NBER WORKING PAPER SERIES

# A MODEL OF RESEARCH, PATENTING, AND PRODUCTIVITY GROWTH

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Working Paper No. 4646

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
February 1994

I have benefitted from the comments of Eric Bartelsman, Russell Cooper, Jonathan Eaton, Zvi Griliches, Glenn Loury, Debraj Ray, Michael Riordan, Jim Schmitz and participants in seminars at Boston University, Hebrew University, the Minneapolis Fed, MIT, the SEDC winter meetings and Yale. I thank the National Science Foundation for financial support. I take responsibility for any errors. This paper is part of NBER's research program in Productivity. Any opinions expressed are those of the author and not those of the National Bureau of Economic Research.

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## ABSTRACT

I use the aggregate behavior of three indicators of technology (employment of research scientists and engineers, patented inventions, and total factor productivity) to identify a plausible model of endogenous technological change. In the US (as well as in other developed countries) research employment and total factor productivity have both grown, while the rate of patenting has remained relatively flat. One interpretation of these facts is that: (i) patentable inventions are becoming increasingly difficult to discover as the quality of techniques in use increases, (ii) inventions which are patented represent percentage improvements on techniques currently in use, and (iii) the size of the economy is growing, making patents increasingly valuable and justifying increased research efforts devoted to discovering them. This paper presents a general equilibrium search theoretic model of invention which formalizes this view.

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# 1 Introduction

There are three widely available indicators of technological change: (i) measures of research input, such as employment of research scientists and engineers, (ii) measures of inventive output, in particular counts of patented inventions, and (iii) measures of the improvement in technologies in use, such as total factor productivity growth. These indicators, if taken seriously, are quite informative as to which models are capable of organizing our empirical understanding of technological change. I propose a model which, in a steady state equilibrium, replicates the observed aggregate trends in research, patenting, and productivity. These aggregate implications follow from a stochastic model of research, patenting, and productivity growth at the level of individual products.

Figure I illustrates the behavior of the three technology indicators in the US over the past three decades (see table 2 for data sources). For ease of comparison, the three series are indexed at zero (on a log scale) in 1957. On the one hand, employment of research and development scientists and engineers in industry (S&E's) has grown dramatically. Moreover, the level of total factor productivity (TFP) in the manufacturing sector has grown more slowly, yet it does display a persistent upward trend. On the other hand, the annual number of successful domestic patent applications has displayed little upward trend relative to its fluctuations. My interpretation of the evidence is that, while S&E's and TFP have grown, the level of patenting has fluctuated around a constant level.[1]

[1] I associate domestic patents with US priority patents: inventions for which patent protection is sought first in the US. The patent data would display a distinct upward trend if I had counted all patents granted in the US, since foreign patenting in the US has expanded rapidly. There is some basis for ignoring foreign patents. For the most part, US priority patents are the result

My interpretation of the trends in the three technology indicators is supported by data from other time periods and from other countries. As for the constancy of patenting: (i) Griliches (1990) shows that domestic patent applications in the US display little trend since the 1920's and (ii) Eaton and Kortum (1993) find no trend between 1952 and 1990 in the annual number of patents granted domestically to residents of France, Germany, UK, and the US. Only Japan, of the countries studied, has experienced a noticeable increase in domestic patenting. As for relative trends in research and patenting: (i) Evenson (1984) has documented a dramatic decline in patenting relative to measures of research for a broad set of countries and (ii) Kortum (1993a) finds that all the US manufacturing industries for which data are available have experienced a decline in patenting per unit of real R&D. Apparently, the aggregate decline in patenting relative to R&D is not simply a result of R&D being performed increasingly in industries (such as Office and Computing Equipment) where we suspect inventions are relatively unlikely to be patented. As for the trends in research and productivity: Jones (1993) documents the rapid growth of research inputs and the relatively constant growth of TFP over the past 25 years in France, Germany, Japan, and the US.

The trends in the three indicators of technology present puzzles for several existing models. A standard assumption in the endogenous technological change literature [Judd (1985), Aghion and Howitt (1992), and Grossman and Helpman (1991)] is that inventive output is proportional to the quantity of labor allocated

of research performed in the US while foreign patents are not. On the other hand, foreign patents may indicate the arrival of productivity improving techniques from abroad [see, Eaton and Kortum (1993)]. The model in the current paper applies to an economy that obtains all its technological advances from its own research.

to research. In Romer (1990), the rate at which a researcher invents new types of goods actually increases over time as a result of a research spillover. If the trend in patents says anything about the rate of invention, the ubiquitous decline in patenting per S&E argues against such formulations. Puzzles remain if we ignore patenting and look directly at the relationship between S&E's and TFP. The same models of endogenous technological change imply that TFP growth is proportional to employment in research. Jones (1993) points out that while research employment has grown dramatically, TFP growth has not increased as the theories would suggest. Finally, the empirical literature, quantifying the effect of research on productivity, generally posits a linear relationship between productivity growth and the growth of a stock of knowledge. If the stock of previous patents is used to measure the stock of knowledge, then the relative constancy of the rate of patenting implies that the stock of knowledge grows at an ever declining rate. As Griliches (1990) notes, the prediction of continuously declining rates of productivity growth has not been observed.

None of these phenomena is a puzzle in the context of the model I construct below. The primitive of the model is a set of distributions from which researchers draw techniques for producing higher quality goods. A technique, once discovered, can be used indefinitely, hence the technical capability of the economy is non decreasing over time. However, the distributions of undiscovered techniques do not evolve over time. Hence, as the quality of existing techniques improves, more research effort (more time drawing from the distributions) is required to find a patentable technique, i.e. one that is better than existing techniques. The model implies that a constantly growing quantity of research input is required to generate a constant flow of new patents. This explains why we observe a constant rate of patenting while the employment of S&E's has risen over time.

To account for constant exponential productivity growth, it must be that the average quality of techniques in use rises at a constant rate. Assume that a fixed percentage of the improvements in technique is patented. Since the rate of patenting is constant, it must be that a patented technique is, on average, a constant percentage improvement on some technique currently in use. This requirement on the average 'size' of a patent can be used to prove that the underlying distributions of undiscovered techniques are from the Pareto family. Given the Pareto restriction, the model implies that productivity growth is proportional to the rate of patenting, which is in turn proportional to the growth in the stock of past research. Thus, a growing path of research input is consistent with a constant rate of patenting as well as a constant rate of productivity growth.

To close the model, I show that if the population of individuals or the stock of human capital in the economy grows at a constant rate, then there is an equilibrium in which a constant fraction of these human resources are devoted to research. Though a researcher is increasingly unlikely to discover a patentable invention, the average value of a patentable invention rises over time. Patentable inventions become more valuable because, in a growing economy, new techniques will be used to produce an ever greater quantity of goods. In equilibrium the value of patents increases at the same rate as the cost of discovering them. This supports an equilibrium with a growing level of research investment.

Having developed a model consistent with the trends, I investigate whether it can account for the observed fraction of human resources devoted to research. I measure this fraction as nominal R&D expenditure (compensation of researchers) divided by nominal compensation of the labor force. I set the parameters of the model to mimic the growth of S&E's and TFP. I find that for moderate rates of discounting, the model overpredicts the R&D share. Next, I ignore productivity growth and fix one of the parameters to mimic econometric estimates of the elasticity of productivity with respect to the stock of research. In this case I am able to fit the R&D share given a reasonable discount rate. But, as with the empirical literature on research and productivity, these parameters imply that industrial research accounts for only 10-20% of observed productivity growth.

## 2 Relation to the Existing Literature

How does my explanation of the behavior of S&E's, TFP, and patenting relate to other arguments in the literature? On the observed fall in patenting relative to S&E's, Kortum (1993a) assesses three explanations: (a) opportunities for discovering patentable inventions have declined, (b) the value of patentable inventions has increased, making researchers willing to expend more effort to get one, and (c) the fraction of inventions that are patented has declined. He concludes, in the context of a theoretical model that differs from the present model, that explanation (c) must be part of the story. Griliches (1990) proposes institutional reasons why the fraction of inventions that are patented may be declining. The present model, on the other hand, assumes that the propensity to patent is constant and explains the observed decline in patenting relative to S&E's by a combination of explanations (a) and (b). The increasing difficulty of discovering patentable inventions is driven by the same search mechanism proposed by Evenson (1984, 1991). The value of patented inventions also rises over time, thus explaining why research activity continues to grow in this environment of declining opportunities.

What should we conclude about a possible decline in the propensity to patent? The direct evidence we have, based on responses from R&D managers, Mansfield (1986), and Henderson and Cockburn (1994), does not support the view that the propensity to patent has declined. Nonetheless, further research needs to be done before we can be certain that explanation (c) is unimportant.

On the behavior of productivity relative to patenting and S&E's, my account is consistent with a number of previous works. In interpreting the trends in patenting and productivity, Griliches (1990, pg. 1698) observes that a constant flow of patenting could yield constant exponential productivity growth if patents are on average percentage improvements. Aghion and Howitt (1992), Grossman and Helpman (1991), and Caballero and Jaffe (1993) develop theoretical models that are consistent with this interpretation of the size of patents. In studying time series of productivity and S&E's, Jones (1993) concludes that the level of productivity has a negative effect on productivity growth conditional on S&E's. When he incorporates this phenomenon into a model of growth, he finds that population growth is needed to support a steady state with endogenous technological change. These features are also present in a model of endogenous technological change developed by Nordhaus (1969). Similarly, in my model, exponential growth of population or human capital is necessary if the value of patents (and research activity) is to grow in steady state.

Caballero and Jaffe (1993) have recently made an ambitious attempt to assess the empirical content of research-driven models of endogenous growth. They put particular emphasis on modeling and measuring technological spillovers. Using data on patent citations, they estimate that there has been a secular decline in the knowledge spillovers generated by successive cohorts of patents. They note that this is a potential explanation of the observed fall in the patent-R&D ratio. While this is a provocative finding, Caballero and Jaffe's theoretical model does not provide an interpretation for it. In particular, their model implies that the patent-S&E ratio is constant in steady state.

The technological spillovers emphasized by Caballero and Jaffe are totally absent in the present model. This absence reflects an attempt to keep the model simple rather than an a priori belief that spillovers are unimportant. In the present model, with no spillovers, there is too much research in equilibrium relative to what a social planner would choose.[2] It is quite likely that this over investment result would be reversed by introducing technological spillovers. A natural way to introduce spillovers is to allow research successes to create general knowledge which shifts the underlying distributions of undiscovered techniques. I leave this as a topic for future research. One challenge will be to introduce research spillovers while maintaining aggregate predictions that are consistent with the data.

The stucture of the theoretical model developed below borrows extensively from existing literature. I start with Evenson and Kislev's (1976) search model of technological change whose implications for patenting are derived in Kortum (1991). Next, I follow Bental and Peled (1992) by assuming that the search distributions are Pareto. Finally, I embed the search model in a general equilibrium framework styled after Grossman and Helpman's (1991, chapt 4) model of rising product quality.[3]

[2]I have not proven this, however it is true for a wide range of parameter values. This result is not caused by racing behavior. Rather, it is a general equilibrium effect. Too many individuals enter the research sector because, due to imperfect competition, they are paid less than their marginal product in the production sector.

