FINANCE IN A TIME OF DISRUPTIVE GROWTH

Nicolae B. Gârleanu
Stavros Panageas

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ABSTRACT

We propose a unified theory of asset price determination encompassing both "conventional" and "alternative" asset classes (private equity, real estate, etc.). The model features disruption of old by young firms and skewness in the distribution of innovative rents among the young innovators. The relative size of asset classes, the dynamics of rich investors' wealth, and the returns of the various asset classes are jointly determined in equilibrium. Besides explaining the observed patterns of returns across asset classes, we analyze the theoretical properties of the most widely used performance-evaluation measure for alternative investments. We also provide connections between the growth of alternative investments, the dispersion of returns across investors, and the turnover inside the ranks of wealthy individuals.

Nicolae B. Gârleanu
Olin Business School
Washington University in St. Louis
1 Brookings Drive
St. Louis, MO 63130
and NBER
garleanu@wustl.edu

Stavros Panageas
Anderson School of Management
University of California, Los Angeles
110 Westwood Plaza
Los Angeles, CA 90095-1481
and NBER
stavros.panageas@anderson.ucla.edu

1 Introduction

Entrepreneurial risk and, more broadly, equity participation in non-publicly traded corporations are many times the origin of spectacular accession to wealth. Indeed, the returns on private-equity (PE) positions are generally recognized as a leading driver behind the wealth dynamics of the ultra-rich. In turn, the allure of these returns spurred the remarkable growth of private equity funds, which have become ever more important over the last four decades. Furthermore, the quest for alternatives to public equities did not stop at private equities, but also led to the growth of such asset classes as real estate and commodities.

Despite the increased importance of all these asset classes, there has not been a lot of work devoted to a unified theoretical understanding of their basic features. In this paper, we wish to provide answers to such questions as: What determines the equilibrium returns of these alternative investments and how do they compare to those on public equities? How should one evaluate the attractiveness of these investments? How does the emergence of this new investment landscape affect wealth dynamics and the return dispersion among investors? What can the joint dynamics of the growing investment share in alternative investments, the dispersion of returns, and the turnover inside the population of rich investors teach us about the fundamental trends affecting the economy?

We propose a unified theory to address these questions. The model has three key features: (a) innovation is disruptive and leads to redistribution from existing-firm owners towards newly arriving innovators, (b) the gains from innovation are extremely skewed and accrue to a very small fraction of the new innovators, and (c) there is imperfect risk sharing that does not allow agents to eliminate the impact of these distributional risks on their wealth. The model is "neoclassical" in that all expected returns reflect exclusively compensation for risk. However, unlike in representative-agent economies, the notion of risk in our paper encompasses both aggregate and distributional risks. In particular, these distributional risks play a central role for the determination of expected returns, the size of the financial industry, and the wealth dynamics of the ultra-rich.

We next provide more detail on the setup of the model and summarize its implications for asset pricing and the dynamics of inequality. The backdrop is an infinite-horizon, stochastic, discrete-time, perpetual-youth, general equilibrium model of creative destruction. A stochastic amount of new blueprints arrives (exogenously) each period and leads to the creation of new firms. These blueprints raise aggregate production, but also displace the profits of old firms. The ownership rights to the arriving blueprints are allocated either to existing publicly traded firms or to newly arriving agents. The allocation of blueprints to new agents is random and highly skewed. A small number of innovators end up with the profitable blueprints, while the rest receive trivial allocations. As a result, the newly arrived innovators are eager to share the allocation risk with (old) investors, by offering their firm's shares for sale. They are, however, subject to an agency friction that prevents them from selling their entire firm. The transfer of fractional ownership from new to old investors is facilitated by financial intermediaries ("private-equity funds") who purchase a portfolio of the new-firm shares on behalf of investors. This diversification is limited, in that each intermediary can only invest in a subset of new firms.

There are therefore two impediments to perfect risk sharing, an inter- and an intra-cohort impediment. At the inter-cohort level, the fraction of the random endowment retained by the newly arriving entrepreneurs is larger than zero. At the intra-cohort level, each investor obtains a different rate of return on their private equity portfolio, depending on the subset of private firms in which they invest. Inside the model this is the most important difference between investments in public and private equity: all investors obtain the same return in their public equities, but their private-equity returns are dispersed. In the next section, where we summarize some empirical facts underpinning the model, we show that this difference in dispersion is a property of the data. At a theoretical level, an attractive feature of the model is that it captures as special cases the essence of several models in the literature, including the perfect risk-sharing limit (Rubinstein (1976), Lucas (1978)), the Constantinides and Duffie (1996) model, and the OLG model of Gârleanu et al. (2012).

The key implications of the model can be summarized as follows. First, the model provides a unified, risk-based explanation of the returns obtained in conventional and alternative asset classes. This is not a straightforward task, because some cross-sectional patterns in the public-equity market appear inconsistent with private-equity markets: On the one hand the value premium requires that the "growth options" embedded in the price of publicly traded growth firms command a comparatively low risk premium. On the other hand, this explanation of the value premium would also seem to imply that investments in venture-capital funds, which are presumably growth-option intensive, should consistently underperform any portfolio of public equities. The intuition is that these dynamic, new firms will displace some of the older, established firms at some point. Accordingly, an investment in new ventures should act as a hedge for the stock market, and should command a rate of return lower than public equities. In the data there is a paucity of such evidence.[1] The model reconciles a positive value premium for public equities with a private-equity premium that can be positive or negative. While the displacement risk of old firms by young firms is a concern to the investor, she also recognizes that the subset of firms in her private-equity portfolio might not be among the small subset of new firms that will end up displacing the old firms. Compared to public equities, investors' private-equity portfolio returns exhibit substantial cross-sectional dispersion (both in the model and the data), which implies that some amount of idiosyncratic risk is retained in these portfolios. Therefore, investments in new ventures expose the investor to idiosyncratic risk, without necessarily being an effective hedge for displacement risk.

Second, the model can be used to study the theoretical properties of some popular methods for "risk-adjusting" and evaluating the "outperformance" of private equity investments. Most popular among those approaches is the public market equivalent approach (PME) of Kaplan and Schoar (2005), which involves discounting private equity cashflows by the cumulative returns of the stock market and then dividing the sum of the discounted values by the amount invested. Values above one are interpreted as an indication of risk-adjusted outperformance. We show that the expected value of the PME must exceed unity, even under the assumptions that the literature identifies as most favorable for the validity of the PME (in particular, unitary risk aversion). The fact that the expected value of the PME is above one (even in the absence of any outperformance) is actually quite general and independent of the specifics of the model. The core intuition involves the fact that the public equity, used as discount factor, does not fully capture a marginal investor's portfolio risk.

Third, the model predicts that asset classes that are immune to displacement, such as commodities or real estate, should have lower expected returns than either public or private equities. The reason is that the profits of such factors of production are not tied to a specific blueprint, but are useful to all blueprints, young and old alike. By being useful to all firms, they are not affected by the distributional shocks that impact either publicly traded or private equity. In that sense, these asset classes are effective hedges against displacement

[1]While there is disagreement in the empirical literature whether the "alpha" of venture capital returns (regressed on the market portfolio) is statistically different from zero or positive, there is no obvious evidence that this alpha is significantly negative. At an even more basic level, if the returns of venture capital are compared to the returns of similar firms in the public equity market (for instance the returns of the "small growth" portfolio), venture capital returns are on average higher.

risk and command a low risk premium. This ranking of expected returns of commodities and real estate at the bottom followed by growth stocks, value stocks, and private equity seems consistent with the performance of these asset classes over the past couple of decades.[2]

Fourth, the same key feature of the model that drives its asset-pricing predictions is also responsible for its ability to explain some intriguing properties of wealth dynamics. As we show in the next section, the ultra-rich investors that are added to the Forbes 400 over every five-year period exhibit a wealth dispersion similar to the investors already in the list. Phrased differently, new entrants don't just enter the distribution of the ultra-rich at its lower ranks, they enter at all ranks. This appears inconsistent with the notion that the wealth growth of these entrants follows a diffusion process, since then one would expect the new entrants to replace predominantly the individuals at the lower ranks of the existing distribution of the rich. Our model can help account for this fact, since the wealth dispersion among the rich occurs predominantly in the early stages of their accession to wealth. More broadly, inequality in the model is primarily driven by the churn of rich investors, i.e., the replacement of old rich by new rich. The old rich don't exhibit spectacular wealth-growth rates, as we discuss in greater detail in the next section.

