NBER WORKING PAPERS SERIES

IRREVERSIBILITY AND AGGREGATE INVESTMENT

Giuseppe Bertola

Ricardo J. Caballero

Working Paper No. 3865

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
October 1991

January 1991. Financial support from the National Science Foundation through grants SES-9010952 and SES-9010443 is gratefully acknowledged. We thank Kevin Hassett for lending us his data, and Hyeng Keun Koo for very competent research assistance supported by the John M. Olin Program for the Study of Economic Organization and Public Policy. This paper is part of NBER's research program in Economic Fluctuations. Any opinions expressed are those of the authors and not those of the National Bureau of Economic Research.

NBER Working Paper #3865
October 1991

IRREVERSIBILITY AND AGGREGATE INVESTMENT

ABSTRACT

Investment is often irreversible, in that installed capital has little or no value unless used in production. In the presence of ongoing uncertainty, an individual firm's irreversible investment policy optimally alternates short bursts of positive gross investment to periods of inaction, when the installed capital stock is allowed to depreciate. The behavior of aggregate investment series is characterized by sluggish, continuous adjustment instead. We argue in this paper that aggregate dynamics should be interpreted in terms of unsynchronized irreversible investment decisions by heterogenous firms, rather than in terms of ad-hoc adjustment cost functions in a representative-agent framework. We propose a closed-form solution for a realistic model of sequential irreversible investment, characterize the aggregate implications of microeconomic irreversibility and idiosyncratic uncertainty, and interpret U.S. data in light of the theoretical results.

Giuseppe Bertola
Department of Economics
Fisher Hall
Princeton University
Princeton, NJ 08544-1021
CEPR and NBER

Ricardo J. Caballero
NBER
1050 Massachusetts Avenue
Cambridge, MA 02138

1. Introduction

Capital accumulation has an essential role both in the theory of production and in the study of macroeconomic fluctuations. If it were possible to rent capital services on a smoothly functioning spot market, investment could be modeled in terms of the user cost of capital defined by Jorgenson (1963). In reality, however, firms own rather than rent their capital stock, and investment can only be studied in an explicitly dynamic framework. Standard investment models assume variations in capital input to entail convex adjustment costs, either internal to the firm and due to increasing costs of installing more capital in shorter intervals of time, or external to it and due to decreasing returns in production of capital goods. Further assumptions are typically necessary to obtain analytically and empirically tractable investment models: firms may be assumed to be perfectly competitive and to operate under constant returns to scale (e.g. Lucas and Prescott, 1971; Hayashi, 1982), or linear-quadratic functional forms may be assumed to obtain certainty equivalence (e.g. Sargent, 1987).

Models of investment based on these assumptions do not provide a convincing interpretation of empirical data (Abel and Blanchard, 1986). In fact, the realism of smooth adjustment costs as the source of investment dynamics is doubtful. From a microeconomic point of view, the cost of additions to an individual firm's capital stock is often linear in investment; the unit price of capital may even be decreasing in investment if lump-sum adjustment costs are present. Disinvestment on the other hand is costly, if at all possible. Many facilities are specific to a particular production process, conversion of industrial real estate is difficult, and markets for used machinery are thin and discount it heavily.

Three largely separate strands of literature have focused on the realism and importance of investment irreversibility at the firm's level. Arrow (1968), Nickell (1974) and others have studied irreversible investment decisions in continuous-time dynamic optimization models, assuming that firms hold point expectations about the cyclical path of exogenous variables, and showing that the marginal revenue product of capital equals the neoclassical user cost of capital whenever gross investment is strictly positive. Investment is not necessarily always positive, of course, if it is irreversible: it ceases before a cyclical peak is reached, and starts again after the cyclical trough. Irreversibility then drives a wedge (negative during booms, and positive during pronounced troughs) between the cost of capital and its marginal contribution to profits.

Other authors have noted that investment irreversibility is especially realistic at the aggregate level: even if capital goods retained their full value on second-hand markets, the direct consumption value of existing productive facilities would clearly be low or nil. This has motivated studies of irreversible capital accumulation in general-equilibrium stochastic growth models (Sargent 1979, Olson 1989). The representative agent, single good framework of this literature, however, is too stylized to allow realistic applications. The dynamics of aggregate production and investment are not variable enough, at least in industrialized countries, to make aggregate irreversibility constraints binding under those assumptions.

Irreversible investment under uncertainty has been studied by financial economists (see McDonald and Siegel 1986 and their references, as well as Ingersoll and Ross, 1987 for the case of interest rate uncertainty). Option pricing techniques provide elegant solutions in the case of a single irreversible investment project with uncertain payoffs: such a project will be adopted only when the expected discounted payoff from investment exceeds the cost by an amount that depends on the level of uncertainty, and can be impressively large for plausible parameter values. Even risk neutral firms are, in a sense, reluctant to invest when projects are irreversible and the future is uncertain, because when the project is adopted the option to wait for some of the uncertainty to be resolved is forsaken, and options are valuable even to risk-neutral agents. These results are clearly relevant to the study of the investment process. Bernanke (1983) notes that the level of uncertainty perceived by firms is likely to vary cyclically, and emphasizes that irreversibility effects are important for understanding the cyclical behavior of aggregate investment. However, most option valuation models only consider the optimal timing for the adoption of an individual project with given characteristics, do not provide a proper dynamic investment function, and are not directly comparable to more familiar dynamic models based on convex adjustment costs. Pindyck (1988) applies option pricing techniques to marginal irreversible investment choices, and Bertola (1988) shows that the solution to problems of this type can be derived by dynamic programming as well as by option evaluation methods; formally similar problems of "singular" stochastic control have been studied in Operations Research by probabilistic and/or analytical methods (see e.g. Chow *et al.*, 1985; Karatzas and Shreve 1984, 1988).

To allow a characterization of aggregate investment series, these microeconomically realistic models need to be combined with a theory of stochastic aggregation. Idiosyncratic uncertainty is on the one hand undeniably realistic, on the other necessary for irreversibility constraints to have empirical relevance in light of the relatively low volatility of aggregate variables. Caballero and Engel (1990) develop a basic framework for the study of endogenous coordination in models of infrequent adjustment in the presence of idiosyncratic sources of uncertainty, and Bertola and Caballero (1990) characterize the aggregate dynamics generated by infrequent microeconomic adjustment.

This paper solves an individual firm's optimal irreversible capital accumulation problem under uncertainty, and characterizes the behavior of aggregate investment when both idiosyncratic and aggregate sources of uncertainty are present. Section 2 characterizes sequential investment decisions at the microeconomic level with and without irreversibility constraints. Section 3 studies aggregate investment behavior when many units like that considered in Section 2 face common as well as idiosyncratic shocks. Section 4 applies the model to an interpretation of postwar U.S. investment series, and Section 5 concludes. Technical derivations are reported in several appendices.

2. Optimal sequential investment under uncertainty

We consider a firm whose cash flows are a constant elasticity function $\Pi(K, Z)$ of K, the installed capital stock, and Z, an index of business conditions:

$$\Pi(K(\tau), Z(\tau)) = K(\tau)^{\alpha} Z(\tau) \qquad 0 < \alpha < 1 \tag{1}$$

This expression may be viewed as loglinear approximation to general functional forms; the approximation is exact if demand for the firm's product has constant elasticity and production is Cobb-Douglas in $K(\tau)$ and other, perfectly flexible factors of production. The business conditions process $\{Z(\tau)\}$ would then depend positively on the strength of demand for the firm's product and on productivity, and negatively on the cost of factors other than capital.

Let $\{Z(\tau)\}$ follow the process

$$dZ(\tau) = Z(\tau)\Big(\vartheta_1 \, d\tau + \sigma_1' \, dW(\tau)\Big) \tag{2}$$

where $\{W(\tau)\}$ is a two-dimensional Wiener process, ϑ_1 is a constant scalar, and σ_1 is a 2×1 constant vector. Equation (2) can again be viewed as a simple and empirically realistic representation of uncertainty; the multiplicative disturbance $\{Z(\tau)\}$ follows the process in (2) under constant elasticity demand and Cobb-Douglas production functions if demand, productivity, and the cost of flexible factors of production grow at some constant mean rate which is perturbed in continuous time by normally distributed random variables, independent over time and possibly contemporaneously correlated.

Capital can be purchased and installed at unit price $P(\tau)$, but installed capital has no resale value if investment is irreversible. By equations (1) and (2), $Z(\tau) > 0$ for all τ and the marginal contribution of installed capital to operating profits is always positive; thus, scrapping is never profitable, and the installed capital stock process $\{K(\tau)\}$ decreases only via depreciation, which is assumed to take place at constant exponential rate δ.

Let the purchase price of capital $\{P(\tau)\}$ follow

$$dP(\tau) = P(\tau)\Big(\vartheta_2 \, d\tau + \sigma_2' \, dW(\tau)\Big) \tag{3}$$

where ϑ_2 and σ_2 are conformable to ϑ_1 and σ_1. The variance-covariance matrix of the proportional increments in the processes exogenous to the firm is then given by

$$\mathrm{Var}\Big(\frac{dZ}{Z}, \frac{dP}{P}\Big) = [\sigma_1 \; \sigma_2]'[\sigma_1 \; \sigma_2] \equiv \Sigma$$

The firm's managers choose the investment policy so as to maximize the market value of the firm, defined as the present discounted value at rate r of expected future cash flows. We assume that the sample path of $\{W(\tau)\}$ contains all the information relevant to the firm's problem. By (2) and (3), the probability distribution of $\{P(\tau), Z(\tau)\}$ as of time t is uniquely determined by $P(t)$

and $Z(t)$ for all $\tau \geq t$, and the optimal investment policy $\{G(\tau)\}$ is to be chosen among the class of nondecreasing processes which depend, at every time t, on the information contained in the sample path of $\{Z_\tau\}$ and $\{P_\tau\}$ up to time t: the investment process cannot predict the future.[1] We then define

$$V(K(t), Z(t), P(t)) \equiv \max_{\{G(\tau)\}} E_t\left\{\int_t^\infty e^{-r(\tau-t)}\left(K(\tau)^\alpha Z(\tau)d\tau - P(\tau)\, dG(\tau)\right)\right\} \tag{4}$$

$$\text{subject to } dK(\tau) = -\delta K(\tau)d\tau + dG(\tau), \qquad dG(\tau) \geq 0 \tag{5}$$

where $\{G(\tau)\}$ is the cumulative gross investment process, restricted to have positive increments, and $E_t\{.\}$ denotes conditional expectation taken, at time t, over the joint distribution of the $\{Z(\tau)\}$, $\{P(\tau)\}$, and $\{K(\tau)\}$ processes. While the first two processes are exogenous to the firm's problem, the distribution of $\{K(\tau)\}$ is determined endogenously by the optimal investment rule.