[3]The search model of patenting does not fit as naturally into a model of expanding variety

The paper proceeds as follows. In the first section I describe the economic environment underlying the model. Next, I solve the model: first, deriving the behavior of patenting conditional on research, second, deriving the behavior of productivity growth conditional on patenting, and finally, solving for an equilibrium path of research. In the fourth section I compare the predictions of the model with measures of technological change in the US. I focus on the prediction of the model for the fraction of human resources devoted to research. The fourth section contains concluding remarks and a look at how the model helps us interpret firm level data on research and productivity growth. The appendices contain a table of notation as well as the more tedious mathematical derivations.

## 3 The Model

The economy consists of a continuum of individuals and a continuum of varieties of goods. Labor is the only input to production. At each moment, each individual chooses to sell its labor or to search for a technique to produce a higher quality good, i.e., an individual can be a laborer or a researcher. A researcher's success is determined by the qualities of existing goods and the probability distributions over the qualities of new ideas. A successful researcher gets an idea for producing a higher quality good, obtains a patent, and hires labor to produce the good. I focus on the relation between research activity, the arrival of new patented goods, and the resulting improvements in product quality, which I relate to productivity growth.

such as Judd (1985), Romer (1990), or Grossman and Helpman (1991, chapt 3).

### 3.1 Research and Patents

During an instant, $dt$, an individual engaged in research will have an idea with probability $dt$. An idea is defined to be knowledge of how to produce a consumption good of variety $j$ and quality $Q$. The variety is drawn from the uniform density on $[0, J]$.[4] The quality, conditional on variety $j$, is drawn from the cumulative distribution function, $F(q) = \Pr[Q < q]$. I assume $F$ is continuously differentiable with density $f(q)$.[5]

I assume that an idea is patentable if and only if its quality exceeds that of all previous patents on variety $j$. The law is somewhat more complicated than this, "Under section 102, a patent is barred for lack of novelty if there is enough in the prior art to enable someone skilled in the area to perform the process or produce the product described in the patent application." (Miller and Davis, 1990, pg. 46). However, in a world where goods have only one quality dimension, it is

[4]Having a continuum of goods allows me to model research as a random process while treating aggregate outcomes as deterministic. This modeling device greatly simplifies the analysis, as does the assumption that the set of varieties is fixed. By assuming that the variety is drawn from the uniform distribution, I have made it impossible for researchers to focus their search on specific varieties. This strong assumption lets me avoid two difficult issues. First, a researcher owning a patent on a given variety may attempt to prevent entry by continuing to search in that variety [see, Reinganum (1989) for a general discussion of this issue]. Second, a researcher may want to direct her research at varieties in which little technological progress has been made in the past. These are both important issues which I would like to address with future incarnations of the model.

[5]The assumption of a Poisson process guarantees that ideas arrive sequentially rather than simultaneously. The assumption of a continuously differentiable density allows me to ignore the possibility of two ideas having exactly the same quality. Both assumption help to simplify the later analysis.

natural to define the prior art as the quality of the previous best idea. Thus, to be patentable, a new idea must surpass the prior art and set a quality standard which becomes the new state of the art. I assume that patenting is costless so that all patentable ideas are patented. I also assume that patent examiners only grant patents on patentable ideas.[6].

Research activity generates an expanding set of patented goods. Let $N(j,t)$ be the number of patents on variety $j$ prior to time $t$. To guarantee that there is always at least one good available in each variety, I assume that there is always an unpatented good (good 0). The entire set of goods at time $t$ is therefore,

$$G(t) \equiv \{(i,j)|i \in \{0,1,\ldots,N(j,t)\}, j \in [0,J]\}.$$

Each good, $(i,j) \in G(t)$, has an associated quality, $Q(i,j) > 0$. The following specification of patent protection justifies keeping track of only the initial unpatented good along with all subsequent patented goods.

Each patent is owned by the individual who had the idea. The patent holder can produce a good using the patented idea and can costlessly prevent others from infringing on the patent. Patent $(i,j)$ is infringed if an individual, other than the owner of the patent, produces the $j$th variety with quality $Q \in (Q(i-1,j), Q(i,j)]$.[7] Patent protection is assumed to last forever, though I show later that patent $(i,j)$ obtains no value from this protection once patent $(i+1,j)$ is discovered. The original unpatented goods, $(0,j)$, for $j \in [0,J]$, can be produced by any individual without causing an infringement.

[6]These issues are discussed more thoroughly in Kortum (1991)

[7]This way of specifying the scope of patent protection was suggested to me by Nobuhiro Kiyotaki.

## 3.2 Production

All goods are produced under constant returns to scale. In particular, to produce a good at rate $x$ requires labor services at rate $cx$, where $c$ is a cost parameter.[8] The producer sets a price, $P$, at which to sell output to other individuals and hires labor to meet demand at that price. Prices are in labor units and the wage is normalized to be unity. The flow profits from patent $(i,j)$ are thus, $C(i,j,t)[P(i,j,t) - c]$, where $C(i,j,t)$ is the quantity demanded of good $(i,j)$.

## 3.3 Preferences

A continuum of infinitely lived individuals is indexed by $h \in [0, e^{nt}]$, where $n > 0$ is the rate of population growth. Each individual is endowed with a unit flow of labor services which it allocates to research, $R(h,t) = 1$, or production, $R(h,t) = 0$.[9] Besides wage income of $1 - R(h,t)$, an individual may get profit income from patents if she has been a successful researcher in the past.

An individual's objective is allocate her labor between research and production and to allocate her consumption across varieties and qualities so as to maximize the expected present discounted value of instantaneous utility,

$$E_t[\int_t^\infty e^{-\rho(s-t)} U(s) ds],$$

[8] For ease of exposition, I interpret technological change as quality improvement and normalize costs of production to be constant. It is easy to show that technological change can also be interpreted as reductions in labor requirements (with qualities unchanged over time). In this alternative interpretation, the behavior of prices (though not price per unit of quality) will be altered in an obvious way.

[9] Alternatively, imagine a fixed measure of individuals, each of whose endowment of labor services grows at rate $n$ due to exogenous accumulation of human capital. If $n = 0$, the economy reaches a steady state where no one chooses research and productivity ceases to grow.

where $\rho$ is the individual's discount rate. I assume that instantaneous utility is homogeneous of degree one in the set of goods, $G(t)$, so that individuals are indifferent to the risk inherent in research. I also choose a convenient functional form which implies that individuals will spread their income evenly across all varieties,

$$U(t) = \exp\{J^{-1} \int_0^J \ln[\sum_{i=0}^{N(j,t)} (Q(i,j)e^{-nt}C(i,j,t))]dj\}.$$

A representative individual obtains utility from quality weighted levels of per capita consumption of the individual goods. With identical preferences that are homogeneous of degree 1, facing the same set of prices, $\{P(i,j,t)|(i,j) \in G(t)\}$, at a point in time each individual will purchase the same basket of goods, scaled up or down according to her income.[10]

## 3.4 The Evolving Set of Goods

The aggregate level of research activity in the economy at time $t$ is given by the measure of individuals engaged in research at that time, $R(t) \equiv \int_0^{e^{nt}} R(h,t)dh$. A path of research $\{R(s)|s \geq t\}$ will determine, probabilistically, future sets of patented goods and their associated qualities. Much of the section 3 will be concerned with describing exactly how the set of patented goods evolves over time. At this point it is necessary only to point out that individuals are assumed to form

[10] Note that,

$$\exp\{J^{-1} \int_0^J \ln[\sum_{i=0}^{N(j,t)} (Q(i,j)\lambda e^{-nt}C(i,j,t))]dj\}$$
$$= \lambda \exp\{J^{-1} \int_0^J \ln[\sum_{i=0}^{N(j,t)} (Q(i,j)e^{-nt}C(i,j,t))]dj\}.$$

Letting $\lambda$ tend to zero shows that an individual with no income, e.g. an unsuccessful researcher, will obtain instantaneous utility of zero by consuming nothing.

expectations taking as given the equilibrium path of research and the implied process for the arrival of new and better goods.

### 3.5 Equilibrium

An equilibrium for this economy in period $t$ is a path of research, $\{R(s)|s \geq t\}$, a consumption allocation, $\{C(i,j,s)|(i,j) \in G(s), s \geq t\}$ and a set of prices, $\{P(i,j,s)|(i,j) \in G(s), s \geq t\}$, such that in all periods $s \geq t$: (a) The consumption allocation is utility maximizing given prices; (b) Each price is profit maximizing for the producer given the prices set by others and the demand schedule for the good (prices are the outcome of Bertrand competition); (c) Each individual's labor allocation maximizes expected utility; (d) The labor market clears: i.e. $\int_0^J \sum_{i=0}^{N(j,s)} C(i,j,s)dj = [e^{ns} - R(s)]/c$ in all periods, $s \geq t$.

## 4 A Solution

The model specified in the previous section is solved in three steps. First, I derive the patent production function: the equation relating the aggregate rate of patenting to aggregate research input. Second, I derive the productivity equation: the equation relating aggregate productivity growth to the aggregate rate of patenting. Finally, I close the model by solving for an equilibrium path of aggregate research. The result is a set of equations for the rate of patenting, the rate of growth of productivity, the rate of growth of research inputs, and the fraction of human resources devoted to research as functions of the parameters of the model.

In solving the model, I impose restrictions on the representative search distribution, $F(q)$. While these restrictions are required for analytical tractability, they are also required if the model is to predict correctly the trends in patenting and productivity growth. This is not true for the patent production function which I derive without restricting the search distribution. Any continuously differentiable search distribution is consistent with the observation that research inputs grow exponentially while the rate of patenting is constant. However, the shape of the search distribution is relevant to the form of the productivity equation. Only a Pareto search distribution is consistent with the observation that a constant rate of patenting produces exponential growth of productivity. Thus, I assume that the representative search distribution is Pareto for the remainder of the paper (except in the derivation of the patent production function). The analytical convenience of the Pareto distribution makes it possible to solve explicitly for the equilibrium level of research.

## 4.1 The Patent Production Function

The model has a striking implication for the aggregate output of patented ideas conditional on a path of research. Let $\dot{N}(t)$ be the rate of patenting, $R(t)$ the rate of research, and $\mu(t) \equiv J^{-1} \int_0^t R(s)ds$ the stock of past research. The model implies,

$$\dot{N}(t) = R(t)\mu(t)^{-1}. \tag{1}$$

The distinguishing feature of (1) is the negative effect on patenting brought about by a rising stock of research. As the stock of past research rises, a decreasing fraction of new research ideas surpass the quality of previous ideas. Hence a decreasing fraction of new ideas are patentable. Consider a path of research where $R(t)$ grows at rate $n$ (i.e., if a constant fraction of human resources are devoted to research). While the number of ideas grows at rate $n$, the percentage of those ideas that are patentable falls over time. Eventually, $\mu(t)$ grows at approximately rate $n$. Hence the patent-research ratio falls at approximately rate $n$ and the rate of patenting approaches a constant, $Jn$. Thus, the patent production function (1) is consistent with the observation that research has grown while the rate of patenting displays little trend.

