NBER WORKING PAPER SERIES

UNDERSTANDING EQUILIBRIUM MODELS WITH A SMALL AND A LARGE NUMBER OF AGENTS

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Working Paper 5792

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
October 1996

The author would like to thank Martin Lettau, Valerie Ramey, Tony Smith, Harold Uhlig, and Hal White for useful comments. This paper is part of NBER's research program in Economic Fluctuations and Growth. Any opinions expressed are those of the author and not those of the National Bureau of Economic Research.

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ABSTRACT

In this paper, I compare a two-agent asset pricing model with the corresponding model with a continuum of agents. In a two-agent economy, interest rates respond to "idiosyncratic" income shocks because each agent represents half of the population. These interest rate effects facilitate consumption smoothing. An agent in a two-agent economy, however, can never lend more than the other agent is allowed to borrow, which prevents him from building a buffer stock of assets. For most parameter values, the first effect is more important. For some parameter values, the interest rate effects in the two-agent economy are so strong that a relaxation of the borrowing constraint reduces an agent's utility. In contrast to these differences, I find that for most parameter values there are no large differences in average interest rates across the two types of economies. In addition, I analyze the business cycle behavior of interest rates in an incomplete markets economy with a continuum of agents. The dynamic response of interest rates to aggregate shocks is a lot more complicated than the response in a complete markets economy and the magnitude of the response is bigger.

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1. INTRODUCTION.

Many papers in the macroeconomics literature argue that heterogeneity is important for explaining the time series behavior of macroeconomic variables. For example, when the lack of frictionless trading in a complete set of financial assets prevents agents from diversifying idiosyncratic income risk, this risk typically affects aggregate variables like asset prices. In this case, the standard assumptions of the representative agent framework are violated and the amount of heterogeneity in agents' wealth and income levels affects the behavior of the variables in the model. Moral hazard, adverse selection, limited liability, and transaction costs can undermine the plausibility of economic models with frictionless complete markets. Of course, the empirical failures of the representative agent framework also have contributed to the popularity of models with heterogeneous agents and incomplete markets. Models with heterogeneous agents have now been used for studying a wide range of topics in macroeconomics including asset pricing,[1] consumption behavior,[2] investment behavior,[3] the monetary transmission mechanism,[4] and the international business cycle.[5]

Quantitative analysis plays a crucial role in economics. Solving dynamic equilibrium models with heterogeneous agents, however, is a challenging problem because the cross-sectional distribution of agents' characteristics typically is a high dimensional endogenous object that is part of the set of state variables. A standard way to avoid this difficulty is to assume that there are only two different (types of) agents in the economy.[6] If the quantitative implications of models with only two heterogeneous agents are to be taken serious as predictions about real world economies, then either the difficulty of solving models with many agents has to be overcome or it has to be shown that the properties that economists are interested in do not depend on the simplifying assumption of having only two agents. In this paper, I use the algorithm developed in Den Haan (1996b) to solve an incomplete markets economy with a continuum of agents and compare its properties to those of a two-agent economy.

Heaton and Lucas (1996) point out that the quantitative asset price predictions from this class of models depend critically on the correlation structure of idiosyncratic shocks and aggregate shocks. In this paper, I will show that this correlation structure is the crucial difference between a two-agent economy and the corresponding economy with a continuum of agents. The reason is that in a two-agent economy all shocks, including individual shocks, have aggregate effects since they affect at least half the population,[7] while in an economy with a large number of agents individual shocks do not affect the amount of cross-sectional dispersion. To understand the implications of this difference consider the required expected rate of return on equity relative to the risk-free rate, i.e., the equity premium. Standard

[1] See, for example, Aiyagari and Gertler (1991), Constantinides and Duffie (1996), Danthine, Donaldson, and Mehra (1992), Den Haan and Spear (1996), Heaton and Lucas (1992,1996), Lucas (1994), Mankiw (1986), Marcet and Singleton (1990), Telmer (1993), Weil (1992), and Zhang (1993).

[2] See, for example, Deaton (1991) and Pischke (1995).

[3] See, for example, Aiyagari (1994).

[4] See, for example, Bernanke and Blinder (1989), Bernanke, Gertler, and Gilchrist (1996), Christiano and Eichenbaum (1995), Fuerst (1992), and Gertler and Gilchrist (1994).

[5] See, for example, Baxter (1995), Baxter and Crucini (1995), and Kouparitsas (1996).

[6] See, for example, Baxter and Crucini (1995), Cole and Obstfeld (1991), Danthine, Donaldson, and Mehra (1992), Den Haan and Spear (1996), Heaton and Lucas (1996), Lucas (1994), Marcet and Singleton (1990), Marcet and Marimon (1992), Rotemberg (1984), Saito (1991), Scheinkman and Weiss (1986), Telmer (1993), and Zhang (1993).

[7] A purely idiosyncratic shock, i.e., a shock that does not affect aggregate quantities, affects both agents, i.e., affects the entire population.

asset pricing theories say that this premium is related to the correlation between the marginal utility of individual consumption and the return on equity. This correlation is likely to be quite different across the two types of economies since in a two-agent economy individual shocks have aggregate effects and thus affect not only individual consumption but also prices and rates of return. For some questions a model with two different types of agents would provide a better description of the economic environment than a model with a continuum of agents. This would be the case, for example, if one studies sectoral shifts. In the macrofinance literature, however, the agents in the model are supposed to represent individual consumers and it is clear that the appropriate model is a model with a large number of agents. Using a continuum as opposed to a large but finite number of agents simplifies the analysis without affecting the results.[8]

To analyze the quantitative importance of these arguments, I analyze an infinite-horizon equilibrium model of the short-term interest rate. Markets are assumed to be incomplete, but agents can smooth their consumption by trading in risk-free bonds. In the first economy that I study, there are only two different agents, while in the second economy, there are a continuum of agents. In all other aspects, the economies are identical. In particular, the stochastic properties of the individual and the per capita driving processes are exactly the same.

As mentioned above, the time-series variation in the amount of cross-sectional wealth dispersion differs considerably across these two economies because in two-agent economies "idiosyncratic" shocks affect the amount of cross-sectional dispersion. This paper documents how and why these differences in the time-series properties of the amount of cross-sectional wealth dispersion affect the time-series properties of interest rates, consumption, and bond trades. Not surprisingly, the higher volatility of cross-sectional dispersion in the two-agent economies implies a much higher volatility of interest rates as well. Heaton and Lucas (1996) mention that "*Typically in models that fit the equity premium, the resulting volatility of the bond return is too high*". This issue has only been addressed in two-agent economies; in economies with a large number of agents the observation may not hold. A key feature of the model is that the interest rate is a concave decreasing function of the amount of cross-sectional dispersion. Jensen's inequality then implies that average interest rates are lower in economies with a smaller number of agents since the volatility of the amount of cross-sectional dispersion is much higher in this type of economy. For most parameter values, however, average interest rates are very similar in the two types of economies. Substantial differences, however, can be found for low values of the borrowing constraint parameter. When agents are not allowed to borrow more than 20% of the per capita endowment and the rate of relative risk aversion is equal to five (three), for example, then the difference in the average interest rates is equal to 1.0 (0.7) percentage points.

Another striking difference between the two-agent economy and the economy with a continuum of agents is the ability of the agent to insure himself against idiosyncratic shocks. In two-agent economies, it is harder to insure against idiosyncratic risk, in one sence, but easier in another. It is harder to smooth

[8] Den Haan (1996b) compares the solution to the economy with a continuum of agents to a simulation with 300,000 agents and finds that the results are very similar.

consumption in a two-agent economy because an agent can never lend more than the other agent is allowed to borrow. This will prevent him from building up a large buffer stock during good times. In contrast, agents in an economy with a continuum of agents can lend to several other agents and at times accumulate assets well in excess over the maximum possible in a two-agent economy.

However, is easier to smooth consumption in a two-agent economy because the interest rate responds to idiosyncratic shocks in such a way that it becomes easier for agents to insure themselves against idiosyncratic shocks. The explanation is the following. Consider the case where the same agent receives a low realization of the idiosyncratic shock for several periods. In the two-agent economy this means that the amount of cross-sectional dispersion increases and this causes the interest rate to drop. This change in the interest rate due to "idiosyncratic" shocks works like a transfer from the rich agent (the lender) to the poor agent (the borrower). In an economy with a continuum of agents, an agent who faces the same sequence of low income realizations is not so lucky to see the interest rate drop at the same time. Consequently his consumption would drop by more than the consumption of the agent in the two-agent economy. Although this second effect usually dominates, there are parameter values for which the first effect is more important.

Quantitatively, these effects are important. When the agent is allowed to borrow twenty percent of the per capita endowment and the parameter of relative risk aversion is equal to five, for example, then the agent in the economy with a continuum of agents would be willing to reduce his consumption permanently by approximately one percent to be in a two-agent economy. At higher levels of the parameter of relative risk aversion the effects of idiosyncratic shocks on interest rates become so strong that a relaxation of the borrowing constraint has a different effect on an agent's welfare in the two types of economies. In economies with a continuum of agents, an agent's welfare is always increasing when the maximum amount he is allowed to borrow increases. In contrast, when the parameter of relative risk aversion is equal to five, an agent's welfare in two-agent economies is at first increasing with a relaxation of the borrowing constraint but then decreasing.

In addition, I analyze the business cycle behavior of interest rates in an incomplete markets economy with a continuum of agents. The business cycle behavior of interest rates in economies with incomplete markets and a large number of agents has not been studied in the literature, because the lack of a solution algorithm made it infeasible to do this. Nevertheless, this is an important topic. For example, market frictions are believed to play an important role in the way monetary shocks affect interest rates and output. I document that the dynamic response of the interest rate to aggregate shocks is a lot more complicated than the response in a complete markets economy and the magnitude of the response is bigger.

The organization of this paper is as follows. In the next section, an infinite-horizon endowment economy with incomplete markets will be discussed. In Section 3, I outline the solution method. In Section 4, I compare the economy with two types with the economy with a continuum of types. The last section concludes.

2. A DIVERSE AND A LESS DIVERSE ECONOMY.

In this section, I develop an infinite horizon model of the short-term interest rate similar to the models in Deaton (1991) and Pischke (1995). In contrast to Deaton (1991) and Pischke (1995), the interest rate is not constant but varies to ensure that the bond market is in equilibrium. If the interest rate is an exogenous variable, then agents do not interact with each other, and the model can just as well be interpreted as a single agent economy. For computational reasons, most papers in the literature that analyze equilibrium models assume that there are only two types of agents for computational reasons. In this paper, two versions of the model are developed. In the first version, there are only two types of agents. In the second version, there are a continuum of agents. In Section 4, I investigate how restrictive this assumption is by comparing an economy with two types of agents to an economy with a continuum of different agents.

2.1. THE INDIVIDUAL AGENT'S PROBLEM.

The first economy considered consists of two infinitely-lived agents, labeled x and y. Ex ante the agents are exactly the same, but ex post they differ due to the presence of idiosyncratic shocks. In particular, the endowment of agent x relative to the per capita endowment, y_t^x, can take on a low value, y^L, and a high value, y^H. The (gross) growth rate of the aggregate endowment, $ya_t = YA_t/YA_{t-1}$, can also take on a low (or recession) value, ya^R, and a high (or boom) value, ya^B.[9] Both processes are assumed to be first-order Markov processes. Markets are assumed to be incomplete, but, as in Deaton (1991) and Pischke (1995), the agents can smooth their consumption by trading in a risk-free one-period bond. Telmer (1993), Heaton and Lucas (1996), and Lucas (1995) focus on the equity premium and also allow agents to invest in equity. I choose to work with a simpler investment environment because this makes it easier to understand why and how the number of agents affects the behavior of asset prices, consumption and investment.

Agent i's maximization problem ($i=\{x,y\}$) is as follows:

$$\max_{\{C_t^i, B_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \, U(C_t^i)$$

s.t.

$$C_t^i + q_t B_t^i = y_t^i YA_t + B_{t-1}^i, \tag{2.1}$$

where C_t^i is the amount of consumption of agent i in period t, B_t^i is the demand for one-period bonds that pay one unit of the consumption commodity in the next period, q_t is the price of this one-period bond, and $B_{-1}^x = (-B_{-1}^y)$ is given. The interest rate, r_t, is defined as $(1-q_t)/q_t$. I assume that the interest rate adjusts to

[9] In this paper, all stationary variables are denoted by small characters and all non-stationary variables are denoted by capital characters.

ensure that the total demand for bonds is equal to zero. In an economy in which all shocks are observed without costs, it would be optimal to write contracts contingent on the realization of the idiosyncratic shock. In a world in which it is impossible for the lender to verify the realization of the borrower's income, however, the optimal one-period contract is a bond with a fixed payment, since the borrower would always report that he received the lowest possible realization.[10]

Next, the question arises whether a lender would want to restrict the amount he is willing to lend to a borrower. It would make sense to limit the amount of debt by the net present value of the borrower's endowment stream. In this economy, in which agents live and receive an endowment stream for ever, this constraint is unlikely to seriously restrict the demand for loans. In reality, however, consumers and firms do seem to face borrowing constraints.[11] More restrictive borrowing constraints arise in our model if one makes the assumption that the borrower faces limited costs of defaulting on the loan. Suppose that the default costs are equal to BC_t. In this case, borrowers will default whenever $-B_t \geq BC_t$. This, of course, implies that the lender will never lend more than BC_t. I assume that BC_t (scaled by the per capita endowment) is a constant. Thus, the following borrowing constraint is added to the maximization problem:

(2.2) $$B_t^i \geq - bcYA_t .$$

Note that in my notation, B_t^i includes the interest payments on the bond. Let $\overline{B}_t^i$ ($= q_t\ B_t^i$) denote the amount of savings net of interest payments. Then Equation (2.2) can be rewritten as follows

(2.3) $$\overline{B}_t^i \geq - \frac{bcYA_t}{1+r_t} .$$

Equation (2.3) is a typical formulation of a borrowing constraint.[12] It has the property that the maximum amount an agent can borrow is decreasing with the interest rate. The first-order conditions for the maximization problem of the agent are the two-part Kuhn-Tucker conditions:

(2.4) $$q_t [C_t^i]^{-\gamma} \geq \beta \mathrm{E}_t [C_{t+1}^i]^{-\gamma} \text{ and}$$
$$(B_t^i + bcYA_t)(q_t [C_t^i]^{-\gamma} - \beta \mathrm{E}_t [C_{t+1}^i]^{-\gamma}) = 0.$$

Note that in the two-agent economy one agent's individual shock is perfectly negatively correlated with the other agent's individual shock. Without this assumption, it would be impossible to simultaneously have plausible values for the amount of idiosyncratic risk and the amount of aggregate risk. In the economy with a continuum of agents, individual shocks are distributed independently across agents and thus truly are idiosyncratic shocks. Note that even if individual shocks would be independent in the two-agent economy, they still can not be viewed as idiosyncratic shocks since they affect half the population. The time-series process for the idiosyncratic shock of an agent and the time-series process for the growth rate of per capita output is exactly the same in both types of economies.

