lambda_t^* = .0000 + 88.55\, V_{1t} + .00\, V_{2t}$,	$\bar{k}^* = -.054,\ \delta = .102.$

[a]SE1 (SE2) is the overall equally weighted standard error of option residuals as a fraction of the underlying asset price, ignoring (adjusting for) estimated serial correlation in option residuals.

Table 2
Maximum likelihood estimates of the process followed by implicit factor realizations.

$$dV = (\alpha - \beta V)dt + \sigma_v \sqrt{V} dW_v$$

	Stochastic volatility parameters						
	α	**β**	**σ_v**	**$\sqrt{\alpha/\beta}$**	**half-life[a] (mths)**	**ln $L_{\{V\}}$**	**Constrained ln $L_{\{V\}}$ (*P*-value)[b]**
Unconstrained models							
SV1	.121 (.030)	4.64 (1.21)	.268 (.011)	.162 (.011)	1.8 (0.5)	18.80	-175.91 (0)
SVJD1	.093 (.024)	4.35 (1.17)	.215 (.009)	.146 (.010)	1.9 (0.5)	58.62	-19.17 (0)
SV2	.031 (.005)	1.98 (1.65)	.502 (.021)	.125 (.051)	4.2 (3.5)	-425.07	-603.44 (0)
	.047 (.008)	4.89 (2.09)	.569 (.023)	.098 (.020)	1.7 (0.7)	-416.80	-438.12 (6e-10)
SVJD2	.061 (.012)	6.08 (1.50)	.237 (.010)	.100 (.008)	1.4 (0.3)	-119.26	-313.33 (0)
	.013 (.005)	1.79 (1.17)	.301 (.012)	.085 (.027)	4.6 (3.0)	-170.26	-175.65
Contrained models							
SVC1	.119 (.030)	4.58 (1.19)	.261 (.011)	.161 (.011)	1.8 (0.5)	25.10	-157.19 (0)
SVJDC1	.090 (.024)	4.32 (1.16)	.209 (.009)	.144 (.010)	1.9 (0.5)	63.61	+12.78 (0)
SVC2	.010 (.006)	.98 (1.00)	.333 (.013)	.099 (.051)	8.4 (8.6)	-193.64	-414.41 (0)
	.071 (.016)	5.97 (1.30)	.139 (.006)	.109 (.005)	1.4 (0.3)	116.16	103.14 (9e-6)
SVJDC2	.057 (.012)	5.42 (1.38)	.215 (.009)	.102 (.008)	1.5 (0.4)	-75.17	-231.35 (0)
	.001 (.001)	.70 (.66)	.174 (.007)	.045 (.025)	11.8(11.2)	-7.38	-11.56 (1e-4)

[a]The half-life $12 \ln 2 / \beta$ is in months. All other parameters are in annualized units.

[b]Constrained log likelihood reflects the imposition of implicit parameter estimates from Table 1. *P*-values are from corresponding χ_2^2 (χ_3^2) likelihood ratio tests for the SV/SVJD (SVC/SVJDC) models.

Table 3
Implicit factor evolution: summary statistics of normalized residuals.

Model	Mean	SD	Skew-ness	Excess Kurtosis	range	SW *P*-value[a]
Normalization using parameters inferred from option prices						
SVC1	.066	.370	.54	4.32	[-1.64, 1.88]	$4 * 10^{-9}$
SVJDC1	.074	.635	.66	4.40	[-2.78, 3.11]	$2 * 10^{-12}$
SVC2	.319	.399	-.60	19.05	[-2.91, 2.82]	
	.055	.870	-.22	15.17	[-6.00, 6.00]	0
SVJDC2	.221	.433	.99	6.98	[-1.69, 2.62]	0
	.019	.879	-1.15	11.67	[-6.00, 4.39]	0
						0
Normalization using maximum likelihood time series estimates						
SVC1	-.007	1.002	.956	4.47	[-4.12, 4.90]	0
SVJDC1	-.006	1.002	.883	4.36	[-4.19, 4.75]	0
SVC2	.097	1.005	.79	9.01	[-5.40, 6.00]	0
	.000	.952	.153	10.25	[-6.00, 6.00]	0
SVJDC2	-.019	1.001	1.48	7.44	[-3.85, 5.64]	0
	.074	.967	-.76	10.87	[-6.00, 5.47]	0

[a]Shapiro-Wilks test of normality.

