AN EXACT SOLUTION FOR
THE INVESTMENT AND MARKET
VALUE OF A FIRM FACING
UNCERTAINTY, ADJUSTMENT COSTS,
AND IRREVERSIBILITY

Andrew B. Abel

Janice C. Eberly

AN EXACT SOLUTION FOR
THE INVESTMENT AND MARKET
VALUE OF A FIRM FACING
UNCERTAINTY, ADJUSTMENT COSTS,
AND IRREVERSIBILITY

ABSTRACT

This paper derives closed-form solutions for the investment and market value,

under uncertainty, of competitive firms with constant returns to scale

production and convex costs of adjustment. Solutions are derived for the case

of irreversible investment as well as for reversible investment. Optimal

investment is a non-decreasing function of q, the shadow value of capital. The

conditions of optimality imply that q cannot contain a bubble; thus, optimal

investment depends only on fundamentals. However, the value of the firm may

contain a bubble that does not affect investment behavior. Relative to the case

of reversible investment, the introduction of irreversibility does not affect q, but

it reduces the fundamental market value of the firm.

Andrew B. Abel   Janice C. Eberly
Department of Finance   Department of Finance
The Wharton School   The Wharton School
University of Pennsylvania   University of Pennsylvania
3620 Locust Walk   3620 Locust Walk
Philadelphia, PA 19104-6367   Philadelphia, PA 19104-6367
and NBER

I Introduction

Most theoretical analyses of capital investment decisions by firms under uncertainty have focused either on irreversibility of investment or on convex costs of adjustment.[1] Recently, Abel and Eberly (1993) have shown that an appropriately specified investment cost function can incorporate convex costs of adjustment as well as irreversibility. In this framework, investment is a nondecreasing function of the shadow price of capital, denoted as q. In the irreversible investment case, investment is a strictly increasing function of q for values of q above a certain threshold value; for values of q below this threshold value, investment equals zero, and negative investment is never optimal.

In this paper we present a parametric example of a firm facing convex costs of adjustment and irreversibility, and we provide closed-form solutions for the investment and market value of the firm. To our knowledge, the existing literature does not contain any closed-form solutions to problems of this type. Specifically, we examine a continuous-time stochastic model of an infinite-horizon, risk-neutral, competitive firm with a constant returns to scale production function. In this case, the value of the firm is a linear function of the firm's capital stock. The slope, q, of the value function with respect to capital is the shadow price of capital which governs investment decisions. The constant term in the value function is the expected present value of rents to the adjustment technology.

We proceed by first analyzing the investment and market value of a competitive firm that faces convex adjustment costs and has the possibility of undertaking negative gross investment. Our motivation for starting with the case of reversible investment is based on substantive as well as expositional considerations. First, the model of reversible investment and market value that we analyze is richer than existing models that have yielded closed- form solutions. Specifically, the models in Abel (1983, 1985) specify convex costs of adjustment but do not include a cost of purchasing capital

---

[1]Eisner and Strotz (1963), Lucas (1967), Gould (1968), and Treadway (1969) examined investment under costs of adjustment. Mussa (1977) and Hayashi (1982) discussed the role of adjustment costs in Tobin's (1969) q theory of investment under certainty, and Abel (1983, 1985) discussed this role under uncertainty. Investment under an irreversibility constraint was introduced by Arrow (1968) and was later studied by Bernanke (1983), McDonald and Siegel (1986), Bertola (1987), Dixit (1989, 1991), and Pindyck (1988). See Pindyck (1991) for a review of the irreversibility literature, and Dixit and Pindyck (1992) for an extended instructive treatment. In addition, Lucas and Prescott (1971) examined investment under uncertainty with both costs of adjustment and irreversibility, though irreversibility was not a focus of their paper. Indeed, they did not even comment on the assumption of irreversibility in their model.

goods. By not including a cost of purchasing capital and by specifying the marginal adjustment cost to be zero at zero investment, those models are set up so that a positive rate of investment is always optimal. However, once we include the realistic assumption that there is a positive purchase price of capital, there will be situations in which it is optimal for the rate of investment to be zero or negative. Caballero (1991) specifies the cost of investment to include a positive purchase price of capital as well as convex costs of adjustment, but does not provide a closed-form solution for investment and the market value of the firm as we do.

Second, in both the reversible and irreversible investment cases, investment can be expressed as a non-decreasing of function of q, the shadow price of capital. Technically, q is derived as the solution to a differential equation. The particular solution to the differential equation is the expected present value of marginal products, but the general solution also contains "bubbles" which are unrelated to the fundamentals. Previous analyses of closed-form solutions for firms facing convex costs of adjustment have ignored this component of the solution. We show that, in fact, the conditions of optimality require these "bubble terms" to be absent from the solution for q. However, the value of the firm may contain bubbles, as we explain.

Third, much of the analytic apparatus needed for the case of irreversible investment is the same as for the case of reversible investment. Because the case of reversible investment is much less complicated, it provides a useful opportunity for presenting the model, its manipulation, and its basic results.

Fourth, the case of reversible investment provides a benchmark against which to compare investment and market value in the presence of irreversibility. We will show that, in our parametric example, the value of q is unaffected by the presence or absence of irreversibility. For values of q high enough to lead to positive investment in the reversible case, the optimal rate of investment is unaffected by the presence of irreversibility. For values of q low enough to lead to negative investment in the reversible case, optimal investment equals zero in the irreversible case. Although irreversibility does not affect the value of q, it does reduce the market value of the firm.

Section II presents the optimization problem of the competitive firm. The optimal rate of investment and the value of the firm in the case of reversible investment are derived in section III. Irreversibility is introduced and analyzed in section IV. Concluding remarks are presented in section V.

II   The Optimization Problem of the Competitive Firm

II.1   The price process

We consider a continuous-time model of a competitive firm that sells its output at time t at an exogenously given price p_t. The price p_t evolves according to geometric Brownian motion

$$\frac{dp_t}{p_t} = \mu\, dt + \sigma\, dz_t \,, \quad p_0 > 0 \tag{1}$$

where μ is the instantaneous drift, σ is the instantaneous standard deviation, and dz_t is an increment to a standard Wiener process.

Later in our analysis it will be convenient to have expressions for the expected present value of p^λ for various λ. Under the geometric Brownian motion assumption in equation (1), $E_t\{p_{t+s}^\lambda\}$ grows at a constant rate $\lambda\mu + \frac{1}{2}\sigma^2\lambda(\lambda-1)$ as s increases for a given t. Thus, the present value of $E_t\{p_{t+s}^\lambda\}$ discounted to time t at the rate R is

$$e^{-Rs}E_t\{p_{t+s}^\lambda\} = p_t^\lambda\, e^{-Rs}\, e^{[\lambda\mu+(1/2)\sigma^2\lambda(\lambda-1)]s} = p_t^\lambda\, e^{-f(\lambda;R)s} \,, \tag{2}$$

where

$$f(\lambda; R) \equiv R - \lambda\mu - \tfrac{1}{2}\sigma^2\lambda(\lambda - 1) \tag{3}$$

is the growth-rate-adjusted discount rate, equal to the discount rate R minus the expected growth rate of p^λ, $\lambda\mu + \frac{1}{2}\sigma^2\lambda(\lambda - 1)$. The growth-rate-adjusted discount rate, $f(\lambda; R)$ is a (concave) quadratic function of λ. Notice that when $R > 0$, the equation $f(\lambda; R) = 0$ has two distinct roots, one positive and one negative.

Define $PV_t[p^\lambda; R]$ to be the present value (discounted at rate R) of p^λ from time t onward. Formally, we have

$$PV_t[p^\lambda; R] \equiv E_t\left\{\int_0^\infty p_{t+s}^\lambda\, e^{-Rs} ds\right\} = p_t^\lambda E_t\left\{\int_0^\infty e^{-f(\lambda;R)s} ds\right\} = \frac{p_t^\lambda}{f(\lambda; R)} \,, \tag{4}$$

where the second equality in equation (4) follows from equations (2) and (3).