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[10] Wang (1993) shows that contracts contingent on the realization of the shock are possible when multi-period contracts are enforceable.
[11] See Jaffee and Stiglitz (1990, Section 5.3) and the references therein and Zeldes (1989).
[12] See, for example, Bernanke, Blinder, and Gilchrist (1996).

In the economy with two types of agents, there are, at each point in time, only two different wealth levels. The economy will go through states where they are very different and through states where they are very similar. In the economy with a continuum of agents, there will be a wide variety of wealth levels at each point in time. The cross-sectional wealth dispersion will only change in response to changes in the aggregate growth rate. In an economy without aggregate uncertainty, the cross-sectional wealth distribution is time-invariant.[13] In contrast, in the economy with two agents the cross-sectional distribution depends on the realizations of the idiosyncratic shock and will change over time even when there is no uncertainty in the driving process of the aggregate endowment.

Several papers in the literature discuss the existence of equilibrium. Existence of equilibrium of equilibrium with borrowing constraints in an economy like ours but with a finite number of agents is discussed in Magill and Quinzii (1994). Den Haan (1996b) compares the numerical solution to the model with a continuum of agents with the simulated results of an economy with 300,000 observations and finds that the results are very similar. This suggests that there are no differences between an economy with a continuum of agents and an economy with a large but finite number of agents. Related papers are Duffie, Geanokoplos, Mas-Colell, and McLennan (1994), Levine (1989), and Levine and Zame (1993).

2.2. EQUILIBRIUM CONDITIONS.

The equations of this model can easily be transformed to a system of equations that contains only stationary variables. To see this, define $z_t = Z_t/YA_t$ for all non-stationary variables Z_t. Equations (2.1), (2.2) and (2.4) can then be written as

(25) $$c_t^i + q_t b_t^i = y_t^i + b_{t-1}^i / ya_t ,$$

(2.6) $$b_t^i \geq -bc ,$$

(2.7) $$q_t [c_t^i]^{-\gamma} \geq \beta \mathrm{E}_t [c_{t+1}^i ya_{t+1}]^{-\gamma} , \quad \text{and}$$
$$(b_t^i + bc) \left(q_t [c_t^i]^{-\gamma} - \beta \mathrm{E}_t [c_{t+1}^i ya_{t+1}]^{-\gamma}\right) = 0.$$

In the economy with two agents, the state variables of agent i are y_t^i, b_{t-1}^i, and ya_t. A solution of the two-agent economy consists of a consumption function, $c(y_t^x, b_{t-1}^x, ya_t)$, an investment function $b(y_t^x, b_{t-1}^x, ya_t)$, and a bond price function, $q(y_t^x, b_{t-1}^x, ya_t)$, that satisfy Equations (2.5) through (2.7) and the equilibrium condition

(2.8) $$b(y_t^x, b_{t-1}^x, ya_t) + b(2 - y_t^x, -b_{t-1}^x, ya_t) = 0 .$$

Note that to solve for the bond price one only has to know the set of state variables of one agent, since the realizations of the endowment (relative to the per capita value) always add up to two and the begin-of-

[13] The last two statements ignore transition dynamics when the economy is not at the ergodic distribution.

period bond holdings always add up to zero. In the economy with a continuum of agents, the state variables of agent *i* are y_t^i, b_{t-1}^i, and the aggregate state variables. The aggregate state variables are ya_t and the joint distribution of endowment levels and bond holdings. Since there are only two possible realizations for the endowment shock, the joint distribution is completely characterized by the cumulative distribution function of the cross-sectional begin-of-period bond holdings of the agents who receive the low income shock, F_t^L, and the cumulative distribution function of the cross-sectional begin-of-period bond holdings of the agents who receive the high income shock, F_t^H.

A solution to the model, in this case, consists of a consumption function, $c(b_{t-1}, y_t, ya_t, F_t^L, F_t^H)$, an investment function, $b(b_{t-1}, y_t, ya_t, F_t^L, F_t^H)$, a bond price function, $q(ya_t, F_t^L, F_t^H)$, and the functionals $F^L(ya_t, ya_{t-1}, F_{t-1}^L, F_{t-1}^H)$ and $F^H(ya_t, ya_{t-1}, F_{t-1}^L, F_{t-1}^H)$ that describe the law of motion of F_t^L and F_t^H, respectively. The only reason why the current value of ya_t is an argument of the transition functionals is that the begin-of-period bond holdings are scaled relative to the per capita endowment. For example, suppose that in period t-1 an agent borrows the maximum amount $bc\ YA_{t-1}$. Thus, $b_{t-1} = -bc$. This means that in period t, begin-of-period bond holdings, relative to the aggregate endowment, are equal to $-bc/ya_t$. This example also makes clear that when the borrowing constraint is binding, F_t^L and F_t^H have point mass at $-bc/ya_t$. The functions $c(\cdot)$, $b(\cdot)$, $q(\cdot)$, and the functionals $F^H(\cdot)$ and $F^L(\cdot)$ have to satisfy Equations (3.5) through (3.7) and the equilibrium condition

$$(2.9)\qquad \int_{-bc/ya_t}^{\infty} b(y^L, b_{t-1}, ya_t, F_t^L, F_t^H)\, d\, F_t^L(b_{t-1}) \quad + \quad \int_{-bc/ya_t}^{\infty} b(y^H, b_{t-1}, ya_t, F_t^L, F_t^H)\, d\, F_t^H(b_{t-1}) \quad = \quad 0.$$

In addition, the functionals $F^L(\cdot)$ and $F^H(\cdot)$ have to describe the law of motion of the cross-sectional dispersion of bond holdings. In particular, if the investment decision is given by $b(\cdot)$ and agents' bond holdings are distributed according to F_t^L and F_t^H, then next period's cross-sectional distribution of bond holdings for the agents who receive the low income draw is given by $F^L(ya_{t+1}, ya_t, F_t^L, F_t^H)$ and is given by $F^H(ya_{t+1}, ya_t, F_t^L, F_t^H)$ for the agents who receive the high income draw. To formalize this condition first denote the largest level of b_{t-1} for which an agent who receives the low (high) endowment draw chooses an investment level x by x^L (x^H). Thus, x^J, $J \in \{L,H\}$ is the solution to the following problem:

$$(2.10)\qquad x^J \quad = \quad \max_{\xi} \{ \xi \mid x = b(y^J, \xi, ya_t, F_t^L, F_t^H) \}.$$

We need the max operator because, for agents at the borrowing constraints, the bond function is not strictly increasing with the begin-of-period bond holdings. The following condition has to be satisfied for $J \in \{L,H\}$, $ya_t, \in \{ya^R, ya^B\}$, $ya_{t+1} \in \{ya^R, ya^B\}$, and for all $x \in [-bc, \infty)$:

$$F_{t+1}^{J}\left(\frac{x}{ya_{t+1}}\right) = F_t^{L}(x^L)\,\text{prob}(y_{t+1} = y^J \mid y_t = y^L) + F_t^{H}(x^H)\,\text{prob}(y_{t+1} = y^J \mid y_t = y^H),$$

(2.11) where

$$F_{t+1}^{J} = F^{J}(ya_{t+1}, ya_t, F_t^L, F_t^H).$$

The meaning of equation (2.11) can be expressed as follows. Of the agents that receive this period an income realization equal to J, the fraction that has begin-of-period bond holdings equal to x/ya_{t+1} is the fraction of agents that received the low-income realization in the last period and invested x times the probability of a low-income agent to receive income realization J plus the fraction of agents that received the high-income realization in the last period and invested x times the probability of a high-income agent to receive income realization J. The difficulty in obtaining numerical solutions for this model is that the cross-sectional distributions of bond holdings are infinite dimensional objects. Moreover, a priori, there is no way of telling what the shape of the distribution is and how it responds to aggregate shocks. In the next section, I develop an algorithm that can handle this problem.

2.3. PARAMETER VALUES.

The parameter values of the stochastic driving processes are from Heaton and Lucas (1996). They obtained estimates for these processes using the income series from the Panel Studies of Income Dynamics (PSID) under the assumption that both processes are two-state first-order Markov processes. The two values for the idiosyncratic shock, y_t, are 0.7554 and 1.2456 and the two values for the aggregate (gross) growth rate, ya_t, are 0.9904 and 1.0470. The probability of receiving in this period the same idiosyncratic shock as was received last period is equal to 0.7412 and the probability of receiving the same aggregate shock is equal to 0.5473. The time period in the model corresponds to a year and the discount rate is set equal to 0.965. The only remaining parameters are the degree of relative risk aversion, γ, and the borrowing constraint parameter, bc. Asset prices and consumption behavior crucially depend on the assumed values for these parameters. I, therefore, consider values for γ equal to -1, -3, and -5 and values for bc ranging from 0.2 to 2.0.

3. THE SOLUTION ALGORITHM.

In this section, I describe the algorithm used to solve the model developed in Section 2 that is characterized by a continuum of agents and aggregate risk. Den Haan (1996b) gives a more detailed description and documents the accuracy of the numerical solution. Solving this type of model has long been considered a challenge by the profession.[14] The first numerical algorithms to solve this type of models with many agents were developed in Den Haan (1996a), Krusell and Smith (1994), and Rios-Rull (1996). Rios-Rull (1996) uses a linear approximation. In my model, however, nonlinearities cannot be ignored because of the high amount of idiosyncratic risk considered and because of the presence of borrowing constraints. Den Haan (1996a) and Krusell and Smith (1994) deal with the infinite dimensional set of state variables by approximating the cross-sectional distribution with a finite set of moments.[15] In Den Haan (1996b) and in this paper, I follow the same approach but associate a density with these moments. This makes it possible to avoid the Monte Carlo integration techniques used in Den Haan (1996a) and Krusell and Smith (1994). This is an important improvement since Monte Carlo integration is known to be less accurate.[16] A notable paper that discusses solution techniques for models with a large (but finite) number of state variables without reducing the dimension of the set of state variables is Judd and Gaspar (1995). The reader who is only interested in the economic results can skip this section without any loss of continuity. In Subsection 3.1, I give an overview of the algorithm, in Subsection 3.2, I describe how to solve for the individual's policy functions, and in Subsection 3.3, I describe how to solve for the bond price and the transition functions. Some concluding comments and information on how to solve the model with two agents are given in Subsection 3.4.

3.1. OVERVIEW.

In the model, the bond price at period t is a function of ya_t and the cross-sectional distribution of bond holdings. A key feature of the proposed algorithm is to approximate the cross-sectional distribution of wealth and income holdings at period t using an $(M\times 1)$ vector, ϕ_t, containing moments of the cross-sectional distribution. To approximate the bond price function I, therefore, use the function $\Theta(ya_t,\phi_t;\ \delta^{\Theta})$, where δ^{Θ} is a vector of parameters and $\Theta(\cdot)$ is chosen from a class of functions that can approximate any function arbitrarily well. Similarly, I use the function $\Phi(ya_{t+1}, ya_t,\phi_t;\ \delta^{\Phi})$ to approximate the transition law of ϕ_t. Since ϕ_t is a vector, $\Phi(\cdot)$ is a vector-valued function. Solving the individual problem requires approximating one more function. Standard choices are the value function, the policy rule for consumption, and the conditional expectation in the Euler equation, $E_t\,[c^i_{t+1}]^{-\gamma}$. I approximate the conditional expectation and denote the approximating function by $\Psi(\,b_{t-1}, y_t, ya_t, \phi_t; \delta^{\Psi})$.[17] In this paper, I use the first two moments of the bond holdings of the agents who receive the low income shock and the first two moments of the bond holdings of the agents who receive the high income shock to approximate the cross-sectional distribution. In this case M equals 3 since the means of the two cross-sectional distributions add up to zero. Because this is a recursive problem, one can restrict one's attention

[14] Cf. Aiyagari and Gertler (1992), Diaz-Gimenez and Prescott (1992), and Weil (1992).
[15] Billingsley (1986, Section 30) shows that a cumulative distribution function is uniquely determined by its moments under very general conditions.
[16] Cf. Press, et.al. (1992, p. 156) and Judd (1991).
[17] From now on, the superscript indicating the agent will be suppressed.

to the current and the next period. I denote the current value of x_t by x and I denote next period's value by x'. Also, let s be the vector $[y, b_{-1}, ya, \phi]$, a be the vector $[ya, \phi]$, and $\bar{a}$ be the vector $[ya, ya_{-1}, \phi_{-1}]$.

For a particular functional form for $\Theta(\cdot)$, $\Phi(\cdot)$, and $\Psi(\cdot)$, the algorithm solves for the parameter values δ^{Φ}, δ^{Θ}, and δ^{Ψ} with the following iteration scheme:

Step 1: Given parameter values for δ^{Φ} and δ^{Θ} solve the individual's problem, that is, obtain parameter values for δ^{Ψ}.

Step 2: Given the decision rules for the individuals, solve the aggregate problem, that is, obtain parameter values for δ^{Φ} and δ^{Θ}. If these parameter values are "close" to the ones used to solve the individual's problem in step 1, then the algorithm has converged. If not, then repeat steps 1 and 2.

3.2. SOLVING FOR THE INDIVIDUAL'S POLICY FUNCTIONS.

Given a bond price function $\Theta(\cdot)$ and a transition function $\Phi(\cdot)$, the individual's problem consist of finding a consumption function $c(s)$ and an investment function $b(s)$ that satisfy, for all possible values of y, b_{-1}, ya, and ϕ, the following three equations.

(3.1) $$c(s) + \Theta(a)b(s) = y + b_{-1} / ya$$

(3.2) $$b(s) \geq -bc,$$

and

(3.3) $$\begin{aligned} &\Theta(a)\,[c(s)]^{-\gamma} \geq \beta \mathrm{E}\left[c(b(s), y', ya', \Phi(\bar{a}'))\, ya'\right]^{-\gamma} \\ &(b(s) + bc)\left(\Theta(a)\,[c(s)]^{-\gamma} - \mathrm{E}\left[c(b(s), y', ya', \Phi(\bar{a}'))\, ya'\right]^{-\gamma}\right) = 0. \end{aligned}$$

Because ϕ is a (3×1) vector, the number of state variables is equal to six. Except for the relatively large number of state variables, this is a standard numerical problem that can be solved in several different ways. The key step in the algorithm used here is to replace the conditional expectation in equation (3.3) by a particular polynomial $\Psi(b_{-1}, y, ya, \phi, \delta^{\Psi})$, where $\Psi(\cdot)$ is chosen from a class of functions that can approximate any function arbitrarily well. It is important that the algorithm can produce an accurate solution at low computational cost, because this problem has to be solved for several choices of $\Theta(\cdot)$ and $\Phi(\cdot)$. To accomplish this, I use several of the techniques proposed in Christiano and Fischer (1994) and Judd (1991). The computational cost of obtaining an accurate solution with this version of the parameterized expectations algorithm (PEA) are a fraction of the cost of the versions of PEA that I have used in my earlier work.[18] Below, I discuss the class of functions from which $\Psi(\cdot)$ is chosen, the construction of the grid space, and the iteration scheme used to find the parameters of the function.