Fifth, the model makes predictions about the *joint* behavior of (a) aggregate growth, (b) inequality, (c) asset returns, and (d) the size of the private-equity industry. The joint nature of these predictions allows one to make qualitative inferences about the type of fundamental changes affecting the economy at different times. For example, an acceleration of disruptive activity raises aggregate growth, increases the investment share of private equity, and also boosts the wealth share of newly-rich entrepreneurs, broadly consistent with the experience of the late nineties. By contrast, the continued growth of the private-equity industry after 2000 was accompanied by a more moderate TFP growth, a deceleration in the rate of displacement of old by new rich, and less dispersed returns for PE investors. The type of fundamental change that could account for these events inside the model would be an increased effectiveness of the PE industry, not a broad acceleration of disruption.

Finally, the model makes a technical contribution. Specifically, in modeling the random endowments of innovators we borrow a construction that is quite popular in non-parametric Bayesian statistics (and more recently in natural language processing), namely the Gamma-Dirichlet-process construction. This construction allows us to model random distributions

[2] See Mauboussin and Callahan (2020), Exhibit 4, which depicts the performance of various asset classes from 1984-2015.

with the property that idiosyncratic risk cannot be easily eliminated, even in large portfolios. The resulting dispersion of the returns of these portfolios matches quite well the dispersion we encounter in the data; this return dispersion also allows idiosyncratic risk to be priced in an entirely neoclassical, arbitrage-free, equilibrium model.

The rest of the paper is structured as follows. In Section 2, we summarize several empirical facts that motivate the model's key assumptions and form the targets of our calibration. Section 3 lays out the model, while Section 4 provides its solution and implications. In Section 5 we derive the model's predictions for a wide set of asset classes and value-growth portfolios, and in Section 6 we calibrate the model and compare its quantitative implications to the data. Finally, Section 7 concludes.

1.1 Literature review

Our paper relates to several strands of the literature.

Methodologically, the paper belongs to the well-developed literature that uses macroeconomic models to price the cross section of returns, and especially the size and value premium. One of the early contributions to this literature is Gomes et al. (2003), which develops a general equilibrium version of Berk et al. (1999), while more recent contributions include Papanikolaou (2011), Gârleanu et al. (2012), Gârleanu et al. (2012), and Kogan et al. (2020). Opp (2019) presents a tractable, macroeconomic model with venture capital, but focuses on different issues than we do in this paper. One of the goals of our paper is to extend this literature to a wider set of asset classes, and provide a resolution to the seemingly inconsistent pricing of growth options across public and private equities. A key role in our paper is played by creative destruction and "displacement risk," as in Gârleanu et al. (2012) and Kogan et al. (2020). There is a small literature that studies the impact of entry and imperfect competition on asset prices. Indicative examples are Loualiche (2021), Corhay et al. (2020), Dou et al. (2021), and Bena et al. (2015). Gârleanu et al. (2012) can be construed as a special case of this paper, obtaining if the market for private equity is shut down. Similarly, relative to Kogan et al. (2020) we introduce a market where entrepreneurs can trade their equity stakes with existing investors.

The lack of both inter-cohort and intra-cohort risk sharing plays an important role for the pricing of risk in our paper. There are two large and developed strands of the literature in asset pricing that pursue the asset-pricing implications of imperfections in both intergenerational[3] and intra-generational risk sharing.[4] Similar to Constantinides and Duffie (1996), higher moments of idiosyncratic risk play a key role in our paper. However, we make entirely different endowment assumptions and allow some limited sharing of these endowment risks through the private-equity market.[5]

The (positively) skewed distribution of blueprints plays an important role in our paper. The skewness of idiosyncratic shocks is also a key element of such papers as Schmidt (2015), Constantinides and Ghosh (2017), and Ai and Bhandari (2021), except that the emphasis is on the *negative* skewness of labor income. Toda and Walsh (2019) and Gomez (2017) focus on the interaction between inequality and asset returns, similar to this paper.

A distinguishing feature of our model (compared to the literature on risk-sharing imperfections) is that we model the financial industry as a vehicle that facilitates transfers both within and across generations of entrepreneurs (behind the "veil of ignorance" about which firms are likely to be profitable). Purely from a technical perspective, our approach of modeling the financial industry as a facilitator of risk sharing resembles Gârleanu et al. (2015). However, the model in the present paper is intertemporal, features lack of both intra- and inter-cohort risk sharing, the arrival of new blueprints follows an extremely skewed distribution (a gamma process, as opposed to the Brownian-bridge construction in Gârleanu et al. (2015)), there is aggregate risk, and the model is amenable to calibration because of the usage of standard, homothetic utilities.[6]

The tradability of private equity shares (which is necessary to discuss the model's asset pricing implications) is also the main feature that distinguishes our model from the large number of models of entrepreneurial equity. In those models each entrepreneur invests exclusively in her own firm.[7] Even if one allowed selling of shares between investors in that literature, the normal distribution of idiosyncratic shocks would make idiosyncratic risk ef-

[3]Indicative examples of asset pricing papers featuring lack of intergenerational risk sharing are Abel (2003), Krueger and Kubler (2006), Geanakoplos et al. (2004), Campbell and Nosbusch (2007), Storesletten et al. (2007), Constantinides et al. (2002), Gomes and Michaelides (2005), Gârleanu and Panageas (2015), Schneider (2022), Maurer (2017), Ehling et al. (2018), Farmer (2018), Gârleanu et al. (2012), Gârleanu and Panageas (2023), Gârleanu and Panageas (2021).

[4]The asset prcing literature featuring uninsurable idiosyncratic shocks is vast and we do not attempt to summarize it here. Panageas (2020) provides a recent survey.

[5]See Krueger and Lustig (2010) on the importance of endowment assumptions in Constantinides and Duffie (1996).

[6]The idea that some investors retain some location-specific risk is reminiscent of Vayanos and Vila (2021), except that in their specification the analog of a "location" is a bond maturity.

[7]We don't attempt to summarize this literature here. An indicative example of an entrepreneurial equity model is Angeletos (2007). See also Di Tella and Hall (2021) for a more recent contribution.

fectively vanish through diversification, therefore making private-equity investments have the same return as public-equity investments.

While the study of inequality is not the primary focus of this paper, our model's assumptions are consistent with some recent studies of wealth inequality. Similar to Gomez (2023), we emphasize the role of entry, displacement, and churn in top wealth shares. Similar to Gabaix et al. (2016), we emphasize the importance of positively skewed, jump-like, idiosyncratic shocks, except that in our framework the positive skewness affects predominantly the entering cohorts. Gouin-Bonenfant et al. (2023) studies the interaction of asset-price determination and wealth dispersion, and in particular the subsets of the population that are affected by asset price fluctuations. Irie (2024) studies the interaction between entrepreneurial finance and wealth inequality, but in a framework that does not feature different returns across asset classes.

Our paper has implications for the evaluation of investments in alternative asset classes. Our finding that the quite popular PME method of Kaplan and Schoar (2005) has an expectation larger than one is novel, to the best of our knowledge. The interesting aspect of this result is that we can sign the direction of the bias, and that the bias obtains despite our making the most favorable assumptions for the validity of the PME.[8] Our model suggests using an investor's return as basis for a simple alternative to the PME measure. Korteweg et al. (2023) implements such an approach.

Our model abstracts from "illiquidity" when valuing alternative asset classes.[9] Illiquidity presents an alternative explanation for the seemingly inconsistent cross-sectional pricing of growth options across public and private equities. We chose to abstract from considerations of illiquidity, because (a) we wish to show that, even in the absence of liquidity considerations, the large cross-sectional dispersion of returns in alternative asset classes is a channel that can both qualitatively and quantitatively account for the observed PME values and (b) the absence of liquidity considerations helps us illustrate that the PME evaluation method is an imperfect risk-adjustment method, since it has an expectation larger than one even in a world where any excess return is exclusively a reward for risk. We also note that the illiquidity of alternative asset classes can have ambiguous effects on their prices: While some

[8]Korteweg and Nagel (2016), Sorensen et al. (2014), Gupta and Van Nieuwerburgh (2021), and Korteweg et al. (2023) present alternatives to the PME approach. Sorensen and Jagannathan (2015) show that the PME validity requires the special assumption of logarithmic utilities. Our result is that even with logarithmic utilities the PME has an expectation larger than one.

