EQUILIBRIUM ASSET PRICES WITH
UNDIVERSIFIABLE LABOR INCOME RISK

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ABSTRACT

In a two-period Lucas tree economy in which *ex ante* identical, but *ex post* dissimilar,

agents face undiversifiable labor income risk, calibrating a (wrong) representative agent model

results in overstating the equilibrium riskfree rate and in understanding the equilibrium equity

premium if the utility function exhibits decreasing absolute risk aversion and decreasing

absolute prudence. These behavioral assumptions provide, as a consequence, a theoretical

rationale for the often advanced conjecture that non-traded risk contributes to the solution of

the riskfree rate and equity premium puzzles.

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Introduction

The representative agent model with complete contingent claims markets has not been very successful in providing a framework within which to explain the equilibrium behavior of stock and bond returns. One of the most striking shortcomings of the paradigm was pointed out by Mehra and Prescott [1985]: in a Lucas [1978] exchange economy with time- and state additive preferences, the representative agent, complete market model dramatically underpredicts the equity premium, and overpredicts the rate of return on safe bonds.[1]

Recent work exploring, within the representative agent framework with complete markets, the possibility that the Mehra-Prescott results might be due to auxiliary restrictions on preferences and technology has not met with unmitigated success. For instance, I show elsewhere[2] that adopting "generalized isoelastic preferences" to relax the parametric restriction on risk aversion and intertemporal substitution imposed by time- and state-additivity does not significantly contribute to the solution of the equity premium puzzle. Constantinides [1988] proposes instead to allow for habit formation: while the introduction of addiction effects helps replicate the equity premium, it also leads to a serious overprediction of the variability of the riskfree rate. Kandell and Stambaugh [1991] argue, on their part, that the time- and state-additive model with complete markets does replicate the stylized facts if one allows the coefficient of relative risk aversion to be as large as 29; but the plausibility of such a parameterization of tastes is still being debated.[3] Introducing richer production structures in which output, dividend and consumption are distinct (see Rouwenhorst [1989]) also increases, but only partially, the descriptive realism of the representative agent model.

In this paper, I wish, therefore, to altogether abandon the representative agent, complete market paradigm, and to directly explore instead the equilibrium asset pricing implications of a model in which consumers face, in addition to aggregate dividend risk, idiosyncratic and undiversifiable labor income risk. The possibility that the incomplete market model might provide a solution to the equity premium and riskfree rate puzzles was first suggested by Mehra-Prescott in the conclusion of their [1985] paper, and explored by Mankiw

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1. I have dubbed the latter fact the "riskfree rate puzzle" in Weil [1989].
2. See Weil [1989]. The following is obviously not a complete survey.
3. These results are conveniently summarized within the very general framework of Hansen and Jagannathan [1988].

[1986] who studied a world in which negative aggregate shocks are concentrated on a small subset of the population.[4] It was also investigated recently by Aiyagari and Gertler [1990] in a model with transactions costs.[5]

It is the theoretical validity of this purported solution which I want to examine here: is it indeed the case that a model with incomplete markets will help us understand the pattern of equilibrium returns, both in terms of the level of equilibrium rates (the riskfree rate puzzle), and discrepancy between equity and bond returns (the equity premium puzzle)?

The answer is: not necessarily. The reason for this ambiguity is twofold. On the one hand, simply assuming that consumers are risk averse does not tell us whether the presence of independent non-insurable labor income risk will lead them to save more or less, and thus whether a model which mistakenly hypothesizes the existence of complete markets will over- or underpredict the level of the riskfree rate: according to the theory of precautionary saving, what will happen depends crucially on the sign of the third derivative of the consumers' utility function. On the other hand, it is immediate from the (partial equilibrium) theory of risk taking in the presence of independent "background" non-insurable risk that there is no *a priori* presumption, in the absence of restrictions on even higher derivatives of the utility function, that additional background risk makes a consumer less willing to bear the risk associated with holding stocks and, therefore, leads to a larger predicted equity premium.[6]

I will show that a sufficient condition under which a model with undiversifiable labor income risk predicts a smaller bond return and larger equity premium than would a representative agent model calibrated on the aggregate consumption series generated by that economy is, fortunately, quite plausible: decreasing absolute risk aversion (i.e., the absolute holding of risky assets rises as the level of wealth rises) and decreasing absolute "prudence" (i.e., the absolute level of precautionary savings declines as wealth rises). The former

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4. Mankiw's model is somewhat peculiar, as it is one in which the "idiosyncratic" shock is correlated with the aggregate shock. The relation between his model and mine is discussed at some length in section 3. It suffices to note, at this juncture, that our models, and thus our results, are very different.

5. Their paper assumes away the existence of dividend risk to highlight the role of liquidity in generating a gap between bond and stock returns in a model with uninsurable labor income risk. I instead concentrate on the interaction between independent aggregate and individual risks in the determination of asset returns.