[18] For example, as in Den Haan (1990), Den Haan and Marcet (1990), Den Haan and Marcet (1994), and Den Haan (1996a).

3.2.1 Chebyshev Polynomials and Chebyshev Grid Points.

To construct the grid space one has to choose upper and lower bounds for the values of b_{-1} as well as for the three elements of ϕ. Initial guesses for the bounds of the elements of ϕ can be found by solving the model without aggregate uncertainty. Unfortunately, some additional trial and error is usually required before sensible bounds are found. For reasons that will become clear below, I use Chebyshev roots to construct a grid. Since Chebyshev roots take on values between -1 and 1, I use a linear operator to transform the state variables so that the lower bound is equal to -1 and the upper bound is equal to 1. For simplicity, I do not introduce new notation for the transformed state variables. Let K_x denote the number of Chebyshev roots used for variable x, and let K denote the total number of grid points.

The function $\Psi(\cdot)$ is chosen from the class of Chebyshev polynomials. In the univariate case, the i-th order Chebyshev polynomial term, $C_i(x)$ can be found with the following iteration scheme:

(3.4)
$$\begin{aligned} C_0(x) &= 1, \\ C_1(x) &= x, \\ C_i(x) &= 2\,x\,C_{i-1}(x) - C_{i-1}(x), \;\; i \geq 2. \end{aligned}$$

In the multivariate case, Chebyshev polynomials are constructed by including the cross-products of the univariate terms. Thus, the approximating function $\Psi(\cdot)$ has the following form

(3.5)
$$\Psi(b_{-1}, y, ya, \phi_1, \phi_2, \phi_3; \delta^{\Psi}) \quad =$$

$$\sum_{n_b=0}^{N_b} \sum_{n_y=0}^{N_y} \sum_{n_{ya}=0}^{N_{ya}} \sum_{n_{\phi1}=0}^{N_{\phi1}} \sum_{n_{\phi2}=0}^{N_{\phi2}} \sum_{n_{\phi3}=0}^{N_{\phi3}} \delta^{\Psi}_{n_b, n_y, n_{ya}, n_{\phi1}, n_{\phi2}, n_{\phi3}} \; C_{n_b}(b_{-1}) C_{n_y}(y) C_{n_{ya}}(ya) C_{n_{\phi1}}(\phi_1) C_{n_{\phi2}}(\phi_2) C_{n_{\phi3}}(\phi_3),$$

where N_x is the highest order of variable x that is included. The total number of coefficients, N, is equal to $(N_b+1)(N_y+1)(N_{ya}+1)(N_{\phi1}+1)\,(N_{\phi2}+1)\,(N_{\phi3}+1)$. Let $X_{i,j}$ be the value of the j-th polynomial term at the i-th grid point, for $i = 1,\cdots, K$ and $j = 1,\cdots, N$. Then, the value of $\Psi(\cdot)$ at the i-th grid point can be written as

(3.6)
$$\Psi_i = \sum_{j=1}^{N} \delta_j^{\Psi} \, X_{i,j} \, .$$

Furthermore, define the $(K \times N)$ matrix X, where $X_{i,j}$ comprises the (i,j)th element of X. To solve for the parameters δ^{Ψ}, one has to set K_x at least equal to $N_x + 1$ for each of the six included variables.

The reason that I use Chebyshev polynomials and Chebyshev roots as grid points is that they satisfy a discrete orthogonality property. In particular, for $n_x, \bar{n}_x < K_x$ and $n_y, \bar{n}_y < K_y$:

$$(3.7) \qquad \sum_{j=1}^{K_y}\sum_{i=1}^{K_x}\left[C_{n_y}(y_j)C_{n_x}(x_i)\right]\left[C_{\bar{n}_y}(y_j)C_{\bar{n}_x}(x_i)\right] = 0, \text{ unless } n_x=\bar{n}_x \text{ and } n_y=\bar{n}_y,$$

where x_i (i=1,⋯,K_x) and y_j (j=1,⋯,K_y) are the Chebyshev roots corresponding with the K_x-th and K_y-th order polynomial, respectively. The orthogonality property improves the convergence properties of the algorithm. For example, it makes it easier to incorporate higher-order terms. Below, we will see that the orthogonality property also implies a considerable reduction in computational cost since it simplifies calculating the inverse of a potentially large matrix.

3.2.2. Iteration Procedure.

Let δ_i^{Ψ} denote the value of δ^{Ψ} at the i-th iteration.[19] If the conditional expectation in Equation (3.3) is replaced by $\Psi(s;\delta_i^{\Psi})$, then it is easy to solve for $c(s)$ and $b(s)$ using $\Theta(\cdot;\delta^{\Theta})$, $\Phi(\cdot;\delta^{\Phi})$, and Equations (3.1) through (3.3). The following three-step iteration scheme can be used to find the parameters δ^{Ψ}.

Step 1: At each grid point, calculate the conditional expectation of next period's marginal utility $[c(s')\,ya']^{-\gamma}$, where $s'=[v',b(s),ya',\phi']$. I use $\Phi(\cdot;\delta^{\Phi})$ to calculate ϕ' and I use equations (3.1) through (3.3), $\Psi(s;\delta_i^{\Psi})$, and $\Theta(a;\delta^{\Theta})$ to calculate $b(s)$. Next period's marginal utility depends on two random variables that can take on two values. Calculating the conditional expectation requires, therefore, calculating the weighted sum of only four values. I use equations (3.1) through (3.3), $\Psi(\cdot;\delta_i^{\Psi})$, and $\Theta(\cdot;\delta^{\Theta})$ to calculate $c(s')$ for the four possible realizations of s'. Denote the calculated value of E $[c(s')\,ya)]^{-\gamma}$ at the i-th grid point by P_i and let P_i comprise the i-th element of the $(K\times 1)$ vector P.

Step 2: Obtain updated values for δ^{Ψ}, $\delta_{\text{new}}^{\Psi}$, by "regressing" the values of P_j on the N Chebyshev polynomial terms. The formula for $\delta_{\text{new}}^{\Psi}$ is given by

$$(3.8) \qquad \delta_{\text{new}}^{\Psi} = (X'X)^{-1}X'P.$$

Because of the discrete orthogonality property of Chebyshev polynomials (see equation 3.7), the matrix $X'X$ is diagonal, which means that calculating the inverse is a trivial operation. Since I have solved problems where the dimension of $X'X$ is as large as 120, this property is very important for the speed of the algorithm.

Step 3: Use the method of successive approximation to update δ^{Ψ}, that is

$$(3.9) \qquad \delta_{i+1}^{\Psi} = \nu\,\delta_{\text{new}}^{\Psi} + (1-\nu)\,\delta_i^{\Psi},$$

[19] Parameter values that solve a related problem are usually good candidates for starting values.

where the adjustment parameter ν is between zero and one. Choosing the value for ν usually requires some experimentation. A higher value of ν has the potential benefit of faster convergence but also carries the risk of global instability.

3.2.3. Differences with Conventional PEA.

This version of PEA differs from the conventional implementation of PEA in how $\delta^{\Psi}_{\text{new}}$ is calculated. The conventional version of PEA simulates a long time-series of the variables in the model and regresses the time-series of $(c_{t+1}\, ya_{t+1})^{-\gamma}$ on the period t polynomial terms. In contrast, this version "regresses" the *conditional expectation* of $(c'\, ya')^{-\gamma}$ on a selectively chosen set of data points. There are, thus, two differences. First, the parameter values for $\delta^{\Psi}_{\text{new}}$ obtained with the conventional version of PEA are subject to sampling variation. The parameter estimates are, therefore, subject to the rather slow square-root rate of convergence. To reduce the effects of sampling uncertainty a large sample has to be used. In contrast, this version uses $\mathrm{E}[(c'\, ya')^{-\gamma}]$, instead of $(c'\, ya')^{-\gamma}$, so there is no sampling variation.[20] It is important to understand that of step 2 is a problem of solving K equation in N unknowns not an estimation problem. Thinking of the problem in this way makes it clear that you do not need many points to solve for the N parameters. The second difference is that, in this version of PEA, the state variables are spread out over a wide range, while in the conventional version of PEA these variables usually are centered around the steady state, because the conventional version of PEA oversamples values that occur frequently. Choosing a wide range of values, however, increases the accuracy just as an increased spread of the explanatory variables decreases the standard error of estimated parameters in least-squares regressions.[21]

3.2.4. The Number of Grid Points and the Order of the Polynomials.

Using the general formulation in Equation (3.5) involves solving for a large number of parameters because an accurate solution requires a fairly high value for N_b. A cheaper, but accurate solution can be obtained by imposing that several of the coefficients of the cross-sectional distribution are zero. In particular, the following polynomial is used:

$$\Psi(b_{-1}, y, ya, \phi_1, \phi_2, \phi_3; \delta^{\Psi}) = \sum_{n_b=0}^{N_b} \sum_{n_y=0}^{N_y} \sum_{n_{ya}=0}^{N_{ya}} \delta^{\Psi}_{n_b,n_y,n_{ya}}\, C_{n_b}(b_{-1})\, C_{n_y}(y)\, C_{n_{ya}}(ya) \;+ \delta'_{\phi}\, \phi. \tag{3.10}$$

[20] Note that if a numerical integration technique is necessary to evaluate $\mathrm{E}[(c'\, ya')^{-\gamma}]$ it is still possible to obtain convergence rates faster than $O(T^{-1/2})$ (see Judd 1991).

[21] This statement assumes that the conditional expectation can be approximated accurately with a polynomial of the specified order. If this is not the case, for example, if one uses a linear function to approximate a quadratic function, then it may very well be better to oversample points that occur frequently.

Recall that y and ya can take on only two values. Without loss of generality, I, therefore, set N_y and N_{ya} equal to 1. Den Haan (1996b) shows that values for N_b equal to 7 and 15 lead to very similar solutions. In this paper N_b, therefore, is set equal to 7. I set K_y and K_{ya} equal to 2, $K_{\phi 1}$, $K_{\phi 2}$, and $K_{\phi 3}$ equal to 5, and K_b equal to 8.

The Chebyshev polynomials can also be used when the arguments are bigger than one or smaller than minus one. The problem of using higher-order approximations, however, is that the behavior of the function may be highly irregular outside the range of grid points considered. For the lower bound this is not a problem, since the lower bound is the borrowing constraint and the agent is not allowed to go below this bound anyway. To improve the efficiency of the algorithm I impose the constraint that the agent can not save more than 4 times the borrowing constraint parameter. In none of the simulated economies, each with a length of 100,000 observations is this constraint ever binding. Moreover, none of the results reported in this paper are sensitive to changes in this upper limit. This suggests that the problem with this very loose constraint on the maximum amount an agent can invest is very similar to the original problem without this upperbound

3.3. SOLVING FOR THE BOND PRICE AND THE TRANSITION FUNCTIONS.

In this section, I discuss how to obtain parameter values for the bond price function $\Theta(ya,\phi)$ and the vector valued transition function $\Phi(ya',ya,\phi)$. As in Section 3.2, I propose to use Chebyshev polynomials as approximating functions. Thus,

$$\Theta(ya,\phi_1,\phi_2,\phi_3;\delta^{\Theta}) =$$

(3.11)
$$\sum_{n_{ya}=0}^{N_{ya}} \sum_{n_{\phi 1}=0}^{N_{\phi 1}} \sum_{n_{\phi 2}=0}^{N_{\phi 2}} \sum_{n_{\phi 3}=0}^{N_{\phi 3}} \delta^{\Theta}_{n_{ya},n_{\phi 1},n_{\phi 2},n_{\phi 3}} \; C_{n_{ya}}(ya) C_{n_{\phi 1}}(\phi_1) C_{n_{\phi 2}}(\phi_2) C_{n_{\phi 3}}(\phi_3)$$

and

$$\Phi(ya',ya,\phi_1,\phi_2,\phi_3;\delta^{\Phi}) =$$

(3.12)
$$\sum_{n_{ya'}}^{N_{ya'}} \sum_{n_{ya}=0}^{N_{ya}} \sum_{n_{\phi 1}=0}^{N_{\phi 1}} \sum_{n_{\phi 2}=0}^{N_{\phi 2}} \sum_{n_{\phi 3}=0}^{N_{\phi 3}} \delta^{\Phi}_{n_{ya'},n_{ya},n_{\phi 1},n_{\phi 2},n_{\phi 3}} \; C_{n_{ya'}}(ya') C_{n_{ya}}(ya) \; C_{n_{\phi 1}}(\phi_1) \; C_{n_{\phi 2}}(\phi_2) \; C_{n_{\phi 3}}(\phi_3).$$

Chebyshev roots are again used as grid points. The values of N_x and K_x ($x = ya$, ϕ_1, ϕ_2, ϕ_3) are equal to the values used in the approximation of the conditional expectation.

3.3.1. Parameterizing the Cross-Sectional Distribution.

To solve for the parameters of the functions $\Theta(\cdot)$ and $\Phi(\cdot)$ more information about the cross-sectional distribution of bond holdings than just a limited set of moments is needed. Den Haan (1996a) and Krusell and Smith (1994) implicitly obtain this information by using Monte Carlo simulation techniques. Monte Carlo techniques, however, introduce sampling uncertainty which makes it hard to obtain accurate solutions. To avoid this problem, I parameterize the cross-sectional distribution of bond holdings with a flexible function form. In particular, I use an element from the class of exponentials. Den Haan (1996b) shows that a given set of n moments uniquely determine the $n+1$ parameters of an n-th order exponential density. Because of the presence of borrowing constraints, I adjust the class of functions to allow for the possibility of having a positive mass at the borrowing constraints.[22] In particular, for $J \in \{L,H\}$, I use the following density as the n-th order approximation for the density of the bond holdings of the agents who receive the income shock J,

$$(3.13)\qquad \begin{aligned} f^J(b) &= f_0^J \exp(f_1^J b + f_2^J b^2 + \cdots + f_n^J b^n), & \bar{b} < b < \infty \\ f^J(b) &= f_0^J \int_{-\infty}^{\bar{b}} \exp(f_1^J \zeta + f_2^J \zeta^2 + \cdots + f_n^J \zeta^n)\, d\zeta, & b = \bar{b} \end{aligned}$$

In this paper, two moments are used to approximate the cross-sectional distribution. Consequently, a second-order exponential is used. To find the parameter values f_0^L, f_1^L, and f_2^L that correspond with particular values of the mean, ϕ_1, and the variance of the distribution of bond holdings of the "low income" agents, ϕ_2, a non-linear equation solver is used to solve the following set of equations:

$$(3.14)\qquad \begin{aligned} 1 &= f_0^L \int_{\bar{b}}^{\infty} \exp(f_1^L b + f_2^L b^2)\, db, \\ \phi_1 &= f_0^L \int_{\bar{b}}^{\infty} b \exp(f_1^L b + f_2^L b^2)\, db, \quad \text{and} \\ \phi_2 &= f_0^L \int_{\bar{b}}^{\infty} (b-\phi_1)^2 \exp(f_1^L b + f_2^L b^2)\, db. \end{aligned}$$

The integrals in Equation (3.14) are calculated numerically. Since, the density is from the exponential family, I use Hermite Gaussian quadrature with 30 quadrature points.[23] Similarly, the parameters of the distribution of bond holdings of the agents with the high income shock are calculated using the values of $-\phi_1$ and ϕ_3.

