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A MODEL OF FIAT MONEY AND BARTER

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ABSTRACT

We present an infinite horizon model with capital in which fiat money and barter are two competing means of payment. Fiat money has value because barter is limited by the extent of a double coincidence of wants. The pattern of exchange generally involves both money and barter. We find that the Chicago rule is sufficient for Pareto efficiency, while nominal interest smoothing is necessary. For a specific utility function we provide a complete characterization of the patterns of exchange and calculate the range of inflation rates over which a stationary monetary equilibrium exists.

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1. Introduction

The question of why fiat money, intrinsically useless pieces of paper, can have value has long intrigued economic theorists. In the cash-in-advance model (see Grandmont and Younes (1972), Lucas (1980)), fiat money is simply assumed to be necessary for transactions purposes. In the infinite horizon economy of Bewley (1980) and in the overlapping generations model of Wallace (1980), money has value only because there are no other stores of value. The "turnpike" model of Townsend (1980) is an explicit model of money demand, but the trading arrangement is rigged so that money is effectively the only means of payment. The random matching models (see Iwai (1988), Oh (1989), Kiyotaki and Wright (1991)) explain why money has value in environments with multiple means of exchange. They, however, do not lend themselves to analysis of inflation because goods and fiat money are assumed indivisible.

Recently, Engineer and Bernhardt (1991) have considered an extension of the Townsend model in which barter is given a non-trivial chance to compete with money. Trades occur simultaneously so that agents cannot use revenues, in goods or money, from trade in one market to augment purchases in other markets.[1] Consequently, barter takes place only if there is a double coincidence of wants and monetary exchange is limited by the cash-in-advance constraint. The Engineer-Bernhardt model, however, is rather special in its specification of preferences. Each agent obtains utility only from her own good and her immediate neighbors' goods, and one of the neighbors' goods and her own good are perfect substitutes, which ensures that there is a unique inflation rate below which the only means of exchange is money and over which

[1] This trading arrangement is similar to, but different from, the market game of Shapley and Shubik (1969). The market game admits a multiplicity of Nash equilibria involving inactive markets because if each player believes that no other player will bring the good to the market, the player's best response is not to bring the good to be traded against. This is not an equilibrium under simultaneous trading because a price is called to ensure that all mutually advantageous pairwise trades are exhausted. Alonso (1991), which came into our attention during the preparation of the first draft of our paper, studies monetary equilibria in the Shapley-Shubik market game.

money has no value.

This paper significantly extends the Engineer-Bernhardt model by considering a general utility function in which all goods are potentially desired. This enables us to entertain a richer pattern of exchange and identify a wider class of optimal monetary policies. We find that in general the emerging pattern of exchange is mixed in two senses. First, goods acquired exclusively through money and those acquired exclusively through barter can co-exist. Second, there can be goods acquired through both means of exchange. Although the Chicago rule ensures monetary equilibria to be Pareto efficient, we find cases in which efficiency is consistent with positive or non-constant nominal interest rates with or without active barter exchange. Despite the rich pattern of exchange it permits, the model remains tractable. For a CES utility function, we can determine which goods are acquired through barter and/or money and calculate equilibrium barter and money prices.

Our model also includes capital. Although not an essential extension, it ensures that money has value despite the existence of assets with superior rates of return, and makes our model a monetary growth model with an explicit micro-foundation of the demand for money. We find that money is super neutral in Tobin's sense.

The rest of the paper is organized as follows. Section 2 presents a barter economy with simultaneous trading where preferences and technology are symmetric in a certain sense. Section 3 introduces fiat money to this symmetric environment. We partially characterize monetary equilibria and examine optimal monetary policy. In Section 4 we provide a complete characterization of monetary equilibria the CES utility function. Section 5 relaxes symmetry and provides examples with patterns of exchange that cannot arise under symmetry. Section 6 contains concluding remarks.

2. Pure Barter Economy without Resale

Before presenting a monetary economy, we briefly study a barter economy without fiat money. The relative prices in the barter economy will serve as a useful benchmark in our characterization of monetary equilibria in Section 4. There are n types of agents, indexed by $i = 0, 1, \cdots, n-1$. Each type consists of a continuum of identical agents. There is a complete specialization in endowments, so agent i is endowed with good i only. We call the good the agent is endowed with the *home good*. In this and the following two sections, we assume the environment is *symmetric* in that each agent is endowed with the same amount, ω, of the home good, and if $u(x_0, x_1, \cdots, x_{n-1})$ is agent 0's utility, agent i's utility from the same consumption bundle is $u(x_i, x_{i+1}, \cdots, x_{n-1}, x_0, x_1, \cdots, x_{i-1})$. The utility function u(.) is assumed to satisfy the usual properties of differentiability, local non-satiation, and strict concavity, over the set of desired goods.[2]

Let x_j be agent 0's consumption of good j and q_j^b be the relative price of good j in terms of good 0 (with super-script b emphasizing the pure barter economy). Then the representative agent's decision problem is:

$$(2.1) \qquad \max \ u(x_0, \cdots, x_{n-1}) \quad \text{s.t.} \quad \sum_{j=0}^{n-1} q_j^b x_j = \omega, \quad x_j \geq 0 \ \ (j = 0, 1, \cdots, n-1).$$

We allow q_j^b to be zero or infinity and adopt the convention $0 \times \infty = 0$, so the agent's decision problem (2.1) is well defined even when some barter prices are infinite: if $q_j^b = \infty$, the budget constraint requires that the agent's demand for good j, x_j, be zero. We focus on symmetric equilibria such that the price of good i+j in terms good i is independent of i.[3] Consequently,

[2] We require concavity rather than quasi-concavity because this utility function is the instantaneous utility function in the infinite-horizon economy in the following sections.

[3] If $i+j > n-1$, then "i+j" is understood to mean "i+j−n". Even if the environment is symmetric, the equilibrium prices may not be symmetric. See Appendix 2 for an example.

q^b_{n-j} equals the price of good 0 in terms of good j, which in turn equals $1/q_j$ by the nature of relative prices. Also, given the symmetric environment, x_j equals agent i's consumption of good i+j and x_{n-j} equals agent j's consumption of good 0 in symmetric equilibria.

The trading arrangement we envision is one in which, as in the standard Walrasian model, there are $n(n-1)/2$ barter markets but in which trading is *simultaneous* with no resale of goods acquired in other markets. Thus, in market (0,j) in which good 0 is exchanged for good j, only two types of agents, agents 0 and j, can participate, and all of the amount transacted gets consumed. The amount of good 0 supplied by agent 0 is $q^b_j x_j$ if x_j units of good j are to be consumed, while the demand for good 0 is the consumption of good 0 by agent j which equals x_{n-j} by symmetry. Thus equilibrium can be defined as:

Definition 2.1: A symmetric barter equilibrium *is* (x_j, q^b_j) $(j = 0, 1, \cdots, n-1)$ *such that:*

(B1) $(x_0, x_1, \cdots, x_{n-1})$ *solves (2.1);*

(B2) markets clear: (market (0, j)) $q^b_j x_j = x_{n-j}$ $(j = 1, \cdots, n-1)$.

Because of simultaneous trading, the market clearing condition is not the usual condition under symmetry, $\sum_j x_j = \omega$, and the equilibrium relative prices need not satisfy the usual triangular relation $q^b_j \cdot q^b_k = q^b_{j+k}$.

Due to the friction imposed on the barter economy, an equilibrium allocation is not necessarily Pareto efficient. A dramatic illustration is the following.

Definition 2.2: There is a complete lack of double coincidence of wants *if, for all* $j = 1, \cdots, n-1$, $u_j(x) > 0$ *for some x implies* $u_{n-j}(x') = 0$ *for all x', where* $u_j(x) \equiv \partial u(x)/\partial x_j$.