II.2   The Operating Profit and Investment Cost Functions

The firm uses capital, K_t, and labor, L_t, to produce output, Y_t, according to a Cobb-Douglas production function $Y_t = L_t^\alpha K_t^{1-\alpha}$, where the labor share α satisfies $0 < \alpha < 1$. The firm pays a fixed wage w so that its operating profit at time t, which equals revenue minus wages, is $p_t L_t^\alpha K_t^{1-\alpha} - wL_t$. Because labor can be costlessly and instantaneously adjusted, the firm chooses L_t

3

to maximize the instantaneous operating profit at time t. The resulting maximized instantaneous operating profit, $\pi(K_t, p_t)$, is

$$\pi(K_t, p_t) \equiv h p_t^\theta K_t , \tag{5}$$

where

$$\theta \equiv \frac{1}{(1-\alpha)} > 1$$

and

$$h \equiv \theta^{-\theta}(\theta-1)^{\theta-1} w^{1-\theta} > 0 .$$

Notice that $h p_t^\theta$ is the marginal revenue product of capital at time t.

The firm's capital stock increases as a result of gross investment I_t, and decreases as a result of depreciation at a constant rate $\delta \geq 0$. Thus, the change in the capital stock is

$$dK_t = (I_t - \delta K_t)dt . \tag{6}$$

Let $c(I_t)$ denote the total cost of investing at rate I_t and assume that $c(I_t)$ is strictly convex.

II.3   The Bellman Equation

We assume that the firm is risk-neutral and maximizes the expected present value of its cash flow. Let $r > 0$ be the constant rate of discount. Let $V(K_t, p_t)$ denote the expected present value of cash flow from time t onward. Thus $V(K_t, p_t)$ is the value of the firm's objective function at time t. Formally, we have

$$V(K_t, p_t) = \max_{\{I_{t+s}\}} E_t \left\{ \int_0^\infty [h p_{t+s}^\theta K_{t+s} - c(I_{t+s})] e^{-rs} ds \right\} . \tag{7}$$

The fundamental value of the firm at time t is $V(K_t, p_t)$.

The fundamental value of the firm satisfies the following Bellman equation (from this point on, we will suppress time subscripts unless they are needed for clarity):

$$rV(K, p) = \max_I \left[h p^\theta K - c(I) + \frac{E\{dV\}}{dt} \right] . \tag{8}$$

The right hand side of equation (8) contains the two components of the expected return on the firm over a short interval of time: the instantaneous net cash flow, $h p^\theta K - c(I)$, and the expected capital gain, $E\{dV\}/dt$. Equation (8) requires that the sum of these components equals the required return $rV(K, p)$.

The expected capital gain is calculated using Ito's Lemma and equations (6) and (1), which describe the evolution of K and p, to obtain

$$\frac{E\{dV\}}{dt} = (I - \delta K)V_K + \mu p V_p + \tfrac{1}{2}\sigma^2 p^2 V_{pp} . \tag{9}$$

4

It will turn out that investment depends on V_K, the marginal valuation of a unit of installed capital. Anticipating this result and emphasizing the relation to the q theory of investment, we define $q \equiv V_K$. Notice that q is the shadow value of installed capital, and is non-negative. Substituting q for V_K in equation (9), and then substituting equation (9) into equation (8) yields

$$rV(K,p) = \max_I \left[hp^\theta K - c(I) + (I - \delta K)q + \mu p V_p + \tfrac{1}{2}\sigma^2 p^2 V_{pp} \right] . \quad (10)$$

We can re-write equation (10) by "maximizing out" the rate of investment to obtain

$$rV(K,p) = hp^\theta K + \phi - \delta K q + \mu p V_p + \tfrac{1}{2}\sigma^2 p^2 V_{pp} , \quad (11)$$

where

$$\phi \equiv \max_I \left[Iq - c(I) \right] . \quad (12)$$

Note that ϕ is the maximized value of rents accruing to the investment technology from undertaking investment at rate I. When the firm invests at rate I over an interval dt of time, it acquires $I dt$ units of capital. Because q is the shadow price of this capital, the firm acquires capital worth $qI dt$, but pays $c(I)dt$ to increase its capital stock by $I dt$. Thus, $qI - c(I)$ gives the value of the rents accruing per unit time to the firm for undertaking investment at rate I.

III   Reversible Investment

In this section we focus on the case of reversible investment. We begin in section III.1 by specifying an investment cost function for which the optimial level of investment can be negative. Also in section III.1, we specify the optimal rate of investment as a function of q. After deriving the differential equation describing the fundamental value of the firm in section III.2 we then obtain q as a function of p in section III.3. The value of the adjustment technology is derived in section III.4.

III.1   The Investment Cost Function and the Optimal Rate of Investment

Specify the total cost of investing at time t, $c(I_t)$ as

$$c(I_t) = bI_t + \gamma I_t^{n/(n-1)} , \quad (13)$$

where $b \geq 0$, $\gamma > 0$, and $n \in \{2, 4, 6, \ldots\}$. The cost of undertaking investment, $c(I_t)$, has two components: (1) bI_t is the cost of purchasing new capital at a fixed price of b per unit; for negative gross investment, $bI_t \leq 0$

5

and represents the proceeds to the firm of selling capital at a price of b per unit. (2) $\gamma I_t^{n/(n-1)}$ is a convex cost of adjustment. Notice that when $n = 2$, the cost of adjustment is γI_t^2 which is quadratic. The assumption that n is an even positive integer insures that the adjustment cost function $\gamma I_t^{n/(n-1)}$ is convex for negative I as well as for positive I. To insure a finite fundamental value of the firm, we assume that $f(n\theta; r) > 0$.

Using the parametric specification of the cost function in (13), we can obtain closed-form solutions for investment and the value of the firm. Using the investment cost function in equation (13) we rewrite equation (12) as

$$\phi = \max_I \big[(q - b)I - \gamma I^{n/(n-1)}\big] \ . \tag{14}$$

The optimal rate of investment is determined by differentiating the term in brackets on the right hand side of equation (14) with respect to I, and setting the derivative equal to zero to obtain

$$I = \left[\frac{n-1}{n\gamma}\right]^{n-1} (q - b)^{n-1} \ . \tag{15}$$

Equation (15) indicates that investment is an increasing function of q. When the shadow price of capital q is greater than the purchase price of capital b, gross investment is positive. When the shadow price q is less than the sale price of capital b, the firm sells capital, and gross investment I is negative.[2]

To determine the value of ϕ, substitute equation (15) into equation (14) to obtain

$$\phi = (q - b)^n \Gamma \tag{16}$$

where

$$\Gamma \equiv (n - 1)^{n-1} n^{-n} \gamma^{(1-n)} > 0 \ .$$

Notice that the maximized value of the rents accruing to the investment technology, ϕ, is positive whenever $q \neq b$. When $q < b$, the firm chooses a negative rate of gross investment and earns rents by selling capital that is worth less to the firm than the price b.

III.2   The Fundamental Value of the Firm

We have derived the optimal rate of investment as a function of q, the marginal value of installed capital. Our next step is to determine q as a function of the price of output p, and then to determine the fundamental value of the firm $V(K, p)$. We proceed by substituting the expression for ϕ from equation (16) into the differential equation (11) to obtain

$$rV(K,p) = hp^\theta K + (q - b)^n \Gamma - \delta K q + \mu p V_p + \tfrac{1}{2}\sigma^2 p^2 V_{pp} \ . \tag{17}$$

---

[2]Recall that with n even, $n - 1$ is odd so that $(q - b)^{n-1}$ has the same sign as $q - b$ and is an increasing function of $q - b$.