6. References to relevant theoretical work are given in the body of the paper.

assumption is widely recognized as realistic; the latter is made plausible by Kimball's [1990b] work, which shows that decreasing absolute prudence is required to explain that the marginal propensity to consume rises with mean-preserving increases in labor income risk.[7]

To keep the analysis as transparent as possible and eliminate the need to track the wealth distribution over time and across states of nature, I will assume that the economy (and consumers) lasts for two periods only, with all consumers identical *ex ante* in terms of endowments and preferences.[8] Undiversifiable idiosyncratic labor income shocks in the second period however create the *ex post* differences between consumers which are at the core of this paper. In section 1, I describe the model and its market structure, and compute the true equilibrium asset returns. In section 2, I characterize the predictions of an economist who ignores the presence of uninsurable risk and mistakenly calibrates the model, as in Mehra and Prescott [1985], on the basis of aggregate data solely, and I derive the conditions under which this misspecification will lead to an overprediction of the riskfree rate and an underprediction of the equity premium. Section 3 interprets the results and contrasts them with earlier literature, while section 4 provides a simple numerical example to illustrate the potentially large magnitude of these miscalibration biases. The conclusion reflects on lessons learned, as yet unresolved difficulties, and directions for future research.

1   Competitive equilibrium

The model, except for the introduction of labor income risk, is a simple two-period version of Lucas [1978].

The economy consists of a continuum of *ex ante* identical two-period lived consumers. Consumers tastes over consumption bundles (c_1, c_2) are represented by a time-additive[9] utility function

$$u(c_1) + v(c_2). \tag{1.1}$$

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7. Kimball [1990b] provides other, more introspective arguments in favor of decreasing absolute prudence.

8. A similar assumption is used by Mankiw [1986].

9. It will become clear from equilibrium conditions below that time-additivity is inessential.

Both $u(.)$ and $v(.)$ are strictly increasing and strictly concave. All the derivatives written in the text are assumed to exist and not to change sign over their domain.[10] Agents are expected utility maximizers.[11]

Every consumer receives a known endowment $y_1 > 0$ of the consumption good (a fruit) in the first period of life, and a random second period endowment $\tilde{y} > 0$. All consumers face the same probability distribution of $\tilde{y}$, but realizations of $\tilde{y}$ are idiosyncratic: agents therefore differ *ex post*, although not *ex ante*, and there is no *ex post* representative agent. Because the number of consumers is infinite, the *ex ante* distribution of second-period endowments also describes the *ex post* distribution of endowments across agents, and $\bar{y} = E\tilde{y}$ thus denotes both the mean of $\tilde{y}$ and the per capita second period endowment. It is therefore assumed—to sharpen the focus of the analysis—that there is no aggregate component to second period endowment risk.[12] With a slight abuse of terminology, I will often refer to second-period non-interest income as labor income.

A consumer's realized second period endowment is assumed to be private information. As a consequence, insurance contracts contingent on unobservable realizations of $\tilde{y}$ cannot be written, and second-period endowment risk cannot be pooled in a decentralized economy. This feature of the model should be, of course, understood as a metaphor for the non-availability of full insurance against fluctuations in labor income—due to both private information and moral hazard considerations.

All consumers are also endowed at birth with the same number $x_0 > 0$ of fruit producing trees. The current fruit crop (dividend) $d_1 \geq 0$ is known, but future fruit production, $\tilde{d}$, is random. Dividend risk is aggregate risk: dividends are the same in every agent's garden. Aggregate dividend risk and idiosyncratic labor income risk are statistically independent.[13] All random variables, functions

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10. For instance, $v'''(.)$ can be either positive or negative everywhere, but it is not allowed to switch sign. Otherwise, the propositions below would not be correct as stated, and would require restrictions on the class of distributions of $\tilde{d}$ and $\tilde{y}$.
11. This is obviously a restrictive assumption. See Kimball and Weil [1991] for a thorough, but partial equilibrium, treatment of precautionary saving with Kreps-Porteus preferences.
12. In other terms, the aggregate second-period endowment is fixed, but its repartition across individuals is random. Various devices are available to ensure that non-degenerate individual endowment i.i.d. random variables sum to a constant. The reader may chose the one he or she prefers.
13. This restriction is natural, given the assumption that there is no aggregate component

of $\tilde{d}$ and $\tilde{y}$, written in the text are assumed to be integrable.

The fruit endowment falling from heaven and the fruit produced by the tree are perfect substitutes in consumption. Both are non-storable.

Since agents are identical *ex ante* (although not *ex post*), they will not trade with one another in equilibrium—a construction which considerably simplifies the characterization of consumption optima. A typical consumer faces the budget constraints:

$$c_1 + px = (p + d_1)x_0 + y_1, \qquad (1.2)$$

$$\tilde{c}_2 = \tilde{d}x + \tilde{y}, \qquad (1.3)$$

$$c_1, c_2 \geq 0, \qquad (1.4)$$

where p denotes the fruit price of a tree, and x the number of trees held in the consumer's portfolio.

The first-order condition for an interior[14] expected utility maximum is simply

$$pu'(c_1) = E\,\tilde{d}v'(\tilde{c}_2), \qquad (1.5)$$

where E is the mathematical expectation operator.

In equilibrium, since agents are identical *ex ante*, each consumer must hold an equal number of trees. Normalizing the initial per capita number of trees, x_0, to be equal to 1, market clearing requires that

$$x = 1. \qquad (1.6)$$

Scaling $u(.)$, without loss of generality, so that

$$u'(d_1 + y_1) = 1, \qquad (1.7)$$

the equilibrium fruit price of a tree is hence, from the budget constraints and equation (1.5),

$$p = E\,\tilde{d}v'(\tilde{d} + \tilde{y}). \qquad (1.8)$$

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to individual labor income fluctuations.