In this case, the only equilibrium is autarchic because no two agents will ever find it advantageous to exchange goods at any terms of trade.

We close this section by providing a parametric example.

Example 2.1: Three goods (n = 3) with logarithmic utility:

$$u(x_0, x_1, x_2) = \alpha_0 \ln(x_0) + \alpha_1 \ln(x_1) + \alpha_2 \ln(x_2).$$

It is easy to show that there exists a unique equilibrium and it is symmetric. The equilibrium barter prices are $q_1^b = \alpha_1/\alpha_2$ and $q_2^b = \alpha_2/\alpha_1$. Unless $\alpha_1 = \alpha_2$, the allocation is not efficient because the marginal rates of substitution are not equalized across agents. If $\alpha_1 > 0$ but $\alpha_2 = 0$, there is a complete lack of double coincidence of wants, with $q_1^b = \infty$ and $q_2^b = 0$.

3. Monetary Economy

3.1. Monetary Equilibrium

We consider an infinite horizon economy with fiat money. In date 0 a monetary authority, which is the sole issuer of fiat money, is established. Thereafter it distributes or extracts money in a lump-sum fashion. Since fiat money has no intrinsic value, agents are willing to accept it only when they intend to get rid of it. In the present model, this requires that the economy is expected to last forever.[4] We study an economy inhabited by infinitely-lived agents with a discount factor of $\beta \in (0, 1)$. Since each type of agents consists of a continuum of individuals and trading is anonymous, no credit arrangements are possible.[5] Goods are perishable, so there can be no commodity money. To highlight the role of fiat money as a means of exchange, rather than a store of value, we also introduce capital accumulation: each agent has access to a technology that converts k units of the home good into f(k) units in the

[4] Recently, Kajii (1991) built a finite horizon economy with incomplete markets in which fiat money is valued in spite of the presence of local non-satiation.

[5] For models of with money and credit, see Bernhardt (1989), Chatterjee and Corbae (1991), and Lacker and Schreft (1993).

following period. To rule out a zero or infinite capital stock equilibrium, we assume that the initial capital stock, k(0), is positive and f(.) satisfies $f'(0) = \infty$, $f'(\infty) = 0$.

With the addition of n markets where money is traded for n goods, there are $n(n+1)/2$ markets with simultaneous trading. There are now two ways to acquire goods, barter and money. If agent 0 obtains $x_j^b(t)$ units of good j through barter in date t and $x_j^m(t)$ units through money, her consumption of good j is $x_j^b(t) + x_j^m(t)$. We focus on symmetric monetary equilibria in which the money price of goods, P(t), is the same across goods. Save the terms relating to barter, the agent's decision problem is the same as in cash-in-advance models:

$$
\text{(3.1)} \qquad \max \sum_{t=0}^{\infty} \beta^t u(x_0^b(t) + x_0^m(t), \cdots, x_{n-1}^b(t) + x_{n-1}^m(t))
$$

$$
\text{s.t.} \quad x_m^b(t)/P(t) + \sum_{j=0}^{n-1} q_j(t) x_j^b(t) + k(t+1) = f(k(t)),
$$

$$
\sum_{j=0}^{n-1} P(t) x_j^m(t) + x_m^m(t) = M(t),
$$

$$
M(t+1) = x_m^b(t) + x_m^m(t) + V(t),
$$

$$
x_j^b(t),\ x_m^b(t),\ x_j^m(t),\ x_m^m(t),\ M(t) \geq 0, \quad M(0),\ k(0) \text{ given},
$$

where $q_j(t)$ is barter price of good j in terms of good 0, $x_m^b(t)$ nominal value of home good sold for money, $x_m^m(t)$ nominal money balances not used for monetary exchange, M(t) nominal money balances at the beginning of date t, and V(t) is nominal monetary transfers from the monetary authority in date t. The first constraint in (3.1) is that the total use of the home good for barter and monetary exchange and storage equals the initial balance f(k(t)), while the second has the interpretation of the *cash-in-advance constraint:* because of simultaneous trading, monetary proceeds from selling the home good cannot be used to augment monetary purchases in other markets. The third constraint describes the sources of cash to be carried over to the next period.

The definition of equilibrium is

Definition 3.1: A symmetric monetary equilibrium associated with a monetary policy (M(0), {V(t)}) *is a time path* $\{P(t), M(t), k(t), q_j(t), x_j^b(t), x_m^b(t), x_j^m(t), x_m^m(t), j = 0, 1, \cdots, n-1\}$ *such that:*

(M1) the time path $\{M(t), k(t), x_j^b(t), x_m^b(t), x_j^m(t), x_m^m(t)\}$ *is a solution to the agent's decision problem (3.1) given* $\{P(t), q_j(t)\}$*;*

(M2) markets clear: (market (0, j)) $\quad q_j(t)x_j^b(t) = x_{n-j}^b(t) \quad (j = 1, \cdots, n-1),$

(market (0, m)) $\quad x_m^b(t)/P(t) = \sum_{i=0}^{n-1} x_i^m(t);$

(M3) active monetary exchange: for all t, $x_m^b(t) > 0$ *and* $x_j^m(t) > 0$ *for some* $j \neq 0$.

To understand the equilibrium condition in (M2) for market (0, m) where the home good is exchanged for money, we note that the left-hand-side, $x_m^b(t)/P(t)$, is the amount of good 0 supplied by agent 0. The amount of good 0 demanded by agent i equals x_{n-i}^m by symmetry. The right-hand-side is the sum of the demands over all agents. This condition and (M3) imply that money has value, *i.e.*, $0 < P(t) < \infty$.

3.2. Partial Characterization of Monetary Equilibrium

What mix of barter and money emerges in a monetary equilibrium? We provide a partial characterization in terms of the inflation rate and the gross nominal interest rate:

$$(3.2) \qquad \pi(t) = [P(t)-P(t-1)]/P(t-1) \text{ and } R(t) = f'(k(t))[1+\pi(t)] = f'(k(t))P(t)/P(t-1).$$

As in cash-in-advance models, the nominal interest rate (the nominal rate of return from capital) can never be negative in a monetary equilibrium, because otherwise agents can expand the budget constraint by substituting money for capital. If, on the other hand, the nominal interest rate is positive, then there is the interest cost of holding money, which prompts agents to limit the initial money balances to what is required for transaction purposes (so $x_m^m(t) = 0$) and avoid using monetary exchange for acquiring the home good (so $x_0^m(t) = 0$).

More interesting is the choice between barter and money. The cost in terms of the home good of obtaining one unit of good j through barter is, of course, $q_j(t)$. That through monetary exchange is not 1 but R(t). The reason is familiar from cash-in-advance models: the initial money balances represent a lost opportunity of saving in the previous period.[6] The more expensive means will not be used in exchange. In particular, $q_j(t)$ must be no greater than R(t) if agent 0 barters with agent j. But by symmetry, agent 0 also barters with agent $n-j$, so $q_{n-j}(t)$ $(= 1/q_j(t)) \leq R(t)$. Thus $q_j(t)$ lies between $1/R(t)$ and R(t). It also follows that there will never be a two-way monetary exchange if the nominal interest rate is positive. When agent 0 uses money over barter to obtain good j from agent j, it must be that $q_j(t) \geq R(t) > 1$. But then agent j will not use money to acquire good 0 (which by symmetry means $x^m_{n-j}(t) = 0$) because the barter price of good 0 in terms of good j, which equals $1/q_j(t)$, is less than R(t). The absence of a two-way monetary exchange means that, if two agents consume each other's goods, at least some of the consumption is acquired through barter. That is, a double coincidence of wants necessarily leads to barter if the nominal interest rate is positive. We have thus proved:

Proposition 3.1:

(a) $R(t) \geq 1$ *for all t. If* $R(t) > 1$, *then* $x^m_m(t) = 0$ *and* $x^m_0(t) = 0$.