6

We will solve this differential equation by hypothesizing that the solution is a linear function of the capital stock. Thus,

$$V(K, p) = q(p)K + G(p) , \qquad (18)$$

where $q(p)$ and $G(p)$ are functions to be determined. To determine these functions, substitute equation (18) into equation (17) to obtain

$$rqK + rG = hp^\theta K + (q - b)^n \Gamma - \delta Kq + \mu p q_p K + \mu p G_p \qquad (19)$$
$$+ \tfrac{1}{2}\sigma^2 p^2 q_{pp} K + \tfrac{1}{2}\sigma^2 p^2 G_{pp} .$$

This differential equation must hold for all values of K. Therefore, the terms multiplying K on the left hand side must equal the terms multiplying K on the right hand side. In addition, the terms not involving K on the left hand side must equal the terms not involving K on the right hand side. These equalities yield

$$rq = hp^\theta - \delta q + \mu p q_p + \tfrac{1}{2}\sigma^2 p^2 q_{pp} , \qquad (20)$$

and

$$rG = (q - b)^n \Gamma + \mu p G_p + \tfrac{1}{2}\sigma^2 p^2 G_{pp} . \qquad (21)$$

There is a recursive structure to these equations. The differential equation for $q(p)$ in equation (20) does not depend on $G(p)$, but the differential equation for $G(p)$ in equation (21) depends on $q(p)$. Thus, we will solve equation (20) for $q(p)$ and then proceed to solve equation (21) for $G(p)$.

III.3   The Marginal Value of Installed Capital, q

The marginal value of installed capital is obtained by solving the differential equation (20). It can be easily verified by direct substitution that a general solution to this differential equation is

$$q(p) = Bp^\theta + A_1 p^{\eta_1} + A_2 p^{\eta_2} , \qquad (22)$$

where

$$B \equiv \frac{h}{f(\theta; r + \delta)} > 0$$

and $\eta_1 > \eta_2$ are the roots of the quadratic equation $f(\eta; r + \delta) = 0$.

The particular solution in equation (22), Bp^θ, equals the expected present value of marginal revenue products of capital, hp^θ, accruing to the undepreciated portion of a unit of currently installed capital.[3]

The terms $A_1 p^{\eta_1}$ and $A_2 p^{\eta_2}$ are solutions to the homogeneous part of the differential equation (20). These solutions involve the roots of the

---

[3]The expected present value of marginal revenue products of capital, hp^θ, accruing to the undepreciated portion of a unit of currently installed

quadratic equation $f(\eta; r + \delta) = 0$. Recall that $f(\eta; r + \delta)$ is a strictly concave function of η. Note that $f(0; r+\delta) = r+\delta > 0$ and, by assumption, $f(n\theta; r + \delta) = f(n\theta; r) + \delta > 0$. Therefore, the concavity of $f(\eta; r + \delta)$ implies that $f(\eta; r + \delta) > 0$ for all η in $[0, n\theta]$. Therefore, one root of $f(\eta; r + \delta) = 0$ is negative and the other root is greater than $n\theta$. Recalling that $\eta_1 > \eta_2$, we have $\eta_1 > n\theta \geq 2\theta > 0 > \eta_2$. Notice that the expected growth rates of $A_1 p^{\eta_1}$ and $A_2 p^{\eta_2}$ are both equal to $r + \delta$. We refer to these terms as bubbles because they are unrelated to the underlying fundamentals (marginal revenue products).

The next task is to determine the values of the coefficients A_1 and A_2. The shadow price of a unit of installed capital, q, is non-negative. Therefore, as we now show, neither A_1 nor A_2 can be negative. Notice that p^{η_1} dominates both p^θ and p^{η_2} as p becomes arbitrarily large. Thus, if A_1 were negative, q would be negative for sufficiently large p which contradicts the fact that q is non- negative. Thus A_1 is not negative. Similarly, p^{η_2} dominates both p^θ and p^{η_1} as p approaches zero. Thus, if A_2 were negative, q would be negative for p sufficiently close to zero which contradicts the fact that q is non-negative. Thus A_2 is not negative.

We have argued that neither A_1 nor A_2 is negative. In Appendix B we prove that a positive bubble in the value of q (corresponding to a positive A_1 or a positive A_2) is inconsistent with the solution of the firm's maximization problem in equation (7). Since A_1 and A_2 can be neither positive nor negative they must equal 0. Thus, the marginal value of capital in equation (22) simplifies to

$$q(p) = Bp^\theta . \tag{23}$$

Notice that the expression for q does not involve any of the parameters of the adjustment cost function. The value of q is simply the present value of expected marginal revenue products, and for a competitive firm with constant returns to scale, the marginal revenue product of capital is exogenous. Because the path of marginal revenue products does not depend on the firm's investment, q is independent of the specification of the adjustment cost function. Also note that because p evolves according to geometric Brownian motion with $p_0 > 0$, p^θ cannot be negative. Thus, as mentioned earlier, q cannot be negative.

Observe from the definition of B in equation (22) that B is an increasing

capital is

$$\int_0^\infty h p_{t+s}^\theta \, e^{-(r+\delta)s} ds = \frac{h p_t^\theta}{f(\theta; r + \delta)}$$

where the equality follows directly from equation (4). Note that $B \equiv h/f(\theta; r + \delta)$ is positive because $h > 0$ and $f(\theta; r + \delta) > 0$. To verify that $f(\theta; r + \delta) > 0$, observe that (1) $f(0; r + \delta) = r + \delta > 0$; (2) $f(n\theta; r + \delta) \geq f(n\theta; r) > 0$ by assumption; (3) $0 < \theta < n\theta$; and (4) $f(\lambda; r + \delta)$ is concave in λ.

function of σ^2 because $\theta > 1$. Thus q is an increasing function of the instantaneous variance σ^2. Because investment is an increasing function of q, investment is an increasing function of σ^2 for a given value of the output price p. This result is the same as in Hartman (1972), Abel (1983), and Caballero (1991).

III.4 The Value of the Adjustment Technology

The intercept term $G(p)$ in the fundamental value of the firm, $V(K,p) = q(p)K + G(p)$, is the present value of the rents accruing to the adjustment technology. The function $G(p)$ is determined by the differential equation (21). Let $GP(p)$ denote a particular solution to equation (21). It can be verified by direct substitution that the following expression is a particular solution to equation (21)

$$GP(p) = \sum_{j=0}^{n} \frac{\Gamma n!}{j!(n-j)!f(j\theta;r)} B^j p^{\theta j}(-b)^{n-j} . \tag{24}$$

Notice that in the special case in which $b = 0$, as in Abel (1983), we have $GP(p) = \Gamma q^n/f(n\theta;r)$ which is equivalent to the intercept of the linear value function in equation (11a) of Abel (1983).

To interpret the particular solution in equation (24) more generally, apply equation (4) with $\lambda = j\theta$ and $R = r$ to obtain

$$PV_t[p^{j\theta};r] = \frac{p_t^{j\theta}}{f(j\theta;r)} . \tag{25}$$

Now multiply both sides of equation (25) by B^j, use equation (23) to obtain $q^j = B^j p^{j\theta}$, and use the fact that $PV_t[\,\cdot\,;r]$ is a linear operator to obtain

$$PV_t[q^j;r] = \frac{B^j p_t^{j\theta}}{f(j\theta;r)} . \tag{26}$$

Substituting equation (26) into equation (24) and again using the fact that $PV_t[\,\cdot\,;r]$ is a linear operator yields

$$GP(p) = \Gamma PV_t\left[\sum_{j=0}^{n} \frac{n!}{j!(n-j)!} q^j(-b)^{n-j};r\right] . \tag{27}$$

Recognizing that the summation on the right hand side of equation (27) is a binomial expansion of $(q-b)^n$, we obtain

$$GP(p) = \Gamma PV_t[(q-b)^n;r] . \tag{28}$$

Thus, the particular solution is the expected present value of the instantaneous rents $(q-b)^n$ accruing to the adjustment technology.

