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DURABLE GOODS: AN EXPLANATION FOR THEIR SLOW ADJUSTMENT

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ABSTRACT

Aggregate expenditure on durable goods responds too slowly to wealth and other aggregate innovations to be consistent with the simplest frictionless version of PIH (permanent income hypothesis). In this paper I present a model of *aggregate* expenditure on durables that builds up from the lumpy nature of microeconomic purchases, and provide evidence supporting its contribution to the resolution of the "slowness" puzzle. The paper also contains several new results on the problem of dynamic aggregation of stochastically heterogeneous units. In particular, I provide a simple characterization of the effects of heterogeneity and microeconomic lumpiness on aggregate dynamics.

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Following Hall's (1978) "random walk" hypothesis for nondurables consumption, Mankiw (1982) shows that a similar pattern should be observv^d for the services yielded by durable goods. Assuming that these services are linear in the stocks implies that changes in the stock of durables should be white noise; and if depreciation is exponential, that changes in real durable goods expenditures, ΔE, should follow an MA(1) process with the MA coefficient equal to negative one minus the depreciation rate:

$$\Delta E_t = e_t - (1 - \delta)e_{t-1} \tag{1}$$

where e_t is a white noise innovation equal to the net change in the stock of durables and δ is the per period depreciation rate of the good. Contrary to the theory, this MA coefficient is not present (at least in the magnitude required) in quarterly post-war U.S. data (the estimated coefficient for the period 59:1–90:1 is -0.08 with a standard deviation of 0.09).[1]

Caballero (1990a) finds, however, that the negative serial correlation implied by consumer optimization does appear after a longer lag —and in the magnitude required— in the covariogram of changes in expenditure on durable goods. This suggests that the basic PIH is the right way to think about the long run response of durables to aggregate shocks, but that further work is required to understand its prolonged dynamic behavior.

Lumpiness with discontinuous adjustment at the microeconomic level has been a primary candidate as an explanation (Bar-Ilan and Blinder 1987, Lam 1989).[2] A well known methodological problem with this appealing explanation, however, arises from the Caplin and Spulber (1987) warning: lumpiness at the microeconomic level needs not produce lumpiness, or even frictions, at the aggregate level. As a consequence, the aggregation process has to be modeled explicitly. Caballero and Engel (1989, 1990a,b) take the first steps in providing a methodology for tracking down the dynamic consequences of heterogeneity (structural and stochastic) for aggregate outcomes in the context of these models.[3] Building on this work, Bertola and Caballero

[1]The expression for ΔE corresponds to changes in the level of expenditures. Throughout the paper, however, I use the logs instead of the levels since the time series properties of both series do not differ substantially and detrending and heteroskedasticity corrections are simpler when the log form is used. The data are per capita. Also notice that under the null time aggregation has little incidence since E is (almost) white noise.

[2]Lumpiness can be an optimal outcome, as it arises naturally from increasing returns in the adjustment technology. This is the sense in which the word "lumpiness" is used in this paper, although most of the discussion here also applies to physical lumpiness.

[3]Blinder (1981) discusses many of the issues involved in the aggregation of these models in the context of lumpy

(1990a) develop a simple framework for implementing a general class of these models empirically, and find suggestive preliminary evidence in favor of lumpiness (infrequent microeconomic actions) as an explanation for the smoothness of the aggregate stock of durables.[4]

In this paper I further develop the theory of stochastic aggregation and study in more detail the consistency of models of infrequent lumpy adjustment with the time series properties of postwar U.S. expenditure on durables. I find that, to a large extent, the results support the claim that the prolonged dynamics of durable goods can be accounted for by lumpiness at the microeconomic level. On the methodological side, I provide a simple characterization of the effects of heterogeneity and microeconomic lumpiness on aggregate dynamics, and show that aggregation in the presence of lumpiness not only makes aggregate variables sluggish, but also changes the variation properties of their sample paths.

This paper is organized in two parts; the first one, the paper itself, contains a description of the main insights and findings. The second one, the appendix, is extensive and includes the formulae and technical results behind the propositions and arguments in the main text of the paper. Section 1 develops the basic model and describes the mechanisms through which lumpiness at the microeconomic level and heterogeneity influence aggregate dynamics. Section 2 presents comparisons of the dynamic behavior of actual expenditures on different types of durables and over different time periods, and provides estimates of a structural model of U.S. purchases of new cars and furniture. Section 3 further describes the time series properties of durables, and Section 4 concludes. The technical part of the paper follows.

1 General Framework

When studying the dynamic behavior of an economic variable, it is convenient to define a "target" or "desired" variable k^* and a "departure" variable z. Typically, the target variable can be represented in terms of some simple theory that disregards (to a first order) dynamic elements. For example, in this paper k^* will be driven by a simple frictionless PIH model. The

inventory policies. And Caplin (1985) characterizes the joint steady state probability density of the inventories of n firms adjusting their stocks discretely. *First steps* refers to the *dynamic* description of a cross sectional distribution whose shape is determined *endogenously*.

[4]Contemporaneously, Caplin and Leahy (1990) provide a stylized model of discontinuous price adjustment in which the aggregate price level exhibits smoothness with respect to changes in the stock of money.

departure variable is just the difference between the actual stock of durables k (unless otherwise indicated, all variables are in logs) and its corresponding target, k^*. From this definition, one can represent the actual stock of durables at time t as follows:

$$k_t = k_t^* + z_t. \tag{2}$$

The dynamic behavior of the U.S. postwar stock of durables can be described in terms of equation (2): When there is a positive wealth shock, k_t^* rises and so does k_t; however the former more than the latter, hence z_t and k_t^* have negative contemporaneous correlation.[5] This yields "excess smoothness" of durable goods with respect to wealth and other aggregate innovations. Over time, z_t increases, generating positive serial correlation in the process for k_t. Since the changes in k_t are the residuals e_t in the equation for changes in expenditure on durables (no longer white noise), equation (1) now reads:

$$\Delta E_t = (1-(1-\delta)B)\Delta k_t^* + (1-(1-\delta)B)\Delta z_t, \tag{1'}$$

where B is the lag operator and the white noise implication of Mankiw's derivation applies only to Δk_t^*. When the dynamic behavior of z_t described above is considered, the MA(1) coefficient of ΔE_t becomes very close to zero.

Hence, an important part of the success of any explanation for the behavior of durable goods —under the maintained assumption that the simple PIH is a good description of what would happen in the absence of frictions— depends on its ability to generate negative contemporaneous correlation between k_t^* and z_t, as well as serial correlation in z_t. I argue below that models of infrequent and lumpy durable purchases at the unit level have these features. In addition, these models put us one step closer to provide a genuine structural interpretation of the adjustment speed and its changes across goods and time periods (Bar-Ilan and Blinder 1987).

Lumpy behavior arises naturally, for example, when purchasing a durable good entails a fixed transaction cost (see e.g. Bather 1966, Harrison et al. 1983, Dixit 1989, Bertola and Caballero 1990a, and Grossman and Laroque 1990). A model of this nature is derived in the appendix.

[5]Typically, however, the long run correlation between k_t^* and z_t is zero since the former is non-stationary whereas the latter is stationary by construction.

Here I describe its main characteristics, stressing the elements that will play an important role at different stages of the empirical implementation of the model.

Let the economy be inhabited by a continuum of individuals indexed by $i \in [0,1]$. Each individual's desired stock of durables is described by:[6]

$$dk_{it}^{*} = dA_t + \sigma_I dW_{it} \tag{3a}$$

and

$$dA_t = \theta dt + \sigma_A dW_{At}, \tag{3b}$$

where W_{it} and W_{At} are independent standard Brownian motions (also independent of the Brownian motions of all other units), and A_t denotes the only common stochastic component across units (the "aggregate").

If there were no transaction costs, $dk_{it} = dk_{it}^{*}$ at all times; however, as in Grossman and Laroque (1990), I assume that the upgrading or downgrading (by more than the natural depreciation of the stock) of each durable good requires selling the old good and buying the new desired amount. Such transaction has a "waste" (fixed cost) equal to a fraction λ of the old stock. Obviously, continuous adjustment of the actual stock to the target one cannot be optimal since it would entail infinite transaction costs. Under certain restrictions (satisfied by the model in this paper), the optimal microeconomic policy takes a simple three-barriers form: Most of the time, an individual i does not match changes in its target stock k_{it}^{*} but lets the actual stock k_{it} be eroded by depreciation: $dk_{it} = -\delta dt$ where δ is the depreciation rate. If, however, the departure z_{it} becomes too large (in absolute value), reaching an upper level U (the stock is too large) or lower level L (the stoch is too small), abrupt action takes place bringing the departure variable z_{it} to the level C, with $U > C > L$. The width of the band depends on parameters like the degree of uncertainty, depreciation rate, drift of the target stock, and convexity of the flow cost of departing from the target stock. Figure 1 presents an example of a sample path of z_{it} under the barrier policy described above (see e.g. Harrison et al.).

