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ASSET DEMAND AND REAL INTEREST RATES

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ABSTRACT

Understanding factors that drive asset demand is central to explaining movements in long-term real interest rates. In this paper, we begin by documenting that much of the increase in the demand for assets in the US in the 30 years prior to Covid represented greater desire to hold assets by households of given age and income levels. For example, if we focus on the 55-64 age group, its wealth-to-income ratio increased by 45-55%, depending on whether housing is included or not. We then develop a model of asset demands which combines retirement motives and inter-temporal substitution motives to quantitatively explore different factors that may have contributed to such an increase. Our findings suggest that decreasing interest rates likely led to a substantial increase in demand for retirement wealth. We also explore some of the across group heterogeneity and show how social security may explain why the lowest income groups did not follow the general trend. Finally, we discuss macroeconomic implications of long-run asset demands that are a decreasing function of interest rates.

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1 Introduction

In most advanced economies, prior to the pandemic, real interest rates had been trending down since the mid to late 1980s (see Figure 1).[1] The most common explanation for this trend is that economies experienced an increased demand for assets that pushed down interest rates and increased prices of stocks and real estate.[2] Important forces cited for inducing such an increase in asset demand include population aging and higher income inequality.

While these factors are certainly relevant, we begin this paper by showing that a key element driving the increased demand for assets over the last thirty years comes from households' desire to hold more assets for given age and income levels. Notably, we document that the increase in the wealth-to-income ratio observed over this period is largely a within-group phenomenon as opposed to resulting mainly from changes in demographics or income distribution. Furthermore, we show that saving behavior supports an interpretation of the observed higher wealth holdings as reflecting a desire to hold more wealth rather than temporarily above target wealth levels due to valuation effects.

When looking to explain why households with similar income and demographic characteristics may have increased their desired asset-to-income ratios over this period, multiple possibilities come to mind. Retirement needs in a low interest rate environment is one of them.[3] This link is nicely expressed by Raghuram Rajan,[4] former governor of the Reserve Bank of India:[5]

[1]The influential empirical study by Laubach and Williams (2003) provides estimates showing that the natural rate of interest r^* has been declining.

[2]A vast literature examines the sources of the decreasing trend in real interest rates. Some of the hypotheses that have been proposed include: demographics (Summers (2014), Eggertsson and Mehrotra (2014), Eichengreen (2015), Goodhart and Pradham (2020) and Auclert et al. (2021)); a productivity slowdown (Gordon (2017)); a global saving glut and/or lack of safe assets (Bernanke (2005), Caballero et al. (2008), Gourinchas et al. (2020), and Acharya and Dogra (2022)); a decline in desired investment (Rachel and Smith (2017)); a rise in inequality (Mian et al. (2020), Auclert and Rognlie (2020), Fagereng et al. (2019), and Rachel and Smith (2017)). Borio et al. (2017) provide an excellent survey of the literature.

[3]While we focus on retirement motives to help explain higher household asset holdings, bequest motives likely play a similar role. See Beaudry and Meh (2021).

[4]See Rajan (2013).

[5]The link between low interest rates and the need for more retirement savings is also often mentioned in the financial industry, as illustrated by the following examples. In the issue from September 2016 – dedicated to living in a low-rate environment – the Economist's briefing on pensions (https://www.economist.com/briefing/2016/09/24/fade-to-grey) noted that for investors who had to buy their own pensions low levels of interest rates and bond yields meant a higher cost of doing so. According to Moneyfacts, a British data firm, in the late 1990s, £100,000 would have bought a 65-year-old British man a lifelong income of £11,170 a year; while more recently this number would have fallen to £4,960. In other words, paying out a given level of income now costs more than twice as much as it did previ-

"...savers put more money aside as interest rates fall in order to meet the savings they think they will need when they retire."

With this in mind, the paper develops a model of asset accumulation in a continuous time overlapping generations (OLG) environment that allows for inter-temporal substitution and retirement motives to compete. The model builds on Blanchard (1985) and Yaari (1965), and is closest to Gertler (1999).[6] The framework is sufficiently tractable to allow the relationship between desired asset-to-income ratios and interest rates to be derived analytically. In particular, we show that if the inter-temporal elasticity of substitution is less than 1 (which is empirically the more plausible case), then long-run asset demands become C-shaped, with lower interest rates motivating households to increase their asset-to-income ratios.

We then use this framework to explore quantitatively how different factors, such as increased longevity, decreased expected aggregate growth and lower interest rates, may have contributed to the observed increases in wealth-to-income ratios. While all these forces may have played a role, the decreases in interest rates appear – from our model's perspective – to have the biggest contribution. In particular, the documented increase in within-group wealth can be readily explained by observed decreases in real interest rates if one believes that the elasticity of inter-temporal substitution is slightly below .5 (i.e., a risk aversion parameter slightly above 2). Moreover, when quantifying the model, we find that retirement incentives start to dominate inter-temporal substitution motives when interest rate falls somewhat below 3%, implying a C-shaped asset demand function with an inflexion point of around 4%. Accordingly, this suggests that the economy may have been operating on a segment of the asset demand function – where asset demand increases as interest rate falls – for most of the 30-year period from 1989 to 2019.

Much of our quantitative analysis focuses on explaining the observed change in wealth holding of the 55-64 age group. While most of the income groups within this age cohort increased their wealth holdings, the lower-income groups – those below 70 000$ per year – appear to have decreased their wealth. We argue that this pattern may reflect the different

ously. Similarly, in January 2022, MoneyRates website (https://www.moneyrates.com/investment/how-lower-interest-rates-ruin-retirement.htm) suggested to its readers that meeting the same retirement goal with reduced rates of return meant that they had to set more of their paychecks aside. Finally, in 2020 the Office of Retirement and Disability Policy of the Social Security Administration in its Perspectives series on "Retirement Implications of a Low Wage Growth, Low Real Interest Rate Economy" (https://www.ssa.gov/policy/docs/ssb/v80n3/v80n3p31.html) provided the following scenario: "suppose the goal is to accumulate $100,000 after 10 years. With a 3 percent interest rate, $8,469 in annual saving is required. If interest rates fall to 1 percent, annual saving must increase to $9,463... That represents an almost 12 percent bump up in annual saving".

[6] Galí (2021) introduces retirement in a similar fashion in a New Keynesian model with bubbles.

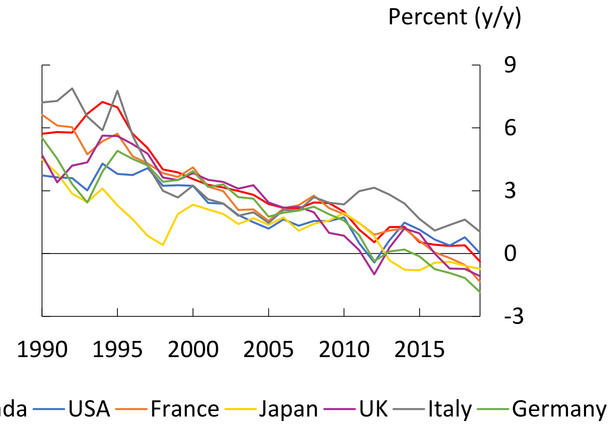
effects of social security on different income groups. For the lower-income groups, social security covers much of their retirement income needs and therefore a fall in interest rates creates mainly an inter-temporal substitution motive favoring consumption over savings. Hence, we argue that the fall in real interest rates can explain both why – on a wealth-weighted basis – households have on average accumulated more wealth, while at the same time lower-income households have decreased their wealth holdings.

One of the insights of our analysis relates to the differences in consumption and saving responses to increases in asset values due to valuation effects, depending on the force behind the re-valuation. If the valuation effect is due to decreased interest rates, this may not change households' consumption or saving behaviours as they do not necessarily view themselves as richer, since their assets will not provide a greater stream of income when in retirement. In contrast, if the valuation effects are due to increased dividend payments, this should stimulate consumption and reduce the need to save for retirement.

In the last section, we present some of the general equilibrium implications of being on the lower portion of a C-shaped asset demand curve. In particular, we discuss how in such an environment (1) a reduction in the supply of safe assets can favor an asset price boom sufficient to increase overall wealth; and (2) a small exogenous increase in asset demand can be strongly amplified. Both of these forces may help explain why real interest rates fell so significantly in the decades prior to Covid.

The remainder of the paper is organized as follows. Section 2 exploits household level data to examine how asset positions changed over the 30 years prior to the pandemic. We show that for households with same age and income levels, asset holdings increased substantially. This remains true when removing housing wealth. Towards the end of this section we focus more specifically on the asset holdings of the 55-64 age group. Section 3 presents an OLG model – similar in spirit to that of Gertler (1999) – that integrates both inter-temporal substitution forces and retirement preoccupations. The model is sufficiently tractable to offer an analytic expression for desired wealth holdings. For pre-retirement individuals these holdings become a C-shaped function of interest rates when the elasticity of inter-temporal substitution is below 1. Section 4 uses the model developed in Section 3 to quantitatively explore the strengths of different forces in influencing asset demand. We find that on their own increases in longevity and decreases in aggregate growth cannot explain the size of the observed within-group changes in wealth over the 1989-2019 period. In contrast, the observed decrease in real interest rates – when the inter-temporal elasticity of substitution is slightly below .5 – is of the right size to help explain the increase in the wealth-to-income ratio of the 55-64 age group over the same period. Section 5 discusses some of the potential macroeconomic implications of our findings, and Section 6 concludes.

Figure 1
Long-term interest rates for G7 countries from 1990 to 2019



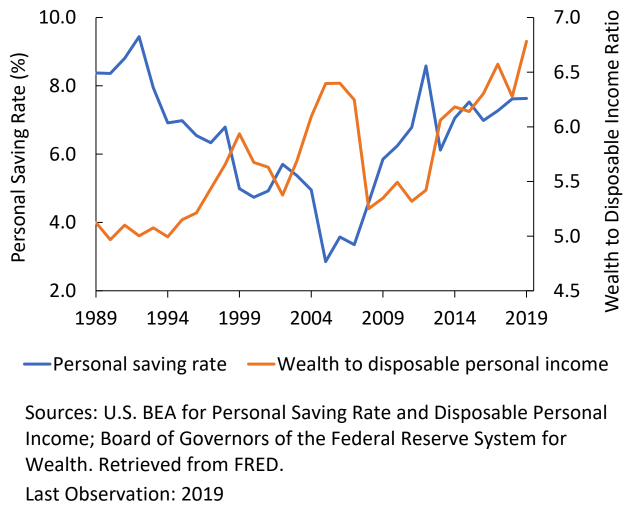
10-year government bond yields minus 3-year moving average inflation (total CPI)
Sources: Jordà-Schularick-Taylor Macrohistory Database until 2016 and OECD Main Economic Indicators from 2017-2019
Last observation: 2019

2 A Between versus Within Household Group Decomposition of Aggregate Asset Holdings over 30 Years: 1989-2019

Figure 2 illustrates that the aggregate wealth-to-income ratio in the US increased significantly and the aggregate saving rate decreased only very mildly over the 1989-2019 period. The question we want to address is how best to interpret and quantify such observation; should it mainly be interpreted as reflecting between-group (compositional) effect or does it instead largely reflect within-group choices.

Between-group (compositional effect) explanation. A potential explanation for the rise in aggregate wealth-to-income ratio is that it reflects increase in demand for assets induced by changes in demographics and income distribution. As the population aged, and more income was concentrated in higher-income groups, the demand for wealth from these groups increased. This put downward pressure on interest rates, which through valuation effects among others, raised the effective supply of wealth. The higher savings of the older and richer population was compensated by a decreased incentive to save by the population at large due to lower interest rates, leaving the overall savings rate relatively flat. Such narrative is essentially a "between"-group narrative which relies on compositional changes in types of individuals to explain the increased demand for wealth.

Figure 2
Household saving rate and aggregate wealth-to-income ratio in the US from 1989 to 2019



Within-group explanation. At the other end of the spectrum, it is possible that within-group asset demand played a significant role in the overall increase in the demand for wealth. This explanation could consider the effects of many forces on asset demand, including increased longevity, decrease in expected growth or decrease in interest rates.

The above discussion underlines the relevance of understanding the relative roles of within versus between-group effects in explaining the increased wealth holdings in the US over the last three decades. To do so, we use the Survey of Consumer Finances (SCF) and focus on the difference in asset holdings across household groups between 1989 and 2019. We choose this period for our analysis as it corresponds quite closely to the period of decreasing real interest rates presented in Figure 1. Furthermore, by looking at the thirty-year difference, we hope to minimize higher frequency movements in wealth accumulation dynamics associated with business cycle forces and crises.

The SCF is the most comprehensive source of data on household-level wealth and its components in the United States. It also has a consistent sampling methodology, over-sampling the rich, in all the survey waves between 1989 and 2019, which is useful for our analysis. The survey has between 3 and 5.5 thousand households, depending on the year, and our results use weights throughout. Given the importance of retirement considerations in the theory sections, we supplement the public-use version of the SCF data with the SCF-based estimates of defined benefit (DB) pensions from Sabelhaus and Volz (2020), which have been widely adopted in the related literature.[7] Thus, our measurement of wealth, including DB pension wealth and excluding social security wealth, also lines up well with that reported in Financial Accounts of the Federal Reserve. In this section, we primarily focus on the findings using the SCF (plus DB pensions) data, but for consistency with the literature, we also provide results using micro data scaled to the aggregates from Financial Accounts (FA) and National Income and Product Account (NIPA).

The aggregate wealth-to-income ratios in 1989 and 2019 we use for our decompositions are calculated from the SCF as the ratio of the sum of the wealth of each household to the sum of incomes of each household, respectively denoted $\left(\frac{w}{y}\right)_{89}$ and $\left(\frac{w}{y}\right)_{19}$. In our baseline measure of wealth, we include all components either directly reported in or constructed from the SCF. To explore robustness, we also provide calculations where we exclude wealth in a primary residence from the baseline measure of wealth. In turn, our measure of income represents the total of components available in SCF, and does not vary with the definition of wealth used in either the baseline or the robustness scenarios.[8]

The aggregate wealth-to-income ratio using the baseline measure of wealth in the SCF increased from 5.61 in 1989 to 8.43 in 2019, which is an increase of about 2.82, or close to 50%. When we exclude net housing wealth from this measure, the increase in the ratio is of similar magnitude at 2.65. Both increases are substantial relative to the 1989 levels.[9] To examine the within versus between-group components of increased wealth holdings, we apply a simple shift-share methodology in the main text, and report robustness results using a regression based decomposition in Appendix B.2.

