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ABSTRACT

This paper describes the efficient allocation of consumption and work effort in an economy in which workers face idiosyncratic employment risk and considerations of moral hazard prevent full insurance. We impose a lower bound on the expected discounted utility that can be assigned to any agent from any date onward, and show, with this feature added, that the efficient unemployment insurance scheme induces an invariant cross sectional distribution of individual entitlements to utility. The paper thus provides a simple prototype model suited to the study of the normative question: what is the tradeoff between equality and efficiency in resource allocation?

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Introduction:

This paper describes the efficient allocation of consumption and work effort in an economy in which workers face idiosyncratic employment risk. Each period, each worker either finds or fails to find a job opportunity. A worker who finds a job opportunity and takes it produces output at the cost of some disutility of effort. A worker who does not have a job opportunity does not work, suffers no disutility of effort, and produces nothing. The presence of a job opportunity is not observable by others, so a worker who has a job opportunity and chooses not to work is indistinguishable from a worker who has no such opportunity. It is this element of moral hazard that precludes the perfect pooling of job risk: in providing insurance against this employment risk, the gains from reducing consumption uncertainty need to be weighed against the costs of reduced work incentives.

In our formulation of this problem we impose a lower bound on the expected discounted utility that can be assigned to any agent from any date onward. With this feature added, we show under fairly general assumptions on preferences that the efficient unemployment insurance scheme induces an invariant cross sectional distribution of individual entitlements to utility: the economy as a whole has an efficient steady state distribution, within which the fortunes of any individual family rise or fall depending on its idiosyncratic luck. The paper thus provides a simple prototype model suited to the study of the *normative* question: what is the tradeoff between equality and efficiency in resource allocation?

The context we use for examining this issue is a simplified version of the frameworks used by Albrecht and Axell (1984) and Hansen and Imrohoroglu (1992) to study unemployment insurance. Our model lacks several realistic features that are incorporated into these earlier studies. Our model, in contrast to Hansen and Imrohoroglu's model, is not capable of matching data on the typical length of employment and unemployment spells, and, in contrast to Albrecht and Axell's model, it is not well suited for the study of the impact of unemployment insurance on the McCall-like determination of a reservation wage. We focus instead on a feature of the efficient allocation of consumption and work effort that these earlier papers do not consider: the dependence of the efficient allocation of current consumption and work effort on a worker's employment history. This dependence was first explored in Townsend (1980) and Radner (1981). Usami (1983) is an early application of these ideas to the problem of unemployment insurance.

Our formulation of this economy's efficiency problem is taken from Atkeson and Lucas (1992). We take as the state of the system a utility distribution — a distribution of households by the expected discounted utility each is to receive from the current period on. We define an allocation, and then define, for any given utility distribution, the cost of that distribution as the smallest (over all allocations) net, constant inflow of resources needed to attain the utilities in the given distribution. Since we deal with a closed system, the set of utility distributions that can be attained with zero cost are on the frontier of the utility possibility set for this economy, and the allocations which attain these distributions are efficient. Following our previous paper, we show that the efficient allocation can be decentralized in a fashion that connects this efficiency problem to the one-on-one principal-agent problem studied by Green (1987) and many others. In this one-on-one principal-agent problem, the state variable is simply the discounted expected utility the agent is to receive from the current period on, and the principal's objective is to minimize the resource cost to himself of providing that utility, where this cost is measured according to some set of intertemporal prices or interest rates. The solution to the original efficiency problem is then found through a process of varying the interest rates the representative principal faces until, period by period, the net resource use implied by the solution to the one-on-one principal-agent problem is set equal to zero. A similar procedure is followed in Taub (1990).

Thomas and Worrall (1990) and Atkeson and Lucas (1992) show that, if agents can give up *all* claims to discounted expected utility for the sake of current consumption, then the solution to the one-on-one principal-agent problem implies that the limiting distribution of agents' wealth and consumption is degenerate, with a vanishing fraction of the population consuming all output in the economy. In light of this result, the efficiency standard used by Albrecht and Axell and Hansen and Imrohoroglu of examining the costs of various unemployment insurance schemes at the steady state distribution of consumption makes little sense in a setting in which each worker's unemployment benefits can depend on his individual history of employment and unemployment and each worker can trade away in the limit all of his claims to future consumption for the sake of current consumption. In this paper, we address this problem by imposing a limit on the extent to which workers can trade future for current consumption. In particular, we impose the constraint that there is a minimum entitlement to discounted expected utility from the beginning of each period on that each agent must receive. We interpret this constraint as a limit on the extent to which living members of an infinitely lived household in our economy can sell the consumption claims of their heirs. It is important to remember that this constraint is an additional constraint imposed upon our efficiency problem and is not derived directly from any efficiency consideration. We find that the solution to the one-on-one principal agent problem with this additional constraint does imply a non-degenerate steady state distribution of consumption.

For any given interest rate for the principals, the resource cost of the steady-state cross-sectional distribution of utility entitlements implied by the solution to the one-on-one principal agent problem is determined by the balancing of two forces, with the relative strength of these forces being determined by the size of the interest rate. When agents have relatively high entitlements, so that the minimum entitlement constraint is not binding, then the first order conditions of the one-on-one principal agent problem imply that the marginal cost to the principal of providing the agent with his entitlement to discounted expected utility follows a sub-martingale. For example, in the case that agents have time additive preferences with momentary utility of the form $u(c) = c^{1/2}$, this implies that agent's current consumption follows a random walk with downward drift. In general, with time additive utility with concave momentary utility, agents' current consumption follows a qualitatively similar process. The downward drift in the process governing agents' current consumption and entitlements is thus a force that pushes all agents in the long run down upon the minimum entitlement constraint, with the push being greater the lower the interest rate.

The second and balancing force is provided by the minimum entitlement constraint itself. When interest rates are very low, this minimum entitlement becomes an absorbing state of the Markov process governing the evolution of individual agents' entitlements, so that, in the limit, all agents end up stuck on this constraint. But if the job opportunity is sufficiently productive, a steady state with all agents at the minimum entitlement entails an excess supply of goods. As the interest rate is increased, the minimum entitlement ceases to be an absorbing state, as agents who have this entitlement and report a job opportunity in essence save some of their earnings and thus raise their entitlement from next period on. In this case, the steady state distribution of entitlements has only some fraction of the population at the minimum entitlement, with that fraction being determined by the interest rate. With these results, we can then pose our original efficiency problem as one of finding the interest rate for the principal in a one-on-one principal agent problem such that the net resource cost of the corresponding steady state distribution of entitlements of utility is zero.

The remainder of paper is organized as follows. In section 2, we present the model, define our efficiency problem, and establish the connection between our original efficiency problem and a one-on-one principal agent problem. In sections 3 and 4, we characterize the solution to that one-on-one principal agent problem. In section 5, we analyze the Markov process of entitlements generated by the solution to the one-on-one principal agent problem and demonstrate that the steady-state level of resource use is a continuous, increasing function of the interest rate, thus establishing the existence of a market clearing interest rate. In section 6, we conclude.

2. The Model:

Time is denoted by $t = 0, 1, 2, \ldots$. Each period, each agent finds a job opportunity with probability π and fails to find such an opportunity with probability $1 - \pi$. An agent who finds a job can work $h \in [0, 1]$ units of time. We assume that job opportunities are independently and identically distributed both across agents and across time. Agents who consume resources c and work h hours within the current period obtain flow utility $(1 - \beta)(U(c) - hv)$, where $U : \mathbf{R}_+ \to D \subseteq \mathbf{R}$ and $v > 0$, the disutility of work, is a fixed parameter. Let $C(u)$, $C : D \to \mathbf{R}_+$ be the inverse of the flow utility function $U(c)$. We assume that C is continuously differentiable, strictly increasing, and strictly convex, with $\inf_{u \in D} C(u) = 0$.