[9]See, e.g., Ang et al. (2014).

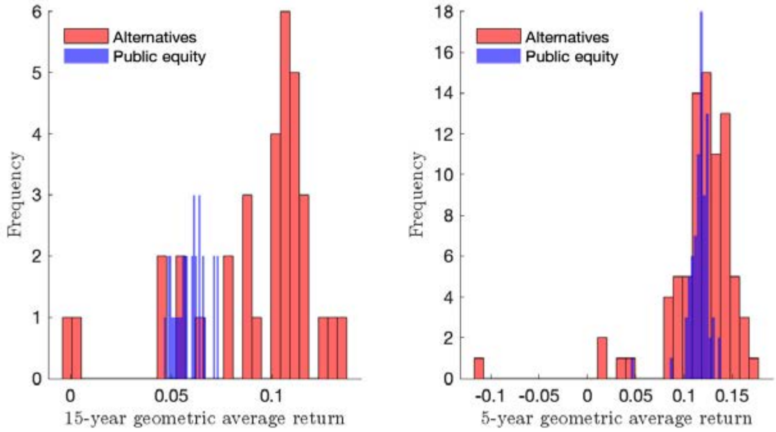

Figure 1: Return dispersion of the investments of public pension plans in public equity and alternative asset classes. Left plot: 15-year geometric averages (2003-2018). Right plot: five-year geometric averages (2013-2018).

investors fear illiquidity, some long-term institutional investors who are subject to regulatory capital requirements and face convex costs of raising equity may be attracted by the fact that private equity positions are not subject to mark-to-market requirements. In addition, during the last decade the market for secondary sales of private equity positions by limited partners has increased dramatically, which has considerably improved the liquidity in the market.[10]

2 Motivating Evidence

Before presenting the model, we briefly highlight three empirical facts to motivate the model assumptions. We revisit these facts (and provide more details on the data) in Section 6, when we calibrate the model.

First, we document that the returns of alternative-asset-class investments tend to be far more dispersed (across investors) than public-equity returns. Second, we show that the wealth dynamics of new entrants into the population of the ultra-rich appear to exhibit jump-like features. Third, we document that the wealth dynamics of a fixed cohort of existing rich

[10]The rapid growth in the market for "secondaries" in the last 10 years has greatly improved the ability of limited partners to liquidate their private-equity positions. Yet, the PME values have not changed dramatically over this period, which suggests that illiquidity is not the primary driver behind the PME values of private equity. Additionally, the data from secondary transactions shows that discounts (compared to net asset value, NAV) are generally larger for venture capital investments than for buyout investments; yet the PME values of buyout funds are generally larger than VC funds. See, e.g., Jefferies (2023).

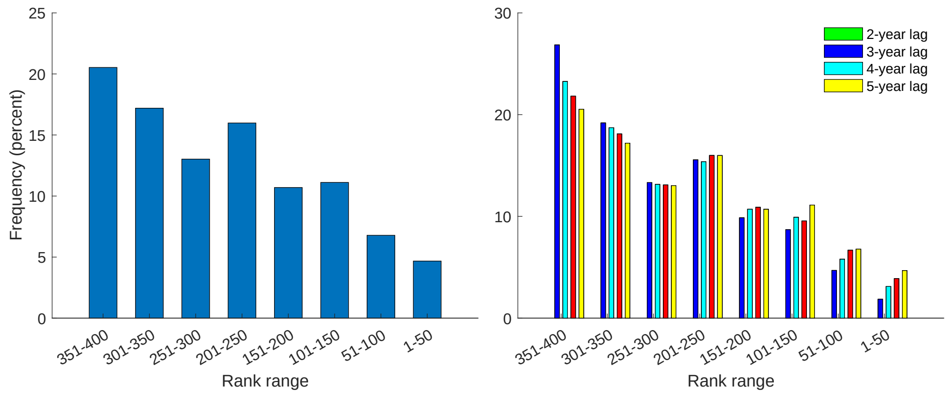

Figure 2: Proportion of self-made billionaires entering the Forbes 400 list in any of eight ranked buckets. The left plot reports results for individuals entering the list during the previous $N = 5$ years. The right plot shortens the window, considering entrants over the previous $N = 2 - 5$ years.

investors roughly align with the S&P 500 returns minus 2% per annum.

To establish the first fact, we examine the returns reported by public pension plans on their public equity and their alternative-asset-class investments. Public pension plans constitute an attractive source of information, since they are required to file comprehensive annual financial reports ("CAFR") about the performance of each of their investments and are subject to "Freedom of Information Act" (FOIA) requests.[11]

Figure 1 shows a histogram of the returns obtained by public pension funds on their public (blue bars) and their alternative (red bars) investments. To mitigate concerns that our results could be driven by the absence of mark-to-market values for the net asset values of unrealized exits, we compute a (geometric) average of the returns in both asset classes over periods of 5 years (right plot) and 15 years (left plot).[12]

The figure shows that the returns obtained by these large institutional investors on their public-equities investments is not very dispersed. However, there is substantial dispersion in the returns that these investors obtain in alternative asset classes. This higher dispersion is quantitatively large: For 15-year geometric averages, the alternative-asset-class returns across pension funds can range from 0% to 13.5%. For comparison, the range of values for public equities is considerably narrower, stretching between 4.7% and 7.3%. In Section 6 we

[11]The source of the data is "Public Plans Data. 2001-2022," Center for Retirement Research at Boston College, MissionSquare Research Institute, National Association of State Retirement Administrators, and Government Finance Officers Association.

[12]To avoid that our results are influenced by the Covid years, we report results for the 5-year and 15-year period ending in 2018. We choose 2018 rather than 2019, because the accounting periods for many pension plans close mid-year.

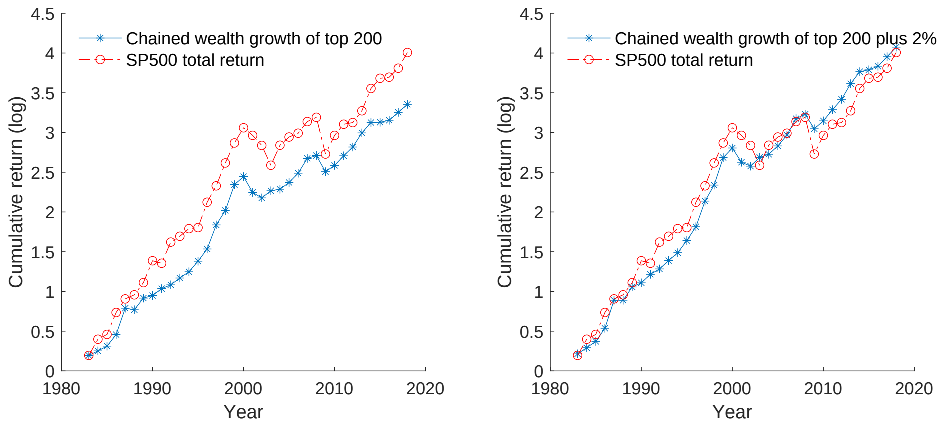

Figure 3: Left panel: Chained wealth growth of the top-200 group of individuals in the Forbes 400 list and S&P 500 total return. Right panel: Cumulative chained return on wealth to top-200 individuals plus 2% and S&P 500 total return. To compute wealth growth and return on wealth we divide the average time-$t+1$ wealth of time-t top-200 individuals still in the Forbes 400 list at time $t+1$ by the average time-t wealth of all time-t top-200 individuals.

confirm this finding using a different data set that contains information on the internal rates of return (IRR) of the private-equity investments of each pension plan. Previewing results, the dispersion of private-equity returns across investors plays an important role in our model for driving a wedge between the expected return of public and private equity.

Since our model contains implications for the dynamics of the ultra-rich, the second fact we document pertains to the dynamics of entrants into the Forbes 400 list. Forbes follows more than one thousand individuals and reports a list of the 400 wealthiest ones each year. Figure 2 reports a remarkable feature concerning the entry of rich individuals into this list. The left plot of the figure depicts the empirical distribution of the ranks of these entrants in the Forbes 400 list upon their entry. Since in our model calibration (Section 6) we use 5-year periods, we consider each of the years $t = 1999, \dots, 2013$ and identify the new entrants into the list over the time interval $(t-5, t]$. We use only entrants tagged by Forbes as "self made." The remarkable feature of Figure 2 for our purposes is the depth of penetration of new entrants even in the right-most tail of the distribution. The individuals that entered the top 400 list don't predominantly occupy the 350-400 ranks, but rather reach even the high ranks of the distribution of the existing rich. For instance, only 20.5% find themselves in the 350-400 range, while 22.6% populate the ranks 150 and above. Remarkably, this penetration of recent entrants into all ranks of the distribution of the ultra-rich does not change much if we shorten the entry interval to cover only the previous $N \in \{2, \dots, 5\}$ years (right plot). This depth of penetration motivates the extreme endowment skewness assumptions (for entering cohorts) that we make in the model.