14. Interiority, as will be shown below, is imposed by equilibrium considerations.

The equilibrium expected gross rate of return on equity (trees) is thus

$$R = E\left(\frac{\tilde{d}}{p}\right) = \frac{E\,\tilde{d}}{E\,\tilde{d}v'(\tilde{d} + \tilde{y})}. \tag{1.9}$$

The equilibrium gross riskfree return R_F can be computed at the margin, since there is no trade on safe claims to consumption in a competitive equilibrium. It is simply

$$R_F = \frac{1}{E\,v'(\tilde{d} + \tilde{y})}. \tag{1.10}$$

The equilibrium gross equity premium is therefore

$$\pi = \frac{R}{R_F} = \frac{E\,\tilde{d}.\,E\,v'(\tilde{d} + \tilde{y})}{E\,\tilde{d}v'(\tilde{d} + \tilde{y})}. \tag{1.11}$$

Risk aversion alone ($v'' < 0$) suffices to guarantee that the net premium on equity is positive ($\pi > 1$).

Notice that because realized labor income is private information, the competitive consumption bundle $(d_1 + y_1, \tilde{d} + \tilde{y})$ is welfare-dominated by the equal treatment, risk-pooling allocation $(d_1 + y_1, \tilde{d} + \tilde{y})$ which could be achieved under, say, omniscient and omnipotent command. This follows from the strict concavity of the utility function, which implies, by Jensen's inequality,

$$u(d_1 + y_1) + E\,v(\tilde{d} + \tilde{y}) < u(d_1 + y_1) + E\,v(\tilde{d} + \tilde{y}). \tag{1.12}$$

2   Misspecification and calibration

Suppose that an outside observer (a calibrator), using only aggregate data and knowing only the stochastic process followed by dividends, were to confront the asset pricing implications of a representative agent model—which ignores the *ex post* heterogeneity of consumers or, equivalently, mistakenly assumes that risk-pooling is competitively achievable—with the observed asset returns data, assumed to have been generated by the "true" model in the fashion described in the previous section. Would our economist over- or underpredict

the (implicit) equilibrium riskfree return? Would he over- or underpredict the equilibrium average rate of return on equity? Would he over- or underpredict the equity premium?

The calibrator believes that *every* consumer has received the non-random average labor income $\bar{y}$, and is thus led to neglect the fact that consumers, when making their portfolio decisions, are facing an additional source of uncertainty: undiversifiable labor income risk. Alternatively, but equivalently, our observer mistakenly thinks that consumers were able to achieve the Pareto-optimal, full-insurance, equal treatment consumption allocation $(d_1 + y_1, \bar{d} + \bar{y})$.

By adapting the argument in the previous subsection, it is straightforward to show that this observer will "predict" that the equilibrium price of equity, expected rate of return on equity, riskfree rate and equity premium are

$$\hat{p} = \mathbf{E}\,\tilde{d}v'(\tilde{d} + \bar{y}), \tag{2.1}$$

$$\hat{R} = \mathbf{E}\left(\frac{\tilde{d}}{\hat{p}}\right) = \frac{\mathbf{E}\,\tilde{d}}{\mathbf{E}\,\tilde{d}v'(\tilde{d} + \bar{y})}, \tag{2.2}$$

$$\hat{R}_F = \frac{1}{\mathbf{E}\,v'(\tilde{d} + \bar{y})}, \tag{2.3}$$

$$\hat{\pi} = \frac{\hat{R}}{\hat{R}_F} = \frac{\mathbf{E}\,\tilde{d}.\mathbf{E}\,v'(\tilde{d} + \bar{y})}{\mathbf{E}\,\tilde{d}v'(\tilde{d} + \bar{y})}. \tag{2.4}$$

To determine the direction of the biases which model misspecification introduces in the calibration exercise, it is useful, to build intuition for the general case, to first characterize true and predicted returns in the absence of dividend uncertainty—a case which highlights the effect of model misspecification on the *level* of equilibrium returns.

2.1 No dividend uncertainty

When there is no dividend uncertainty ($\tilde{d} = \bar{d} > 0$ in all states), we immediately obtain

Proposition 1 *When there is no dividend risk, a calibrator who is not aware of the existence of undiversifiable labor income risk will overpredict the magnitude of the riskfree rate if and only if $v''' > 0$.*

Proof: Remembering that $\bar{y} = E\,\tilde{y}$ by definition and using Jensen's inequality, $E\,v'(\bar{d} + \tilde{y}) > v'(\bar{d} + \bar{y})$ if and only if $v'''(.) > 0$. Therefore, $\hat{R}_F > R_F$ if and only if $v'''(.) > 0$.

This result is intuitive: consumers engage in precautionary saving only when marginal utility is convex.[15] Neglecting labor income uncertainty thus leads to underpredicting savings, and thus—since individual first-period consumption is fixed in equilibrium—to overpredict the equilibrium riskfree rate.[16] An observer who mistakenly believes that labor income risk is tradeable would thus conclude to the presence of a "riskfree rate puzzle:" he could not explain why the riskfree rate is so low.