Using the exponential family when moments are used to characterize the distribution is motivated by information theory. In particular, this class of functions is the solution to the problem that maximizes Shannon's entropy among all possible densities subject to the specified moment conditions.[24] Using the exponential family, thus, satisfies the Maximum Entropy principle, which states that we should find a distribution in a way as to minimize the use of information that is not explicitly available.[25] Without any moment conditions, the uniform distribution satisfies the Maximum Entropy principle. Of

[22] For notational convenience, I only include the constraint on the amount borrowed. The constraint on the maximum amount invested is treated in exactly the same way.

[23] All quadrature points that correspond with points below the truncation point are relocated at the truncation point. Accuracy requires that there are enough quadrature points below the truncation point.

[24] Cf. Zellner and Highfield (1988).

[25] Cf. Maasoumi (1993).

course, after a solution has been obtained, the researcher has to check whether the distributional assumption is accurate.

3.3.2. Iteration Procedure.

In this subsection I describe how parameter estimates for the bond price function and the transition function are calculated. I will first discuss the procedure for the bond price. Calculating the parameters of the bond price function, δ^{Θ} requires the following two steps.

Step 1: At each grid point in the state space calculate the bond price. Recall that the state variables for the bond price are ya and the three elements of ϕ. The bond price is calculated as follows. First, use Equation (3.14) to find the parameters of the distribution of the bond holdings that correspond with the value of ϕ. Next, find the bond price as the value for which the aggregate demand for bonds is equal to zero. That is, find the bond price q that solves the following equation.

$$\text{(3.15)} \qquad \int_{-bc/ya}^{\infty} b(y^L, b_{-1}, a; q, \delta^{\Psi}) f^L(b_{-1})\, db_{-1} + \int_{-bc/ya}^{\infty} b(y^H, b_{-1}, a; q, \delta^{\Psi}) f^H(b_{-1})\, db_{-1} = 0.$$

The investment function $b(s;q,\delta^{\Psi})$ is implicitly defined in equations (3.1) through (3.3) with the conditional expectation replaced by $\Psi(s;\delta^{\Psi})$. To evaluate the integrals in Equation (3.15) Hermite Gaussian Quadrature with 30 quadrature points is used. Let K^{Θ} be the number of grid points and let the calculated bond price at the i-th grid point comprise the i-th element of the $(K^{\Theta}\times 1)$ vector P^{Θ}.

Step 2: Obtain new values for δ^{Θ}. As in Section 3.2, let X^{Θ} be the matrix of Chebyshev polynomial terms. Values for δ^{Θ}_{new} are given by

$$\text{(3.16)} \qquad \delta^{\Theta}_{new} = (X^{\Theta\prime} X^{\Theta})^{-1} X^{\Theta\prime} P^{\Theta}.$$

Step 3: Use the method of successive approximation to update δ^{Ψ}, that is

$$\text{(3.17)} \qquad \delta^{\Theta}_{i+1} = \nu \delta^{\Theta}_{new} + (1-\nu)\delta^{\Theta}_{i},$$

where the adjustment parameter ν is between zero and one.

Finally, I discuss how to find the parameters of the transition function. The idea is very similar to the procedure used to find the parameters of the bond price function. In the first step, the values of ϕ_1', ϕ_2', and ϕ_3' are calculated at a particular grid point, where a grid point consists of values for ya', ya, ϕ_1, ϕ_2, and ϕ_3. For each of the three moments, the second step consists of regressing the calculated values on the Chebyshev polynomials. In the third step, the method of successive approximation is used to update the parameters. Since the last two steps are identical to the procedure for the bond price, I only give the details of the first step. Recall that ϕ_1 is the average amount of bond holdings of the agents who receive the low income shock, ϕ_2 is the variance of the bond holdings of the agents who receive the low income shock, and ϕ_3 is the variance of the bond holdings of the agents who receive the high income shock. At each grid point, the values of ϕ_1', ϕ_2', and ϕ_3' are calculated with the following equations.

$$\phi_1' = \text{prob}(y' = y^L | y = y^L) \int_{-bc/ya'}^{\infty} \frac{b(y^L, b_{-1}, ya, \phi_1, \phi_2, \phi_3; q, \delta^\Psi)}{ya'} f^L(b_{-1})\, db$$

$$+ \text{prob}(y' = y^L | y = y^H) \int_{-bc/ya'}^{\infty} \frac{b(y^H, b_{-1}, ya, \phi_1, \phi_2, \phi_3; q, \delta^\Psi)}{ya'} f^H(b_{-1})\, db \; ,$$

$$\phi_2' = \text{prob}(y' = y^L | y = y^L) \int_{-bc/ya'}^{\infty} (\frac{b(y^L, b_{-1}, ya, \phi_1, \phi_2, \phi_3; q, \delta^\Psi)}{ya'} - \phi_1')^2 f^L(b_{-1})\, db$$

$$+ \text{prob}(y' = y^L | y = y^H) \int_{-bc/ya'}^{\infty} (\frac{b(y^H, b_{-1}, ya, \phi_1, \phi_2, \phi_3; q, \delta^\Psi)}{ya'} - \phi_1')^2 f^H(b_{-1})\, db \; , \text{ and}$$

$$\phi_3' = \text{prob}(y' = y^H | y = y^L) \int_{-bc/ya'}^{\infty} (\frac{b(y^L, b_{-1}, ya, \phi_1, \phi_2, \phi_3; q, \delta^\Psi)}{ya'} + \phi_1')^2 f^L(b_{-1})\, db$$

$$+ \text{prob}(y' = y^H | y = y^H) \int_{-bc/ya'}^{\infty} (\frac{b(y^H, b_{-1}, ya, \phi_1, \phi_2, \phi_3; q, \delta^\Psi)}{ya'} + \phi_1')^2 f^H(b_{-1})\, db \, ,$$

where $q = \Theta(ya, \phi_1, \phi_2, \phi_3; \delta^\Theta)$. Again, Equation (3.14) is used to solve for the parameters of the exponential density that corresponds with the moments ϕ_1, ϕ_2, and ϕ_3 and Hermite Gaussian Quadrature with 30 quadrature points is used to evaluate the integrals.

3.4. REMAINING REMARKS.

The solution obtained using the procedures described in the last three subsections is an approximation to the true solution. Den Haan (1996b) discusses in detail the accuracy of the solution. In particular, this work reports that the results are robust when higher-order polynomials are used and when the skewness and the kurtosis of the distribution of bond holdings are included as state variables as well. Using a simulation with 300,000 agents, it is also shown that the class of second-order exponentials defined in Equations (3.11) and (3.12) together with the transition functions provide a fairly accurate description of the cross-sectional distribution and its time-series fluctuations.

A few words about the speed of the algorithm are appropriate. When the solution for the case with bc equals 0.4 is used as the initial guess for the case where bc equals 0.2, then it takes the algorithm around one hour on a 486DX-66MHz personal computer. This number is misleading for the following reasons. First, I am sure that I can reduce the computation speed of this particular experiment by trying different values for parameters such as the parameter ν used to update the parameters and report the fastest time that I found. Second, occasionally the algorithm does not converge for the initial conditions used. In this case, the homotopy idea has to be used to find suitable initial conditions. More importantly, this time estimate grossly underestimates the actual time involved since I used lower and upper bounds for ϕ that are appropriate for the case where bc equals 0.2. In reality, these bounds were found only after a process of trial and error during which the model had to be solved several times.[26] Nevertheless, this estimate indicates that solving this type of problem is computationally feasible with the commonly available technology.

Solving the two-agent economy is more straightforward. For this economy, one agent's state variables describe the state of the whole economy. The algorithm to solve the two-agent economy is similar to the procedure used to solve the individual policy rules described in Subsection 3.2. The main difference is that the individual problem is not separated from the aggregate problem. That is, I do not use an approximating function for the bond price. Whenever I have to calculate the bond price for a particular set of state variables, then I use a nonlinear equation solver to find the value of the bond price for which the sum of the demand for bonds is equal to zero. In the two-agent economy, the conditional expectation is a function of the individual's idiosyncratic shock, the individual's bond holdings, and the aggregate shock. Again, it is approximated with a Chebyshev polynomial. The two exogenous shocks can take on only two values. I, therefore, set N_y and N_{ya} equal to 1. I set N_b equal to 29.[27] K_y, K_{ya} , and K_b are equal to 2, 2, and 30, respectively. The procedure to find the parameters of the approximating function is the same as the procedure to find the parameters in the model with a continuum of agents.

4. COMPARING DIVERSE AND LESS DIVERSE ECONOMIES.

In this section, I compare the properties of the two equilibrium models described in Section 2. In the first economy, there are only two different types of agents. In the second economy, there are a continuum of different agents. Agents in the two economies face exactly the same amount of idiosyncratic and aggregate endowment risk. In Section 4.1, I analyze the time-series properties of the interest rate and the amount of cross-sectional dispersion. In Section 4.2, I analyze the time-series properties of consumption and bond trades in the two economies. In Section 4.3, I compare the ability to insure against idiosyncratic shocks in the two economies. In Section 4.4, I investigate the business cycle behavior of the interest rate in an incomplete markets economy with a continuum of agents.

4.1. INTEREST RATES IN DIVERSE AND LESS DIVERSE ECONOMIES.

In this section, I compare the behavior of the interest rate in the two-agent economy with the behavior of the interest rate in the economy with a continuum of agents. Important for the time-series properties of the interest rate are the time-series properties of the amount of cross-sectional dispersion and

[26] Of course, during such a process, the model does not have to be solved to the same degree of accuracy.

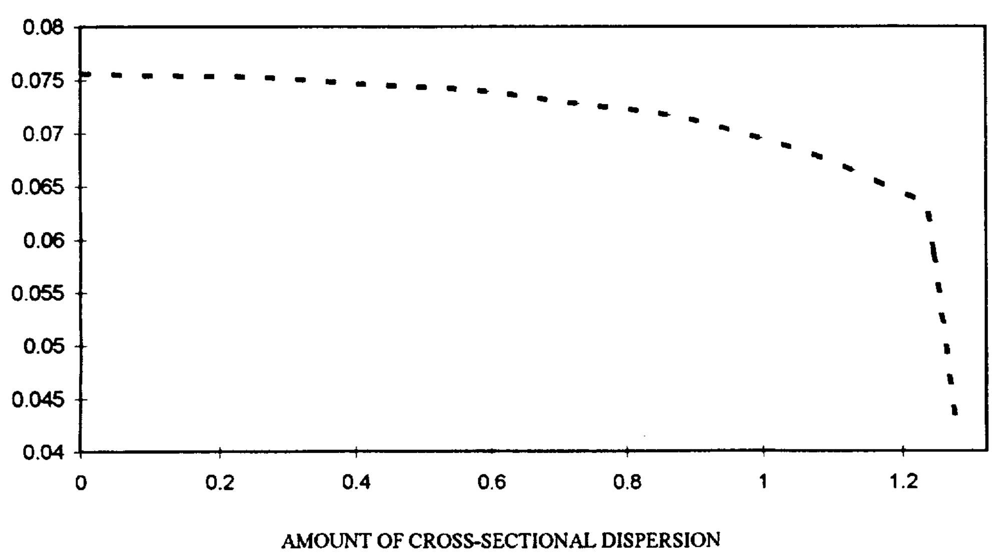
[27] I know that using a 29-th order polynomial is kind of overdoing it, but there is no reason not to nail the eighth digit when you are solving a cheap problem.

the relationship between the amount of cross-sectional dispersion and the interest rate. In Section 4.1.1, I discuss the relationship between the amount of cross-sectional dispersion and the interest rate. In Section 4.1.2, I discuss the time-series properties of the amount of cross-sectional dispersion. In Section 4.1.3, I compare the behavior of interest rates in the two models.

4.1.1. The Amount of Cross-Sectional Dispersion and the Interest Rate.

In this section, I discuss the relationship between the amount of cross-sectional dispersion and the interest rate. An example of this relationship is given in Figure 1 that plots the interest rate as a function of the low-income agent's begin-of-period debt in the two-agent economy. Note that as the debt of the low-income agent increases, the amount of cross-sectional dispersion increases. The decreasing concave relationship between the interest rate and the amount of cross-sectional dispersion depicted in Figure 1 is robust to changes in the parameter values. This relationship can be easily understood when one recalls the property that the marginal propensity to save is increasing in wealth.[28] Consider a wealth transfer from a "poor" agent to a "rich" agent. In response to this transfer the poor agent would want to save less (or borrow more) and the rich agent would want to save more. Since the rich agent's marginal propensity to save is higher than the poor agent's marginal propensity to save, the interest rate has to decrease to keep the bond market in equilibrium. Moreover, the effect becomes smaller when the amount of cross-sectional dispersion decreases, i.e., the relationship is concave. To understand this, consider the extreme case when there is no cross-sectional dispersion. In this case, the marginal propensities to save are equal for all agents and a marginal wealth transfer would increase the amount of dispersion but would have no effect on the interest rate.

Figure 1: Cross-Sectional Dispersion and the Interest Rate.



NOTE: This figure plots the interest rate as a function of the low-income agent's begin-of-period debt in the two-agent economy. As the debt of the low-income agent increases, the amount of cross-sectional dispersion increases. The borrowing constraint parameter is equal to 1.4 and the parameter of relative risk-aversion is equal to 3. The graph is drawn for the low realization of the aggregate growth rate.

[28] Carroll and Kimball (1996) showed that this property holds in intertemporal optimization models under very general conditions.