Recalling that the definition of the departure variable is $z_{it} \equiv k_{it} - k_{it}^{*}$, it is possible to see

[6]The driving variables are made explicit in the empirical section.

that when no action is taken:

$$dz_{it} = -\delta dt - dk^*_{it}.$$

It is then apparent that during the inaction periods k^*_{it} and z_{it} have negative contemporaneous correlation, satisfying the smoothness requirement discussed above. Stopping here, and disregarding the sporadic overreaction of z_{it} to changes in k^*_{it} can be highly misleading, however. This is particularly true when the realistic consideration that consumers are not all in the same position of their state space (i.e. do not have the same z_{it}) is taken into account; since the overreaction of some may counteract the negative correlation due to the vast majority.[7]

The requirement is then to generate negative correlation between the aggregates Z_t and K^*_t (defined to match the integral over all i of the corresponding lowercase microeconomic variables). The variable Z_t represents the average of the departures of individual consumers' actual stock of durables from their desired level. Caballero and Engel (1989, 1990a,b) show that in order to characterize the dynamic behavior of Z_t it is not necessary to describe the path of the joint *probability* distribution of the individual departures but only their *cross sectional* distribution (henceforth the expression *cross sectional distribution* refers to the empirical distribution of the z_i's).

For this, let $f(z_0)$ represent the initial probability density of each individual z_i;[8] if the realization of the aggregate process (a path) in the time interval $[t_0, t_0 + \Delta t]$ is denoted by $\{A\}_{t_0}^{t_0+\Delta t}$, the conditional probability density of each z_i at $t_0 + \Delta t$ is $f(z_t|\{A\}_{t_0}^{t_0+\Delta t})$. It is natural to let $f(z_0)$ coincide with the initial cross sectional density; this, plus the fact that after conditioning on the aggregate the only stochastic elements left are specific to the individual units, determines that $f(z_t|\{A\}_{t_0}^{t_0+\Delta t})$ also represents the cross sectional density, or the empirical density, of the z_i's at time $t_0 + \Delta t$ (by the Glivenko-Cantelli Theorem). Put differently, if one is careful about conditioning on the aggregate process A_t, much can be learned about the aggregate from the microeconomic problem, since probability statements about the latter translate into

[7]The most prominent and extreme example of this is given in Caplin and Spulber (1987), where units are arranged in the state space in a way such that the negative correlation of the many is exactly offset by those that overreact, eliminating the impact of microeconomic frictions on aggregate dynamics. Caballero (1990b) discusses other dynamic fallacies arising from the direct application of microeconomic arguments to aggregate phenomena in the context of these models.

[8]Each individual knows his own position at each point in time; the probability density refers to the one relevant for an outside observer that does not have information specific to the individual units.

statements about the fraction of units in each position of the state space in the former.[9]

Proposition 1: *Let individual units follow similar (L,C,U)-policies, exogenous uncertainty be characterized by equations (3a) and (3b), $\sigma^2 \equiv \sigma_A^2 + \sigma_I^2$, the cross sectional density $f(z_t|\{A\}_{t_0}^{t_0+\Delta t})$ be denoted by $f(z,t)$. Let also $\lim_{t\to 0} f(z,t) = g(z)$, with $g(z)$ continuous for all $z \in (L,U)$ and satisfying the boundary conditions below. Then for all $t > 0$ and $\{z \in (L,U)/C\}$,*

$$df(z,t) = \left[(\theta+\delta)\frac{\partial f(z,t)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 f(z,t)}{\partial z^2}\right] dt + \sigma_A \frac{\partial f(z,t)}{\partial z} dW_{At},$$

with boundary conditions:

$$f(U,t) = f(L,t) = 0,$$

$$\frac{\partial f(C^+,t)}{\partial z} - \frac{\partial f(C^-,t)}{\partial z} = \frac{\partial f(U^-,t)}{\partial z} - \frac{\partial f(L^+,t)}{\partial z},$$

and

$$f(C^+,t) = f(C^-,t).$$

Proof: See the appendix. †

The discussion of Proposition 1 is left for the appendix, however it is important to notice at this point that the stochastic nature of the PDE (partial differential equation) describing the path of the cross sectional density depends exclusively on the presence of aggregate uncertainty; if the last term is shut down, the density eventually converges to a stationary cross sectional density where, by definition, $dZ = 0$ and no interesting dynamics arise from the microeconomic adjustment costs.[10] In general, the more important are the drift and total variance relative to aggregate uncertainty, the stronger will be the effect of the deterministic part of the PDE, and therefore the less Z will move to offset fluctuations in K^*.

Intuitively, the ergodic density plays the role of an *attractor*; the cross sectional density tends to return to the ergodic one, and it is only prevented from reaching it by the continuous aggregate shocks.[11] The intensity of the attractor —and therefore the dynamic implication of

[9] An important difference with respect to the methodology developed in Caballero and Engel (1989, 1990a,b) is that there conditioning requires knowledge of the value of the aggregate shock at time t only, while here the entire path of the aggregate shock up to time t is required.

[10] Of course the frictionless case is not very interesting in this case either, since it is fully deterministic.

[11] Note that this refers to the ergodic *probability* distribution of an individual. As long as aggregate uncertainty

microeconomic lumpiness for aggregate dynamics (with opposite sign)— depends on the relative importance of the deterministic part of the PDE.

The path of the mean of the cross sectional density —summarizing the dynamic difference of models of the type studied here and their frictionless counterpart— can be conveniently expressed in terms of the stock upgrading and downgrading in each infinitesimal time interval. This is shown in the next proposition.

Proposition 2: *Under the assumptions of Proposition 1:*

$$dZ_t = -\delta dt - dK_t^* + \frac{\sigma^2}{2}\left\{f_z(L^+,t)(C-L) + f_z(U^-,t)(U-C)\right\} dt. \tag{4}$$

Proof: See the appendix. †

Unless the third term in (4) dominates, the (finite time interval) correlation between increments in Z and K^* is unambiguously negative, yielding the desired smoothing.[12] The most interesting features of the model described here, however, lie precisely in the third term, as this captures the effect of those units that are adjusting their stock of durables in the time interval dt.

Equation (4) shows that the dynamics of this complex economy can be characterized in terms of the slopes of the cross sectional density at each point in time evaluated at the boundaries. In order to understand this suppose, for a moment, that no consumer upgrades or downgrades (by more than the depreciation) his stock of durables in the time interval dt; then $dZ_t = -\delta dt - dK_t^*$ and no matter what has happened to K_t^*, the actual stock is only driven by depreciation, $dK_t = -\delta dt$. Next, consider the case in which some units upgrade their stocks; then, dK_t declines by less than the depreciation, and this is reflected one to one in dZ_t. By how much less depends on the size of the increase in the stock of durables of those that decide to upgrade, $(C-L)$, times the fraction of units that do it. The latter depends on the density in the "neighborhood" of L, captured by its (right) derivative at the boundary, $f_z(L,t)^+$, and by the total uncertainty faced by each unit, σ.[13] The latter matters since as σ rises, L can be reached by units farther away within the time interval dt, hence the "neighborhood" is increasing with

is present, the cross sectional density does not have a stationary state.

[12] Note that the third term in (4) is only locally predictable.

[13] Remember that the density is zero at L and U (trigger points).

respect to σ. The arguments run in the same way, although with the opposite sign, when one considers the units that decide to downgrade their stocks. Proposition 3 below transforms the expression describing the path of dZ_t into an expression for dK_t:

Proposition 3: *Under the assumptions of Proposition 1, and letting $f(z)$ represent the ergodic probability distribution of an individual:*

$$dK_t = \theta dt + \frac{\sigma^2}{2}\{(f_z(L^+,t) - f_z(L^+))(C - L)$$
$$+(f_z(U^-,t) - f_z(U^-))(U - C)\}\, dt,$$

or

$$dK_t = -\delta dt + \frac{\sigma^2}{2}\{f_z(L^+,t)(C - L) + f_z(U^-,t)(U - C)\}\, dt.$$

Proof: See the appendix. †

Interestingly, Proposition 3 shows that in spite of all sources of uncertainty in the economy being infinite variation Wiener processes, the path of the aggregate stock of durables has *finite variation.* Contrary to Brownian motion, the sample path of K_t is differentiable; the fluctuation in the cross sectional distribution not only attenuates the volatility of the aggregate stock variable, but it also changes the fundamental stochastic nature of this stock.