I derive the patent production function below. Since the derivation does not depend on the form of the search distribution, I postpone imposing the Pareto distribution. In Kortum (1993b), I derive the patent production function under more general conditions and then test its implications. The empirical findings support the model's prediction that the rate of patenting is positively related to current research and negatively related to a proxy for the stock of past research.

### 4.1.1 Ideas

Remember that an idea is the knowledge of how to produce a good of variety $j$ and quality $Q$. Although most ideas are unpatentable, it will be useful to keep track of all the ideas for producing a given variety. Let $I(j,t)$ be the number of ideas for variety $j$ that occur before time $t$. The sequence of random variables $\{I(j,s)|s \geq 0\}$ is a nonhomogenous Poisson process with an arrival rate $R(s)/J$.[11] The fraction of varieties for which there have been exactly $I$ ideas by time $t$ is given by the Poisson density, $\mu(t)^I e^{-\mu(t)}/I!$, where $\mu(t) \equiv J^{-1}\int_0^t R(s)ds$. Note

[11] This result can be derived as the limit of a model where a finite number of researchers have ideas about a finite number of varieties. For example, imagine $R/d$ researchers and $J/d$ varieties. A given individual has ideas at rate 1, hence the individual has ideas about a given variety at rate $1/(J/d) = d/J$. Since $R/d$ researchers are generating ideas, the rate at which all researchers have ideas about a given variety is $(R/d)(d/J) = R/J$. This result remains true as $d$ becomes arbitrarily small.

that $J\mu(t)$ is the undepreciated stock of past research.

### 4.1.2 Patentable Ideas

Ideas are patentable if and only if they are the highest quality in their variety. In the following argument, I will hold the variety fixed and solve for the expected number of patentable ideas in that variety conditional on the total number of ideas in that variety.

I assume that the search process begins at time 0, and that there have been $I$ ideas prior to time $t$. Denote the times at which those ideas occurred by, $\tau_1 < \tau_2 < \ldots < \tau_I$, where $0 < \tau_1$ and $\tau_I \leq t$. Let the qualities of the ideas be $Q_{\tau_1}, Q_{\tau_2}, \ldots, Q_{\tau_I}$, respectively. Since no patents exist before time 0, the first idea is always patentable. The second idea is patentable if and only if $Q_{\tau_2} > Q_{\tau_1}$, an event that occurs with unconditional probability 1/2. Similarly, the $I$th draw is patentable if and only if,

$$Q_{\tau_I} > \max\{Q_{\tau_1}, \ldots, Q_{\tau_{I-1}}\},$$

an event that occurs with probability $1/I$. The argument above leads to a simple equation for expected cumulative patents of variety $j$, $N(j,t)$, conditional on the number of ideas of variety $j$,

$$E[N(j,t)|I(j,t) = I] = \sum_{i=1}^{I} 1/i \approx \ln I + \psi,$$

where $\psi = .5772\ldots$ is Euler's constant.[12]

[12]There are simple and tight bounds for the expectation formula,

$$\ln(I) < \sum_{i=1}^{I} 1/i \leq \ln(I) + 1$$

### 4.1.3 Aggregate Patenting

I define the aggregate stock of patents as the integral over all varieties of the stock of patents in each variety, $N(t) \equiv \int_0^J N(j,t)dj$.[13] Conditional on the stock of past research, $J\mu(t)$, the measure of varieties for which exactly $I$ ideas have occured is proportional to the Poisson density, $J\frac{\mu(t)^I e^{-\mu(t)}}{I!}$. But, among the continuum of varieties for which $I$ ideas have occured, the average number of patentable ideas will be exactly $\sum_{i=1}^{I} 1/i$. Combining these results we get an equation for the aggregate stock of patents conditional on the aggregate stock of research,

$$N(t) = J\sum_{I=1}^{\infty} \frac{\mu(t)^I e^{-\mu(t)}}{I!} \sum_{i=1}^{I} 1/i.$$

I show in appendix A.2 that this equation simplifies considerably when the stock of research is large,

$$\lim_{\mu(t)\to\infty} \{N(t) - J\ln(\mu(t)) - J\psi\} = 0.$$

I will always assume that $\mu(t)$ is large enough so that the error of approximation may be ignored. Taking time derivatives of the asymptotic approximation above, we obtain equation (1) for the rate of patenting, $\dot{N}(t) = R(t)\mu(t)^{-1}$.

## 4.2 The Productivity Growth Equation

Economists' interest in patent data is ultimately based on the belief that patents may indicate something about technological change and productivity growth. Un-

and an asymptotic result,

$$\lim_{I\to\infty} (E[N|I] - \ln I) = \psi.$$

These results are described in more detail in Kortum (1991).

[13]Note that while the stock of patents in each variety is an integer, the aggregate stock of patents is not confined to the integers.

fortunately, there is little empirical work linking productivity growth to patenting.[14] The trends in the data suggest that if such a link exists, it is a proportional relation between productivity growth and the rate of patenting. Below, I show that if productivity growth is proportional to the rate of patenting, the representative search distribution (giving the probability that a new idea will have quality less than q) must be the Pareto distribution,

$$F(q) = 1 - q^{-1/\lambda},$$

with parameter $\lambda < 1$.[15] Letting $\dot{A}(t)/A(t)$ be productivity growth, I show that,

$$\dot{A}(t)/A(t) = \lambda \dot{N}(t)/J, \tag{2}$$

where $\lambda$ is the parameter of the Pareto distribution.

Combining the productivity equation (2) with the patent production function (1), and noting that $\dot{\mu}(t) = J^{-1}R(t)$, we get an equation relating productivity growth and growth in the stock of research,

$$\dot{A}(t)/A(t) = \lambda \dot{\mu}(t)/\mu(t). \tag{3}$$

Equation (3) is the empirical specification traditionally used to quantify the impact of research on productivity [Griliches (1973, 1979)]. However, while in the traditional derivation, $\lambda$ is a parameter of the production function, in the present derivation it is the parameter of the underlying search distributions. Econometric estimates of $\lambda$ are generally less than 0.1.

[14] Kortum and Lach (1994) provide some preliminary estimates using industry level data.

[15] I have imposed an arbitrary normalization so that the lower support of the distribution is unity. I assume that the original unpatented goods all have unit quality, formally, $Q(0, j) = 1$ for $j \in [0, J]$. This implies that the first patent in each variety will raise productivity.

### 4.2.1 A Productivity Index

The productivity index that I use in this section is the geometric mean of the quality of goods currently produced. Let the state of the art quality, $Z(j,t)$ of a good of variety $j$ be the highest quality of that variety that can currently be produced: formally, $Z(j,t) \equiv Q(N(j,t),j)$. It is well known that in a Bertrand equilibrium, only goods with state of the art quality will be produced. Therefore, the productivity index is the geometric mean of the state of the art qualities,

$$\ln A(t) \equiv J^{-1} \int_0^J \ln Z(j,t) dj.$$

Alternatively, I can define a productivity index based on real aggregate output per worker. In appendix A.3, I show that the two alternative indices of productivity grow at the same rate. This is a justification for using the index defined above which depends only on state of the art qualities.

### 4.2.2 The Search Distribution

With productivity so defined, constant exponential productivity growth results from the state of the art quality of each variety rising at a constant rate on average. If quality improvements are associated with patents and if the rate of patenting is constant then on average a patent must represent a percentage improvement in quality. I define the percentage improvement in quality made possible by a patented idea as the inventive step of the patent, denoted as $Y$. In a search model, as described above, $Y$ is a random variable. Some patents will represent substantial increases in the state of the art while others will be minor improvements. However, the randomness in inventive steps will average out across varieties. If the distribution of inventive steps does not change over time, a constant rate of aggregate productivity growth will result from a constant rate of patenting.

Since the state of the art within each variety rises stochastically over time, time stationarity of the inventive step requires a search distribution such that the implied distribution of the inventive step for new patentable inventions does not depend on the previous state of the art.

Consider a specific variety of good. Let $Q$ be the quality of a new idea in that variety and let $x$ be the previous state of the art in that variety. If the idea is patentable then $Q > x$ and the patent has an inventive step $Y = Q/x$. The distribution of the inventive step for patentable inventions, conditional on the previous state of the art, is

$$\begin{aligned} G(y|x) &\equiv \Pr(Y \leq y|Q \geq x) = \Pr(x \leq Q \leq xy)/\Pr(Q \geq x) \\ &= [F(xy) - F(x)]/[1 - F(x)]. \end{aligned}$$

I want to find conditions on the distribution function, $F(.)$, such that $G(y|x)$ does not depend on $x$.

It is convenient to normalize the left support of the distribution function at unity: formally $F(q) > 0$ for $q > 1$ and $F(1) = 0$. This implies, $G(y|1) = F(y)$. Thus, we want a distribution function which satisfies the functional equation,

$$F(y) = [F(xy) - F(x)]/[1 - F(x)],$$

for all $x \geq 1$. The unique solution to this functional equation is the Pareto distribution,

$$F(y) = G(y|x) = 1 - y^{-1/\lambda},$$

for $y \geq 1$.[16] With the Pareto distribution, the inventive step of patentable ideas is not only independent of the previous state of the art, it also inherits the Pareto form of the underlying search distribution. Throughout the remainder of the paper, I assume that the search distribution is Pareto.[17]

### 4.2.3 Productivity and Patents

We can now derive an equation for productivity growth as a function of the rate of patenting. Consider the increase in the log of productivity between time $t$ and $t + s$. This increase is due to the increase in the state of the art qualities of

[16] I follow a proof in Billingsley (1986, pg. 191) which solves a similar equation to obtain the exponential distribution. We want a function satisfying, $1 - F(xy) = (1 - F(x))(1 - F(y))$. Equivalently, we want a function $H(.)$ that satisfies $H(\ln x + \ln y) = H(\ln x)H(\ln y)$ where, $H(\ln q) = 1 - F(q)$. Cauchy's equation implies $H(w) = e^{aw}$ for some number $a$. Thus, $1 - F(q) = e^{a \ln q} = q^a$, which is the desired result.