9

We can obtain a general solution to equation (21) by adding the particular solution in equation (24) and the solution to the homogeneous part of the differential equation. The solution to the homogeneous part of equation (21) is $C_1 p^{\omega_1} + C_2 p^{\omega_2}$, where $\omega_1 > \omega_2$ are the roots of the quadratic equation $f(\omega; r) = 0$. Observe that $f(0; r) = r > 0$, $f(n\theta; r) > 0$, and $f(\omega; r)$ is concave in ω, so that $f(\omega; r) > 0$ for ω in $[0, n\theta]$. Therefore, the roots of $f(\omega; r) = 0$ satisfy $\omega_1 > n\theta \geq 2\theta > 0 > \omega_2$.

Notice that the expected growth rates of $C_1 p^{\omega_1}$ and $C_2 p^{\omega_2}$ are both equal to r. These terms are bubbles in the sense that they are unrelated to the underlying fundamentals (cash flows) of the firm. Although the optimality conditions associated with the maximization problem in equation (7) rule out bubbles in q, they do not rule out bubbles in the market value of the firm. Thus, the market value of the firm can differ from the fundamental value by the bubble terms $C_1 p^{\omega_1}$ and $C_2 p^{\omega_2}$. Imposing the condition that the market value of the firm is non-negative for all $p > 0$ and all $K \geq 0$ rules outs negative values of C_1 and C_2. However, we are still left with the possibility of positive C_1 and/or positive C_2, which correspond to positive bubbles in the market value of the firm.

If we define the value of the firm to be the present value of expected cash flows as in equation (7), then we have ruled out bubbles by assumption/definition. This is to be contrasted with the finding that a bubble on q is ruled out by optimality.

IV   The Case of Irreversible Investment

Now we consider the case of irreversible investment. Rather than simply assume that it is impossible for gross investment to be negative, we modify the investment cost function $c(I)$ for negative values of gross investment so that it is never optimal for the firm to undertake negative gross investment. In this case, optimal behavior is observationally equivalent to a case in which investment is physically irreversible. Our goal in this section is to understand the impact of irreversibility on the firm's value and on its investment decisions. Fortunately, much of the mathematics of this case is the same as for the case of reversible investment discussed in earlier sections. We take advantage of this overlap to abbreviate the derivations in this section and to focus on aspects that are specific to irreversibility. We can also use the results of earlier sections as a benchmark for comparison to understand the impact of irreversibility on the firm's decisions and its value.

IV.1 The Modified Investment Cost Function and the Optimal Rate of Investment

The only modification that we will make to the firm's decision problem is to change the investment cost function to

$$c(I_t) = \begin{cases} bI_t + \gamma I_t^{n/(n-1)} \,, & \text{for } I_t \geq 0 \\ g(I_t) > 0 \,, & \text{for } I_t < 0 \end{cases} \tag{29}$$

where we continue to assume that $b \geq 0$, $\gamma > 0$, and n is an even positive integer. Note that for non-negative values of I_t, the investment cost function in equation (29) is identical to that in equation (13). Again, we assume that $f(n\theta; r) > 0$ to insure a finite fundamental value of the firm. For all negative values of gross investment, we now assume that the cost of investment $c(I_t)$ is positive.

Even with this modification to the investment cost function, the fundamental value of the firm $V(K, p)$ must still satisfy the Bellman equation in equation (10). However, using the investment cost function in equation (29) changes the optimal rate of investment and the maximized value of rents $qI - c(I)$. Notice that with the investment cost function in equation (29), the rents $qI - c(I)$ are negative for any $I < 0$ (since $c(I) > 0$ for $I < 0$ and $q > 0$). Because the firm can always attain a value of zero for $qI - c(I)$ by setting I equal to zero, the firm will never choose a negative rate of investment. Thus equation (14) can be modified as

$$\phi = \max_{I \geq 0} \left[(q - b)I - \gamma I^{n/(n-1)} \right] . \tag{30}$$

Now observe that if $q < b$, the maximand in brackets on the right hand side of equation (30) is negative for any positive value of I. Therefore, if $q < b$, the firm can maximize its rents by setting $I = 0$. In this case, the optimal rate of investment is zero, and $\phi = 0$.

For values of $q \geq b$, the maximization problem in equation (29) is identical to that in the case of reversible investment in equation (14); in this case the optimal rate of investment is given by equation (15) and ϕ is given by equation (16). We summarize these findings for the case of irreversible investment as

$$I = \max \left\{ 0, \left[\frac{n-1}{n\gamma} \right]^{n-1} (q - b)^{n-1} \right\} , \tag{31}$$

and

$$\phi = (\max[0, q - b])^n \Gamma . \tag{32}$$

IV.2 The Fundamental Value of the Firm

Equation (31) gives the optimal rate of investment as a function of the shadow price of capital q. The next task, as in the case of reversible investment in section III, is to determine q as a function of the price of output

11

p. We proceed by substituting the expression for ϕ from equation (32) into the Bellman equation (11) to obtain

$$rV(K,p) = hp^{\theta}K + (\max[0, q-b])^n\Gamma - \delta Kq + \mu pV_p + \tfrac{1}{2}\sigma^2 p^2 V_{pp} \ . \quad (33)$$

We will solve this differential equation by hypothesizing that $V(K,p)$ is a linear function of the capital stock, and that there are two regimes: regime H applies for values of q greater than or equal to b; regime L applies for values of q less than b. Thus,

$$V^{(i)}(K,p) = q^{(i)}(p)K + G^{(i)}(p)\ , \qquad \begin{array}{ll} i = L\ , & \text{for } q < b \\ i = H\ , & \text{for } q \geq b \end{array} \quad (34)$$

where $q^{(i)}(p)$ and $G^{(i)}(p)$ are functions to be determined. To determine these functions, substitute equation (34) into equation (33). As in the reversible case in section III, the terms multiplying K on the left hand side must equal the terms multiplying K on the right hand side. In addition, the terms not involving K on the left hand side must equal the terms not involving K on the right hand side. Setting these terms equal yields

$$rq^{(i)} = hp^{\theta} - \delta q^{(i)} + \mu p q_p^{(i)} + \tfrac{1}{2}\sigma^2 p^2 q_{pp}^{(i)}\ , \quad i = L, H \quad (35)$$

$$rG^{(i)} = \left(\max[0, q-b]\right)^n\Gamma + \mu p G_p^{(i)} + \tfrac{1}{2}\sigma^2 p^2 G_{pp}^{(i)}\ , \quad i = L, H \quad (36)$$

These equations correspond to equations (20) and (21) in the reversible investment case. As in section III, we exploit the recursive structure of these equations by solving equation (35) for $q^{(i)}(p)$ and then solving equation (36) for $G^{(i)}(p)$.

IV.3 The solution for q and the optimal rate of investment

The solution for $q^{(i)}(p)$ is determined from equation (35), which is identical to equation (20). Therefore, a general solution to this equation is

$$q^{(i)}(p) = Bp^{\theta} + A_1^{(i)}p^{\eta_1} + A_2^{(i)}p^{\eta_2}\ , \quad i = L, H \quad (37)$$

where B, η_1 and η_2 are identical to their values in equation (22) (the reversible investment case). Notice that the particular solution, Bp^{θ}, is identical to that in equation (22). The only new aspect of equation (37) is that the coefficients $A_1^{(i)}$ and $A_2^{(i)}$ in the homogeneous part of the solution can potentially differ across the two regimes L and H. We now explore this possibility.