When there is no dividend uncertainty, the true and predicted equity premia coincide ($\hat{\pi} = \pi = 1$) since fruit output is deterministic.

2.2 General case

The previous subsection has highlighted the existence of a link between the theory of precautionary saving in the presence of a *single* risk and the direction of the misspecification bias. To characterize, more generally, the market equilibrium implications of dividend risk in the presence of uninsurable background labor income risk, it will be necessary, very naturally, to refer to the well-developed theory of risk taking in the presence of *multiple* independent risks—developed, most recently, by Pratt and Zeckhauser [1987], Kimball [1990b], and Kimball [1991].

As a preliminary, it is useful to provide the generalization—which relies only on the convexity of the marginal utility function—of the results of proposition 1:

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15. See Sandmo [1970].
16. This is well-known: Bewley [1980], Bewley and Radner [1980], Clarida [1987], and Aiyagari and Gertler [1990] have already noted that non-traded risk depresses the equilibrium riskfree rate.

Proposition 2 *In the presence of dividend risk, a calibrator who is not aware of the existence of undiversifiable labor income risk will overpredict the magnitude of the riskfree rate and of the expected equity return if and only if $v''' > 0$.*

Proof: If $v'''(.) > 0$, it follows by Jensen's inequality and the assumed independence of $\tilde{d}$ and $\tilde{y}$, that

$$
\begin{aligned}
E\,v'(\tilde{d} + \tilde{y}) &= E_d[E_y\,v'(\tilde{d} + \tilde{y})] \\
&\geq E_d\,v'(\tilde{d} + \tilde{y}) \\
&= E\,v'(\tilde{d} + \tilde{y}),
\end{aligned}
$$

where E_j, $j = d, y$, denotes expectation taken with respect to random variable j only. This immediately implies that $\hat{R}_F > R_F$ if and only if $v'''(.) > 0$.

The same argument, adapted to $\tilde{d}v'(\tilde{d} + \tilde{y})$, demonstrates that $\hat{R} > R$ if and only if $v'''(.) > 0$.

This proposition makes a simple point: if agents behave prudently because their marginal utility function is convex, then mistakenly believing that there exists a representative agent (or that all labor income risk is diversifiable) will lead us to overpredict both the expected rate of return on stocks and the rate of return on a safe bond—because we will have overlooked the existence of a precautionary saving motive.

What happens is simply that, when there is a positive precautionary saving motive, the presence of uninsurable risk increases the demand for both the safe asset (bonds, in zero net supply) and the risky asset (equity, in fixed positive supply). In equilibrium, this increased demand for the two assets must be defeated—which requires, in a world in which first-period consumption must remain fixed at $c_1 + d_1$, a *de*crease in the equilibrium rate of return on both assets.

But since both the riskfree rate and the rate of return on equity are over-predicted if v''' is positive, there is an *a priori* ambiguity as to the effect of misspecification on the equity premium. To lift this ambiguity, it is necessary to know to which asset, bond or equity, the increased savings is predominantly allocated—i.e., which rate of return falls *more* in equilibrium in the presence of uninsurable risk.

This task of course requires to go beyond the assumptions made *supra* about the convexity or concavity of marginal utility, as what we need is a more refined

description of precautionary saving behavior. A powerful and useful concept, due to Kimball [1990b], is that of *prudence*: by analogy with the Arrow-Pratt measure of absolute risk aversion, measure the curvature of the marginal utility function by

$$\mathcal{P}(x) = -v'''(x)/v''(x),$$

and call $\mathcal{P}$ the *coefficient of absolute prudence*. Following Kimball [1990b], we will say that v exhibits *decreasing absolute prudence* if precautionary savings decrease as deterministic base wealth rises, i.e., if and only if $\mathcal{P}' < 0$.

Armed with this concept, I now show that two behaviorally plausible restrictions on the utility function are sufficient to guarantee that the equity premium will be understated as a result of misspecification:

Proposition 3 *If $v(.)$ exhibits decreasing absolute risk aversion (DARA) and decreasing absolute prudence (DAP), then a calibrator who is not aware of the existence of undiversifiable labor income risk will understate the magnitude of the equity premium.*

Proof:

In order to establish the veracity of the proposition, it is useful to think in terms of comparative statics and to prove, instead of the proposition itself, the equivalent result that if $v(.)$ exhibits DARA and DAP, then the true risk premium is larger in an economy with uninsurable labor income risk $\tilde{y}$ than in the same economy with constant labor income $\bar{y} = \mathbf{E}\,\tilde{y}$.

Following Kihlstrom et al. [1981], define the "indirect" utility functions

$$w(d) = \mathbf{E}v(d + \tilde{y}),$$

and let

$$\hat{w}(d) = v(d + \bar{y}).$$

By inspection of equations (1.11) and (2.4), the true and predicted equity premia can be rewritten as

$$\pi = \frac{\mathbf{E}\,\tilde{d}.\,\mathbf{E}\,w'(\tilde{d})}{\mathbf{E}\,\tilde{d}w'(\tilde{d})},$$

and

$$\hat{\pi} = \frac{\mathbf{E}\,\tilde{d}.\,\mathbf{E}\,\hat{w}'(\tilde{d})}{\mathbf{E}\,\tilde{d}\hat{w}'(\tilde{d})}.$$

To prove the proposition, I will first show that $w(d)$ is more risk averse ($\succ$), in absolute terms, than $\hat{w}(d)$. I will then establish that the equity premium is larger the more risk averse, in absolute terms, the utility function.