For the shift share analysis, we place households in I bins, with N_i households in a bin $i = 1, ..., I$. The change in the aggregate wealth-to-income ratio can be decomposed as follows:

[7]SCF only directly measures pensions in defined contribution plans. Defined benefit pension entitlements calculated by Sabelhaus and Volz (2020) are measured by their termination value. This value represents the legal obligation of employer plans, and corresponds to the measure of defined benefit pension entitlements (both funded and unfunded) in Financial Accounts. We thank the authors for sharing their estimates with us.

[8]Following Fagereng et al. (2019) and Eika et al. (2020) we have also examined the case where we include in the SCF definition of income a measure of imputed housing rents of homeowners, constructed by distributing NIPA reported rents according to the value of housing of SCF respondents. When applying this definition of income with the baseline measure of wealth, we find that the contribution of the within-group component to the overall change in the wealth-to-income ratio is largely unchanged. For this reason, we keep the original measure of income from SCF in our baseline results.

[9]In the scaled wealth and income data, the ratio corresponding to the baseline measure of wealth increases by 171pp from 4.27 in 1989 to 5.98 in 2019. The literature has also used other definitions of wealth ratios, for example, normalized by GDP. While the exact changes in these ratios may depend on what goes into their numerator/denominator, they all have increased substantially over time.

$$\left(\frac{w}{y}\right)_{19} - \left(\frac{w}{y}\right)_{89} = \underbrace{\sum_i \left(\frac{\bar{w}_i}{\bar{y}_i}\right)_{89} \left[\left(\frac{y_i}{y}\right)_{19} - \left(\frac{y_i}{y}\right)_{89}\right]}_{\text{between-group or compositional effect}} + \underbrace{\sum_i \left(\frac{y_i}{y}\right)_{19} \left[\left(\frac{\bar{w}_i}{\bar{y}_i}\right)_{19} - \left(\frac{\bar{w}_i}{\bar{y}_i}\right)_{89}\right]}_{\text{within-group}}, \quad (1)$$

where the first summation term represents the between-group component, using 1989 as the base year for income and wealth profiles, and the second one represents the within-group component.[10] In this expression, y_i is the total income in bin i, $\bar{y}_i$ is the average income in bin i, $\bar{w}_i$ is the average wealth in bin i and finally y is the total income across all bins. All nominal variables are converted into real values indexed in 2019 dollars.

As can be seen from Equation (1), the changes in the total wealth-to-income ratio can be divided into the between-group component determined by the shift in the share of income going to each of the individual groups (y_i/y) and the within-group component determined by changes in the (average) wealth-to-income ratio of each group $(\bar{w}_i/\bar{y}_i)$. If the wealth-to-income ratios of individual groups were stable across time (i.e., $(\bar{w}_i/\bar{y}_i)_{19} = (\bar{w}_i/\bar{y}_i)_{89}$ for all groups i), the change in the aggregate wealth-to-income ratio would need to be fully explained by the between-group component (i.e., by the change in income shares alone). However, at the other extreme, if the income and age distributions remained stable across time (i.e., if $(y_i/y)_{19} = (y_i/y)_{89}$ for all groups i), then the within-group components would need to account for all of the change in the aggregate wealth-to-income ratio.

We start by dividing the population of households into age groups, defined by the age of the household head, to look at the effects of demographic changes in isolation.[11] Then, we divide the population of households into income groups to examine only the effects of changes in the current income distribution.[12] Finally, in our preferred specification, we combine the two and place households into age-income specific bins. The results of the shift share analysis for these different groupings are presented in Tables 1 and 2.

In Table 1, we report results for the more narrow focus on either only age or income

[10] The discussion of the shift-share and the regression based decompositions in Appendix B.2 also helps highlight under what conditions the first term represents the between-group component and the second term represents a within-group component.

[11] Work by Auclert et al. (2021) is the closest to this paper in terms of quantifying the contribution of population aging, i.e. between-group component with 5-year age groups in place of i, to the change in the wealth-to-income ratio in the US between 1950 and 2016.

[12] In their shift-share analysis of the changes in saving to national income ratio, Mian et al. (2021) test the relative importance of aging versus income inequality drivers over the 1950-2019 period, including the period from 1995 to 2019, when the natural rate of interest fell to an extremely low level. They focus separately on age groups and within-birth-cohort income distribution groups defined by 10th, 50th, and 90th income percentiles.

Table 1
Shift Share Decomposition of the Change in the Aggregate Wealth-to-Income Ratio Between 1989 and 2019: Separate Income and Age Groups, SCF Data

Definition	Groups	Total Change	Within (%)	Between (%)
			Age	
Wealth (baseline)	5	2.819	66.5	33.5
Wealth (baseline)	12	2.819	65.1	34.9
			Income	
Wealth (baseline)	6	2.819	93.8	6.2
Wealth (baseline)	12	2.819	93.6	6.4

Note: The 5 age groups are: 18-34, 34-35, 35-44, 45-54, 55-64, 65+ and the 12 age groups are: <25, 25-29, 30-34, 35-39, 40-44, 45-49, 50-54, 55-59, 60-64, 65-69, 70-74, and 75+. The 6 income groups (in thousands of real 2019 dollars) are: 0-20, 20-40, 40-60, 60-80, 80-120, 120+ and the 12 income groups: 0-20, 20-40, 40-60, 60-80, 80-120, 120-160, 200-250, 250-500, 500-750, 750-1250, 1250+, where the top end of income intervals is excluded. Wealth includes defined benefit pensions.

groups, using our baseline measure of wealth.[13] With respect to the results focusing on demographics the table presents two breakdowns: one based on 5 age groups and one based on 12 age groups. For these two breakdowns, we get very similar results: the within-group component explains about 65 percent of the change in the wealth-to-income ratio.[14] Then, we look at two groupings based only on income: one using 6 groups and another one using 12 groups. In both of these cases, the between-group component explains only about 7 percent of the change, leaving 93 percent of the change to the within-group component.

In Table 2, we present results for our preferred approach, where we allow for 30 groups as

[13] Appendix B.1 also presents the results of the decomposition of the changes in wealth-to-income ratio between 1989 and 2019 into within and between-group components with additional income groups at the higher end of the income distribution. The results using additional groups are similar to the benchmark 6 and 12 income groups.

[14] In Auclert et al. (2021), the compositional age effect using 2016 for base profiles of labor earnings and wealth is responsible for 105 out of 118 percentage points increase in the wealth-to-GDP ratio over the 1950-2016 period. Over the period 1989-2019 that compositional effect falls by about half, while the actual change in wealth-to-GDP ratio in the same period is similar to 1950-2016. In what follows, we also discuss the results of using 2019 as the base year for the between-group component calculation, which also helps to explain some of the difference in our results relative to theirs.

the product of 5 age groups and 6 income groups. These results use two different measures of wealth: our baseline measure inclusive of all wealth and the baseline measure less net housing wealth (primary residence). For comparison between the survey and aggregate data, we also report the results of the shift-share analysis when rescaling SCF estimates of wealth and income to match the FA and NIPA aggregates ("scaled" estimates). The latter approach is used in the literature, such as Feiveson and Sabelhaus (2019), Mian et al. (2020), and Bauluz and Meyer (2019). It builds each group's wealth using its shares of different assets and liability classes in SCF and values of their counterpart FA classes.The same is done on the income side where SCF reports income from different sources, which are matched to their corresponding aggregates in NIPA.[15] As shown in Panels A and B, the two sets of results are quite similar. The within-group component — that is, the component associated with changes in the wealth-to-income ratio of different groups — accounts for between 57 and 65 percent of the change with the between-group component explaining around 40 percent.[16]

The results in Table 2 are obtained using 1989 as the base year for each group i's initial profiles. In Appendix B.1 we also check the robustness of these results when changing the base year to 2019. When doing so, as suggested by Mian et al. (2021), the importance of the between-group component increases. However, even with the change in the base year the within-group component accounts for more than 50% of the change in the aggregate wealth-to-income ratio between 1989 and 2019.[17]

It must be immediately noted that these decomposition results — by themselves — do not necessarily imply that within-group *desired* wealth holdings have gone up. Instead,

[15] In the benchmark, for the scaled results we use the definition of gross NIPA income less imputations for owner-occupied housing rents, given that this component of income is not included in the SCF. However, we have also used other measures of income, including with imputed owner-occupied rents, and the results using these other measures are similar.

[16] Feiveson and Sabelhaus (2019) also look at within-birth-cohort permanent income groups which are only available for the 1995-2019 period. When using normal income for the formation of income groups and income measure itself, we find that over the period between 1995 and 2019, the within-group component is responsible for 55% of the change in our benchmark measure of the wealth-to-income ratio.

[17] It is worth noting that the sensitivity of results to base year choice tends to decrease as we increase the number of groups. However, as we go above 30 groups, some of the groups start to have rather few observations, which explains why we do not exceed that number in our baseline results. Nonetheless, we did explore the consequences of considerably increasing the number of groups. For example, when we allow for 75 groups (15 income groups and 5 age groups), our within-group estimate declines to around 48% when using 1989 as the base year, and does not change much if we change the base year to 2019. Given this, we are comfortable interpreting our results as suggesting that both the within and between-group components are close to being equally important in explaining the increase in wealth-to-income ratios between 1989 and 2019.

Table 2
Shift Share Decomposition of the Change in the Aggregate Wealth-to-Income Ratio Between 1989 and 2019: Combined Income and Age Groups, SCF and Scaled Data

Definition	Groups	Total Change	Within (%)	Between (%)
		Panel A: Raw SCF Data		
Wealth (baseline)	30	2.819	61.6	38.4
Wealth less housing	30	2.649	61.4	38.6
		Panel B: Scaled SCF Data		
Wealth (baseline)	30	1.71	57.4	42.6
Wealth less housing	30	1.64	65.9	34.1

Note: The decomposition is done for 30 groups which are the product of 5 age groups and 6 income groups. The age groups are: 18-34, 34-35, 35-44, 45-54, 55-64, 65+ and the income groups (in thousands of real 2019 dollars) are: 0-20, 20-40, 40-60, 60-80, 80-120, 120+, where the top end of income intervals is excluded..

within-group increases in wealth holdings could simply reflect higher valuation of wealth due to lower interest rates. Subsequently, households in 2019 could be holding much more wealth than they desire relative to similar households in 1989. This could be the case if households face constraints on adjusting assets, such as housing, in their portfolios. As we saw, however, the results are not driven by housing wealth. Nonetheless, to more throughly explore the possibility that household wealth holdings exceed their desired levels, we need to examine the changes in saving rates by age-income groups. We do so focusing on total saving rates.

2.1 Within-group Saving Behavior: Are Households in 2019 Trying to Shed Their Increased Wealth?

In the previous section, we documented that increase in the aggregate wealth-to-income ratio in the US over the 1989-2019 period is in large part accounted for by increases in wealth for given levels of age and income. That is, it is predominantly a within-group phenomenon. There are at least two potential interpretations of such an observation. On the one hand, increases in wealth-to-income ratio could reflect higher desired wealth holdings due to low expected returns on assets, higher longevity or decreased economic growth. On the other hand, they could reflect unanticipated valuation effects, where the observed higher wealth reflects holdings that are above their desired levels. To help discriminate between these two possibilities, in this section we look at the changes in within-group saving rates over the same period. In particular, if the observed within-group increases in wealth-to-income ratios reflect wealth holdings in 2019 that are above their desired levels, then we should see household groups with large increases in wealth wanting to spend more and save less to get their wealth back down to its target level. Accordingly, we should see them decrease their savings rates.[18] Hence, the absence of a negative relationship between changes in wealth and changes in saving rates would indicate that the extra wealth holdings are likely desired not excessive.

In line with the previous section, we focus on within-group changes in saving rates for the 30 groups we used for our analysis of changes in wealth-to-income ratios. We measure saving in the SCF using synthetic saving approach, widely adopted in the literature, which approximates saving by each group by netting out valuation effects from changes in the group's wealth between two SCF waves.[19] Our saving rates are calculated over a three-year window. Saving rates for 1989-92 and 2016-2019 periods, respectively, correspond to the start and the end of our 30-year period. In our robustness exercises, we also exclude net inheritances from changes in wealth, which does not materially change the results.[20]

We follow the approach of the previous section in using both raw and scaled to the aggregates SCF data to construct group saving rates and their changes. For valuation effects in both cases we apply Mian et al. (2020) asset/debt inflation factors, which are aggregate in nature and are available until 2016 inclusively. We extend these factors to 2019 using the authors' methodology. In comparison to the aggregate measures of saving rates in FA/NIPA and by extension in scaled SCF data, we find somewhat lower saving rates in raw SCF data. Appendix A provides further details of the saving rate construction.

Table 3 presents correlations between changes in wealth-to-income ratios and changes in saving rates. Using raw SCF data results in a coefficient of correlation of -0.05, while with scaled SCF data it is 0.16. Both of these numbers suggest that groups that faced greater increases in wealth-to-income ratios do not appear to systematically reverse this

[18] Fagereng et al. (2019) ask a similar question whether households who experience capital gains sell off the assets subject to price increases to consume. They find evidence against such behavior and show that it is consistent with a model where asset price increases are driven by declining asset returns, as opposed to growing dividends.

[19] For other papers using synthetic saving approach see Mian et al. (2021) and references therein.

[20] Accounting for inheritances has a zero net effect in the aggregate, as inheritances received should equal inheritances bequeathed, but within groups these inflows and outflows may not be equal, potentially affecting group-wise changes in saving rates.

accumulation by decreasing their saving rates.[21]

Table 3
Correlation between Group Changes in Wealth-to-Income Ratios and Changes in Saving Rates: Raw and Scaled SCF Data, 30 Age-Income Groups

	Raw SCF	Scaled SCF
corr($\Delta(s/y)$, $\Delta(w/y)$)	-0.05	0.16

Note: Correlation is computed using 30 age-income groups constructed using SCF data as defined previously.