Reversible investment

If capital could be purchased or sold at the same price $P(t)$, then $dG(t)$ would be unconstrained and the first-order condition for choice of capital stock at every point in time would be

$$\partial_K \Pi(K(t), Z(t)) = (r + \delta - \vartheta_2)\, P(t) \quad \forall t, \tag{6}$$

the Jorgenson (1963) optimality condition, equating the marginal revenue product of capital to its *rental cost.* Intuitively, if the purchase and sale price of capital were equal to each other (though random over time) it would be possible to *rent* capital services; if risk-neutral arbitrageurs are present in the market, the expected opportunity cost of carrying a stock of progressively depreciating capital available for rental should equal the flow operating profits from its use in production.

Under the assumed functional forms, (6) yields an expression for the *frictionless* capital stock,

$$K^f(Z(t), P(t)) = \left(\frac{(r + \delta - \vartheta_2)}{\alpha}\frac{P(t)}{Z(t)}\right)^{\frac{1}{\alpha - 1}} \qquad \forall t \tag{7}$$

When investment is unconstrained, $K(t)$ is not a state variable and the value of the firm's investment strategy is given by

$$V^f(Z(t), P(t)) = E_t\left\{\int_t^\infty e^{-r(\tau-t)}\left(K^f(\tau)^\alpha Z(\tau)\, d\tau - P(\tau)\, \left(dK^f(\tau) + \delta K^f(\tau)\, d\tau\right)\right)\right\} \tag{8}$$

By (2), (3), and (7), $K^f(\tau)^\alpha Z(\tau)$ is lognormally distributed, and the integral in (8) converges if

$$r > \left(\frac{\vartheta_1}{1-\alpha} - \frac{\alpha\vartheta_2}{1-\alpha} + \tfrac{1}{2}\|\Sigma\|\frac{\alpha}{(\alpha-1)^2}\right): \tag{9}$$

---

[1] Formally, what is required is that $\{G(\tau)\}$ be progressively measurable with respect to the filtration $F_t^W \equiv \sigma(W(s); 0 \leq s \leq t)$, the nondecreasing family of sigma-fields generated on the space of continuous function $t \rightarrow R^2$ by observation of $W(\tau)$. By the accumulation constraint (5), the installed capital stock process $\{K(\tau)\}$ is also adapted to $\{F_t^W\}$.

by Ito's lemma, the right-hand side is the expected rate of increase of revenues (the determinant of the variance-covariance matrix, $\|\Sigma\|$, appears because of Jensen's inequality). Intuitively, for the firm's infinite horizon value to be finite the required rate of return must be large relative to the growth rate of operating profits for given capital (ϑ_1) and (minus) the expected rate of increase of the capital purchase price ($-\vartheta_2$).

Characterization of irreversible investment

The installed capital stock $\{K(\tau)\}$ may depend on the whole past history of $\{Z(\tau), P(\tau)\}$ if investment is irreversible; thus, $K(t)$ is a state variable at time t. However, history dependence does not extend past the last time of positive gross investment. It is convenient to define a *desired capital* process $K^d(P(t), Z(t))$ such that

$$K(t) \geq K^d(P(t), Z(t)); \qquad dG(t) > 0 \quad \Rightarrow \quad K(t) = K^d(P(t), Z(t))$$

To see that an optimal irreversible investment program can be characterized in terms of the "desired capital" construct, imagine momentarily lifting the irreversibility constraint at some time $\tilde{t}$. At $\tilde{t}$, the installed capital stock would be unconstrained, thus not a state variable; if an optimal choice of capital stock at $\tilde{t}$ exists, it must then be a function of $Z(\tilde{t})$ and $P(\tilde{t})$. If $dG(\tilde{t}) > 0$ in the optimal irreversible process, the irreversibility constraint is not binding at $\tilde{t}$ and removing it has no effect: hence $dG(t) > 0$ implies $K(t) = K^d(P(t), Z(t))$. In general, the firm may choose to decrease its capital stock when given an opportunity to do so, to imply that $K(t) \geq K^d(P(t), Z(t))$.

The investment rule which achieves the maximum in (4) can be computed explicitly. In Appendix A, we derive differential equations which are necessarily satisfied by the value function $V(.)$ and by its derivative with respect to K, denoted $v(.)$, along the optimal capital accumulation path. The investment policy that solves these functional relationships has a simple and intuitive form: the marginal revenue product of capital should never be allowed to exceed a *constant* proportion c of the purchase price of capital P,

$$\partial_K \Pi(K(t), Z(t), P(t)) \begin{cases} \leq cP(t) & \forall t; \\ = cP(t) & \forall t \text{ such that } dG(t) > 0 \end{cases} \tag{10}$$

Using the results reported in the Appendix, the ratio of flow marginal profits to purchase price of capital which triggers investment can be shown to equal

$$c \equiv r + \delta - \vartheta_2 + \tfrac{1}{2}\|\Sigma\| A \tag{11}$$

We show in Appendix A that $A > 0$ if the condition in (9) is satisfied and the irreversible investment problem admits a solution. Thus, when $\|\Sigma\| > 0$ the marginal revenue product of capital that triggers irreversible investment is larger than the neoclassical user cost of capital.

The marginal condition in (10) can be inverted to obtain an expression for the firm's desired capital stock as a function of the current values of $Z(t)$ and $P(t)$:

$$K^d(Z(t), P(t)) = \left(\frac{c}{\alpha}\frac{P(t)}{Z(t)}\right)^{\frac{1}{\alpha-1}} \tag{12}$$

The optimal irreversible investment policy can then be characterized quite simply in terms of the closed form expression (12). If the currently installed capital stock $K(t)$ is smaller than $K^d(t)$, the firm should immediately invest so as to obtain $K(t) = K^d(t)$; otherwise, $K(t)$ should be allowed to depreciate. The firm's managers need to form expectations for the distant future when deciding when and how much to invest, because irreversible investment decisions, unlike reversible ones, are relevant to future cash flows: the firm may find itself stuck with an excessive stock of capital. The desired capital stock, however, is by definition a function of current observables only and, under the assumed functional forms, the features of the $\{Z(t)\}$ and $\{P(t)\}$ stochastic processes which are relevant to the firm's problem can be summarized by the scalar constant c. It can be shown that $c/(r + \delta - \vartheta_2)$ is decreasing in ϑ_1, increasing in ϑ_2, δ, and $\|\Sigma\|$. Intuitively, current decisions are likely to make the irreversibility constraint binding over the relevant planning horizon, determined by the discount rate r, if the rate of increase of desired capital is expected to be lower (relative to the depreciation rate of installed capital), or if it is more volatile.

Figure 1 plots the ratio of desired to frictionless capital stocks against $\|\Sigma\|$ for several values of the other parameters. As long as $\|\Sigma\| > 0$, the "desired" irreversible capital stock $K^d(Z(t), P(t))$ is smaller than the frictionless capital stock $K^f(Z(t), P(t))$ of equation (7). It does not follow, however, that the installed stock of capital should generally be smaller when investment is irreversible than when it is reversible. In fact, the risk-neutral firm under consideration is *ex-ante* reluctant to undertake irreversible investment only because adverse realizations in the process it takes as exogenous may *ex-post* leave it stuck with excess capital. Thus, although $K(t) < K^f(t)$ at times of positive gross investment, we should observe $K(t) > K^f(t)$ when the realizations of $\{Z(t)\}$ and $\{P(t)\}$ are such as to make the firm regret having invested in the past. On average, the capital intensity of production under investment irreversibility is actually *higher* than it would be if equation (7) applied at all times. This can be verified by the long-run average expressions derived in the next section, and is due to the discounting effects discussed in Bertola (1988) and Bentolila and Bertola (1990). In this paper, we disregard these effects on the level of capital to focus on the dynamic implications of irreversibility for the investment series.

3. Aggregate Investment

Our stylized model of homogeneous capital accumulation yields a closed-form solution under reasonably flexible functional form assumptions. Many real-life investment projects are well approximated by models similar to that of Section 2, and the simplicity of the solution makes it possible to explore the empirical implications of the model. However, aggregation issues need to be addressed before confronting the model with available data. Investment decisions are not taken in isolation: different firms' decisions to purchase similar equipment depend on each other through market interactions; capital equipment is in general heterogeneous even within the same productive process, with different types of capital being substitutable or complementary to each other; and it may be possible to reconvert capital on hand to new uses, or to sell it to other users (at a price, of course, which reflects the equipment's current profitability and replacement cost).

We deal with these problems by focusing on investment in *new* capital goods, as is appropriate from a macroeconomic point of view, and by modeling aggregate investment in terms of stochastic aggregation of a continuum —approximating a very large number— of individual units indexed by $i \in [0, 1]$. Each "unit" should be understood to correspond to a specific type of homogeneous capital, owned by a specific economic agent. We assume each unit to provide its owner with cash flows approximated by the constant-elasticity function of equation (1), disturbed by a unit-specific stochastic process $Z_i(t)$, and we let the capital stock installed in it be (irreversibly) accumulated by paying the stochastic, unit-specific purchase price $P_i(t)$. In this framework, then, all linkages across investment decisions —both those due to common ownership of heterogeneous types of capital, and those deriving from market interactions among distinct decision makers— are modeled in terms of cross sectional correlation of innovations in the unit-specific stochastic processes.