[17] The following are some notes related to the Pareto distribution. The mean of the Pareto distribution is $1/(1-\lambda)$. If the random variable $Q$ has the Pareto distribution, then the distribution of $\ln Q$ is,

$$\Pr(\ln Q \leq x) = \Pr(Q \leq e^x) = 1 - e^{-x/\lambda},$$

which is the exponential distribution. In Kortum (1991) I note that only the exponential distribution has the property that the absolute increase in the state of the art is independent of the initial state of the art. Now, we have the companion result, that only the log exponential distribution has the property that the inventive step (defined as a percentage improvement) is independent of the initial state of the art. The Pareto distribution is used by Bental and Peled (1992) to illustrate their model of endogenous growth based on a search process. While Scherer (1965) reports fitting the Pareto distribution to patent values (i.e. private values), the result here concerns the distribution of social values of patents. The distribution of private values will be discussed below. To examine the distributional assumption directly, one would want micro data of the type being studied by Henderson and Cockburn (1993).

individual varieties,

$$\ln A(t+s) - \ln A(t) = J^{-1} \int_0^J \ln(Z(j,t+s)/Z(j,t))dj.$$

The increase in the state of the art within variety $j$, $Z(j,t+s)/Z(j,t)$, is equal to the product of the inventive steps of any patents on variety $j$ which were discovered between $t$ and $t+s$. Let $\chi(k,t,s)$ be the measure of varieties in which exactly $k \in \{0,1,2,\ldots\}$ patents are discovered between $t$ and $t+s$. Then,

$$\ln A(t+s) - \ln A(t) = J^{-1} \sum_{k=1}^{\infty} \chi(k,t,s) E[\ln(\prod_{i=1}^{k} Y_i)].$$

Since the $Y_i$ are drawn from the Pareto distribution, $\ln Y_i$ is drawn from the exponential distribution and therefore has a mean of $\lambda$. Thus,

$$E[\ln(\prod_{i=1}^{k} Y_i)] = E[\sum_{i=1}^{k} \ln Y_i] = k\lambda.$$

Furthermore,

$$\sum_{k=1}^{\infty} k\chi(k,t,s) = N(t+s) - N(t).$$

Combining results,

$$\ln A(t+s) - \ln A(t) = J^{-1} \sum_{k=1}^{\infty} \chi(k,t,s)k\lambda = \lambda(N(t+s) - N(t))/J.$$

Dividing both sides of the equation by $s$ and taking the limit as $s$ goes to zero, we get equation (2) for productivity growth.

### 4.3 Equilibrium Research

In the derivations above I have conditioned on a path of research. I can now solve for an equilibrium path of research. Individuals in the model always have the choice of being researchers or earning a wage of unity. A researcher has a chance of discovering a patentable idea which will yield a stream of future earnings. As long as a positive measure of individuals choose to be researchers, a given researcher knows that patentable ideas will arrive to her at a stochastic rate $\dot{N}(t)/R(t)$. Our patent production function (1) implies that this arrival rate is $\mu(t)^{-1}$. Let $V(t)$ be the expected present discounted value of a patentable idea discovered at time $t$. In an equilibrium where $0 < R(t) < e^{nt}$, all individuals must be indifferent between research and earning a wage of unity, i.e. $V(t)\mu(t)^{-1} = 1$ or $V(t) = \mu(t)$.

I will concentrate on the steady state equilibrium in which a constant fraction of individuals choose research, $R(t)/e^{nt} = \alpha$. Along the steady state, the stock of research is given by $\mu(t) = \alpha e^{nt}/(nJ)$. It is increasing in $\alpha$ because a larger fraction of individuals engaged in research leads to a higher stock of past research. Below, I show that the expected value of a new patent is given by $V(t) = (1-\alpha)Be^{nt}$ where $B$ is a complicated function of several of the parameters of the model. Intuitively, $V(t)$ is decreasing in $\alpha$ because in an economy where a large fraction of individuals are engaged in research, levels of production will be low and hence patents will be less profitable. The steady state equilibrium is the unique value of $\alpha$ such that $\mu(t) = V(t)$. I obtain an expression for $\alpha$ and calculate it numerically for a range of values of the model's parameters. I conclude the section by comparing $\alpha$ with the fraction of human resources which a social planner would devote to research.

### 4.3.1 The Value of a Patent

The value of a patent is the expected presented discounted value of the flow of profits obtained by selling the patented good. A first step in determining the value of a patent is to calculate this flow of profits. It was noted above that in a Bertrand equilibrium between producers, only state of the art goods are consumed.

Given the simple specification of preferences, individuals will distribute expenditure uniformly across all state of the art goods. Let $C(j,t) \equiv C(N(j,t),j,t)$ be the quantity of the state of the art good of variety $j$ and let $P(j,t) \equiv P(N(j,t),j,t)$ be its price. Define aggregate expenditure as $X(t) \equiv \int_0^J P(j,t)C(j,t)$. Then, we have $P(j,t)C(j,t) = J^{-1}X(t)$.

The producer of a state of the art good faces competition from the producer of the next highest quality pricing at marginal cost. She maximizes profits by charging a price of $cY$ where $Y$ is her inventive step. Thus, the flow of profits to the producer of the state of the art good of variety $j$ is,

$$P(j,t)C(j,t)[1 - c/P(j,t)] = J^{-1}X(t)[1 - Y(j,t)^{-1}],$$

where, $Y(j,t) \equiv Z(j,t)/Q(N(j,t)-1,j)$ is the inventive step of the state of the art good of variety $j$. Note that profits depend on the variety, $j$, via the inventive step. A patent with a greater inventive step is more valuable because the producer can maintain a greater markup of price over marginal cost.

Now consider a state of the art patent with an inventive step of $y$. Denote the quality of the previous state of the art patent by $x$. Thus, the state of the art patent has quality $z = xy$. This patent generates a flow of profits, $\pi(t,y) \equiv J^{-1}X(t)[1 - y^{-1}]$ at time $t$. This equation governs its flow of profits until the current state of the art patent is made obsolete by a new patent with quality exceeding $z$. Due to the scope of patent protection, the patent holder does not see its markup eroded by subsequent unpatentable ideas with qualities between $x$ and $z$.

At time $t$, the discounted future profits from a patent with inventive step $y$ are, $\int_t^{t+L} e^{-\rho(s-t)}\pi(s,y)ds$. This is a random variable because the future profitable life of the patent, $L$, is a random variable. Each patent faces an instantaneous hazard given by the rate at which new ideas with higher qualities are being discovered. At time $s$, ideas on a given variety are produced at rate $J^{-1}R(s)$. For a patent with a state of the art $z$ the fraction of these ideas that are patentable is $z^{-1/\lambda}$. Thus, the hazard rate for a patent with state of the art $z$ is $J^{-1}R(s)z^{-1/\lambda}$, which is decreasing in $z$. Combining results, the expected value of a patentable idea at time $t$ with a state of the art $Z = z$ and inventive step $Y = y$ is,

$$V(t,z,y) = \int_t^{\infty} e^{-\int_t^s (\rho + J^{-1}R(v)z^{-1/\lambda})dv} \pi(s,y).$$

The value of a patent depends on time (through its effect on aggregate expenditure and aggregate research), the inventive step (through its effect on the price which the producer can charge), and the state of the art (through its effect on the expected profitable life of the patent). Conditional on these three factors, the date at which a patent is invented is irrelevant.

For a researcher hoping to discover a patentable idea, the inventive step and state of the art are random variables. Thus the value of a patent is also a random variable, $V(t,Z,Y)$. The expected value of a patent, conditional on it being discovered at time $t$ is,

$$V(t) = E[V(t,Z,Y)] = \int_1^{\infty} \int_1^{z} V(t,z,y) f(z,y|t) dy dz, \tag{4}$$

where $f(z,y|t)$ is the joint density of the state of the art and inventive step of patentable ideas discovered at time $t$. That density is derived in appendix A.5.

### 4.3.2 Aggregate Expenditure

The profits from marketing a patentable good are proportional to aggregate expenditure in the economy, $X(t)$. At each instant, the aggregate expenditure on goods is equal to aggregate income. Aggregate income consists of wage income and profit income. Since the wage is unity, wage income is equal to the measure of individuals who are not researchers, $e^{nt} - R(t)$. Aggregate profit income is equal to the integral over varieties of the profits derived from the state of the art good of each variety,

$$\int_0^J \pi(t, Y(j,t))dj = J^{-1}X(t)\int_0^J (1 - Y(j,t)^{-1})dj = X(t) - X(t)\theta(t),$$

where, $\theta(t) \equiv J^{-1}\int_0^J Y(j,t)^{-1}dj$. Therefore, $X(t) = e^{nt} - R(t) + X(t) - X(t)\theta(t)$, or $X(t) = (e^{nt} - R(t))/\theta(t)$.

The term $\theta(t)$ is the average value across varieties at time $t$ of the inverse of the inventive step. Since there are a continuum of varieties, this average is equal to the expected value of the inverse of the inventive step, $\theta(t) = \int_1^\infty y^{-1}g_2(y|t)dy$, where $g_2(y|t)$ is the density of the inventive step for patents which are profitable at time $t$. I derive this density in appendix A.6 and show that it does not depend on time. Therefore, $\theta(t)$ must not depend on time either. In summary, aggregate expenditures are

$$X(t) = \frac{e^{nt} - R(t)}{\theta(\lambda)}, \tag{5}$$

where the average inverse inventive step is now denoted by $\theta(\lambda)$ to indicate that it depends only on the parameter $\lambda$. In the second column of table 1, I use Mathematica to calculate $\theta(\lambda)$ for various values of $\lambda$.

### 4.3.3 The Steady State Solution

In order to solve for an equilibrium path of research, I assume that the economy is in a steady state. I define a steady state to be an equilibrium such that the fraction of individuals engaged in research is constant, i.e. $R(s) = \alpha e^{ns}$ for all

$s \geq t$. Furthermore, I assume that the stock of past research has adjusted to this steady state: formally $J^{-1}R(t)/\mu(t) = n$.

First, I calculate $V(t)$ subject to this restriction on the path of research. Second, I solve for the value of $\alpha$ which is consistent with equilibrium. In solving for $V(t)$, there is a complicated interaction between patent values (as a function of $z$, $y$ and $t$) and the joint distribution of the state of the art and the inventive step (conditional on time) for new patents. This interaction remains complicated even after applying the steady state results that $R(s)$ and $X(s)$ both grow at rate $n$. The derivation of the following equation is found in appendix A.7,

$$V(t) = \frac{X(t)}{J\rho}[1 - \phi(\rho/n, \lambda)] = \frac{(1-\alpha)e^{nt}}{J\rho\theta(\lambda)}[1 - \phi(\rho/n, \lambda)],$$

where, $\phi(\rho/n, \lambda)$ is a complicated function. Numerical values of $\phi$, for different settings of $\rho/n$ and $\lambda$, are shown in table 1.

Two observations can be made which do not depend on the numerical values of $\phi$. First, the equation for $V(t)$ conditional on $X(t)$ does not depend on the fraction of individuals who are researchers, $\alpha$, though $X(t)$ itself is a function of $\alpha$. Second, $V(t)$ grows at rate $n$. This second result comes out of the interaction (noted above) between the value over time of a patent conditional on $Y$ and $Z$ and the joint distribution of $Y$ and $Z$ for each new cohort of patents. The value of any existing patent grows at a rate less than $n$ since, with rising research, existing patents face increasing hazard rates. On the other hand, the marginal distribution of $Z$ is stochastically increasing over time, since ideas with lower qualities are less likely to be patentable as the state of the art rises. In a steady state, these two effects cancel each other.