Although this explanation does not rely on borrowing constraints, the effects are magnified in the presence of borrowing constraints. The reason is that borrowing constraints reduce the marginal propensity to save of the less wealthy even further. When a poor agent has reached his debt limit, a wealth transfer from this agent to a wealthier agent requires a large drop in the interest rate since the new interest rate has to be such that the wealthier agent is willing to consume the total wealth transfer. This explains the kink in Figure 1. At the kink, the poor agent has reached his borrowing constraint and the slope of the function sharply decreases.[29]

4.1.2. Time-Series Behavior of the Cross-Sectional Dispersion.

In this section, I analyze the time-series behavior of the cross-sectional dispersion in the two types of economies. It is obvious that the time-series variation in the amount of cross-sectional wealth dispersion is higher in the economy with two types than in the economy with a continuum of types. The reason is that in the economy with two types the amount of cross-sectional dispersion is affected by idiosyncratic shocks as well as by aggregate shocks, while in the economy with a continuum of types the amount of cross-sectional dispersion is only affected by aggregate shocks. It is not clear, however, how other statistics like the average amount of cross-sectional dispersion, the autocorrelation coefficient, or the correlation with the aggregate growth rate differ.

There are several candidate statistics that can measure the amount of cross-sectional dispersion. Some of the statistics that are sensible in an economy with a continuum of agents, however, do not make sense in an economy with two agents. The variance of the bond holdings of the high-income agents, for example, is always equal to zero in an economy with only two types of agents. Two cross-sectional statistics that avoid this type of problem are the average of the bond holdings of the low-income agents and the variance of the bond holdings across all agents. In the economy with two agents, the first statistic is the amount of bond holdings of the low-income agent and this statistic has non-trivial time-series properties. To characterize the time-series properties of these two measures of cross-sectional dispersion I calculate the average, the standard deviation, the first-order autocorrelation coefficient, and the correlation with the growth rate of the aggregate endowment. The results are reported in Table 1. Panel A reports the time-series statistics for the average bond holdings of the low-income agents and Panel B reports the time-series statistics for the cross-sectional variance of the bond holdings.

[29] The possibility of being constrained in the future is likely to affect the marginal propensity to save when the agent is close to but not yet at the borrowing constraint.

Table 1: Time-Series Behavior of Cross-Sectional Dispersion.

A. Cross-sectional average of the bond holdings of the low-income agents.

		Average		Standard Deviation		Autocorrelation		Correlation with ya_t	
γ	bc	∞ agents	2 agents	∞ agents	2 agents	∞ agents	2 agents	∞ agents	2 agents
	.2	-0.074	-0.128	0.0020	0.0766	0.044	0.053	0.996	-0.008
1	1.	-0.166	-0.340	0.0048	0.5412	0.042	0.396	0.974	-0.002
	2.	-0.194	-0.406	0.0058	1.0690	0.205	0.451	0.978	-0.003
	.2	-0.074	-0.150	0.0020	0.0752	-0.040	0.031	0.989	-0.012
3	1.	-0.162	-0.338	0.0046	0.5612	-0.018	0.395	0.973	-0.006
	2.	-0.190	-0.384	0.0054	1.1286	0.033	0.450	0.975	-0.010

B. Cross-sectional variance of the bond holdings of all agents.

		Average		Standard Deviation		Autocorrelation		Correlation with ya_t	
γ	bc	∞ agents	2 agents	∞ agents	2 agents	∞ agents	2 agents	∞ agents	2 agents
	.2	0.0178	0.0140	0.0010	0.0080	0.236	0.204	-0.989	0.003
1	1.	0.2526	0.2146	0.0162	0.1866	0.525	0.853	-0.880	-0.021
	2.	0.7281	0.7106	0.0572	0.6726	0.707	0.942	-0.728	-0.022
	.2	0.0192	0.0138	0.0012	0.0080	0.382	0.233	-0.955	0.000
3	1.	0.2690	0.2042	0.0199	0.1962	0.648	0.855	-0.800	-0.027
	2.	0.7584	0.6538	0.0782	0.6410	0.851	0.942	-0.563	-0.037

NOTE: This table reports time-series statistics for the indicated measure of cross-sectional dispersion. γ denotes the parameter of relative risk aversion and bc indicates the amount the agent is allowed to borrow relative to the per capita endowment. The other parameter values are reported in Section 2.3. Population moments are approximated using the sample moments of a simulated draw of 100,000 observations.

From the table the following observations can be made. The first observation is that, as expected, cross-sectional dispersion is much more volatile in the economy with two agents. Second, as documented in Panel A, the average amount of debt of the low-income agent is considerably higher in the economy with two agents. For example, when γ is equal to 3 and bc is equal 0.1, then the average amount of debt of a low income agent is equal to 0.075 in the economy with two agents and is equal to 0.037 in the economy with a continuum of agents. The intuition for this difference is the following. Consider an agent who receives the low realization of the idiosyncratic shock for several periods. In the two-agent economy this means that the other agent receives the high-income shock for several periods. Since the marginal propensity to save is increasing in wealth this means that the net demand for bonds would increase if there would be no change in the interest rate. To keep the bond market in equilibrium, the interest rate has to drop which increases the amount borrowed by the agent who received the low-income shocks.

Using this measure of cross-sectional dispersion one would conclude that the amount of cross-sectional dispersion is on average higher in the economy with two agents. The third observation, however, is that the variance of bond holdings is on average higher in the economy with a continuum of agents. The reason is that in the economy with two agents, an agent can never accumulate more than the amount the other agent is allowed to borrow, while this is not the case in the economy with a continuum of agents.

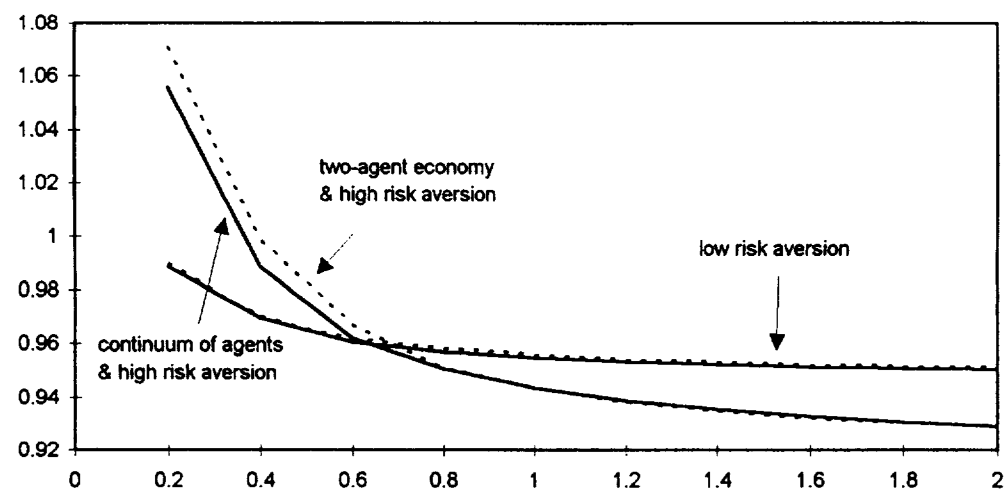
Fourth, the correlation between the amount of cross-sectional dispersion and the aggregate growth rate is close to zero in the economy with two agents, while there is an almost perfect correlation in the economy with a continuum of agents. The intuition is as follows. In the economy with two agents, the amount of cross-sectional dispersion is mainly influenced by the idiosyncratic income shock, while in the economy with a continuum of agents, the amount of cross-sectional dispersion is only affected by shocks to the aggregate endowment. In Section 4.4, I will continue the discussion of the relationship between aggregate shocks and the amount of cross-sectional dispersion.

Fifth, there are similarities and differences across the two types of economies for the autocorrelation coefficient. For example, in both types of economies the autocorrelation of the second measure of cross-sectional dispersion is higher than the autocorrelation of the first measure. The relation between the autocorrelation coefficient and the borrowing constraint parameter, however, differs considerably across the two types of economies. Consider the case when γ is equal to 3. In the economy with a continuum of agents, the average bond holdings of the low income agent display virtually no serial correlation for any level of the borrowing constraint parameter. In contrast, in the economy with two agents, the autocorrelation coefficient increases sharply when the borrowing constraint parameter increases.

4.1.3. Time-Series Behavior of the Interest Rate.

In this section, I compare the time-series behavior of the interest rate in an economy with two types of agents with the behavior of the interest rate in an economy with a continuum of agents. In particular, I consider differences in the average, the standard deviation, the autocorrelation coefficient, and the correlation with the growth rate of the aggregate endowment. I start with a comparison of the averages of the bond price and the interest rate. Figure 2 plots the average bond price in both economies as a function of the borrowing constraint parameter. I choose to plot the average bond price because the differences are bigger for the bond price than for the interest rate.[30] Nevertheless, the results are remarkably similar for most parameter values. This is good news for several papers in the macrofinance literature that mainly focused on average returns.[31] As documented in the graph, exceptions can be found for low values of the borrowing constraint and high values of the parameter of relative risk aversion. When the borrowing constraint parameter is equal to 0.2 and the parameter of relative risk aversion is equal to three (five), for example, then the average bond price in the economy with two agents is 1.4 (2.9) percent higher than the average bond price in the economy with a continuum of agents.[32] When one is particularly interested in calculating the average interest rate in an economy with a large number of agents at these parameter values, then these errors are substantial. For many other exercises the differences are of less importance. When one wants to know the effect of a change in the borrowing constraint parameter from 0.2 to 2.0, for example, then the two-agent economy predicts a decrease in the average bond price with 13.3 percent and the economy with a continuum of agents predicts a decrease of 12.0 percent. In both cases one would conclude that borrowing constraints have an enormous effect on average interest rates.

Figure 2: The Average Bond Price in Diverse and Less Diverse Economies.



NOTE: This graph plots the average bond price as a function of the borrowing constraint parameter. The borrowing constraint parameter indicates the maximum amount an agent is allowed to borrow relative to the per capita endowment. The parameter of relative risk aversion is equal to one (three) in the case of low (high) risk-aversion. The other parameter values are reported in Section 2.3. Population moments are approximated using the moments calculated with a simulated draw of 100,000 observations.

The results from the last two subsections provide an intuitive explanation for the finding that the average interest rate (bond price) is lower (higher) in the economy with two agents. In Section 4.1.1, it was shown that the interest rate is a concave function of the amount of cross-sectional dispersion. In

[30] Jensen's inequality implies that the differences are smaller for the average interest rates than for average bond prices because the interest rate is a concave function of the bond price.
[31] Examples are Heaton and Lucas (1996), Lucas (1994), and Telmer (1993).
[32] When the borrowing constraint parameter is equal to 0.2 and the parameter of relative risk aversion is equal to three (five), then the difference in average interest rates is equal to 0.7 (1.0) percentage points.

Section 4.1.2, it was shown that the amount of cross-sectional dispersion is considerably more volatile in economies with two types of agents. Jensen's inequality then suggests that the average interest rate should be lower in the economy with two agents.[33]

Table 2 documents the differences across the two types of economies for the following time series statistics of the interest rate: the standard deviation, the first-order autocorrelation coefficient, and the correlation with the growth rate of the aggregate endowment. The following observations can be made from the table. First, big differences are observed when one compares the standard deviation of interest rates across the two types of economies. This can be explained by the large differences in the volatility of the amount of cross-sectional dispersion across the two types of economies.

Table 2: Time-Series Behavior of the Interest Rate.

		Standard Deviation		Autocorrelation		Correlation with ya_t	
γ	bc	∞ agents	2 agents	∞ agents	2 agents	∞ agents	2 agents
	.2	0.00403	0.03197	0.144	0.551	0.997	0.137
1	1.	0.00586	0.01560	0.151	0.674	0.996	0.286
	2	0.00499	0.01047	0.152	0.593	0.994	0.380
	.2	0.01001	0.07867	0.101	0.532	1.000	0.117
3	1.	0.01276	0.03747	0.129	0.711	0.999	0.277
	2	0.01280	0.02423	0.111	0.635	1.000	0.417
	.2	0.01304	0.11306	0.085	0.489	1.000	0.114
5	1.	0.01704	0.05463	0.106	0.705	1.000	0.285
	2	0.02017	0.03337	0.097	0.623	1.000	0.472

NOTE: This table reports time-series statistics of the interest rate on a one-period bond. γ denotes the parameter of relative risk aversion and bc indicates the amount the agent is allowed to borrow relative to the per capita endowment. The other parameter values are reported in Section 2.3. Population moments are approximated using the sample moments of a simulated draw of 100,000 observations.

Second, the autocorrelation of the interest rate is much higher in the two-agent economy. The reason is that in this type of economy the interest rate is influenced by realizations of the idiosyncratic shock which are much more persistent than shock to the aggregate growth rate. Also note that the autocorrelation is a humped shaped function of the borrowing constraint parameter in the two-agent economy, while the amount of serial correlation is not affected by the borrowing constraint parameter in the economy with a continuum of agents.

[33] Note that other factors are important as well. First, cross-sectional dispersion does not have exactly the same interpretation in both economies. Second, even if this would be the case, the relationship between the interest rate and the amount of cross-sectional dispersion does not have to be the same across the two types of economies. Third, the average amount of cross-sectional dispersion differs across the two types of economies.

Third, the correlation with the growth rate of the aggregate endowment is close to one in the economy with a continuum of agents because this random variable is the only shock that affects the interest rate. In the economy with only two agents this coefficient is much lower because the interest rate is also affected by idiosyncratic shocks.

Note that in the presence of complete markets, the interest rates would be exactly the same in both types of economies. Consistent with this is the observation that the differences across the two types of economies become smaller when the borrowing constraint parameter increases, that is, when the financial frictions become smaller.

4.2. CONSUMPTION AND BOND TRADES IN DIVERSE AND LESS DIVERSE ECONOMIES.

In this section, I compare the time-series behavior of consumption and bond trades in the two-agent economy with the behavior of these variables in the economy with a continuum of agents. The consumption variable that I focus on is the percentage change in individual consumption, $\Delta \ln C_t = \Delta \ln (c_t YA_t)$. Since bond trades can take on negative values and values close to zero, it does not make sense to use the percentage change. To study the behavior of bond trades I, therefore, focus on the amount of bonds purchased relative to the per capita endowment, $b_t = B_t / YA_t$. This variable has the disadvantage that its unconditional covariance with any aggregate variable is by construction equal to zero because agents are ex ante identical. The statistics considered for the consumption variable are the standard deviation, the correlation with individual endowment growth rate, the correlation with aggregate consumption growth rate, and the correlation with the interest rate. The statistics considered for the bond trade variable are the standard deviation, the first-order autocorrelation coefficient, the correlation with the idiosyncratic endowment shock, and the fraction of times the agents is at the constraint. The results are reported in Tables 3 and 4 for consumption and bond trades, respectively.