I. The proof that $w(d) \succ \hat{w}(d)$ is adapted from an argument in the appendix of Kimball [1988]. Let θ_1 and θ_2 be such that

$$v'(d + \bar{y} - \theta_1) = E_y \, v'(d + \tilde{y}),$$

$$v''(d + \bar{y} - \theta_2) = E_y \, v''(d + \tilde{y}).$$

By analogy with the standard definition of the risk premium in consumer theory, θ_1 has the interpretation of a risk premium for the "pseudo" (but well-behaved under the conditions of the proposition) utility function $-v'(.),$[17] while θ_2 has the same meaning for the pseudo utility function $v''(.).$

The function $v(d + \bar{y})$ has a positive third derivative because $v(d)$ is DARA; therefore, because consumers do engage in precautionary saving, $\theta_1 > 0$. On the other hand, DAP implies, by definition, that

$$-\frac{v''''(c)}{v'''(c)} > -\frac{v'''(c)}{v''(c)} \quad \forall c.$$

Hence, $v'' \succ -v'$, and, as a consequence (applying theorem 7 in Pratt [1964] to the functions $-v'$ and v''),

$$\theta_2 > \theta_1 > 0.$$

Using these inequalities and remembering that DARA requires $v''' > 0$, it follows that, for all d,

$$-\frac{w''(d)}{w'(d)} \quad = \quad -\frac{v''(d + \bar{y} - \theta_2)}{v'(d + \bar{y} - \theta_1)} \tag{2.5}$$

$$> \quad -\frac{v''(d + \bar{y} - \theta_1)}{v'(d + \bar{y} - \theta_1)} \tag{2.6}$$

$$> \quad -\frac{v''(d + \bar{y})}{v'(d + \bar{y})} \tag{2.7}$$

$$= \quad -\frac{\hat{w}''(d)}{\hat{w}'(d)}, \tag{2.8}$$

which establishes that $w(d) \succ \hat{w}(d)$.

II. The proof that the equity premium rises as the utility function becomes more risk averse is—despite the intuitive nature of the result—more involved. Consider any two increasing and strictly concave utility functions ψ_i, $i = 1, 2$, and an individual with unit initial wealth which she can allocate between a safe asset with gross return a and a risky asset with gross return $\tilde{d}$. If our hypothetical consumer's utility function is ψ_i, she will choose her investment in the risky asset, z, to maximize expected utility

$$\Psi_i(z, a) = E \, \psi_i[(1 - z)a + z\tilde{d}].$$

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17. More interestingly, but less usefully for this proof, θ_1 is a precautionary saving premium for the utility function $v(.)$.

Because of the strict concavity of $\psi(., a)$,

$$\partial^2 \Psi_i / (\partial z)^2 < 0,$$

and the optimal investment in the risky asset will satisfy

$$\frac{\partial}{\partial z} \Psi_i(z_i^*, a) = E(\tilde{d} - a)\psi_i'[(1 - z_i^*)a + z_i^*\tilde{d}] = 0. \tag{2.9}$$

Now we know, from Arrow [1965] and Pratt [1964], that investment in the risky asset declines as the utility function becomes more risk averse.

Therefore,

$$\psi_1 \succ \psi_2 \iff z_1^* < z_2^*.$$

But z_2^* maximizes $\Psi_2(., a)$. Hence

$$\frac{\partial}{\partial z} \Psi_2(z_1^*, a) > \frac{\partial}{\partial z} \Psi_2(z_2^*, a) = 0. \tag{2.10}$$

Now let $\psi_1(d) = w(d)$, $\psi_2(d) = \hat{w}(d)$, and $a = E \tilde{d}/\pi$. By inspection of equation (2.9), and using (1.11), it follows that

$$z_1^* = 1,$$

so that, using (2.10), we find, since $w(d) \succ \hat{w}(d)$, that

$$E\left\{ \left(\tilde{d} - \frac{E \tilde{d}}{\pi} \right) \hat{w}'(\tilde{d}) \right\} > 0. \tag{2.11}$$

Using the definitions of the true and predicted equity premia in equations (1.11) and (2.4) and that of $\hat{w}(d)$, it immediately follows from (2.11) that

$$\pi > \hat{\pi},$$

a result which proves the proposition.

To ascertain the relevance of the proposition, it is worth noting that DARA implies DAP for utility functions member of the HARA class:

Corollary 1 *If $v(.)$ has linear risk tolerance and exhibits DARA, then a calibrator who is not aware of the existence of undiversifiable labor income risk will underpredict the magnitude of the equity premium.*

Proof: If $v(.)$ has linear risk tolerance, then

$$-\frac{v'(c)}{v''(c)} = A + Bc.$$

Now DARA requires B > 0. But then, differentiating and rearranging, the coefficient of absolute prudence is

$$\mathcal{P}(c) \;=\; -\frac{v'''(c)}{v''(c)}$$
$$=\; \frac{1 + B}{A + Bc};$$

it thus declines when c rises. Therefore, any DARA utility function with linear risk tolerance also exhibits DAP, and it satisfies the condition of proposition 3.