At each date $t \geq 0$ agents are distinguished by their names

$w_0 \in D$ and their history of reported job opportunities $z^t = (z_0, z_1, \ldots, z_t)$. We assume $z_t \in \{0, 1\}$ for all t and we use $z_t = 0$ to indicate a report of no job opportunity and $z_t = 1$ to indicate a report of a job opportunity in the current period t. We use w_0 interchangeably to denote an agent's name and his initial entitlement to discounted expected utility. At each date t, a hypothetical social planner assigns each agent of each type some current level of consumption $C(x_t)$ (some current flow utility from consumption x_t) and assigns each agent who reports a job opportunity some hours of work h_t. An allocation in this environment is thus a sequence of functions

$$\sigma = \{x_t(w_0, z^t), h_t(w_0, z^t)\}_{t=0}^{\infty}$$

where x_t maps agents' initial entitlements w_0 and histories of reports z^t into levels of current utility in D, while h_t maps these same variables into the interval $[0, 1]$.

Given an allocation σ, an agent chooses a strategy for reporting job opportunities to maximize the discounted expected utility he obtains under that allocation. This strategy is denoted by $z = \{z_t(\theta^t)\}_{t=0}^{\infty}$, where $\theta^t = (\theta_0, \theta_1, \ldots, \theta_t)$, $\theta_t \in \{0, 1\}$ for all $t \geq 0$, denotes the agent's true job experience. We use $\Theta^{t+1} = \{0, 1\} \times \{0, 1\} \times \ldots \times \{0, 1\}$ to denote the space of possible job histories up through time t, μ^{t+1} to denote the distribution of

$\theta^t \in \Theta^{t+1}$ generated by π, $z_t(\theta^t)$ to denote the choice of report at time t as a function of history θ^t, and $z^t(\theta^t)$ to denote the history of reports up through time t induced by a reporting strategy z and history θ^t. We assume that an agent cannot report a job opportunity if he does not have one. That is, we assume that for all t, $z_t(\theta^t) = 0$ if $\theta_t = 0$.

An agent's initial discounted expected utility can be written as a function of w_0, σ, and z as

$$U(w_0, \sigma, z) =$$

$$(1-\beta)\sum_{t=0}^{\infty} \beta^t \int_{\Theta^t} \{x_t(w_0, z^t(\theta^t)) - z_t(\theta^t) h_t(w_0, z^t(\theta^t)) v\} d\mu^{t+1}.$$

Let $z^* = \{z_t^*(\theta^t)\}_{t=0}^{\infty}$, where $z_t^*(\theta^t) = \theta_t$ for all $t \geq 0$ and $\theta^t \in \Theta^{t+1}$, denote the truthful reporting strategy. We use the notation $U_t(w_0, \sigma, z^*, \theta^{t-1})$ to denote the discounted expected utility from date t on received under the allocation σ by an agent who was originally entitled to $w_0 \in D$, has reported employment history $\theta^{t-1} \in \Theta^t$ up to date t, and who uses the truthful reporting strategy z^*.

We impose four conditions on allocations. The first requires that σ delivers w_0 to those entitled to w_0:

$$w_0 = U(w_0, \sigma, z^*), \tag{2.1}$$

for all $w_0 \in D$. The second is incentive compatibility:

$$U(w_0, \sigma, z^*) \geq U(w_0, \sigma, z) \tag{2.2}$$

for all $w_0 \in D$ and all reporting strategies z. The third is a lower bound on the discounted expected utility that an agent can receive after any employment history:

$$U_t(w_0, \sigma, z^*, \theta^{t-1}) \geq \underline{w} \tag{2.3}$$

for all $t \geq 1$, $w_0 \in D$, $\theta^{t-1} \in \Theta^t$. The fourth is an upper bound on the discounted expected utility that an agent can expect in the tail of the allocation:

$$\lim_{t \to \infty} \beta^t \sup_{\theta^{t-1} \in \Theta^t} U_t(w_0, \sigma, z^*, \theta^{t-1}) = 0 \tag{2.4}$$

An allocation σ is said to *attain* a distribution of entitlements ψ_0 with transfers τ if (2.1)-(2.4) are satisfied and if the allocation σ never requires a net infusion of resources greater than τ:

$$\int_{D \times \Theta^{t+1}} \{C[x_t(w_0, \theta^t)] - \theta_t h_t(w_0, \theta^t) y\} d\mu^{t+1} d\psi_0 \leq \tau \tag{2.5}$$

for all $t \geq 0$. An allocation is said to be *efficient* if it attains a distribution ψ_0 with transfers τ and there is no other allocation that attains ψ_0 with transfers less than τ.

Following Atkeson and Lucas (1992), prices can be used to decentralize the overall problem of finding efficient allocations into component planning problems. To define what we mean by a "component planning problem", consider a planner responsible for allocating resources *only* to those who are initially entitled to expected utility w_0. He assigns an allocation (specific to w_0) $\sigma(w_0) = \{x_t(w_0, \theta^t), h_t(w_0, \theta^t)\}_{t=0}^{\infty}$, $x_t : \Theta^{t+1} \rightarrow D$, $h_t : \Theta^{t+1} \rightarrow [0, 1]$ in such a way as to minimize the *value* of the total resources he allocates, with resources at each date valued at prices determined by the sequence $\{q_t\}_{t=0}^{\infty}$, $q_t \in (0, 1)$. The objective for this planner is to choose $\sigma(w_0)$ to minimize

$$(1 - q_0) \int_{\Theta} \{C[x_0(w_0, \theta)] - \theta h_0(w_0, \theta)y\} d\mu + \tag{2.6}$$

$$\sum_{t=1}^{\infty} (1 - q_t) \prod_{s=0}^{t-1} q_s \int_{\Theta^{t+1}} \{C[x_t(w_0, \theta^t)] - \theta_t h_t(w_0, \theta^t)y\} d\mu^{t+1}$$

where $\sigma(w_0)$ is chosen subject to the constraints (2.1)-(2.4). It is as if consumers are grouped by their initial w_0 values, with each group represented by its own social planner, and then these planners trade certain claims to current and future resources among themselves at prices given by $\{q_t\}_{t=0}^{\infty}$.

The next result, essentially Theorem 1 of Atkeson and Lucas (1992), provides one connection between these component planning problems (2.6) and the problem of finding efficient allocations.

Theorem 1. Suppose there exist an allocation $\sigma = \{x_t(w_0, \theta^t), h_t(w_0, \theta^t)\}_{t=0}^{\infty}$, prices $\{q_t\}_{t=0}^{\infty}$, a distribution of entitlements ψ_0, and transfers τ such that

(i) at prices $\{q_t\}_{t=0}^{\infty}$, for all $w_0 \in D$, $\sigma(w_0)$ minimizes (2.6) subject to (2.1)-(2.4);

(ii) for all t, (2.5) holds with equality;

(iii) $(1 - q_0) + \sum_{t=1}^{\infty}(1 - q_t)\prod_{s=0}^{t-1} q_s < +\infty$.

Then the allocation σ attains ψ_0 with transfers τ and is efficient.

Proof: That σ attains ψ_0 with transfers τ is immediate. We prove that σ is efficient by contradiction. Suppose that there exists some other allocation $\tilde{\sigma} = \{\tilde{x}_t(w_0, \theta^t), \tilde{h}_t(w_0, \theta^t)\}_{t=0}^{\infty}$ that attains ψ_0 with transfers $\tilde{\tau} < \tau$. Thus, by (ii),

$$\int_{D\times\Theta^{t+1}} \{C[\tilde{x}_t(w_0, \theta^t)] - \theta_t \tilde{h}_t(w_0, \theta^t) y\} d\mu^{t+1} d\psi_0$$

$$< \int_{D\times\Theta^{t+1}} \{C[x_t(w_0, \theta^t)] - \theta_t h_t(w_0, \theta^t) y\} d\mu^{t+1} d\psi_0$$

for all t, with the difference between these two quantities being at least $\tau - \tilde{\tau}$. Then,

$$(1 - q_0) \int_{\Theta} \{C[\tilde{x}_0(w_0, \theta)] - \theta \tilde{h}_0(w_0, \theta) y]\} d\mu +$$

$$\sum_{t=1}^{\infty}(1-q_t)\prod_{s=0}^{t-1}q_s\int_{\Theta^{t+1}}\{C[\tilde{x}_t(w_0,\theta^t)]-\theta_t\tilde{h}_t(w_0,\theta^t)y\}d\mu^{t+1}$$

$$<(1-q_0)\int_{\Theta}\{C[x_0(w_0,\theta)]-\theta h_0(w_0,\theta)y]\}d\mu+$$

$$\sum_{t=1}^{\infty}(1-q_t)\prod_{s=0}^{t-1}q_s\int_{\Theta^{t+1}}\{C[x_t(w_0,\theta^t)]-\theta_t h_t(w_0,\theta^t)y\}d\mu^{t+1}<+\infty$$

where the last inequality follows from (iii). This contradicts the hypothesis (i) that $\sigma(w_0)$ minimizes (2.6) for each w_0. ∎

Theorem 1 is an analogue to the first theorem of welfare economics, with conditions (i)-(iii) defining the counterpart to a competitive equilibrium. Condition (i) requires quantities to be optimal (cost minimizing) for each w, given prices; condition (ii) is market clearing; and condition (iii) is a boundedness condition on prices.