(b) $x^b_j(t) > 0 \Rightarrow 1/R(t) < q_j(t) \leq R(t)$; $x^m_j(t) > 0 \Rightarrow q_j(t) \geq R(t)$.

(c) If $R(t) > 1$, *then* $x^m_j(t) > 0$ *and* $x^m_{n-j}(t) > 0$ *never hold simultaneously.*

(d) If $R(t) > 1$, *then* $x^b_j(t) + x^m_j(t) > 0$ *and* $x^b_{n-j}(t) + x^m_{n-j}(t) > 0$ *(so agents 0 and j consume each other's goods) imply that agents 0 and j barter.*

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[6] To obtain one additional unit of good j through money in date t, the initial money balances M(t) have to be increased by P(t) dollars, which is possible if the sale of the home good for money in date $t-1$, $x^b_m(t-1)$, is increased by P(t) dollars. But this reduces saving in date $t-1$, k(t), by $P(t)/P(t-1)$ units, so the initial balance of the home good in date t, f(k(t)), is less by $f'(k(t))[P(t)/P(t-1)] = R(t)$.

Since the agent can obtain capital through barter with herself at the exchange rate of one,[7] it is not surprising that money is *super neutral:*

Proposition 3.2: (Super Neutrality) In a stationary monetary equilibrium in which $x_j^b(t)$, $x_j^m(t)$, $q_j(t)$, $k(t)$, *and* $\pi(t)$ *are constant over time, the gross real interest rate,* $f'(k(t))$, *equals* $1/\beta$.

Proof: See Appendix 1.

3.3. Optimal Monetary Policy

Now we turn to the issue of optimal monetary policy. We say that a monetary policy is *optimal* if there exists an associated symmetric monetary equilibrium whose allocation $\{x_j(t), k(t)\}$ with $x_j(t) = x_j^m(t) + x_j^b(t)$ is Pareto efficient. In the present symmetric environment, there is only one efficient allocation, and it is the solution to the planner's problem:

$$\max \sum_{t=0}^{\infty} \beta^t u(x_0(t), \cdots, x_{n-1}(t)) \quad \text{s.t.} \quad x_0(t) + \cdots + x_{n-1}(t) + k(t+1) = f(k(t)),\ t = 0, 1, 2, \cdots$$

Therefore, a necessary condition for optimal monetary policy is that the marginal rate of substitution between any two goods be 1 if both goods are consumed. Since our definition of monetary equilibria requires active monetary exchange, we assume that the efficient allocation in any date is not autarchic.

The question of our concern is whether the Chicago rule of setting the nominal interest rate to zero is optimal. The answer is yes.

Proposition 3.3: (Optimality of Chicago Rule) There exists an optimal monetary policy with $R(t) = 1$ *for all* t.

Proof: See Appendix 1.

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[7] In the terminology of Lucas and Stokey (1983), capital goods are "credit goods".

Grandmont and Younes (1973) have proved for a cash-in-advance economy that a monetary policy that provides just enough money for agents to purchase the efficient allocation valued at Arrow-Debreu prices brings about zero nominal interest rates. The crux of the proof of Proposition 3.3 is that the same monetary policy admits barter prices that leave agents indifferent between the two means of exchange.

The Chicago rule, however, is not necessary for optimality. If there is a complete lack of double coincidence of wants, thus effectively reducing the model to a pure cash-in-advance model, and if agents do not desire the home good, then there exist efficient monetary equilibria with positive interest rates.[8] The absence of barter guarantees that the terms of trade between the home good and non-home goods is given by the cost of monetary exchange which, as shown in Proposition 3.1, is the gross nominal interest rate. But a positive nominal interest rate is not distortionary if the home good is not desired. Conversely, if the home good is consumed or if there is a double coincidence of wants, then the Chicago rule is necessary as well as sufficient:

Proposition 3.4: Suppose the Pareto efficient allocation is such that either (a) $x_0(t) > 0$ *for some t, or (b)* $x_j(t) > 0$ *and* $x_{n-j}(t) > 0$ *for some* $j \neq 0$ *(so agents 0 and j consume each other's goods) for some t. Then the Chicago rule is necessary as well as sufficient for optimal monetary policy.*

Proof: As already noted, if the home good is consumed, a positive interest rate is distortionary. If two agents consume each other's goods, the two agents barter by Proposition 3.1(d). So the marginal rate of substitution between the two goods is set to the barter price, which has to be one by efficiency. In a monetary equilibrium, there exists a good $v \neq 0$ such that $x_v^m(t) > 0$. If $v = j$, then $R(t) = 1$ by Proposition 3.1(a) and (b). If $v \neq j$, the marginal rate of substitution

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[8] This point has been noted in Woodford (1990). Section 4 has an example of this for an arbitrarily high inflation rate. If symmetry is relaxed, even active barter exchange is not an obstacle for efficiency under arbitrarily high inflation rates, see Example 5.2.

between v and j is not 1 unless $R(t) = 1$. ■

A separate question is what is a necessary condition for optimal monetary policy applicable to any Pareto efficient allocation. One such condition is nominal interest rate smoothing:

Proposition 3.5: If a monetary equilibrium is Pareto efficient, then $R(t) = R \geq 1$.

Proof: We consider the case in which the same good, j, is obtained through money in two successive dates t and t+1, so that $x_j^m(t) > 0$ and $x_j^m(t+1) > 0$. (The proof of the more general case is in Appendix 1.) Since the home good cost of monetary exchange is the gross nominal interest rate, a reduction of the cash purchase of good j by one unit allows the agent to increase the cash sales of the home good by $R(t)P(t)$ dollars which can be used to increase the cash purchase of good j in date t+1 by $R(t)\cdot P(t)/P(t+1)$ units. So the agent sets the marginal rate of substitution between dates t and t+1 for good j to $\beta\cdot R(t)\cdot P(t)/P(t+1)$. But efficiency requires this to equal $\beta\cdot f'(k(t+1))$. This and (3.2) imply $R(t) = R(t+1)$. ■

4. The CES Utility Function

This section presents a complete characterization of the pattern of monetary and barter exchange that endogenously emerges for a specific utility function. As a by-product, we obtain a condition on the growth rate of money under which a stationary monetary equilibrium exists.

The utility function we entertain is a CES function given by

$$(4.1)\qquad u(x_0,\cdots,x_{n-1}) = \sum_{j=0}^{n-1} \alpha_j (x_j)^{1-\gamma}/(1-\gamma),\ \ 0 < \gamma < 2,\ \ \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{n-1} > 0.$$

Whether the home good is desirable ($\alpha_0 > 0$) or not will not affect the equilibrium prices, although, as elucidated in Proposition 3.4, the efficiency of monetary equilibria critically depends on it. That all α's (except possibly α_0) are positive implies that all non-home goods are essential in that $\partial u(x)/\partial x_j \to \infty$ as $x_j \to 0$, which, given the assumed existence of barter markets, ensures positive consumption of all non-home goods. Later we show that it is straightforward to allow some α's to be zero. Cases in which $\gamma = 2$ or $\gamma > 2$ are covered in Appendix 3.