The two regimes L and H come together when $q = b$. Let p^* denote the value(s) of p for which $q = b$. The value matching condition requires that $q^{(L)}(p^*) = q^{(H)}(p^*)$. Applying this condition to $q^{(i)}(p)$ in equation (37) yields

$$A_1^{(L)}p^{*\eta_1} + A_2^{(L)}p^{*\eta_2} = A_1^{(H)}p^{*\eta_1} + A_2^{(H)}p^{*\eta_2}\ . \quad (38)$$

What is often called the smooth pasting condition requires that $q_p^{(L)}(p^*) = q_p^{(H)}(p^*)$. Applying this condition to $q^{(i)}(p)$ in equation (37), and multiplying both sides of the resulting equation by p^*, yields

$$\eta_1 A_1^{(L)} p^{*\eta_1} + \eta_2 A_2^{(L)} p^{*\eta_2} = \eta_1 A_1^{(H)} p^{*\eta_1} + \eta_2 A_2^{(H)} p^{*\eta_2} . \qquad (39)$$

Multiplying equation (38) by η_1 and subtracting the resulting equation from equation (39) yields $A_2^{(L)} = A_2^{(H)}$. Similarly, multiplying equation (38) by η_2 and subtracting the resulting equation from equation (39) yields $A_1^{(L)} = A_1^{(H)}$.

We have just shown that the coefficients $A_1^{(i)}$ and $A_2^{(i)}$ are the same across the regimes L and H. Thus, the general solution for $q^{(i)}(p)$ is the same in both regimes and is identical to equation (22). From this point on, the solution method for $q(p)$ is identical to the reversible case. In particular, the same arguments used in section III can be applied to show that $A_1^{(i)} = 0$ and $A_2^{(i)} = 0$. Thus, $q = Bp^\theta$ just as in the reversible case.

To summarize our comparison of the reversible and irreversible investment cases so far, we have shown that the value of q is identical in both cases, and that there are no bubbles in the value of q in either case. Furthermore, for values of q greater than or equal to b, investment is the same in the reversible case and in the irreversible case. The only difference in investment behavior occurs when $q < b$. In this situation, investment is negative in the reversible case and is zero in the irreversible case.

IV.4   The Value of the Adjustment Technology

Although the shadow price q is unaffected by whether or not investment is reversible, the rents to the adjustment technology, represented by the intercept $G(p)$, depend on whether or not investment is reversible. The function $G(p)$ is determined by the differential equation in equation (36). Notice that this differential equation contains the term $\left(\max[0, q - b]\right)^n \Gamma$ which equals 0 in regime L but equals $(q - b)^n \Gamma$ in regime H. We need to solve the differential equation separately for each regime. We begin with the simpler case, which is regime L.

Recall that in regime L we have $q < b$. In this case, $\max[0, q - b] = 0$, and the differential equation (36) is homogeneous. Specifically,

$$rG^{(L)} = \mu p G_p^{(L)} + \tfrac{1}{2} \sigma^2 p^2 G_{pp}^{(L)} . \qquad (40)$$

The general solution to this homogeneous differential equation is

$$G^{(L)}(p) = C_1^{(L)} p^{\omega_1} + C_2^{(L)} p^{\omega_2} , \qquad (41)$$

where $\omega_1 > \omega_2$ are the roots of $f(\omega; r) = 0$ as in the reversible case, and $C_1^{(L)}$ and $C_2^{(L)}$ are constants to be determined.

We have already shown that $\omega_1 > n\theta \geq 2\theta > 0 > \omega_2$. Notice that p^{ω_2} becomes arbitrarily large as p approaches zero. However, the fundamental value of the firm approaches zero as p approaches zero. Thus, if we assume that there are no bubbles, $C_2^{(L)}$ must equal zero and $G^{(L)}(p)$ can be written as

$$G^{(L)}(p) = C_1^{(L)} p^{\omega_1} . \tag{42}$$

Equation (42) gives the fundamental value of a firm with no capital in regime L. Even though a firm in this situation has no capital and is not currently undertaking gross investment, it will have a positive value because of the prospect that one day q may rise above b, and it will become profitable for the firm to invest. Thus $G^{(L)}$ is positive, which implies that $C_1^{(L)}$ is positive. We will tie down the value of $C_1^{(L)}$ after we solve for $G^{(H)}$. Notice that in this case, $C_1^{(L)} p^{\omega_1}$ is not a bubble. As we have said, when $q < b$, a firm with no capital will have positive value if it has access to the adjustment technology and faces the possibility that eventually q will be greater than b, so that positive rents will accrue to the adjustment technology.

In regime H, $q \geq b$ so that $\left(\max[0, q - b]\right)^n \Gamma = (q - b)^n \Gamma$. Thus, the differential equation (36) becomes

$$rG^{(H)} = (q - b)^n \Gamma + \mu p G_p^{(H)} + \tfrac{1}{2}\sigma^2 p^2 G_{pp}^{(H)} . \tag{43}$$

This differential equation is identical to the differential equation (21) in the reversible case. Therefore, the particular solution, $GP(p)$, in equation (24) is also the particular solution of the differential equation in equation (43).

We can obtain a general solution to equation (43) by adding the particular solution in equation (24) and the solution to the homogeneous part of the differential equation. Notice that the homogeneous part of the differential equation (43) is identical to the differential equation (40). Thus the solution to the homogeneous part of equation (43) is $C_1^{(H)} p^{\omega_1} + C_2^{(H)} p^{\omega_2}$. Notice that $\omega_1 > n\theta$ so that p^{ω_1} dominates $GP^{(H)}(p)$ and p^{ω_2} as p grows without bound. Imposing the condition that the market value of the firm is non-negative requires $C_1^{(H)} \geq 0$. As in the case of reversibility we cannot rule out a positive bubble ($C_1^{(H)} > 0$) in the market value of the firm, even if we assume that the firm maximizes its fundamental value in equation (7). However, from this point on, we *assume* that there is no bubble in the market value of the firm. Thus, $C_1^{(H)} = 0$.

The coefficients $C_1^{(L)}$ and $C_2^{(H)}$ still remain to be determined. These two coefficients can be tied down by using the value matching condition $G^{(L)}(p^*) = G^{(H)}(p^*)$ and the smooth pasting condition $G_p^{(L)}(p^*) = G_p^{(H)}(p^*)$. Appendix A shows how these two conditions lead to the following solutions for $G^{(L)}$ and $G^{(H)}$. It is more convenient to write these expressions as

functions of $q \equiv Bp^\theta$ rather than of p:

$$G^{(L)}(q) = \left[(\omega_1 - \omega_2)\prod_{j=0}^{n}(\omega_1 - j\theta)\right]^{-1} \frac{2\Gamma b^n \theta^n n!}{\sigma^2}\left(\frac{q}{b}\right)^{\omega_1/\theta} \qquad (44)$$

$$G^{(H)}(q) = \Gamma PV_t[(q-b)^n, r] \qquad (45)$$
$$+ \left[(\omega_1 - \omega_2)\prod_{j=0}^{n}(\omega_2 - j\theta)\right]^{-1} \frac{2\Gamma b^n \theta^n n!}{\sigma^2}\left(\frac{q}{b}\right)^{\omega_2/\theta} \ .$$

Recall that $\omega_1 > n\theta > 0 > \omega_2$. Therefore, equation (44) implies that $G^{(L)}(q) > 0$ because $\omega_1 - \omega_2 > 0$ and $\omega_1 - j\theta > 0$ for $j = 0, 1, \ldots, n$. Thus, as suggested earlier, even when $q < b$ so that it is not currently profitable for the firm to undertake positive gross investment, the prospect that q will eventually exceed b means that the present value of the rents accruing to the adjustment technology is positive.