The final motivating fact pertains to the average wealth dynamics of the existing rich. Specifically, the average return obtained by the extremely rich is quite similar to the S&P 500 total return, as illustrated by Figure 3. This figure graphs, with a continuous line, a measure of return on wealth computed as follows. Every year t, we take the top-200 individuals in the list and compute their average wealth in year t as well as, for the ones still on the list in year $t+1$, their average wealth in year $t+1$. (We focus on the top 200 to mitigate the effects of attrition.[13]) The left plot shows that the wealth growth of the already-rich individuals is somewhat smaller than the total return of the S&P 500. The right plot shows that the difference is approximately 2%: If we interpret this 2% as a consumption-to-wealth ratio, then the figure shows that the pre-consumption aggregate wealth growth of the existing rich lines up closely with the return of the market portfolio.

Taken together, Figures 2 and 3 motivate the model's assumption that the rents from new firm creation are skewed and cause entry into the distribution of the ultra rich by new entrepreneurs (even into the highest ranks of this distribution). By contrast, the aggregate wealth growth of the existing rich does not differ substantially from the returns of the assets already contained in the index.

3 Model

We next present the model. For ease of reference, we compile a list of the definitions of the main mathematical symbols at the end of paper.

3.1 Agent preferences and demographics

We consider a model with discrete and infinite time: $t = \{\ldots, 0, 1, 2, \ldots\}$. The size of the population is normalized to one. At each date a mass λ of agents are born, and each pre-existing agent may die with probability λ, independently of all other agents. Consequently, a mass λ of agents die and the population remains constant. We denote by $V_{t,s}$ the time-t utility of an agent born at time s. Investors have Epstein-Zin-Weil preferences with a unitary

[13] Since Forbes reports the 400 richest individuals, there is a high likelihood that individuals in the top 200 list in year t will still be in the top 400 list in year $t+1$, so we can observe their wealth. This group of 200 individuals may include more than 200 names, due to ties.

intertemporal elasticity of substitution and a risk aversion equal to γ:

$$\log V_{t,s} = (1 - \beta(1-\lambda)) \log c_{t,s} + \beta(1-\lambda) \log \mathcal{R}(V_{t+1,s}), \tag{1}$$

where $\mathcal{R}(x) = (\mathrm{E}_t[x^{1-\gamma}])^{\frac{1}{1-\gamma}}$. We focus on the empirically relevant case in which the risk aversion, γ, exceeds one, $\gamma > 1$. The subjective discount factor, β, lies in $(0,1)$. The quantity $c_{t,s}$ is the agent's consumption at time t. Since the second term on the right hand side of (1) depends on the product of β and $(1-\lambda)$, it is useful to define $\hat{\beta} = \beta(1-\lambda)$ as the agent's effective discount factor.[14] In the limit case $\gamma = 1$ the life-time utility of the agent takes the simpler form $\mathrm{E}_s \sum_{t=s}^{\infty} \hat{\beta}^{t-s} \log c_{t,s}$.

3.2 Technology

To expedite the presentation of the main results, we follow the common practice in asset pricing of simply making assumptions on the dynamics of aggregate output, the share of profits and the share of labor as simple endowments. Appendix B shows how the postulates of this section can be micro-founded by utilizing a standard "creative destruction" model featuring production and an expanding variety of "blueprints" for the production of new intermediate goods.

Specifically, we assume an increasing, stochastic process A_t of blueprints. Total output, Y_t, increases in the number of blueprints,

$$Y_t = Z_t A_t^{1-\alpha} \text{ with } \alpha \in (0,1). \tag{2}$$

The logarithm of the process Z_t is a random walk with dynamics $\Delta \log Z_{t+1} = \varepsilon_{t+1}$, where ε_{t+1} is normally distributed. A fraction $\alpha(1-\alpha)$ of the output is paid out as profits to the owners of blueprints. The remaining fraction is paid out as labor income. Accordingly, the stream of profits, π_t, per blueprint is

$$\pi_t = \frac{\alpha(1-\alpha) Y_t}{A_t} = \alpha(1-\alpha) Z_t A_t^{-\alpha}. \tag{3}$$

Equation (2) shows that output, Y_t, is an increasing function of A_t, while by equation (3) profits per blueprint, π_t, are decreasing in A_t. This is the sense in which our simple specifi-

[14] Gârleanu and Panageas (2015), Online Appendix D, discusses recursive utilities with mortality risk and provides a justification for equation (1), i.e., specifying recursive utility as if investors were infinitely lived, but with a discount factor equal to $\hat{\beta} = \beta(1-\lambda)$.

cation captures the idea of displacement of old blueprints by new ones.

The owners of blueprints create firms that own the profit stream from those blueprints. Shares of these firms can be traded in financial markets.

3.3 Allocation of blueprints

The measure λ of newly born agents are of two types: a fraction θ are entrepreneurs and a fraction $1-\theta$ are workers. Workers collect the labor share of output, $1-\alpha(1-\alpha)$. Since workers are not the focus of the paper, we assume that they are "hand-to-mouth" consumers, i.e., they simply consume their endowment and don't participate in financial markets. This assumption is not essential for the results, and we relax it in Section 5.4.

The agents who participate in financial markets are the entrepreneurs, whose endowment takes the form of blueprints. Specifically, each period a total mass

$$\Delta A_{t+1} = A_{t+1} - A_t = \eta A_t \Gamma_{t+1} = \eta A_t \left(\Gamma^N_{t+1} + \Gamma^E_{t+1} + \Gamma^U_{t+1}\right) \tag{4}$$

of new blueprints arrives. The proportional increment $\frac{\Delta A_{t+1}}{A_t}$ is captured by the random variable $\eta\Gamma_{t+1}$ and consists of three components, $\eta\Gamma^E_{t+1}$, $\eta\Gamma^N_{t+1}$, and $\eta\Gamma^U_{t+1}$. Here, Γ^l_{t+1}, $l \in \{E, N, U\}$ are independent gamma distributed variables with shape parameters a^l and rate parameters b^l, and η is a constant. Since we want to focus on the asset-pricing implications of the model, we simplify matters and assume an exogenous arrival of blueprints, which are randomly allocated to the agents in the economy. We next specify the allocation of the three categories of blueprints.

The new blueprints in the amount of $\eta A_t \Gamma^E_{t+1}$ capture innovation that occurs within existing firms. Specifically, we assume that the A_t blueprints of time-t firms increase to $A_t(1+\eta\Gamma^E_{t+1})$ next period.[15]

The blueprints $\eta A_t \Gamma^N_{t+1}$ are the main focus of our analysis. They capture the arrival of new, private-equity-backed firms. Specifically, these blueprints accrue at time $t+1$ to entrepreneurs born at time t. The crucial aspect for our analysis is that the new blueprints are allocated randomly to these entrepreneurs. To capture this randomness, we index entrepreneurs by a "location" $i \in [0,1)$ on a circle with circumference normalized to one. The number of blueprints distributed to entrepreneurs in location i is $\eta A_t d\Gamma^N_{i,t+1}$, where $d\Gamma^N_{i,t+1}$

[15] Given our assumptions of frictionless trading of existing firms, the distribution of these blueprints to existing firms is irrelevant. Since the representative investor holds all existing firms, the stochastic discount factor is affected only by the total number of blueprints allocated to existing firms, not the distribution.

denotes the increments of a gamma process, so that the total number of Γ_{t+1}^N blueprints is $\int_0^1 d\Gamma_{i,t+1}^N = \Gamma_{t+1}^N$. (For technical reasons, we assume that there is a continuum of entrepreneurs of mass 1 in each location i sharing equally the blueprints accruing to location i.[16]) Since the gamma process is not commonly used in economics, we summarize briefly some of its properties. To build intuition, we consider a discrete construction. We split the interval $[0, 1)$ into K equal intervals, and think of the gamma process at the location $\frac{k}{K}$ as a sum, $\sum_{n=1}^k \xi_{\frac{n}{K}}$, of gamma-distributed, independent increments $\xi_{\frac{n}{K}}$, where the pdf of the increment ξ_i is given by the gamma distribution:

$$Pr(\xi_i \in dx) = \frac{b^{\frac{a}{K}}}{\Gamma\left(\frac{a}{K}\right)} x^{\frac{a}{K}-1} e^{-bx}\, dx. \tag{5}$$