We first examine the pure barter economy. As shown in Appendix 2, the only equilibrium in this symmetric environment with $0 < \gamma < 2$ is the symmetric barter equilibrium studied in Section 2. The first-order condition for the agent's optimization (2.1) and the reciprocal nature of barter prices under symmetry ($q_j^b = 1/q_{n-j}^b$) imply

$$q_j^b = (\alpha_j/\alpha_{n-j})^{1/2}(x_{n-j}/x_j)^{\gamma/2} \quad (j = 1,\cdots,n-1). \tag{4.2}$$

Use of the barter equilibrium condition (B2) on (4.2) yields

$$q_j^b = (\alpha_j/\alpha_{n-j})^{1/(2-\gamma)} \quad (j = 1,\cdots,n-1). \tag{4.3}$$

Thus we have proved

Proposition 4.1: The pure barter economy has a unique equilibrium and it is symmetric. The barter prices are given by (4.3) and are independent of the endowed amount, ω, of the home good.

We now turn to the monetary economy. Since all non-home goods are consumed, Proposition 3.1(d) guarantees that $x_j^b(t) > 0$ for $j = 1,\cdots,n-1$. Therefore the marginal rates of substitution between any pairs of non-home goods are controlled by the barter prices. Thus

$$q_j(t) = (\alpha_j/\alpha_{n-j})^{1/2}[x_{n-j}(t)/x_j(t)]^{\gamma/2} \quad (j = 1,\cdots,n-1), \tag{4.2'}$$

where $x_j(t) = x_j^b(t) + x_j^m(t)$. It is then easy to establish a relationship between the barter prices in the pure barter economy and those in the monetary economy.

Lemma 4.1: Suppose the nominal interest rate is positive (so $R(t) > 1$). Then the following holds for $j = 1, 2,\cdots,n-1$: (a) $x_j^m(t) = 0 \Rightarrow q_j(t) \geq q_j^b$; (b) $x_j^m(t) > 0 \Rightarrow q_j(t) < q_j^b$, where q_j^b is the symmet-

ric pure barter equilibrium price defined in (4.3).

Proof: If $x_j^m(t) = 0$, then $q_j(t)x_j(t) = q_j(t)x_j^b(t) = x_{n-j}^b(t) \le x_{n-j}(t)$. Dividing both sides by $x_j(t) > 0$, we obtain $q_j(t) \le x_{n-j}(t)/x_j(t)$. This and (4.2') imply $q_j(t) \ge (\alpha_j/\alpha_{n-j})^{1/(2-\gamma)} = q_j^b$ since $0 < \gamma < 2$. If $x_j^m(t) > 0$, then $q_j(t)x_j(t) > q_j(t)x_j^b(t) = x_{n-j}^b(t) = x_{n-j}(t)$; the last equality follows because $x_{n-j}^m(t) = 0$ by Proposition 3.1(c). So $q_j(t) > x_{n-j}(t)/x_j(t)$. Combine this with (4.2') to obtain the desired result. ■

This lemma lends itself to a complete characterization of symmetric monetary equilibria. Since $\{\alpha_j\}$ is non-increasing in j, so is $\{q_j^b\}$. Also, $q_1^b \ge 1 \ge q_{n-1}^b$. Thus if $R(t) > 1$ there is a unique integer $j \le n-1$, call it $j^*(t)$, such that

$$(4.4) \qquad q_j^b > R(t) \text{ for } 1 \le j \le j^*(t) \text{ and } q_j^b \le R(t) \text{ for } j^*(t) < j \le n-1$$

(If there is no such $j^*(t) \ge 1$, set $j^*(t) = 0$.) By definition, we have $j^*(t) < n-j^*(t)$ and hence case (b) below is possible.

Proposition 4.2: Suppose the nominal interest rate is positive. Then:

(a) for $1 \le j \le j^*(t)$, $x_j^m(t) > 0$ and $q_j(t) = R(t)$;

(b) for $j^*(t) < j < n-j^*(t)$, $x_j^m(t) = 0$ and $q_j(t) = q_j^b$;

(c) for $n-j^*(t) \le j \le n-1$, $x_j^m(t) = 0$ and $q_j(t) = 1/R(t)$.

Proof: (a) Suppose $1 \le j \le j^*(t)$ and $x_j^m(t) = 0$. Then by Lemma 4.1 and (4.4), $q_j(t) \ge q_j^b > R(t)$, which contradicts Proposition 3.1(b) since $x_j^b(t) > 0$. Thus $x_j^m(t) > 0$ for $1 \le j \le j^*(t)$, which, combined with Proposition 3.1(b), implies that $q_j(t) = R(t)$.

(c) Since $x_j^m(t) > 0$ for $1 \le j \le j^*(t)$ and $R(t) > 1$, Proposition 3.1(c) implies that $x_{n-j}^m(t) = 0$, or $x_j^m(t) = 0$ for $n-j^*(t) \le j \le n-1$. Since by symmetry $q_{n-j}(t) = 1/q_j(t)$, we get $q_j(t) = 1/R(t)$ for $n-j^*(t) \le j \le n-1$.

(b) Suppose $x_j^m(t) > 0$ for some j with $j^*(t) < j < n-j^*(t)$. By Lemma 4.1 and (4.4), $q_j(t) < q_j^b \leq R(t)$, which contradicts Proposition 3.1(b). So $x_j^m(t) = 0$ for all j such that $j^*(t) < j < n-j^*(t)$. Thus (4.2') holds for such j with $x_j(t) = x_j^b(t)$ and $x_{n-j}(t) = x_{n-j}^b(t)$. This and the barter market equilibrium condition $q_j(t)x_j^b(t) = x_{n-j}^b(t)$ imply that $q_j(t) = q_j^b$. ■

That is, agent 0 obtains goods $[1, j^*(t)]$, which are the most desirable goods to agent 0, through *both* barter *and* money, and obtains the rest of the goods exclusively through barter, while selling her home good for cash to agents $[n-j^*(t), n-1]$. This pattern of exchange is illustrated in Figure 1 (set $v = 0$ in the Figure).

This characterization explains why the demand for money is an increasing function of the interest rate. If the nominal interest rate is positive, there will be no idle money balances (Proposition 3.1). Thus from the second constraint in (3.1) real money balances $M(t)/P(t)$ equal $\sum_{j=1}^{j^*(t)} x_j^m(t)$. Money demand depends on the interest rate for two reasons. First, for any good $j \in [1, j^*(t)]$ for which $x_j^m(t) > 0$, the amount acquired through money depends on the price $R(t)$. Second, the *range* $[1, j^*(t)]$ of those goods shrinks as the nominal interest rate increases. Since all non-home goods are essential, the demand for money remains positive as long as the range remains non-empty, *i.e.*, as long as $q_1^b \geq R(t)$. In a stationary equilibrium, $R(t) = (1+\sigma)/\beta$ where σ is the growth rate of money, since the constant inflation rate π equals σ and the gross real interest rate is given by $1/\beta$ by super neutrality. It follows that

Proposition 4.3: Let σ be the constant growth rate of the money supply. There exists a unique symmetric stationary monetary equilibrium if $1 \leq (1+\sigma)/\beta \leq q_1^b$ $(= (\alpha_1/\alpha_{n-1})^{1/(2-\gamma)})$. Otherwise, there exists no symmetric stationary monetary equilibrium.

Extension to the case in which the last v goods are not desired (*i.e.*, $\alpha_{n-v} = \alpha_{n-v+1} = \cdots = \alpha_{n-1} = 0$) is immediate. If $v \geq n/2$, then there is a complete lack of double coincidence of wants, and all goods are acquired through monetary exchange. Since the marginal rate of substitution between any two non-home goods acquired through money is one, the equilibrium is efficient for any inflation rate if the home good is not desired. The Chicago rule is not necessary for efficiency here.