The following observations can be made. First, in both types of economies, the correlation between individual consumption growth and income growth decreases and the correlation between individual and aggregate consumption growth increases when the borrowing constraint parameter increases. Results not reported here show that the amount of serial correlation in consumption growth reduces when the borrowing constraint parameter increases. These findings document that as financial frictions weaken agents are better able to smooth consumption.[34] Second, for the higher borrowing constraint parameters, an agent in the two-agent economy spends more time at the borrowing constraint, than an agent in the economy with a continuum of agents. When the borrowing constraint parameter is equal to two, for example, then the agent living in the two-agent economy is three times more likely to be at the borrowing constraint. The reason is that in the two-agent economy the interest rate drops in response to a series of negative shocks that hit the same agent. The third observation is that in the two-agent economy the amount of consumption variability is smaller and the amount of variability in bond

[34] Note that the correlation between (scaled) bond purchases and the realization of the endowment process decreases with an increase in the borrowing constraint parameter. Presumably this is due to the increased persistence and the increased volatility of bond purchases when the borrowing constraint parameter increases.

purchases is higher. The ability to smooth consumption in the two types of economies will be discussed in more detail in the next subsection. Fourth, the correlation of consumption growth with the interest rate in the two-agent economy is much smaller than the correlation in the economy with a continuum of agents. Given the much higher volatility of interest rates in the two-agent economy, it is no surprise that the correlation coefficients differ across economies. Fifth, the qualitative change in the time-series statistics in response to changes in the parameter values is very similar. For example, the correlation between individual consumption growth and the interest rate increases in both types of economies with an increase in the borrowing constraint parameter, although the amount of correlation differs substantially in both types of economies.

Table 3: Time-Series Behavior of Individual Consumption Growth.

		Standard Deviation		Correlation with $\Delta\ln(Y_t)$		Correlation with ya_t		Correlation with r_t	
γ	bc	∞ agents	2 agents	∞ agents	2 agents	∞ agents	2 agents	∞ agents	2 agents
	.2	0.280	0.270	0.887	0.924	0.198	0.206	0.198	0.059
1	1.	0.140	0.128	0.724	0.786	0.402	0.442	0.400	0.149
	2.	0.102	0.092	0.537	0.683	0.540	0.614	0.537	0.255
	.2	0.262	0.244	0.890	0.927	0.211	0.226	0.211	0.055
3	1.	0.142	0.120	0.761	0.808	0.398	0.464	0.397	0.146
	2.	0.112	0.090	0.713	0.736	0.489	0.615	0.489	0.265

NOTE: This table reports the time-series statistic for the percentage change in individual consumption growth, $\Delta\ln C_t$. The variable $\Delta\ln Y_t$ denotes the percentage change in the individual endowment. ya_t is the growth rate of aggregate endowment which equals the growth rate of aggregate consumption. r_t is the interest rate. γ denotes the parameter of relative risk aversion and bc indicates the amount the agent is allowed to borrow relative to the per capita endowment. The other parameter values are reported in Section 2.3. Population moments are approximated using the sample moments of a simulated draw of 100,000 observations.

Table 4: Time-Series Behavior of Individual Bond Purchases.

		Standard Deviation		Autocorrelation		Correlation with y_t		Fraction at Constraint	
γ	bc	∞ agents	2 agents	∞ agents	2 agents	∞ agents	2 agents	∞ agents	2 agents
	.2	0.396	0.332	0.844	0.788	0.794	0.887	0.308	0.294
1	1.	1.550	1.276	0.967	0.954	0.456	0.532	0.068	0.082
	2.	2.696	2.274	0.986	0.981	0.303	0.355	0.016	0.038
	.2	0.380	0.334	0.831	0.781	0.823	0.893	0.311	0.298
3	1.	1.496	1.310	0.966	0.956	0.466	0.517	0.085	0.100
	2.	2.638	2.408	0.968	0.984	0.305	0.331	0.020	0.058

NOTE: This table reports the time-series statistic for the bond purchases relative to the current per capita endowment, b_t. The variable y_t denotes the agent's endowment relative to the per capita endowment. γ denotes the parameter of relative risk aversion and bc indicates the amount the agent is allowed to borrow relative to the per capita endowment. The other parameter values are reported in Section 2.3. Population moments are approximated using the sample moments of a simulated draw of 100,000 observations.

4.3. INSURANCE AGAINST IDIOSYNCRATIC RISK IN DIVERSE AND LESS DIVERSE ECONOMIES.

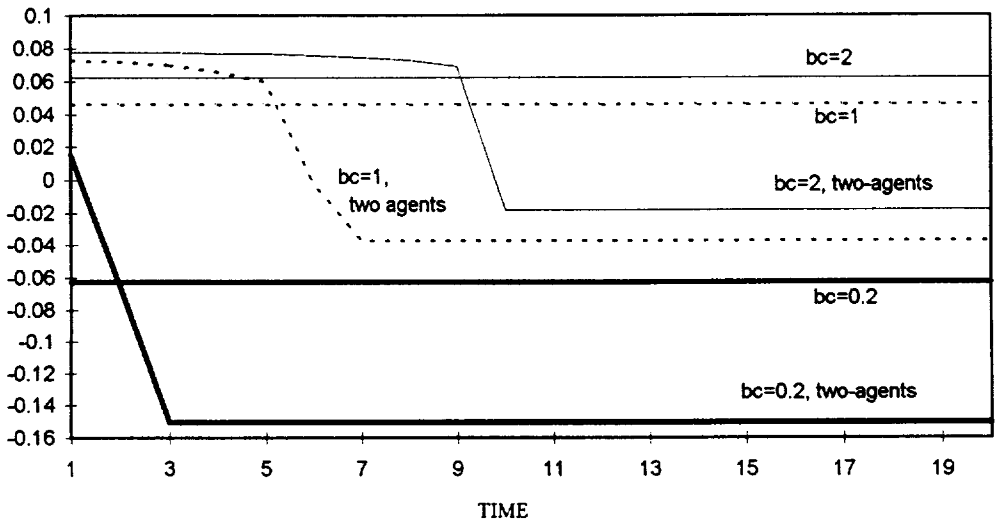
In the last section, it was reported that consumption volatility is smaller in two-agent economies. A priori, it is not clear in which type of economy consumption variability should be higher. Consider the response to an idiosyncratic endowment shock in the two-agent economy when the amount of begin-of-period bond holdings is the same for both agents. The agent who receives the low realization would like to consume and save less. The agent who receives the high realization would like to do the opposite. Since the marginal propensity to save is increasing with income, the interest rate decreases to keep the bond market in equilibrium. Because the interest rate decreases, the agent who receives the low realization reduces his consumption by less than he would otherwise. This suggests that consumption volatility is less in the two-agent economy than in the economy with a continuum of agents, since in the economy with a continuum of agents, the interest rate does not respond to idiosyncratic shocks. The outcome for the agent who receives the high realization, however, suggests the opposite. This agent increases his consumption by more than he would without the change in the interest rate. It is, therefore, not clear whether consumption variability should be higher or lower in the two-agent economy. The results for savings are ambiguous as well.

The example in the last paragraph does illustrate, however, that changes in the interest rates help to insure agents in the two-agent economy against idiosyncratic shocks. In the two-agent economy, the interest rate drops when an agent receives the low realization for several periods because the amount of cross-sectional dispersion increases. This works like a transfer from the rich agent (the lender) to the poor agent (the borrower) and suggests that agents in two-agent economies are better off than agents in economies with a continuum of agents. There is another reason, however, that makes it harder to smooth consumption in two-agent economies. It is harder to smooth consumption in a two-agent economy because an agent can never lend more than the other agent is allowed to borrow. This will prevent him from building up a large buffer stock during good times. Agents in an economy with a continuum of agents can lend to several other agents and at times accumulate assets well in excess over the maximum amount that is possible in a two-agent economy. For most, but not all, of the parameter values considered the first effect dominates.

To document the quantitative importance of the interest rate effect, I plot in Figure 3 the impulse response function of the interest rate when the same agent receives the low-income realization for several periods. The graph plots the interest rate for three levels of the borrowing constraint parameter and for the case when the parameter of relative risk aversion is equal to three. The agents in the two-agent economy start out with zero bond holdings, and without loss of generality, I consider the case where the economy is in a recession. The graph also plots the interest rate in the economy with a continuum of agent. These interest rates, of course, do not respond to the realizations of an individual's income shock.[35] Consider the initial period in the two types of economies. Since the initial levels of bond holdings are equal to zero in the two-agent economy, there is little cross-sectional dispersion when the agent receives his first low-income realization. In the economy with a continuum of agents there always is a certain amount of dispersion. Initially, therefore, the interest rate is lower in the economy with a continuum of agents, since higher cross-sectional dispersion corresponds to lower interest rates. When the agent receives additional negative shocks, the amount of cross-sectional dispersion in the economy with a continuum of agents is not affected. However, the amount of cross-sectional dispersion in the two-agent economy increases and at some point passes the level of the economy with a continuum of agents. As documented in the graph, this causes the interest rate to drop. Quantitatively, these effects are enormous. When the agent reaches his constraint in the two-agent economy, then the interest rate is equal to *minus* 15%, *minus* 3.7%, and *minus* 1.8%, for values of the borrowing constraint parameter equal to 0.2, 1.0, and 2.0, respectively. Not bad, to live in a world with this kind of insurance! These negative interest rates imply not only that the agent consumes more than his income level when he reaches the constraint, but the more restrictive the borrowing contraint, the more the constrained agent consumes. I will return to this issue below.

[35] It is assumed that the economy has been in a recession for several periods so that there are no more changes in the amount of cross-sectional dispersion and the interest rate in the economy with a continuum of agents.

Figure 3: The Response of the Interest Rate to Idiosyncratic Shocks.

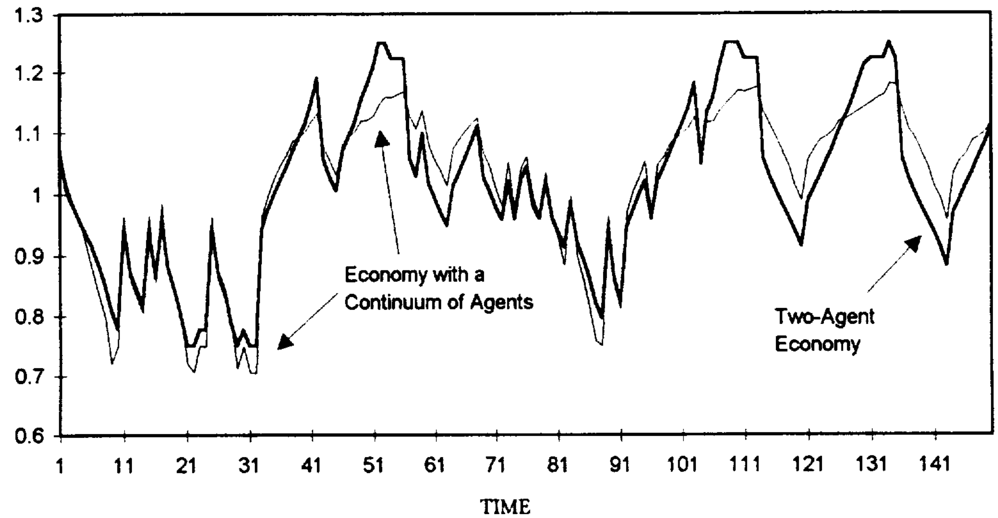


NOTE: This graph plots the realization of the interest rate in the two-agent economy when the agent receives the low-income realization for several periods. The straight line with the same thickness indicates the interest rate in the corresponding economy with a continuum of agents. The parameter of risk aversion is equal to three and the aggregate growth rate is always equal to the low value. The parameter *bc* indicates the amount an agent is allowed to borrow relative to the per capita endowment. The value of the other parameters are reported in Section 2.3.

The graph does not reveal the ability of an agent in the economy with a continuum of agents to accumulate more assets during good times. Figures 4 and 5 reveal both this effect and the interest rate effect discussed above. In particular, these figures plot for the economy with a continuum of agents and the two-agent economy a realization for consumption (relative to per capita income) and bond purchases (relative to per capita income). The parameter of risk-aversion is equal to one and the borrowing constraint parameter is equal to one. During the first 30 periods, the agent is hit several times by negative shocks and is repeatedly at the borrowing constraint. As documented in the graph, the behavior of bond purchases is very similar when the agent has received a series of low realizations and the agent is close to the borrowing constraint.[36] In contrast, the consumption level in the two-agent economy is higher than the consumption level in the economy with a continuum of agents during this period with frequent negative shocks as documented in Figure 4. In the two-agent economy, the drop in consumption is less because the interest rate decreases. The behavior of bonds is quite different across the two types of economies when the agent has received a series of high realizations. This happens around periods 50, 110, and 130. In the two-agent economy this means that the other agent has received a series of low realizations and has reached his borrowing constraint. Consequently, equilibrium on the bond market limits the amount of savings that the agent with the positive shocks can accumulate. In the economy with a continuum of agents, there is no such limitation and the agent accumulates assets well in excess of the borrowing constraint parameter. The graph for consumption shows that the agent in the economy with a continuum of agents is better able to smooth his consumption in the period after which he has build up a large buffer stock of savings.

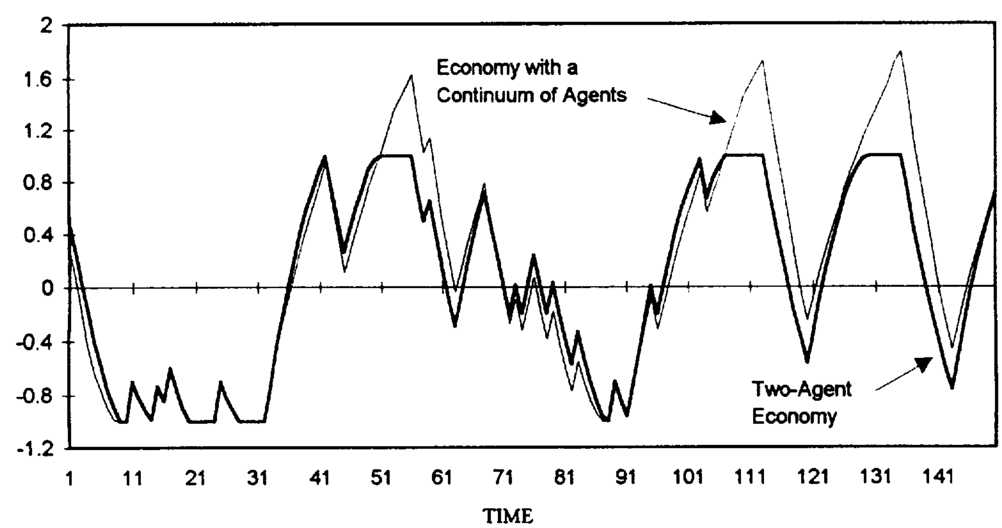
[36] This particular drawing does not make clear that agents in the two-agent economies spend more time at the constraint than agents in the economy with a continuum of agents.

Figure 4: A Realization for Individual Consumption.