We can therefore rest assured that proposition 3 is far from vacuous: all power and logarithmic utility functions satisfy the condition which it sets forth. More generally, using results in Pratt [1964] and Kimball [1990b], it is straight-forward to show that all non-degenerate exponential mixtures[18] —a category which includes, but is not limited to, the DARA members of the HARA class and mixtures thereof— exhibit both DARA and DAP.

One should also note that exponential or quadratic utility provide special cases in which miscalibration is irrelevant for the equity premium (but, as shown above, not for the *level* of equilibrium returns).[19] In the exponential case, this follows immediately from the fact that $w(d)$ has the same coefficient of absolute risk aversion as $v(d + \bar{y})$ when $v(.)$ is exponential. For quadratic utility, this is a direct consequence of the property often referred to as "certainty equivalence."

Finally, lest anyone should be tempted to conjecture that miscalibration always leads to understating the equity premium irrespective of the sign and magnitude of the high-order (greater than 3) derivatives of the utility function, it is useful to establish the following

Corollary 2 *If $v''' > 0$ and $v'''' > 0$, then a calibrator who is not aware of the existence of undiversifiable labor income risk will overpredict the magnitude of the equity premium.*

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18. I.e., linear mixtures which are not limited to a single exponential.

19. Note the role of the definition of the equity premium as the ratio of the expected return on stocks over the rate of return on safe bonds. With exponential utility, miscalibration implies that both returns are overpredicted in the same proportion, so that their ratio is unaffected by the calibration error. This implies, however, that the *difference* between rates—another possible definition—will be overpredicted. But, of course, this qualitative difference in conclusions is almost immaterial for quantitative purposes as $(1 + x)/(1 + y) \approx x - y$ for x and y small.

Proof: By Jensen's inequality, the conditions of the corollary imply that, for all d,

$$w'(d) = E_y v'(d + \tilde{y}) > v'(d + \bar{y}),$$

and

$$w''(d) = E_y v''(d + \tilde{y}) > v''(d + \bar{y}).$$

Hence, for all d,

$$-\frac{w''(d)}{w'(d)} < -\frac{v''(d + \bar{y})}{v'(d + \bar{y})},$$

which, using Theorem 1 in Pratt [1964], is equivalent to stating that $w(d)$ is *less* risk averse than $v(d + \bar{y})$. From the second part of the proof of proposition 3, it then follows that miscalibration lead to *over*stating the equity premium if $v''' > 0$ and $v'''' > 0$.

It is worthwhile to note that, because the utility function has a positive fourth derivative, this "unpleasant" case is one in which tastes exhibit *in*creasing absolute prudence.

3 Discussion

I now take stock, to interpret them more broadly and to provide some further economic intuition, of the results of the previous section. I then contrast the lessons of the exercise carried through in this paper with Mankiw's [1986] differing conclusions.

3.1 Intuition

Proposition 3 is obviously good news for the incomplete market explanation of the riskfree rate and equity premium puzzles. The existence of a precautionary saving motive, required to contribute to the solution of the former, is a weak and plausible requirement. The much stronger restrictions imposed on tastes by DARA and DAP—which are necessary if non-traded labor income risk is to increase the equity premium relative to the full-insurance case— are satisfied by all commonly used utility functions but exponential and quadratic utility). Moreover, on empirical grounds, both DARA and DAP are necessary to rationalize observed consumption and portfolio behavior under uncertainty.[20] One can therefore feel confident that theoretical case built here

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20. Arrow [1965] and Pratt [1964] adduce arguments in favor of DARA, while Kimball [1990b] convincingly presents the case of DAP.

in favor of the incomplete markets explanation of the Mehra-Prescott puzzles is at least plausible.

The economic intuition underlying the main result (proposition 3) is almost transparent from its proof: if consumers' tastes exhibit DARA and DAP, then non-traded labor income risk makes them more unwilling, *ceteris paribus*, to bear dividend risk. Since the stock of trees must be held for markets to clear, the equilibrium return premium on equity must rise, relative to the full-insurance case, to overcome this increased reluctance to hold stocks.

To make this argument even more limpid, it remains to explain why the two assumptions of DARA and DAP are sufficient to guarantee that non-traded labor income risk makes consumers less willing to hold stocks. The proof of this result is provided, in a partial equilibrium setting, by Kimball [1991], who has shown that DARA and DAP taken together are sufficient to guarantee that the utility function is "proper" in the sense of Pratt and Zeckhauser [1987]. Properness—the property that "an undesirable lottery can never be made desirable by the presence of an independent, undesirable lottery" (*ibid.*, p. 143)—is an attribute of a utility function required, in partial equilibrium, for the availability of insurance against one risk to make a consumer more willing to bear another, independent risk.