Aggregation of reversible investment

The parameters of the individual unit's problem could all be indexed by i without substantially affecting the results. For simplicity, however, let α, r, ϑ_1, ϑ_2, σ_1 and σ_2 be the same for all units. *Reversible* investment policies are then easily aggregated, even allowing for cross-sectional heterogeneity in the realizations of unit-specific stochastic processes. Consider the logarithm of unit i's revenue-maximizing capital stock in the absence of irreversibility constraints,

$$k_i^f(t) \equiv \ln K_i^f(t) = \frac{1}{\alpha - 1} \ln \left(\frac{(r + \delta - \vartheta_2)}{\alpha} \frac{P_i(t)}{Z_i(t)} \right) \qquad (13)$$

If $Z_i(t)$ and $P_i(t)$ follow geometric Brownian motion processes (equations (2) and (3)), then an application of Itô's lemma yields

$$dk_i^f(t) = \Theta \, dt + \sigma \, dW_i(t) \qquad (14)$$

where $W_i(t)$ is a univariate Brownian motion process constructed as a combination of the processes driving $P_i(t)$ and $Z_i(t)$, and

$$\Theta = \frac{\vartheta_1 - \vartheta_2 - \frac{1}{2}(\sigma_1'\sigma_1 - \sigma_2'\sigma_2)}{1 - \alpha}, \qquad \sigma = \frac{\sqrt{(\sigma_1 - \sigma_2)'(\sigma_1 - \sigma_2)}}{1 - \alpha}$$

We denote with $d\bar{k}(t)$ the rate of growth of aggregate capital, and we let unit i's share in $\bar{k}(t)$ be a function $\omega_i(t)$ on $i \in [0,1]$ at time t:[2]

$$d\bar{k}(t) = \int_0^1 \omega_i(t) dk_i(t)\, di$$

We then define the *aggregate* component of the uncertainty facing an individual unit,

$$\sigma_A d\bar{W}(t) \equiv \int_0^1 \omega_i(t)\, \sigma\, dW_i(t)\, di,$$

and rewrite (14) as

$$dk_i^f(t) = \Theta\, dt + \sigma_A\, d\bar{W}(t) + \sigma_I\, dW_i(t). \tag{15}$$

Noting that the *idiosyncratic* component of uncertainty

$$\sigma_I\, dW_i(t) \equiv \sigma\, dW_i(t) - \sigma_A\, d\bar{W}(t)$$

averages to zero by definition in aggregate data, we can multiply (15) by $\omega_i(t)$ and integrate over i on $[0,1]$ to obtain the dynamics of the aggregate capital stock under reversible investment:

$$d\bar{k}^f(t) = \Theta\, dt + \sigma_A\, d\bar{W}(t). \tag{16}$$

Thus, if investment were reversible idiosyncratic uncertainty would be irrelevant to aggregate outcomes, and microeconomic investment theory could be directly applied to aggregate data. Investment functions derived from equations like (13), however, perform poorly when confronted with actual data, at all levels of aggregation. In particular, their error terms are strongly serially correlated, prompting researchers to include lags in their investment equations, rationalized by *ad hoc* adjustment cost functions.

Irreversible Investment

We prefer to interpret the empirical shortcomings of equations like (13) in terms of unit-level irreversibility. Each unit's irreversible capital accumulation path is determined by unit-specific $Z_i(t)$ and $P_i(t)$ processes through its desired capital stock process as defined in equation (12). Since the desired and reversible capital stocks differ only by a constant of proportionality, the dynamics of $k_i^d(t) \equiv \ln K_i^d(t)$ coincide with those of $k_i^f(t)$. Aggregating, we obtain

$$d\bar{k}^d(t) = \Theta\, dt + \sigma_A\, dW(t). \tag{17}$$

Consider next the logarithm of unit i's installed capital stock, $k_i(t)$. Roughly speaking, we have $dk_i(t) = dk_i^d(t)$ at times when unit i is investing, and $dk_i(t) = -\delta dt$ at all other times.[3] Let

[2] As is customary in general equilibrium and macroeconomic models, we approximate a large number of finitely-sized individuals by a continuum of infinitesimally small units.

[3] This statement is formally correct since the processes under consideration have continuous sample paths. Note, however, that dk_t is (infinitesimally) positive only on a measure-zero set of time points, reflecting the *singular* character of the optimal investment policy.

$s_i(t) \equiv k_i(t) - k_i^d(t)$ denote the log deviation of a unit's actual capital stock from its desired one. Clearly,

$$ds_i = \begin{cases} -\delta dt - dk_i^d(t) & \text{when } dG_i(t) = 0, \\ 0 & \text{otherwise} \end{cases}$$

Integrating over i and defining $\bar{s}(t) = \int_0^1 \omega_i(t) s_i(t)\, di$, we obtain the dynamics of the aggregate installed capital stock:

$$d\bar{k}(t) = d\bar{k}^d(t) + d\bar{s}(t), \tag{18}$$

Noting that

$$d\bar{k}(t) = d\ln \bar{K}(t) = \frac{d\bar{G}(t) - \delta \bar{K}(t)\, dt}{\bar{K}(t)},$$

where $d\bar{G}(t)$ and $\bar{K}(t)$ denote aggregate gross investment and aggregate capital, we can rewrite equation (18) in terms of

$$\Gamma(t) \equiv \frac{d\bar{G}(t)}{\bar{K}(t)}, \qquad \Gamma^*(t) \equiv \frac{d\bar{K}^f(t)}{\bar{K}^f(t)} + \delta = \frac{d\bar{K}^d(t)}{\bar{K}^d(t)} + \delta,$$

to obtain

$$\Gamma(t) = \Gamma^*(t) + d\bar{s}(t). \tag{19}$$

In our stochastic aggregation framework, the *actual* and *desired* gross investment/capital ratios differ by $d\bar{s}(t)$, the change in the average difference between installed and desired capital stocks at the individual units' level.[4]

While only the mean of the empirical cross-sectional distribution of $k_i(t) - k_i^d(t)$ is directly relevant to aggregate phenomena, its dynamics are determined by all moments of the $s_i(t)$ distribution. To study aggregate investment, it is then necessary to track the whole cross sectional distribution, whose behavior depends crucially on the *idiosyncratic* component of unit-level uncertainty.[5]

The role of idiosyncratic uncertainty

We consider first the simple special case $\sigma_A = 0$, in which innovations in the $\{Z_i(t)\}$ and $\{P_i(t)\}$ processes are independent across units as well as over time. We are then studying a large number (approximated by a continuum) of units, driven by independent sources of uncertainty with common probability law. By the Glivenko-Cantelli theorem (see e.g. Billingsley, 1986), the empirical

[4] For most purposes, we might equivalently work with capital stock (log) levels which, by (18), obey $\bar{k}(t) = \bar{k}^d(t) + \bar{s}(t)$. We choose to work with first differences because we feel that the dynamics to be explained, at business-cycle frequencies, are those of the investment rate. However, we make use of the (cointegrated) relationship in levels to estimate the $k^d(t)$ series (see Section 4 below).

[5] Since the identity of units at different points in the state space is irrelevant from the aggregate point of view, it is not necessary to study the *joint* probability distribution of individual units (a process of much higher dimensionality). Caballero and Engel (1990) discuss this point further.

distribution converges to the common probability distribution of the individuals as the number of units considered becomes larger; in the limit, it is therefore possible to characterize the behavior of the former by that of the latter.

Appropriate methods for the study of probability distributions generated by continuous time, continuous state space Markov processes are readily available in the literature. Let $\{F(s,t)\}_{t=0}^{\infty}$ denote the (nonstochastic) path of the cross sectional distribution for a given initial condition $F(s,0)$. We study the path of the "empirical density" $f(s,t) \equiv \partial F(s,t)/\partial s$, which we define as the density function associated to the limit of the empirical distribution.

The cross sectional distribution $f(s,t)$ must satisfy the same Kolmogorov transition equation and boundary conditions as the probability density of individual $s_i(t)$ deviations, namely those appropriate for a Brownian motion process with drift $\vartheta = -(\Theta + \delta)$, standard deviation σ, and (since $k_i(t) \leq k^d(t)$) a reflecting barrier at zero.

In Appendix B, we solve the appropriate functional equations to obtain

$$f(s,t) = \xi\, e^{-\xi s} + \int_{0+}^{\infty} A(\beta) e^{-\lambda(\beta) t} e^{-\frac{1}{2}\xi s} \left(\cos(\beta s) - \frac{\xi}{2\beta} \sin(\beta s) \right) d\beta \tag{20}$$

where $\xi \equiv -\frac{2\vartheta}{\sigma^2}$, $\lambda(\beta) = (-\beta^2 + \xi^2/4)\vartheta/\xi$, and

$$A(\beta) = \frac{2}{\pi(1 + \xi^2/4\beta^2)} \int_0^{\infty} f(s,0)\, e^{\frac{1}{2}\xi s} \left(\cos(\beta s) - \frac{\xi}{2\beta} \sin(\beta s) \right) ds.$$

If the initial cross-sectional distribution $f(s,0)$ is different from

$$\lim_{t \to \infty} f(s,t) = \xi e^{-\xi s}, \tag{21}$$

then its mean, given by

$$\bar{s}(t) = \frac{1}{\xi} - \int_{0+}^{\infty} \frac{A(\beta) e^{-\lambda(\beta) t}}{\beta^2 (1 + \xi^2/4\beta^2)} d\beta \tag{22}$$

is not constant over time. As $t \to \infty$, however, $\bar{s}(t)$ converges to $\frac{1}{\xi}$, the mean of the ergodic distribution in (21), and $d\bar{s}(t)$ approaches zero from any initial condition. Figure 2 plots the ergodic distributions for two different values of ξ (solid lines), and some of the distributions encountered as the empirical densities converges to the flatter one starting from the steeper one (dashed lines).

By equations (18) and (19), actual investment should then track desired investment exactly in the aggregate if all uncertainty is idiosyncratic. Idiosyncratic uncertainty cancels at all times in the definition of aggregate desired capital as well, thus it should eventually be the case that $\Gamma(t) = \Gamma^*(t) = (\Theta + \delta)\, dt$: the rate of investment should be constant as well as nonstochastic, and the capital stock should grow exponentially at the deterministic rate Θ.

Aggregate uncertainty

In realistic situations, of course, capital accumulation does not follow a steady exponential path, and aggregate uncertainty is not negligible. In the context of our model, this implies that $\bar{s}(t)$ should fluctuate over time in equation (19), producing the rich dynamics often captured by ad-hoc lags in empirical investment equations. Recalling that the theory of Section 2 would apply exactly only to accumulation of perfectly homogeneous capital, these remarks apply to empirical studies of data at all levels of aggregation.

When $\sigma_A > 0$, the empirical distribution over $[0, \infty)$ of the $s_i(t)$ deviations has no steady state, and its evolution over time is governed by *stochastic* versions of the functional equations solved in Appendix B. While solutions to such stochastic partial differential equations can be shown to exist (see e.g. Krylov and Rozovskii, 1977), explicit analytical solution methods are not available; and, even if they were, additional steps would be necessary for empirical work on discrete-time observations. The solution can be characterized quite precisely by heuristic arguments, however, which also suggest an empirically useful approximation method.