Equation (45) allows a direct comparison of the present value of rents to the adjustment technology in the reversible and irreversible cases. Recall that the first term on the right hand side of equation (45), $\Gamma PV_t[(q-b)^n, r]$, equals $G(p)$ in the reversible case. Thus, the difference between between $G(p)$ in the irreversible and reversible cases is given by the second term in equation (45). Observe that $\omega_2 - j\theta < 0$ for $j = 0, 1, \ldots, n$. Since the second term in equation (45) contains an odd number $(n + 1)$ of such terms, the second term in equation (45) is negative. That is, $G^{(H)}$ in the irreversible case is smaller than $\Gamma PV_t[(q-b)^n, r]$. The reason for this result is clear. In the case of reversible investment the current rents to the adjustment technology are $(q-b)^n$ regardless of whether q is greater than, less than, or equal to b. However, in the irreversible case, current rents to the adjustment technology are 0 (which is less than $(q-b)^n$) whenever $q < b$. Even when q is currently greater than b, the prospect that q may eventually fall below b means that the expected present value of rents to the adjustment technology is smaller in the case of irreversible investment than in the reversible case.

V Conclusions

We have presented a closed-form solution for the optimal investment and market valuation of a competitive firm under uncertainty. This solution was obtained under the assumptions that the firm has a constant returns to scale production technology and convex costs of investing. We have solved for investment and market valuation in both a reversible and an irreversible investment case. Optimal investment is a non- decreasing function of q, the shadow value of capital, and we have shown that bubbles on q are ruled out

15

by optimality. Optimal investment is therefore driven only by fundamentals and will be invariant to bubbles. Bubbles can exist, however, on the market value of the firm.

The shadow value of an additional unit of capital, q, is unaffected by irreversibility in our model. The only effect of irreversibility on investment behavior is to set investment to zero when it would otherwise be negative. Irreversibility does, however, affect the market valuation of the firm. Since rents accrue to the firm when it is investing or disinvesting, irreversibility reduces these rents and therefore the value of the firm. An implication of this finding is that average q, computed as the ratio of market valuation to the capital stock, $V(K,p)/K$, and often used as an empirical proxy for q, is reduced by irreversibility despite the fact that (marginal) q is unaffected. Irreversibility therefore reduces the difference between marginal and average q, since average q exceeds marginal q by the ratio of the present value of rents to the capital stock, $G(p)/K$. Also, because the value of the firm may contain bubbles, average q may contain bubbles even though marginal q cannot contain bubbles.

The invariance of (marginal) q to the imposition of irreversibility is in contrast to results reported in the irreversible investment literature,[4] where imposition of a non-negativity constraint on investment reduces the marginal value of additional capital as a result of what is often called the "option value of waiting". In our model, the value function is linear in capital, so q does not depend on the capital stock. Consistent with Pindyck's (1993) argument, the firm does not "kill an option" when it invests, since its investment behavior does not affect the current or future return to capital.

We have confined our attention to a competitive firm with constant returns to scale so that the operating profit function of the firm is linear in capital. Therefore, the marginal operating profit of capital is invariant to the capital stock. However, if the firm has some monopoly power and/or if the production function exhibits decreasing returns to scale, the operating profit function of the firm will be strictly concave in the capital stock, and the marginal operating profit of capital will be strictly decreasing in the capital stock. The case with a strictly concave operating profit function is substantially more difficult, but would allow analysis of the "option value" found in some irreversible investment models. This issue is the subject of ongoing research.

---

[4]For example, McDonald and Siegel (1986), Dixit (1989), Bertola (1987), Pindyck (1988).

Appendix A

Applying Value Matching and Smooth Pasting
to $G^{(H)}(p)$ and $G^{(L)}(p)$

Recall that we have argued in the text that $C_1^{(H)} = 0$ so that

$$G^{(H)}(p) = GP(p) + C_2^{(H)} p^{\omega_2} . \tag{A1}$$

Now evaluate the particular solution $GP(p)$ in equation (24) and its derivative at $p = p^*$. In evaluating these expressions at $p = p^*$, it is helpful to note that $Bp^{*\theta} = b$.

$$GP^{(H)}(p^*) = \Gamma b^n \sum_{j=0}^{n} (-1)^{n-j} \frac{n!}{j!(n-j)!f(j\theta;r)} \tag{A2}$$

$$GP_p^{(H)}(p^*) = \left(\frac{\theta}{p^*}\right) \Gamma b^n \sum_{j=0}^{n} (-1)^{n-j} j \frac{n!}{j!(n-j)!f(j\theta;r)} . \tag{A3}$$

The value matching condition $G^{(L)}(p^*) = G^{(H)}(p^*)$ implies, using equations (42) and (A1) that

$$C_1^{(L)} p^{*\omega_1} = GP(p^*) + C_2^{(H)} p^{*\omega_2} . \tag{A4}$$

The smooth pasting condition $G_p^{(L)}(p^*) = G_p^{(H)}(p^*)$ implies, using equations (42) and (A1) that

$$\omega_1 C_1^{(L)} p^{*\omega_1 - 1} = GP_p^{(H)}(p^*) + \omega_2 C_2^{(H)} p^{*\omega_2 - 1} . \tag{A5}$$

Equations (A4) and (A5) are two linear equations in the two unknown variables $C_1^{(L)} p^{*\omega_1}$ and $C_2^{(H)} p^{*\omega_2}$. Solving these two linear equations yields

$$C_1^{(L)} p^{*\omega_1} = \frac{p^* GP_p^{(H)}(p^*) - \omega_2 GP^{(H)}(p^*)}{\omega_1 - \omega_2} , \tag{A6}$$

and

$$C_2^{(H)} p^{*\omega_2} = \frac{p^* GP_p^{(H)}(p^*) - \omega_1 GP^{(H)}(p^*)}{\omega_1 - \omega_2} . \tag{A7}$$

Now use the fact that $q/b = (p/p^*)^\theta$ to observe that $C_1^{(L)} p^{\omega_1} = C_1^{(L)} p^{*\omega_1}(p/p^*)^{\omega_1} = C_1^{(L)} p^{*\omega_1}(q/b)^{\omega_1/\theta}$ so that equation (A6) implies

$$C_1^{(L)} p^{\omega_1} = \left\{ \frac{p^* GP_p^{(H)}(p^*) - \omega_2 GP^{(H)}(p^*)}{\omega_1 - \omega_2} \right\} \left(\frac{q}{b}\right)^{\omega_1/\theta} . \tag{A8}$$

17

Similarly, observe that $C_2^{(H)}p^{\omega_2} = C_2^{(H)}p^{*\omega_2}(p/p^*)^{\omega_2} = C_2^{(H)}p^{*\omega_2}(q/b)^{\omega_2/\theta}$ so that equation (A7) implies

$$C_2^{(H)}p^{\omega_2} = \left\{ \frac{p^* GP_p^{(H)}(p^*) - \omega_1 GP^{(H)}(p^*)}{\omega_1 - \omega_2} \right\} \left(\frac{q}{b}\right)^{\omega_2/\theta} . \qquad (A9)$$

Next use equations (A2) and (A3) to calculate

$$p^* GP_p^{(H)}(p^*) - \omega_i GP^{(H)}(p^*) = b^n \sum_{j=0}^{n} (j\theta - \omega_i)D_j(-1)^{n-j} \qquad (A10)$$

where $D_j = \Gamma n!/[j!(n-j)!f(j\theta; r)]$.

Substituting equation (A10) into equation (A8), and recalling that $G^{(L)}(p) = C_1^{(L)}p^{\omega_1}$, we obtain

$$G^{(L)}(p) = \left(\frac{q}{b}\right)^{\omega_1/\theta} b^n \sum_{j=0}^{n} \frac{(j\theta - \omega_2)D_j(-1)^{n-j}}{\omega_1 - \omega_2} . \qquad (A11)$$

Substituting equation (A10) into equation (A9), and recalling that $G^{(H)}(p) = GP^{(H)}(p) + C_2^{(H)}p^{\omega_2}$, we obtain

$$G^{(H)}(p) = GP^{(H)}(p) + \left(\frac{q}{b}\right)^{\omega_2/\theta} b^n \sum_{j=0}^{n} \frac{(j\theta - \omega_1)D_j(-1)^{n-j}}{\omega_1 - \omega_2} . \qquad (A12)$$

We can simplify the expressions in equations (A11) and (A12) using the following Lemma.