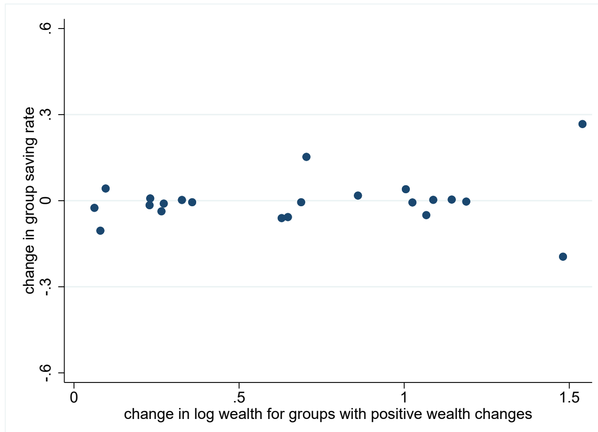
In Figure 3, we complement the evidence on correlations from Table 3 by plotting the changes in saving rates against the changes in log wealth for all the groups that experienced increases in wealth. Given the lower aggregate saving rates in raw SCF data, for this figure we are using results based on the scaled SCF data. The average change in savings rates for this subset of age-income groups is slightly positive. Moreover, as can be seen in the figure (and is confirmed by the correlation), higher increases in wealth are not on average associated with larger decreases in saving rates. It must be recognized that our measure of saving rates, which is common to the literature, is quite noisy. Accordingly, we witness substantial variation in saving rates. Nonetheless, we view these patterns as providing support to the notion that increases in within-group wealth-to-income ratios documented in the previous section more likely reflect changes in desired wealth holdings as opposed to wealth holdings that exceed their desired levels.

2.2 Focusing on the 55-64 Age Cohort

In this section, we focus on the change in wealth for the 55-64 age group, using our baseline measure of wealth inclusive of defined benefit pensions. Since this is mainly a pre-retirement group, its asset holdings most likely reflect a combination of life-long saving decisions which had to balance different forces, including retirement needs. We start by considering the relationship between income and wealth in each of the 1989 and 2019 SCF

[21] Amongst our 30 benchmark groups, we find that none of the groups in the top income grouping, except one, decreased their saving rates, which is consistent with findings in Mian et al. (2021) using averages for 1963-1982 and 1995-2019 periods and the top 10% of the within-cohort income distribution. However, the time periods and the income group definitions in the two studies are not fully comparable.

Figure 3
Change in saving rates vs. change in log wealth for age-income groups with wealth increases between 1989 and 2019



Sources: Survey of Consumer Finances values scaled using aggregates from the Financial Flow Accounts and National Income and Product Accounts.

survey years, focusing on the range of incomes between 40 000$ and 400 000$ in 2019 dollars. In both years, it covers over 90% of the group's total income and helps avoid outliers. In Appendix B.3, we also extend income coverage to the 20 000$ to 1 000 000$ range. The patterns documented below are very similar in this range as well.

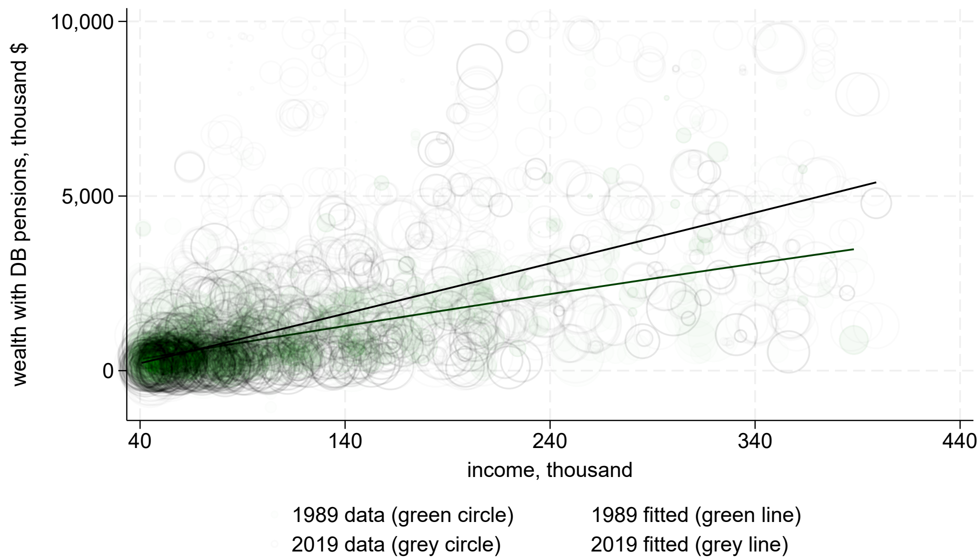
In Figure 4, we provide the scatterplot of the raw data on baseline household wealth against household income for both 1989 (green circles) and 2019 (grey circles). We also superimpose a quadratic regression line of the relationship for each of the two years. These are depicted in the figure in solid green and grey lines respectively. The coefficients for the regressions are reported in Table 4. As can be seen in the figure, and confirmed in the table, the relationship between wealth and income is approximately linear in both 1989 and 2019.[22] However, the magnitude of this relationship changed considerably from 1989 to 2019. In 1989, each additional dollar of income was associated with an increase of 9.2$ of wealth. By 2019, each additional dollar of income was associated with additional 14.3$ of wealth.[23] In Figure 5 we present a similar plot but using our baseline measure of wealth less housing. The relationship is again approximately linear with the slope now going from 8.43 to 12.32, representing an increase of 46% compared to 55% when housing is included.

[22]Note that the observation of a linear relationship between wealth and income does not imply that the wealth-to-income ratio is constant across income levels since the intercept for this relation is non-zero. This will be discussed later.

[23]For incomes above 1 000 000$ a year, we do find some indication of non-linearities, but the number of observations for this group in our sample is very small, especially in 1989.

Finally, in Figure 6 we present the plot of wealth on income, but where we have removed from the baseline measure of wealth both housing and defined benefit pension components. For this more restricted measure of wealth we do see some degree of non-linearity with the wealth level increasing more than proportionally as income increases in both years.

Figure 4
Relationship Between Baseline Wealth and Income: 1989 and 2019



Sources: Survey of Consumer Finances, 1989 and 2019. This figure plots the raw observations on wealth and income from SCF and the fitted lines from the quadratic regression of wealth on income, separately for 1989 and 2019.

3 Deriving the Asset-demand Interest-rate Relationship: When Inter-temporal Substitution Competes with Retirement Motives

In the previous section we documented that desired asset holdings of households within the same age and real income groups appear to have increased substantially over the 1989-2019 period. In this section, we propose a tractable framework aimed at capturing different potential forces that could help explain this observed change. Our approach builds on a model similar to that of Gertler (1999) that integrates both inter-temporal substitution forces and retirement preoccupations in wealth accumulation.[24] In particular, these two

[24]We depart from Gertler (1999) by adopting the more common CRRA utility specification instead of RINSE preferences. Carvalho et al. (2016) use the model of Gertler (1999) to examine equilibrium rates. More recently, Galí (2021) introduces retirement in a New Keynesian model with logarithmic utility in which there are multiple steady state real rates that are related to the size of bubbles. In a two-period

Table 4
Regression of Different Measures of Wealth on Income: Linear and Quadratic Fit (1989 and 2019)

	Income (t-stat)	Income^2 (t-stat)	Intercept
	Baseline Wealth		
1989	9.22 (15.3)	–	-27 092
1989	9.83 (5.6)	-1.8*10^{-6} (.29)	-61 952
2019	14.3 (24.5)	–	-370 409
2019	14.1 (7.9)	7.7*10^{-7} (.14)	-353 925
	Baseline Wealth Less Housing		
1989	8.43 (5.6)	-2.6*10^{-6} (.39)	-103 767
2019	12.32 (7.35)	8.1*10^{-7} (.15)	-364 139
	Baseline Wealth Less Housing and Defined Benefits		
1989	3.77 (2.2)	1.24*10^{-5} (2.0)	47 673
2019	6.19 (4.52)	1.39*10^{-4} (3.17)	74 644

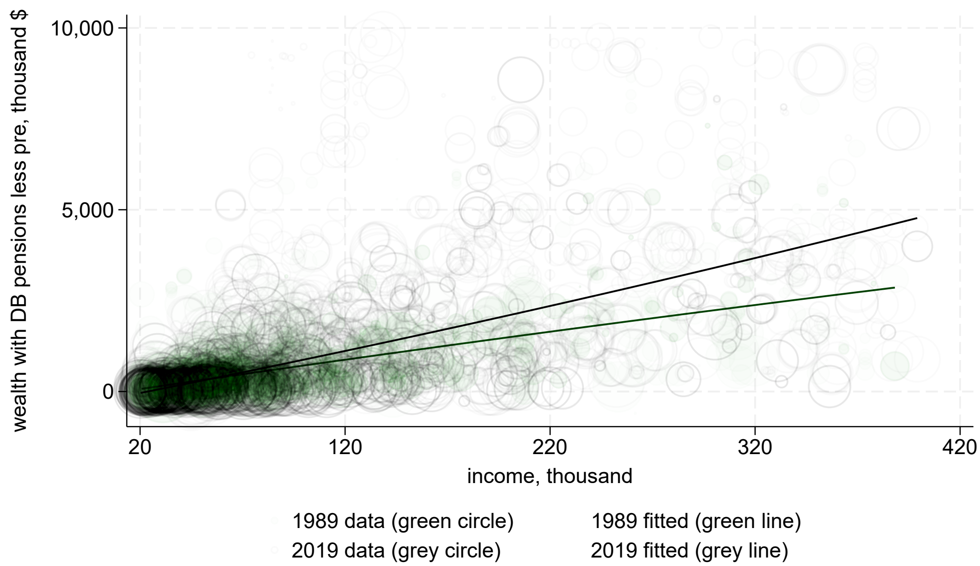
forces will be shown to interact in a manner that gives rise to rich specifications of asset demand as a function of interest rates. A key insight of this section is that the effect of interest rates on long-run asset demand is likely to be C-shaped.

3.1 The Household Wealth Accumulation Problem

When thinking about consumption and wealth accumulation decisions, it is common to think about people in different states. As is standard in simple OLG models, we can think of a household in one of three states: an active work state, a retirement state and a death state. Following Blanchard (1985), Yaari (1965) and Gertler (1999) we want to think of these states as evolving stochastically. To be more precise, let us assume that a person starts life in an active work state and transits out of it with instantaneous probability δ_1. In the absence of fixed retirement dates, this transition can be thought as a health shock.

OLG model with nominal rigidities, Plantin (2022) also examines the case where a Taylor rule may create monetary bubbles. We do not explore the possibility of bubbles in our analysis.

Figure 5
Relationship Between Baseline Wealth Less Housing and Income: 1989 and 2019



Sources: Survey of Consumer Finances, 1989 and 2019. This figure plots the raw observations on wealth and income from SCF and the fitted lines from the quadratic regression of wealth on income, separately for 1989 and 2019.

After that, with probability q, the person retires and with probability $(1-q)$, the health shock is severe, and the person dies. If the person retires, they will die with instantaneous probability $\delta_2 \geq \delta_1$. If we denote the expected discounted utility of entering the retirement state at time t by V_t, we can express the utility of a household in an active work state as:

$$\int_0^\infty e^{-(\delta_1+\rho)t}\left[\frac{c_t^{1-\sigma}}{1-\sigma}+\delta_1 q V_t\right]dt, \qquad \sigma > 0$$

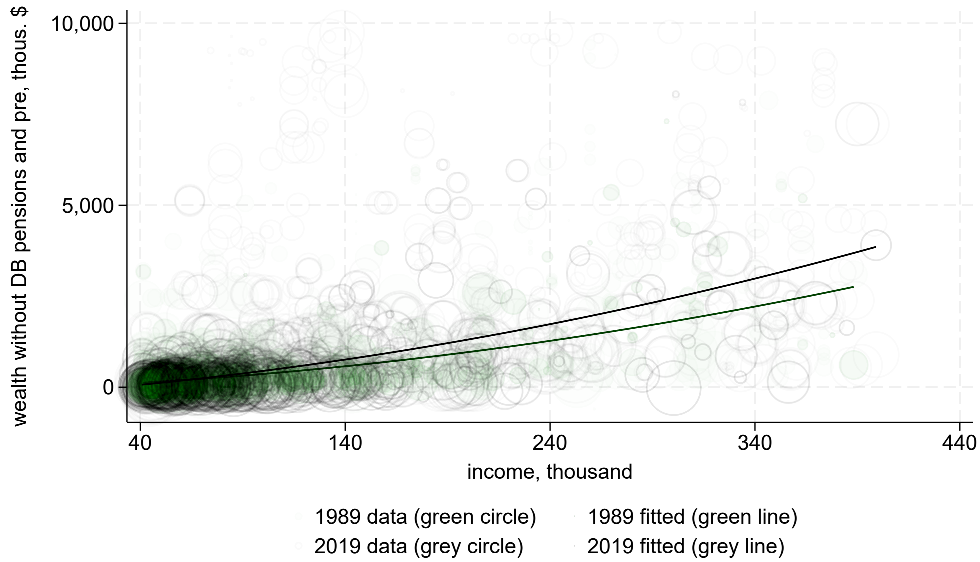
where c_t is consumption, ρ is the subjective discount rate and $\sigma > 0$ is the inverse of the inter-termporal elasticity of substitution $(1/\sigma)$, or alternatively the risk aversion parameter.

The retiree's decision problem. For the household in the retirement state, the preferences are given by:

$$\int_0^\infty e^{-(\delta_2+\rho)\tau}\frac{c_\tau^{1-\sigma}}{1-\sigma}d\tau, \qquad \sigma > 0$$

The budget constraint facing the retired household is given by:

Figure 6
Relationship Between Baseline Wealth Less Housing and Defined Benefits and Income: 1989 and 2019



Sources: Survey of Consumer Finances, 1989 and 2019. This figure plots the raw observations on wealth and income from SCF and the fitted lines from the quadratic regression of wealth on income, separately for 1989 and 2019.

$$\dot{a}_t = a_t r_t + pb_t - c_t,$$

where a_t is the asset directly held by a retired person at time t, r_t is the return on the asset a_t and pb_t is a pension benefit.

Given our structure, the expected discounted utility of a household who retires at time t_1, V_{t_1}, can be solved explicitly and expressed as[25]

$$V_{t_1} = \frac{(a_{t_1}^T)^{1-\sigma}}{1-\sigma}\left[\int_{t_1}^{\infty} e^{-\int_{t_1}^{t}\frac{1}{\sigma}[(\rho+\delta_2)-(1-\sigma)r(\tau)]d\tau} dt\right]^{\sigma},$$

where $a_{t_1}^T$ is the total value of assets held by the household at time of retirement, including assets directly accumulated by the household a_{t_1} plus the present discounted value of pension benefits, i.e.