At every point in time, the realizations of idiosyncratic stochastic processes tend to shape the empirical distribution into the exponential form of equation (21). Thus, the empirical distribution fluctuates around the form that would be stationary if all uncertainty were idiosyncratic, converging towards it when aggregate developments are dominated by cross sectional ones, but never coinciding with it. From the empirical point of view, only the mean of the cross sectional distribution can be observed, at discrete time points. The mean, of course, is consistent with an infinite variety of shapes for the cross-sectional distribution $f(s,t)$. Each of these shapes has different implications for the mean's responsiveness to further aggregate shocks, and must in turn be consistent with the pattern of aggregate shocks observed in the past. The empirical problem is then one of inferring, from the observed dynamics of endogenous and exogenous variables, the shape of empirical distributions at every observation point — which depends on the history of aggregate shocks and, given the assumed probability structure, on the relative importance of aggregate and idiosyncratic sources of uncertainty.

In Bertola and Caballero (1990), we adopted a discrete approximation to both the state and time dimensions of the problem. That approximation allows the empirical distribution's evolution to be characterized by a vector difference equation, which is straightforward (if cumbersome) to implement on a computer; the possible patterns of aggregate shocks between observations are finite in number in a discrete framework, and the shape of the empirical distribution can be determined either choosing the path that minimizes some empirical criterion function, or (more restrictively) considering only the average across all the possible paths. We implemented both procedures on U.S. aggregate durables consumption data, obtaining quite similar results.

In this paper we propose an alternative approach. Since continuous information is not available as to aggregate developments, we assume the realizations of aggregate uncertainty to be evenly spread within each observation period. Namely, if we can infer from aggregate data that the *average* desired stock of capital increases by $x\%$ between t and $t+h$, then we model aggregate dynamics *as if* the increase occurred at a constant rate x/h in the continuous time interval between observations. This is only an approximation, of course. Investment being irreversible, the time-aggregated investment rate is path-dependent and the variability of desired capital at higher frequencies is, in principle, relevant for the observed path of installed capital. The approximation also neglects the infinite variation property of Brownian paths; we believe, however, that any empirical importance of these issues is overshadowed by the substantial simplification of the analytical and estimation problems: the proposed approach makes it possible for us, as external observers, to reduce the intractable stochastic partial differential equation to a sequence of deterministic linear PDEs similar to the one solved in Appendix B.

Specifically, let observations on desired and actual capital stocks be available at the times $(t_0, t_1, t_2, \ldots)$, with $t_h - t_{h-1} = \Delta t$, $h = 0, 1, 2, \ldots$. The discrete counterpart of equation (19) is then

$$\Gamma(t_h) = \Gamma^*(t_h) + \Delta\bar{s}(t_h)$$

where $\Gamma(t_h)$ and $\Gamma^*(t_h)$ denote, respectively, the observed and desired ratios of gross investment to capital between t_{h-1} and t_h.

Since aggregate developments are approximated by a sequence of nonstochastic trends, the results of Appendix B can be applied within each observation period. Defining

$$\xi_h = -\frac{2\,\Gamma^*(t_h)}{\sigma_I^2},$$

we can treat the cross-sectional distribution at the end of each period as the initial condition for the next period, and compute cross sectional densities at each observation point by the following recursive relationships:

$$f(s,h) = \xi_h e^{-\xi_h s} + \int_{0+}^{\infty} A(\beta;h)\, e^{-\lambda(\beta)t} e^{-\frac{1}{2}\xi_h s} \left(\cos(\beta s) - \frac{\xi_h}{2\beta}\sin(\beta s)\right) d\beta \qquad (23)$$

$$A(\beta;h) = \frac{2}{\pi(1+\xi_h^2/4\beta^2)} \int_0^{\infty} f(s,h-1)\, e^{\frac{1}{2}\xi_h s} \left(\cos(\beta s) - \frac{\xi_h}{2\beta}\sin(\beta s)\right) ds. \qquad (24)$$

If the investment/capital ratio were constant over time, as would be the case if $\sigma_A = 0$, then we would have $\xi_h = \xi_{h-1}$ for all h, and the recursion would track at discrete times the convergent path of the empirical distribution to its stable form in equation (21). If aggregate investment fluctuates over time, however, the ξ_h values relevant to each observation are different, and the recursion generates a sequence of distributions linked by initial and final conditions.

The change in the mean of the cross sectional distribution, $\Delta\bar{s}(t)$, is then readily computed by the expression in equation (22):

$$\Delta\bar{s}(h) = \frac{1}{\xi_h} - \int_{0+}^{\infty} \frac{A(\beta,h)e^{-\lambda(\beta)t}}{\beta^2(1+\xi_h^2/4\beta^2)}\, d\beta - \left(\frac{1}{\xi_{h-1}} - \int_{0+}^{\infty} \frac{A(\beta,h-1)e^{-\lambda(\beta)t}}{\beta^2(1+\xi_{h-1}^2/4\beta^2)}\, d\beta\right) \qquad (25)$$

4. Empirical implications and evidence

Given a value of σ_I and a $\Gamma^*(t)$ sequence, we can construct a predicted path of aggregate investment, $\Gamma^*(t) + \Delta\bar{s}(t)$, and compare it to the observed actual investment rate $\Gamma(t)$.

This section interprets the behavior of U.S. investment in light of our theoretical results. An annual $\Gamma(t)$ data series is constructed as the ratio of gross investment to the relevant capital stock over the 1954-1986 period. The corresponding $\Gamma^*(t)$ is not directly observable, of course. We approximate it in two different ways, both based on the simple neoclassical model of investment. Appendix C reports some derivations along with data sources and definitions, and the two approximation methods are described below. These series correspond to the *hypothetical* investment rate that would be observed if disinvestment were possible at the individual units' level, *and* demand, prices and interest rates were those actually observed in the U.S. economy. In our partial equilibrium exercise, these series simply summarize aggregate effects in the *partial equilibrium* investment problems of firms, and do not represent counterfactual general equilibrium experiments.

The actual investment/capital ratio $\Gamma(t)$ and the two alternative $\Gamma^*(t)$ series are plotted in Figure 3. The observed series is clearly much less variable than the theoretical constructs: the standard deviation of the former is 0.017, those of the latter are 0.046 and 0.047. The contemporaneous correlations between actual investment and the two series are only 0.29 and 0.30. The first-order serial correlation coefficients of the $\Gamma^*(t)$ series are 0.25 and 0.23, while $\Gamma(t)$ exhibits substantially higher (0.68) first-order serial correlation. Before proceeding, we need to verify the realism of the constant drift and variance assumptions we made in our theoretical sections. In fact, there is no evidence of different dynamics in the data: the estimated $\Gamma^*(t)$ series is statistically indistinguishable from white noise.

The dynamic effects of irreversibility

In previous research, these facts have been rationalized by postulating smooth and convex adjustment cost functions and have led researchers to estimate partial-adjustment equations of doubtful microeconomic realism. Irreversibility of investment decisions, like more familiar forms of adjustment costs, reduces the responsiveness of endogenous variables to exogenous shocks: at the individual unit's level gross investment is completely unresponsive to the forcing variables when the irreversibility constraint is binding. The extent to which microeconomic inaction affects aggregate dynamics depends on the degree of synchronization of individual actions, which is in turn determined by the form of the adjustment policy on the one hand, and by the importance of aggregate developments relative to that of idiosyncratic uncertainty (see Bertola and Caballero 1990 and Caballero and Engel 1990).

In the problem we are considering, aggregate uncertainty is small relative to the drift: the sample mean of the gross investment/capital ratio, which approximates $\Theta + \delta$, is 0.16, almost four times as large as its standard deviation. At the *aggregate* level, then, the irreversibility constraint should almost never be binding if aggregate innovations are normally distributed as we assumed above; in fact, desired gross investment is always strictly positive in our sample. If idiosyncratic uncertainty were negligible, then, all units would be bunched in a spike at the $k_i(t) = k_i^d(t)$ point, and actual investment should track desired investment exactly.

If σ_I is large, on the other hand, the irreversibility constraint is binding (for some of the units, in some of the periods), and it is possible for changes in the cross-sectional distributions to smooth out the response of $\Gamma(t)$ to movements in $\Gamma^*(t)$. The investment actually observed in a short period of time is that undertaken by those units which are in the neighborhood of the $k(t) = k^d(t)$ point: the width of the relevant neighborhood increases with total uncertainty, since units within a given distance from the investment point are more likely to hit it if $k^d(t)$ is highly volatile.

As σ_I becomes larger, an aggregate shock which shifts all units by a given amount in this space triggers investment by fewer units, since the cross-section distribution of $k(t) - k^d(t)$ deviations becomes flatter (see Figure 2, where a large σ_I yields a smaller ξ), and the proportion of units within the relevant distance from the investment point which experience a positive change in their desired capital stock tends to become constant as σ_I/σ_A increases.[6] Thus, investment per unit time is unresponsive to aggregate shocks when aggregate developments are drowned by idiosyncratic ones and investment is irreversible at the microeconomic level.

Empirical results

We proceed to compare the observed sample path of investment with that implied by our model. Since the unbounded state space implied by investment irreversibility is computationally impractical, we implement the solution derived for the case of a bounded state space in Appendix B, choosing a value S for the upper reflecting barrier so large as to obtain a $f(S)$ value smaller than the precision of the numerical routines we use. We also need to truncate the Fourier series representation of equation (B12). Taking $S = 10$ (i.e., allowing actual capital to be over twenty thousand times larger than desired capital) and considering 15 terms in equation (B12) yields a more than satisfactory approximation.

We show in Appendix C that, under assumptions of Cobb-Douglas production and isoelastic demand, the hypothetical desired investment series can be computed as

$$\Gamma_i^*(t) = \nu\Big(\Delta \ln Y(t) - \Delta \ln r_k(t)\Big) + (1-\nu)\Gamma(t) + \nu\delta \tag{26}$$

[6] A more detailed discussion of this insight is in Bertola and Caballero (1990). In the framework considered there, however, another effect also came into play: if there is reflection at an upper (disinvestment) boundary as well as at the $k^d(t) = k(t)$ point, similar considerations apply to disinvestment and, since the effects are symmetric inasmuch as idiosyncratic uncertainty is concerned, *net* adjustment tends to reflect aggregate shocks fully as idiosyncratic uncertainty increases. In the symmetric driftless case we emphasized in that paper, this effect is the predominant one; in the presence of drift, however, the relationship between idiosyncratic uncertainty and aggregate smoothing is not monotonic even when reflection occurs at both boundaries of the inaction region.

where $Y(t)$ is real value added and $r_k(t)$ is the neoclassical "rental" cost of capital, an empirical counterpart of $(r+\delta-\vartheta_2)P(t)$ in equation (6) constructed from price and interest rate data. We set the parameter ν, defined in Appendix C, to 1.1 (reasonable alternative values leave the results unchanged).