Recalling that the cross sectional density is equal to the conditional (on the aggregate path) density of an individual, one can we see that the non-trivial dynamic behavior of the stock of durables is described by the difference between the slopes at the boundaries of the conditional and the unconditional densities, weighted by the size of the upward and downward jumps. Figure 2 illustrates this mechanism. The solid curve represents the ergodic probability distribution of an individual and the solid lines its slopes at the boundaries. Suppose the economy starts from a depressed situation where dK_t is less than its average, θdt: this is reflected in the Figure by the fact that $f_z(L^+,t) < f_z(L^+)$ and $f_z(U^-,t) < f_z(U^-)$. If good times follow, initially relatively few units replenish their durables, thus the rate of growth of durables is still below its long run average; however the slopes start increasing as more units approach the neighborhood of L and fewer remain in the neighborhood of U. Eventually, if the upward pressure continues, the rate

Table 1: Example's Statistics

	$\lambda = 0.00$	$\lambda = 0.05$	$\lambda = 0.25$
$\sigma_{\Delta K}$	0.023	0.022	0.013
$\rho_{\Delta K}$	0.037	0.091	0.305

The corresponding values for $U - L$ are 0.00, 0.54 and 0.98 respectively.

of growth of the stock of durables exceeds its long run average.

In building up a framework for the empirical section, it is also worth highlighting the impact of the size of transaction costs on aggregate dynamics. A common result in the microeconomic literature on the problem addressed here is that as the adjustment cost rises, the size of the jumps $U - C$ and $C - L$, and therefore the width of the inaction range, increases; in his attempt to avoid transaction costs, the consumer reduces the expected frequency of hitting the trigger barriers. The equivalent statement at the aggregate level is that as transaction costs rise, the fraction of units near the barriers at each point in time decreases.[14] Since these are the units responsible for offsetting the negative correlation between K^* and Z, the increase in the transaction cost tends to increase the excess smoothness of the actual stock of durables for a given path of K^*. The counterpart of this is an increase in the (positive) serial correlation of the stock of durables, and a further reduction (in absolute value) of the MA(1) coefficient in the expenditure (its changes) series.

Figure 3 illustrates this: Using the model described in the appendix and in the next section, I generate a sample path for the aggregate shocks A_t and plot the path of ΔK_t under three different values for λ (0.00, 0.05, and 0.25). It is apparent that as the adjustment cost rises, the path of the actual stock of durables becomes more sluggish, reflecting the delayed dynamic response due to the endogenous microeconomic lumpiness. Table 1 reports several statistics arising from this example; they reveal the smoothing and serial correlation implications of an increase in the adjustment cost parameter, λ.

Summarizing, given the trigger points, increases in the drift (e.g. by an increase in the rate of depreciation) and relative importance of idiosyncratic uncertainty (given aggregate uncertainty)

[14] It is important to notice that typically the fraction of units near the barriers decrease by more than the bandwidth increase (except in the one sided uniform case).

tend to reduce the impact of microeconomic lumpiness on aggregate dynamics, by strengthening the *attractor* effect.[15] On the other hand, given the drift and variance parameters, an increase in the transaction costs faced by individual units introduce more sluggishness into K_t.

2 Empirical Evidence

Figure 3 above as well as the evidence in Bertola and Caballero (1990a) show that the type of models discussed here have the potential to account for the sluggishness of the aggregate stock of durables. In this section I try to go one step further and look at whether these models can explain more subtle characteristics of the evolution of aggregate durables purchases and stocks. In particular, I look at the differences in the dynamic behavior of different goods within consumer durables (cars and furniture), and across time (within cars).[16] I then try to provide an interpretation to these differences within the context of the model presented in the previous section.

2.1 Diagnostic

It seems natural to start the diagnostic of the data by estimating MA(1) models for each of the goods. Under this metric cars are clearly closer to the PIH than furniture. For the period 59:1-90:1, cars have an MA coefficient of -0.39 while furniture have a positive MA coefficient equal to 0.28, both of them significant.

A problem of the simple MA(1) comparison is that both cases fail to match the PIH (where the MA coefficient should be approximately between -0.95 and -1.00 under Mankiw's assumptions), thus it is possible than even though the initial response of cars is faster than that of

[15] It is worth noticing that in this case, total uncertainty rises thus the *neighborhood* effect is also strengthened. Both, the attractor and neighborhood effects tend to reduce the impact of microeconomic inaction on aggregate dynamics. However, there is a third effect that points in the opposite direction: When total uncertainty faced by consumers rises, the barriers are reached more often; in order to reduce the impact of this on transaction costs, units widen their bands. Under fairly general conditions, however, the expected frequency with wich the barriers are hit is larger than when total uncertainty is lower; thus the neighborhhod effect dominates the bandwidening effect. Finally, when total unit level uncertainty is low relative to the drift, there is an additional effect of an increase in idiosyncratic uncertainty that may be called "beating the drift". That is, given aggregate uncertainty, an increase in indiosyncratic uncertainty may introduce more dynamics rather than less by letting the movements in k_i^* be non-monotonic, moving the problem away from the Caplin and Spulber paradox.

[16] Cars correspond to new cars, and Furniture to furniture and furnishings as described in CITIBASE for the period going from the first quarter of 1959 to the first quarter of 1990.

Table 2: Sample Instability (Cars)

	59:1–72:4	73:1–79:4	80:1–90:1
MA(1)	-0.447	-0.147	-0.571
	(0.124)	(0.195)	(0.133)

Standard errors in parenthesis.

furniture, this may be overturned later along the adjustment process. To see whether this is the case I look at the plot of the sum of the autocorrelations of both series; having in mind that under the simple PIH model these should be flat and at about -0.5. Figure 4 shows that the sum of the autocorrelations for cars is consistently smaller, and closer to -0.5, than that of furniture.[17] Interestingly, however, both sums of autocorrelations eventually settle at about -0.5; providing further support to the idea that the PIH is the right way to think about the long run (Caballero 1990a).

As discussed above, the dynamic behavior of the aggregate depends in intricate ways on parameters like the amount and composition (idiosyncratic versus common) of uncertainty. Whatever the sign of these effects, one would not expect them to be constant during the 70's, especially for cars where the price of gasoline is an important determinant of its user cost. Table 2 presents MA(1) coefficients for three sample periods 59:1-72:4, 73:1-79:4 and 80:1-90:1, for cars. Although the coefficients are not very precisely estimated, it seems sensible to claim that the departure from the PIH is accentuated during the 70s.

In sum, both furniture seems to depart more from the PIH than cars, and within cars the departure seems to be more important during the 70's. In the next subsections I provide a structural interpretation to these findings.

2.2 Estimation Strategy

The empirical implementation of the model developed in Section 1 and the appendix requires —at least conceptually— of two distinctive steps: In the first one, the frictionless model and the realizations af the aggregate shocks are estimated; and in the second one, the dynamic

[17] Similarly, the spectra show that the action is concentrated at much lower frequencies in the case of furniture than in the case of cars.

component of the model and its parameters are identified. The interpretation of the results, on the other hand, requires to define the limits of what can be inferred from the type of aggregate data used; I postpone the discussion of this issue until the end of this subsection.

As in Bertola and Caballero (1990a,b), I first estimate the target stock of each of the goods by exploiting the cointegrating properties between the observed stock and the desired one. For this, I postulate that the desired stock of each good follows a simple PIH type relation, enriched by price effects:

$$K_t^* = \beta_0 + \beta_1 H_t + \beta_2 P_t + \beta_3 R_t,$$

where H_t is the log of wealth,[18] P_t is the log of the ratio of the price of the durable good in question and the price of nondurables, and R_t is a user cost index equal to the log of the price of gasoline over the price of nondurables in the case of cars, and equal to zero in the case of furniture.[19] The coefficients β_1, β_2 and β_3 are determined from the cointegrating relationship: $K_t = K_t^* + Z_t$.

An important caveat is that models with important dynamic elements due to adjustment costs are prone to a strong bias of its coefficients towards zero when estimated with static OLS procedures (Caballero 1990c); fortunately, the novel Stock and Watson (1989) dynamic OLS approach can be used to reduce the importance of this bias. In implementing their procedure I include in the right hand side 4 leads and 12 lags of the first difference of the variables driving K^*. Table 3 reports the results together with the statistics of the variables of main concern.

I have imposed the constraint $\beta_1 = 1$ so the (frictionless) share of each stock remains constant if relative prices do not change. The rest of the coefficients are precisely estimated. Relative price effects are important for both goods and the price of gasoline plays a significant role in explaining the the path of the target stock of cars. The statistics are also quite interesting. Consistent with the arguments presented up to now, they indicate that excess smoothness is more important in the case of furniture than in that of cars (a ratio of the standard deviation of the actual stock growth series versus the target series of 0.26 for furniture versus 0.55 for

[18] Constructed as the expected present value of per-capita disposable income, under the (non-rejected) assumption that the logarithm of this follows a random walk. The main results of the paper are unchanged by the use of nondurables consumption as a proxy for wealth, instead of H_t.