Table 4
Summary statistics for log-differenced noon S&P 500 futures prices, 1988-93

	Daily	**Weekly**
observations	1515	307
mean	.0003	.0011
standard deviation	.0089	.0183
skewness	-.316	-.381
Excess kurtosis	3.52	.696
minimum	-.058	-.062
maximim	+.050	+.055
H_0: normal (P-value)	.000	.702

Table 5
Maximum likelihood estimates of the S&P 500 futures price process *conditional* on stochastic variance parameters and weekly factor realizations inferred from options prices.[a]

Jump frequency: $\lambda_t = l_0 + l_1 \lambda_t^*$

Instantaneous variance: $V_t = V_0 + d_1 V_{1t}^{options} + d_2 V_{2t}^{options}$

	Jump parameters				Variance parameters			
Model	l_0	l_1	$\bar{k}$	δ	V_0	d_1	d_2	$\ln L_{\{F\}}$
SV1					.00360	.465		813.91
					(.00188)	(.105)		
SVJD1	0	1	-.096[b]	.108[b]	.00120	.671		812.47
					(.00227)	(.139)		
SVJD1	.291	.000	.064	.000[c]	.00000[c]	.707		815.11
	(.300)	(.002)	(.023)			(.072)		
SV2					.03207	.455	.469	815.58
						(.114)	(.164)	
SVJD2	0	1	-.057[b]	.102[b]	.00000[c]	.719	.772	815.06
						(.217)	(.131)	
SVJD2	.368	.05	.063	.000[c]	.00000[c]	.679	.656	817.71
	(.297)	(1.02)	(.017)			(.152)	(.117)	

[a]Wednesday noon log-differenced future prices, 1988-93, 309 observations. Asymptotic standard errors are in parentheses.

[b]Parameter set equal to value inferred from option prices.

[c]Nonnegativity constraint binding.

Table 6
Correlation estimates for weekly S&P 500 futures returns and implicit factor shocks

$$Corr(\Delta \ln F,\ \Delta V_1,\ \Delta V_2) \equiv \begin{pmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_{vv} \\ \rho_2 & \rho_{vv} & 1 \end{pmatrix}$$

Model	One-factor models (ρ_1)	Two-factor models		
		ρ_1	ρ_2	ρ_{vv}
SV	-.614	-.518	.004	-.580
SVJD	-.615	-.559	.035	-.515
SVC	-.616	-.561	-.179	-.152
SVJDC	-.615	-.612	.037	-.324