In Figure 4 we plot the aggregate investment path predicted by our model for the forcing process in (26), choosing $\sigma_I = 0.6$ to match the observed character of the actual investment series. Although the fit is far from perfect, this simple implementation of our model is clearly capable of smoothing the dynamics of the investment process, and of increasing the persistence of aggregate events' effects on capital accumulation. On the one hand, the standard deviation of investment is reduced from 0.046 (the standard deviation of frictionless investment) to 0.015 (to be compared to the actual investment's standard deviation, 0.017). On the other, the first order serial correlation is raised from 0.25 (for frictionless investment) to 0.66 (recall that the this parameter is 0.68 for actual investment).

A more flexible specification

The constant-elasticity functional forms assumed in Section 2 and in Appendix C imply unit coefficients for the rate of growth of output and the change in the rental cost of capital in equation (26). Because these functional forms should be regarded as approximations to more general ones, however, it may be desirable to relax this restriction. We consider the alternative specification

$$\Gamma_i^*(t) = \nu\Big(b_1\Delta \ln Y(t) + b_2\Delta \ln r_k(t)\Big) + (1-\nu)\Gamma(t) + \nu\delta. \tag{26'}$$

The left hand side variable is unobservable, making estimation of the coefficients b_1 and b_2 somewhat problematic. However, estimation can exploit the cointegration properties of the model, which are more easily understood considering the integral version of equation (26'), derived in Appendix C:

$$k^f(t) = \nu\,(b_1 \ln Y(t) + b_2 \ln r_k(t)) + (1-\nu)k(t) + \nu \ln(\gamma(1-\eta)). \tag{27}$$

Recalling that $k(t) = k^f(t) + \bar{s}(t)$, (27) can be written in terms of installed capital:

$$k(t) = b_1 \ln Y(t) + b_2 \ln r_k(t) + \ln(\gamma(1-\eta)) + \frac{1}{\nu}\bar{s}(t). \tag{28}$$

The unobservable component of equation (28), $\bar{s}(t)$, is stationary if our model is correctly specified (see Bertola and Caballero 1990). The $k(t)$, $\ln Y(t)$, and $\ln r_k(t)$ series, which are all integrated of order one, must then be cointegrated with cointegrating vector $(1, -b_1, -b_2)$.

Our model implies a strong negative covariance between $k^f(t)$ and $\bar{s}(t)$. Although this moment is of lower asymptotic order than the variance of the nonstationary regressors, the small sample biases in the estimates of b_1 and b_2 are likely to be large.[7] The first row of Table 1 shows that the

[7] Basic projection theory implies that the estimate of $b_1 \ln Y(t) + b_2 \ln r_k(t)$, the most important

TABLE 1

PROCEDURE	b_1	b_2
Static OLS	1.35	-0.23
	(-)	(-)
Stock-Watson	1.19	-0.87
	(0.12)	(0.39)

All equations include a constant.
Standard errors in parenthesis.

conventional cointegrating approach (static OLS) yields an output coefficient slightly larger than one, while the coefficient of cost of capital is very close to zero.

The small sample bias can be reduced if the cointegrating vector is estimated by the procedure proposed in Stock and Watson (1989) which, in its simplest form, requires adding leads and lags of the first differences of $\ln Y(t)$ and $\ln r_k(t)$ to the right hand side of (28) until the leftover residual becomes approximately orthogonal, at all leads and lags, to the integrated regressors.[8] The second row of Table 1 reports the estimates of b_1 and b_2 when two leads and lags of the first difference of each integrated regressor are added; results are not very sensitive to the number of leads and lags included. The coefficients we obtain by this procedure are not significantly different from the theoretical unitary ones implied by constant-elasticity functional form. The coefficient of the cost of capital variable is larger in absolute value than that obtained by static OLS estimation, and is significant: this is interesting in itself, since finding significant effects of cost of capital variables on investment is not always easy.

Figure 5 reports the path of $\Gamma(t)$ along with those of the $\widehat{\Gamma}(t)$ and $\Gamma^*(t)$ estimates obtained by the Stock and Watson procedure. Once again, there is clear evidence of excess smoothness and persistence in actual investment as compared with frictionless investment, and the path of investment predicted by our model tracks actual investment fairly closely.[9]

---

component of our proxy for $k^f(t)$, must have smaller variance than $k(t)$ if estimated from equation (28). Our model implies that the variability of $k(t)$ should be smaller than that of $k^f(t)$ instead: thus, the small sample bias of the conventional cointegrating regression is particularly harmful in the context of models like the one we propose.

[8] The leads and lags are added for estimation purposes only. Also note that the projection theory result mentioned in footnote 7 no longer applies: here the fitted values are obtained multiplying a *subset* of the regressors by their respective coefficients, and their variance can be larger than the variance of the left hand side variable.

[9] The overall fit of the flexible model is slightly better, for a given σ_I, than that of the more restrictive model. With $\sigma_I = 0.60$, the R^2 rises from 0.36 to 0.41. The R^2 and other goodness-of-fit measures are, of course, much higher if computed on the levels of the capital stock series rather than on their first differences (i.e. investment rates).

On the results

It would be possible, in principle, to estimate σ_I by maximizing the fit of the $\widehat{\Gamma}(t)$ series to the observed $\Gamma(t)$ series. When we search over larger values of σ_I, however, the R^2 improves but the predicted series becomes too smooth. Monte Carlo experiments suggest that measurement errors in $\Gamma^*(t)$ might at least in part account for this. When an "observable" $\Gamma(t)$ series is generated by our model from a $\Gamma^*(t)$ series with the same moments as the ones we consider, the fit is of course maximized at the σ_I the value used in generating the data; if serially uncorrelated noise is added to the artificial $\Gamma^*(t)$ series, however, then the fit of the original $\Gamma(t)$ tends to be maximized at much higher values of σ_I, although the serial correlation properties of the two series remain by and large unchanged. This suggests that our procedure is quite sensitive to measurement errors. We have therefore chosen to match the volatility and serial correlation of the observed series rather than maximizing the fit.

This criterion yields a σ_I in the neighborhood of the 0.6 value used in generating the series reported in the Figures, to imply very high unit-level volatility of desired capital: if capital were perfectly mobile across units instead of being irreversibly allocated to a specific production process, the capital stock in use at a given unit would vary by as much as 60% on a yearly basis with very high probability.[10]

Since Section 2 provides a completely specified model of unit-level investment, we can characterize the microeconomic importance of investment irreversibility by specifying its parameters so as to match the observed rate of growth of capital and of capital costs as well as the degree of idiosyncratic uncertainty implied by the smoothness of aggregate investment. The choice of parameters

$$\vartheta_1 = 0.05,\ \vartheta_2 = -0.02,\ \sigma_1 = \begin{pmatrix} .55 \\ 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 \\ 0.50 \end{pmatrix}$$

is realistic, in that when combined with $\alpha = 0.10$ they yield $\sigma = 0.614$ and $\Theta = 0.049$ for the parameters of equation (14) above, matching the empirical evidence quite closely.[11] The depreciation

[10] Once again, the "desired capital" construct simply reflects external influences on the firm's problem in our partial equilibrium setting. The volatility of installed capital would not necessarily be so high in a world without irreversibility constraints: if, counterfactually, capital were perfectly homogeneous and all investment decision were reversible, the process followed by the price of new capital $P(t)$ would of course be quite different.

[11] The parameter α equals 0.10 if, for example, the share of equipment capital in value added is 13% and the markup coefficient is 24%. Note that we have assumed all units to have the same structural parameters, and the only source of comovement to be exogenous aggregate shocks. In a sense, structural heterogeneity plays a role very similar to that of exogenous idiosyncratic uncertainty (Caballero and Engel 1990), while strategic interactions may exacerbate (strategic complementarities) or dampen (strategic substitutability) the relative importance of aggregate shocks (Caballero and Engel 1989). These factors should be taken into account in order to interpret

rate implied by the capital and investment series we use (U.S. business equipment investment) is large (0.12 per year), and we set $r + \delta = 0.20$ in our empirical work. As shown in Figure 1 above, these parameters imply that when investment is positive at the unit level the stock of capital is about 33% lower than it would be implied by equality of marginal revenue product and conventional user cost of capital.

the magnitude of the sources of the uncertainty faced by individual units.

5. Conclusions

This paper proposes a closed-form sequential irreversible investment rule, studies its implications for aggregate investment behavior when both idiosyncratic and aggregate sources of uncertainty are present, and provides some empirical evidence for the model using postwar U.S. private equipment investment data.

We have shown that aggregate data are broadly consistent with a model in which microeconomic units rationally choose to install less capital than it would be implied by a frictionless neoclassical model when they invest, and allow the installed capital stock to depreciate when the irreversibility constraint becomes binding. In our model, microeconomic investment aims to keep installed capital close to a moving target which depends on the level of activity as well as on the cost of capital; the microeconomic irreversibility constraint, interacting with idiosyncratic sources of uncertainty, yields a smooth, highly persistent response of aggregate investment to innovations in activity and in the cost of capital.

Our model of aggregate investment has sound microeconomic foundations, and should be used to characterize tax policy experiments and business cycle dynamics in future applied work. The results of this paper do not yet provided a complete interpretation of investment dynamics, however: the fit of our specification is satisfactory, but leaves unexplained a nontrivial and serially correlated error component. Future research should explore the role of delivery lags, of other non-convexities, and of quasi-fixed factors other than capital. For this purpose, it will be important to study more disaggregated data, and it might also be useful to allow the volatility of both aggregate and idiosyncratic sources of uncertainty to vary within the sample.

Appendix A

The value function is defined as a present discounted value of expected values, and heuristic arguments (based, for example, on a discrete time approximation) suggest that optimal irreversible investment should satisfy a Bellman equation in the form

$$\begin{aligned} r V(K(t), Z(t), P(t))\, dt = \max_{dG(t)} K(t)^{\alpha} Z(t) dt - P(t) dG(t) + E_t\{dV(K(t), Z(t), P(t))\} \\ \text{subject to } dG(t) \geq 0 \end{aligned} \tag{A1}$$

at all times t.