I can now solve for the fraction of human resources devoted to research. Assume that $\alpha$ is strictly between 0 and 1.[18] To support $0 < \alpha < 1$ as an equilibrium requires that $V(s) = \mu(s)$ for all $s \geq t$. But, since $R(s) = \alpha e^{ns} = J\dot{\mu}(s)$ and $n = \dot{\mu}(s)/\mu(s)$ we have $\mu(s) = \frac{\alpha e^{ns}}{nJ}$. Equating our expressions for $V(s)$ and $\mu(s)$ yields,

$$\alpha/n = (1-\alpha)(1-\phi(\rho/n,\lambda))/(\theta(\lambda)\rho).$$

Solving for the fraction of the population who are researchers,

$$\alpha = \frac{1}{1+\frac{\theta(\lambda)\rho/n}{1-\phi(\rho/n,\lambda)}}. \tag{6}$$

We have found an equilibrium where the number of researchers and the stock of past research both grow at rate $n$. According to the patent production function, equation (1), the rate of patenting in this equilibrium will be $Jn$. According to the equation relating productivity growth and patenting, equation (2), productivity will grow at rate $\lambda n$ in this equilibrium.

In table 1, I show numerical calculations of $\alpha$ for various values of the parameters, $\lambda$ and $\rho/n$. The fraction of human resources devoted to research is increasing in the richness of the search distributions ($\lambda$) and decreasing in the value of the discount rate relative to population growth.

A natural question for a policy maker is whether the equilibrium value of $\alpha$ is above or below the value, $\alpha^*$, which a social planner would choose. We would not expect equality since, in the decentralized equilibrium, goods are not sold at marginal cost. The social planner's problem, worked out in appendix A.10, yields

[18] Consider the possibility of $\alpha = 1$. In that case, $s > t$ implies $V(s) = 0$, hence individuals will not choose research. Therefore, $\alpha = 1$ cannot be an equilibrium. Now, consider the possibility of $\alpha = 0$. In that case $V(s)$ grows at rate $n$ while $\mu(s)$ is constant. Eventually $V(s)$ will exceed $\mu(s)$ which will cause individuals to choose research. Therefore $\alpha = 0$ cannot be an equilibrium.

the simple result that $\alpha^* = n\lambda/\rho$. In table 1, I compare values of $\alpha$ and $\alpha^*$ for various values of the parameters. In all cases, the equilibrium leads to over investment in research.

To gain an intuition for this result, let me begin by debunking an alternative explanation. Does over investment result from racing behavior (since researchers are competing for the prize of having the best invention)? The answer must be 'no' because, within any variety, search is sequential with ideas arriving as a non homogeneous Poisson process. At each instant, a researcher knows the probability of making a marginal improvement in quality. Since a researcher earns profits based on the inventive step of her idea, her payoffs are closely related to the marginal social value of her idea. The correct intuition for over investment arises from the general equilibrium effect of markups on the wages of production workers. Since output is proportional to labor input, if production workers were paid their marginal product, then aggregate wage income would be sufficient to purchase aggregate production. In fact, aggregate wage income will purchase only a fraction $\theta(\lambda) < 1$ of aggregate production (with the rest puchased with profit income). Since production workers are under paid, too many individuals choose to be researchs instead. This is only an intuition since researchers, by not being able to price discriminate, are also under compensated. In the numerical examples, the under compensation of production workers is quantitatively more important.

# 5 Calibration

I have shown that the search model can account for the trends in the data which are observed in the US and several other technologically advanced countries. In this section I take the model literally and look for values of the parameters which cause the model to make quantitative predictions which are similar to observations from the US economy from 1957-1989.[19] The data and sources are shown in table 2.

I equate $R(t)$ with the number of research and development scientists and engineers employed in US industry. Since S&E's have increased at an annual rate of 3.6%, I take $n = .036$. This is a lower bound on $n$ since it ignores any growth in human capital per scientist and engineer.

I now consider calibrating productivity growth in the model, $\dot{A}/A = \lambda n$ based on measures of productivity growth in the US. There are a range of values of $\lambda$ implied by different measures. TFP growth in the manufacturing sector has been 2.0% which is also the growth rate of output per hour in the private business sector. This implies $\lambda n = .02$ or $\lambda = .55$. Alternatively if we use the growth of TFP in the private business sector, 1.3%, we get $\lambda = .36$.

Alternatively, we can calibrate $\lambda$ based on the econometric estimates of the effect of research on productivity growth. As noted earlier, these estimates tend to be no greater than 0.10. It is well known that these estimates of $\lambda$ imply that less than 0.5% annual productivity growth can be attributed to research, with the balance of productivity growth left unexplained [see, Sveikauskas (1989)]. The model developed above can easily be adapted to allow for exogenous productivity growth. Assume that the cost of production parameter, $c$, falls exogenously over time at an exponential rate: $c(t) = c(0)e^{-gt}$. Since $c$ does not show up in any other equations, this change will have no effect on the other predictions of

[19]The data on research and development is a consistent series beginning in 1957.

the model. However, it does create a wedge between the growth of labor productivity, as defined in appendix A.3, and the growth of our productivity index. In particular, labor productivity grows at rate $g + \dot{A}(t)/A(t)$, where $g$ is exogenous productivity growth and $\dot{A}(t)/A(t)$ is productivity growth accounted for by industrial research.[20].

I now look at data on the fraction of human resources devoted to research. The obvious measure is the number of R&D scientists and engineers as a fraction of US civilian employment. That number has increased from 0.38% in 1957 to 0.61% in 1989. Alternatively, we may interpret the model's prediction to be about the fraction of human capital devoted to research. If we assume that the wage per unit of human capital is the same in research and production then we should look at the ratio of R&D expenditures (which is roughly the compensation of researchers) to total compensation of employees. This ratio is equal to 3.0% in 1957 and 3.3% in 1989. The fact that the ratio of compensation is higher than the ratio of employment reflects the fact that researchers have above average skills. I will take .033 as the appropriate empirical counterpart to the fraction of human capital devoted to research.

The model's prediction for the fraction of human capital devoted to research, $\alpha$, depends on two functions, $\theta(\lambda)$ and $\phi(\rho/n, \lambda)$ which I calculate numerically using Mathematica. Table 1 shows the value of $\alpha$ for a range of parameter values. If we take $\lambda = .36$ then the model comes closest to fitting the data when the

[20]This follows from the fact that

$$\ln\{[X(t)/P(t)]/[e^{nt} - R(t)]\} = -\ln c(t) - E[\ln Y] + \ln A(t) - \ln \theta(\lambda),$$

see appendix A.3

discount rate is the highest, $\rho/n = 3$ (so that $\rho = .11$). Yet even then, the model over predicts the observed value of $\alpha$ by a factor of 4. The situation would be even worse for larger values of $\lambda$. The situation would also be worse if we attempted to fit the fraction of the labor force employed in research (which suggests $\alpha$ is less than 1%).

Now, consider the values of $\lambda$ implied by econometric estimates. If we take $\lambda = .1$ then for $\rho = .11$ we come very close to matching the data. If $\lambda = .05$, then we can match the data even for a reasonable value of $\rho/n = 2$ (so that $\rho = .07$). The model tells a story which is consistent with the empirical literature on productivity and R&D. The observed fraction of human resources devoted to research is small and this is consistent with productivity growth being primarily exogenous. Possibly as little as 0.2% annual productivity growth ($\lambda n = (.05)(.036) \approx 0.002$) can be attributed to industrial research.

There is another, more speculative, explanation for the low fraction of human resources devoted to research. This alternative explanation does not rely on small values of $\lambda$. In the model, inventors face only one hazard in appropriating the benefits of their inventions: the success of subsequent inventors. In reality, inventors face many other hazards, such as imitation, which hamper their ability to appropriate returns; see Mansfield, Schwartz and Wagner (1981) and Levin et. al. (1987). If in reality appropriability is more difficult than is implied by the model, then it is not surprising that the model over predicts the level of research investment. Actually modeling this may be difficult because a constant hazard of imitation will interact in a complicated way with the existing hazard rates which are not exponential (in the present model, a given patent faces an increasing hazard).

Data on patent renewels provides some evidence for this view. In countries with annual patent renewal fees, we observe the rate at which owners of patents fail to renew their patents. Following a cohort of patents invented in year $t$, one can calculate the number of patents which fail to renew in year $s+1$ divided by the number of patents which renewed in year $s > t$. Lanjouw (1993) plots these hazard rates, averaged over cohorts of German patents. For patents of age 5 or more, the hazard rates are almost always above 0.1. Data in Pakes (1986) indicate that these high hazard rates are also found in France and the UK.

Below I derive the model's prediction for the hazard rate from a cohort of patents that are all invented at the same time. For simplicity I assume that the renewel fees are small enough that their effect on the decision to patent may be ignored. Furthermore, I assume that the failure to renew marks the age at which a patent is surpassed by a better patent.[21]

Let the random variable $L$ equal the length of the profitable life of a patent. If the state of the art of the patent is $Z = z$ and the patent is invented at time $t$,

$$\Pr(L \leq x|z,t) = 1 - e^{-z^{-1/\lambda}(\mu(t+x)-\mu(t))}.$$

To get a prediction of the rate at which patents in a given cohort lose value, we need to integrate over the density of the state of the art for patents invented in $t$.

[21]These assumptions are, at best, approximations. The model implies that there are some patents with arbitrarily small inventive steps and therefore arbitrarily small values. Thus, for any given positive renewal fee, some patents with positive values will not renew. Putnam (1991) has a provocative finding from his study of 'patent families' where the same invention is patented in many countries. While the patents in the different countries face different renewal fees and presumably have different values, all patents in a given family tend to stop renewing in the same year. This suggests that it is the arrival of a better invention that eliminates the private value of the entire patent family.

I show in appendix A.8 that,

$$\Pr(L \leq x|t) = \int_1^\infty \Pr(L \leq x|z,t) f_1(z|t) dz = 1 - \mu(t)/\mu(t+x).$$

In the steady state equilibrium with the stock of research growing at rate $n$, $\Pr(L \leq x|t) = 1 - e^{nx}$. Thus, patents are expected to drop out of a cohort at rate $n$. From the calibration I determined that $n = .036$. Notice that the model underpredicts (by at least a factor of 3) the observed hazard rates. I view this as evidence that the private value of patents is being undercut by imitation, a factor which is absent in the theoretical model.

# 6 Conclusion

The progress economists have made in quantifying the economic importance of research can be attributed to a vast number of empirical studies of the link between productivity and research expenditures. The results of those studies suggest that R&D investments contribute significantly to productivity growth. Yet productivity growth remains, to a large extent, unexplained. The current paper is sympathetic to that conclusion: the set of parameters that are consistent with the observed growth of research as well as the observed fraction of resources devoted to research also imply that research investments account for only a small share of productivity growth.

The present paper is also consistent with the empirical literature in the sense that it provides an alternative justification for regressing productivity growth on the growth of the stock of research. The resulting estimate of the elasticity of productivity with respect to the stock of research is the parameter of the underlying Pareto search distributions. This interpretation is valid when the estimates are obtained from aggregate data and is probably also valid when the estimates are obtained from industry data.