NOTE: This graph plots the consumption levels relative to the per capita endowment for the indicated economy. The parameter of relative risk aversion and the borrowing constraint parameter are equal to one. The values of the other parameters are reported in Section 2.3.

Figure 5: A Realization for Individual Savings.



NOTE: This graph plots the savings decision relative to the per capita endowment for the indicated economy. The parameter of relative risk aversion and the borrowing constraint parameter are equal to one. The values of the other parameters are reported in Section 2.3.

The question arises how these differences in the ability to insure against idiosyncratic shock affect the agents' welfare. In Table 5, I report the welfare differences between the different economies. To do this I calculate the (unconditional) expected discounted utility for the economies considered. The expected utility is calculated as the average of the actual utility across 50,000 replications of 250 observations.[37] In each replication, the initial conditions are drawn from the ergodic distribution. Panel A of Table 5 compares the incomplete markets economies relative to the complete markets economies. In particular, it reports the permanent percentage increase in consumption that would make the agents in the incomplete markets economies as well off as the agents in the complete markets economy. The table

[37] The discounted utility of consumption in periods 251 and higher is so small that ignoring it does not affect the results.

documents that agents in the incomplete markets economies are substantially worse off at the lower levels of the borrowing constraint parameters, especially for the higher levels of risk aversion.

Table 5: Welfare Comparisons.

A. Permanent percentage increase of consumption to become as well off as in complete markets economy.

	$\gamma = 1$		$\gamma = 3$		$\gamma = 5$	
bc	∞ agents	2 agents	∞ agents	2 agents	∞ agents	2 agents
0.2	1.70	1.71	4.43	4.16	6.10	5.09
1.0	0.70	0.71	2.13	1.98	3.45	2.95
2.0	0.46	0.36	1.68	1.66	3.20	3.68

B. Permanent percentage increase of consumption to become as well off as in the two-agent economy.

bc	$\gamma = 1$	$\gamma = 3$	$\gamma = 5$
0.2	-0.01	0.27	0.96
1.0	-0.01	0.15	0.49
2.0	0.07	0.02	-0.45

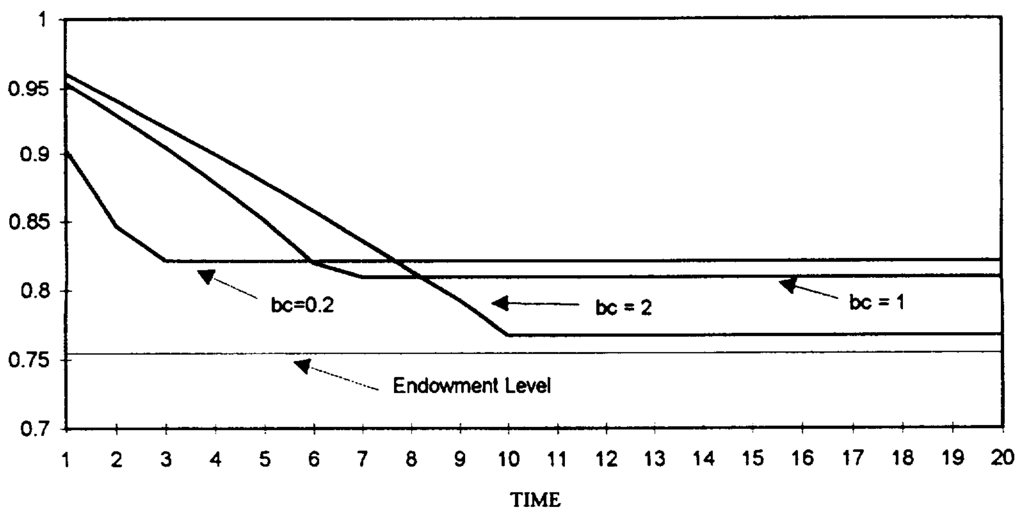
NOTE: Panel A reports the permanent change in consumption that is required to make an agent in the indicated economy as well off as an agent in the complete markets economy. Panel B reports the permanent change in consumption that is required to make an agent in the economy with a continuum of agents as well off as an agent in the two-agent economy. To calculate the (unconditional) expected discounted utility the average of the actual utility across 50,000 replications of 250 observations is used. In each replication, the initial conditions are drawn from the ergodic distribution. γ is the parameter of relative risk aversion and *bc* indicates the amount an agent is allowed to borrow relative to the per capita endowment. The values of the other parameters are reported in Section 2.3.

Panel B reports the permanent percentage change in consumption that would make an agents in the economy with a continuum of agents as well off as an agent in the two-agent economy. For most parameter values, this number is positive, which means that an agent is better off in an two-agent economy. In those cases, the interest rate effects are more important than the ability to accumulate high levels of assets. Moreover, the results are quantitatively important. When the parameter of relative risk aversion is equal to five (three) and the borrowing constraint parameter is equal to 0.2, then the agent in the two-agent economy is willing to permanently reduce his consumption by 0.96 (0.27) percent to avoid being in an economy with a continuum of agents.[38] An important exception is the case when the parameter of risk aversion is equal to five and the borrowing constraint parameter is equal to two. In this case, the agent is better off in the economy with a continuum of agents.

[38] There may be some politically correct arguments against diversification after all.

One more interesting observation can be made about the two-agent economy. As documented in Panel A, when the parameter of risk aversion is equal to five, then an increase in the borrowing constraint parameter from 0.2 to 1.0 makes the agent better off, but a further increase leads to a decrease in the agent's expected utility. The reason is that for higher levels of risk-aversion, the interest rate effects are so helpful in smoothing consumption that being in an economy with a higher borrowing constraint parameter does not make you necessarily better off. To understand this result, I plot in Figure 6 the impulse response function of consumption (relative to the per capita endowment) when the agent receives the low-income realization for several periods. This figure is the analog of Figure 3 for consumption.

Figure 6: The Response of Consumption to a Series of Negative Idiosyncratic Shocks.



NOTE: This graph plots the realization of consumption in the two-agent economy when the agent receives the low-income realization for several periods. The parameter *bc* indicates the amount an agent is allowed to borrow relative to the per capita endowment. The parameter of risk aversion is equal to five and the aggregate growth rate is always set at the low value.

As documented in the graph, during the first few periods, the agent reduces his consumption the most in the economy with the most restrictive borrowing constraint. When the agent continues to receive the low-income realization, the interest rate increases. This works like a transfer from the rich agent (the lender) to the poor agent (the lender). The graph makes clear how it can be possible that an agent's expected utility is higher when the borrowing constraint parameter is equal to 1.0 than when the borrowing constraint parameter is equal to 0.2 or 2.0. Note that when the agent is hit just a few times by a low realization of the income shock, then he benefits from facing a more relaxed borrowing constraint but the difference between the case where the parameter is equal to 1.0 and the case where the parameter is equal to 2.0 is small. When he is more often hit by the low realization, he benefits from a less relaxed borrowing constraint because of the corresponding low interest rate. Now the differences between the case where the parameter is equal to 1.0 and the case where the parameter is equal to 0.2 is small.

Note that when the agent reaches his constraint in the two-agent economy, he consumer more than his endowment level because the interest rate is negative. One might think that the agent that is allowed to borrow less would benefit less from the negative interest rates. The opposite is true because, reported in Figure 3, the interest rate is so much more negative in the economy with the low level of the borrowing constraint.

4.4. INCOMPLETE MARKETS AND BUSINESS CYCLE BEHAVIOR OF INTEREST RATES.

The business cycle behavior of interest rates in economies with incomplete markets and a large number of agents has not been studied in the literature, because the lack of a solution algorithm made it infeasible to do this.[39] Nevertheless, this is an important topic. For example, frictions in the credit market are believed to play an important role in the way monetary shocks affect interest rates and output.[40] In this section, I analyze the business cycle behavior of the interest rate in the economy described in section 2 with incomplete markets and a large number of agents. This model is clearly much too simple to describe the actual behavior of US interest rates. The simple nature of the model, however, makes it possible to clarify the interaction between the business cycle behavior of interest rates and incomplete markets. In particular, I will show that the business cycle behavior of interest rates in an economy with incomplete markets differs considerably from the behavior in an economy with complete markets and that changes in the amount of cross-sectional dispersion over the business cycle play a crucial role. It is important to note that, in the model used in this paper, all changes in the amount of cross-sectional dispersion over the business cycle are due to changes in the dispersion of bond holdings, an endogenous variable. In the model, the amount of cross-sectional dispersion in endowments and the amount of financial frictions do not change over the business cycles. There are reasons to question both assumptions.[41] The effects discussed in this section are, therefore, likely to underestimate the importance of changes in the amount of cross-sectional dispersion for interest rate behavior.

The outline of this section is as follows. First, I discuss how cross-sectional dispersion changes over the business cycle. Next, I discuss the relation between the interest rate and the amount of cross-sectional dispersion in an economy with a continuum of agents. Finally, I discuss the relationship between the interest rate and the growth rate of the aggregate endowment. Note that this relationship would be very simple in the complete markets version of the economy used in this paper. With complete markets, there are only two possible values for the interest rate, a low value when the aggregate growth rate is low and a high value when the aggregate growth rate is high. Adjustment takes place immediately. This section shows that in the incomplete markets version, the dynamics are a lot more complicated and that changes in the amount of cross-sectional dispersion play a crucial role.

The relevant measure of heterogeneity in this model is the cross-sectional dispersion of begin-of-period bond holdings *relative* to the per capita endowment. The initial effect of an increase in the aggregate growth rate, therefore, is that all measures of cross-sectional dispersion decrease since the per capita endowment increases. It is important to understand that this is not just a rescaling of the bond

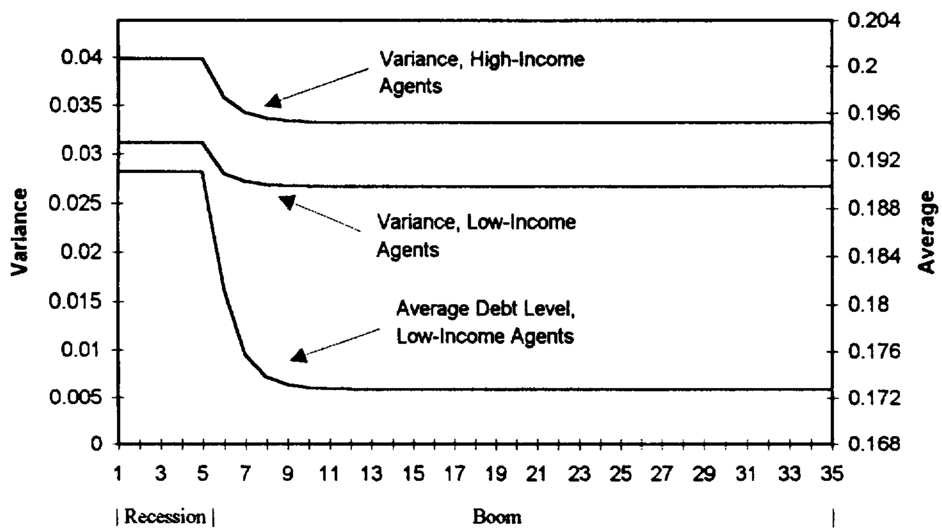
[39] It does not make much sense to study the business cycle behavior of interest rates in a two-agent economy since fluctuations in interest rates are dominated by idiosyncratic shocks in these types of economies.

[40] See, for example, Bernanke, Gertler, and Gilchrist (1996) and the references therein.

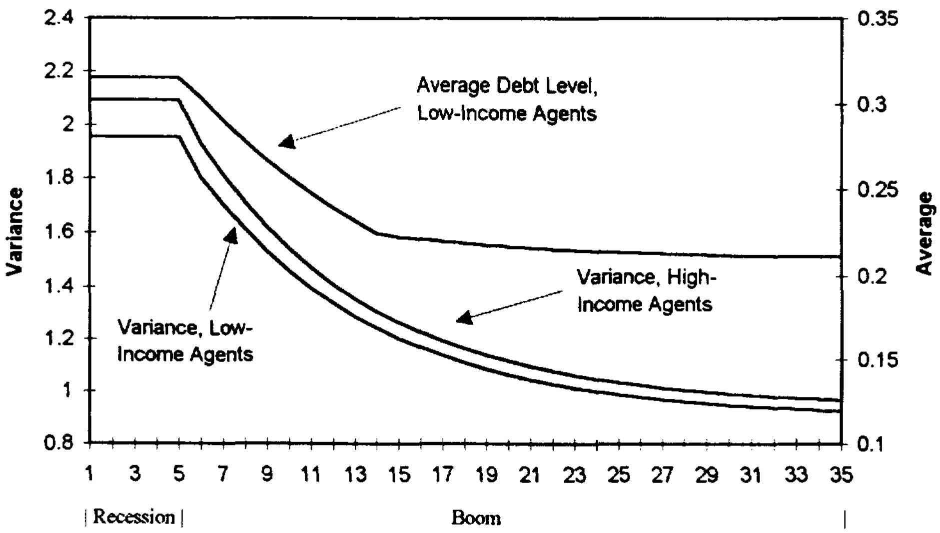
[41] Burtless (1990) argues that earnings inequality increases over the business cycle. See Den Haan and Spear (1996) and the references therein for empirical and theoretical arguments that the amount of financial frictions change over the business cycle.

holdings. When there is a positive shock to the aggregate endowment, then the relative differences between a borrower who starts the period with a liability of X units of debt and a lender who starts with an asset of X units decrease. Moreover, I will show below that this decrease in the amount of cross-sectional dispersion has an effect on the interest rate. Figures 7A and 7B plot the dynamic response of the amount of cross-sectional dispersion in response to a prolonged change in the growth rate of the aggregate endowment. Figure 7A plots the response for the economy with a tight borrowing constraint (bc = 0.2) and Figure 7B plots the response for the economy with a loose borrowing constraint (bc =2.0). In the economy with a tight borrowing constraint, the adjustment is relatively rapid. Within three periods, all three measures of cross-sectional dispersion have completed at least 93% of the total adjustment. In contrast, in the economy with a loose borrowing constraint the adjustment is very slow. It takes 20 periods before all three measures of dispersion have completed 93% of the total adjustment. Similar results are obtained for higher values of the parameter of relative risk-aversion.

Figure 7: Cross-Sectional Dispersion and the Business Cycle.
A. Tight Borrowing Constraint (bc = 0.2).