Figure 1

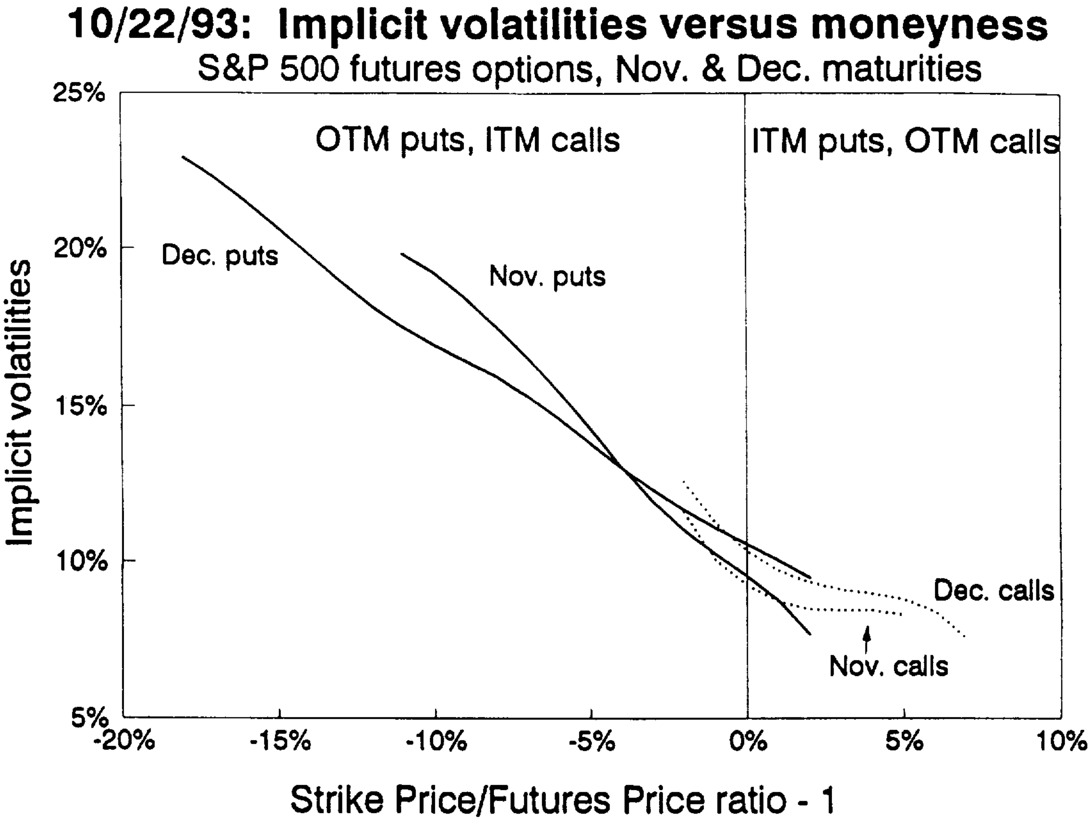


Figure 2