The more interesting case is one in which $v < n/2$. There is a lack of double coincidence of wants for market (0, j) for $j = 1, \cdots, v$, so that $x_j^b(t) = 0$ and $q_j^b = \infty$ for $j = 1, \cdots, v$ and 0 for $j = n-v, \cdots, n-1$. But this is covered by (4.3) since $\alpha_j/\alpha_{n-j} = \infty$ for $j = 1, \cdots, v$ and 0 for $j = n-v, \cdots, n-1$. Accordingly, Proposition 4.1 holds for this case. Proposition 4.2 also holds, with the additional property that $v \leq j^*(t)$ and $x_j^b(t) = 0$ for $j = 1, \cdots, v$ and $j = n-v, \cdots, n-1$. That is, goods [1, v], which are most desirable, are acquired exclusively through money, goods $[v+1,\ j^*(t)]$ are acquired through both money and barter, and goods $[j^*(t)+1,\ n-v-1]$ are acquired exclusively through barter. The home good is still sold for cash to agents $[n-j^*(t),\ n-1]$. This pattern of exchange is graphically illustrated in Figure 1. Proposition 4.3 also holds. Since $q_1^b = \infty$, there exists a monetary equilibrium for an arbitrarily high inflation rate.

5. Non-symmetric Economies

Given the repeated use of symmetry, it is of some importance to see which results in Section 3 remain valid when symmetry is relaxed. To permit non-symmetry, we denote agent i's variables by adding super-script i, and write the barter price of good j in terms of good i as q_{ji}. The decision problem of agent i is

$$(5.1) \qquad \max \sum_{t=0}^{\infty} \beta^t u^i(x_0^{ib}(t) + x_0^{im}(t), \cdots, x_{n-1}^{ib}(t) + x_{n-1}^{im}(t))$$

$$\text{s.t.} \quad x_m^{ib}(t)/P_i(t) + \sum_{j=0}^{n-1} q_{ji}(t) x_j^{ib}(t) + k^i(t+1) = f^i(k^i(t)),$$

$$\sum_{j=0}^{n-1} P_j(t)x_j^{im}(t) + x_m^{im}(t) = M^i(t),$$

$$M^i(t+1) = x_m^{ib}(t) + x_m^{im}(t) + V^i(t),$$

$$x_j^{ib}(t),\ x_m^{ib}(t),\ x_j^{im}(t),\ x_m^{im}(t),\ M^i(t) \geq 0,\ \ M^i(0),\ k^i(0) \text{ given.}$$

Accordingly, the definition of monetary equilibrium becomes

Definition 5.1: A monetary equilibrium associated with a monetary policy $(M^i(0), \{V^i(t)\})$ *is a time path* $\{P_i(t), M^i(t), k^i(t), q_{ji}(t), x_j^{ib}(t), x_m^{ib}(t), x_j^{im}(t), x_m^{im}(t), i, j = 0, 1, \cdots, n-1\}$ *such that:*

(M1) the time path $\{k^i(t), x_j^{ib}(t), x_m^{ib}(t), x_j^{im}(t), x_m^{im}(t), M^i(t)\}$ *is a solution to agent i's decision problem (5.1) given* $\{P_j(t), q_{ji}(t)\}$;

(M2) markets clear: (market (i, j)) $\quad q_{ji}(t)x_j^{ib}(t) = x_i^{jb}(t) \quad (i, j = 0, 1, \cdots, n-1;\ i \neq j),$

(market (i, m)) $\quad x_m^{ib}(t)/P_i(t) = \sum_{j=0}^{n-1} x_i^{jm}(t) \quad (i = 0, 1, \cdots, n-1);$

(M3) active monetary exchange: for all (i, t), $x_m^{ib}(t) > 0$ *and* $x_j^{im}(t) > 0$ *for some* $j \neq i, m$.

The partial characterization of monetary equilibria in Proposition 3.1 easily carries over to the non-symmetric case. The main change is that in the non-symmetric case the nominal interest rate is good-specific.

Proposition 5.1: Let $R_i(t) = f^{i\prime}(k^i(t))P_i(t)/P_{i-1}(t)$ *be the gross nominal interest rate on good i. Then:*

(a) $R_i(t) \geq 1$ *for all (i, t). If* $R_i(t) > 1$, *then* $x_m^{im}(t) = 0$ *and* $x_i^{im}(t) = 0$.

(b) $x_j^{ib}(t) > 0 \Rightarrow 0 < q_{ji}(t) \leq R_i(t)P_j(t)/P_i(t);\ \ x_j^{im}(t) > 0 \Rightarrow q_{ji}(t) \geq R_i(t)P_j(t)/P_i(t)$.

(c) If $R_i(t) > 1$, *then* $x_j^{im}(t) > 0$ *and* $x_i^{jm}(t) > 0$ *never hold simultaneously.*

(d) If $R_i(t) > 1$, *then* $x_j^{ib}(t) + x_j^{im}(t) > 0$ *and* $x_i^{jb}(t) + x_i^{jm}(t) > 0$ *imply agents i and j barter.*

It is also easy to show that super neutrality (Proposition 3.2) and the optimality of the Chicago rule (Proposition 3.3) can be extended with obvious modifications. (The proof of

Propositions 3.2 and 3.3 in Appendix 1 is actually for this more general non-symmetric case.)

Propositions 3.4 and 3.5 do not generalize, as illustrated by the following two examples. In the first example, the monetary equilibrium is Pareto efficient, but nevertheless an asymmetric distribution of initial money balances causes the nominal interest rates to oscillate rather than remain constant over time. In the second example, the equilibrium is Pareto efficient with positive nominal interest rates even if there is active barter exchange due to a double coincidence of wants. This example also illustrates the point that money and barter can co-exist under arbitrarily high inflation rates.

Example 5.1: There are four types of agents ($n = 4$) with identical technology $f^i(k) = 2\sqrt{k}$ and identical initial capital stock $k^i(0) = \beta^2$ (which also is the steady state capital stock since $f^{i\prime}(k^i(0)) = 1/\beta$). Preferences exhibit a complete lack of double coincidence of wants: $u^i(x^i) = v^i(x^i_{i+1})$. So there is no active barter exchange, which is supported by any barter prices as long as $q_{i+1,i}(t) = \infty$ and $q_{i,i+1}(t) = 0$. Consider a Pareto efficient allocation that is stationary: $x^i_{i+1}(t) = c$, $k^i(t) = \beta^2$ ($i = 0,1,2,3$), where $c = 2\beta - \beta^2$. This can be supported by a monetary policy with zero growth of money but with an asymmetric distribution of initial money balances: $M^0(0) = M^2(0) = 2c$, $M^1(0) = M^3(0) = c$; $V^i(t) = 0$. The associated monetary equilibrium is: $P_0(t) = P_2(t) = 1$, $P_1(t) = P_3(t) = 2$ (for t even), $P_0(t) = P_2(t) = 2$, $P_1(t) = P_3(t) = 1$ (for t odd); $x^{im}_{i+1}(t) = c$. So $R_0(t) = R_2(t) = 1/(2\beta)$, $R_1(t) = R_3(t) = 2/\beta$ (for t even), $R_0(t) = R_2(t) = 2/\beta$, $R_1(t) = R_3(t) = 1/(2\beta)$ (for t odd).