[25]The expected utility associated with the retirement state is found by first solving for the optimal consumption path, which is governed by the Euler equation $\frac{\dot{c}_t}{c_t} = \frac{r_t-\rho-\delta_2}{\sigma}$ and then integrating the implied utility flow over the expected duration of retirement.

$$a^T_{t_1} = a_{t_1} + \int_{t_1}^{\infty} pb_t e^{-\int_{t_1}^{t}(r(\tau)+\delta_2)d\tau} dt.$$

For convenience, we will also express V_{t_1} as

$$V_{t_1} = V(a^T_{t_1}, \Gamma_{t_1}) = \frac{(a^T_{t_1})^{1-\sigma}}{1-\sigma}\left[\Gamma_{t_1}\right]^{\sigma},$$

where

$$\Gamma_{t_1} = \int_{t_1}^{\infty} e^{-\int_{t_1}^{t}\frac{1}{\sigma}[(\rho+\delta_2)-(1-\sigma)r(\tau)]d\tau} dt$$

is a function of the whole future path of returns $\{r_t\}_{t_1}^{\infty}$. Expressing utility of someone who retires at time t_1 as $V_{t_1} = V(a^T_{t_1}, \Gamma_{t_1})$ makes it clear that it depends both on the total asset at the time of retiring and the entire path of asset returns over the retirement period. As we shall see, the degree of inter-temporal substitution $\frac{1}{\sigma}$ will play an important role in controlling how asset returns affect marginal value of assets.

For future reference, it is useful to note that Γ_{t_1} obeys the following differential equation

$$\dot{\Gamma}_t = -1 + \Gamma_t\left[\frac{\rho+\delta_2}{\sigma} - \frac{1-\sigma}{\sigma}r_t\right]. \tag{2}$$

To see more easily how asset returns affect retirement utility, note that if the return on asset is constant, $r_t = r$, then V_{t_1} can be expressed as

$$V_{t_1} = \frac{(a^T_{t_1})^{1-\sigma}}{1-\sigma}\left[\frac{\rho+\delta_2}{\sigma} - \frac{1-\sigma}{\sigma}r\right]^{-\sigma} \tag{3}$$

We can see from (3) that higher r increases utility in both cases when $\sigma < 1$ and $\sigma > 1$, that is, retired individuals like higher interest rates as they give them a superior income stream. However, what will play an important role in our analysis is how higher r affects the marginal value of $a^T_{t_1}$ to a retiree. This is given by the following key lemma.

Lemma 1. *For a fixed r, the marginal value of assets to a retiree is decreasing in r when $\sigma > 1$ and is increasing in r when $\sigma < 1$, since* $\frac{\partial^2 V_{t_1}}{\partial a^T_{t_1}\partial r} = (a^T_{t_1})^{-\sigma}(1-\sigma)\left[\frac{\rho+\delta_2}{\sigma} - \frac{1-\sigma}{\sigma}r\right]^{-\sigma-1}$.

In general, as noted in Lemma 1, the effect of asset returns on the marginal value of assets to a retiree depends on σ.[26] This marginal value is decreasing in r when $\sigma >$

[26] Lemma 1 can be trivially extended to include the case of log preferences. In this case, the marginal

1. In other words, when a retiree has limited opportunities to inter-temporally substitute consumption across time, the retiree will view assets at time of retirement to have greater marginal value when interest rates are low than when they are high.[27,28,29]

Another comparative static one obtains from (3) is a non-surprising result that the marginal value of assets is decreasing in δ_2, that is, higher expected longevity in retirement increases the marginal value of assets. Both of these properties of the marginal value of assets will be important drivers of the wealth accumulation decisions of active households.

The active household's decision problem. Let us now turn to the decision problem of an active household. Replacing $V_t = V(a_t^T, \Gamma_t)$ as the continuation value of assets in retirement, we can now re-write utility of an active household as:

$$\int_0^\infty e^{-(\delta_1+\rho)t}\left[\frac{c_t^{1-\sigma}}{1-\sigma} + \delta_1 q V(a_t^T, \Gamma_t)\right] dt,$$

subject to

$$\dot{a}_t = y_t - c_t \tag{4}$$

where $a_t^T = a_t + \int_t^\infty p b_{t'} e^{-\int_t^{t'}(r(\tau)+\delta_2)d\tau} dt'$, $y_t = w_t + r_t a_t - T_t$ is disposable income, w_t is labor income and T_t are taxes.

The consumption Euler equation for the active household is:

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value of assets is independent of interest rates, i.e., $\frac{\partial^2 V_{t_1}}{\partial a_{t_1} \partial r} = 0$.

[27]When $\sigma > 1$, a rise in interest rates causes the optimal path of post-retirement consumption to be higher at all dates and hence the marginal value of assets is lower. This is easily understood and intuitive. In contrast, when $\sigma < 1$ different interest rates cause optimal paths of post-retirement consumption to cross; with retirees consuming initially less in a higher interest rates environment but having their consumption decline more slowly over time. Because of this crossing property, the effect of interest rates on the marginal value of assets is not straightforward when $\sigma < 1$. Lemma 1 indicates that higher interest rates increase the marginal value of assets when $\sigma < 1$ due to this crossing feature.

[28]It is worth noting that, although we have not explicitly introduced an annuity market for transforming asset a_t into a guaranteed income stream, the content of Lemma 1 would remain identical if we were to allow for an annuity market similar to that in Blanchard (1985).

[29]Like in Gertler (1999), a key assumption is the absence of a pension system which acts as a perfect insurance market against loss of labor income. It implies that consumption in retirement relies at least in part on the accumulated savings when active.

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho - \delta_1}{\sigma} + \frac{c_t^{\sigma}}{\sigma}\delta_1 q V_a(a_t^T, \Gamma_t) \tag{5}$$

Relative to a standard infinitely lived agent Euler equation, this Euler equation incorporates forces associated with both inter-temporal substitution and retirement preoccupations as in Gertler (1999) and Grandmont (1985). The first term in this Euler equation maintains the standard substitution effect of interest rates on consumption. However, this effect now relates to short-term interest rate movements holding the future path of interest rates constant. When both short-term and long-run interest rates move together the net effect is more involved. The additional term in the Euler equation — $\frac{c_t^{\sigma}}{\sigma}\delta_1 q V_a(a_t^T, \Gamma_t)$ — represents the incentive to save due to retirement motives. Given this term is always positive, it implies that retirement adds a force towards postponing consumption and favoring asset accumulation.[30] The key element for us is that the retirement incentive to save is affected by long-run returns to savings. In particular, when interest rates are constant, $r_t = r$, we have seen that $V_{a,r}(a_t^T) < 0$ when $\sigma > 1$. Hence, interest rates have two opposing effects in our set-up when $\sigma > 1$. Low interest rates will favor higher consumption today due to inter-temporal substitution forces, while at the same time, low interest rates are an incentive for greater retirement savings if they are viewed as persistent.

To help further highlight implications of this Euler equation, it is helpful to examine the implied long-run asset holdings of the active household when the return on asset is constant and therefore $\Gamma_t = [\frac{\rho+\delta_2}{\sigma} - \frac{1-\sigma}{\sigma}r]^{-1}$. We will denote an active household's steady state total asset holdings by $a^{T,ss}$. Proposition 1 indicates that $a^{T,ss}$ is attractive and describes the key properties of the steady state asset-to-income ratio $\frac{a^{T,ss}}{y}$.

Proposition 1. *With a fixed r, the asset-to-income ratio of active households will converge to*

$$\frac{a^{T,ss}}{y} = (\delta_1 q)^{\frac{1}{\sigma}} \left[\frac{\rho + \delta_2}{\sigma} - \frac{1-\sigma}{\sigma}r\right]^{-1} [\rho + \delta_1 - r]^{\frac{-1}{\sigma}}, \tag{6}$$

when r is in the interval defined by $[\frac{\rho+\delta_1-r}{\delta_1 q}(\frac{\rho+\delta_2}{\sigma} - \frac{(1-\sigma)r}{\sigma})^{\sigma}]^{\frac{1}{\sigma}} > \max[0, r]$.[31]

This steady state asset-to-income ratio is increasing in longevity (i.e. decreasing in δ_2*) and decreasing in* ρ*. Moreover, if* $\sigma \leq 1$*, then* $\frac{a^{T,ss}}{y}$ *is monotonically increasing in asset*

[30] This force is also present in models with warm-glow bequest motives, but in that case it does not depend on interest rates, which is the key feature for our purposes.

[31] If r is not in the interval, asset holdings do not converge.

return r, while if $\sigma > 1$, it is C-shaped in r.

See Appendix C.1 for the proof.

The most important element of Proposition 1 relates to the effect of r on desired long-run asset holdings. In particular, we see that if $\sigma \leq 1$, then desired long-run asset holdings would be monotonically increasing in r because the substitution effect always dominates retirement savings effect. In contrast, when $\sigma > 1$ the effects of r on long-run asset holdings are non-monotonic. For high levels of returns, desired holdings are increasing in r, while for low returns they are decreasing in r. To understand this effect, recall that interest rates have two effects in this model. At low interest rates, households are encouraged to consume more, and accumulate less, through the standard inter-temporal substitution channel. However, retirement preoccupations play a counterbalancing role. When long-run interest rates are low and $\sigma > 1$, active households have an increased marginal incentive to accumulate assets for retirement needs. What Proposition 1 indicates is that there will be a point of reversal of the effect of steady state r on accumulation incentives. When r is sufficiently high, a marginal increase in steady state r would lead to more accumulation as the positive substitution effect dominates the decreased retirement need effect even if $\sigma > 1$. When interest rates are low, the marginal value of asset becomes very high. This causes the need for retirement wealth to dominate the inter-temporal substitution effect and gives rise to the C-shaped asset demand.[32]

The C-shape of the active households' steady state asset-to-income ratio $\frac{a^T}{y}$ for the case when $\sigma > 1$ is illustrated in Figure 7. Moreover, we can see that the asset-to-income ratio is delimited by two levels of r. As r tends to $\rho + \delta_1$ from below, the steady sate asset-to-income of active households will tend to infinity. As r tends to $\frac{\rho+\delta_2}{1-\sigma} < 0$ from above, $\frac{a^T}{y}$ will tend again toward infinity. When $\sigma > 1$, there also exists a threshold or point of inflexion

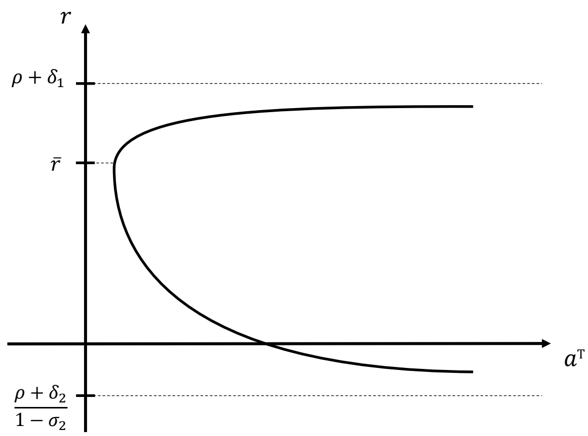
$$\bar{r} = \left[\frac{\sigma(\sigma-1)(\rho+\delta_1) - (\rho+\delta_2)}{(\sigma-1)(\sigma+1)}\right],$$

such that the asset-to-income ratio of active households is increasing in interest rates when r is above $\bar{r}$ and is decreasing in interest rates when r is below $\bar{r}$.

Up to now, we have not allowed for growth in labor income. Extending this model to growing labor income – when seen as due to an aggregate growth trend – is straightforward.

[32] Our paper has some similarities with the work of Abadi et al. (2023) on the reversal interest rate, which refers to the rate below which interest rates become contractionary. Their reversal rate results from banking frictions. Our set-up can also be thought of as having a reversal rate, which we denote $\bar{r}$. It arises from expected income effects in retirement that drive up households' desired savings while working and therefore depress consumption.

Figure 7
Active households' long-run asset-to-income ratio



In fact, the household problem then inherits balanced growth properties, with the system converging to constant asset-to-income ratio growth path. In particular, if labor income grows at the instantaneous rate g, then it is easy to verify that optimal consumption decision will generate a steady state asset-to-income ratio given by:

$$\frac{a^{T,ss}}{y} = (\delta_1 q)^{\frac{1}{\sigma}} \left[\frac{\rho+\delta_2}{\sigma} - \frac{1-\sigma}{\sigma} r\right]^{-1} [\rho+\delta_1+\sigma g - r]^{\frac{-1}{\sigma}}. \tag{7}$$

Equation (7) will form the basis for examining what force may best help explain the observed changes in asset-to-income ratio. This will include considering the potential role of increased longevity (decreased δ_2), fall in economic growth g, and changes in real interest rates r. As can be seen from Equation (7), slower growth favors a higher asset-to-income ratio, as does increased longevity, while the effect of a change in r is ambiguous and depends on σ.

3.2 Valuation Effects on Consumption and Saving Behavior

In our baseline model, households had only one savings vehicle: a short-term bond. In this section we will briefly discuss how the framework extends to allow for interest-rate sensitive assets. Our goal is to illustrate why different valuation scenarios can have very different effects on consumption decisions depending on the source of the valuation effects. To this end, let us introduce a Lucas tree into our setting where the tree produces a flow f of goods every period. A household can now hold a combination of short-term bonds and trees. If we denote by z_t the price of a unit of trees at time t, then arbitrage between the two assets will cause z_t to satisfy the standard asset pricing relationship

$$\frac{\dot{z}_t}{z_t} = \frac{f}{z_t} - r_t,$$

and households will be indifferent between holding bonds or trees. The active households' consumption Euler equation in this case can be re-written as

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho - \delta_1}{\sigma} + \delta_1 q^s \frac{c_t^\sigma}{\sigma} V_a(\Omega_t, \Gamma_t),$$

where $\Omega_t = b_t + s_t z_t$ denotes household wealth that now includes bonds b_t and trees s_t. Note that the desired steady state asset-to-income ratio, where total assets are now given by Ω, maintains all the previous properties.

Now we can use this framework to consider two different types of changes that can lead to valuation effects, that is, two types of changes in Ω that are induced by increases in z_t. Consider an active household that is initially in steady state (at fixed interest rate) with a portfolio composed of bonds and trees. We want to contrast the household's response to a one-time permanent change in dividend f – an increase from f to f' – with a one time permanent change in r – a decrease from r to r'. As a result, Ω will jump to a new value of either $b + s_t \frac{f'}{r}$ or $b + s_t \frac{f}{r'}$. In both cases, the impact effect is increased wealth due to a valuation effect. However, the impact on consumption and saving incentive can be very different even if the initial effect on Ω is of the same size.