Lemma *Let ω_i and ω_k be the two real roots of the quadratic equation $f(x; r) \equiv r - \mu x - \frac{1}{2}\sigma^2 x(x-1) = 0$. Define*

$$S_n = \sum_{j=0}^{n} (j\theta - \omega_i) \left[\frac{n!}{j!(n-j)!f(j\theta; r)}\right] (-1)^{n-j} .$$

Then

$$S_n = \frac{2n!\theta^n/\sigma^2}{\left[\prod_{j=0}^{n}(\omega_k - j\theta)\right]} .$$

Proof. Observe that the quadratic function $f(x; r)$ can be written as $f(x; r) = \frac{1}{2}\sigma^2(\omega_k - x)(x - \omega_i)$, so that $f(j\theta; r) = \frac{1}{2}\sigma^2(\omega_k - j\theta)(j\theta - \omega_i)$. Therefore, $(j\theta - \omega_i)/f(j\theta; r) = \frac{1}{\frac{1}{2}\sigma^2(\omega_k - j\theta)}$ and we can rewrite S_n as

$$S_n = \frac{2}{\sigma^2} \sum_{j=0}^{n} \left[\frac{1}{\omega_k - j\theta}\right] \left[\frac{n!}{j!(n-j)!}\right] (-1)^{n-j} . \qquad (L1)$$

18

Applying equation (L1) for $n - 1$ yields

$$S_{n-1} = \frac{2}{\sigma^2} \sum_{j=0}^{n-1} \left[\frac{1}{\omega_k - j\theta} \right] \left[\frac{(n-1)!}{j!(n-1-j)!} \right] (-1)^{n-1-j} \ . \qquad \text{(L2)}$$

Multiply both sides of equation (L2) by $-n\theta$ to obtain

$$- n\theta S_{n-1} = \frac{2}{\sigma^2} \sum_{j=0}^{n-1} \left[\frac{\theta}{\omega_k - j\theta} \right] \left[\frac{n!}{j!(n-1-j)!(n-j)} \right] (n-j)(-1)^{n-j} \ .$$
$$\text{(L3)}$$

Observe that when $j = n$, the summand on the right hand side of equation (L3) is zero, so we can increase the upper limit on the summation index j from $n - 1$ to n. Performing this change and rearranging yields

$$- n\theta S_{n-1} = \frac{2}{\sigma^2} \sum_{j=0}^{n} \left[\frac{\theta(n-j)}{\omega_k - j\theta} \right] \left[\frac{n!}{j!(n-j)!} \right] (-1)^{n-j} \ . \qquad \text{(L4)}$$

Now replace $\theta(n - j)$ by $(\theta n - \omega_k) + (\omega_k - j\theta)$ to obtain

$$- n\theta S_{n-1} \ = \ \frac{2}{\sigma^2} \sum_{j=0}^{n} \left[\frac{\theta n - \omega_k}{\omega_k - j\theta} \right] \left[\frac{n!}{j!(n-j)!} \right] (-1)^{n-j} \qquad \text{(L5)}$$

$$+ \ \frac{2}{\sigma^2} \sum_{j=0}^{n} \left[\frac{\omega_k - j\theta}{\omega_k - j\theta} \right] \left[\frac{n!}{j!(n-j)!} \right] (-1)^{n-j} \ .$$

The second summation on the right hand side of equation (L5) is simply the binomial expansion of $(1-1)^n$, which is, of course, zero. Using equation (L1) to simplify the first term on the right hand side of equation (L5) we obtain

$$- n\theta S_{n-1} = (\theta n - \omega_k) S_n \qquad \text{(L6)}$$

Therefore,

$$S_n = \left[\frac{n\theta}{\omega_k - \theta n} \right] S_{n-1} \ . \qquad \text{(L7)}$$

To solve the difference equation in (L7), we need a boundary condition. Observe (using equation (L1)) that when $n = 1$, we have

$$S_1 = \frac{2}{\sigma^2} \left\{ \left[\frac{1}{\omega_k} \right] (-1) + \frac{1}{\omega_k - \theta} \right\} = \frac{2\theta/\sigma^2}{\omega_k(\omega_k - \theta)} \ . \qquad \text{(L8)}$$

19

Equations (L7) and (L8) together imply

$$S_n = \frac{2n!\theta^n/\sigma^2}{\left[\prod_{j=0}^{n}(\omega_k - j\theta)\right]} .$$

(L9)

q.e.d.

Now use the Lemma to re-write equations (A11) and (A12) as

$$G^{(L)}(p) = \left(\frac{q}{b}\right)^{\omega_1/\theta} b^n \frac{2\Gamma n!\theta^n/\sigma^2}{(\omega_1 - \omega_2)\prod_{j=0}^{n}(\omega_1 - j\theta)}$$

(A13)

and

$$G^{(H)}(p) = GP^{(H)}(p) + \left(\frac{q}{b}\right)^{\omega_2/\theta} b^n \frac{2\Gamma n!\theta^n/\sigma^2}{(\omega_1 - \omega_2)\prod_{j=0}^{n}(\omega_2 - j\theta)} ,$$

(A14)

which are identical to equations (44) and (45) in the text.

Appendix B
Proof That q Cannot Have a Positive Bubble

Consider a firm at time t with capital stock K_t. Assume that the firm solves the maximization problem on the right hand side of equation (7). Let $\{I^*_{t+s}, K^*_{t+s}\}_{s\geq 0}$ be the path of investment and capital stock if $q_{t+s} = Bp^\theta_{t+s}$. Observe that the first-order condition for investment implies that $c'(I^*_{t+s}) \geq q_{t+s} = Bp^\theta_{t+s}$. (This condition always holds with equality under reversibility; under irreversibility it holds with equality when $I^*_{t+s} > 0$.) Define V^*_t as the value of the objective function under this path of investment and capital stock so that

$$V^*_t \equiv E_t \left\{ \int_0^\infty [hp^\theta_{t+s} K^*_{t+s} - c(I^*_{t+s})]e^{-rs} ds \right\} . \tag{B1}$$

Let $\{I^{**}_{t+s}, K^{**}_{t+s}\}_{s\geq 0}$ be the path of investment and capital stock if $q_{t+s} = Bp^\theta_{t+s} + A_1 p^{\eta_1}_{t+s} + A_2 p^{\eta_2}_{t+s}$. Define V^{**}_t as the value of the objective function under this path of investment and capital stock so that

$$V^{**}_t \equiv E_t \left\{ \int_0^\infty [hp^\theta_{t+s} K^{**}_{t+s} - c(I^{**}_{t+s})]e^{-rs} ds \right\} . \tag{B2}$$

Suppose that $A_1 \geq 0$ and $A_2 \geq 0$ and either A_1 or A_2 is strictly positive. Thus, $Bp^\theta_{t+s} + A_1 p^{\eta_1}_{t+s} + A_2 p^{\eta_2}_{t+s} > Bp^\theta_{t+s}$ so that $I^{**}_{t+s} \geq I^*_{t+s}$. (Under reversibility $I^{**}_{t+s} > I^*_{t+s}$ always; under irreversibility, $I^{**}_{t+s} > I^*_{t+s}$ if $Bp^\theta_{t+s} + A_1 p^{\eta_1}_{t+s} + A_2 p^{\eta_2}_{t+s} > b$.)