Much of the recent empirical work on research and productivity has been done on firm level data sets [Mairesse and Sassenou (1991) survey this literature]. I conclude by examining, in the context of the present model, what can be learned from these firm level regressions. I obtain the provocative result that productivity growth at the firm level is orthogonal to research at the firm level. Thus, the firm level regressions of productivity growth on measures of research do not identify the parameter of the Pareto search distribution.

I define firms as countable sets of researchers. Assume there is a continuum of such firms indexed by $m \in [0, M(t)]$. At time $s$ firm $m$ consists of $R_m(s)$ researchers. The firm's value is the value of all the patents invented by researchers while they were in the firm. The firm's revenues are the receipts from marketing the varieties on which the firm owns patents. Let $S_m(t)$ be the set of varieties that are produced by firm $m$ at time $t$ and let the integer $J_m(t)$ be the corresponding number of varieties. I define the productivity index of firm $m$ as the (geometric) average of the qualities of the varieties it produces, $\ln A_m(t) = J_m(t)^{-1} \sum_{j \in S_m(t)} \ln(Z(j,t))$. This index is only calculated for firms which are currently producing at least one variety.

How does $E[\ln A_m(t) | \{R_m(s)\}_{s=0}^{t}]$ depend on the firm's path of research? The path of research will determine (stochastically) the timing of the firm's patents. To see how this alters the index of productivity, consider the probability density of the state of the art for patents which are profitable at time $t$ but were invented at time $s \leq t$. In appendix A.9, I show that this density is not a function of $s$, the cohort of the patent. In particular $E[\ln(Z)|t] = \lambda \ln(\mu(t)) + \lambda\psi$ for patents discovered at any previous date. This implies that expected productivity growth for firm $m$, conditional on its past research, is,

$$E[\ln(A_m(t)/A_m(t-1)) | \{R_m(s)\}_{s=0}^{t}] = \lambda \ln(\mu(t)/\mu(t-1)),$$

where $\mu(t)$ is the aggregate stock of research. Thus, while productivity growth is random across firms, it is orthogonal to their paths of research. Returning to our initial question, the present model predicts a coefficient of zero in a firm level cross-sectional regression of productivity growth on any measure of research.

Productivity growth for a firm occurs through the introduction of new high quality goods as well as by the abandonment of old low quality goods. The former is a consequence of the firm's own research while the latter is due to the research of other firms. In the present model these forces act at the same rate so that the productivity growth of a firm does not depend on its own research.

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# A Mathematical Appendix

## A.1 List of Symbols

| | |
|---|---|
| $j \in [0, J]$ | Index of variety. |
| $I(j,t)$ | Stock of ideas (patentable or not) on variety $j$ by time $t$. |
| $N(j,t)$ | Stock of patents on variety $j$ by time $t$. |
| $i \in [0, \ldots, N(j,t)$ | index of goods (ordered by quality). |
| $Q(i,j)$ | Quality of the ith good of variety $j$. |

| | |
|---|---|
| $C(i,j,t)$ | Consumption of the ith quality of variety $j$. |
| $P(i,j,t)$ | Price of the ith quality of variety $j$. |
| $c$ | Labor requirement equals marginal cost (wage=1). |
| $h \in [0, e^{nt}]$ | Index of individuals. |
| $n$ | Population growth. |
| $\rho$ | Discount rate. |
| $R(t)$ | measure of individuals who are researchers. |
| $\mu(t)$ | $(= J^{-1} \int_0^t R(s)ds)$: The stock of research. |
| $\phi$ | Euler's constant ($\approx$ .5772.) |
| $Z(j,t)$ | $(= Q(N(j,t),j))$: The state of the art in variety $j$. |
| $C(j,t)$ | Consumption of the state of the art good of variety $j$. |
| $P(j,t)$ | Price of the state of the art good of variety $j$. |
| $Y(j,t)$ | $(= Q(N(j,t),j)/Q(N(j,t)-1,j))$: The inventive step. |
| $A(t)$ | Index of productivity, geometric mean quality. |
| $\lambda$ | Parameter of the Pareto distribution. |
| $\pi(t,y)$ | Flow profits at $t$ to a patent with inventive step $y$. |
| $V(t,z,y)$ | Value of a patent with state of art $Z = z$ and inventive step $Y = y$. |
| $V(t)$ | $(= E[V(t,Z,Y))$: The expected value of a patent discovered at time $t$. |
| $X(t)$ | Expenditures at time $t$. |
| $\theta$ | Expected value (across varieties) of the inverse inventive step. |
| $\alpha$ | Equilibrium fraction of human capital devoted to research. |

## A.2 Derivation of the Invention Production Function

I want to prove that

$$\lim_{\mu(t)\to\infty}\{N(t) - J\ln(\mu(t)) - J\psi\} = 0.$$

Remember that,

$$N(t) = J\sum_{x=1}^{\infty}\frac{\mu(t)^x e^{-\mu(t)}}{x!}\sum_{i=1}^{x} 1/i = JE[f(I)],$$

where, $f(I) = \sum_{i=1}^{I} 1/i$. Since $f(I)$ is concave and $I$ is a random variable with mean $\mu(t)$, Jensen's inequality implies $E[f(I)] \leq \sum_{i=1}^{int(\mu(t))+1} 1/i$, where $int(y)$ is the largest integer which is less than $y$. We can also derive a lower bound for $E[f(I)]$. For any $\varepsilon > 0$,

$$1 - \Pr[(1-\varepsilon)\mu(t) \leq I \leq (1+\varepsilon)\mu(t)]$$

$$= 1 - \Pr[-\varepsilon\mu(t) \leq I - \mu \leq \varepsilon\mu(t)]$$

$$= \Pr(|I - \mu(t)| \geq \varepsilon\mu(t)) \leq \frac{1}{\mu(t)\varepsilon^2},$$

where the last line follows from Chebyshev's inequality, Thus,

$$\Pr[(1-\varepsilon)\mu(t) \leq I \leq (1+\varepsilon)\mu(t)] \geq 1 - \frac{1}{\mu(t)\varepsilon^2}.$$

Using this result,

$$E[f(I)] \geq (1 - \frac{1}{\mu(t)\varepsilon^2})\sum_{i=1}^{int((1-\varepsilon)\mu(t))} 1/i.$$

Now let $\varepsilon = \mu(t)^{-1/3}$ and let $\mu(t) \to \infty$. We trap $E[f(I)]$ between two quantities, each of which is becoming arbitrarily close to $\ln(\mu(t)) + \psi$. The approximation is quite accurate, even for $\mu(t)$ as small as 20.

## A.3 An Alternative Productivity Index

Consider a productivity index defined as real output per production worker, $[X(t)/P(t)]/[e^{nt} - R(t)]$. From equation (5) we see that $X(t)/[e^{nt} - R(t)]$ is constant over time. Therefore, real output per production worker grows at the same rate at which $P(t)$ falls. The price index, $P(t)$, is the geometric mean, over varieties, of price per unit of quality,

$$\ln P(t) \equiv J^{-1} \int_0^J \ln(P(j,t)/Z(j,t))dj = E[\ln P(j,t)] - \ln A(t),$$

where $P(j,t)$ is the price of the state of the art good of variety $j$. In calculating the value of a patent, I show that $P(j,t) = cY(j,t)$. Furthermore, in deriving aggregate expenditures, I show that the distribution of $Y$ across varieties is stationary over time. Thus, $E[\ln P(j,t)]$ is a constant over time, which proves the result.

## A.4 Derivation of the Productivity-Research Equation

We saw that by combining the patent production function and the productivity-patent equation we could obtain a function relating productivity to the stock of research. In this appendix I derive this relation directly. The strategy is to derive the cross-sectional distribution of the state of the art across varieties as a function of the stock of research. I then integrate over this distribution to obtain the level of productivity. In particular,

$$\ln A(t) = J^{-1} \int_0^J \ln Z(j,t)dj = J^{-1} \int_1^\infty \ln(z) g_1(z|t)dz,$$

where, $g_1(z|t)$ is the cross sectional density of the state of the art. Below, I derive that density, $g_1(z|t) = (\mu(t)/\lambda) z^{-(1+\lambda)/\lambda} e^{-\mu(t) z^{-1/\lambda}}$. Performing the integration and letting $\mu(t)$ become large,

$$\ln A(t) = \lambda \ln(\mu(t)) + \lambda\psi,$$

or,

$$A(t) = e^{\lambda\psi}\mu(t)^{\lambda},$$

which is the desired result.

I now derive the cross-sectional distribution used above. Since the outcome of search is independent across a continuum of varieties, the cross-sectional distribution of the state of the art conditional on the aggregate stock of research is identical to the distribution of the state of the art within a given variety conditional on $\mu(t) \equiv J^{-1}\int_0^t R(s)ds$. The latter can be derived quite easily. The trick is to condition on the number of ideas (patentable or not) which might have occured and then to sum over the probability distribution for the number of ideas, which is Poisson. For an arbitrary variety $j$, consider $z \geq 1$,

$$\begin{aligned}
\Pr(Z(j,t) \leq z|\mu(t)) &= \sum_{m=0}^{\infty} \Pr(I = m|\mu(t))F(z)^m \\
&= \sum_{m=0}^{\infty} \frac{\mu(t)^m e^{-\mu(t)}}{m!}(1 - z^{-1/\lambda})^m \\
&= e^{-\mu(t)z^{-1/\lambda}} \sum_{m=0}^{\infty} \frac{[\mu(t)(1 - z^{-1/\lambda})]^m e^{-\mu(t)(1-z^{-1/\lambda})}}{m!} \\
&= e^{-\mu(t)z^{-1/\lambda}}.
\end{aligned}$$

Note that $\Pr(Z(j,t) \leq z|\mu(t)) = 0$ for $z < 1$. There is a discontinuity at $z = 1$ which corresponds to the probability that no ideas have arrived before time $t$. However, the probability of no ideas approaches zero as $\mu(t)$ becomes large. I will assume throughout that $\mu(t)$ is large enough that this issue may be ignored.

Thus, the density of the state of the art across varieties is,

$$\lambda^{-1}\mu(t)z^{-(1+\lambda)/\lambda}e^{-\mu(t)z^{-1/\lambda}}.$$

Thus,

$$\ln A(t) = \int_1^\infty \ln(z)\lambda^{-1}\mu(t)z^{-(1+\lambda)/\lambda}e^{-\mu(t)z^{-1/\lambda}}dz.$$

Changing the variable of integration to $x = \mu(t)z^{-1/\lambda}$,

$$\ln A(t) = \lambda \int_0^{\mu(t)} \ln(\mu(t)/x)e^{-x}dx = \lambda \ln(\mu(t))(1 - e^{-\mu(t)}) - \lambda \int_0^{\mu(t)} \ln(x)e^{-x}dx.$$

The following equation is an arbitrarily good approximation for large enough $\mu(t)$,

$$\ln A(t) = \lambda \ln \mu(t) - \lambda \int_0^\infty \ln(x)e^{-x}dx.$$

The Laplace transform of $-\psi - \ln t$ is $s^{-1}\ln s$, where $\psi$ is Euler's constant. Evaluating the Laplace transform at $s = 1$ implies,

$$\int_0^\infty \ln(x)e^{-x}dx = -\psi.$$

This gives us the desired result that,

$$\ln A(t) = \lambda \ln \mu(t) + \lambda\psi.$$

## A.5 The Joint Density of $Z$ and $Y$ for New Patents

Let $W \equiv Z/Y$ denote the random level of the state of the art which a new patent surpasses. I first derive the density of $W$ for patents invented in $t$. The joint density of $W$ and $Y$, $h(w, y|t)$, is then easily obtained since the inventive step, $Y$, is independent of the state of the art surpassed, $W$. Finally, the joint density of $Z$ and $Y$, $f(z, y|t)$, is obtained as the derivative of a certain integral over $h(w, y|t)$.