We conjecture (and verify later) that the function $V(K, Z, P))$ is bounded and twice continuously differentiable in all its arguments for $\{K, Z, P\} \in (0, \infty)^3$. Itô's change-of-variable formula then yields (omitting time indexes and the arguments of $V(.)$):

$$\begin{aligned} E_t\{dV(.)\} =& E_t\{\partial_K V(.) dK + \partial_Z V(.) dZ + \partial_P V(.) dP \\ &+ \tfrac{1}{2}\partial_{ZZ} V(.)(dZ)^2 + \tfrac{1}{2}\partial_{PP} V(.)(dP)^2 + \partial_{PZ} V(.)(dZ)(dP)\} \\ =& \partial_K V(.)(-\delta K dt + dG) + \partial_Z V(.) Z \vartheta_1 dt + \partial_P V(.) P \vartheta_2 dt \\ &+ \tfrac{1}{2}\left(\partial_{ZZ} V(.) Z^2 \sigma_1' \sigma_1 dt + \partial_{PP} V(.) P^2 \sigma_2' \sigma_2 dt + \partial_{PZ} V(.) P Z(\sigma_1' \sigma_2 + \sigma_2' \sigma_1) dt\right) \end{aligned} \tag{A2}$$

where $\partial_{XY} f(x, y)$ denotes the partial derivative of a function $f(.)$ with respect to x and y, evaluated at $X = x$, $Y = y$.

Using (A2) in (A1), elementary Kuhn-Tucker formalism suggests that optimal irreversible investment should satisfy the complementary slackness conditions

$$\begin{aligned} \partial_K V(K(t), Z(t), P(t)) \leq P(t) \quad \forall t; \\ \partial_K V(K(t), Z(t), P(t)) = P(t) \quad \forall t : dG(t) > 0 \end{aligned} \tag{A3}$$

If $\partial_{KK} V(.)$ exists and is not zero, the second line of (A3) implicitly defines $K^d(Z(t), P(t))$.

Along the optimal irreversible accumulation path, the maximization in (A2) can be taken for granted, and (A1,A2,A3) imply that $V(.)$ should satisfy the relationship

$$\begin{aligned} r V(.) =& K^{\alpha} Z + \partial_K V(.)(-\delta K) + \partial_Z V(.) Z \vartheta_1 + \partial_P V(.) P \vartheta_2 \\ &+ \tfrac{1}{2}\left(\partial_{ZZ} V(.) Z^2 \sigma_1' \sigma_1 + \partial_{PP} V(.) P^2 \sigma_2' \sigma_2 + \partial_{PZ} V(.) P Z(\sigma_1' \sigma_2 + \sigma_2' \sigma_1)\right) \end{aligned} \tag{A4}$$

This functional equation holds identically along the optimal path, and can be differentiated term-by-term with respect to K if the relevant derivatives exist. Defining $v(K, Z, P) \equiv \partial_K V(K, Z, P)$, we obtain

$$\begin{aligned} (r + \delta) v(.) =& \alpha K^{\alpha - 1} Z + \partial_K v(.)(-\delta K) + \partial_Z v(.) Z \vartheta_1 + \partial_P v(.) P \vartheta_2 \\ &+ \tfrac{1}{2}\partial_{ZZ} v(.) Z^2 \sigma_1' \sigma_1 + \tfrac{1}{2}\partial_{PP} v(.) P^2 \sigma_2' \sigma_2 + \partial_{PZ} v(.) P Z(\sigma_1' \sigma_2 + \sigma_2' \sigma_1) \end{aligned} \tag{A5}$$

By (A3), the investment policy prevents $v(K, Z, P)$ from ever exceeding P. Thus, $v(.,.,.)$ must satisfy (A5) as well as the boundary condition

$$v(K^d(Z, P), Z, P) = P \tag{A6}$$

If the relevant derivatives exist (and, in particular, if $K^d(Z,P)$ is differentiable), differentiation of (A6) yields

$$\begin{aligned} \partial_K v(K^d(Z,P),Z,P) &= 0 \\ \partial_P v(K^d(Z,P),Z,P) &= 1 \\ \partial_Z v(K^d(Z,P),Z,P) &= 0 \end{aligned} \tag{A7}$$

Further, it must be the case that

$$\lim_{Z\to 0} v(K,Z,P) = 0 \tag{A8}$$

since $Z = 0$ is absorbing for the $\{Z(t)\}$ process.

The solution to (A5) is

$$v(K,Z,P) = \begin{cases} \frac{\alpha K^{\alpha-1}Z}{r+\alpha\delta-\vartheta_1} - \frac{c^{-A}}{A-1}\left(\alpha K^{\alpha-1}Z\right)^A P^{1-A}, & K \geq K^d(Z,P) \\ P, & K < K^d(Z,P) \end{cases} \tag{A9}$$

where

$$A \equiv \frac{-\left(\vartheta_1 + (1-\alpha)\delta - \vartheta_2 - \frac{1}{2}\|\Sigma\|\right) + \sqrt{\left(\vartheta_1 + (1-\alpha)\delta - \vartheta_2 - \frac{1}{2}\|\Sigma\|\right)^2 + 2(r+\delta-\vartheta_2)\|\Sigma\|}}{\|\Sigma\|}$$

is the positive solution in x of

$$\tfrac{1}{2}\|\Sigma\|x^2 + \left(\vartheta_1 + (1-\alpha)\delta - \vartheta_2 - \tfrac{1}{2}\|\Sigma\|\right)x - (r+\delta-\vartheta_2) = 0.$$

and c is given in equation (11) in the text.

It can be verified that $A > 1/(1-\alpha)$ if condition (9) in the main text holds true, and that $c = r + \delta - \vartheta_2$ only if $\|\Sigma\| = 0$ and $\vartheta_1 - (1-\alpha)\delta - \vartheta_2 > 0$. This would be a degenerate special case of Arrow's (1968) nonstochastic model, in which disinvestment is never desirable and the irreversibility constraint is completely irrelevant.

The investment policy characterized by (10) and (11) in the text is associated to a bounded, twice continuously differentiable value function $V(.)$, confirming the conjectures that led to (A1) and (A5). Over the region $K > K^d(Z,P)$, the derivative of the value function with respect to capital is $v(K,Z,P)$ given by (A9); thus,

$$V(K,Z,P) = V\left(K^d(Z,P),Z,P\right) + \int_{K^d(Z,P)}^{K} v(k,Z,P)\,dk \qquad \text{if } K > K^d(Z,P) \tag{A10}$$

Over the region $K < K^d(Z,P)$, by the definition of the desired capital stock, we have

$$V(K,Z,P) = V(K^d(Z,P),Z,P)) - \left(K^d(Z,P) - K\right)P \tag{A11}$$

Equation (A10) can be integrated to yield

$$V(K,Z,P) = V_0(Z,P) + \begin{cases} \dfrac{K^\alpha Z}{r+\alpha\delta-\vartheta_1} - \dfrac{K^{(A(\alpha-1)+1)}c^{-A}(\alpha Z)^A P^{1-A}}{(A-1)((\alpha-1)A+1)} & \text{if } K \geq K^d(Z,P) \\ -(K^d(Z,P)-K)P & \text{if } K < K^d(Z,P) \end{cases}$$

where $V_0(Z,P) = V\left(K^d(Z,P), Z, P\right)$. Since the irreversibility constraint can only decrease the value of the firm, $V(.)$ is bounded for P, Z in $(0,\infty)$ if the condition in (9) is satisfied.

Appendix B

This appendix derives the dynamic density of a Brownian motion with reflecting barriers at zero and at $S > 0$, and takes the appropriate limit as $S \to \infty$.

Finite state space

Let $f(s,t)$ denote the probability density of a process $s(t)$ with stochastic differential

$$ds(t) = \vartheta\, dt + \sigma\, dW(t), \qquad \vartheta < 0, \quad \sigma > 0$$

where $\{W(t)\}$ a standard Wiener process, and let $\{s\}$ be reflected at 0 and at $S > 0$. The function $f(s,t)$ can be derived by solving the forward Kolmogorov equation

$$\partial_t f(s,t) = \tfrac{1}{2}\sigma^2 \partial_{ss} f(s,t) - \vartheta \partial_s f(s,t), \tag{B1}$$

with boundary conditions

$$\tfrac{1}{2}\sigma^2 \partial_s f(0,t) = \vartheta f(0,t) \quad \forall t, \tag{B2}$$

$$\tfrac{1}{2}\sigma^2 \partial_s f(S,t) = \vartheta f(S,t) \quad \forall t, \tag{B3}$$

and given initial condition

$$f(s,0) = \bar{g}(s), \qquad \int_0^S \bar{g}(s) ds = 1 \tag{B4}$$

Separating the variables, we write $f(s,t) = g(s)h(t)$ and obtain a couple of ordinary differential equations. In the t direction,

$$h'(t) + \lambda h(t) = 0$$

has general solution $h(t) = Ae^{-\lambda t}$, A a constant of integration. In the s direction,

$$g''(s) + \xi g'(s) - \lambda \frac{\xi}{\vartheta} g(s) = 0 \tag{B5}$$

$$g'(0) = -\xi g(0) \tag{B6}$$

$$g'(S) = -\xi g(S) \tag{B7}$$

where $\xi \equiv -2\vartheta/\sigma^2$, $\xi > 0$.