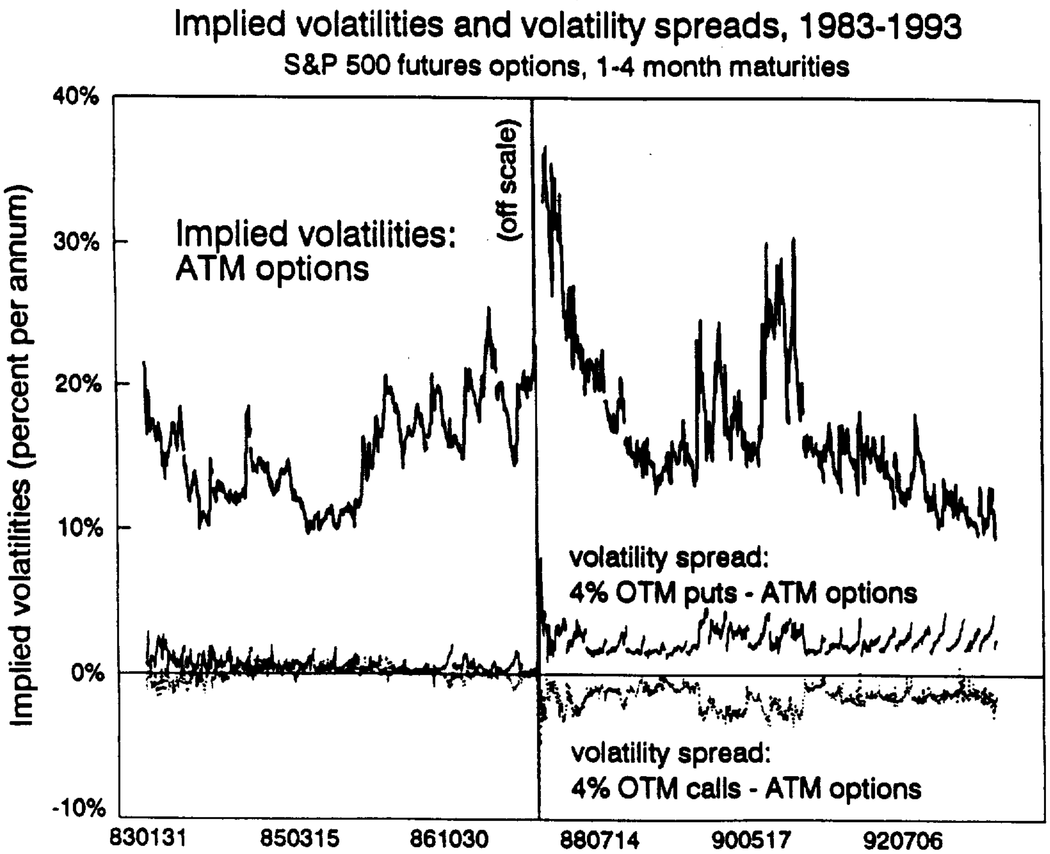
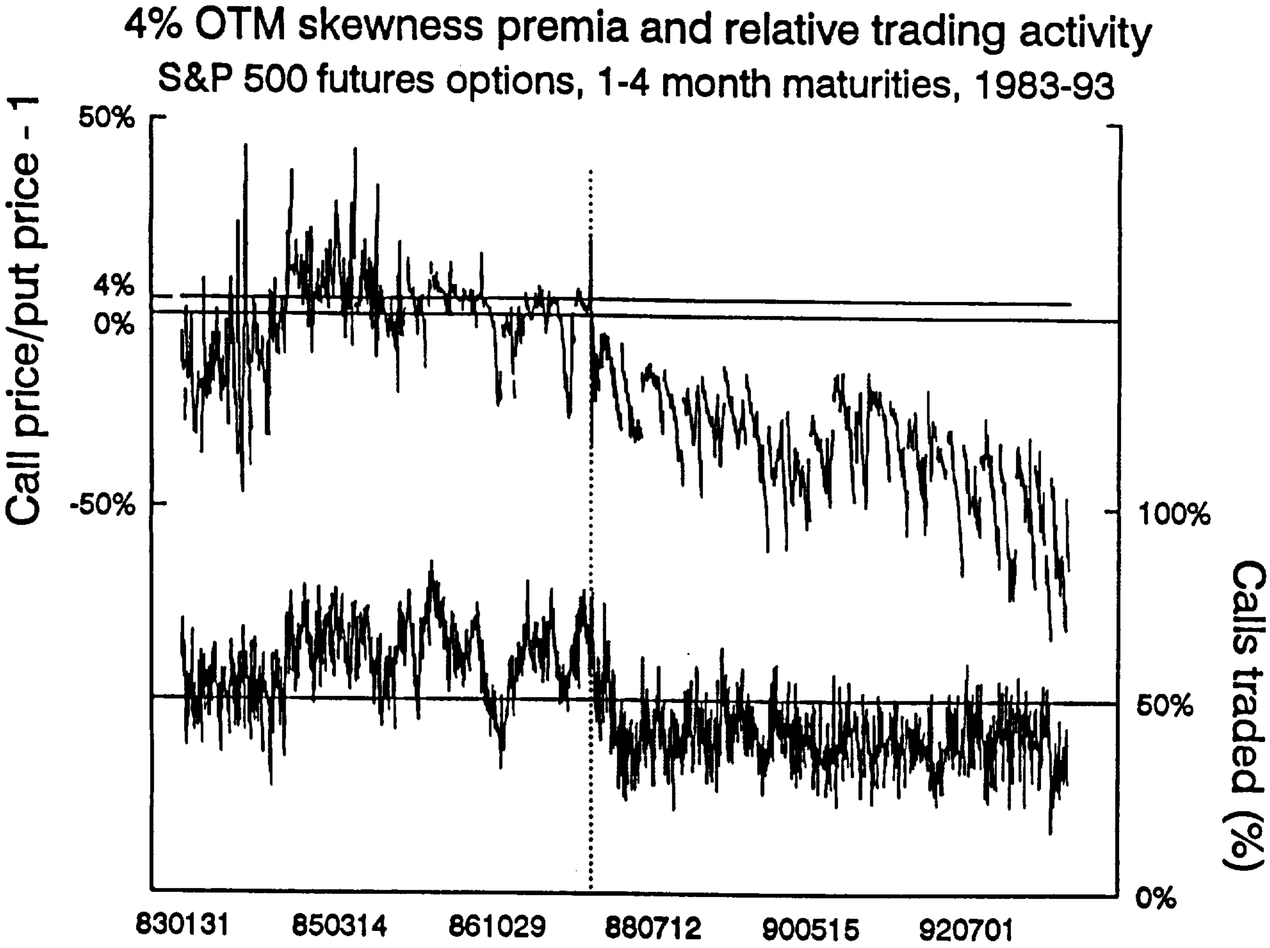