In the first case, the change in f does not affect the desired steady state asset-to-income ratio. Accordingly, such a valuation effect will lead to an upward jump in consumption as the household's asset-to-income ratio is above target, and the household can therefore take advantage of the valuation effect to consume more and run down assets. However, in the second case, the effect on consumption will be more muted if $\sigma > 1$ and the household is on the lower branch of the asset demand relationship. In fact, it is not clear whether the valuation effect will cause an increase or a decrease in consumption when it is driven by a change in r. The reason being that the change in interest rates now can increase the desired steady state asset-to-income ratio. This can cause the new steady state level of desired wealth to be greater that the wealth effect generated by the fall in rates.

Hence, in this framework, the consumption effect of a re-valuation of assets can be very different depending on the driving force behind increased valuation. In particular, valuation effects due to decreases in interest rates will likely have more muted and even possibly perverse effects on consumption when compared to valuation effects that are due to an asset providing a greater flow of income.

4 A Quantitative Exploration of the Drivers of Increased Wealth-to-Income Ratio

In this section, we will use Equation (7) to explore different forces which could have led to the observed within-household increases in wealth-to-income ratio over the 1989-2019 period. We will mainly focus on the observed accumulation pattern of the 55-64 age group as households in this group are most likely to have wealth levels close to the steady state level implied by Equation (7). In fact, for the range of parameters we will discuss below, we have found that agents in our model would be very close to their steady state wealth holding by the average age of retirement.[33]

We will examine the effects of three sources of changes due to longevity, growth rate and interest rate. In our baseline, parameters are set as follows, with a period corresponding to a year. The working life is set at 40 years, so $\delta_1 = .025$, $\delta_{2,89}$ is set at .058, which corresponds to the 17.25 years of life expectancy at age 65 in 1989. The fraction of people that make it to retirement is set at $q = .81$ to match overall life expectancy. The growth rate g_{89} is set at .02. For the real interest rate in 1989, we consider two values, either $r_{89} = .04$ or $r_{89} = .03$. For 2019, we set the real interest rate $r_{19} = 0$, reflecting the large fall in real interest rates between 1989 and 2019. To reflect increased longevity between 1989 and 2019, we set $\delta_{2,19}$ to .051, which corresponds to the life expectancy of 19.6 years at age 65 in 2019. Finally, we set the growth rate g_{19} at .015 as it corresponds to the growth rate in per capita income in the US from 1989 to 2019.

Our aim will be to examine the average (within-group) asset-to-income ratios predicted by the model for 1989 and 2019 and compare them with those found in our earlier empirical exploration. In particular, our regression of wealth on income suggests that the wealth-to-income ratio increased from 9.2 to 14.3 over the 30-year period. The theoretical levels are given by:

[33] The results of the quantitative evaluation using wealth holdings implied by the transitional dynamics after 40 years of work, as opposed to their steady state values, are available in Appendix E.

$$\frac{a^T}{y}_{89} = (\delta_1 q)^{\frac{1}{\sigma}} \left[\frac{\rho + \delta_{2,89}}{\sigma} + \frac{\sigma - 1}{\sigma} r_{89} \right]^{-1} [\rho + \delta_1 + \sigma g_{89} - r_{89}]^{\frac{-1}{\sigma}}$$

$$\frac{a^T}{y}_{19} = (\delta_1 q)^{\frac{1}{\sigma}} \left[\frac{\rho + \delta_{2,19}}{\sigma} + \frac{\sigma - 1}{\sigma} r_{19} \right]^{-1} [\rho + \delta_1 + \sigma g_{19} - r_{19}]^{\frac{-1}{\sigma}}$$

These asset-to-income ratios will depend on the values of σ and ρ in addition to the parameters noted above. Since there is considerable uncertainty regarding the appropriate values for σ and ρ, our approach will be to search for both $\sigma \in [1, 3]$ and $\rho \in [.02, .05]$ to match our empirical counterparts for $\frac{a^{T,ss}}{y}_{89}$ and $\frac{a^{T,ss}}{y}_{19}$, when all three sources of change are allowed to happen at once. We then look at the respective roles of changes in longevity, growth and interest rates (one by one) in explaining the observed changes, holding σ and ρ at the values needed to explain the aggregate change.[34]

Table 5
Contribution of longevity, growth and interest rates to change in asset-to-income ratio: 1989-2019

9.2-14.3 target			
	σ	ρ	$\frac{a}{y}_{89}$-$\frac{a}{y}_{19}$
$\Delta r = .04$	2.5	.034	9.35-14.34
$\Delta r = .03$	2.74	.049	9.55-14.35
	longevity	growth	interest rates
	Percentage point contribution		
$\Delta r = .04$	4.6	7.6	31.2
$\Delta r = .03$	4.5	5.5	35.9

Table 5 presents the results of the first exercise, where we consider separately 2 cases, which differ with respect to the size of the change in r. We consider real interest rate changes from 4% to 0%, or from 3% to 0%. Recall that we used the change in slope of the wealth on income relationship to establish our targets. For each case, we report the

[34]Note that, from an ex-ante perspective, it is not clear if there exist $\sigma \in [1, 3]$ and $\rho \in [.02, .05]$ that would allow us to match the data.

Table 6
Predicted levels of the asset-to-income ratio in 1989 and 2019, change in asset-to-income ratios from 1989 to 2019 as a function of (ρ, σ)

σ	1.0001	1.8	2	2.2	2.4	2.6	2.8	3.0
	Predicted asset-to-income ratio in 1989 (r=.03)							
$\rho = 0.02$	7.33	10.49	11.17	11.80	12.40	12.96	13.49	13.99
$\rho = 0.03$	5.05	8.65	9.40	10.11	10.70	11.38	11.97	12.51
$\rho = 0.04$	3.71	7.30	8.07	8.79	9.47	10.12	10.72	11.30
$\rho = 0.05$	2.85	6.27	7.03	7.75	8.43	9.08	9.69	10.27
	Predicted asset-to-income ratio in 2019 (r=0)							
$\rho = 0.02$	4.70	12.44	14.55	16.69	18.87	21.09	23.33	25.59
$\rho = 0.03$	3.53	10.15	11.98	13.85	15.76	17.70	19.67	21.66
$\rho = 0.04$	2.75	8.47	10.08	11.74	13.43	15.15	16.90	18.68
$\rho = 0.05$	2.20	7.21	8.64	10.12	11.64	13.18	14.75	16.35
	Implied percentage change in asset-to-income ratios in 1989-2019							
$\rho = 0.02$	-35.91	18.62	30.26	41.43	52.21	62.68	72.90	82.91
$\rho = 0.03$	-30.15	17.31	27.39	37.06	46.41	55.50	64.38	73.08
$\rho = 0.04$	-25.95	16.09	24.98	33.51	41.76	49.80	57.64	65.35
$\rho = 0.05$	-22.77	14.98	22.93	30.57	37.96	45.15	52.19	59.10

Note: The changes in the implied asset-to-income ratio in 2019 are calculated for the cases when real interest rate falls to 0, growth rate falls to 1.5%, and post-retirement life expectancy $(1/\delta_2)$ rises to 19.6 years.

parameters σ and ρ needed to approximately match the targets and the values of $\frac{a}{y}_{89}$ and $\frac{a}{y}_{19}$ implied by these parameters. For example, when r falls by 4 percentage points, to match the within household change in wealth-to-income ratio implied by the change in the slope of the wealth on income relationship for the 55-64 age group (9.2-14.3), we need $\sigma = 2.5$ and $\rho = .034$. Both of these values are within the bounds typically associated with these parameters. When looking at the individual contributions of each of the three sources of change – longevity, growth and interest rates – we see that longevity can explain a 4.6 percentage point increase in the wealth-to-income ratio, that is, an increase from 9.2 to 9.66. Reduced growth can explain a 7.6 percentage point increase in the wealth-to-income ratio. Both of these taken together explain less than half of what is implied by the observed increase in the slope of the wealth on income relationship. The biggest explanatory power

Table 7
Percentage changes in the asset-to-income ratios (A/Y) in 1989 and 2019 when real rate falls to 0, growth rate falls to 1.5%, and post-retirement life expectancy rises to 19.6 years as a function of (ρ, σ)

σ	1.0001	1.8	2	2.2	2.4	2.6	2.8	3.0
	Decline in real rate from 3% to 0%							
$\rho = 0.02$	-46.15	1.13	11.38	21.24	30.80	40.11	49.22	58.15
$\rho = 0.03$	-39.99	1.91	10.92	19.58	27.99	36.17	44.19	52.06
$\rho = 0.04$	-35.29	2.35	10.39	18.12	25.62	32.94	40.10	47.13
$\rho = 0.05$	-31.57	2.60	9.85	16.84	23.62	30.22	36.69	43.06
	Decline in growth rate from 2% to 1.5%							
$\rho = 0.02$	16.67	11.39	10.55	9.83	9.20	8.65	8.16	7.72
$\rho = 0.03$	12.50	9.27	8.71	8.21	7.77	7.37	7.01	6.69
$\rho = 0.04$	10.00	7.82	7.42	7.05	6.72	6.42	6.15	5.90
$\rho = 0.05$	8.33	6.76	6.46	6.18	5.93	5.69	5.47	5.27
	Increase in post-retirement longevity from 17.2 to 19.6 yrs							
$\rho = 0.02$	9.86	7.37	6.93	6.54	6.19	5.88	5.60	5.34
$\rho = 0.03$	8.64	6.67	6.31	5.98	5.69	5.43	5.19	4.96
$\rho = 0.04$	7.69	6.09	5.79	5.51	5.26	5.04	4.83	4.64
$\rho = 0.05$	6.93	5.60	5.34	5.11	4.90	4.70	4.52	4.35

Note: Changes in the implied asset-to-income ratios are reported when changing only one parameter at a time. In each panel of the table, the remaining parameters are kept constant at the 1989 levels.

comes from the fall in interest rates, as it explains a 31.2 percentage point increase in the wealth-to-income ratio.[35]

To see more clearly the implications of Equation (7) for both level and change in asset-to-income ratios, Tables 6 and 7 provide extra information. In particular, Table 6 reports what different values of σ and ρ imply for the asset-to-income ratios in 1989 and 2019 when values of other parameters are set at levels described earlier, while Table 7 provides individual contributions due to interest rate, growth rate and longevity changes. As can be seen in Table 7, if σ is set to 1 – in which case asset supplies are not C-shaped – the fall in interest rates would imply that asset-to-income ratio would be predicted to decrease

[35] The sum of these effects does not add up to the total effect because of interaction terms.

substantially, even though increased longevity and the fall in growth rate favor greater accumulation.

Overall, the patterns documented in Table 7 suggest that the change in longevity observed over 1989-2019 can explain around 4 to 6 percentage points increase in the wealth-to-income ratio. Similarly, a potential fall in the per-capita growth rate of the economy from 2% to 1.5% can explain another 5 to 7 percentage points increase in the ratio. Both of these forces are significant, but they fall short of offering an explanation to most of the increase in wealth-to-income ratio observed for the 55-64 age cohort. Most of the explanatory power is attributed to the accumulation incentives induced by the fall in interest rates.

It is worth noting that the asset demand curves implied by values of σ and ρ reported in the Table 5 are C-shaped – since $\sigma > 1$. Moreover, we can calculate the "reversal rate" $\bar{r}$, that is the rate at which the asset-interest-rate relationship (as illustrated in Figure 7) switches sign. For the parameter values that allow us to match the data, this rate is around 4%. This essentially implies that the economy would have been operating on the downward segment of the asset demand for active households throughout the 1989-2019 period.

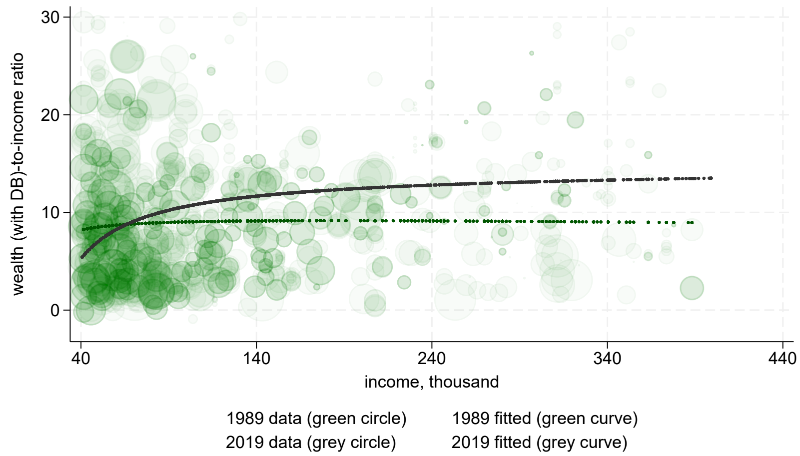
4.1 The Heterogenous Changes in Wealth-to-income Ratios for Lower- versus Higher-Income Households: the Potential Role of Social Security

Although Figure 4 shows that, for most income levels, the 55-64 age group increased its wealth holdings, it is important to recognize that the regression lines in Figure 4 actually cross around an income level of approximately 67 000$. Therefore, for households with income below 67 000$, average wealth actually fell between the two periods. This implies that wealth-to-income ratios also declined for lower-income groups, while they increased for higher-income groups. This can be seen more clearly in Figure 8, where we now plot wealth-to-income ratio against income for both 1989 and 2019. We again include fitted quadratic lines of the relationship for each year. These lines show that, as income increases, the wealth-to-income ratio converges to around 9 in 1989, while it converges to around 14 in 2019. On this figure, the fall in the wealth-to-income ratios for lower-income groups becomes evident.[36]

The pattern presented in Figure 8 raises the question: why did observed wealth levels decrease for lower-income groups, while they increased for higher-income groups? Although

[36] While increased longevity could have contributed to a desire by higher-income groups to accumulate additional wealth, this factor is unlikely to explain the behavior of the lower-income groups as their life expectancy remained largely unchanged over this period.

Figure 8
Wealth-to-income ratios constructed using raw SCF data and prediction from a quadratic regression of wealth-to-income ratio on income for 1989 and 2019



Sources: Survey of Consumer Finances, 1989 and 2019.

there are many potential explanations, here we want to focus on the distributive aspect of social security. Social security payments are a very important source of income in retirement for many American households, especially lower-income. In our wealth measure we do not have the value of social security payments. These payments depend on the whole history of work which we do not see in the SCF.[37] While we do not directly observe social security payments, we can use our model to explore the extent to which social security may be driving the different accumulation behaviour observed between lower- and higher-income groups over the 1989-2019 period.