Equations (B5-B7) define a Sturm-Liouville problem (see e.g. Churchill and Brown, 1987), with characteristic equation

$$\alpha^2 + \xi\alpha - \frac{\lambda}{\vartheta}\xi = 0$$

If $\lambda \leq \frac{-\xi\vartheta}{4} = \frac{\xi^2\sigma^2}{8}$, the roots are real and solutions take the general form

$$g(s) = A_1 e^{\alpha_1 s} + A_2 e^{\alpha_2 s}. \tag{B8}$$

Solutions in this form need to be considered only if they can satisfy the boundary conditions with A_1 and/or A_2 different form zero: (B6) and (B7) require

$$A_1(\alpha_1 + \xi) + A_2(\alpha_2 + \xi) = 0 \qquad A_1 e^{\alpha_1 S}(\alpha_1 + \xi) + A_2 e^{\alpha_2 S}(\alpha_2 + \xi) = 0$$

All solutions in the form (B8) are then identically zero except the one corresponding to $\lambda = 0$, with $\alpha_1 = -\xi,\ \alpha_2 = 0, A_2 = 0$:

$$g(s; \lambda = 0) = Ae^{-\xi s}$$

We then consider the solutions obtained for $\lambda > \frac{\xi^2\sigma^2}{8}$. The roots are complex, and the solution has the form

$$g(s;\lambda) = e^{-\frac{\xi}{2}s}\left(A\cos(\beta(\lambda)s) + B\sin(\beta(\lambda)s)\right) \tag{B9}$$

where $\beta(\lambda) = \sqrt{-\xi\left(\frac{\lambda}{\vartheta} + \frac{\xi}{4}\right)}$. Imposing (B6), we obtain

$$\frac{\xi}{2}A + \beta(\lambda)B = -\xi A$$

to imply that $B = -A\frac{\xi}{2\beta}$; for (B7) to be satisfied, we then need

$$A\left(\frac{\xi^2}{4\beta(\lambda)} - \beta(\lambda) - \frac{\xi^2}{2\beta(\lambda)}\right)\sin(\beta(\lambda)S) = 0$$

Using the definition of β and simplifying, we get

$$A\,\xi\frac{\lambda}{\vartheta}\sin(\beta(\lambda)S) = 0$$

Thus, all solutions in the form of (B9) are identically zero except those for which $\sin(\beta(\lambda)S) = 0$, or

$$-\xi\left(\frac{\lambda}{\vartheta} + \frac{\xi}{4}\right) = \frac{n^2\pi^2}{S^2}, \quad n = 1, 2, \ldots$$

Combining the results, we find that the general solution to (B1-B3) can be written

$$f(s,t) = \sum_{n=0}^{\infty} A_n f_n(s) e^{-\lambda_n t}$$

$$\lambda_0 = 0; \qquad \lambda_n = \frac{\sigma}{2}^2\left(\frac{n^2\pi^2}{S^2} + \frac{\vartheta^2}{\sigma^4}\right), \quad n = 1, 2, \ldots$$

$$f_0(s) = e^{-\xi s}; \qquad f_n(s) = e^{-\frac{\xi}{2}s}\left(\cos(\frac{n\pi}{S}s) - \frac{S}{n\pi}\frac{\xi}{2}\sin(\frac{n\pi}{S}s)\right), \quad n = 1, 2, \ldots$$

The initial condition

$$\sum_{n=0}^{\infty} A_n f_n(s) = \bar{g}(s) \tag{B10}$$

determines the constants A_k, $k = 0, 1, 2, \ldots$. Multiplying both sides of (B10) by $f_k(s)e^{\xi s}$, integrating between 0 and S, and exploiting the fact that $\int_0^S f_n(s)f_k(s)e^{\xi s}ds = 0$ for $n \neq k$, we obtain

$$A_0 = \xi/(1 - e^{-\xi S})$$

$$A_k = \frac{\int_0^S \bar{g}(s) f_k(s) e^{\xi s} ds}{\int_0^S (f_n(s)^2)\, e^{\xi s} ds}, \qquad k = 0, 1, 2, \ldots \tag{B11}$$

The integral in the numerator of (B11) can be computed numerically for general $\bar{g}(.)$, and the denominator has the closed form

$$\int_0^S (f_n(s))^2 \, e^{\xi s} ds = \left(\tfrac{S}{2} + \tfrac{S^3 \xi^2}{8 n^2 \pi^2}\right)$$

We note at this point that as $n \to \infty$ the constants converge to zero, and the λs diverge to positive infinity. Thus, truncation of the infinite series yields a satisfactory approximation to $f(s, t)$.

The mean of $f(s, t)$ can be computed analytically,

$$\begin{aligned} \bar{s}(t) &= \sum_{n=0}^{\infty} A_n e^{-\lambda_n t} \int_0^S f_n(s)\, ds \\ &= \frac{S}{1 - e^{\xi S}} + \frac{1}{\xi} + \sum_{n=0}^{\infty} A_n e^{\lambda_n t} \left(F(S, -\xi/2, n\pi/S) - \frac{S\xi}{2n\pi} G(S, -\xi/2, n\pi/S) \right) \end{aligned} \tag{B12}$$

where

$$\begin{aligned} F(z, a, b) &= \int_0^z e^{ax} \cos(bx) x \, dx \\ &= \left(z \frac{a\cos(bz) + b\sin(bz)}{a^2 + b^2} - \frac{(a^2 - b^2)\cos(bz) + 2ab\sin(bz)}{(a^2 + b^2)^2} \right) e^{az} + \frac{a^2 - b^2}{(a^2 + b^2)^2} \\ G(z, a, b) &= \int_0^z e^{ax} \sin(bx) x \, dx \\ &= \left(\frac{2ab\cos(bz) + (b^2 - a^2)\sin(bz)}{(a^2 + b^2)^2} - z \frac{b\cos(bz) - a\sin(bz)}{a^2 + b^2} \right) e^{az} - \frac{2ab}{(a^2 + b^2)^2} \end{aligned}$$

Unbounded state space

Given that $\xi > 0$,

$$\lim_{S \to \infty} f(S, t) = 0$$

and the boundary condition corresponding to (B6) is satisfied identically. All solutions in the form

$$f(s; \beta) = e^{-\frac{1}{2}\xi s} \left(\cos(\beta s) - \frac{\xi}{2\beta} \sin(\beta s) \right)$$

are therefore admissible, as well as $f_0(S)$ defined above, and the solution of the PDE takes the integral form

$$f(s, t) = A_0 f_0(s) + \int_{0^+}^{\infty} A(\beta)\, e^{-\lambda(\beta) t} f(s; \beta)\, d\beta.$$

where $\lambda(\beta) = \beta^{-1}(\lambda)$. In the limit as $S \to \infty$, the expressions for $A(\beta)$ and for the mean of the distribution take the form reported in the main text.

Appendix C

This appendix discusses how the desired stock of capital can be inferred from observable data under specific functional form assumptions, chosen to be consistent with the microeconomic optimization model of Section 1 on the one hand, and to be readily approximated by available data on the other.

Let every "unit" produce a homogeneous good by a constant-elasticity production function, $Y(t) = K(t)^{\gamma}\epsilon_s(t)$, and let the unit price of output be determined by a constant elasticity demand function, $P(t) = Y(t)^{-\eta}\epsilon_d(t)$. Here $\epsilon_s(t)$ denotes the level effect on production of disembodied productivity as well as that of inputs other than "capital," taken as given in this paper, and $\epsilon_s(t)$ indexes the strength of demand for the unit's output.

If investment were not irreversible, the capital stock should be chosen at every point in time to maximize

$$P(t)Y(t) - r_k(t)K(t) = K(t)^{\gamma(1-\eta)}\epsilon_s(t)^{1-\eta}\epsilon_d(t) - r_k(t)K(t)$$

where $r_k(t)$ denotes the rental cost of capital at time t. As long as $0 < \gamma(1-\eta) < 1$, the maximization problem has a unique solution $K^f(t)$; in logarithmic form, the first order condition yields

$$k^f(t) = \frac{1}{1-\gamma(1-\eta)}\Big((1-\eta)\ln\epsilon_s(t) + \ln\epsilon_d(t) + \ln(\gamma(1-\eta)) - \ln r_k(t)\Big)$$

From the production and demand functions, we have

$$\epsilon_s(t) = Y(t)K(t)^{-\gamma}, \qquad \epsilon_d(t) = P(t)Y(t)^{\eta},$$

where $K(t)$ denotes the *installed* capital stock; using these relationships to eliminate the unobservable $\epsilon_d(t)$ and $\epsilon_s(t)$, and defining

$$\nu = \frac{1}{1-\gamma(1-\eta)} > 1,$$

we obtain

$$k^f(t) = \nu(\ln Y(t) - \ln r_k(t)) + (1-\nu)k(t) + \nu\ln(\gamma(1-\eta)).$$

The individual unit's desired gross investment to capital ratio can then be computed by the simple expression in equation (26) in the main text.

Data definition

Our aggregate is the U.S. economy, and we use nonresidential private equipment for our capital and investment series.

We construct a $\Gamma^*(t)$ series setting $\nu = 1.1$ in (26) (we experimented with other values obtaining essentially the same results), and identifying:

$Y(t)$ with Gross National Product;

$K(t)$ with the stock of nonresidential private equipment (as reported by the Department of Commerce of the B.E.A.);

$r_k(t)$ with the projection (in the logs) of $(r+\delta)(1-T(t))P_k(t)/((1-\tau(t))P(t))$ on $\tau(t)$ and $P_k(t)/P(t)$, where $(r+\delta)$ is constant at 0.2,

$\frac{P_k(t)}{P(t)}$ is the ratio of National Account investment and GNP deflators,

$T(t)$ is the perfect foresight present value of tax saving from investment credits constructed by Auerbach and Hassett (1990), and

$\tau(t)$ is the corporate tax rate, also from Auerbach and Hassett.

REFERENCES

Abel, Andrew B. and Olivier J. Blanchard (1986), "The Present Value of Profits and Cyclical Movements in Investment," *Econometrica* **1986**, 249-273

Abel, Andrew B. (1983), "Optimal Investment Under Uncertainty," *American Economic Review* **1983**, 228-233

Arrow, Kenneth J.(1968), "Optimal Capital Policy with Irreversible Investment," in J. N. Wolfe (ed.), *Value, Capital and Growth. Papers in Honour of Sir John Hicks*, Edinburgh University Press 1968.

Auerbach, Alan J. and Kevin Hassett (1990), "Tax Policy and Business Fixed Investment in the United States," working paper.

Bernanke, Ben S.(1983), "Irreversibility, Uncertainty and Cyclical Investment," *Quarterly Journal of Economics* **1983**, 85-106

Bentolila, Samuel and Giuseppe Bertola (1990), "Firing Costs and Labor Demand: How Bad Is Eurosclerosis?," *Review of Economic Studies* **57**, 381-402

Bertola, Giuseppe (1988), "Adjustment Costs and Dynamic Factor Demands: Investment and Employment under Uncertainty," Ph.D. dissertation, M.I.T.