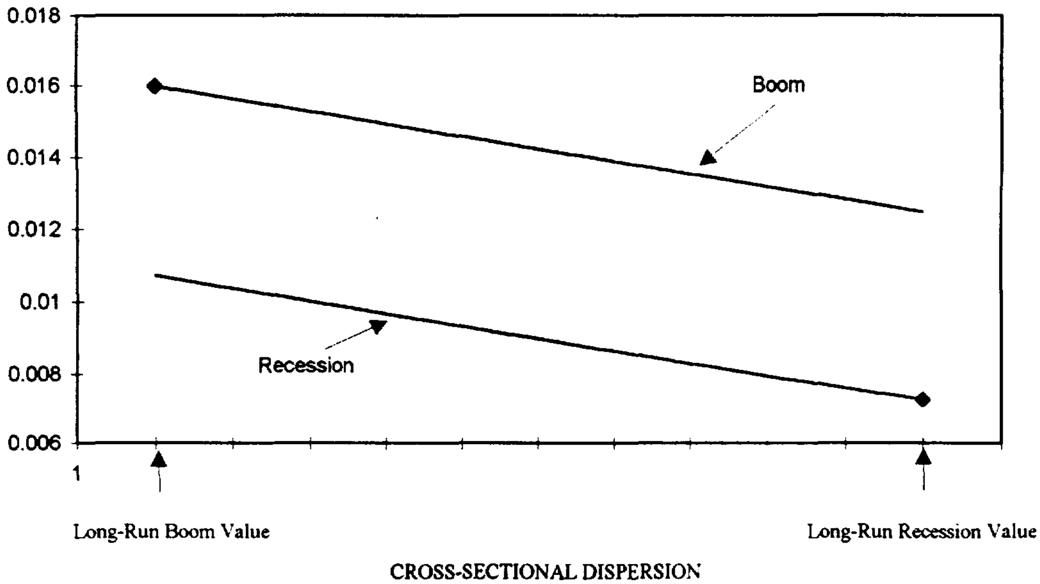
B. Loose Borrowing Constraint (bc = 2.0).



NOTE: These figures plot the impulse response of the indicated measure of cross-sectional dispersion in response to a change in the aggregate growth rate from a long period of low growth rates to a long period of high growth rates. The parameter of relative risk aversion is equal to one. The values of the other parameters are reported in Section 2.3.

Before discussing the adjustment of the interest rate to a change in the aggregate growth rate of the endowment, I discuss the relationship between the interest rate and the amount of cross-sectional dispersion. In Section 4.1.1, it was shown that the interest rate in a two-agent economy is a decreasing concave function of the amount of cross-sectional dispersion. Den Haan (1996b) shows that in an economy with a continuum of agents the changes in the amount of cross-sectional dispersion are small enough to approximate the bond function as a linear function of the amount of cross-sectional dispersion. Figure 8 plots the interest rate as a function of the amount of cross-sectional dispersion for the realization of the low aggregate growth rate and the high growth rate. The parameter of relative risk-aversion is equal to one and the borrowing constraint parameter is equal to 0.2. The variable on the x-axis indicates the amount of cross-sectional dispersion. The highest x-value correspond to the point where all three measures of cross-sectional dispersion are at the value they obtain when the economy has been in a recession for a long time. The lowest x-value correspond to the point at which all three measures are at the values when the economy has been in a boom for a long time period. The x-values in-between correspond to linear interpolations of the three measures. The graph documents that - as in Subsection 4.1.1 - the interest rate is a decreasing function of the amount of cross-sectional dispersion. The graph also documents that if the economy changes from a period of low growth rates to a period of high growth rates then an important part of the change is due to a change in the amount of cross-sectional dispersion. When the risk aversion parameter is equal to one and the borrowing constraint parameter is equal to 0.2, for example, then the total change in the interest rate is equal to an increase of 0.87 percentage points. If the amount of cross-sectional dispersion would have stayed at the long-run recession level, then the increase would have been only 0.52 percentage points. This last number is similar to the change in the complete markets economy in which cross-sectional dispersion has no effect on the dynamics of the interest rate.

Figure 8: The Interest Rate and the Amount of Cross-Sectional Dispersion.

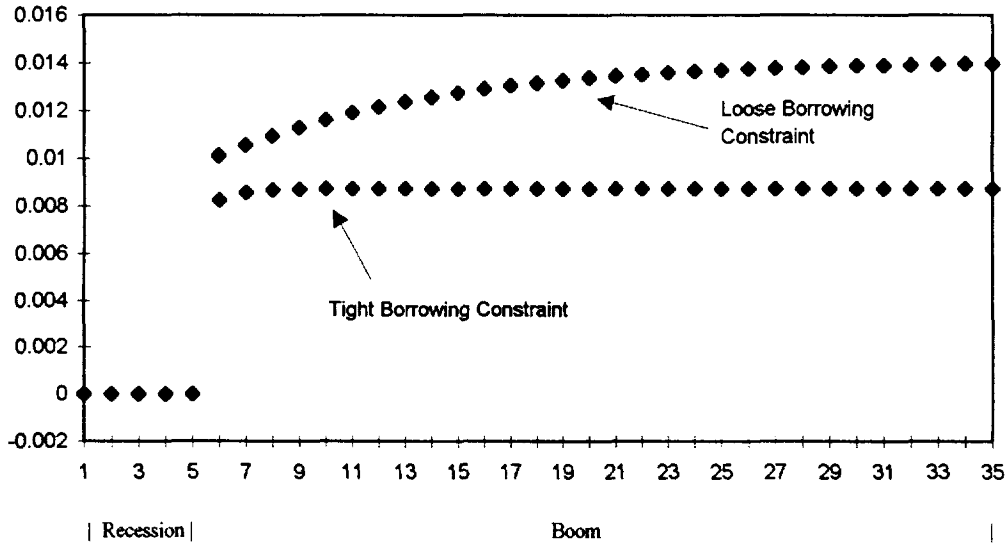


NOTE: This graph plots the interest rate in the indicated state as a function of the amount of cross-sectional dispersion. The diamonds indicate the value of the interest rate when the economy has been in the indicated state for a "long" time period and the amount of cross-sectional dispersion has reached its long-run values. The parameter of relative risk aversion is equal to 1 and the borrowing constraint parameter is equal to 0.2. The values of the other parameters are reported in Section 2.3.

The graph is similar for other parameter values. What is different is the speed at which the interest rate moves from its long-run recession value to its long-run boom value. As documented in Figure 7, the cross-sectional dispersion adjusts quickly in the economy with a tight borrowing constraint and slowly in the economy with a loose borrowing constraint. Consequently the interest rate adjusts much faster in the economy with a tight borrowing constraint. This is documented in Figure 9, which plots the impulse response of the interest rate in response to a prolonged period of economic growth. As documented in the figure, it takes much longer for the interest rate to adjust to its long-run value for the high borrowing constraint parameter. Also, the response of the interest rate in response to a change in the aggregate growth rate is substantially higher in the economy with the higher borrowing constraint parameter. However, for the parameter values used in this paper, persistent changes in the aggregate growth rate are not very common. This means that the business cycle behavior is less sensitive to the borrowing constraint parameter than Figure 9 seems to indicate.

In the complete markets version and in the incomplete markets version where agents are not allowed to borrow at all, the interest rate fully adjusts within one period to the change of the aggregate growth rate. Moreover, the magnitude of the change is exactly the same when the interest rate is defined as minus the natural logarithm of the bond price. For the parameter values corresponding to the case shown in Figure 9, the magnitude of the change is approximately equal to 0.5 percentage points. For all non-zero borrowing constraint parameters considered, the change is always bigger in the incomplete markets economy. The reason is the change in the cross-sectional dispersion. It is important to note that this economy does not get arbitrarily close to a complete markets economy when the borrowing constraint parameter increases. When the borrowing constraint parameter increases,[42] the magnitude of the change in the interest rate in response to a change in the aggregate growth rate levels off but does not become smaller and, thus, does not approach the value observed in the complete markets economy.

Figure 9: Change of the Interest Rate in Response to an Aggregate Shock.



Note: This figure plots the impulse response of the interest rate in response to a change in the aggregate growth rate from a long period of low growth rates to a long period of high growth rates. The parameter of relative risk aversion is equal to one. In the case of a tight (loose) borrowing constraint the borrowing constraint parameter is equal to 0.2 (2.0). The values of th e other parameters are reported in Section 2.3.

5. CONCLUDING COMMENTS.

In this paper, I compare the properties of an asset-pricing model with a continuum of agents to the properties of the corresponding two-agent model. Besides the number of agents, the two models are exactly identical. In particular, the amount of aggregate uncertainty is the same for the two models. What makes the two-agent economy different from the economy with a continuum of agents is that idiosyncratic shocks affect the amount of cross-sectional dispersion and thus the interest rate. In particular, the following differences in the time-series behavior of interest rates are found.

- Interest rates are much more volatile in two-agent economies because an important part of the fluctuation in interest rates is due to idiosyncratic shocks.
- Interest rates are more persistent in the two-agent economies because they are affected by idiosyncratic shocks which are more persistent than shocks to the aggregate growth rate. Moreover, the level of the borrowing constraint parameter has little affect on the amount of persistence in the economy with a continuum of agents but has a nonlinear effect on the amount of persistence in the economy with two-agents.
- Interest rates are on average lower in two-agent economies because the interest rate is a concave function of the amount of cross-sectional dispersion and cross-sectional dispersion is more volatile in

[42] I tried values for *bc* as high as 8.

two-agent economies. Whether the difference is quantitatively important depends crucially on the amount of risk aversion and the borrowing constraint parameter.

The different time-series behavior of interest rates causes the behavior of consumption and bond trades to differ as well. The following differences are found.

- In a two-agent economy, each agent can never invest more than the other agent is allowed to borrow. In an economy with a continuum of agents, equilibrium on the bond market does not create such a constraint. Consequently, in the economy with a continuum of agents, agents accumulate during good times assets well in excess of the amount they are allowed to borrow. This makes it easier to smooth consumption.

- There is also a reason why it is easier to smooth consumption in the economy with only two agents. If, in a two-agent economy, the same agent is hit by a series of negative "idiosyncratic" shocks, the amount of cross-sectional dispersion increases which leads to a drop in the interest rate. The reduction in the interest rate works like a transfer from the rich agent (the lender) to the poor agent (the borrower). For most parameter values, this effect dominates the limit on the maximum amount an agent can save.

- For higher values of the parameter of relative risk aversion these interest rate effects become so strong that the effect of financial frictions on agent's welfare has a different sign in the two types of economies. In economies with a continuum of agents, an agent's welfare is always increasing when the maximum amount he is allowed to borrow increases. There are parameter values, however, for which an agent's welfare in the two-agent economy is decreasing when the borrowing constraint is relaxed.

- Because interest rates drop if the amount of cross-sectional dispersion increases, the average amount of debt of low-income agents and the fraction of times spend at the constraint are higher in two-agent economies.

Despite these differences, there are many qualitative and quantitative similarities between the two types of economies. For example, both types of models indicate that, relative to the representative agent model, interest rates are substantially lower in the presence of incomplete markets and borrowing constraints. Moreover, the effect of a decrease in the borrowing constraint parameter on the average interest rate is very similar.

In future research, I hope to analyze models in which agents can invest in equity and in particular compare the equity premium across the two types of economies. The equity premium is related to the correlation between the marginal utility of individual consumption and the return on equity. This correlation is likely to be quite different across the two types of economies since in a two-agent economy individual shocks have aggregate effects and thus affect not only individual consumption but also prices and rates of return. Note that solving the model with equity would be a non-trivial extension since in this case the cross-sectional distribution of begin-of-period asset holdings depends on the stock price, an endogenous variable that just like the bond price in the model in this paper is likely to depend on the amount of cross-sectional dipsersion.

In this paper, I also compare the business-cycle behavior of interest rates in an incomplete markets economy with a continuum of agents to the behavior of interest rates in a complete markets economy. The findings are as follows.

- In the complete markets economy, the interest rate adjusts within one period to a change in the aggregate growth rate. Due to changes in the amount of cross-sectional dispersion, the interest rate takes longer to adjust fully in an incomplete markets economy. The speed of adjustment is negatively related to the value of the borrowing constraint parameter.
- The magnitude of the change in the interest rate is bigger in the economy with incomplete markets than in the economy with complete markets. When the parameter of relative risk aversion is equal to one and the amount agents are allowed to borrow is equal to 20 percent of the per capita endowment, for example, then the initial increase in the interest rate, in response to the transition from the "low growth rate" state to the "high growth rate" state, is equal to 0.83 percentage points in the incomplete markets economy. In contrast, the initial (and total) increase is equal to 0.52 percentage points in the complete markets economy. The additional increase in the incomplete markets economy is due to a reduction in the amount of cross-sectional dispersion.

The model used in this paper is likely to underestimate the importance of cross-sectional dispersion for the behavior of the interest rate over the business cycle. For example, the amount of cross-sectional dispersion in the idiosyncratic shock is assumed not to change over the business cycle.[43] Furthermore, the borrowing constraint does not vary with the business cycle, although several papers argue that financial frictions do vary with the business cycle.[44]

It would be interesting to investigate whether the predictions of this model for the changes in the amount of cross-sectional wealth dispersion are comparable to changes in empirical measures of cross-sectional dispersion. Unfortunately, it is hard to obtain time-series data on cross-sectional wealth dispersion. Moreover, the abstract definition of wealth used in this model is unlikely to correspond closely to empirical measures of wealth. To get some idea on how plausible the reported changes in the amount of cross-sectional dispersion of bond holdings relative to the per capita endowment are, I report the cross-sectional dispersion in total family income relative to the per capita level over the period from 1968 to 1989. The data are from the Panel Studies of Income Dynamics (PSID). Included in the cross-section are all the heads of households in the SRC sample.[45] The PSID replaces negative values by a zero and truncates large positive values. The estimated cross-sectional dispersion, thus, underestimates the actual cross-sectional dispersion. The results are reported in Figure 10. In the model with a continuum of agents, a change from the low-growth state to the high-growth state reduces the cross-sectional standard

[43] Burtless (1990), however, points out that income inequality is countercyclical. A willing eye can detect this countercyclical pattern in Figure 10.
[44] See, for example, Bernanke and Blinder (1989), Bernanke, Gertler, and Gilchrist (1996), and Den Haan and Spear (1996).
[45] The data set is not a panel since heads of households die and children start their own household.

deviation of begin-of-period bond holdings by five percent in the first period. This reduction of the cross-sectional dispersion leads to an additional increase of the interest rate of approximately 0.3 percentage points when the parameter of relative risk aversion is equal to one.[46] As documented in Figure 10, much larger changes in the amount of cross-sectional income dispersion are observed during the post-war period.

Figure 10: The Amount of Cross-Sectional Dispersion in Total Family Income.

NOTE: This graph plots the cross-sectional standard deviation of total family income relative to the per capita level using data from the PSID. The cross-section includes all head of households in the SRC sample. In this data set negative values and large positive values are truncated.

[46] The model predicts much larger changes in the amount of cross-sectional dispersion when the aggregate growth rate changes for a long time period. These persistent changes occur infrequently, however, since shocks to the aggregate growth rate are assumed to be not very persistent.

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