Figure 3


4% OTM skewness premia and relative trading activity
S&P 500 futures options, 1-4 month maturities, 1983-93
Call price/put price - 1
50%
4%
0%
-50%
100%
50%
0%
Calls traded (%)
830131
850314
861029
880712
900515
920701

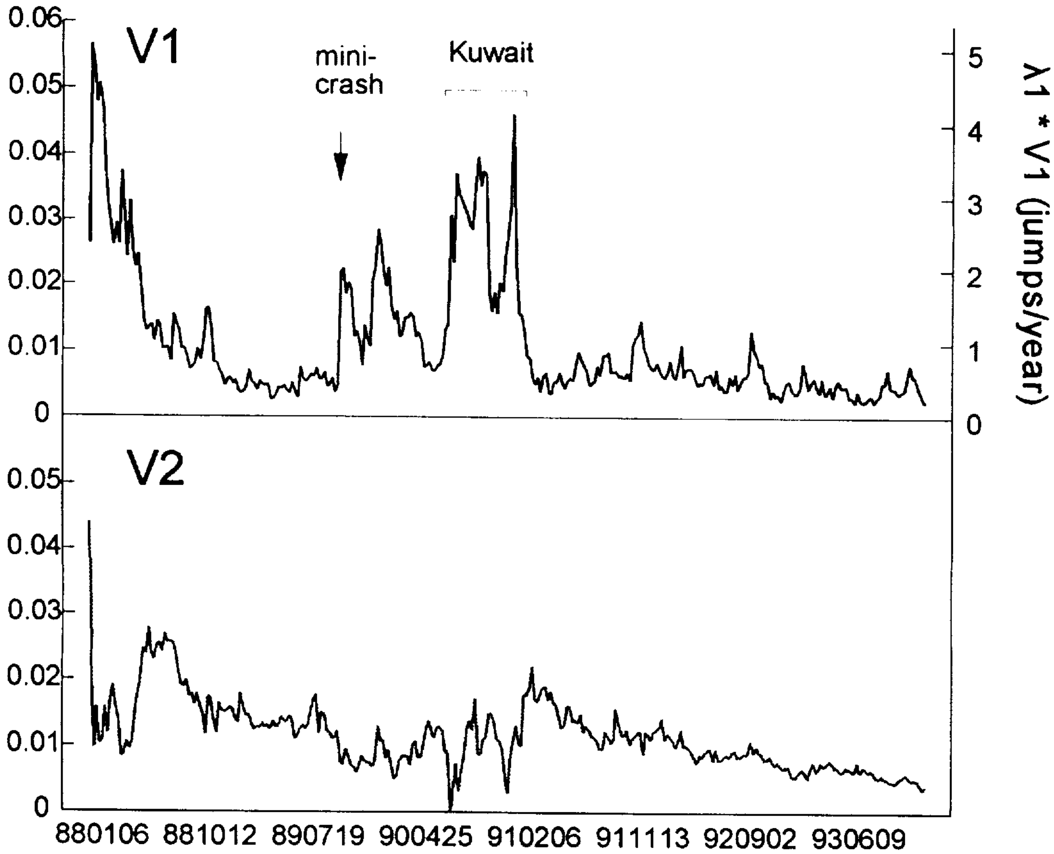


Figure 4. Implicit factor estimates from constrained (smoothed) stochastic volatility/jump-diffusion model (SVJDC2).

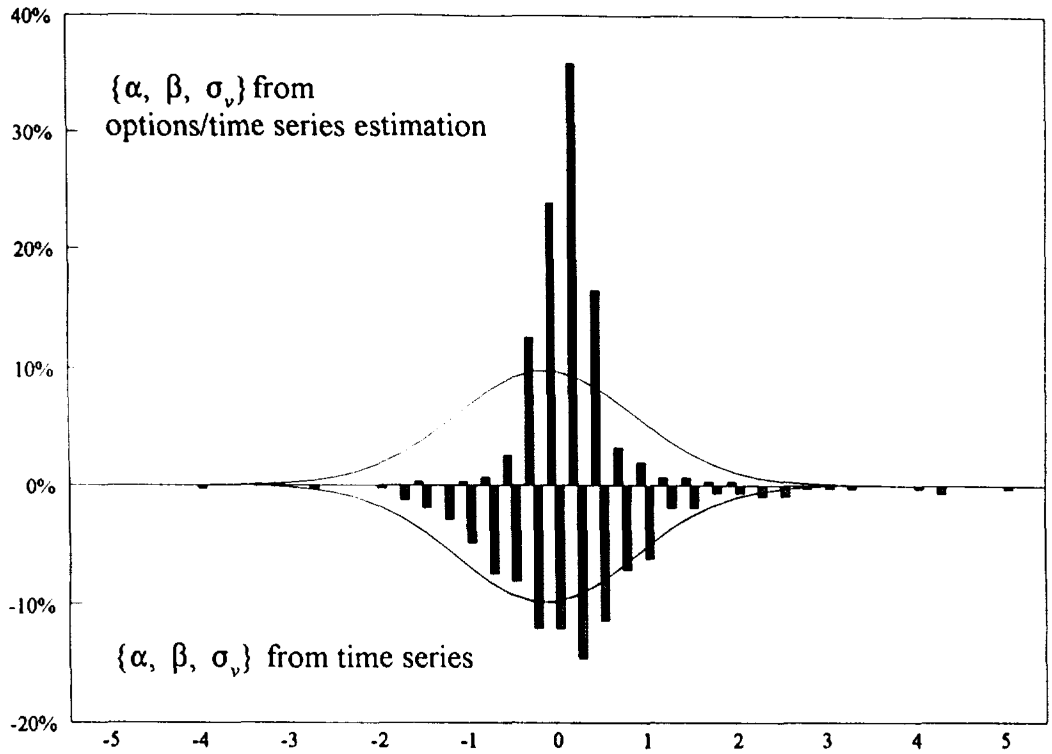


Figure 5. Distribution of "normalized" implicit spot variance innovations: 1- factor constrained stochastic volatility model (SVC1).