We do not have a general method for obtaining price sequences $\{q_t\}_{t=0}^{\infty}$ that will clear these markets among planners. In the sections that follow, we develop a procedure for finding a price q and a distribution of entitlements ψ_q that prevails in a steady state. That is, we develop a procedure to find a constant sequence of prices all equal to q, an allocation σ_q, and a distribution of entitlements ψ_q, such that these objects satisfy the conditions of Theorem 1 with $\tau = 0$.

3. A Recursive Formulation of the Problem:

In this section we define and study a Bellman equation that characterizes solutions to the component planning problem (2.6) when the price sequence $\{q_t\}_{t=0}^{\infty}$ is constant at fixed $q \in [\beta, 1)$. In section 4, we use this Bellman equation to characterize the dynamics of consumption and employment under the cost minimizing unemployment insurance scheme at constant price q. In section 5, we use this characterization of consumption and employment dynamics to find a distribution of entitlements ψ_q that prevails in the steady state at constant price q. This result gives us, for each $q \in [\beta, 1)$, a constant price sequence $\{q_t\}_{t=0}^{\infty}$, an allocation σ_q, a distribution of entitlements ψ_q, and a level of transfers $\tau(q)$ that satisfy the hypotheses of Theorem 1.

The function $\hat{V}_q$ on D defined by setting $\hat{V}_q(w_0)$ equal to the infimum of (2.6) over allocations that satisfy (2.1)-(2.4) when prices are constant at q is clearly unbounded. In defining our Bellman equation, our approach will be to set an upper bound $\overline{w}$ on entitlements and to consider cost functions on the bounded set $\bar{D} = [\underline{w}, \overline{w}]$. We formulate and study the Bellman equation on $C(\bar{D})$, where $C(\bar{D})$ is the space of bounded, continuous functions on $\bar{D}$, and obtain the corresponding optimal policy functions. Then we will show that if the bound $\overline{w}$ is chosen large enough, it will not be binding for any initial entitlements $w_0 \in \bar{D}$, so that our optimal policy functions are also cost minimizing for the original, unbounded problem (2.6) for any such w_0.

The Bellman Equation is specified as follows. Define the operator T_q on $C(\bar{D})$ by:

$$(T_qV)(w) = \inf_{u,l,g}(1-q)[\pi C(u(1)) + (1-\pi)C(u(0)) - \pi ly] + \tag{3.1}$$

$$q[\pi V(g(1)) + (1-\pi)V(g(0))]$$

where $u : \{0,1\} \to D$, $l \in [0,1]$, and $g : \{0,1\} \to \bar{D}$ satisfy the constraints:

$$w = (1-\beta)[\pi u(1) + (1-\pi)u(0) - \pi lv] +$$

$$\beta[\pi g(1) + (1-\pi)g(0)], \tag{3.2}$$

$$(1-\beta)[u(1) - lv] + \beta g(1) \geq (1-\beta)u(0) + \beta g(0). \tag{3.3}$$

The constraint (3.2) requires that an agent entitled to discounted expected utility w from the current period on receives that utility. It is analogous to (2.1). The constraint (3.3) is a single-period version of the incentive compatibility constraint (2.2). It requires that agents who plan to report their future job opportunities truthfully also find it optimal to report their current job opportunity truthfully.

Lemma 3.1 The operator T_q has a unique fixed point V_q in $C(\bar{D})$, and, for all $V \in C(\bar{D})$, $\lim_{n\to\infty} T_q^n V = V_q$. The function

V_q is increasing and convex. For all $w \in \bar{D}$, the infimum on the right hand side of (3.1) is attained.

Proof: The proof is standard. Applying T_q involves minimizing a continuous function over a compact set and Theorems 4.6, 4.7, and 4.8 of Stokey, Lucas, and Prescott (1989) apply. ∎

We obtain further facts about the value function V_q and about the minimizing policies u_q, l_q, g_q by studying the sequence of functions $\{T_q^n C\}_{n=0}^{\infty}$, where C is the inverse of the momentary utility function U, and then applying the fact that this sequence converges to V_q. To this end, we establish some facts about $T_q V$ when V is *any* convex function in $C(\bar{D})$.

Lemma 3.2 For any convex $V \in C(\bar{D})$, the minimum on the right hand side of (3.1) can be attained at a point at which the incentive constraint (3.3) holds with equality.

Proof: Suppose (3.1) is minimized at a point (u, l, g) such that (3.3) is slack. Then, since C is strictly convex, $u(1) = u(0)$, and thus, by (3.3), $\beta g(1) - (1-\beta) l v > \beta g(0)$. Since V is convex, with l fixed it is possible to choose $a > 0$ to raise $\hat{g}(0) = g(0) + a$ and lower $\hat{g}(1) = g(1) - a$ until (3.3) binds without raising the right hand side of (3.1). ∎

In view of this lemma, the two constraints (3.2) and (3.3) can be replaced by the equalities

$$w + (1-\beta)lv = (1-\beta)u(1) + \beta g(1) \tag{3.4}$$

$$w = (1-\beta)u(0) + \beta g(0) \tag{3.5}$$

It then follows that for any convex $V \in C(\bar{D})$ and $w \in \bar{D}$, $T_q V$ satisfies

$$T_q V(w) = \tag{3.6}$$

$$\min_{l,g}(1-q)[\pi C(\frac{w-\beta g(1)}{1-\beta} + lv) + (1-\pi)C(\frac{w-\beta g(0)}{1-\beta})] -$$
$$(1-q)lv + q[\pi V(g(1)) + (1-\pi)V(g(0))]$$

with $g : \{0,1\} \to \bar{D}$ and $l \in [0,1]$.

Lemma 3.3: For any strictly increasing and strictly convex $V \in C(\bar{D})$ and any $w \in \bar{D}$, the minimum on the right hand side of (3.6) is attained by a unique $l(w), g(w,\theta)$ (and hence $u(w,\theta)$) and these functions are continuous. Furthermore, $T_q V$ is continuous, strictly increasing, and strictly convex.