[19] Note that I am implicitly assuming that the expected interest rate is approximately constant. Also note that throughout the paper I am adopting the very simplifying assumption that the system is decoupled; i.e. that it can be solved separately across goods.

Table 3: Frictionless Model and Basic Statistics (63:1-89:1)

	CARS	FURNITURE
θ	0.028	0.027
$\sigma_{\Delta K}$	0.012	0.005
$\sigma_{\Delta K^*}$	0.022	0.020
$\frac{\sigma_{\Delta K}}{\sigma_{\Delta K^*}}$	0.546	0.259
$\rho_{\Delta K}$	0.695 (0.071)	0.894 (0.039)
$\rho_{\Delta K^*}$	0.175 (0.098)	0.083 (0.100)
β_1	1.000	1.000
β_2	-0.283 (0.018)	-0.346 (0.020)
β_3	-0.103 (0.041)	-

All equations include a constant.
Standard errors in parenthesis.
The standard deviations and θ's are annualized using the relations $\sigma\sqrt{dt}$ and θdt (the same applies to the rest of the tables).

cars). Furthermore, the positive serial correlation of the rate of growth of the stock of furniture is stronger than for cars (0.89 versus 0.70), and this is the serial correlation that counteracts the negative MA(1) coefficient in the series of changes in expenditures. It is also interesting to notice that in spite of the significant price effects, changes in the frictionless stocks do not display significant serial correlation.[20]

These estimates, together with an exogenous depreciation rate of 15% per year for both goods, yield estimates of the discrete time analog of the terms $(\theta + \delta)dt$ and $\sigma_A dW_{At}$ in Proposition 1's Kolgomorov equation (see the appendix).

The next step in estimating the path of Z_t is to find an initial cross sectional density —which I arbitrarily choose to be the ergodic density of an individual,[21] the variance of idiosyncratic shocks, and the parameters L, U and C. Three caveats concerning what can be inferred from

[20]This provides some support to using a Brownian motion as an approximation of the true driving process.

[21]The first ten observations are excluded from the sum of squared residuals in order to reduce the impact of this arbitrary selection..

aggregate data alone are in order, however: First, given the free constant in the equation for the desired durables' stock, the location of the cross sectional density is not identified; thus I set $C = 0$. Second, the asymmetry in the jumps at the microeconomic level is not likely to have strong implications for aggregate time series; at least when looked at with a model which is driven by smooth processes (Caballero 1990b). Given that the size of the asymmetry depends too heavily on the nature of the objective function postulated in the microeconomic problem, and that symmetry simplifies the numerical routine for tracking down the path of the cross sectional density, I impose symmetry. Moreover, once the exact parametric implications of the microeconomic models are left inactive, it is not clear whether the estimate of U represents the size of the jumps or one half of the inaction range.[22] Microeconomic data are required to resolve these issues; the results obtained here must therefore be interpreted only as a rough index of inaction, one of the main determinants of aggregate sluggishness. And third, for technical reasons, identifying the total uncertainty faced at the individual level is difficult in the range of parameters obtained. As a result, I use a two steps approach: First, I estimate both U and σ freely, to then repeat the estimation with σ fixed at its lowest value within the flat region of the likelihood.[23]

I should stress, however, that these are not limitations of the model presented in this paper but just an acknowledgement of what can and cannot be inferred from aggregate data alone if one does not want to impose the exact constraints of a specific microeconomic model.

2.3 Results (Cars)

Column 1 in Table 4 provides the full sample results for Cars. The main results are obtained under the assumption that total good-specific uncertainty at the individual level is 40% per year.[24] Once total uncertainty is fixed, the estimates of the inaction range are very precise; for

[22] Furthermore, in the Proposition $A1$ I extend the model in this paper to a four band (L, C_L, C_U, U)-policy. Again, however, there is little hope of distinguishing among the many possible rules from aggregate data alone.

[23] Where "flat" is defined as no change (up to two digits) in the R^2 of the model when σ is changed by 10% points.

[24] It is important to notice that this is the uncertainty as seen by the econometrician. Individuals could, and are likely to, be able to foresee many of the things that appear as idiosyncratic shocks for an outside observer. Moreover, structural differences, as generated by heterogeneous wealth and price elasticities of the frictionless stocks across individuals, are also likely to have implications very similar to those of idiosyncratic shocks (Caballero and Engel 1990b). Finally, taste shocks also form part of idiosyncratic shocks, and these may be quite large for individual consumption goods.

Table 4: Structural Estimates ($\sigma = 0.4$)

	CARS 63:1–90:1	CARS 70's	CARS Rest	FURNITURE 63:1–90:1
U	1.021	1.029	1.028	1.818
	[-0.192]	[-0.183]	[-0.182]	[-0.594]
	{0.202}	{0.156}	{0.205}	{0.719}
γ	0.056	0.063	0.051	0.050
	[0.018]	[0.021]	[0.017]	[0.017]
	{-0.011}	{-0.013}	{-0.010}	{-0.010}
σ_A	0.022	0.025	0.020	0.020
$R^2_{\Delta Z}$	0.81	0.90	0.75	0.92
	[-0.01]	[0.00]	[-0.01]	[-0.03]
	{0.00}	{0.00}	{0.00}	{0.02}

All equations include a constant. Results in square and curly brackets represent the difference of the cases $\sigma = 0.3$ and $\sigma = 0.5$, respect to the basis case, respectively. The 70's corresponds to the period 73:1-79:4 and the *Rest* to the periods 63:1-72:4 and 80:1-89:1. All the estimates use the full sample to generate the path of the cross sectional density. Five Fourier coefficients were used.

this reason, instead of reporting standard errors, I have chosen to report the impact of changing σ by 10%. The numbers in square and curly brackets are the differences with respect the basis case when σ equals 30% and 50%, respectively.

The estimate of U (or L) —if interpreted as the size of the control jumps— implies that the (service) value of the car when upgraded is about 2.8 times the value of the old car,[25] and that on average individuals change cars after less than six years from the previous purchase. The estimate of γ indicates that aggregate uncertainty accounts for about 6% of total uncertainty.

Overall, the fit of the model is good; it accounts for 90% of the departure in the rate of growth of actual and frictionless stocks. The reduction in the R^2 when σ is lowered and the negligible change on it (zero to a two digit) when σ is raised by 10%, reflect the two stage estimation strategy discussed above (looking for the minimum σ within the range of (almost) flat likelihood).

Interestingly, the next two columns seem to indicate that there was no significant economic impact of the 70's (for given total uncertainty) on the inaction index. It is therefore the surge in

[25] As said above, however, I prefer to interpret the estimate of U as just an inaction index.

the relative importance of aggregate uncertainty —thus the reduction in the *attractor* effect— that is principally to blame for the reduction in the MA(1) coefficient during the 70s.

Finally, Figure 5 decomposes the rate of change of the stock into the fraction of units that upgrade their stocks and those that downgrade it plus depreciation. It is apparent that given the strong drift, units seldom exercise control to reduce their stocks.

2.4 Results (Furniture)

The last column in Table 4 reports the results for furniture. The model explains 96% of the departure of the path of the actual growth series from the path implied by the PIH model with no dynamics. The estimate of U is substantially larger than that of Cars. Given that total uncertainty has been assumed to be the same across goods, this suggests that transaction costs are larger in the case of Furniture; perhaps reflecting the less developed secondary markets in the case of Furniture than in Cars. Furthermore, given the parameters assumed and estimated, aggregate shocks are relatively less important in Furniture than Cars;[26] thus, in order to explain the relative departures of the MA coefficient from that implied by the PIH, the effect of a wider band must dominate the stronger attractor effect.[27]

Figure 6 illustrates the decomposition of the rate of change of the stock into the fraction of units that upgrade their stocks and those that downgrade it plus depreciation. That the source of fluctuations comes from consumers' stock upgrades rather than downgrades, is even more apparent than in the case of Cars; and this is consistent with the idea of a less developed secondary market for furniture than for cars.

3 Arma Representation and Impulse Responses

The model presented in this paper is non-linear; there is no ARMA representation, at least with stable parameters, even when the underlying parameters remain unchanged. The impulse responses depend on past history on intricate ways; the previous section provides this "historical

[26]Furthermore, I have used the same total uncertainty in both goods using Cars as the benchmark to determine the flat region of the likelihood; the R^2s for furniture show, however, that at $\sigma = 0.4$ the likelihood is still steep, suggesting an even larger estimate of total uncertainty.