Example 5.2: Let $n = 6$, $f^i(k) = 2\sqrt{k}$, $u^0(x^0) = \log(x^0_1) + \log(x^0_3)$, $u^1(x^1) = \log(x^1_2)$, $u^2(x^2) = \log(x^2_0)$, $u^3(x^3) = \log(x^3_4) + \log(x^3_0)$, $u^4(x^4) = \log(x^4_5)$, $u^5(x^5) = \log(x^5_3)$. Note that there is a double coincidence of wants between agents 0 and 3. The steady state capital stock $\bar{k}^i$ equals β^2 and the gross nominal interest rate is $R = (1+\pi)/\beta$ for all i. Let $c = f^i(\bar{k}^i) - \bar{k}^i = 2\beta - \beta^2$. It can be shown that the following is a stationary monetary equilibrium: $x_3^{0b} = x_0^{3b} = Rc/(1+R)$, $x_1^{0m} = x_2^{1m} = x_4^{3m} = x_5^{4m} = c$, $x_0^{2m} = x_3^{5m} = c/(1+R)$, $x_m^{1b}(t) = x_m^{2b}(t) = x_m^{4b}(t) = x_m^{5b}(t) = c$, $x_m^{0b}(t) = x_m^{3b}(t) = c/(1+R)$, zero for all other x's; $q_{03} = q_{30} = 1$, $q_{10} = q_{21} = q_{02} = q_{43} = q_{54} = q_{35} = \infty$, $q_{01} = q_{12} = q_{20} = q_{34} = q_{45} = q_{53} = 0$, arbitrary for all other q_{ji} with $j \neq i$; $M^i(t) = (1+\pi)^t M(0) \equiv M(t)$, $V^i(t) = \pi(1+\pi)^t M(0)$ for arbitrary $M(0)$; $P_1(t) = P_2(t) = P_4(t) = P_5(t) = M(t)/c$, $P_0(t) = P_3(t) = (1+R)M(t)/c$. The flow of goods and money is shown in Figure 2. Note that every agent participates in monetary exchange, satisfying the stringent requirement (M3). The allocation is efficient for arbitrarily high inflation rates, because no pairs of goods are desired by more than one agents. ■

6. Concluding Remarks

We presented a tractable monetary model in which divisible fiat money competes with barter as a means of payment and with capital as a store of value. Barter exchange is limited by the extent of a double coincidence of wants, while monetary exchange is costly due to the cash-in-advance constraint. We provided an extensive characterization of the emerging mix of money and barter exchange.

This model could be extended in several directions. The CES model could be calibrated to measure the welfare cost of inflation. A two-country extension would provide a model of exchange rate determination. It would be interesting to examine what patterns of exchange emerge if other means of payment such as credit and bearer notes are allowed to compete. These are left for future work.

Appendix 1: Proofs

The Kuhn-Tucker conditions for agent i's optimization (5.1) are:

(KT1) $\beta^t u_j^i(x^{ib}(t)+x^{im}(t)) \leq q_{ji}(t)\cdot\lambda^i(t)$, "=" if $x_j^{ib}(t) > 0$ $(j = 0,1,\cdots,n-1)$,

(KT2) $\beta^t u_j^i(x^{ib}(t)+x^{im}(t)) \leq P_j(t)\cdot\mu^i(t)$, "=" if $x_j^{im}(t) > 0$ $(j = 0,1,\cdots,n-1)$,

(KT3) $$\lambda^i(t)/P_i(t) = \mu^i(t+1),$$

(KT4) $$\mu^i(t) \geq \mu^i(t+1), \text{ "=" if } x_m^{im}(t) > 0,$$

(KT5) $$\lambda^i(t) = \lambda^i(t+1)\cdot f^{i\prime}(k^i(t+1)).$$

Since $x_m^{ib}(t) > 0$ in a monetary equilibrium, the first and the third constraints in (5.1) can be combined to eliminate $x_m^{ib}(t)$. $\lambda^i(t)$ here is the Lagrange multiplier for the combined constraint. Since $M^i(t) > 0$ in a monetary equilibrium, (KT3) holds with equality.

Proposition 3.2: (Super Neutrality) In a stationary monetary equilibrium in which the inflation rate $P_i(t+1)/P_i(t) - 1$ is constant over time and the same across goods, $f^{i\prime}(k^i(t)) = 1/\beta$ for all $i = 0, 1, \cdots, n-1$.

Proof: In a stationary monetary equilibrium, there exists a $j \neq i$ such that $x_j^{im}(t) > 0$ for all t. For such j, (KT2) holds with equality for t and t+1. This and (KT3) and (KT5) imply

$$\frac{u_j^i(x^i(t))}{u_j^i(x^i(t+1))} = \beta\cdot\frac{P_i(t)}{P_i(t-1)}\cdot\frac{P_j(t+1)}{P_j(t)}\cdot f^{i\prime}(k^i(t))$$

Since $x^i(t)$ and $P_i(t)/P_i(t-1)$ are constant over time and $P_i(t+1)/P_i(t) = P_j(t+1)/P_j(t)$ by stationarity, the desired result follows. ■

Let $\{x_j^{i*}(t), k^{i*}(t)\}$ $(i,j = 0,1,\cdots,n-1;\ t = 0,1,\cdots)$ be a Pareto efficient allocation. By the Second Welfare Theorem, there exists Arrow-Debreu prices $\{p_j^*(t)\}$ and a transfer scheme $\{y^i(t)\}$ with $y^i(t) \in \mathbf{R}^n$ and $\sum_{i=0}^{n-1} y_j^i(t) = 0$, such that $\{x^{i*}(t), k^{i*}(t)\}$ is a solution to the decision problem:

(A1.1) $$\max \sum_{t=0}^{\infty} \beta^t u^i(x^i(t))$$

$$\text{s.t.}\quad \sum_{t=0}^{\infty}[\{\sum_{j=0}^{n-1} p_j^*(t)(x_j^i(t)+y_j^i(t))\} + p_i^*(t)(k^i(t+1) - f^i(k^i(t)))] = 0,\ x^i(t) \geq 0.$$

If θ^i is the Lagrange multiplier associated with the constraint, we have

$$\beta^t u_j^i(x_j^{i*}(t)) \leq \theta^i \cdot p_j^*(t),\ \text{"="}\ \text{if}\ x_j^{i*}(t) > 0, \tag{A1.2}$$

$$p_i^*(t) = p_i^*(t+1)\cdot f^{i\prime}(k^{i*}(t+1)). \tag{A1.3}$$

By the assumption that $f^{i\prime}(\infty) = 0$, the capital stock is bounded from above, so that

$$\lim_{t\to\infty} p_i^*(t)k^{i*}(t) = 0 \ \text{ and } \ \lim_{t\to\infty}\sum_{j=0}^{n-1} p_j^*(t)x_j^{i*}(t) = 0. \tag{A1.4}$$

Proposition 3.3: (Optimality of the Chicago Rule) For every Pareto efficient allocation such that, for any (i, t), there exists a j with $x_j^{i*}(t) > 0$, there exists an optimal monetary policy in which $R_i(t) = 1$ and $\sum_{i=0}^{n-1}\sum_{t=0}^{\infty} V^i(t) = -\sum_{i=0}^{n-1} M^i(0)$.

Proof: We construct a candidate equilibrium as follows. For prices, set: $q_{ji}(t) = p_j^*(t)/p_i^*(t)$, $P_j(t) = p_j^*(t)$. For quantities, set: $k^i(t) = k^{i*}(t)$, $x_j^{ib}(t) = 0$ $(j = 0, 1, \cdots, n-1)$, $x_m^{ib}(t) = P_i(t)[f^i(k^i(t)) - k^i(t+1)]$, $x_j^{im}(t) = x_j^{i*}(t)$ $(j = 0, 1, \cdots, n-1)$, and $x_m^{im}(t) = 0$. Thus the home good is acquired exclusively though money. For the monetary policy, choose $(M^i(0), \{V^i(t)\})$ so that $M^i(t) = \sum_{j=0}^{n-1} p_j^*(t)x_j^{i*}(t)$.