Let $Q$ be the quality of a new idea of a given variety. Conditional on the idea being patentable, the probability that the state of the art, $W$, which it exceeds is less than $w$ is,

$$\begin{aligned}
\Pr(W \leq w | Q \geq W) &= \Pr(W \leq w, Q \geq W) / \Pr(Q \geq W) \\
&\approx \frac{\int_1^w x^{-1/\lambda}(\mu(t)/\lambda)x^{-(1+\lambda)/\lambda}e^{-\mu(t)x^{-1/\lambda}}dx}{\int_1^\infty x^{-1/\lambda}(\mu(t)/\lambda)x^{-(1+\lambda)/\lambda}e^{-\mu(t)x^{-1/\lambda}}dx} \\
&\approx \int_1^w (\mu(t)^2/\lambda)x^{-(2+\lambda)/\lambda}e^{-\mu(t)x^{-1/\lambda}}dx,
\end{aligned}$$

where the second line integrates over all states of the art $x$, the product of the probability of surpassing that state of the art and the (approximate for finite $\mu(t)$) measure of varieties with that state of the art. The approximations in the second and third lines become arbitrarily close as $\mu(t)$ gets large.

Multiplying the density of the state of the art surpassed by the density of the inventive step (since the associated random variables are independent),

$$h(w, y|t) = (\mu(t)/\lambda)^2 y^{-(1+\lambda)/\lambda} w^{-(2+\lambda)/\lambda} e^{-\mu(t)w^{-1/\lambda}}.$$

The joint distribution of the state of the art and inventive step for a patentable idea discovered at time $t$ is,

$$\Pr(Z \leq \bar{z}, Y \leq \bar{y}) = \int_1^{\bar{y}} \int_1^{\bar{z}/y} h(w, y|t) dw dy.$$

Therefore, the corresponding density is

$$f(z, y|t) = y^{-1}h((z/y), y|t) = (\mu(t)/\lambda)^2 y^{(1-\lambda)/\lambda} z^{-(2+\lambda)/\lambda} e^{-\mu(t)(z/y)^{-1/\lambda}},$$

which is the result we were seeking.

## A.6 The Joint Distribution of $Z$ and $Y$ for Existing Profitable Patents

To derive the density of the inventive step, I first derive a more complicated object, the joint distribution, $G(z, y|t)$, of the state of the art and the inventive step for patents which are profitable at time $t$. This joint distribution can be used to derive the density of the inventive step (and could also be used to derive various moments of distribution of patent values).

Start with the joint density, $f(z, y|s)$, of the state of the art and the inventive step for each cohort of patents, $s \in [0, t]$ (that density was derived in appendix A.5). To derive the joint distribution of the state of the art and the inventive step of patents which have remained profitable through time $t$, I integrate over all past cohorts taking into account the probability, $e^{-(\mu(t)-\mu(s))z^{-1/\lambda}}$, that a patent invented in $s$ will still be profitable at time $t$. This consideration leads to the expression,

$$G(\bar{z}, \bar{y}|t) = \int_0^t \frac{\dot{N}(s)}{J} \int_1^{\bar{y}} \int_y^{\bar{z}} f(z, y|s) e^{-(\mu(t)-\mu(s))z^{-1/\lambda}} dz dy ds,$$

where $\frac{\dot{N}(s)}{J}$ is the size of cohort $s$ relative to the measure of profitable patents. Note that,

$$G(\bar{z}, \bar{y}|t) = \int_0^t \frac{\dot{N}(s)}{J} D(s) ds,$$

where,

$$D(s) = \int_1^{\bar{y}} \int_y^{\bar{z}} f(z, y|s) e^{-(\mu(t)-\mu(s))z^{-1/\lambda}} dz dy.$$

Thus, $D(s)$ is the fraction of patents from cohort $s$ which are still profitable in $t$ and which have state of the art below $\bar{z}$ and inventive step below $\bar{y}$. We can write,

$D(s) = D_1 + D_2$ where,

$$D_1 = \int_1^{\bar{y}} \int_1^{z} f(z, y|s) e^{-(\mu(t)-\mu(s))z^{-1/\lambda}} dy dz,$$

and,

$$D_2 = \int_{\bar{y}}^{\bar{z}} \int_1^{\bar{y}} f(z, y|s) e^{-(\mu(t)-\mu(s))z^{-1/\lambda}} dy dz.$$

Using our expression for $f(z, y|s)$,

$$D_1 = \int_1^{\bar{y}} (\mu(s)/\lambda)^2 z^{-(2+\lambda)/\lambda} e^{-(\mu(t)-\mu(s))z^{-1/\lambda}} B_1(z) dz,$$

where,

$$B_1(z) = \int_1^{z} y^{(1-\lambda)/\lambda} e^{-\mu(s)(z/y)^{-1/\lambda}} dy,$$

and,

$$D_2 = \int_{\bar{y}}^{\bar{z}} (\mu(s)/\lambda)^2 z^{-(2+\lambda)/\lambda} e^{-(\mu(t)-\mu(s))z^{-1/\lambda}} B_2(z) dz,$$

where,

$$B_2(z) = \int_1^{\bar{y}} y^{(1-\lambda)/\lambda} e^{-\mu(s)(z/y)^{-1/\lambda}} dy.$$

Changing the variable of integration to $x = \mu(s)(z/y)^{-1/\lambda}$ we obtain,

$$B_1(z) = (\lambda/\mu(s)) z^{1/\lambda} (e^{-\mu(s)z^{-1/\lambda}} - e^{-\mu(s)}),$$

and,

$$B_2(z) = (\lambda/\mu(s)) z^{1/\lambda} (e^{-\mu(s)z^{-1/\lambda}} - e^{-\mu(s)(z/\bar{y})^{-1/\lambda}}).$$

Solving for $D_1$ under the assumption that $\mu(s)$ is large (so that $e^{-\mu(s)} \approx 0$),

$$D_1 = \int_1^{\bar{y}} (\mu(s)/\lambda) z^{-(1+\lambda)/\lambda} e^{-\mu(t)z^{-1/\lambda}} dz.$$

Changing the variable of integration to, $x = \mu(t)z^{-1/\lambda}$, and solving,

$$D_1 = \frac{\mu(s)}{\mu(t)} (e^{-\mu(t)\bar{y}^{-1/\lambda}} - e^{-\mu(t)}).$$

We can write $D_2 = D_3 + D_4$ where,

$$D_3 = \int_{\bar{y}}^{\bar{z}} (\mu(s)/\lambda) z^{-(1+\lambda)/\lambda} e^{-\mu(t) z^{-1/\lambda}} dz,$$

and,

$$D_4 = -\int_{\bar{y}}^{\bar{z}} (\mu(s)/\lambda) z^{-(1+\lambda)/\lambda} e^{-[\mu(t)+\mu(s)(\bar{y}^{1/\lambda}-1)] z^{-1/\lambda}} dz.$$

The integral $D_3$ can be solved in the same way we solved $D_1$,

$$D_3 = \frac{\mu(s)}{\mu(t)} (e^{-\mu(t)\bar{z}^{-1/\lambda}} - e^{-\mu(t)\bar{y}^{-1/\lambda}}).$$

It is convenient to define,

$$h(\mu(s)) \equiv \mu(t) + \mu(s)(\bar{y}^{1/\lambda} - 1).$$

To solve $D_4$, change the variable of integration to, $x = h(\mu(s))\bar{z}^{-1/\lambda}$,

$$D_4 = -\frac{\mu(s)}{h(\mu(s))} (e^{-h(\mu(s))\bar{z}^{-1/\lambda}} - e^{-h(\mu(s))\bar{y}^{-1/\lambda}}).$$

Using the fact that, $\lim_{\mu(t)\to\infty} e^{-\mu(t)} = 0$,

$$D(s) = D_1 + D_3 + D_4 = \frac{\mu(s)}{\mu(t)} e^{-\mu(t)\bar{z}^{-1/\lambda}} - \frac{\mu(s)}{h(\mu(s))} (e^{-h(\mu(s))\bar{z}^{-1/\lambda}} - e^{-h(\mu(s))\bar{y}^{-1/\lambda}}).$$

Note that $\dot{N}(s)/J = \dot{\mu}(s)/\mu(s)$, thus our equation for the joint distribution of the state of the art and the inventive step, evaluated at $z$ and $y$, is,

$$G(z, y|t) = \int_0^t \frac{\dot{\mu}(s)}{\mu(s)} D(s) ds = e^{-\mu(t) z^{-1/\lambda}} - C_1,$$

where,

$$C_1 = \int_0^t \frac{\dot{\mu}(s)}{\mu(t) + \mu(s)(y^{1/\lambda} - 1)} (e^{-[\mu(t)+\mu(s)(y^{1/\lambda}-1)] z^{-1/\lambda}} - e^{-[\mu(t)+\mu(s)(y^{1/\lambda}-1)] y^{-1/\lambda}}) ds.$$

To derive the marginal cumulative distribution function of the state of the art, we evaluate the joint distribution function at $y = z$ to get $G(z, z|t) = e^{-\mu(t) z^{-1/\lambda}}$ (note that we derived this same distribution using a different approach in Appendix A.4.) To obtain the marginal distribution of the inventive step, we evaluate the joint distribution at an arbitrarily large value of $z$,

$$\lim_{z\to\infty} G(z,y|t) = 1 - C_2 + C_3,$$

where,

$$C_2 = \int_0^t \frac{\dot{\mu}(s)}{\mu(t)+\mu(s)(y^{1/\lambda}-1)} ds,$$

and,

$$C_3 = \int_0^t \frac{\dot{\mu}(s)}{\mu(t)+\mu(s)(y^{1/\lambda}-1)} e^{-[\mu(t)+\mu(s)(y^{1/\lambda}-1)]y^{-1/\lambda}} ds.$$

Integrating out the first expression,

$$C_2 = \ln(y^{1/\lambda})/(y^{1/\lambda}-1).$$

On the other hand, $C_3 \leq e^{-\mu(t)y^{-1/\lambda}} C_2$, thus, $\lim_{\mu(t)\to\infty} C_3 = 0$. Therefore, for large $\mu(t)$,

$$G_2(y|t) = 1 - \ln(y^{1/\lambda})/(y^{1/\lambda}-1).$$

Differentiating this function gives us the density of the inventive step,

$$g_2(y|t) = \left(\frac{y^{1/\lambda - 1}}{(y^{1/\lambda}-1)^2}\ln(y^{1/\lambda}) - \frac{y^{-1}}{y^{1/\lambda}-1}\right)/\lambda.$$

This density has the important property that it does not depend on $t$.