[27]Aggregation across goods brings about issues very similar to those of aggregation across individuals; thus the path of the composite furniture good for each individual is likely to have a smoother behavior than that of each individual good.

Table 5: MA(1) Models (66:1-89:1)

	CARS	FURNITURE
ΔE	-0.328 (0.100)	0.142 (0.104)
ΔE^*	-0.904 (0.051)	-0.968 (0.040)
$\Delta \tilde{E}$	-0.101 (0.105)	-0.344 (0.099)

All equations include a constant.
Standard errors in parenthesis.
The statistics are in annual rates.

filter". In this section, on the other hand, I disregard the non-linearities and look for an "average" impulse response in terms of conventional ARMA models. This facilitates the comparison with previous results.

As mentioned in the introduction, the simplest PIH model implies that if only wealth shocks exist, changes in the stock of durables should be white noise, or equivalently, changes in the expenditure series should follow an MA(1) process with the MA coefficient being determined by the depreciation rate, $(\delta - 1)$. The first row in Table 5 shows, as argued above, that observed changes in expenditure on durables are far from meeting this condition, especially so for furniture (the difference with the results in Section 2.1 are due to the different sample period; here 66:1-89:1). Row 2 presents the MA coefficients obtained from constructing artificial expenditure series from the estimated frictionless series; the proximity to the theoretical value is apparent, supporting the the first stage procedure. And row 3 presents the MA coefficients obtained from constructing expenditure series from the estimated stock of durables. The success of the model can be measured by the proximity of these values to the observed values (row 1), especially for cars.

Finally, since the rate of growth of the actual stocks is equal to the rate of growth of the frictionless stocks plus the change in the mean of the cross sectional density; and the latter is a function of the current and past rates of growth of the frictionless stocks, one may construct an approximate (average) impulse response function in which the innovations to the frictionless stock are the impulses. For this I run non-parsimonious regressions of ΔK on its own lag, and the current and six lags of ΔK^*. Figure 7 portraits the response of the actual and frictionless rate of growth of cars and furniture, to an impulse yielding a 1% long run increase in the stock (approximately the area under the curves). The shapes are fairly consistent with the description given in the paper.

4 Conclusion

In this paper I have shown how lumpiness at the microeconomic level can aid explaining different features of the time series behavior of durable goods. For the sake of clarity, factors introducing serial correlation in the frictionless stock of capital growth series, other than prices, have been excluded. Allowing for other realistic features like habit formation (e.g. Constantinides 1990), non-separabilities across goods and time (e.g. Eichenbaum and Hansen 1987) and precautionary savings (e.g. Caballero 1990d) should enrich the characterization of the target stock, K^*, and reduce the need for large inaction range estimates (especially in the case of furniture) to account for the large departure of durables from the simplest PIH.

The model presented here exhibits history dependence and nonlinearities, and yields a framework to interpret the ARMA representations of durables and their sample instability. I have found sharp differences in the behavior of furniture and cars, that I have tentatively attributed to the degree of development of secondary markets and the relative importance of aggregate shocks. Further research on the nature and extent of transaction costs is much needed. Microeconomic evidence and case studies should provide a natural complement to this work. Eberly (1990) has taken important steps along these lines, finding strong microeconomic empirical support for the type of models discussed in this paper.

Within cars, sample instability is apparent and it is most likely generated by the oil shocks. The increase in the relative importance of aggregate uncertainty during the 70's seems to have increased the sluggishness of durable purchases by reducing the strength of the *attractor effect*.

On the methodological side, the paper has provided an intuitive characterization of the dynamic behavior of a large number of units subject to transaction costs, in terms of the behavior of the slopes of the cross sectional density at the trigger points. It has also shown that transaction costs together with stochastic heterogeneity can introduce radical changes in the stochastic nature of aggregate variables.

APPENDIX

A The individual consumer problem

Grossman and Laroque (1990) develop a model of portfolio choice and durable purchases in the presence of transaction costs. Since the only objective of the microeconomic model in this paper is to motivate a general type of inaction policies used, here I develop a less realistic but simpler version of their model.

Let each individual i have a stock (level) of durables D_{it} and a strictly positive desired (or target) stock D^*_{it}. In terms of dollars, the flow utility cost of departing from the target stock level is summarized by the expression $aD_{it}[\ln D_{it}/D^*_{it}]^2$, $a > 0$; i.e. the cost is quadratic in the percentage departure and scaled by the level of the durable held by the individual. The transaction cost incurred when changing the durable at any given time τ is proportional to the old durable sold: $\lambda D_{i\tau^-}$. At any normalized time 0, the consumer's problem is to minimize the present value cost:

$$V(D_{i0}, D^*_{i0}) = \inf_{\tau, D_{i\tau}} \mathrm{E}_0 \left[a \int_0^\tau e^{-\rho t} D_{it} [\ln (D_{it}/D^*_{it})]^2 \, dt + \lambda D_{i\tau^-} e^{-\rho\tau} \right.$$

$$\left. + e^{-\rho\tau} V(D_{i\tau}, D^*_{i\tau}) \right]$$

where ρ is the discount rate and τ is the first stopping time.[28]

Letting $z_{it} \equiv \ln(D_{it}/D^*_{it})$ $(\equiv k_{it} - k^*_{it})$ it is possible to write $V(D_{it}, D^*_{it})$ as $D_{it}J(z_{it})$. Assuming that the durable depreciates exponentially at the rate δ, and defining $M = \inf_z e^z J(z)$, the problem can be written in terms of a single state variable:

$$J(z_{i0}) = \inf_{\tau} \mathrm{E}_0 \left[a \int_0^\tau e^{-\eta t} z^2_{it} \, dt + \lambda e^{-\eta\tau} + e^{-\eta\tau} e^{-z_{i\tau^-}} M \right]$$

where $\eta \equiv \rho + \delta$.

If the target stock level, D^*_{it}, is assumed to follow a geometric Brownian Motion with $d\ln D^*_{it} = \theta dt + \sigma dW_{it}$, so:

$$dz_{it} = \mu dt - \sigma dW_{it}$$

[28] Note that $D_{it} > 0$ a.s. since $\lim_{D_{it} \to 0} D_{it} [\ln D_{it}/D^*_{it}]^2 = \infty$.

with $\mu \equiv -(\delta+\theta)$, it is easy to verify after repeated application of Ito's lemma that $J(z)$ satisfies the linear non-homogenous ODE (ordinary differential equation):

$$\frac{\sigma^2}{2}J'' + \mu J' - \eta J + az^2 = 0.$$

The stationarity of this problem yields a stationary solution in the space of the state variable z. The optimal policy consists in lower and upper trigger points, denoted by L and U respectively, and a common target point denoted by C (see e.g. Grossman and Laroque 1990, Bertola and Caballero 1990a, Harrison et al. 1983, etc.), with $L < C < U$.

The general solution of the ODE is:

$$J(z) = A_1 e^{\alpha_1 z} + A_2 e^{\alpha_2 z} + v_0(z),$$

where

$$\alpha_1 = \frac{-\mu + \sqrt{\mu^2 + 2\eta\sigma^2}}{\sigma^2}$$

$$\alpha_2 = \frac{-\mu - \sqrt{\mu^2 + 2\eta\sigma^2}}{\sigma^2}$$

$$v_0(z) = \frac{a}{\eta}\left[z^2 + \frac{2\mu z}{\eta} + \frac{\eta\sigma^2 + 2\mu^2}{\eta^2}\right]$$

and A_1 and A_2 are constants of integration to be determined simultaneoulsy with L, U and C from the boundary conditions of the problem. The latter are given by the Value Matching and Smooth Fit conditions that, after some rearranging, yield the following non-homogenous full rank system:

$$J'(L) + J(L) = \lambda,$$

$$J'(U) + J(U) = \lambda,$$

$$J'(C) + J(C) = 0,$$

$$J'(L) + e^{(C-L)}J(C) = 0$$

and

$$J'(U) + e^{(C-U)}J(C) = 0.$$

This system of non-linear (algebraic) equations can be solved numerically.

B Aggregation

In studying the evolution of the cross sectional density of the z_{it}'s, given identical (L, C, U)-policies for all agents, it is convenient to recall the decomposition of the driving processes into components that affect all units and those that are consumer specific:

$$dk^*_{it} = \theta dt + \sigma dW'_{it} = \theta dt + \sigma_A dW_{At} + \sigma_I dW_{it}.$$

Integrating over i (and multiplying by the appropriate weights) yields the rate of growth of the aggregate target stock:

$$dK^*_t = \theta dt + \sigma_A dW_{At}.$$

The final objective is to determine the path of the actual stock of durables: Integrating over i the identity $dk_{it} = dk^*_{it} + dz_{it}$, yields $dK_t = dK^*_t + dZ_t$. Given dK^*_t, a characterization of the change in the average of the cross sectional density of departures is all that is left to track down the actual stock of durables. Noticing that

$$\int_0^1 z_{it}\, di = \int_L^U z f(z,t)\, dz = Z_t,$$

where $f(z,t)$ denotes the cross sectional density at time t, suggests that the problem is solved by tracking the path of this cross sectional density. A derivation from first principles determines that the Kolmogorov equation characterizing such path. This is the proof of Proposition 1.