Proof: For each $w \in \bar{D}$, the right hand side of (3.6) is continuous in the arguments $l, g(\theta)$. The arguments l and $g(\theta)$ are restricted to lie in closed intervals. Since C is strictly convex and V is convex, these choices are unique. By the Theorem of the Maximum, $T_q V$ is continuous. That $T_q V$ is strictly increasing and strictly convex follows from the assumed properties of V and C.∎

For any convex, differentiable $V \in C(\bar{D})$, the first order conditions that characterize the minimizing choice of $l, g(\theta)$ include

$$C'(\frac{w-\beta g(w,1)}{1-\beta}+l(w)v)-\frac{y}{v} = 0 \quad \text{if } l(w) \in (0,1),$$

$$C'(\frac{w-\beta g(w,1)}{1-\beta}+l(w)v)-\frac{y}{v} \leq 0 \quad \text{if } l(w) = 1, \tag{3.7}$$

$$C'(\frac{w-\beta g(w,1)}{1-\beta}+l(w)v)-\frac{y}{v} \geq 0 \quad \text{if } l(w) = 0.$$

The intertemporal first order conditions are given by

$$C'(\frac{w-\beta g(w,\theta)}{1-\beta}+\theta l(w)v) \leq \tag{3.8}$$

$$\frac{q}{\beta}\frac{(1-\beta)}{(1-q)}V'(g(w,\theta)) \quad \theta = 0,1,$$

where (3.8) holds with equality when $g(w,\theta) > \underline{w}$. The envelope condition is given by

$$\frac{d(T_qV)(w)}{dw} = \tag{3.9}$$

$$\frac{(1-q)}{(1-\beta)}[\pi C'(\frac{w-\beta g(w,1)}{1-\beta}+l(w)v)+(1-\pi)C'(\frac{w-\beta g(w,0)}{1-\beta})]$$

In the next two lemmas, we show that if $\overline{w}$ is large enough, the upper bound on entitlements does not bind the optimal policy functions g_q. To this end we analyze the sequence of functions $\{V_q^n\}_{n=0}^{\infty}$, $V_q^n \in C(\bar{D})$ for all n, defined by $V_q^0(w) = C(w)$ and

$V_q^{n+1}(w) = (T_q V_q^n)(w)$, $n \geq 0$. On each iteration n, we denote the optimal choice of l, g by l_n, g_n.

Lemma 3.4: Assume that $q \in [\beta, 1)$. Define w^0 to be the solution to $C'(w^0) = y/v$ and assume $w^0 > \underline{w}$. Then the optimal policy functions g_n satisfy:

(i) $g_n(w, 1)$ and $g_n(w, 0)$ are non-decreasing functions with $g_n(w, 1) \geq g_n(w, 0)$ for all w, and

(ii) for all n, $g_n(w, \theta) \leq w$ when $w \geq w^0$.

Proof: Given that C and V_q^n are convex, (3.8) gives the result (i). Result (ii) is proved by induction. Begin with the case with $n = 0$ and $V_q^0 = C$. If $g_0(w, 1) = \underline{w}$ for $w \geq w^0$ then the induction hypothesis is immediate. When $w \geq w^0$ and $g_0(w, 1) > \underline{w}$, then (3.8) holds with equality. Since $q \geq \beta$, substituting from the right hand side of (3.8) into the left hand side of (3.7) implies that $l(w) = 0$ if $g_0(w, 1) \geq w^0$. Thus (3.8) implies that $g_0(w, 1) \leq w$ for $w \geq w^0$. Since $g_0(w, 1) \geq g_0(w, 0)$ for all w, this proves the induction hypothesis for $n = 0$.

Now assume that result (ii) holds for $n-1$. Again, if it is the case that $g_n(w, 1) = \underline{w}$ for $w \geq w^0$ then the induction hypothesis is immediate. When $w \geq w^0$ and $g_n(w, 1) > \underline{w}$, then (3.8) holds with equality. From (3.9), we have that

$$V_n'(g_n(w, 1)) = \tag{3.10}$$

$$\frac{(1-q)}{(1-\beta)}[\pi C'(\frac{g_n(w,1)-\beta g_{n-1}(g_n(w,1),1)}{1-\beta}+l_{n-1}(w)v)+$$
$$(1-\pi)C'(\frac{g_n(w,1)-\beta g_{n-1}(g_n(w,1),0)}{1-\beta})]$$

By the induction hypothesis, $g_{n-1}(g_n(w,1),\theta) \leq g_n(w,1)$ when $g_n(w,1) \geq w^0$, so that, since $q \geq \beta$

$$\frac{q}{\beta}\frac{(1-\beta)}{(1-q)}V_n'(g_n(w,1)) \geq C'(g_n(w,1)) \tag{3.11}$$

when $g_n(w,1) \geq w^0$. Thus, by (3.7) and (3.8), $l_n(w) = 0$ when $g_n(w,1) > w^0$. Finally, by (3.8) and (3.11), $g_n(w,1) \leq w$ when $g_n(w,1) \geq w^0$. Thus, the induction hypothesis is proved for n. ∎

We sum up the implications of these results for the Bellman equation (3.1) in the following result.

Lemma 3.5 Assume that $C'(\underline{w}) < y/v$ and $C'(\overline{w}) > y/v$. Then for $q \in [\beta, 1)$, V_q – the unique fixed point of T_q – is strictly increasing, strictly convex, and continuously differentiable, with a derivative given by (3.9). The policy functions $\rho_q = (u_q, l_q, g_q)$ are continuous and satisfy properties (i) and (ii) of Lemma 3.4.

Proof: That V_q is strictly increasing and strictly convex follows from Lemmas 3.1 and 3.3, as do the existence and continuity of the policy functions.

By Theorem 3.8 of Stokey, Lucas, and Prescott, the sequence $\{u_n, l_n, g_n\}$ defined above in Lemma 3.4 converges uniformly to $\rho_q = (u_q, l_q, g_q)$. Hence g_q satisfies properties (i) and

(ii) of Lemma 3.4. It also follows that the sequence of derivatives $\{dV_q^n/dw\}$ defined in (3.9) converges uniformly to the expression given on the right hand side of (3.9) evaluated at g_q. Since $\{V_q^n\}$ converges uniformly to V_q, the expression (3.9) is the derivative of V_q. It is evidently continuous.∥

We now relate the solution to the Bellman equation (3.1) to the solution to the component planning problem (2.6). First note that a policy function $\rho = (u(w,\theta), l(w), g(w,\theta))$ can be used to generate an allocation σ in the following manner. Let $x_0(w_0,\theta) = u(w_0,\theta)$ and $h_0(w_0,\theta) = \theta l(w_0)$ for all $w_0 \in \bar{D}$, $\theta \in \{0,1\}$. Define $w_1(w_0,\theta) = g(w_0,\theta)$ for all $w_0 \in \bar{D}$, $\theta \in \{0,1\}$. Iterating on this procedure to complete the definition, set $x_t(w_0,\theta^t) = u(w_t(w_0,\theta^{t-1}),\theta_t)$, $h_t(w_0,\theta^t) = \theta_t l(w_t(w_0,\theta^{t-1}))$, and $w_{t+1}(w_0,\theta^t) = g(w_t(w_0,\theta^{t-1}),\theta^t)$ for all $t \geq 1$, $w_0 \in D$, $\theta^t \in \Theta^{t+1}$.

Lemma 3.6 Assume that $C'(\overline{w}) > y/v$. For any $q \in [\beta, 1)$, let ρ_q be the policy function that minimizes (3.1) subject to the constraints (3.2) and (3.3) and σ_q be the allocation generated by ρ_q. Then, for any $w_0 \in \bar{D}$, $w_0 < \overline{w}$, $\sigma_q(w_0)$ minimizes (2.6) subject to constraints (2.1)-(2.4).

Proof: By Stokey, Lucas, and Prescott, Theorems 4.3, 4.4, and 4.5, for all $w_0 \in \bar{D}$, $V_q(w_0)$ equals the infimum of (2.6), with prices constant at q, over the set of allocations $\sigma(w_0)$ which satisfy

$$U_t(w_0, \sigma, z^*, \theta^{t-1}) \in \bar{D} \tag{3.12}$$

$$U_t(w_0, \sigma, z^*, \theta^{t-1}) = \tag{3.13}$$

$$(1-\beta)[\pi(x_t(w_0, (\theta^{t-1}, 1)) - h_t(w_0, (\theta^{t-1}, 1))v)] +$$

$$(1-\beta)[(1-\pi)x_t(w_0, (\theta^{t-1}, 0))] +$$

$$\beta[\pi U_{t+1}(w_0, \sigma, z^*; (\theta^{t-1}, 1)) + (1-\pi)U_{t+1}(w_0, \sigma, z^*, (\theta^{t-1}, 0))],$$

and

$$(1-\beta)[x_t(w_0, (\theta^{t-1}, 1)) - h_t(w_0, (\theta^{t-1}, 1))v] +$$

$$\beta U_{t+1}(w_0, \sigma, z^*, (\theta^{t-1}, 1)) \geq \tag{3.14}$$

$$(1-\beta)x_t(w_0, (\theta^{t-1}, 0)) + \beta U_{t+1}(w_0, \sigma, z^*, (\theta^{t-1}, 0))$$

for all $t \geq 0$, $\theta^{t-1} \in \Theta^t$. Moreover, the allocation $\sigma_q(w_0)$ uniquely attains the minimum on the set satisfying (3.12)-(3.14).