Bertola, Giuseppe and Ricardo J. Caballero (1990), "Kinked Adjustment Costs and Aggregate Dynamics," in O. J. Blanchard and S. Fischer (eds.) *NBER Macroeconomic Annual 1990*, Cambridge (MA): MIT Press

Billingsley, P. (1986), *Probability and Measure*, *2nd* ed. New York: John Wiley & Sons

Caballero, Ricardo J. and Eduardo M.R.A. Engel (1989), "Heterogeneity and Output Fluctuations in a Dynamic Menu Cost Economy," Columbia Working Paper #453

Caballero, Ricardo J. and Eduardo M.R.A. Engel (1990), "Dynamic (S, s) Economies: Aggregation, Heterogeneity and Coordination," Columbia Working Paper #476

Chow, Pao-Liu, José Luis Menaldi and Maurice Robin (1985), "Additive Control of Stochastic Flow Systems with Finite Horizon," *SIAM Journal of Control and Optimization* **1985**, 858-899

Churchill, Ruel V., and James W. Brown (1987), *Fourier Series and Boundary Value Problems*, 4^{th} ed. New York: McGraw-Hill

Hayashi, Fumio (1982), "Tobin's Marginal q and Average q: A Neoclassical Interpretation," *Econometrica* **1982**, 213-224

Ingersoll, Jonathan E. Jr, and Stephen A. Ross (1987), "Waiting to Invest: Investment and Uncertainty," mimeo, Yale University

Jorgenson, Dale W. (1963), "Capital Theory and Investment Behavior," *American Economic Review Papers and Proceedings* **1963**, 247-259

Karatzas, Ioannis and Steven E. Shreve (1984), "Connections between Optimal Stopping and Singular Stochastic Control I: Monotone Follower Problem," *SIAM Journal of Control and Optimization* **1984**, 856-877

Karatzas, Ioannis and Steven E. Shreve (1988), "Brownian Motion and Stochastic Calculus," New York: Springer-Verlag

Krylov, N.V. and B.L. Rozovskii (1977), "On the Cauchy Problem for Linear Stochastic Partial Differential Equations," *Mathematics USSR Izvestija* **11** # **6**, 1267-1284

Lucas, Robert E. Jr and Edward C. Prescott (1971), "Investment Under Uncertainty," *Econometrica* **1971**, 659-681

McDonald, Robert and Daniel Siegel (1986), "The Value of Waiting to Invest," *Quarterly Journal of Economics* **1986**, 707-727

Nickell, Stephen (1974), "On the Role of Expectations in the Pure Theory of Investment," *Review of Economic Studies* **1974**, 1-19

Olson, Lars J. (1989), "Stochastic Growth with Irreversible Investment," *Journal of Economic Theory* **47**, 101-129

Pindyck, Robert S. (1988), "Irreversible Investment, Capacity Choice, and the Value of the Firm," *American Economic Review* **1988**, 969-985

Sargent, Thomas J. (1979), "Tobin's Q and the Rate of Investment in General Equilibrium," in *On the State of Macroeconomics*, edited by K.Brunner and A.H.Metzler, North Holland 1979

Sargent, Thomas J. (1987), "Macroeconomic Theory," Second Edition, Academic Press

Stock, James H. and Mark W. Watson (1989), "A Simple MLE of Cointegrating Vectors in Higher Order Integrated Systems," NBER Technical Working Paper #83, December.

Figure 1

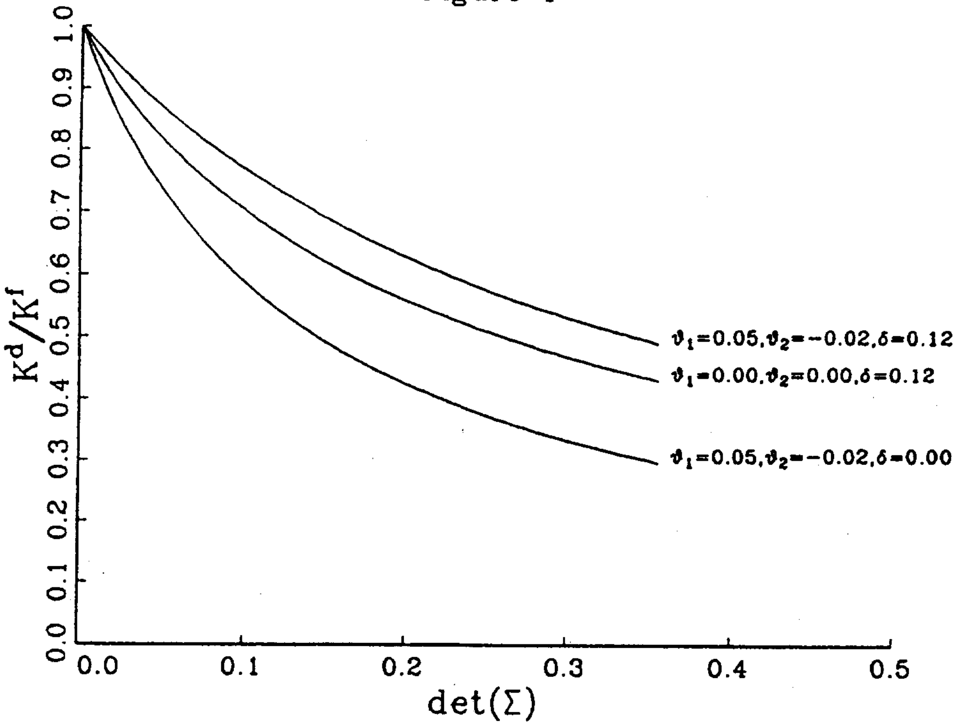


Figure 2

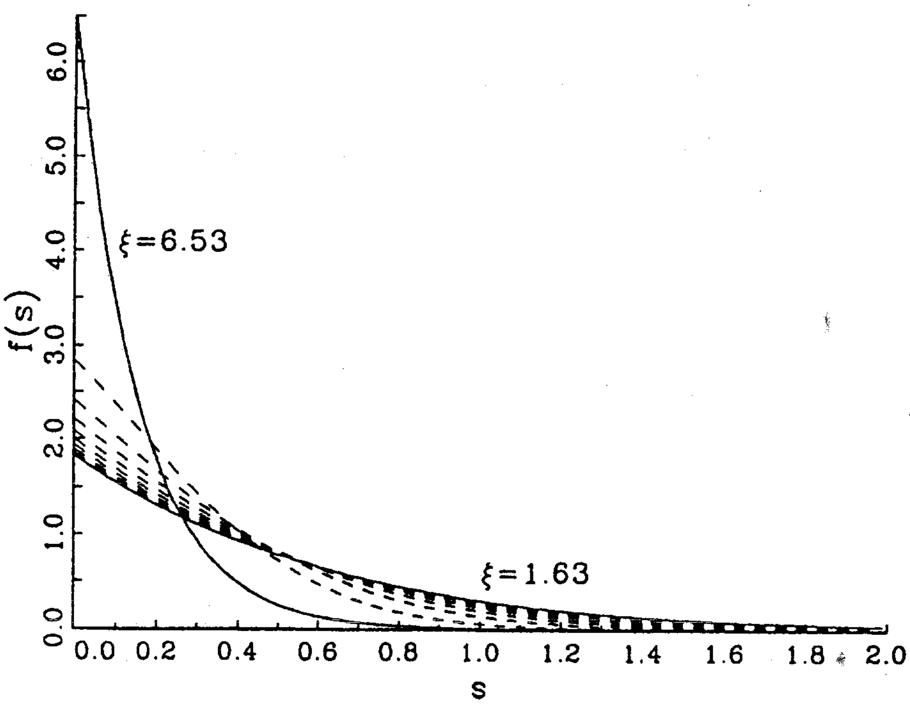
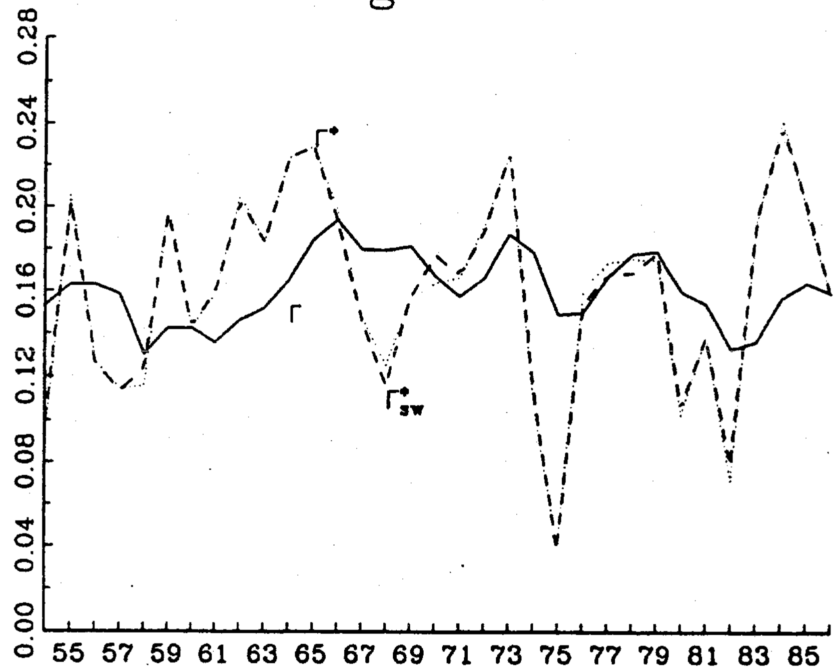


Figure 3



0.00
0.04
0.08
0.12
0.16
0.20
0.24
0.28
55 57 59 61 63 65 67 69 71 73 75 77 79 81 83 85
Γ*
Γ
Γ*sw

Figure 4

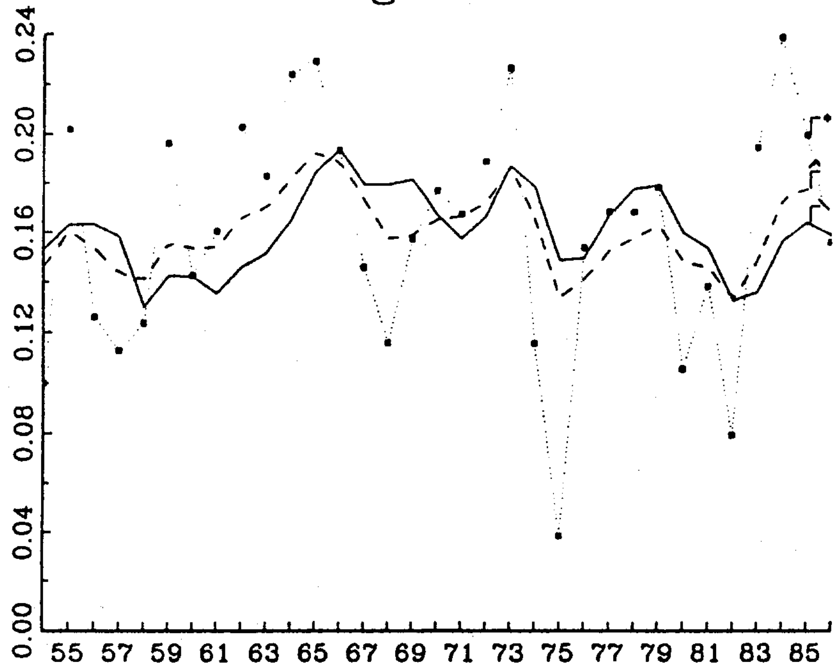


Figure 5