Proof of Proposition 1

The discrete time-space analog of the driving processes used in this paper are:

$$\Delta A_{t+dt} = \begin{cases} \sigma_A\sqrt{\Delta t} & \text{w.p. } \frac{1}{2}\left(1 + \frac{\theta\sqrt{\Delta t}}{\sigma_A}\right) \\ -\sigma_A\sqrt{\Delta t} & \text{w.p. } \frac{1}{2}\left(1 - \frac{\theta\sqrt{\Delta t}}{\sigma_A}\right). \end{cases}$$

If $\Delta A_{t+dt} = \sigma_A\sqrt{\Delta t}$,

$$\Delta k^*_{i,t+dt} = \begin{cases} \sigma\sqrt{\Delta t} & \text{w.p. } \frac{1}{2}(1+\gamma) \\ -\sigma\sqrt{\Delta t} & \text{w.p. } \frac{1}{2}(1-\gamma) \end{cases}$$

and if $\Delta A_{t+dt} = -\sigma_A\sqrt{\Delta t}$,

$$\Delta k^*_{i,t+dt} = \begin{cases} \sigma\sqrt{\Delta t} & \text{w.p. } \frac{1}{2}(1-\gamma) \\ -\sigma\sqrt{\Delta t} & \text{w.p. } \frac{1}{2}(1+\gamma) \end{cases}$$

where $\gamma \equiv \sigma_A/\sigma$.

Except for the boundary and center points, the probabilities are communicated across time and space by the Kolmogorov equation:

$$f(z,t) = 1[\Delta A_t > 0]\left\{f(z-\Delta z, t-\Delta t)\frac{1}{2}(1-\gamma) + f(z+\Delta z, t-\Delta t)\frac{1}{2}(1+\gamma)\right\}$$
$$+1[\Delta A_t < 0]\left\{f(z-\Delta z, t-\Delta t)\frac{1}{2}(1+\gamma) + f(z+\Delta z, t-\Delta t)\frac{1}{2}(1-\gamma)\right\},$$

where $1[x]$ is an indicator function that takes value 1 when x is satisfied.

Rearranging this balance equation, using the relation $(\Delta z)^2 = \sigma^2\Delta t$, and taking the limit of Δt as this converges to the infinitesimal quantity dt, yields the Kolmogorov equation in the text:

$$df(z,t) = \left[-\mu\frac{\partial f(z,t)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 f(z,t)}{\partial z^2}\right]dt + \sigma_A\frac{\partial f(z,t)}{\partial z}dW_{At}.$$

In the discrete state space case, the first set of boundary conditions are obtained directly from the fact that L and U are trigger points so no unit ever spends any time at them. In the continuous state space case, this is not sufficient since the density is only determined up to a measurable set. The proof, however, is easily obtained from evaluating the discrete Kolmogorov equation at the boundaries, noticing that the density is identically zero beyond these boundaries, and taking the limit as Δz goes to zero. The boundary condition involving the derivatives, on the other hand, is derived from the Kolgomorov equation for the case $z = C$, and recognizing the additional terms due to the jumps from the trigger points. And the last boundary condition is just a result of the continuous nature of Brownian motion sample paths. †

The proof of Propositions 2 and 3 follow.

Proof of Proposition 2:

Replacing the stochastic PDE directly in the definition of dZ, yields:

$$dZ = \int_{L}^{C-} zdf(z,t)\,dz + \int_{C+}^{U} zdf(z,t)\,dz$$

to obtain, after applying Fubini's theorem:

$$dZ = -\mu\left\{\int_{L}^{C-} zf_z(z,t)\,dz + \int_{C+}^{U} zf_z(z,t)\,dz\right\}dt$$

$$+\frac{\sigma^2}{2}\left\{\int_{L}^{C-} zf_{zz}(z,t)\,dz + \int_{C+}^{U} zf_{zz}(z,t)\,dz\right\}dt+$$

$$\sigma_A\left\{\int_{L}^{C-} zf_z(z,t)\,dz + \int_{C+}^{U} zf_z(z,t)\,dz\right\}dW_{At}.$$

Solving the integrals and replacing the boundary conditions in the solution, proofs the Proposition. †

Proof of Proposition 3:

Direct substitution of the relation $dK_t = dK_t^* + dZ_t$ in Proposition 2, yields:

$$dK_t = -\delta dt + \frac{\sigma^2}{2}\left\{f_z(L^+,t)(C-L) + f_z(U^-,t)(U-C)\right\}dt.$$

Noticing that if the ergodic density replaces the cross section density on the right hand side, $dK_t = \theta dt$, proves the proposition. †

An interesting extension is obtained for the case in which the return (or target) point from L and U are different: C_L and C_U respectively.

Proposition A1: *Let individual units satisfy the assumptions in Proposition 1 but follow (L, C_L, C_U, U)-policies. Then Propositions 1, 2 and 3 hold, with the following modifications:*

The boundary conditions of the PDE are:

$$f(U,t) = f(L,t) = 0,$$

$$\frac{\partial f(C_L^+,t)}{\partial z} - \frac{\partial f(C_L^-,t)}{\partial z} + \frac{\partial f(L^+,t)}{\partial z} = 0$$

$$\frac{\partial f(C_U^+,t)}{\partial z} - \frac{\partial f(C_U^-,t)}{\partial z} - \frac{\partial f(U^-,t)}{\partial z} = 0,$$

$$f(C_L^+,t) = f(C_L^-,t),$$

and

$$f(C_U^+,t) = f(C_U^-,t).$$

The path of the mean of the cross sectional distribution is:

$$dZ_t = -\delta dt - dK_t^* + \frac{\sigma^2}{2}\left\{f_z(L^+,t)(C_L - L) + f_z(U^-,t)(U - C_U)\right\} dt.$$

And the path of K_t is given by:

$$dK_t = \theta dt + \frac{\sigma^2}{2}\{(f_z(L^+,t) - f_z(L^+))(C_L - L)$$

$$+(f_z(U^-,t) - f_z(U^-))(U - C_U)\} dt.$$

Proof: It follows trivially from applying the same steps used in the proof of Propositions 1, 2 and 3. †

Implementation

In practice, data are only observed at discrete time intervals, but the discrete change in the aggregate stock of durables can be obtained by integrating dK_s over the time interval $(t - \Delta t, t]$.

$$\Delta K_t = \theta \Delta t + \frac{\sigma^2}{2}\int_{t-\Delta t}^{t}\{(f_z(L^+,s) - f_z(L^+))(C - L)$$

$$+(f_z(U^-,s) - f_z(U^-))(U - C)\} ds.$$

A more serious problem is that the shape of the cross sectional density at time t depends on the realization of the entire path of the aggregate shocks; by having information only at discrete intervals one cannot know the exact position of the distribution. One possibility is to solve a filtering problem. Although appropriate, this is rather intractable. Instead, I take a short cut that simplifies the problem substantially: I assume that the realization of the aggregate is homogeneously distributed within the observation periods. This not only simplifies matters by avoiding the filtering problem, but also makes the stochastic PDE deterministic within each observation period.[29]

In this context, the drift for period $h = (t-\Delta t, t]$ is defined as ϕ_h and is computed by adding to the depreciation the observed change in the frictionless stock divided by the time-intervals length.

$$\phi_h = \delta + \frac{K_t^* - K_{t-\Delta t}^*}{\Delta t},$$

and the PDE for period h simplifies to:

$$\frac{\partial f(z,t)}{\partial t} = \phi_h \frac{\partial f(z,t)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 f(z,t)}{\partial z^2}.$$

Even after reducing the complexity of the PDE by letting it be deterministic in the small time interval $h = (t-\Delta t, t]$, the problem is not simple since it involves non-homogenous (due to the connection of the PDE's across time) and non-local (due to the finite jumps implied by the optimal microeconomic policies) boundary conditions. The fact that the first and second derivatives of the empirical density are only piecewise continuous (due to the kink at the return point) also makes the algebra more tedious.