The reason why social security could help explain this pattern relates to the different replacement rates it offers to different income groups. For lower-income groups, social security offers a high replacement rate of pre-retirement earnings, covering much more of their retirement needs (see, for example, Poterba (2014)). Therefore, the main effect of reduced interest rates for these groups is to favor early consumption, because inter-temporal substitution motives dominate. So we could expect observable wealth (not inclusive of social security) to fall. In contrast, for higher-income households, social security offers a rather low income replacement rate. For these households, as interest rates decrease, there is a need to accumulate more assets if one wants to maintain a similar lifestyle in retirement.

[37] Sabelhaus and Volz (2020) provide some estimates of social security wealth using SCF, in addition to defined benefit pension wealth, using both termination and expected values of such wealth. However, they are not quite suitable for our analysis.

Accordingly, we should see a desire to accumulate more wealth.

To explore the quantitative plausibility of this mechanism, recall that our model implies that target asset holdings inclusive of social security should obey the relationship $a^T + \frac{ss}{r+\delta_2} = h(r)y$, where a^T would be the measure of wealth from our data set, ss are social security payments and $h(r) = (\delta_1 q)^{\frac{1}{\sigma}} \left[\frac{\rho+\delta_2}{\sigma} + \frac{\sigma-1}{\sigma} r\right]^{-1} [\rho + \delta_1 + \sigma g - r]^{\frac{-1}{\sigma}}$. Hence, the ratio of assets a^T – where a^T is wealth exclusive of social security – to income y should satisfy

$$\frac{a^T}{y} = h(r) - \frac{ss}{y}\frac{1}{r+\delta_2}$$

.

It is increasing in income because the social security replacement rate $\frac{ss}{y}$ is generally decreasing in income. Based on this relation we can ask at what cutoff of social security replacement rate, $\frac{ss}{y}$, would our setup predict the wealth-to-income ratio to remain constant as longevity increased, growth decreased and interest rates decreased. Building on our previous quantification, if we have $h(r)$ going from 9.2 to 14.3 and $\frac{1}{r+\delta_2}$ is going from $\frac{1}{.04+.058}$ to $\frac{1}{0+.051}$, then the cutoff replacement rate would be 54%. This implies that we should have witnessed increases in wealth-to-income ratio – by our measure of wealth – for households with social security replacement rates above 54%, while we should have observed decreases in our measure of wealth-to-income ratio for households with social security replacement rates below 54%.

While the mapping between income near retirement and eventual social security payments is not straightforward, the work by Arapakis et al. (2023) can give us an idea of the level of income at which the social security replacement rate is close to 54%. In their study, the authors report that for a white household with average contributing income of 65 736$ in maximum income years, the yearly social security benefits average 37 920$, which gives a replacement rate of 57%. Interestingly, this would imply a cutoff of income below which wealth-to-income ratio would be predicted to decline very close to the 67 000$ level seen in our data. While such calculations are approximate, they do suggest that our model of savings – where inter-temporal substitution competes with retirement needs – can help explain both Figures 4 and 8.

5 General Equilibrium

Our quantitative exploration suggests that the fall in interest rates from 1989 to 2019 likely played a prominent role in the observed within-group increase in wealth-to-income ratios. The reason we put forward is the need to accumulate more wealth for retirement when interest rates are low. This is a consequence of a C-shaped demand for assets induced when inter-temporal and retirement motives compete. However, in all our discussions up to now, we have been treating interest rates as exogenous. In general equilibrium, interest rates will need to adjust to equate the supply and the demand for assets. The aim of this section is to briefly highlight some of the potential general equilibrium implications of having households whose wealth targets may be C-shaped.

To look at the general equilibrium implications of C-shaped asset demands, we first need to verify whether the C-shaped property we derived for active households in Section 3.1 is likely to be maintained when we look at the aggregate demand for assets in the economy, which combines the demands emanating from both active and retired households. In Appendix D we show how we can use the model of Section 3.1 to derive a steady state aggregate demand for assets. This aggregate demand, denoted A^d, is given by Equation (8):[38]

$$A^d = \frac{Yh(r)}{\phi}\left(\frac{\phi + (1-\phi)g(r)}{\phi + (1-\phi)g(r)(\delta_1 q)(\rho + \delta_1 - r)^{\frac{1}{\sigma}}}\right) \tag{8}$$

where Y is aggregate income, $h(r) = (\delta_1 q)^{\frac{1}{\sigma}}\left[\frac{\rho+\delta_2}{\sigma} - \frac{1-\sigma}{\sigma}r\right]^{-1}[\rho+\delta_1-r]^{\frac{-1}{\sigma}}$, $\phi = \frac{\delta_2}{\delta_1 q+\delta_2}$ and $g(r) = \frac{\sigma\delta_2}{\rho+\delta_2-r+\sigma\delta_2}$.

It is worth noting that this representation of the aggregate asset demand should be viewed as mainly illustrative as its derivation involves assumptions on how inheritances are shared among new households and on who the government taxes. These assumptions are made to make aggregation tractable. Notwithstanding this caveat, Equation (8) is quite informative as it helps clarify why the steady state aggregate demand for assets likely echoes the property derived for asset demand by active households. The term $h(r)$ in Equation (8) is the identical term used to capture the steady state asset-to-income ratio of active households. Although retired households have different consumption-savings incentives than active households, the assets with which they start their retirement reflect the decisions of active households. This explains why the term $h(r)$ plays such a prominent role in the aggregate demand for assets. In effect, the aggregate demand for assets tends to depart from that of active households only due to the term $\frac{\phi+(1-\phi)g(r)}{\phi+(1-\phi)g(r)(\delta_1 q)(\rho+\delta_1-r)^{\frac{1}{\sigma}}}$ in Equation (8). While it is difficult to derive simple analytical properties for Equation (8), it is easy to explore its quantitative properties. In fact, when using the range of parameters presented

[38] To express this aggregate demand in levels, we have assumed away aggregate growth. It is straightforward to relax this assumption, but requires detrending level variables by the growth path.

in Section 4, we have found that the aggregate demand for assets given by Equation (8) mimics very closely the properties of $h(r)$, in particular it maintains a C-shape when $\sigma > 1$.[39] Accordingly, this motivates us to discuss general equilibrium implications under the assumption that the aggregate demand for assets as given by Equation (8) is C-shaped.

Since we aim to highlight only quantitative general equilibrium implications of having an aggregate C-shaped asset demand, we can adopt a rather simple structure for the aggregate asset supply. To this end, let us allow for only two types of assets: a short-term government bond in quantity B[40] and a mass one of Lucas trees, where the Lucas trees pay dividend f. By arbitrage, the price of the Lucas tree will be given by $\frac{f}{r}$ in steady state and hence the aggregate supply of assets, denoted A^s, can be expressed as

$$A^s = B + \frac{f}{r} \tag{9}$$

This aggregate supply of assets can be thought of as the bond equivalent supply of assets, where the Lucas trees are converted to bonds at price $\frac{f}{r}$. In this environment, aggregate income will be given by $\frac{w}{\phi} + f$, that is, labor income plus dividends from trees. The equilibrium determination of interest rates and of aggregate wealth $(B + \frac{f}{r})$ is therefore found by equating expressions in (8) and (9).[41] When $\sigma > 1$, the aggregate demand for assets takes the C-shaped form. The aggregate supply of assets is monotonically decreasing in interest rates as higher interest rates decrease the price of the Lucas trees. Figure 9 depicts a situation where the supply of assets can be cutting the C-shaped demand for asset in either its upper or lower portion. When the asset supply cuts it in the upper portion (as illustrated by brown curves), this can be thought of as a conventional configuration where locally, the demand curve for assets is sloping up and the supply curve is sloping down as functions of interest rates. For this configuration, our framework has nothing especially new to highlight. However, when the demand for assets is sufficiently strong, or the supply of safe asset equivalents is sufficiently weak, the aggregate supply of assets will cut the

[39]Furthermore, when σ is in the 2-3 range, the inflexion point is around 3%.

[40]In our derivation of the aggregate demand for assets, we assumed that payments on government bonds were supported by a tax on active households.

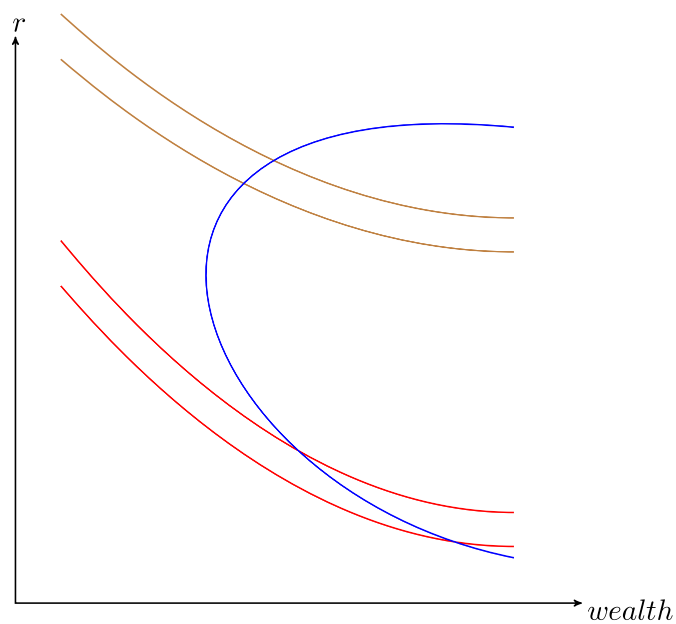
[41]The equilibrium condition for interest rates can be expressed as

$$\frac{B}{Y} + \frac{f}{Yr} = \frac{h(r)}{\phi}\left(\frac{\phi + (1-\phi)g(r)}{\phi + (1-\phi)g(r)(\delta_1 q)(\rho + \delta_1 - r)^{\frac{1}{\sigma}}}\right)$$

.

This formulation highlights the potential roles of the government debt-to-GDP ratio, $\frac{B}{Y}$, and the capital share of income, $\frac{f}{Y} = \frac{f}{\frac{w}{\phi} + f}$, in affecting interest rates.

Figure 9
The different effects of changes in asset supply on wealth



demand on the lower portion of the C-shape. This is illustrated by the red curves and is the configuration we want to focus on here. Note that it is also possible for the supply curve to cut the demand curve more than once, which implies the existence of multiple steady state levels of real interest rates. While this configuration opens the door to many intriguing issues, we will not pursue it further here.[42]

5.1 Steady State General Equilibrium Comparative Statics When Asset Demands Are C-shaped

Let us start by focusing on the red lines in Figure 9 and contrast how the effects of a reduction in asset supply – say due to a reduction in B – differ in this configuration relative to the case where the reduction happens when the supply curve cuts the demand curve in the upper portion of the C-shape (brown lines). In both cases, a reduction in safe asset supply B will lead to a reduction in equilibrium interest rates. However, in the case depicted

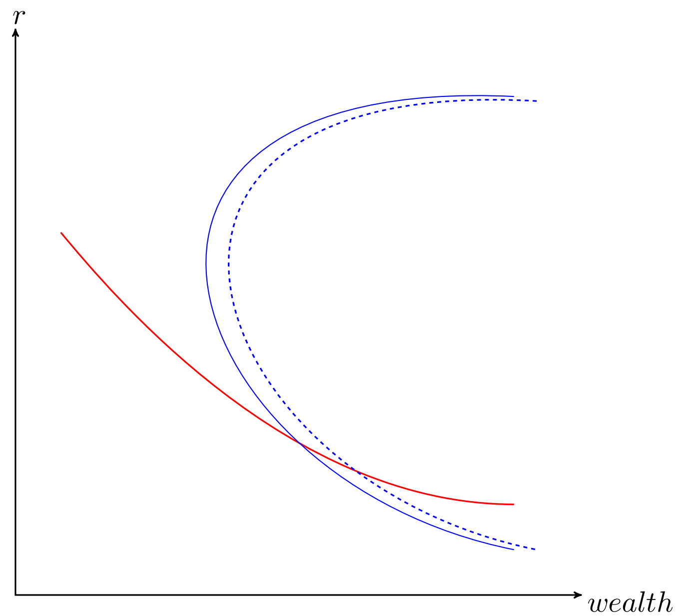
[42]Some of the implications of multiple steady state interest rates are explored in Beaudry et al. (2023). Using the environment with sticky prices the authors also discuss potential role for monetary policy to influence the long-run evolution of real rates.

in red, this reduction in supply actually induces a sufficiently strong reduction in interest rates that, in equilibrium, total wealth increases. This increase in wealth arises despite the direct effect on wealth being negative. Such an outcome is potentially interesting given the large literature that purports to explain the observed decrease in real interest rates in the first two decades of the 2000s as reflecting a decrease in the supply of safe assets (see, for example, Caballero and Farhi (2018)). If a reduction in the supply of assets occurred in the upward (as a function of r) sloping portion of the demand for assets, this would imply that total wealth should have been observed to decrease due to the reduction in safe assets. However, during the period of decreasing interest rates in the 2000s, as we have discussed, total wealth increased substantially. A C-shaped asset demand offers a potential reconciliation whereby a reduction in safe asset supply could explain the decrease in interest rates while at the same time being consistent with an induced valuation effect that is large enough to explain the observed increase in wealth.

The second general equilibrium implication of C-shaped demand curves we want to highlight relates to their potential role in amplifying the effects of demand changes on interest rates and wealth. For example, consider an increase in longevity, as captured by a decrease in δ_2. Increased longevity implies a rightward shift in the aggregate demand for assets. If this shift happens when the economy is operating on the upper portion of the C-shaped asset demand, then the general equilibrium effect of such a change on household wealth holdings will be smaller than the partial equilibrium effect (i.e., when holding interest rates constant). However, as we noted in Section 4, the partial equilibrium effect on asset demand of the increase in longevity observed over the last thirty years is likely rather small, so in general equilibrium, it would be even smaller in such a case. In contrast, if the economy is operating on the lower segment of a C-shaped asset demand, then even a small partial equilibrium effect on asset demand can potentially cause a large change in interest rates and thereby induce a large change in household-level asset holdings. This is depicted in Figure 10. While from a partial equilibrium perspective we found that increased longevity was an unlikely explanation for the observed within-group increase in wealth holdings, from a general equilibrium perspective, it may have played a larger role as the potential source of the decrease in interest rates, which we argued favored more accumulation of wealth.