By construction, (M2) and (M3) are satisfied. For (M1), all the three constraints in (5.1) are satisfied by construction. The Kuhn-Tucker conditions (KT1)-(KT5) are satisfied with $\mu^i(t) = \theta^i$ and $\lambda^i(t) = p_i^*(t)\theta^i$ by (A1.2) and (A1.3). By (A1.4) the transversality conditions, $\lim_{t\to\infty}\lambda^i(t)k^i(t) = 0$ and $\lim_{t\to\infty}\mu^i(t)M^i(t) = 0$, are satisfied. The Kuhn-Tucker conditions and the transversality conditions constitute a sufficient condition for (M1).

That $R_i(t) = 1$ for all (i, t) follows from (A1.3). That $\sum_{i=0}^{n-1}\sum_{t=0}^{\infty} V^i(t) = -\sum_{i=0}^{n-1} M^i(0)$ follows from (A1.4) and the budget constraint in (A1.1). ■

Appendix 2: Symmetry of Exchange Equilibrium under the CES Utility Function

This appendix proves the following claim.

Claim: For the pure barter economy with the CES utility function in Section 4, the equilibrium is unique and is symmetric (i.e., $q^b_{i+j,i}$ is independent of i) if $0 < \gamma < 2$.

We will also show by a counter example that this claim is not true if $\gamma > 2$. As explained in Section 4 and as clear from the proof below, there is no equilibrium if $\gamma = 2$.

Agent i's decision problem is

$$\max \sum_{j=0}^{n-1} \alpha_j \left(x^i_{i+j}\right)^{1-\gamma}/(1-\gamma) \quad \text{s.t.} \quad \sum_{j=0}^{n-1} q^b_{i+j,i}\, x^i_{i+j} = \omega,$$

where we employ the modular notation for subscripts: $i+j = \mathrm{mod}(i+j, n)$. The demand function for good $i+j$ is

$$\text{(A2.1)} \qquad x^i_{i+j} = (\alpha_j)^{1/\gamma}\left(q^b_{i+j,i}\right)^{-1/\gamma}\omega/\xi_i,$$

where

$$\text{(A2.2)} \qquad \xi_i = \sum_{j=0}^{n-1} (\alpha_j)^{1/\gamma}\left(q^b_{i+j,i}\right)^{1-1/\gamma}.$$

The market equilibrium condition for market (i, i+j) is

$$\text{(A2.3)} \qquad q^b_{i+j,i}\, x^i_{i+j} = x^{i+j}_i.$$

Substitute (A2.1) with (A2.2) into (A2.3) and use the reciprocal relation $q^b_{i,i+j} = 1/q^b_{i+j,i}$ to obtain

$$\text{(A2.4)} \qquad (\alpha_j)^{1/\gamma}\left(q^b_{i+j,i}\right)^{1-1/\gamma}/\xi_i = (\alpha_{n-j})^{1/\gamma}\left(q^b_{i+j,i}\right)^{1/\gamma}/\xi_{i+j}.$$

Solve this for $q^b_{i+j,i}$ to obtain

$$\text{(A2.5)} \qquad q^b_{i+i,i} = (\alpha_j/\alpha_{n-j})^{1/(2-\gamma)}(\xi_{i+j}/\xi_i)^{\gamma/(2-\gamma)}.$$

Thus the equilibrium barter prices are functions of $\{\xi_j\}$. We can obtain the system of equations for $\{\xi_j\}$ by substituting (A2.5) into (A2.2):

$$\text{(A2.6)} \qquad z_i = \sum_{j=0}^{n-1} b_j\, z_{i+j}{}^{\rho} \quad (i = 0, 1, \cdots, n-1),$$

where

$$\text{(A2.7)} \qquad z_i = (\xi_i)^{1/(2-\gamma)},\ b_j = (\alpha_j)^{1/[(2-\gamma)\gamma]}(\alpha_{n-j})^{(1-\gamma)/[(2-\gamma)\gamma]},\ \rho = \gamma - 1.$$

Coefficients $\{b_j\}$ can be mapped back to $\{\alpha_j\}$ one-to-one because

(A2.8) $$\alpha_0 = (b_0)^{\gamma},\ \alpha_j = b_j(b_{n-j})^{\gamma-1} \quad (j = 1,2,\cdots,n-1).$$

Written out in full, (A2.6) is a system of nonlinear simultaneous equations in z's:

(A2.9)
$$\begin{aligned}
z_0 &= b_0 z_0^{\rho} + b_1 z_1^{\rho} + \cdots + b_{n-2} z_{n-2}^{\rho} + b_{n-1} z_{n-1}^{\rho} \\
z_1 &= b_{n-1} z_0^{\rho} + b_0 z_1^{\rho} + \cdots + b_{n-3} z_{n-2}^{\rho} + b_{n-2} z_{n-1}^{\rho} \\
&\cdots\cdots\cdots\cdots\cdots \\
z_{n-1} &= b_1 z_0^{\rho} + b_2 z_1^{\rho} + \cdots + b_{n-1} z_{n-2}^{\rho} + b_0 z_{n-1}^{\rho}.
\end{aligned}$$

Since α's are non-negative and unique up to a scalar, they can be normalized so that

(A2.10) $$b_j = 1,\ b_j \geq 0.$$

To prove the claim, it is sufficient to show that the unique solution to (A2.9) is that $z_j = 1$, $\forall j$, because then $q^{b}_{i+j,i}$ is independent of i by (A2.5). The proof is trivial if $\gamma = 1$. For $0 < \gamma < 1$ or $1 < \gamma < 2$, we prove by deriving contradiction. Suppose, then, that $z_j \neq 1$ for some j. If $z_i = z_j$ for all i, j ($= 0, 1, \cdots, n-1$), then evidently $z_j = 1$, $\forall j$ by (A2.9). So there must exist some i, j such that $z_i \neq z_j$. Without loss of generality, we can suppose that

(A2.11) $$z_0 \leq z_1 \leq \cdots \leq z_{n-1} \text{ and } z_0 < z_{n-1}.$$

Case 1: $1 < \gamma < 2$. Since by definition $\rho = \gamma - 1$, we have $0 < \rho < 1$, so that

(A2.12) $$z_0^{\rho} \leq z_1^{\rho} \leq \cdots \leq z_{n-1}^{\rho}.$$

Suppose $z_0 \geq 1$. Then by (A2.11) $z_{n-1} > 1$. From the last equation in (A2.9) (the one for z_{n-1}), we can derive $z_{n-1} \leq z_{n-1}^{\rho}$ by using (A2.10) and (A2.12). But this is a contradiction since $z_{n-1} > 1$ and $0 < \rho < 1$. So $z_0 < 1$. Then from the first equation in (A2.9), we can derive $z_0 \geq z_0^{\rho}$ by using (A2.10) and (A2.12). This is a contradiction since $z_0 < 1$ and $0 < \rho < 1$.

Case 2: $0 < \gamma < 1$. Since $-1 < \rho < 0$, we have

(A2.13) $$z_0^{\rho} \geq z_1^{\rho} \geq \cdots \geq z_{n-1}^{\rho}.$$

Suppose $z_{n-1} \leq 1$. Then by (A2.11) $z_0 < 1$. From the first equation in (A2.9) we can derive $z_0 \geq z_{n-1}^{\rho}$ by using (A2.10) and (A2.13). Thus $z_{n-1}^{\rho} < 1$, which is a contradiction since $\rho < 0$.