The parameters $\frac{a}{K}$ and b are sometimes refered to as the shape and the rate of the gamma distribution, and $\Gamma(\cdot)$ is the gamma function. The increments ξ_i are indepenent of each other, and the properties of the gamma distribution imply

$$Pr\left(\sum_{n=1}^{K} \xi_{\frac{n}{K}} \in dx\right) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}\, dx, \tag{6}$$

which is the distribution of a gamma variable with shape a and rate b. In short, the gamma process exploits the "infinite divisibility" of gamma distributions.[17]

Using the gamma process is technically attractive for our purposes, since it captures in a stylized way the notion that entrepreneurship is very risky. This is illustrated in the left plot of Figure 4. The figure shows a sample of increments ξ_i for the case where the interval $[0, 1]$ is split into $K = 1000$ subintervals. The figure illustrates that only a small and random subset of locations exhibit big spikes (of random height), while in all other locations the increments are so small that they are not even visible in the plot. From an economic point of view, this means that only the lucky few entrepreneurs, who happen to find themselves in the locations exhibiting the large spikes, obtain a non-trivial allocation of blueprints.

The limit of the sum $\sum_{n=1}^k \xi_{\frac{n}{K}}$ as the number of locations K goes to infinity is a gamma process. It is a positive and increasing process, whose paths are nowhere continuous (they are, however, right continuous with left limits). The process increases on a measure-zero, but

[16] This assumption ensures price-taking behavior, the ability of annuity companies to break even by an appropriate law of large numbers, etc.

[17] The right-hand side of (6) does not depend on the number of partitions, K. This illustrates the property of infinite divisibility.

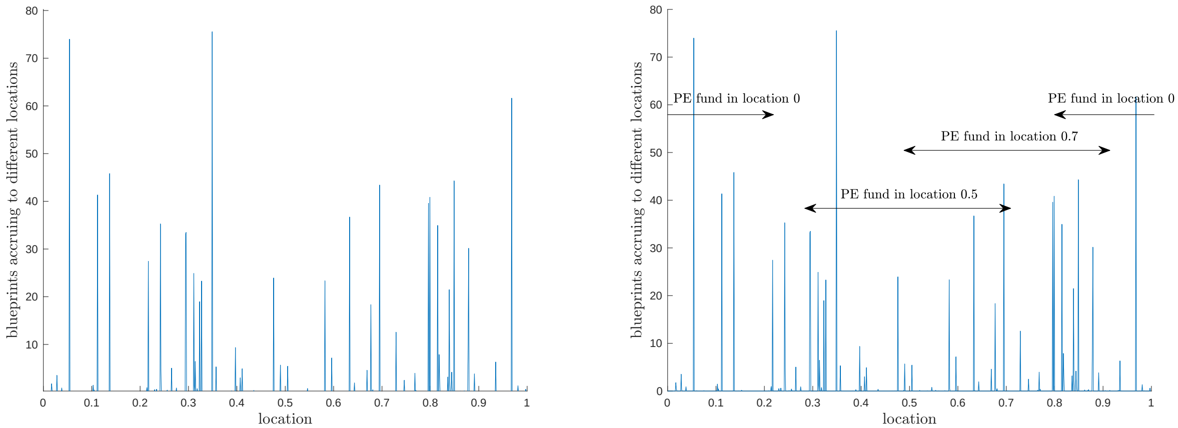

Figure 4: Left plot: An illustration of the increments ξ_i, for the case $K = 1000$, $a = 15$, and $b = 2$. Right plot: An illustration of how private equity funds help with risk sharing, when $\Delta = 0.4$. The private equity fund in position 0.5 provides an equal weighted portfolio of the increments in [0.3, 07]. The private equity fund in location 0 averages the increments in $[0, 0.2] \cup [0.8, 1)$.

dense, subset of the interval $[0, 1)$. This implies that even though the process increases over any given subinterval of $[0, 1)$, most increments are small.[18] However, a countable measure of locations receive a non-trivial allocation of blueprints, and the entrepreneurs who find themselves in these locations become quite wealthy.[19]

Before proceeding, we would like to note that this extreme-inequality setup is mostly for illustrative purposes and technical convenience. Less extreme distributions would preserve the economic insights, assuming a sufficiently strong notion of distributional risk.

A crucial assumption is that no agent knows at time t the realization of the path of the gamma process Γ^N_{t+1} at time $t + 1$. In other words, no one knows which locations will receive a non-trivial endowment of blueprints. Because of this uncertainty, entrepreneurs are approached by "private-equity funds" at time t. (We choose the broader term "private-equity funds" rather than "VC funds" for reasons that we explain in Section 5.1.) These private-equity funds offer to buy an equity share in the entrepreneur's blueprints at time t behind the "veil of ignorance" about which locations on the circle will obtain the valuable blueprints at time $t + 1$. Because the probability that any given location will receive a non-trivial endowment of blueprints is zero, the risk-averse entrepreneurs are eager to share their endowment risk by selling shares to private-equity funds. These shares entitle private-equity investors to a fraction υ of the profits that will be produced by the newly arriving blueprints

[18]More precisely, only a finite number of locations experience increments exceeding ε for any $\varepsilon > 0$.
[19]A similar outcome obtains for the non-PE backed blueprints given by Γ^U, which we describe shortly.

in perpetuity. A fraction $1 - \upsilon$ is "inalienable," a reduced form way of capturing incentive effects of equity retention.

Finally, we specify the allocation of the blueprints $\eta A_t \Gamma_{t+1}^U$. These blueprints are meant to capture entrepreneurship that is not backed by private equity. For simplicity, we assume that these blueprints accrue to entrepreneurs born at time $t+1$ (rather than t), so that there is no room for trade with private equity funds behind the veil of ignorance. We note that (a) the inclusion of blueprints that are not backed by private equity is mostly for realism and with an eye towards calibrating the model later on, and (b) the distribution of $\eta A_t \Gamma_{t+1}^U$ across entrepreneurs arriving at time $t+1$ is irrelevant for the description of equilibrium. Only the total size of $\eta A_t \Gamma_{t+1}^U$ matters.[20]

3.4 Markets

At each point in time, an investor can trade a zero net-supply bond. To complete the market with respect to the random death events, we follow Blanchard (1985) and assume that agents can also trade annuities with competitive insurance companies that break even. These annuity contracts entitle an insurance company to collect the wealth $W_{i,t}$ of an agent i in the event that she dies at time t and in exchange provide her with an income stream $\lambda W_{i,t}$ while she is alive. We refer to Blanchard (1985) for further details.

Investors at time t can trade costlessly in the shares of all companies created prior to time t ("public equity"). Aside from shares in existing companies, investors can also buy shares of portfolios sold by competitive, private-equity funds. Each private-equity fund is positioned in a location i on the circle. This fund approaches the entrepreneurs in an arc of length $\Delta \leq 1$ centered at location i and purchases an equally weighted portfolio of ownership rights to the random amount of blueprints that the entrepreneurs will be receiving at time $t+1$. The parameter Δ captures the contractual and monitoring difficulties of signing contracts with a multitude of innovators.

The private equity finances its purchases by selling shares to old investors. (Investors are entrepreneurs from previous periods). Each investor is allocated to a position on the circle and purchases shares in the portfolio offered by the private equity fund positioned in that location. Private equity funds are competitive and break even by selling their portfolios to investors for $\frac{1}{\Delta_i} \int_{i-\frac{\Delta_i}{2}}^{i+\frac{\Delta_i}{2}} \Pi_{j,t}^N dj$, where $\Pi_{j,t}^N$ is the price that the private equity fund must pay

[20]However, this distribution matters for the wealth dispersion. See Section 6.