In summary, embedding C-shaped asset demands into general equilibrium has the potential to explain better how decreases in safe asset supply and/or factors such as increases in longevity may have played an important role in causing simultaneous large falls in interest rates and large increases in household-level wealth holdings. In both of these cases, if the asset demand curve is the more conventional positively sloped function of interest rates,

Figure 10
The multiplier effect on wealth and interest rates of changes in asset demand



these changes are unlikely to explain such joint patterns.[43]

6 Conclusion

A good comprehension of long-run asset demand is necessary for a better understanding of low-frequency movements in interest rates. In this paper, we have used a quantitative theory to put forward the notion that long-run asset demands are plausibly C-shaped and that the gradual fall in interest rates observed from the late 1980 up until the COVID period likely favored an increase in household-level demand for wealth. To support this point, we began by documenting how households – with the same demographics and income – appear to have targeted greater wealth-to-income ratios in 2019 than in 1989. We then presented

[43]An alternative perspective on equilibrium determination in this economy involves the goods market. We can define a (steady state) aggregate goods demand function AG^d as

$$AG^d = [\frac{h(r)}{\phi}(\frac{\phi + (1-\phi)g(r)}{\phi + (1-\phi)g(r)(\delta_1 q)(\rho + \delta_1 - r)^{\frac{1}{\sigma}}})]^{-1}[B + \frac{f}{r}]$$

In the case of a unique equilibrium, this function is monotonically decreasing in r even if asset demands are C-shaped. In the case of multiple equilibria, it becomes non-monotonic.

a tractable model of wealth accumulation to explore different potential explanations for this observation. The model combines both inter-temporal substitution and retirement motives. An attractive feature of our simple setup is that it allows for a straightforward quantification of competing mechanisms. While increased longevity and decreased aggregate growth can partly explain the observed increases in desired wealth levels, we found their effects to be too small. In contrast, we found the fall in real interest rates observed between 1989 and 2019 to be of the right size to potentially account for the majority of the observed increase in wealth holdings. This quantitative finding arises when adopting estimates of the elasticity of inter-temporal substitution slightly below .5. Finally, on the aggregate front, we discussed how C-shaped asset demands can help explain, among others, how a decrease in the availability of safe assets could have caused simultaneous decreases in interest rates and increases in overall wealth, and how small increases in longevity could explain large falls in real interest rates.

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Appendix

A Data

For the main analysis we use four waves of the US Survey of Consumer Finances for 1989, 1992, 2016 and 2019. The 1989 and 2019 SCFs are used for the wealth-to-income ratio decomposition into between- and within-group components, while the 1989-1992 and 2016-2019 SCFs are used in the construction of saving rates corresponding to the beginning (1989) and the end (2019) of our period of interest for the joint analysis of changes in wealth-to-income ratios and saving rates. We further supplement the findings based solely on SCF micro-data with the results that combine SCF with household sector aggregates reported in the Fnancial Accounts (FA) of the Federal Reserve and the National Income and Product Accounts (NIPA).

Household wealth in the SCF is defined to include all assets of households (both real and financial) net of their liabilities. On the one hand, household non-financial (real) assets include primary and other residential real estate, non-residential real estate equity, as well as equity holdings in privately held businesses (both corporate and non-corporate) and other non-financial assets. Financial assets, on the other hand, include directly held stocks and bonds, defined contribution pension accounts and investment funds, for which we can define the split into fixed-income and equity components, and other financial assets. While SCF collects information about the types of pensions households are entitled to (account or traditional pensions), the estimates of the wealth in defined benefit plans are not directly available. Given the importance of these plans in household pension wealth, we use estimates from Sabelhaus and Volz (2020) to construct a measure of wealth in SCF that includes defined benefit pensions, and use aggregate shares from detailed FA pension accounts to split them into fixed-income vs. equity components, similar to defined contribution account pensions. Unlike other papers, we also do not exclude vehicles as a measure of consumer durables from household wealth in the SCF, given its importance for less wealthy households, which makes our measure of saving closer to the concept used by the Financial Accounts.[44] On the liability side, we include both mortgage and non-mortgage household debt obligations. While in the main text we conduct analysis using measures of wealth, inclusive of all components and net of housing wealth, our preferred measure of wealth includes housing. This is also the measure of wealth we use to construct saving rates in the SCF. While it is possible to exclude the components of housing wealth – both in assets and liabilities – from the construction of saving rates, the relevance of this measure in comparison to other studies using standard measures of saving from the data is less clear. Saving in housing expressed as the net new housing also represents a non-trivial component of saving.

When combining SCF with household-sector aggregates from the Financial Accounts, we follow the literature in consistently defining detailed asset and liability classes in SCF and aggregate data, and then creating a larger number of asset/liability classes (see, for example, Mian et al. (2021)). The same grouping into a larger number of asset/liability classes is also useful for the construction of saving rates in raw SCF data, given that pure inflation factors from Mian et al. (2021) are defined for the same asset and liability classes. To create scaled SCF estimates we then construct each group's share in the total value of each asset/liability category and distribute FA aggregates between groups using these shares. Each group's net worth is summed up using the

[44] In comparison to NIPA saving rate, the FA rate is more noisy. However, the dynamics over time of these saving rates are quite similar.

values for each component. On the income side, we follow a similar approach by aggregating each group's income from its components, e.g., wages, business income, interest and dividend income, etc. Similar to the assets/liabilities we do adjustments to the income components reported in SCF to make them consistent with their aggregate counterparts. See Feiveson and Sabelhaus (2019) for the discussion of the comparison between different components of wealth/income reported in FA/NIPA and SCF.[45]

Overall, we prefer working with the SCF-based estimates given that they allow us to construct consistent wealth-to-income ratios and saving rates (in particular, adjusting for net bequests, which can only be constructed in SCF) from the same data source. However, as shown in the main text our wealth-to-income ratio decomposition results are largely unchanged when we use scaled SCF estimates, as in the literature. The scaled results in the aggregate do provide a better fit with the saving rates obtained from FA/NIPA. This is why together with the main results for correlations between group-wise changes in wealth-to-income ratios and changes in saving rates using raw SCF data, we provide additional evidence using scaled data as well.

Other data we use for the empirical analysis include pure price inflation factors from Mian et al. (2020), whose replication package provides them until 2016. We extend the series until 2019 using their methodology for different asset categories.[46] Since Mian et al. (2020) measures of wealth and saving do not include consumer durables, we also use an additional factor for consumer durables, and test the results for robustness to its different values.

B Robustness Results for the Wealth-to-income Ratio Change Decomposition

B.1 Shift-share Decomposition: Alternative Groupings

In this section, we present robustness results associated with using a different number of income-age groups (in Table B1) and using 2019 as a base-year (in Table B2) for the shift-share decomposition. We use raw SCF data and our baseline measure of wealth, inclusive of all the components.

The 10 income groups are defined as follows: 0-20, 20-40, 40-60, 60-80, 80-120, 120-160, 160-200, 200-250, 250-500, 500+ (000, in 2019 \$); while in the 12 income groups the top group is also split into the following additional groups: 500-750, 750-1250, 1250+ (000, in 2019 \$). The 15 income groups further split the top 1250+ bracket into 1250-1750, 1750-3000, 3000-15000, and 15000+ (000, in 2019 \$). The six age groups split the 65+ age category into 65-74 and 75+ years.

In the first panel of Table B2 for comparison with Auclert et al. (2021) we present results for 12 age groups; in the second panel of the table we report results for different combinations of age and income groups.

[45] We only use the aggregates for different balance sheet and income categories directly reported in FA and NIPA, instead of combining some measures from the aggregate accounts and others from SCF, as done in some papers, which argue that the aggregate survey-based measures may be preferred for certain categories. But our results in the main text do not substantially change when we also use a combination approach.

[46] For the pure inflation factors on the liability side, however, we are unable to extend the series, and use the last available data point from 2016 for the additional years of interest.

Table B1
Shift Share Decomposition of the Change in the Aggregate Wealth-to-Income Ratio Between 1989 and 2019: Robustness to number of age-income groups

Grouping	Total Change	Within %	Between %
10 inc gr x 6 age gr	2.82	59.4	40.6
12 inc gr x 6 age gr	2.82	54.9	45.1
15 inc gr x 5 age gr	2.82	51.8	48.2

Table B2
Shift Share Decomposition of the Change in the Aggregate Wealth-to-Income Ratio Between 1989 and 2019: Robustness to base year of income/wealth profiles

Definition	Total Change	Within (%)	Between (%)
	12 Age groups		
1989 base	2.82	65.1	34.9
2019 base	2.82	52.1	47.9
	30 Income-age groups		
1989 base	2.82	61.6	38.4
2019 base	2.82	42.9	57.1
	60 Income-age groups		
1989 base	2.82	59.4	40.6
2019 base	2.82	45.5	54.5
	72 Income-age groups		
1989 base	2.82	54.9	45.1
2019 base	2.82	46.2	53.8
	75 Income-age groups		
1989 base	2.82	51.8	48.2
2019 base	2.82	42.8	57.2

B.2 Regression-based Decomposition Approach

As the alternative approach to the simple shift-share decomposition presented in the main text, we use the 1989 cross section to estimate a wealth holding function, which we denote by $F_{89}(age, y)$, where as previously *age* represents the age of the household head and y represents real income of a household. Function F can take different forms. In this section, we focus on the polynomial function F in income and age. Then, for each household in the 2019 cross section, we use estimated function $F_{89}(age, y)$ to create a predicted wealth holding, which we denote by $\hat{w}_{19}$. These predicted wealth levels allow us to create a predicted wealth-to-income ratio in

2019 by adding up $\hat{w}_{19}$ across households, and by dividing it by the aggregate income in 2019 (denoted $\left(\frac{\hat{w}}{y}\right)_{19}$). By using the same prediction function for the wealth in 2019, as in 1989, the predicted ratio reflects only the changes in the proportions of different groups in the population. Accordingly, the fraction of the change in the wealth-to-income ratio explained by the within component can be expressed as

$$1 - \left[\frac{\left(\frac{\hat{w}}{y}\right)_{19} - \left(\frac{w}{y}\right)_{89}}{\left(\frac{w}{y}\right)_{19} - \left(\frac{w}{y}\right)_{89}} \right]. \tag{B1}$$

In Table B3, we report the results of using expression (B1) with our baseline measure of wealth as well as baseline wealth less housing. Using a fifth order polynomial in income and age to build predicted wealth, we find that the between-group component accounts for between 40 and 42 percent of the increase in the aggregate wealth-to-income ratio, leaving slightly under 60 percent for the within-group component.[47] In Table B4, we also show that proportions implied by (B1) are similar when using a regression of wealth on a set of thirty dummy variables for income and age groups (6 income and 5 age groups as in the main text), which we refer to as a step-function regression approach.

Table B3
Total Change in the Aggregate Wealth-to-Income Ratio Between 1989 and 2019 and the Fraction of the Change due to Within and Between Effects: Decomposition Using Polynomial Regression Approach

Definition	Total Change	Within (%)	Between (%)
Wealth (baseline)	2.819	59.8	40.2
Wealth less housing	2.649	57.2	42.8

Table B4
Total Change in the Aggregate Wealth-to-Income Ratio Between 1989 and 2019 and the Fraction of the Change due to Within and Between Effects: Decomposition Using Step-function Regression Approach

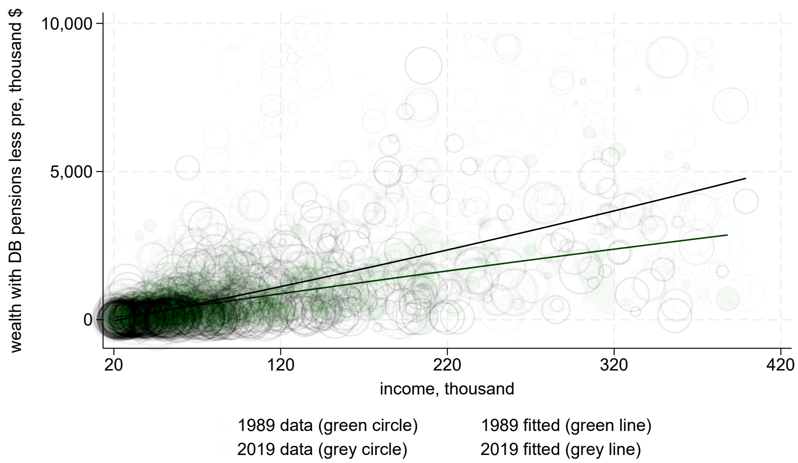
Definition	Total Change	Within (%)	Between (%)
Wealth (baseline)	2.819	64.8	35.2
Wealth less housing	2.649	63.7	36.3

[47] We have run our predictive regressions using polynomials of order 3, 4, and 5. Polynomial function of order 5 delivers the best prediction.

While together these findings still support an important role of changes in demographics and income inequality in explaining movements in the wealth-to-income ratio, they indicate that an even greater share is due to changes in wealth holdings keeping income and age constant.

B.3 Relationship between Wealth and Income for Incomes from 20 000$ to 1 000 000$

Figure B1



C Proofs of Propositions and Lemmas

C.1 Proof of Proposition 1

We first prove that asset holdings of active households converge to the long-run asset holdings $a^{a,ss}(y,r)$ and then characterize the properties of $a^{a,ss}(y,r)$.

Convergence of active households' asset holdings to $a^{a,ss}(y,r)$. Let's recall the dynamics of the optimization problem when r is fixed.

$$\dot{c}_t = \left(\frac{r-\rho-\delta_1}{\sigma}\right)c_t + \frac{c_t^{\sigma+1}}{\sigma}\delta_1 q a_t^{-\sigma}\Gamma^{\sigma},$$

$$\dot{a}_t = r_t a_t + w_t - T_t - c_t,$$

Linearizing this system around the steady state ($\dot{c}_t = 0$, and $\dot{a}_t = 0$) with $r_t = r$ leads to the dynamic system:

$$\begin{pmatrix} \dot{\hat{c}}_t \\ \dot{\hat{a}}_t \end{pmatrix} = \underbrace{\begin{bmatrix} \rho+\delta_1-r & -\frac{c}{a}(\rho+\delta_1-r) \\ -1 & r \end{bmatrix}}_{J \text{ Jacobian evaluated at the steady state}} \begin{pmatrix} \hat{c}_t \\ \hat{a}_t \end{pmatrix},$$

where $\hat{x}_t \equiv x_t - x$ means the deviation of a variable x_t from its steady state x, and $\rho + \delta_1 - r = \delta_1 q c^{\sigma} a^{-\sigma} \Gamma^{\sigma}$.