Define $\Delta \equiv V^{**}_t - V^*_t$, so that equations (B1) and (B2) imply that

$$\Delta \equiv E_t \left\{ \int_0^\infty \{hp^\theta_{t+s}[K^{**}_{t+s} - K^*_{t+s}] - [c(I^{**}_{t+s}) - c(I^*_{t+s})]\}e^{-rs} ds \right\} . \tag{B3}$$

Observe that

$$K_{t+s} = K_t e^{-\delta s} + \int_0^s I_{t+v} e^{-\delta(s-v)} dv , \tag{B4}$$

which implies that

$$K^{**}_{t+s} - K^*_{t+s} = \int_0^s (I^{**}_{t+v} - I^*_{t+v})e^{-\delta(s-v)} dv . \tag{B5}$$

Equation (B5) can be used to obtain

$$E_t \left\{ \int_0^\infty \{hp^\theta_{t+s}[K^{**}_{t+s} - K^*_{t+s}]\}e^{-rs} ds \right\} \tag{B6}$$

$$= E_t \left\{ \int_0^\infty \int_0^s \{hp^\theta_{t+s}[I^{**}_{t+v} - I^*_{t+v}]\}e^{-\delta(s-v)} e^{-rs} dv\, ds \right\} .$$

Using the law of iterated projections $(E_t\{\,\cdot\,\} = E_t\{E_{t+v}\{\,\cdot\,\}\}$ for $v \geq 0)$, and changing the order of integration, equation (B6) becomes

$$E_t \left\{ \int_0^\infty \{hp_{t+s}^\theta [K_{t+s}^{**} - K_{t+s}^*]\}e^{-rs}ds \right\} \tag{B7}$$

$$= E_t \left\{ \int_0^\infty E_{t+v} \left\{ \int_v^\infty hp_{t+s}^\theta [I_{t+v}^{**} - I_{t+v}^*]e^{-\delta(s-v)}e^{-rs}ds \right\} dv \right\}$$

$$= E_t \left\{ \int_0^\infty [I_{t+v}^{**} - I_{t+v}^*]E_{t+v} \left\{ \int_v^\infty hp_{t+s}^\theta e^{-(r+\delta)(s-v)}ds \right\} e^{-rv}dv \right\} \, .$$

Now use the fact that

$$Bp_{t+v}^\theta = E_{t+v} \left\{ \int_v^\infty hp_{t+s}^\theta e^{-(r+\delta)(s-v)}ds \right\} \tag{B8}$$

to rewrite equation (B7) as

$$E_t \left\{ \int_0^\infty \{hp_{t+s}^\theta [K_{t+s}^{**} - K_{t+s}^*]\}e^{-rs}ds \right\} \tag{B9}$$

$$= E_t \left\{ \int_0^\infty Bp_{t+v}^\theta [I_{t+v}^{**} - I_{t+v}^*]e^{-rv}dv \right\} \, .$$

Recall that $c(I)$ is convex and that $I_{t+s}^{**} \geq I_{t+s}^*$ so that

$$c(I_{t+s}^{**}) - c(I_{t+s}^*) \geq (I_{t+s}^{**} - I_{t+s}^*)c'(I_{t+s}^*) \tag{B10}$$

with strict inequality whenever $I_{t+s}^{**} > I_{t+s}^*$. It follows from equation (B10) and the fact that there is a positive measure on the event $I_{t+s}^{**} > I_{t+s}^*$ that

$$E_t \left\{ \int_0^\infty [c(I_{t+s}^{**}) - c(I_{t+s}^*)]e^{-rs}ds \right\} \tag{B11}$$

$$> E_t \left\{ \int_0^\infty (I_{t+s}^{**} - I_{t+s}^*)c'(I_{t+s}^*)e^{-rs}ds \right\} \, . \tag{B12}$$

Using equations (B9) and (B11) along with equation (B3) we obtain

$$\Delta < E_t \left\{ \int_0^\infty (I_{t+s}^{**} - I_{t+s}^*)[Bp_{t+s}^\theta - c'(I_{t+s}^*)]e^{-rs}ds \right\} \leq 0 \, , \tag{B13}$$

where the second inequality in equation (B12) follows from the facts that $I_{t+s}^{**} - I_{t+s}^* \geq 0$, $Bp_{t+s}^\theta - c'(I_{t+s}^*) \leq 0$, and $e^{-rs} > 0$.

We have shown that $\Delta < 0$ which means that the path $\{I_{t+s}^{**}, K_{t+s}^{**}\}_{s \geq 0}$ is strictly dominated by the path $\{I_{t+s}^*, K_{t+s}^*\}_{s \geq 0}$. Therefore, neither A_1 nor A_2 can be positive. Since we have already shown in the text that neither A_1 nor A_2 can be negative, it follows that A_1 and A_2 must be zero.

References

Abel, A. B., 1983, "Optimal Investment under Uncertainty," *American Economic Review*, 73(1), March, 228–233.

Abel, A. B., 1985, "A Stochastic Model of Investment, Marginal q, and the Market Value of the Firm," *International Economic Review*, 26(2), June, 305–322.

Abel, A. B., and J. C. Eberly, 1993, "A Unified Model of Investment Under Uncertainty," Working Paper 4296, National Bureau of Economic Research, March.

Arrow, K. J., 1968, "Optimal Capital Policy with Irreversible Investment," in J. N. Wolfe (ed.), *Value, Capital and Growth, Papers in Honour of Sir John Hicks*, Edinburgh University Press, pp. 1–19.

Bernanke, B., 1983, "Irreversibility, Uncertainty, and Cyclical Investment," *Quarterly Journal of Economics*, 98(1), February, 85–106.

Bertola, G., 1987, "Irreversible Investment," unpublished manuscript, M.I.T., July.

Caballero, R. J., 1991, "On the Sign of the Investment-Uncertainty Relationship," *American Economic Review*, 81(1), March, 279–288.

Dixit, A., 1989, "Entry and Exit Decisions under Uncertainty," *Journal of Political Economy*, 97(3), June, 620–638.

Dixit, A., 1991, "Irreversible Investment with Price Ceilings," *Journal of Political Economy*, 99(3), June, 541–557.

Dixit, A., and R. S. Pindyck, 1992, "Investment under Uncertainty," unpublished manuscript.

Eisner, R., and R. H. Strotz, 1963, "Determinants of Business Investment," in *Impacts of Monetary Policy*, Commission on Money and Credit, Prentice-Hall, Inc., Englewood Cliffs, NJ, pp. 59–337.

Gould, J. P., 1968, "Adjustment Costs in the Theory of Investment of the Firm," *Review of Economic Studies*, 35(1), January, 47–55.

Hartman, R., 1972, "The Effect of Price and Cost Uncertainty on Investment," *Journal of Economic Theory*, 5, 258–266.

Hayashi, F., 1982, "Tobin's Marginal and Average q: A Neoclassical Interpretation," *Econometrica*, 50(1), January, 213–224.

Lucas, Jr., R. E., 1967, "Adjustment Costs and the Theory of Supply," *Journal of Political Economy*, 75(4, Part I), August, 321–334.

Lucas, Jr., R. E., and E. C. Prescott, 1971, "Investment under Uncertainty," *Econometrica*, 39(5), 659–681.

McDonald, R., and D. Siegel, 1986, "The Value of Waiting to Invest," *Quarterly Journal of Economics*, 101, November, 707–728.

Mussa, M., 1977, "External and Internal Adjustment Costs and the Theory of Aggregate and Firm Investment," *Economica*, 44, May, 163–178.

Pindyck, R. S., 1988, "Irreversible Investment, Capacity Choice, and the Value of the Firm," *American Economic Review*, 78(5), December, 969–985.

Pindyck, R. S., 1991, "Irreversibility, Uncertainty, and Investment," *Journal of Economic Literature*, 29(3), September, 1110–1148.

Pindyck, R. S., 1993, "A Note on Competitive Investment under Uncertainty," *American Economic Review*, 83(1), March, 273–277.

Tobin, J., 1969, "A General Equilibrium Approach to Monetary Theory," *Journal of Money, Credit and Banking*, 1, February, 15–29.

Treadway, A. B., 1969, "On Rational Entrepreneurial Behaviour and the Demand for Investment," *Review of Economic Studies*, 36(2), April, 227–239.