The deterministic PDE can be solved through the method of Separation of Variables. For this, let me postulate that the particular solutions take the form $f(z,t) = H(z)M(t)$. Plugging this back into the PDE it is possible to decompose the problem into two ODE linked by a real-valued parameter ψ:

$$M'(t) - \psi M(t) = 0$$

and the *Sturm-Liouville problem*:

$$H''(z) + \xi_h H'(z) - \frac{\xi_h}{\phi_h}\psi H(z) = 0,$$

with $\xi_h \equiv 2\phi_h/\sigma^2$ and boundary conditions:

$$H(L) = H(U) = 0,$$

[29]A similar short cut is taken in Bertola and Caballero (1990b) for the case of irreversible investment. Some of the problems of this approximation in terms of the quadratic variation of the aggregate process are discussed there.

$$H'(C)^{+} - H'(C)^{-} = H'(U)^{-} - H'(L)^{+},$$

$$H(C)^{+} = H(C)^{-}$$

and the intial condition

$$f(z,0) = g(z),$$

where $g(z)$ is the empirical density at the end of period $h-1$.

The roots, β_1 and β_2 of the *Sturm-Liouville problem* are given by:

$$\beta_1 = -\frac{\xi_h}{2} + \frac{1}{2}\sqrt{\xi_h^2 + \frac{4}{\phi_h}\psi}$$

and

$$\beta_2 = -\frac{\xi_h}{2} - \frac{1}{2}\sqrt{\xi_h^2 + \frac{4}{\phi_h}\psi}.$$

When $\psi \geq \frac{-\phi_h \xi_h}{4}$ the two roots are real and the solutions are of the form:

$$H(z) = \begin{cases} A_1 e^{\beta_1 z} + A_2 e^{\beta_2 z} & \text{for } L \leq z \leq C \\ A_3 e^{\beta_1 z} + A_4 e^{\beta_2 z} & \text{for } C \leq z \leq U \end{cases}$$

where the constants A_1, A_2, A_3 and A_4 are to be determined from the boundary conditions. It is easy to verify that in the case of real roots the only non-trivial solution for these constants occurs when $\psi = 0$. Moreover, this solution gives the unconditional or long run distribution under the drift ϕ_h. Simple algebra shows that $H(z;\psi = 0)$ takes the form:

$$H(z;\psi = 0) = \begin{cases} A_0\left(e^{-\xi_h z} - e^{-\xi_h L}\right) & \text{for } L \leq z \leq C \\ A_0\frac{(e^{-\xi_h L} - e^{-\xi_h C})}{(e^{-\xi_h U} - e^{-\xi_h C})}\left(e^{-\xi_h z} - e^{-\xi_h U}\right) & \text{for } C \leq z \leq U \end{cases}$$

where A_0 is to be determined later.

On the other hand, when $\psi < \phi_h \xi_h / 4$, the roots are imaginary:

$$\beta_1 = -\frac{\xi_h}{2} + i\gamma(\psi)$$

and

$$\beta_2 = -\frac{\xi_h}{2} - i\gamma(\psi),$$

where i stands for $\sqrt{-1}$ and

$$\gamma \equiv \gamma(\psi) = \frac{1}{2}\sqrt{-\left(\xi_h^2 + \frac{4}{\phi_h}\psi\right)}.$$

In this case the solutions of the *Sturm-Liouville* problem have the form:

$$H(z) = \begin{cases} e^{-\frac{\xi_h}{2}z}\left(A_1\cos(\gamma z) + A_2\sin(\gamma z)\right) & \text{for } L \le z \le C \\ e^{-\frac{\xi_h}{2}z}\left(A_3\cos(\gamma z) + A_4\sin(\gamma z)\right) & \text{for } C \le z \le U \end{cases}$$

and the boundary conditions reduce to:

$$A_1\cos(\gamma L) + A_2\sin(\gamma L) = 0$$

$$A_3\cos(\gamma U) + A_4\sin(\gamma U) = 0$$

$$(A_1 - A_3)\cos(\gamma C) + (A_2 - A_4)\sin(\gamma C) = 0$$

and

$$(A_1 - A_3)\sin(\gamma C) - (A_2 - A_4)\cos(\gamma C) =$$

$$(A_1\sin(\gamma L) - A_2\cos(\gamma L))\, e^{\frac{\xi_h}{2}(C-L)} - (A_3\sin(\gamma U) - A_4\cos(\gamma U))\, e^{\frac{\xi_h}{2}(C-U)}.$$

Lengthy algebra shows that this system has non-trivial solutions only when the condition:

$$e^{\frac{\xi_h}{2}(C-L)}\sin(\gamma(U-C)) + e^{-\frac{\xi_h}{2}(U-C)}\sin(\gamma(C-L)) - \sin(\gamma(U-L)) = 0$$

is met. The parameters ψ_n for which this condition is satisfied are called the eigenvalues of the Sturm-Liouville problem, and there are a countable infinity of them: if $\psi^{-1}(\gamma_1)$ denotes the solution associated to the smallest positive γ that solves the condition above, then it is straight forward to verify that all the eigenvalues $\psi^{-1}(n\gamma_1)$, for positive integers n, satisfy the same condition.

Associated with each eigenvalue there is a solution of the Sturm-Liouville problem:

$$H(z;\psi_n) = \begin{cases} A_n e^{-\frac{\xi_h}{2}z}(\cos(\gamma_n z) - \cot(\gamma_n L)\sin(\gamma_n z)) & \text{for } L \leq z \leq C \\ A_n B e^{-\frac{\xi_h}{2}z}(\cos(\gamma_n z) - \cot(\gamma_n U)\sin(\gamma_n z)) & \text{for } C \leq z \leq U \end{cases}$$

where $A_n \equiv A(\psi_n)$ for $n = 1, ...\infty$, $\psi_n = -\frac{\sigma^2}{2}(\gamma_1^2 n^2 + \frac{\phi_h^2}{\sigma^4}) < 0$, and

$$B = \frac{(\cot(\gamma_n C) - \cot(\gamma_n L))}{(\cot(\gamma_n C) - \cot(\gamma_n U))}.$$

Consistently, the ODE for the time component of the PDE has a solution:

$$M(t) = e^{\psi_n t},$$

and the general solution of the PDE has the form:

$$f(z,t) = \sum_{n=0}^{\infty} H(z;\psi_n)e^{\psi_n t},$$

with $\psi_0 = 0$ and

$$H(z;\psi = 0) = \begin{cases} A_0\left(e^{-\xi_h z} - e^{-\xi_h L}\right) & \text{for } L \leq z \leq C \\ A_0\frac{(e^{-\xi_h L} - e^{-\xi_h C})}{(e^{-\xi_h U} - e^{-\xi_h C})}\left(e^{-\xi_h z} - e^{-\xi_h U}\right) & \text{for } C \leq z \leq U \end{cases}$$

where A_0 is determined by the adding up constraint of the probability distribution:

$$A_0 = \frac{e^{-\xi_h U} - e^{-\xi_h C}}{(C-L)e^{-\xi_h(C+L)} + (U-C)e^{-\xi_h(C+U)} - (U-L)e^{-\xi_h(L+U)}}$$

and the rest of the constants are determined from the non-homogeneous boundary condition:

$$\sum_{n=0}^{\infty} H(z;\psi_n) = g(z),$$

or, equivalently,

$$\sum_{n=1}^{\infty} H(z;\psi_n) = p(z),$$

where $h(z) = g(z) - H(z;\psi = 0)$.

Multiplying each side by $e^{\xi_h z} H(z; \psi_k)/A_k$, and noticing that the eigenfunctions of the Sturm-Liouville problem are orthogonal under $e^{\xi_h z}$, yields:

$$A_n = \frac{\int_L^U p(z)(H(z;\psi_n)/A_n)e^{\xi_h z}\, dz}{\int_L^U (H(z;\psi_n)/A_n)^2 e^{\xi_h z}\, dz}.$$

The denominator has a closed form that I omit since it is not informative. The path of the cross sectional density together with its derivatives at the boundaries can now be computed numerically.

In the empirical implementation of the model I normalize the problem so $C = 0$ and make the simplifying assumption: $L = -U$. In this case $H(z;\psi_n)$ becomes:

$$H(z;\psi_n) = \begin{cases} A_n e^{-\frac{\xi_h}{2} z} \sin(\gamma_n z) & \text{for } L \leq z \leq C \\ A_n (-1)^{n+1} e^{\frac{\xi_h}{2} U} e^{-\frac{\xi_h}{2} z} \sin(\gamma_n z) & \text{for } C \leq z \leq U \end{cases}$$

where $A_n \equiv A(\psi_n)$ for $n = 1, \ldots \infty$, and $\psi_n = -\frac{\sigma^2}{2}\left(\frac{\pi^2 n^2}{U^2} + \frac{\phi_h^2}{\sigma^4}\right) < 0$.