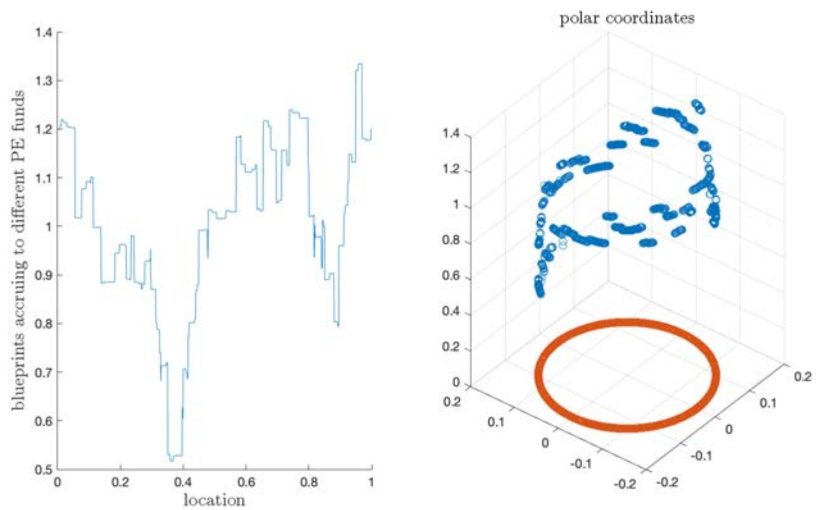

Figure 5: The distribution of equal weighted returns. The left figure depicts the blueprints accruing to the portfolio formed by the private equity fund in each location i, which is simply an equal weighted average of the blueprints accruing to locations in an arc Δ around the private equity fund's location. The right figure is identical to the left figure except that the results are now depicted in polar coordinates.

to the entrepreneur in location j for purchasing an equity share to the blueprints she sells. Assuming a location-invariant equilibrium, such that $\Pi_{j,t}^N = \Pi_t^N$, the price of each portfolio share is just $\frac{1}{\Delta_i}\int_{i-\frac{\Delta_i}{2}}^{i+\frac{\Delta_i}{2}} \Pi_{j,t}^N dj = \Pi_t^N$.

The right plot of Figure 4 and the two plots in Figure 5 illustrate how private equity funds can facilitate risk sharing in this economy. By purchasing an equal-weighted portfolio of shares to blueprints on an arc of length Δ, the private equity funds are able to "smooth out" the spikes of the gamma process. Indeed, as the two figures illustrate, they can offer their investors a portfolio of blueprints that has the same mean as the number of blueprints that arrive in each location, but is second-order stochastically dominant.[21]

In summary, the private equity funds facilitate risk sharing between the old and the new investor-cohorts behind "the veil of ignorance" about future endowments. The parameter υ controls the fraction of shares sold by young to old investors (and thus the extent of "inter-

[21]Specifically, by using properties of the gamma distribution, one can show that $\frac{1}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_j^N$ is gamma distributed with shape $a^N\Delta$ and rate $b^N\Delta$. Accordingly, $\frac{1}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_j^N$ has mean equal to $\frac{a^N}{b^N}$, which is independent of Δ, and standard deviation equal to $\frac{\sqrt{a^N}}{b^N\sqrt{\Delta}}$, which is declining in Δ. Indeed, if $\Delta_2 > \Delta_1$, then $\frac{1}{\Delta_2}\int_{i-\frac{\Delta_2}{2}}^{i+\frac{\Delta_2}{2}} d\Gamma_j^N$ second-order stochastically dominates $\frac{1}{\Delta_1}\int_{i-\frac{\Delta_1}{2}}^{i+\frac{\Delta_1}{2}} d\Gamma_j^N$ (Bawa (1975)). For future reference, we note that the correlation between $\frac{1}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_j^N$ and $\Gamma^N = \int_0^1 d\Gamma_j^N$, is $\sqrt{\Delta}$.

cohort" risk sharing), whereas the parameter Δ controls the dispersion of private equity returns across existing investors (therefore, the extent of "intra-cohort" risk sharing).

3.5 Budget constraints

With these assumptions, the dynamic budget constraint of an investor who resides in location i can be expressed as

$$
\begin{aligned}
W_{i,t} &= S^E_{i,t} A_t \Pi^E_t + B_{i,t} + S^N_{i,t}\Pi^N_t + c_{i,t}, && (7)\\
W_{i,t+1} &= S^E_{i,t} A_t \left(1+\eta\Gamma^E_{t+1}\right)\left(\Pi^E_{t+1}+\pi_{t+1}\right) + (1+r^f_t)B_{i,t} + && (8)\\
&\quad S^N_{i,t} A_t \left(\Pi^E_{t+1}+\pi_{t+1}\right)\frac{\upsilon\eta}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma^N_{i,t+1} + \lambda W_{i,t+1},
\end{aligned}
$$

where Π^E_t is the market value of a blueprint at time t, $S^E_{i,t}$ is the number of shares of the market portfolio, $B_{i,t}$ is the amount invested in bonds, r^f_t the interest rate, $S^N_{i,t}$ is the number of shares purchased in the private equity fund in location i, and Π^N_t was defined earlier as the price of a private-equity share. We note also that $W_{i,t+1}$ is the agent's wealth at $t+1$ conditional on survival and the term $\lambda W_{i,t+1}$ in the second line of (8) represents annuity income. A convenient way to express (8) is

$$
\frac{W_{i,t+1}}{W_{i,t}} = \frac{1-\frac{c_{i,t}}{W_{i,t}}}{1-\lambda}\left(\phi^B_i\left(1+r^f_t\right) + \phi^E_i R^E_{t+1} + \phi^N_i R^N_{i,t+1}\right), \tag{9}
$$

where $\phi^B_i \equiv \frac{B_{i,t}}{W_{i,t}-c_{i,t}}$, $\phi^E_i \equiv \frac{S^E_{i,t}A_t\Pi^E_t}{W_{i,t}-c_{i,t}}$, and $\phi^N_i = \frac{S^N_{i,t}\Pi^N_t}{W_{i,t}-c_{i,t}}$ are the post-consumption wealth shares invested by investor i in bonds, existing firms, and newly arriving firms respectively. $R^E_{t+1} \equiv \frac{\Pi^E_{t+1}+\pi_{t+1}}{\Pi^E_t}\left(1+\eta\Gamma^E_{t+1}\right)$ is the gross return of investing in existing firms, which is the product of the gross return from investing in a fixed blueprint, $\frac{\Pi^E_{t+1}+\pi_{t+1}}{\Pi^E_t}$, and the growth factor $\left(1+\eta\Gamma^E_{t+1}\right)$, which reflects the new blueprints accruing to existing firms at time $t+1$. Finally, $R^N_{i,t+1} \equiv \frac{A_t}{\Pi^N_t}\frac{\eta\upsilon}{\Delta}\left(\Pi^E_{t+1}+\pi_{t+1}\right)\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma^N_{i,t+1}$ is the gross return from investing in newly arriving blueprints, which is the product of the number of new blueprints accruing to the private equity fund, $\eta A_t \int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma^N_{i,t+1}$, the ownership share, υ, and the value $\Pi^E_{t+1}+\pi_{t+1}$ of a blueprint at time $t+1$, divided by the purchase cost of a share, Π^N_t.

An important observation about (9) is that, as long as $\Pi^N_{j,t} = \Pi^N_t$ for all $j \in [0,1)$, the portfolio choices ϕ^B_i, ϕ^E_i, and ϕ^N_i, as well as $\frac{c_{i,t}}{W_{i,t}}$, are the same for all investors, irrespective of their level of wealth and the location where they reside at time t. Phrased differently, if

$\Pi^N_{j,t}$ is the same at all locations, the equilibrium is location invariant.

To ensure that $\Pi^N_{j,t} = \Pi^N_t$ for all $j \in [0,1)$ we make one final, "free entry" assumption: investors can relocate prior to the start of trading in each period, so that the total wealth of all the investors positioned in each location becomes the same.[22] Given the location-invariant nature of the distribution of new blueprints across the circle, the investors have the incentive to position themselves in locations that offer lower prices for a share to the portfolio of new firms. The assumption of free entry equalizes prices across locations, a situation that occurs when the wealth of investors positioned in every location is identical across locations.

3.6 Location-invariant equilibrium

The definition of a location-invariant equilibrium is standard. We normalize the supply of shares to unity and define an equilibrium as a collection of prices Π^E_t and Π^N_t, portfolio allocations ϕ^B, ϕ^E, and ϕ^N, and consumption processes for all agents $c_{j,t}$ such that a) given prices, ϕ^B, ϕ^E, ϕ^N, and $c_{j,t}$ are choices that maximize (1) subject to (9), b) the consumption market clears:[23] $\int_j dc_{j,t} = A_t\pi_t$, c) the markets for all shares (both new and existing) clear: $\int_j dS^E_{j,t} = \int_j dS^N_{j,t} = 1$, and d) the bond market clears: $\int_j dB_{j,t} = 0$.