Properness, however, is itself too strong a condition for our purpose—as one is only dealing in this model with the addition of a *mean zero* risk (the deviation of individual income from the population average). The necessary and sufficient condition for the presence of non-traded labor income risk to raise the equity premium is thus the condition under which the non-availability of insurance against an additional *mean zero* risk makes consumers less willing to bear dividend risk.[21]

The search for this necessary and sufficient condition presents, however, no interest beyond that of theoretical elegance—because the sufficient condition of proposition 3 is satisfied by all commonly used utility functions, and because the only gray area left unexplored by proposition 3 and corollary 2 is the case when $v''' > 0$ but v'''' is not negative enough to ensure DAP. This search is

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21. Pratt and Zeckhauser [1987] show that analytical necessary and sufficient conditions such as properness are unavoidably complicated. Their counterpart in a general equilibrium setting is even more untractable.

therefore not pursued here.[22]

3.2 Related literature

Corollary 2 proves that one simply cannot presume, as is done in much of the literature on asset pricing following Mehra and Prescott [1985], that the introduction of incomplete markets for labor income risk is sufficient, when there is positive precautionary saving, to solve the equity premium puzzle:[23] in the absence of any correlation between aggregate and idiosyncratic shocks, it might well be the case (if for instance the utility function has positive third and fourth derivatives) that the presence of non-traded labor income risk deepens, rather than solves, the equity premium puzzle.

This conclusion is in sharp contrast with Mankiw's [1986] result that a positive third derivative is sufficient to guarantee that the equity premium rises in the presence of uninsurable risk. The reason for this apparent contradiction is that Mankiw does not describe the physical sources of uncertainty and budget constraints facing his consumers, but instead directly writes stochastic processes for aggregate and individual consumption, assuming that *negative* aggregate shocks are concentrated on a subset of the population. As a consequence, he implicitly assumes that consumers face only *one* uninsurable source of uncertainty—the effects of which are governed solely by the sign of the third derivative of the utility function—while I explicitly consider, in a general equilibrium model, the very specific issues introduced by the simultaneous presence of *two independent* risks—the effects of which depend in a crucial way on the higher derivatives of $v(.)$.

In other terms, Mankiw's model allows no role for anything but risk aversion and precautionary saving because it is a model about the concentration of a single aggregate shock. The model developed here deals, in contradistinction, with the general equilibrium effects on stock and bond returns of non-traded labor income risk in a Lucas tree economy with random dividends.

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22. Miles Kimball has however kindly worked out the following necessary and sufficient condition (see Kimball [1990a]): $\hat{w}(z) = \mathbf{E} w(z + \tilde{\epsilon})$ is more risk averse than $w(z)$ for all mean-zero random variables $\tilde{\epsilon}$ if and only if $[w'(z+\epsilon) - w'(z) - zw''(z)]/w'(z)$ is a decreasing function of z for all ϵ. To go from this consumer theory result to a general equilibrium statement, a straightforward adaptation of the proof of proposition 3 is all that is needed.
23. As showed above, it is nevertheless correct to assert that incomplete markets contribute, when consumers are prudent, to the solution of the riskfree rate puzzle.

One should also note that Mankiw's model, as he himself recognizes, is one in which concentrated *positive* aggregate shocks lead an observer who mistakenly assumes that there exists an *ex post* representative agent to overstate the risk premium if $v'' < 0$ and $v''' > 0$. No such ambiguity arises here: proposition 3 holds for all independent distributions of labor income and dividends.

In a thought-provoking paper, Mankiw and Zeldes [1989] observe, using PSID data, that the covariance of the equity return with the consumption of stockholders (who represent only one fourth of families) is about seven times larger than the covariance of the equity return with the consumption of non-stockholders. This finding should be interpreted as one more indication of the inappropriateness of the standard asset pricing framework—as such differences in individual consumption patterns are totally inconsistent with the complete markets paradigm. The challenge is of course to build a tractable, yet rich enough, model which can rationalize the fact that 77% of families do not hold equity in excess of $1,000.

Following Losq [1978], Svennson [1988] studies the effects of the presence of non-traded assets on optimal portfolio choice and asset pricing in a continuous time framework. Svennson's paper generalizes the "fundamental valuation equation" of Cox et al. [1985], and provides a useful delineation of the cases in which equilibrium pricing functions can be written explicitly. Svennson's model relies, however, on a representative agent construct: therefore, it does not shed any direct light on the miscalibration and comparative statics issues explored here.

4   An example

The results of the preceding section can be illustrated qualitatively (without attempting precise calibration) by a simple numerical example.

Suppose that

$$v(c) = c^{1-\gamma}/(1 - \gamma) \qquad (4.1)$$

where $\gamma > 0$ ($\gamma \neq 1$) denotes the constant coefficient of relative risk aversion. This utility function exhibits decreasing absolute risk aversion and decreasing absolute prudence.

The first one is well-understood: in the presence of a precautionary saving motive, a representative agent model which hypothesizes that insurance markets are complete underestimates private savings, and thus overestimates equilibrium returns—be they on stock or on bonds. The existence of non-diversifiable labor income risk thus undoubtedly helps solve the "riskfree rate puzzle" brought out by representative agent calibration exercises.

The second conclusion is novel: the sign of the "bias" on the equity premium introduced by miscalibration cannot be determined from risk aversion and precautionary saving (second and third derivatives) alone. To tell into which direction one is misled by the representative agent assumption when there is idiosyncratic labor income risk, it is necessary to know how consumers' attitudes towards dividend risk are affected by the existence of background uninsurable labor income risk—an aspect of behavior determined by the signs and magnitudes of higher order (i.e., larger than 3) derivatives of the utility function.