The determinant of J is given by

$$det(J) = (\rho + \delta_1 - r)\left(r - \frac{c}{a}\right)$$

If $r < \frac{c}{a}$ in the steady state, then $det(J) < 0$, implying that the steady state is saddle stable since $det(J) = \lambda_1 \lambda_2$ and the eigenvalues (λ_1, λ_2) have opposite signs.

Combining $\rho + \delta_1 - r = \delta_1 q c^{\sigma} a^{-\sigma} \Gamma^{\sigma}$ and $\Gamma^{-1} = \frac{\rho+\delta_2}{\sigma} - \frac{1-\sigma}{\sigma} r$ leads to

$$\frac{c}{a} = \left[\left(\frac{\rho + \delta_1 - r}{\delta_1 q}\right)\left(\frac{\rho + \delta_2}{\sigma} - \frac{1-\sigma}{\sigma} r\right)^{\sigma}\right]^{\frac{1}{\sigma}}.$$

Note that this equation also defines the long-run wealth-to-income ratio $\frac{a^{a,ss}}{y}(r)$ when disposable income y equals c.

Therefore, a necessary condition for the convergence toward $\frac{a^{a,ss}}{y}(r)$ is

$$r < \left[\left(\frac{\rho + \delta_1 - r}{\delta_1 q}\right)\left(\frac{\rho + \delta_2}{\sigma} - \frac{1-\sigma}{\sigma} r\right)^{\sigma}\right]^{\frac{1}{\sigma}}.$$

The sufficient condition is

$$\max\{r, 0\} < \left[\left(\frac{\rho + \delta_1 - r}{\delta_1 q}\right)\left(\frac{\rho + \delta_2}{\sigma} - \frac{1-\sigma}{\sigma} r\right)^{\sigma}\right]^{\frac{1}{\sigma}},$$

where $\max\{r, 0\}$ guarantees consumption to be non-negative.

Properties of $\frac{a^{a,ss}}{y}(r)$. Recall the steady state wealth-to-income ratio

$$\frac{a^{a,ss}}{y}(r) = (\delta_1 q)^{\frac{1}{\sigma}} \left[\frac{\rho + \delta_2}{\sigma} - \frac{1-\sigma}{\sigma} r\right]^{-1} [\rho + \delta_1 - r]^{\frac{-1}{\sigma}}.$$

Taking the derivative of $\frac{a^{a,ss}}{y}(r)$ with respect to r, we have

$$\frac{d\frac{a^{a,ss}}{y}(r)}{dr} = (\delta_1 q)^{\frac{1}{\sigma}} (\rho + \delta_1 - r)^{\frac{-1}{\sigma} - 1} \left(\frac{1}{\rho + \delta_2 + (\sigma - 1) r}\right)\left[1 - \frac{\sigma(\sigma-1)(\rho + \delta_1 - r)}{\rho + \delta_2 + (\sigma - 1) r}\right].$$

If $\sigma \leq 1$, $\frac{\frac{a^{a,ss}}{y}(r)}{dr} \geq 0$ and hence the steady state asset wealth-to-income ratio of active households is increasing in the interest rate.

Now let us assume that $\sigma > 1$. When $r = \bar{r}$, we have $\frac{d\frac{a^{a,ss}}{y}(r)}{dr} = 0$ where

$$\bar{r} \equiv \frac{\sigma(\sigma-1)(\rho + \delta_1) - (\rho + \delta_2)}{(\sigma - 1)(\sigma + 1)}.$$

If $r > \bar{r}$, $\frac{d\frac{a^{a,ss}}{y}(r)}{dr} > 0$. And if $r < \bar{r}$, $\frac{d\frac{a^{a,ss}}{y}(r)}{dr} < 0$. As a result, $\frac{a^{a,ss}}{y}(r)$ is increasing (decreasing) in the interest rate when r is above (below) $\bar{r}$. Hence, $\frac{a^{a,ss}}{y}(r)$ is C-shaped in the space (r, a).

Q.E.D.

D Appendix: Deriving the Steady State Aggregate demand for assets for both active and retired households

To derive the aggregate demand for assets, let us begin by normalizing the population to have a measure 1 of households, with the implied fraction $\phi \equiv \frac{\delta_2}{\delta_1 q + \delta_2}$ who are active and the fraction $1-\phi$ who are retired. When a household dies it is replaced by a new active household. Since we have not introduced annuity markets, private agents will have positive asset holdings when they die and therefore there will be unintended bequests. We assume that the unintended bequest of a household goes to the newborn household replacing that household. We also assume away defined benefit pension plans and treat each household as solely responsible for their retirement savings. To keep the structure more tractable, we assume that the government ensures — through a tax T_{2t} on active households — that all newborn households receive the same bequest.[48] Under this assumption, if asset holdings are equal across active households at a point in time, then the system inherits a representative agent structure for active households.[49,50]

We can now determine total asset demands in this economy in a steady state with constant interest rates and taxes. This demand is comprised of both the long-run (per household) asset demand function of active households, $a^{a,ss}(y, r)$, and that of retired households, which in its total amount will be denoted $a^{r,ss}$. For simplicity, let us consider the case with growth first. The steady state asset demand function of active households when interest rates are constant is given in Proposition 1 and can be stated as $a^{a,ss}(y, r) = h(r)y$, where $h(r) = \sigma(\delta_1 q)^{\frac{1}{\sigma}}(\rho + \delta_2 + (\sigma - 1)r)^{-1}(\rho + \delta_1 - r)^{\frac{-1}{\sigma}}$, with $y = c$ in steady state. Since long-run asset demands relative to consumption of active households go to ∞ when r goes to either $\rho + \delta_1$ or $-\frac{\rho+\delta_2}{\sigma-1}$, we restrict attention to situations where $r \in \left(-\frac{\rho+\delta_2}{\sigma-1}, \rho + \delta_1\right)$ as this is the only feasible range for a steady state equilibrium.

To get the steady state asset demand for retired households, we need to aggregate the asset holdings across the different retirement cohorts. With $r < \rho + \delta_1 \leq \rho + \delta_2$, retired households will be depleting their asset holdings as they age. In particular, this will cause the current asset

[48] We are also assuming away pensions.This is without much loss of generality, since in general equilibrium, pensions must be paid for and therefore are just an alternative form of accumulation.

[49] Assuming that active households act like a large family as in Gertler et al. (2020) would also lead to maintaining the tractability of the representative agent structure.

[50] The tax on active households needed to ensure that all newborns receive the same bequest is determined by the budget constraint

$$\delta_1(1-q)\phi a_t + \delta_2(a^r) = [\delta_1(1-q)\phi + \delta_2(1-\phi)]\, a_t - \phi T_{2t}.$$

where a^r denotes the aggregate asset holding of retirees. The first term on the left hand side of this equation refers to the total funds received from accidental bequests. On the right hand side, the first term refers to the funds needed to be given to newborn active households, while the second term is the tax levied on all active households to equalize wealth inherited by newborns from retired and active households. Rearranging the equation, we obtain that $T_{2t} = \delta_2(a^r_t)/\phi$.

holdings of a retired household who retired τ periods ago with a assets to be given by $ae^{-(\frac{\rho+\delta_2-r}{\sigma})\tau}$. Furthermore, note that the aggregate consumption of retirees satisfies the relationship $c_t^r = a_t^r\Gamma^{-1}$, where a_t^r is the total assets held be retirees at time t. Since in steady state, each retiree starts retirement with the same amount of assets, which is equal to the steady state asset holdings of active households ($a^{a,ss}$), the aggregate asset holdings of retirees ($a^{r,ss}$) are given by

$$a^{r,ss} = a^{a,ss}(y,r)(1-\phi)\frac{\delta_2}{\frac{\rho+\delta_2-r}{\sigma}+\delta_2} = a^{a,ss}(y,r)(1-\phi)g(r)$$

where $g(r) \equiv \frac{\delta_2}{\frac{\rho+\delta_2-r}{\sigma}+\delta_2}$.

As a result, total asset demand in the steady state of this economy can be expressed as

$$a^{tot,ss}(y,r) = \phi a^{a,ss}(y,r)\left(1+\frac{g(r)(1-\phi)}{\phi}\right).$$

This expresses aggregate asset demand as a function of the total income y of active households. However, y itself depends on a^{ss} and therefore to get an expression for the demand of households that depends on fundamentals, we need to use the goods market clearing condition. The goods market equilibrium condition is given by

$$\phi c = \phi w_+ f - a^r\Gamma^{-1}$$

where $a^r\Gamma^{-1}$ is the aggregate consumption of retirees. Using goods market equilibrium condition to replace $c = y$ in the active households' asset demand, the aggregate asset demand can now be expressed as

$$h(r)(w+\frac{f}{\phi})(\frac{\phi+(1-\phi)g(r)}{\phi+(1-\phi)g(r)(\delta_1 q)(\rho+\delta_1-r)^{\frac{1}{\sigma}}})$$

where $\phi w + f$ is aggregate income. Recall that we have not introduced aggregate growth in this formulation. Allowing for aggregate growth is straightforward, but requires expressing aggregate asset demand in deviation from a growth path.[51]

[51] The out of steady state aggregate dynamics for this economy can be represented by the following set of equations. This formulation continues to assume the use of taxes that ensure that all newborn households receive the same inheritance and that active households are taxed to pay for government spending and public debt payments.

$$\frac{\dot{c}_t}{c_t} = \frac{r_t-\rho-\delta_1}{\sigma} + \frac{c_t^\sigma}{\sigma}\delta_1 q V_a(a_t,\Gamma_t) \qquad \dot{\Gamma}_t = -1+\Gamma_t\left[\frac{\rho+\delta_2}{\sigma} - \frac{1-\sigma}{\sigma}r_t\right] \qquad \frac{\dot{z}_t}{z_t} = \frac{f}{z_t} - r_t$$

$$\dot{a}_t = w + r_t a_t - T_t - c_t \qquad \text{where taxes are} \qquad T_t = \frac{(a_t - B - z_t)\delta_2}{\phi} + \frac{G+Br_t}{\phi}$$

plus the goods market clearing condition

$$\phi c_t = \phi w + f - G - (B + z_t - \phi a_t)\Gamma_t^{-1}$$

where c_t is the consumption of the representative active household and a_t is its asset holdings. The aggregate consumption of retirees is given by $(B + z_t - \phi a_t)\Gamma_t^{-1}$ and their aggregate wealth holdings are given by $(B + z_t - \phi a)$.

E Transitional dynamics

The results of the quantitative evaluation using wealth holdings implied by the transitional dynamics after 40 years of work, as opposed to their steady state levels, are summarized in Tables E5 and E6. As can be seen from these tables, the implied asset-to-income ratios after 40 periods give similar insights as their steady-state counterparts presented in Tables 6 and 7.

Table E5

Predicted levels of the asset-to-income ratio in 1989 and 2019, change in asset-to-income ratios from 1989 to 2019 as a function of (ρ, σ): After 40 periods

σ	1.0001	1.8	2	2.2	2.4	2.6	2.8	3.0
	Predicted asset-to-income ratio in 1989 (r=.03)							
$\rho = 0.02$	6.98	9.92	10.55	11.13	11.69	12.21	12.70	13.17
$\rho = 0.03$	4.94	8.31	9.00	9.65	10.26	10.83	11.37	11.87
$\rho = 0.04$	3.67	7.10	7.82	8.49	9.12	9.72	10.28	10.80
$\rho = 0.05$	2.84	6.15	6.87	7.55	8.19	8.79	9.36	9.90
	Predicted asset-to-income ratio in 2019 (r=0)							
$\rho = 0.02$	4.66	11.97	13.88	15.80	17.74	19.68	21.62	23.56
$\rho = 0.03$	3.52	9.88	11.58	13.30	15.03	16.76	18.50	20.24
$\rho = 0.04$	2.74	8.32	9.85	12.39	12.95	14.52	16.09	17.67
$\rho = 0.05$	2.20	7.12	8.50	9.90	11.32	12.75	14.19	15.63
	Implied percentage change in asset-to-income ratios in 1989-2019							
$\rho = 0.02$	-33.29	20.65	31.61	41.94	51.76	61.16	70.22	78.98
$\rho = 0.03$	-28.76	18.95	28.62	37.74	46.72	54.73	62.73	70.48
$\rho = 0.04$	-25.31	17.29	25.97	34.15	41.95	49.41	56.61	63.57
$\rho = 0.05$	-22.50	15.82	23.68	31.11	38.19	44.98	51.53	57.87

Note: The changes in the implied asset-to-income ratio in 2019 are calculated for the cases when real interest rate falls to 0, growth rate falls to 1.5%, and post-retirement life expectancy $(1/\delta_2)$ rises to 19.6 years.

Table E6

Percentage changes in the asset-to-income ratios (A/Y) in 1989 and 2019 when real rate falls to 0, growth rate falls to 1.5%, and post-retirement life expectancy rises to 19.6 years as a function of (ρ, σ): After 40 periods

σ	1.0001	1.8	2	2.2	2.4	2.6	2.8	3.0
	Decline in real rate from 3% to 0%							
$\rho = 0.02$	-43.76	3.93	13.82	23.18	32.12	40.69	48.96	56.98
$\rho = 0.03$	-38.71	4.07	12.93	21.31	29.31	36.99	44.40	51.57
$\rho = 0.04$	-34.69	3.93	11.94	19.54	26.80	33.77	40.49	47.01
$\rho = 0.05$	-31.32	3.70	11.01	17.95	24.60	30.98	37.15	43.13
	Decline in growth rate from 2% to 1.5%							
$\rho = 0.02$	15.03	10.12	9.36	8.72	8.16	7.67	7.24	6.86
$\rho = 0.03$	11.59	8.26	7.72	7.25	6.84	6.48	6.15	5.87
$\rho = 0.04$	9.57	7.08	6.65	6.28	5.95	5.66	5.40	5.16
$\rho = 0.05$	8.15	6.24	5.89	5.58	5.31	5.06	4.84	4.64
	Increase in post-retirement longevity from 17.2 to 19.6 yrs							
$\rho = 0.02$	13.60	15.97	17.01	18.23	19.61	21.15	22.87	24.76
$\rho = 0.03$	10.14	12.34	13.32	14.44	15.71	17.14	18.71	20.44
$\rho = 0.04$	8.25	9.78	10.61	11.58	12.71	13.97	15.38	16.94
$\rho = 0.05$	7.12	7.98	8.63	9.44	10.40	11.49	12.73	14.12

Note: Changes in the implied asset-to-income ratios are reported when changing only one parameter at a time. In each panel of the table, the remaining parameters are kept constant at the 1989 levels.