4 Solution

Next we construct a location-invariant, time-invariant, and symmetric equilibrium in the sense that all agents choose the same portfolio and consumption-to-wealth ratio. Since all investors choose the same portfolio shares ϕ^B, ϕ^E and ϕ^N, bond-market clearing implies $\phi^B = 0$. The i.i.d. nature of shocks also imples that interest rate r^f, the ratios $P^E \equiv \frac{\Pi^E_t}{\pi_t}$, $P^N \equiv \frac{\Pi^N_t}{A_t\pi_t}$, and the consumption-to-wealth ratio $c \equiv \frac{c_{i,t}}{W_{i,t}}$ are the same for all agents and

[22] Mathematically, such a relocation is always possible; one of the infinitely many ways to achieve it is to assign the investor with wealth $W_{j,t}$ to location $F^{-1}(W_{j,t})$, where $F(\cdot)$ is the wealth distribution. Because of the assumption that there is a continuum of investors in each location, this assignment is well defined.

[23] Note that we only need to clear the consumption market for all generations of entrepreneurs, who collectively consume aggregate profits. The remaining consumption accrues to the hand-to-mouth workers.

constant across time. Assuming existence of such an equilibrium, the return R^E_{t+1} is

$$\begin{aligned}
R^E_{t+1} &= \frac{\pi_{t+1} + \Pi^E_{t+1}}{\Pi^E_t}\left(1 + \eta\Gamma^E_{t+1}\right) = \frac{\pi_{t+1}}{\pi_t}\left(\frac{1 + P^E}{P^E}\right)\left(1 + \eta\Gamma^E_{t+1}\right) \\
&= \frac{Z_{t+1}}{Z_t}\left(\frac{1 + P^E}{P^E}\right)\left(\frac{A_{t+1}}{A_t}\right)^{-\alpha}\left(1 + \eta\Gamma^E_{t+1}\right) \\
&= \frac{Z_{t+1}}{Z_t}\left(\frac{1 + P^E}{P^E}\right)\left(1 + \eta\Gamma_{t+1}\right)^{-\alpha}\left(1 + \eta\Gamma^E_{t+1}\right). \qquad (10)
\end{aligned}$$

Moreover, using similar reasoning, the return on a private-equity investment is

$$\begin{aligned}
R^N_{i,t+1} &= \frac{\upsilon A_t\left(\Pi^E_{t+1} + \pi_{t+1}\right)\frac{\eta}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma^N_{i,t+1}}{A_t \pi_t P^N} = \frac{R^E_{t+1}}{1 + \eta\Gamma^E_{t+1}}\frac{P^E}{P^N}\frac{\eta\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma^N_{i,t+1} \\
&= \frac{P^E}{P^N}R^E_{t+1}H_{i,t+1}, \qquad (11)
\end{aligned}$$

where $H_{i,t+1}$ is defined as

$$H_{i,t+1} \equiv \left(1 + \eta\Gamma^E_{t+1}\right)^{-1} \times \frac{\eta\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma^N_{i,t+1}. \qquad (12)$$

Proposition 1 describes explicitly the symmetric time- and location-invariant equilibrium.

Proposition 1 *There exists a location-invariant, time-invariant, and symmetric equilibrium with a constant consumption-to-wealth ratio for all agents given by $c = 1 - \hat{\beta}$, and constant portfolio shares for all agents given by $\phi^B = 0$, $\phi^N = 1 - \phi^E$, and*

$$\phi^E = \frac{E_t\left[\left(R^E_{t+1}\right)^{1-\gamma}\left(1 + H_{i,t+1}\right)^{-\gamma}\right]}{E_t\left[\left(R^E_{t+1}\right)^{1-\gamma}\left(1 + H_{i,t+1}\right)^{1-\gamma}\right]}. \qquad (13)$$

The equilibrium values of P^E and P^N are constant and given by

$$P^E = \phi^E(1 - \hat{\beta})^{-1}\hat{\beta} \qquad (14)$$

$$P^N = \left(1 - \phi^E\right)(1 - \hat{\beta})^{-1}\hat{\beta} \qquad (15)$$

and the interest rate is constant and equals

$$1 + r^f = \frac{E_t\left[\left(R^E_{t+1}\right)^{1-\gamma}\left(1 + H_{i,t+1}\right)^{-\gamma}\right]}{E_t\left[\left(R^E_{t+1}\right)^{-\gamma}\left(1 + H_{i,t+1}\right)^{-\gamma}\right]}. \qquad (16)$$

We analyze the properties of the equilibrium in steps. First, we derive the implications of the equilibrium for risk sharing both within and across cohorts of entrepreneurs. Then we present results on the equilibrium expected returns of existing firms (R^E_{t+1}) and private-equity shares ($R^N_{i,t+1}$), and highlight implications for performance evaluation of private equity. We also discuss implications for the portfolio share of private equity, which we refer to as the (relative) size of private equity investments.

4.1 Risk-sharing implications

To derive the implications of the model for risk sharing we start with a proposition.

Proposition 2 *Aggregate wealth growth is given by*

$$\frac{W_{t+1}}{W_t} = \frac{Z_{t+1}}{Z_t} \left(1 + \eta \Gamma_{t+1}\right)^{1-\alpha}, \tag{17}$$

while an individual investor's wealth growth (conditional on survival) is given by

$$\frac{W_{i,t+1}}{W_{i,t}} = \frac{W_{t+1}}{W_t} \left(\frac{1}{1-\lambda}\right) \left(\frac{1+P^E}{1+P^E+P^N}\right) \left(\frac{1+\eta\Gamma^E_{t+1}+\eta\upsilon\Gamma^N_{t+1}}{1+\eta\Gamma_{t+1}}\right) X_{i,t+1}, \tag{18}$$

where

$$X_{i,t+1} \equiv \frac{1+\eta\Gamma^E_{t+1}+\eta\upsilon\Gamma^N_{t+1}\frac{1}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_{j,t+1}}{1+\eta\Gamma^E_{t+1}+\eta\upsilon\Gamma^N_{t+1}} \text{ and } dL_{j,t+1} \equiv \frac{d\Gamma^N_{j,t+1}}{\Gamma^N_{t+1}}. \tag{19}$$

Equation (18) shows that risk is imperfectly shared both within and across cohorts. The impairment of within-cohort risk sharing is captured by the term $X_{i,t+1}$, which reflects heterogenous returns experienced by existing investors. This heterogeneity is driven by the investors' inability to invest in all arriving blueprints. In the limiting case where $\Delta = 1$, and investors can purchase rights to all blueprints across the circle, the term $X_{i,t+1}$ equals 1, and the within-cohort lack of risk sharing disappears.

However, risk-sharing is limited not only along the intra-cohort dimension, but also across cohorts. Even if $\Delta = 1$, equation (18) shows that individual wealth $\frac{W_{i,t+1}}{W_{i,t}}$ and aggregate wealth $\frac{W_{t+1}}{W_t}$ are not perfectly correlated as long as either (a) entrepreneurs have to retain some equity ($\upsilon < 1$) or (b) some blueprints accrue to entrepreneurs who have no access to the private equity funds ($\Gamma^U_{t+1} \neq 0$). The term $\frac{1+\eta\Gamma^E_{t+1}+\eta\upsilon\Gamma^N_{t+1}}{1+\eta\Gamma^E_{t+1}+\eta\Gamma^N_{t+1}+\eta\Gamma^U_{t+1}}$ in equation (18) captures the inter-cohort lack of risk sharing. In general this term is random and smaller than one, except in the special case where $\upsilon = 1$ and $\Gamma^U_{t+1} \equiv 0$.