While it is difficult to get introspective knowledge of those high derivatives, it is fortunate that all commonly used utility functions (except for the watershed case of exponential utility, and for quadratic utility) guarantee—because they exhibit DARA and DAP—that the representative agent model underpredicts, and sometimes by a very large factor, the size of the equity premium. There is thus, at last, a theoretical underpinning to the notion that incomplete insurance markets must be part of the solution to the equity premium puzzle.

To go beyond the qualitative results of this paper and to assess their likely quantitative and empirical importance, a good starting point is the remark that the uninsurable individual risk present in the two-period model I have built here can be interpreted as either a permanent shock to future labor income, or as a temporary shock against which one cannot self-insure because of transactions costs which prevent trade with future periods. This interpretation reflects the fact that a two-period model can be thought of as a metaphor either of the present vs. the future, or of a long-lived consumer with alternating high and low income subject to borrowing constraints (as suggested by Woodford [1986]). The empirical impact of labor income risk on equilibrium asset prices thus can stem from either the presence of permanent uninsurable labor income shocks, and/or the impossibility to effectively self-insure against

temporary individual shocks because of the existence of transactions costs.[25] I conjecture that empirical work will establish the importance of both channels.

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25. The interaction between i.i.d. uninsurable shocks and transactions costs has recently been studied numerically by Heaton and Lucas [1991].

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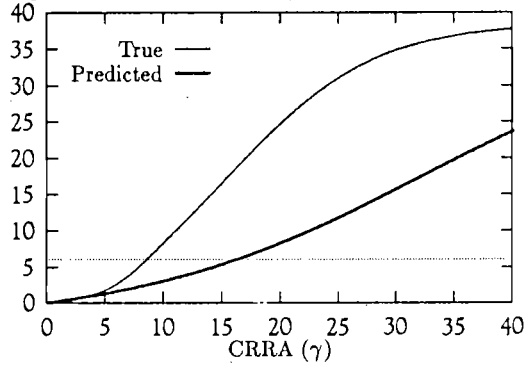Fig. 1a: Net risk premium (1/3 income when employed)

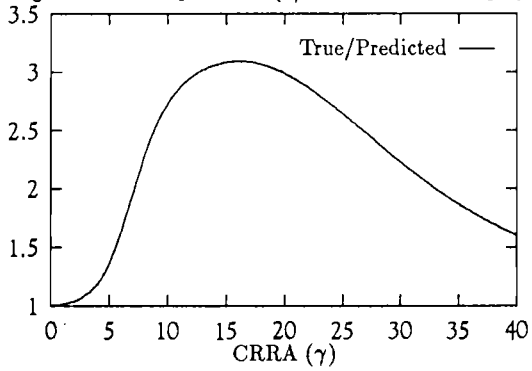Fig. 1b: Net risk premium (1/3 income when employed)

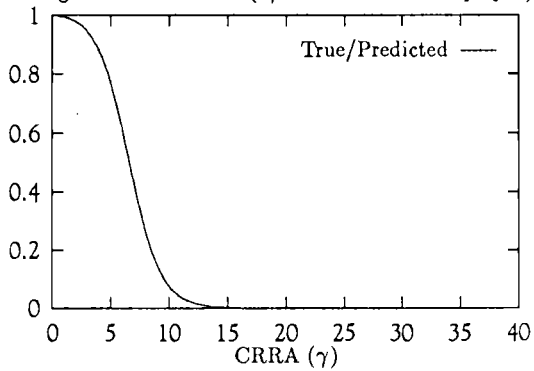Fig. 1c: Riskfree rate (1/3 income when employed)

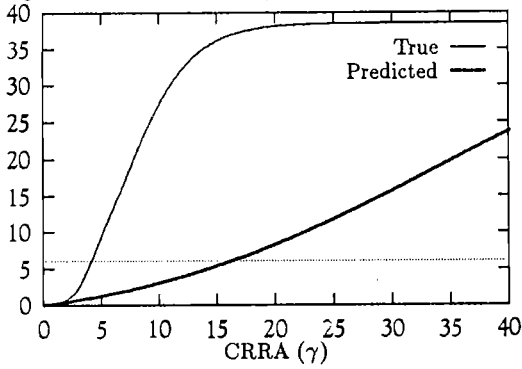Fig. 2a: Net risk premium (zero income when employed)
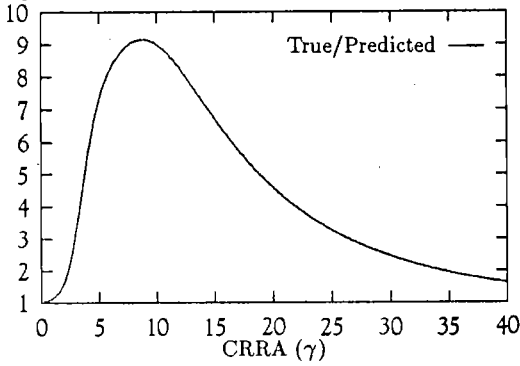

Fig. 2b: Net risk premium (zero income when employed)

Fig. 2c: Riskfree rate (zero income when employed)