By Lemma 3.4, $\sigma_q(w_0)$ satisfies (2.4). By the argument of Atkeson and Lucas (1992), Lemmas 3.1 and 3.2, the set of values of (2.6) (at price sequence $\{q_t\}_{t=0}^{\infty}$) that can be attained with allocations satisfying (2.1)-(2.4) is the same as the set of values of (2.6) than can be attained with allocations satisfying (3.13), (3.14), and (2.4). Thus, $\sigma_q(w_0)$ uniquely attains the minimum of (2.6) subject to (2.1)-(2.4) and the additional constraint

$$U_t(w_0, \sigma, z^*, \theta^{t-1}) \leq \overline{w} \tag{3.15}$$

for all $t \geq 0$, $\theta^{t-1} \in \Theta^t$.

Finally, Lemma 3.5 and the hypothesis $C'(\overline{w}) > y/v$ imply that the inequality (3.15) is never binding on $\sigma_q(w_0)$ for $w_0 < \underline{w}$. Since the constraint set (2.1)-(2.4) is convex, this implies that $\sigma_q(w_0)$ minimizes (2.6) among the set of allocations satisfying (2.1)-(2.4).∎

4. Characterization of the Policy Functions

In this section we characterize the policy functions $g_q(w, \theta)$ and $l_q(w)$ and then analyze the Markov process in entitlements generated by these policy functions.

The behavior of $g_q(w, \theta)$ and $l_q(w)$ as functions of w on the interval $[\underline{w}, \infty)$ is drawn in Figure 1 and can be described as follows. Provided that

$$C'(\underline{w} + v) < y/v, \tag{4.1}$$

there is an interval $[\underline{w}, w_q^1]$ on which $l_q(w) = 1$, an interval (w_q^1, w_q^2) on which $l_q(w) \in (0, 1)$, and an interval $[w_q^2, \infty)$ on which $l_q(w) = 0$. The value w_q^1 is determined as the solution to the equation

$$C'(\frac{w_q^1 - \beta g_q(w_q^1, 1)}{1 - \beta} + v) = y/v, \tag{4.2}$$

where $g_q(w_q^1, 1)$ solves the equation

$$\frac{q}{\beta} \frac{(1 - \beta)}{(1 - q)} V'(g) = y/v. \tag{4.3}$$

Condition (4.1) guarantees that $w_q^1 > \underline{w}$. The value w_q^2 is determined as the solution to the equation

$$C'(\frac{w_q^2 - \beta g_q(w_q^2, 1)}{1-\beta}) = y/v, \tag{4.4}$$

where $g_q(w_q^2, 1)$ solves the equation (4.3) (and is thus equal to $g_q(w_q^1, 1)$). For entitlements $w \geq w_q^2$, the resource cost of compensating an agent for the disutility of work is less than that agent's product on the job. Thus, no agent with such high entitlements works and $g_q(w, 1) = g_q(w, 0)$ for all $w \geq w_q^2$. For all $w \in (w_q^1, w_q^2)$, $g_q(w, 1)$ is constant at the value that solves equation (4.3) and $l_q(w)$ decreases in w from 1 to 0 so as to preserve the equality (3.4).

Figure 1 is drawn so that the curve $g_q(w, 0)$ lies everywhere below the 45 degree line, while $g_q(w, 1)$ lies above the 45 degree line on the interval $[\underline{w}, w_q^1]$ and crosses that line in the interval (w_q^1, w_q^2). The first of these features is necessary given the assumptions we have already made; the second feature is not. The next two lemmas describe the possibilities.

Lemma 4.1: Assume $q > \beta$. If $g_q(w, 1)$ and $g_q(w, 0)$ are the policy functions that minimize (3.6), then

(i) there exists some $\delta > 0$ and $k > 0$ such that $g_q(w, 0) = \underline{w}$ for all $w \in [\underline{w}, \underline{w} + \delta)$ and $g_q(w, 0) \leq w - k$ for all $w \in (\underline{w} + \delta, w_q^2]$;

(ii) $g_q(\underline{w}, 1) > \underline{w}$ if and only if

$$C'(w+v) > \frac{q}{\beta}[\pi C'(w+v) + (1-\pi)C'(w)] \qquad (4.5)$$

holds at $w = \underline{w}$.

Proof: To verify (i), use (3.8), (3.9) and the result that $g_q(w,1) \geq g_q(w,0)$ for all w to conclude that if $g_q(w,0) > \underline{w}$ then

$$V'(w) = \frac{q}{\beta}[\pi V'(g_q(w,1)) + (1-\pi)V'(g_q(w,0))]. \qquad (4.6)$$

If $g_q(w,0) \geq w$ for $w > \underline{w}$, then (4.6) implies that $V'(w) \geq \frac{q}{\beta}V'(w)$. Since $q > \beta$, this is a contradiction. Thus $g_q(w,0) < w$ for all $w > \underline{w}$. To prove (i), observe that, by the continuity of $V'(\cdot)$, there must exist some $k > 0$ such that

$$V'(w) \leq \frac{q}{\beta}V'(w-k)$$

for all $w \in [\underline{w}, w_q^2]$. Using the same argument above, for all $w \in [\underline{w}, w_q^2]$, (4.6) cannot hold with $g_q(w,0) > \underline{w}$ unless $g_q(w,0) \leq w-k$. Thus $g_q(w,0) > \underline{w}$ implies $g_q(w,0) \leq w-k$ and $g_q(w,0) = \underline{w}$ for $w \in [\underline{w}, \underline{w}+k)$. Since $g_q(w,0)$ is continuous and non-decreasing in w, $g_q(w,0) = \underline{w}$ over an interval and result (i) is proved.

To prove (ii), note that $g_q(\underline{w},0) = \underline{w}$. Thus, (3.8) and (3.9) imply that if $g_q(\underline{w},1) = \underline{w}$ as well, then (4.5) fails to hold at

$w = \underline{w}$. Thus, if (4.5) does hold at $w = \underline{w}$, then $g_q(\underline{w}, 1) > \underline{w}$. Conversely, if (4.5) fails to hold at $w = \underline{w}$, then it cannot be the case that (3.8) and (3.9) are satisfied with some $g_q(\underline{w}, 1) > \underline{w}$.∥.

Lemma 4.1 does not cover the case $q = \beta$. We treat this case separately in the next result.

Lemma 4.2: Assume $q = \beta$. If $g_q(w, 1)$ and $g_q(w, 0)$ are the policy functions that minimize (3.6), then

(i) there exists $k > 0$ such that $g_q(w, 1) \geq w + k$ for all $w \in [\underline{w}, w_q^1]$;

(ii) $g_q(w, 1) = w_q^2$ for all $w \in [w_q^1, w_q^2]$; and

(iii) $g_q(w, 1) = g_q(w, 0) = w$ for all $w \geq w_q^2$.

Proof: Equations (3.8) and (3.9) imply

$$V'(w) \leq \pi V'(g_q(w, 1)) + (1 - \pi)V'(g_q(w, 0)). \qquad (4.7)$$

For $w \in [\underline{w}, w_q^1]$, $l_q(w) = 1$, and thus (3.8) and the strict concavity of C and V imply that $g_q(w, 1)$ is uniformly bounded above $g_q(w, 0)$ on this interval. Thus, if $g_q(w, 1) = w$ for some $w \in [\underline{w}, w_q^1]$, then $g_q(w, 0) < w$ and by the strict convexity of V_q, (4.7) implies the contradiction $V_q'(w) < V_q'(w)$. Since condition (4.5) is automatically satisfied when $q = \beta$, $g_q(\underline{w}, 1) > \underline{w}$. Thus, by the continuity of $g_q(w, 1)$, $g_q(w, 1) > w$ for all $w \in [\underline{w}, w_q^1]$. Since $g_q(w, 1)$ is uniformly bounded above $g_q(w, 0)$, for all $w \in [\underline{w}, w_q^1]$, (4.7) implies that $g_q(w,1)$ is uniformly bounded above w on this interval. Thus, (i) is proved.