## A.7 The Expected Value of a Newly Discoved Patent

The value of patent with state of the art $z$ and inventive step $y$ is,

$$V(t,z,y) = \int_t^\infty e^{-\int_t^s (\rho + J^{-1}R(v)z^{-1/\lambda})dv} \frac{(1-y^{-1})X(s)}{J} ds.$$

We are assuming that the economy is in a steady state, so, $X(s) = X(t)e^{n(s-t)}$ and $\mu(s) - \mu(t) = (e^{n(s-t)} - 1)\mu(t)$. Applying these steady state results and changing the variable of integration to $x = (e^{n(s-t)} - 1)\mu(t)z^{-1/\lambda}$, we get,

$$V(t,z,y) = \frac{(1-y^{-1})X(t)}{Jn\mu(t)z^{-1/\lambda}} \int_0^\infty e^{-x}(\frac{x}{\mu(t)z^{-1/\lambda}} + 1)^{-\rho/n} dx.$$

We are actually interested in the expected value of a new invention, where we integrate over the joint density of the state of the art and the inventive step for new patents,

$$V(t) = E[V(t,Z,Y)] = \int_1^\infty \int_1^z V(t,z,y) f(z,y|t) dy dz,$$

where, from appendix A.5,

$$f(z,y|t) = y^{-1} g((z/y), y|t) = (\mu(t)/\lambda)^2 y^{(1-\lambda)/\lambda} z^{-(2+\lambda)/\lambda} e^{-\mu(t)(z/y)^{-1/\lambda}}.$$

Combining these results and changing a variable of integration from $z$ to $w = \mu(t)z^{-1/\lambda}$ we get,

$$V(t) = \frac{X(t)}{Jn\lambda} \int_0^{\mu(t)} \int_0^\infty e^{-x}((x/w)+1)^{-\rho/n} dx \int_1^{(\mu(t)/w)^\lambda} (1-y^{-1}) y^{(1-\lambda)/\lambda} e^{-wy^{1/\lambda}} dy dw.$$

By changing the variable of integration to $u = wy^{1/\lambda}$ we simplify the integral over $y$,

$$\int_1^{(\mu(t)/w)^\lambda} (1-y^{-1}) y^{(1-\lambda)/\lambda} e^{-wy^{1/\lambda}} dy = (\lambda/w)e^{-w} - \lambda w^{\lambda-1}\Gamma(1-\lambda, w),$$

where, $\Gamma(a,x) \equiv \int_x^\infty e^{-s} s^{a-1} ds$ is the incomplete gamma function. By changing the variable of integration to $u = x + w$ we simplify the integral over $x$,

$$\int_0^\infty e^{-x}((x/w)+1)^{-\rho/n} dx = w^{\rho/n} e^w \Gamma(1-\rho/n, w).$$

Combining these results, and letting $\mu(t)$ tend to infinity,

$$V(t) = \frac{X(t)}{Jn\lambda} \int_0^\infty w^{\rho/n-1}\Gamma(1-\rho/n, w) dw - \frac{X(t)}{Jn\lambda} \int_0^\infty w^{\rho/n+\lambda-1} e^w \Gamma(1-\rho/n, w)\Gamma(1-\lambda, w) dw.$$

Using the identity, $\int_0^\infty t^{a-1}\Gamma(b,t)dt = \Gamma(a+b)/a$, where $\Gamma(a)$ is the complete gamma function, the first integral reduces to $\frac{X(t)}{J\rho}$. Thus, we have the desired result,

$$V(t) = \frac{X(t)}{J\rho}[1 - \phi(\rho/n, \lambda)],$$

where,

$$\phi(\rho/n, \lambda) = \rho/n \int_0^\infty w^{\rho/n+\lambda-1} e^w \Gamma(1-\rho/n, w)\Gamma(1-\lambda, w)dw.$$

## A.8 The Drop-Out Rate from a Cohort of Patents

We want an expression for,

$$\Pr(L \leq x|t) = \int_1^\infty \Pr(L \leq x|z,t) f_1(z|t)dz,$$

where,

$$\Pr(L \leq x|z,t) = 1 - e^{-z^{-1/\lambda}(\mu(t+x)-\mu(t))},$$

and $f_1(z|t)$ is the marginal density of the state of the art for patents invented at time $t$. We can integrate over the joint density we derived in appendix A.5 to obtain,

$$f_1(z|t) = \int_1^z f(z,y|t)dy = (\mu(t)/\lambda)z^{-(1+\lambda)/\lambda} e^{-\mu(t)z^{-1/\lambda}}.$$

Combining these results,

$$\Pr(L \leq x|t) = 1 - \int_1^\infty (\mu(t)/\lambda)z^{-(1+\lambda)/\lambda} e^{-\mu(t+x)z^{-1/\lambda}} dz.$$

Changing the variable of integration to $w = \mu(t+x)z^{-1/\lambda}$, and letting $e^{\mu(t+x)} \rightarrow 0$, we get the desired result.

## A.9 The Distribution of $Z$ for Surviving Patents, by Cohort

From appendix A.6, $D(s)$ is the fraction of patents from cohort $s$ which are still profitable in $t$ and which have state of the art below $z$ and inventive step below $y$. We were able to show that,

$$D(s) = \frac{\mu(s)}{\mu(t)} e^{-\mu(t)z^{-1/\lambda}} - \frac{\mu(s)}{h(\mu(s))}(e^{-h(\mu(s))z^{-1/\lambda}} - e^{-h(\mu(s))y^{-1/\lambda}}),$$

where $h(\mu(s)) \equiv \mu(t) + \mu(s)(y^{1/\lambda} - 1)$. We want to ignore the distribution of the inventive step so we evaluate $D(s)$ at $y = z$ i.e. at the maximum value of $y$,

$$D(s)|_{y=z} = \frac{\mu(s)}{\mu(t)} e^{-\mu(t)z^{-1/\lambda}}.$$

Dividing by the unconditional probability, $\mu(s)/\mu(t)$, that a patent invented in $s$ will be alive in $t$ we obtain the distribution function for the state of the art of patents invented in $s$ which are profitable in $t$,

$$e^{-\mu(t)z^{-1/\lambda}},$$

which has the interesting property that it does not depend on $s$. In fact, the corresponding density is $g_1(z|t)$, which is derived by a different route in appendix A.4.

## A.10 The Social Planner's Problem

I assume that the social planner's objective is to maximize the utility of the representative individual weighted by the size of the population:

$$\int_t^{\infty} e^{-\rho(s-t)} U(s) e^{ns} ds],$$

where,

$$U(s)e^{ns} = \exp\{J^{-1} \int_0^J \ln[Z(j,t)C(j,t)]dj\}.$$

Implicit in the utility function is the result that the social planner will only have state of the art goods produced. The planner's control variable is $\alpha = R(t)e^{-nt}$, i.e. the fraction of the labor force devoted to research.

The planner cannot influence the underlying research technology, thus the levels version of equation (3), which is derived in appendix A.4, gives the (geometric) mean state of the art conditional on the stock of past research. The planner can determine quantities produced and consumed. She will maximize utility by having equal quantities produced of each variety. Given the quantity of production labor determined by a choice of $\alpha$, this implies, $C(j,t) = (1-\alpha)e^{nt}/(Jc)$. Combining these results, we have, $U(t)e^{nt} = e^{\lambda\psi}\mu(t)^{\lambda}(1-\alpha)e^{nt}/(Jc)$. The state variable is the stock of research, whose law of motion is, $\dot{\mu}(t) = R(t)/J = \alpha e^{nt}/J$.

The current value Hamiltonian is, $H(t) = U(t)e^{nt} + \Lambda(t)\dot{\mu}(t)$, where $\Lambda(t)$ is the shadow value of research capital. The first order conditions are, $\Lambda(t) = e^{\lambda\psi}\mu(t)^{\lambda}/c$ and $\lambda e^{\lambda\psi}\mu(t)^{\lambda-1}(1-\alpha)e^{nt}/(Jc) = \rho\Lambda(t) - \dot{\Lambda}(t)$. The transversality condition requires that $\lim_{s\to\infty} e^{-\rho s}\Lambda(s)\mu(s) = 0$. Combining the first order conditions and using the fact that in steady state, $\mu(t) = \alpha e^{nt}/(Jn)$, we get the result that, $\alpha^{*} = \lambda n/\rho$. The transversality condition is satisfied if $\rho > (1+\lambda)n$.

Table 1: Simulation Results

| $\lambda$ | $\theta(\lambda)$ | $\rho/n = 1.5$ $\phi(\rho/n,\lambda)$ | $\alpha$ | $\alpha^*$ | $\rho/n = 2.0$ $\phi(\rho/n,\lambda)$ | $\alpha$ | $\alpha^*$ | $\rho/n = 3.0$ $\phi(\rho/n,\lambda)$ | $\alpha$ | $\alpha^*$ |
|---|---|---|---|---|---|---|---|---|---|---|
| .05 | .923 | .934 | .046 | .033 | .939 | .032 | .025 | .944 | .020 | .017 |
| .10 | .857 | .876 | .088 | .067 | .885 | .063 | .050 | .893 | .040 | .033 |
| .36 | .616 | .655 | .272 | .240 | .676 | .208 | .180 | .696 | .141 | .120 |
| 1.0 | .355 | .397 | .531 | | .420 | .450 | | .446 | .342 | .333 |

*Notes:* The numerical calculations are performed using Mathematica.
Remember that $\alpha$ is the equilibrium fraction of the workforce devoted to research, while $\alpha^*$ is the social planner's choice.
The transversality condition for the social planner's problem is violated when $\lambda = 1$ and $\rho/n$ equals either 1.5 or 2.

Table 2: Data Used in Calibration Exercise

| | level in 1957 | level in 1989 | growth rate 1957-1989 |
|---|---|---|---|
| TFP, Manufacturing | | | 2.0 |
| TFP, Private Business | | | 1.3 |
| Output per Hour, Private Business | | | 2.0 |
| Industry R&D Scientists & Engineers (000's) | 229.4 | 720.2 | 3.6 |
| Civilian Employment (000's, over age 16) | 64,071 | 117,342 | 1.9 |
| Total Industry R&D Expenditure ($ billions) | 7.7 | 101.9 | |
| Compensation of Employees ($ billions) | 256.5 | 3,100 | |
| Successful US Priority Patent Applications (000's) | 39.2 | 58.5 | 1.3 |

*Sources:* Productivity measures are from the Bureau of Labor Statistics.
S&E's are from the National Science Foundation (1987 and 1990).
R&D expenditure is from the National Science Foundation (1990).
Employment and Compensation are from,
*Economic Report of the President* (1982 and 1993).
Patents are from Kortum (1993a).

# Figure 1:

Research, Patenting, and Productivity