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FIGURE 1

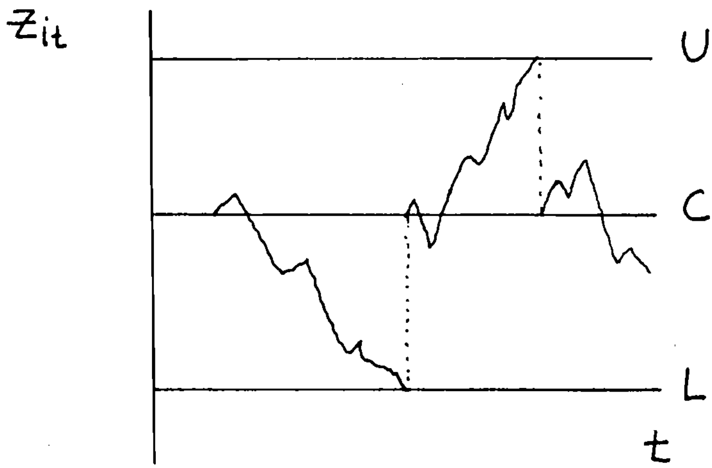

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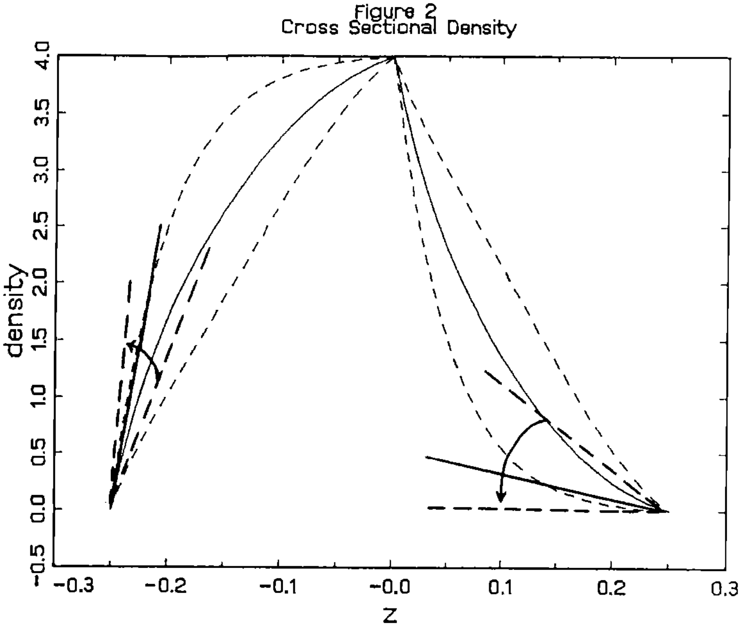
Figure 2



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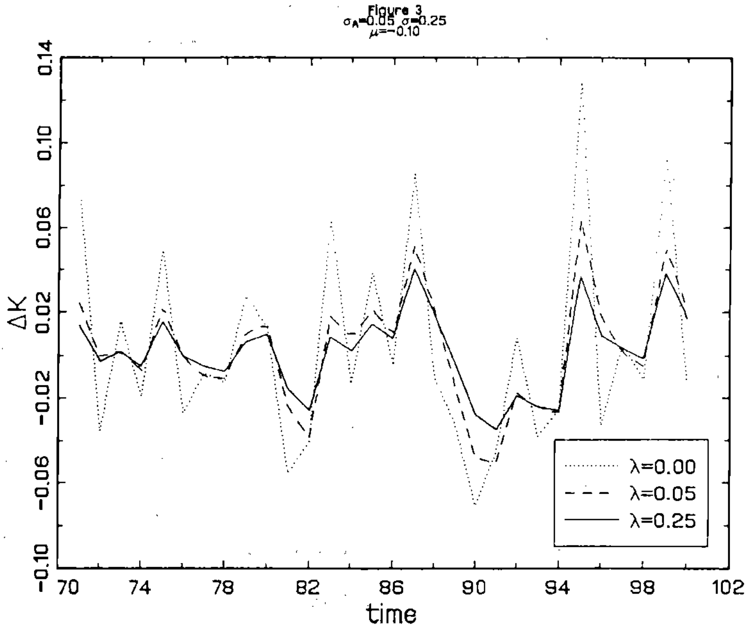

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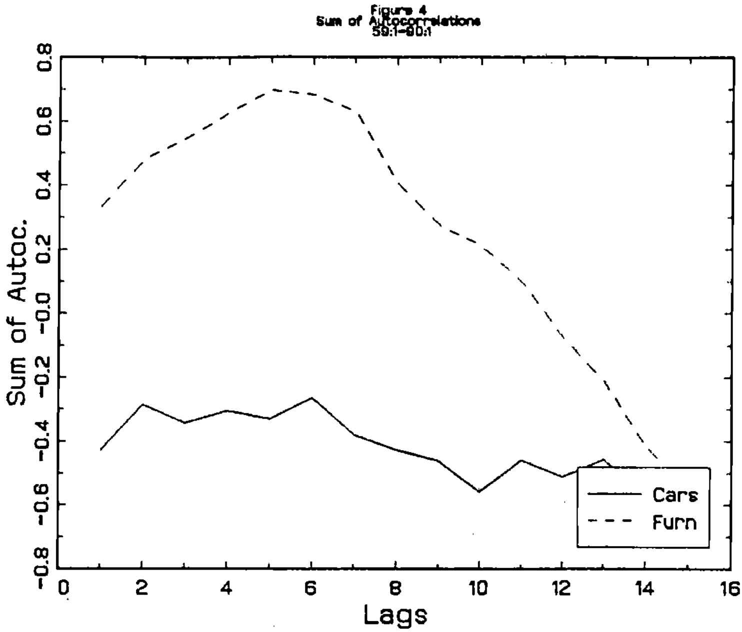

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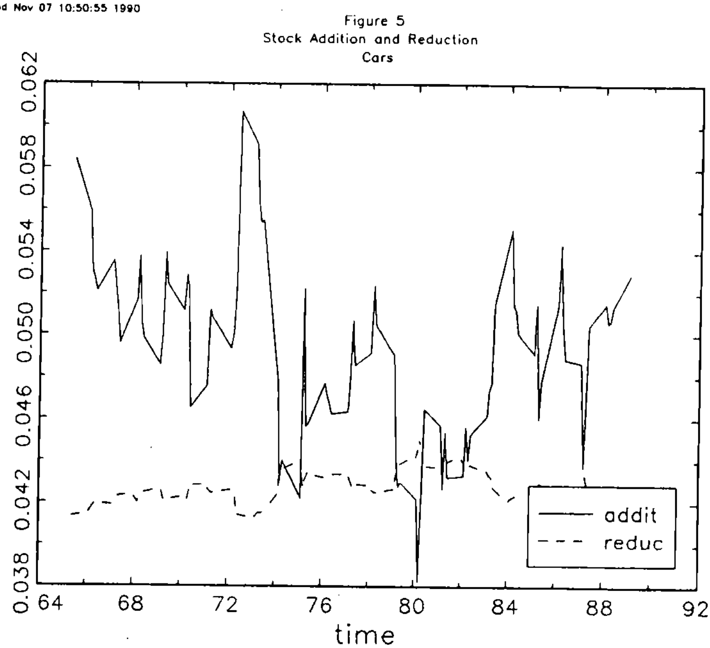

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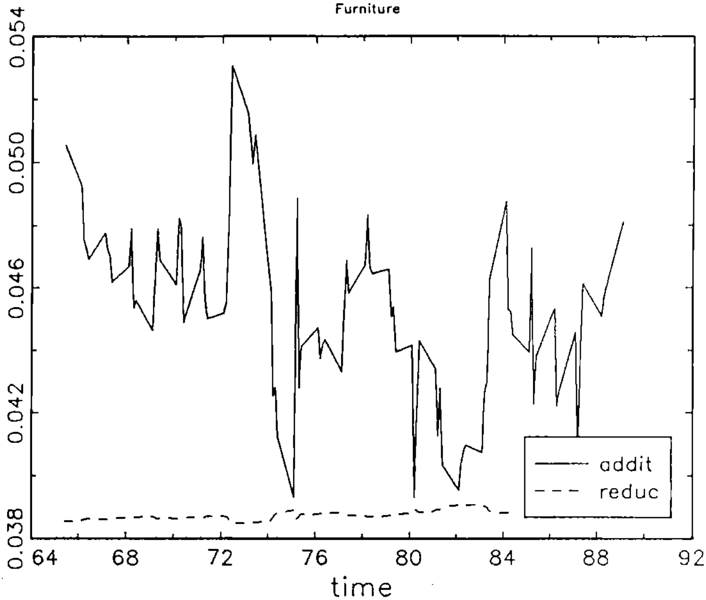
Figure 6
Stock Addition and Reduction
Furniture



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