Result (iii) is proved by the observation that $V(w) = C(w)$ with $g_q(w,1) = g_q(w,0) = w$ for all $w \geq w_\beta^2$ solves the Bellman equation (3.6) when $q = \beta$.

Equations (3.7) and (3.8) imply that $g_q(w,1)$ is constant for all $w \in [w_q^1, w_q^2]$. Thus (ii) is implied by (iii).∎

5. Analysis of the Entitlement Process

The previous section completes the characterization of the value and the policy functions which solve the Bellman equation (3.1) for any fixed $q \in [\beta, 1)$. We now turn to the study of the Markov processes defined by the job finding probability π and the policy functions $g_q(w,\theta)$. For any $q \in [\beta, 1)$, the state space of the entitlement process is $[\underline{w}, \infty)$. If $q > \beta$, the results in section 3 imply that the ergodic sets of this process must be subsets of the set $[\underline{w}, w_q^2]$. If $q = \beta$, then there is at least one ergodic set in the interval $[\underline{w}, w_\beta^2]$, but it also the case that every point $w \geq w_\beta^2$ is also an ergodic set. To take all of these possibilities into account, we first study the processes generated by $(\pi, g_q(w,\theta))$ on the set $S = [\underline{w}, w_\beta^2]$ and then deal with the additional possibilities that arise when $q = \beta$.

Let λ be any probability measure on the Borel sets $\mathbf{S}$ of S, and define the Markov operator P_q by

$$(P_q\lambda)(A) = \pi \int_{\{g_q(w,1)\in A\}} d\lambda + (1-\pi) \int_{\{g_q(w,0)\in A\}} d\lambda$$

for any $A \in \mathbf{S}$.

Lemma 5.1: Assume $q \in [\beta, 1)$. The process $(\pi, g_q(w,\theta))$ has a unique invariant distribution ψ_q in $(S, \mathbf{S})$, the unique fixed point of P_q, and for any probability measure λ, $P_q^n\lambda$ converges to ψ_q in the total variation norm.

Proof: The proof is divided into two cases: $q \in (\beta, 1)$ and $q = \beta$. In both cases, the proof is an application of Theorem 11.12 of Stokey, Lucas, and Prescott (1989).

Part 1: Assume $q \in (\beta, 1)$. Let λ_w be the probability measure that concentrates mass on the point w. We show that there exist $N \geq 1$ and $\epsilon > 0$ such that $(P_q^N\lambda_w)(\underline{w}) \geq \epsilon$ for all $w \in S$. By result (i) of Lemma 4.1, there exists $k > 0$ such that either $g_q(w,0) \leq w - k$ or $g_q(w,0) = \underline{w}$ for all $w \in [\underline{w}, w_q^2]$. Choose N large enough so that $w_q^2 - Nk < \underline{w}$. Then the probability of passing from the point w_q^2 to the point $\underline{w}$ in N steps (that is $(P_q^N\lambda_{w_q^2})(\underline{w})$) is at least $(1-\pi)^N$. Since $g_q(w,0)$ is non-decreasing in w, this transition to $\underline{w}$ is at least as probable from any other point in S, so letting $\epsilon = (1-\pi)^N$, the proof that the Markov process under study satisfies the hypotheses of Theorem 11.12 in Stokey, Lucas, and Prescott (1989) is complete.

Part 2: The proof in the case with $q = \beta$ is slightly different. In this case, we show that there exist $N \geq 1$ and $\epsilon > 0$ such that $(P_q^N \lambda_w)(w_\beta^2) \geq \epsilon$ for all $w \in S$. By result (i) of Lemma 4.2, there exists $k > 0$ such that $g_\beta(w, 1) \geq w + k$ for all $w \in [\underline{w}, w_q^1]$. By result (ii), $g_\beta(w, 1) = w_\beta^2$ for all $w \in [w_\beta^1, w_\beta^2]$. Choose N large enough so that $\underline{w} + (N-1)k > w_\beta^2$. Then the probability of passing from the point $\underline{w}$ to the point w_β^2 in N steps (that is $P_q^N(\lambda_{\underline{w}})(w_\beta^2)$) is at least π^N. Since $g_q(w, 1)$ is non-decreasing in w, this transition to w_β^2 is at least as probable from any other point in S, so letting $\epsilon = \pi^N$, the proof that the Markov process under study satisfies the hypotheses of Theorem 11.12 in Stokey, Lucas, and Prescott (1989) is complete.∎

In view of Lemma 5.1, the function

$$\tau(q) = \int_D V_q(w) d\psi_q$$

is well-defined for $q \in [\beta, 1)$. The function τ has the interpretation as the constant, net inflow of resources required to attain the entitlement distribution ψ_q in the steady state. In the rest of this section we provide conditions under which $\tau(q) = 0$ is satisfied for a price $q \in [\beta, 1)$. We first examine the values of this function at $q = \beta$ and q near one. Then we establish the continuity of τ (Lemmas 5.2-5.4). Finally, we establish that $\tau(q)$ is decreasing in q (Lemmas 5.5 and 5.6).

In the case that $q = \beta$, the point w^2_β is an absorbing state, and, as shown in the proof of Lemma 5.1, part 2, the probability of transiting from any other point in the state space to w^2_β in N steps is strictly positive. Thus the unique invariant distribution in this case is concentrated at the point w^2_β. At this point, no one works, so the cost of attaining this distribution in the steady state is $\tau(\beta) = C(w^2_\beta)$. This cost is clearly greater than zero.

In the case that $q > \beta$ and

$$C(\underline{w} + v) \leq \frac{q}{\beta}[\pi C'(\underline{w} + v) + (1 - \pi)C'(\underline{w})]$$

for q close to one, then $g_q(\underline{w}, 1) = \underline{w}$, and $\underline{w}$ is an absorbing state. As shown in the proof of Lemma 5.1, part 1, the probability of transiting from any other point in the state space to $\underline{w}$ in N steps is strictly positive. Thus, the unique invariant distribution in this case is concentrated at the point $\underline{w}$. At this point, everyone who has a job opportunity works $l(\underline{w}) = 1$, so the cost of attaining this distribution in the steady state is

$$\lim_{q \to 1} \tau(q) = \pi C(\underline{w} + v) + (1 - \pi)C(\underline{w}) - \pi y.$$

If $\underline{w}$ is too large relative to y and π, then this cost is also positive and there is no market clearing price q. We assume that this quantity is negative.

Lemma 5.2 Let $q \in [\beta, 1)$ and $\{q_n\}_{n=0}^{\infty}$, $q_n \in [\beta, 1)$ be a sequence of prices converging to q. Then $\{V_{q_n}\}_{n=0}^{\infty}$ converges uniformly to V_q on $[\underline{w}, w_\beta^2]$.

Proof: We show that $\|V_{q'} - V_q\| \to 0$ as $q' \to q$. For all $n \geq 1$,

$$\|V_{q'} - V_q\| \leq \|V_{q'} - T_{q'}^n V_q\| + \|T_{q'}^n V_q - V_q\|.$$

Since $T_{q'}$ is a contraction mapping with fixed point $V_{q'}$, $\|V_{q'} - T_{q'}^n V_q\| \to 0$ as $n \to \infty$ for all values of q and q'. The term

$$\|T_{q'}^n V_q - V_q\| \leq \sum_{k=1}^{n} \|T_{q'}^k V_q - T_{q'}^{k-1} V_q\| \leq \sum_{k=1}^{n} q'^k \|T_{q'} V_q - V_q\|,$$

where the last inequality follows from the fact that $T_{q'}$ is a contraction mapping with modulus q'. Thus, as $n \to \infty$, $\|T_{q'}^n V_q - V_q\|$ converges to a quantity less than or equal to $\frac{1}{1-q'}\|T_{q'} V_q - V_q\|$. Since $V_q = T_q V_q$, the result that $\|T_{q'} V_q - V_q\| \to 0$ as $q' \to q$ follows from the observation that the operator T_q is continuous in q by the Theorem of the Maximum.‖

Lemma 5.3 Let $\{(q_n, w_n)\}_{n=0}^{\infty}$, $q_n \in [\beta, 1)$, $w_n \in [\underline{w}, w_\beta^2]$, be a sequence converging to the point (q, w). Then the sequence $\{(g_{q_n}(w_n, 1), g_{q_n}(w_n, 0))\}_{n=0}^{\infty}$ converges to $(g_q(w, 1), g_q(w, 0))$.