So $z_{n-1} > 1$. By (A2.10) the first equation in (A2.9) can be written as

(A2.14) $$1 - z_0 = b_0(1 - z_0^{\rho}) + \cdots + b_{n-1}(1 - z_{n-1}^{\rho}).$$

Since $1 - z_j^{\rho} \leq 1 - z_{n-1}^{\rho}$, $\forall j$ by (A2.13), this and (A2.10) imply $1 - z_0 \leq 1 - z_{n-1}^{\rho}$, or

(A2.15) $$z_{n-1}^{\rho} \leq z_0.$$

By (A2.10) the last equation in (A2.9) can be written as

(A2.16) $$1-z_{n-1} = b_1(1-z_0^{\rho}) + \cdots + b_0(1-z_{n-1}^{\rho}).$$

Since $1-z_j^{\rho} \geq 1-z_0^{\rho}$, $\forall j$ by (A2.13), this and (A2.10) imply $1-z_{n-1} \geq 1-z_0^{\rho}$, or $z_0^{\rho} \geq z_{n-1}$. Since $\rho < 0$, this last inequality implies that

(A2.17) $$z_0 \leq (z_{n-1})^{1/\rho}.$$

Combining (A2.15) and (A2.17), we obtain: $z_{n-1}^{\rho} \leq (z_{n-1})^{1/\rho}$. Since $z_{n-1} > 1$ and $-1 < \rho < 0$, this is a contradiction. This completes the proof of the claim. ■

We now show that this claim is not true if $\gamma > 2$ by providing the following counter-example. Let $n = 3$, $\gamma = 3$, and $\alpha_0 = (2/3)^3$, $\alpha_1 = \alpha_2 = (1/6)^3$. So $b_0 = 2/3$, $b_1 = b_2 = 1/6$ by (A2.7). It is easy to verify that $(z_0, z_1, z_2) = (2/3, 2/3, 4/3)$ is *a* solution to (A2.9). (Other solutions includes (4/3, 2/3, 2/3), (2/3, 4/3, 2/3), (1, 1, 1).) A set of equilibrium barter prices, given by (A2.5), is: $q^b_{1,0} = 1$, $q^b_{2,0} = 8$, $q^b_{2,1} = 8$; $q^b_{0,1} = 1$, $q^b_{0,2} = 1/8$, $q^b_{1,2} = 1/8$.

Appendix 3: The CES Model with $\gamma = 2$ and $\gamma > 2$

If $\gamma = 2$, (4.2) implies $q^b_j x_j/x_{n-j} = \alpha_j/\alpha_{n-j}$, so unless α's are all equal, there is no symmetric barter equilibrium, so q^b_j cannot be defined. For the monetary economy, an argument similar to the proof of Lemma 4.1 shows that Proposition 4.2 holds with $j^*(t)$ replaced by $n/2$, except that if n is even, $x^m_j(t) = 0$ and $q_j(t) = 1$ for $j = n/2$. Proposition 4.3 holds with $q^b_1 = \infty$.

Finally, consider case $\gamma > 2$. As shown in Appendix 2, for the pure barter economy, there exists a non-symmetric equilibrium. If symmetry is imposed, (4.2) and (4.3), and hence Proposition 4.1 still hold. However, this equilibrium has a rather strange property. (4.2) and (4.3) imply

$$q^b_j x_j/x_{n-j} = (q^b_j)^{1-2/\gamma}(\alpha_j/\alpha_{n-j})^{1/\gamma}.$$

If $\gamma > 2$, the "Marshall-Lerner condition" for market (0, j) does not hold: $\partial(q^b_j x_j/x_{n-j})/\partial q^b_j > 0$, which is responsible for the property that the more desirable goods (small j's) are *less* expensive. Accordingly, for the monetary economy, Proposition 4.2 for case $\gamma > 2$ states that less desirable goods are acquired through money. The upper bound for $(1+\sigma)/\beta$ in Proposition 4.3 is now given by $q^b_{n-1} = (\alpha_{n-1}/\alpha_1)^{1/(2-\gamma)}$.

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Figure 1
Pattern of Exchange

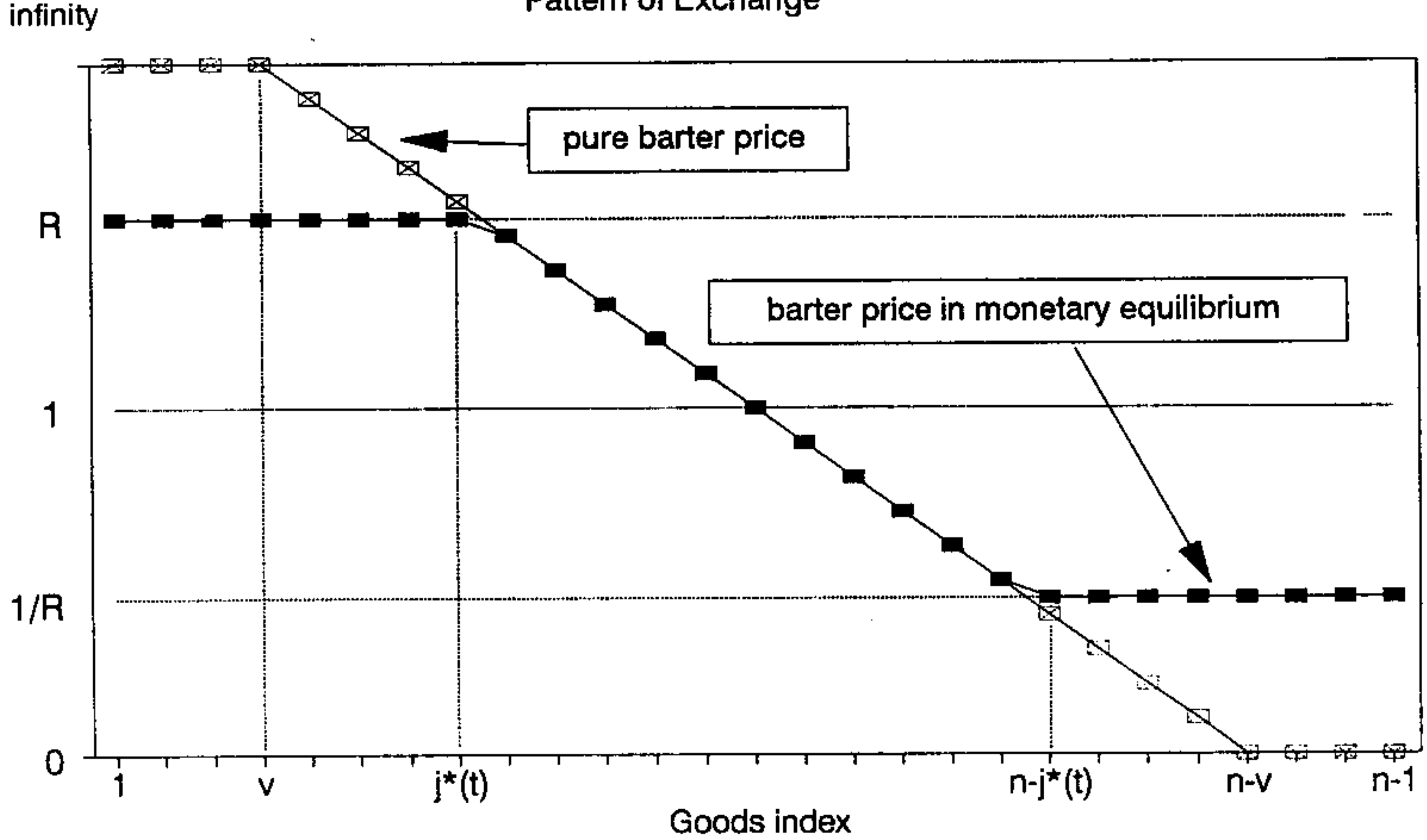

Figure 2
Flow of Goods and Money