Proof: Since the policy functions $g_q(w, \theta)$ are continuous in w, for all $\epsilon > 0$, there exists an $N \geq 1$ such that $|g_q(w_n, \theta) - g_q(w, \theta)| < \epsilon$, $\theta = 0, 1$, for all $n \geq N$. By Lemma 5.2 and Stokey, Lucas, and Prescott (1989), Theorem 3.8, for all $\epsilon > 0$, there exists an $N \geq 1$ such that for $n \geq N$, $|g_{q_n}(w,\theta) - g_q(w,\theta)| < \epsilon$, $\theta = 0, 1$, for all $w \in [\underline{w}, w_\beta^2]$. Thus, for all $\epsilon > 0$, there exists an $N \geq 1$ such that $|g_{q_n}(w_n,\theta) - g_q(w,\theta)| < \epsilon$ for all $n \geq N$.∥

Lemma 5.4 $\tau(q)$ is continuous on $[\beta, 1)$.

Proof: By Lemma 5.1 and Stokey, Lucas, and Prescott (1989), Theorem 12.13, if $\{q_n\}_{n=0}^{\infty}$ converges to q, then $\{\psi_{q_n}\}_{n=0}^{\infty}$ converges to weakly to ψ_q. By Lemma 5.2, $\{V_{q_n}\}_{n=0}^{\infty}$ converges uniformly to V_q on $[\underline{w}, w_\beta^2]$. Thus the lemma is proved.∥

Lemma 5.5 If $q' > q$, then $\psi_{q'} \leq \psi_q$ (where "$\leq$" here denotes first order stochastic dominance).

Proof: We first use the first order conditions to show that the optimal policies $g_q(w,\theta)$ are decreasing in q. Then we use this fact to prove the lemma.

Let $q' > q$. For $n \geq 1$, let $V_{q'}^n = T_{q'}^n V_q$, and u_n, l_n, g_n be the associated optimal policies. We first use an induction to show that for all (w,θ) and $n \geq 1$,

$$C'(\frac{w - \beta g_n(w,\theta)}{1-\beta} + \theta l_n(w)v) \geq \tag{5.1}$$

$$C'(\frac{w - \beta g_q(w,\theta)}{1-\beta} + \theta l_q(w)v).$$

For $n = 1$, $q' > q$ implies

$$\frac{q'}{\beta}\frac{(1-\beta)}{(1-q')}\frac{dV_q}{dw} \geq \frac{q}{\beta}\frac{(1-\beta)}{(1-q)}\frac{dV_q}{dw}. \tag{5.2}$$

Hence (3.8) implies that $g_1(w,0) \leq g_q(w,0)$ for all w and (5.1) is proved for $\theta = 0$ and $n = 1$. From (4.2) – (4.4) and the fact that the functions $g_1(w,\theta)$ and $g_q(w,\theta)$ are non-decreasing in w, the points w_q^1 and w_q^2 decrease as q is increased, so that $l_1(w) \leq l_q(w)$ for all w. If $l_1(w) = l_q(w) = 1$ or $l_1(w) = l_q(w) = 0$, then (3.8) and (5.2) imply $g_1(w,1) \leq g_q(w,1)$ and (5.1). If $l_1(w) < l_q(w) = 1$, then (5.1) follows directly from (3.7). If $l_1(w), l_q(w) \in (0,1)$, then (3.7) implies that (5.1) holds as an equality. Finally, if $l_1(w) = 0$ and $l_q(w) > 0$, then (5.1) follows again from (3.7). Hence, (5.1) holds for all $w \in D$ when $n = 1$.

Now assume (5.1) holds for $n = N$. Then from (3.9) and the induction hypothesis

$$\frac{q'}{\beta}\frac{(1-\beta)}{(1-q')}\frac{dV_{q'}^{N+1}(w)}{dw} =$$

$$\frac{q'}{\beta}[\pi C'(\frac{w-\beta g_N(w,1)}{1-\beta} + l_N(w)v) + (1-\pi)C'(\frac{w-\beta g_N(w,0)}{1-\beta})]$$

$$\geq \frac{q}{\beta}[\pi C'(\frac{w-\beta g_q(w,1)}{1-\beta} + l_q(w)v) + (1-\pi)C'(\frac{w-\beta g_q(w,0)}{1-\beta})]$$

$$= \frac{q}{\beta}\frac{(1-\beta)}{(1-q)}\frac{dV_q(w)}{dw},$$

for all $w \in D$. Then, (3.8) implies that $g_{N+1}(w,0) \leq g_q(w,0)$. When $\theta = 1$, the reasoning used in the case $n = 1$ implies $l_{N+1}(w) \leq l_q(w)$, $g_{N+1}(w,1) \leq g_q(w,1)$, and that (5.1) holds for $n = N+1$. Thus, if $q' > q$, then $g_{q'}(w,\theta) \leq g_q(w,\theta)$ for all $w \in \bar{D}$ and $\theta = 0,1$.

Now let $\lambda^n_{q'} = (P^n_{q'}\psi_q)$ where ψ_q is the invariant distribution of entitlements corresponding to price q. From the result above, $\lambda^1_{q'} = (P_{q'}\psi_q) \leq (P_q\psi_q) = \psi_q$. Since $P_{q'}$ is monotone and $\lambda^1_{q'} \leq \psi_q$, for $n \geq 1$, $\lambda^{n+1}_{q'} \leq \lambda^n_{q'} \leq \psi_q$. By Lemma 5.1, $\{\lambda^n_{q'}\}^{\infty}_{n=0} \to \psi'_q$ in the total variation norm, so the lemma is proved.‖

Lemma 5.6 $\tau(q)$ is decreasing in q.

Proof: By definition,

$$\tau(q) = \int_D V_q(w)d\psi_q = \tag{5.3}$$

$$\int_{D\times\Theta} \{(1-q)[C(u_q(w,\theta)) - \theta l_q(w)y] + qV_q(g_q(w,\theta))\}d\mu d\psi_q.$$

Since ψ_q is the invariant distribution,

$$\psi_q = S_{g_q}\psi_q \tag{5.4}$$

and

$$\int_D V_q(w)d\psi_q = \int_{D\times\Theta} V_q(g_q(w,\theta))d\mu d\psi_q. \tag{5.5}$$

Thus,

$$\int_D V_q(w)d\psi_q = \int_{D\times\Theta} \{C(u_q(w,\theta)) - \theta l_q(w)y\}d\mu d\psi_q. \tag{5.6}$$

Now consider changing the price q to q' and calculating the integral $\int (T_{q'}V_q)(w)d\psi_q$. Let u_0, l_0, g_0 be the optimal controls associated with the determination of $T_{q'}V_q$, so that

$$\int_D (T_{q'}V_q)(w)d\psi_q =$$

$$\int_{D\times\Theta} \{(1-q')[C(u_0(w,\theta)) - \theta l_0(w)y] + q'V_q(g_0(w,\theta))\}d\mu d\psi_q.$$

Since u_0, l_0, g_0 minimize the right hand side of (3.1),

$$\int_{D\times\Theta} \{(1-q')[C(u_0(w,\theta)) - \theta l_0(w)y] + q'V_q(g_0(w,\theta))\}d\mu d\psi_q$$

$$\leq \int_{D\times\Theta} \{(1-q')[C(u_q(w,\theta)) - \theta l_q(w)y] + q'V_q(g_q(w,\theta))\}d\mu d\psi_q.$$

Thus, by (5.5) and (5.6), $\int (T_{q'}V_q)(w)d\psi_q \leq \int V_q(w)d\psi_q$. Let $V^1_{q'} = T_{q'}V_q$ and $V^{n+1}_{q'} = T_{q'}V^n_{q'}$ for all $n \geq 1$. Let u_n, l_n, g_n be the optimal policies associated with evaluating $T_{q'}V^n_{q'}$.

We prove $\int_D V_{q'}(w)d\psi_q \leq \int_D V_q(w)d\psi_q$ by induction. As an induction hypothesis, assume that $\int V^n_{q'}(w)d\psi_q \leq \int V_q(w)d\psi_q$. By (5.4),

$$\int_{D\times\Theta} V^n_{q'}(g_q(w,\theta))d\mu d\psi_q = \tag{5.7}$$

$$\int_D V^n_{q'}(w)dS_{g_q}(\psi_q) = \int_D V^n_{q'}(w)d\psi_q.$$

By the definition of $T_{q'}$

$$\int_D (T_{q'}V^n_{q'})(w)d\psi_q =$$

$$\int_{D\times\Theta}\{(1-q')[C(u_n(w,\theta))-\theta l_n(w)y]+q'V_{q'}^n(g_n(w,\theta))\}d\mu d\psi_q.$$

Since u_n, l_n, g_n minimize the right hand side of (3.1),

$$\int_{D\times\Theta}\{(1-q')[C(u_n(w,\theta))-\theta l_n(w)y]+q'V_{q'}^n(g_n(w,\theta))\}d\mu d\psi_q$$

$$\leq \int_{D\times\Theta}\{(1-q')[C(u_q(w,\theta))-\theta l_q(w)y]+q'V_{q'}^n(g_q(w,\theta))\}d\mu d\psi_q$$

$$= \int_D\{(1-q')V_q(w)+q'V_{q'}(w)\}d\psi_q$$

where the last equality follows from (5.4) and (5.5). By the induction hypothesis,

$$\int_D\{(1-q')V_q(w)+q'V_{q'}(w)\}d\mu d\psi_q \leq \int_D V_q(w)d\psi_q.$$

Thus,

$$\int(T_{q'}V_q^n)(w)d\psi_q \leq \int V_q(w)d\psi_q.$$

Since $\lim_{n\to\infty} T_{q'}^n V_q = V_{q'}$ we have proved that

$$\int_D V_{q'}(w)d\psi_q \leq \int_D V_q(w)d\psi_q.$$

By Lemma 5.5, ψ_q first order stochastically dominates $\psi_{q'}$ when $q' > q$. Then since $V_{q'}$ is increasing in w,

$$\int_D V_{q'}(w)d\psi_{q'} \leq \int_D V_{q'}(w)d\psi_q \leq \int_D V_q(w)d\psi_q$$

and we are done.∎

6. Conclusion:

In this paper, we have presented a model of the long run consequences of efficient unemployment insurance for the distribution of welfare and consumption in a simple environment in which workers face idiosyncratic, serially uncorrelated employment risk. Under the assumptions of the preceding sections, there exists an invariant distribution of utility entitlements and an associated invariant distribution of consumption and employment. The invariant distribution has a mass point at the lower bound on utility entitlements ($\underline{w}$) and also distributes probability over higher entitlement levels. The existence of a mass point follows from Thomas and Worrall's (1990) proof that entitlements to utility must converge to their lower bound with probability one. In our case, the lower bound on entitlements serves as a reflecting barrier rather than an absorbing one as in Thomas and Worrall. It is clear, then, that if the lower bound on entitlements were removed, no steady state would exist.

The dynamics of an individual's entitlement within the invariant distribution are described by the solution to a one-on-one principal agent contracting problem between the individual worker and an unemployment insurance intermediary. In that contracting problem, the intermediary minimizes the resource cost, evaluated at a fixed intertemporal price given by q, of providing incentive compatible unemployment insurance to the worker. The first order conditions of that problem indicate that, when the minimum utility constraint is not binding, the worker's entitlements to discounted expected utility are set so that the ratio of the expected marginal cost to the intermediary of the worker's entitlement next period and the marginal cost to the intermediary of the worker's entitlement in the current period is set equal the intertemporal price set by q. When the minimum entitlement constraint is binding, this ratio of marginal costs exceeds the intertemporal price given by q. These first order conditions, and the dynamics of individual consumption implied by these first order conditions, are qualitatively very similar to those obtained from a model like Hansen and Imrohoroglu's in which consumption smoothing is achieved through pure credit markets with uncontingent borrowing and lending. In particular, as in Hansen and Imrohoroglu, when the minimum entitlement constraint is not binding, idiosyncratic movements in individual consumption in response to realizations of employment risk are highly persistent and in equilibrium follow a downward drift until the minimum entitlement constraint binds. When the minimum entitlement constraint is binding, then workers who fail to find a job opportunity experience a transitory fall in consumption while agents who find a job opportunity experience a persistent rise in their consumption. The quantitative differences between the equilibrium in this model and that in Hansen and Imrohoroglu would manifest themselves in the divergence between the lifetime utility and discounted present value of consumption of agents who are lucky in finding employment and agents who are unlucky.

In this model we have assumed that workers experience serially uncorrelated employment risk. It is clear that if one were to use this model to establish a benchmark against which to judge the efficiency of existing unemployment insurance schemes, it would be necessary to adapt the techniques used here to the analysis of the efficient invariant distribution of a model like Hansen and Imrohoroglu's in which this employment risk is serially correlated so as to match the risk in the model to data on the distribution of the length of employment and unemployment spells. We leave this to future work. At this point, we conclude with remarks on two questions that we have left aside in the body of the paper. The first of these questions concerns the role of our assumption that agents in the model cannot enter into contracts that would leave them at any point in time with discounted expected utility in some states of nature below some minimal entitlement to discounted expected utility. The second question concerns the possibilities for decentralizing the efficient allocation found in this model.

Regarding the role of the minimum entitlement constraint, we motivate our assumption prohibiting agents from entering agreements that require that they forgo all claims to future consumption in certain states of nature with the idea that ancestors in a dynasty have limited rights to sell the consumption of their heirs. Phelan (1993) motivates the same assumption in a similar model with the idea that workers cannot legally commit to remain in a contract that delivers them a discounted expected utility below the level that they could obtain by entering a new unemployment insurance contract with another insurance intermediary. In either case, by introducing this form of contract incompleteness into the model, we get the result that there is a non-degenerate steady state distribution of entitlements. This result introduces the following trade-off between equilibrium efficiency and equality into the model: stricter limits on agents' rights to trade away claims to future consumption reduce steady state inequality at the expense of limiting possibilities for insuring idiosyncratic employment risk and thus reducing welfare ex-ante. Relaxing these limits on contracting enhances efficiency at the cost of widening the spread in the long run distribution of entitlements and consumption. This tension between equality and efficiency arises in the model because considerations of efficiency dictate that movements in individual consumption be persistent in response to uninsured idiosyncratic shocks while equality requires that the movement of individual consumption in response to idiosyncratic shocks be bounded or show some mean reversion.

Regarding the possibilities for decentralizing the efficient allocation found here, the technique we use for finding the efficient allocation itself suggests one decentralization in which financial intermediaries compete in offering unemployment insurance contracts to clients who are then bound to work and consume as instructed by the intermediary for the rest of time, subject to the minimum entitlement constraint imposed on contracts. These intermediaries can be thought of as trading resources with each other through time at the price q in their competition to develop the low-cost dynamic unemployment insurance contract. One might think of the market as one in which workers join risk pools to sign long term contracts with insurance companies who then control the worker's consumption and work effort over time. The difficulty with this market interpretation, as is well known, is that, to implement the contract solved for here, the insurance intermediary must have the ability to prevent the worker from participating in any other asset market activity.

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Figure 1