In summary, Δ controls the extent of intra-cohort risk sharing, while υ and Γ^U_{t+1} control inter-cohort risk sharing. If $\Delta = \upsilon = 1$, and all entrepreneurs have access to private equity

($\Gamma_{t+1}^U = 0$), then risk is perfectly shared both within and accross cohorts; individual wealth growth and aggegate wealth growth are perfectly correlated. However, even in that case individual and aggregate wealth-growth rates differ by a negative constant. Indeed, aggregating the wealth growth of all investors surviving into $t+1$, we obtain

$$\log\left((1-\lambda)\frac{\int W_{i,t+1}di}{W_t}\right) - \log\left(\frac{W_{t+1}}{W_t}\right) = \log\left(\frac{1+P^E}{1+P^E+P^N}\right) < 0. \tag{20}$$

The negative constant reflects the fact that the existing investors have to pay the arriving entrepreneurs to purchase their blueprints. If all blueprints accrued to existing firms ($\Gamma_{t+1}^N = \Gamma_{t+1}^U = 0$) then the entire portfolio of the investor is invested in existing firms ($\phi^E = 1, \phi^N = 0$); therefore $P^N = 0$ and aggregate and individual growth rates are identical.

4.2 Implications for the stochastic discount factor (SDF)

Since the wealth-to-consumption ratio is constant, our conclusions on wealth changes apply without modification to consumption changes of individual investors: an individual investor's consumption change is given by the right hand side of (18). The SDF $M_{i,t+1}/M_{i,t}$ of an individual investor is given by

$$\frac{M_{i,t+1}}{M_{i,t}} = \hat{\beta}\left(\frac{W_{i,t+1}}{W_{i,t}}\right)^{-\gamma} \propto (1+\eta\Gamma_{t+1})^{\gamma\alpha}\left(1+\eta\Gamma_{t+1}^E + \frac{\eta\upsilon}{\Delta}\Gamma_{t+1}^N\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_{j,t+1}\right)^{-\gamma}\left(\frac{Z_{t+1}}{Z_t}\right)^{-\gamma}. \tag{21}$$

For the markets where all investors are participating (in particular, the market for existing stocks and the risk-free asset), any $\frac{M_{i,t+1}}{M_{i,t}}$ is a valid SDF, and so is the (cross-sectional) average of $\frac{M_{i,t+1}}{M_{i,t}}$, defined as

$$\frac{M_{t+1}}{M_t} \equiv \mathrm{E}_t\left[\frac{M_{i,t+1}}{M_{i,t}}\middle|\Gamma_{t+1}^E, \Gamma_{t+1}^N, \Gamma_{t+1}^U\right], \tag{22}$$

where the expectation is taken over all agents i that participate in financial markets at time t. For our purposes, the interesting property of $\frac{M_{t+1}}{M_t}$ is its covariance with the growth shocks Γ_{t+1}^N and Γ_{t+1}^U. To isolate this covariance, we assume that Z_t is non-stochastic.

Proposition 3 *Assume that Z_t is deterministic.*

(i) When $\Delta = 1$ and $\upsilon = 1$, Cov$\left(\frac{M_{t+1}}{M_t}, \Gamma_{t+1}^N\right) < 0$.

(ii) By contrast, Cov$\left(\frac{M_{t+1}}{M_t}, \Gamma_{t+1}^N\right) > 0$ when either υ or Δ is sufficiently small.

(iii) For all parameter values, Cov$\left(\frac{M_{t+1}}{M_t}, \Gamma_{t+1}^U\right) > 0$.

Proposition 3 shows how risk sharing imperfections can determine whether the SDF (i.e., the marginal utility of the representative investor) rises or declines as the innovative activity by new entrants, Γ_{t+1}^{N}, increases. If $\upsilon = \Delta = 1$, the uncertainty associated with Γ_{t+1}^{N} is perfectly shared both within and across investor cohorts, and large realizations of Γ_{t+1}^{N} are "good news" for the representative investor (the SDF declines). The gains in the value of the portfolio of new firms are enough to offset the losses from the reduced value of the existing assets owned by the representative investor.[24] However, away from the perfect risk-sharing limit, large realizations of Γ_{t+1}^{N} are "bad news." For instance, if risk is shared perfectly within cohorts ($\Delta = 1$) but imperfectly across cohorts ($\upsilon < 1$ and sufficiently small), then the gains from a large innovation shock Γ_{t+1}^{N} accrue predominantly to new entrepreneurs. A low value of υ means that entrepreneurs retain a large share of their blueprints, and therefore the share sold to investors, $1 - \upsilon$, is not enough to offset the losses on their existing assets, whose profits are displaced. This intuition is clearest in the case where there is no trade whatsoever between existing investors and new entrepreneurs through private-equity funds (statement (iii) of Proposition 3); a large increase in the number of those blueprints (Γ_{t+1}^{U}) is unambiguously "bad news" for the representative investor (high marginal utility state).

Even if risk is perfectly shared across cohorts ($\upsilon = 1, \Gamma_{t+1}^{U} \equiv 0$), large realizations of Γ_{t+1}^{N} may still be (unconditionally) perceived as high-marginal-utility states when Δ is sufficiently small. In this situation existing investors *as a group* gain from increased innovation, since they buy all the shares of the newly arriving entrepreneurs before the realization of Γ_{t+1}^{N}. As *individual investors,* however, they do not know ex ante whether they will receive a large or a small allotment of the new blueprints. Since any given investor is risk averse, she worries about events where Γ_{t+1}^{N} is large (so that her existing assets will lose significant value), but her own personal allotment of new blueprints is not large enough to offset the losses. As a result, she views high realizations of Γ_{t+1}^{N} as high marginal-utility states.

In short, the model allows for a positive covariance between the SDF and the increments to blueprints that do not accrue to existing firms (Γ_{t+1}^{N} and Γ_{t+1}^{U}). As we argue in Section 5.3, this positive covariance is important for generating a value premium among traded stocks.

[24] Note that equation (10) implies that a large realization of Γ_{t+1}^{N} implies a higher realization of Γ_{t+1} and therefore a lower return on existing assets, R_{t+1}^{E}.

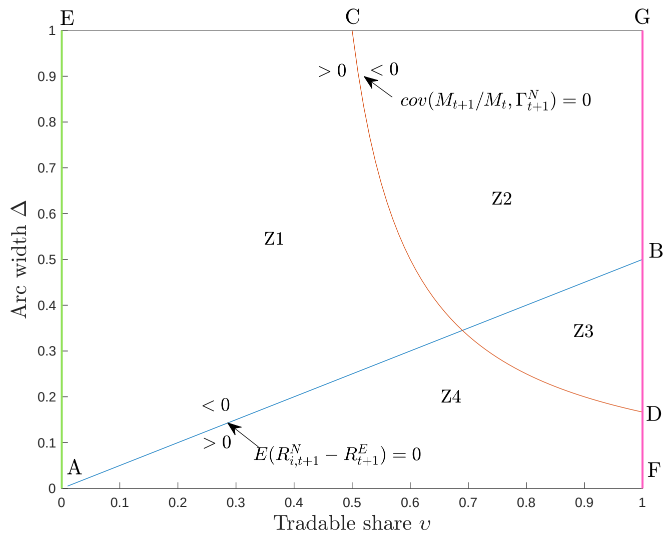

Figure 6: This figure depicts (a) the sign of the covariance between marginal-utility growth $M_{i,t+1}/M_{i,t}$ and the innovation shock Γ^N_{t+1} and (b) the sign of the expected-return differential between private and public equity investments, $\mathrm{E}_t[R^N_{i,t+1} - R^E_{t+1}]$, for each pair of the tradable share, υ, and arc width, Δ.

4.3 Equilibrium excess returns and public market equivalents

In this section we compare the expected returns on existing equities, $\mathrm{E}_t[R^E_{t+1}]$, with those on private-equity investments, $\mathrm{E}_t[R^N_{i,t+1}]$. The main result of this section is that $\mathrm{E}_t[R^N_{i,t+1} - R^E_{t+1}]$ could be of either sign. Moreover, the sign of $\mathrm{E}_t[R^N_{i,t+1} - R^E_{t+1}]$ neither determines, nor is determined by, the sign of $\mathrm{Cov}\left(\frac{M_{t+1}}{M_t}, \Gamma^N_{t+1}\right)$. To that end, we first state a formal result.

Proposition 4 *Let Z be independent of all other random variables.*
(i) Fixing $\Delta > 0$, for υ small enough it holds that $E_t[R^N_{i,t+1} - R^E_{t+1}] < 0$.
(ii) Fixing $\upsilon > 0$, if Δ is small enough then $E_t[R^N_{i,t+1} - R^E_{t+1}] > 0$.

Proposition 4 captures the two forces behind the determination of the sign of $\mathrm{E}_t[R^N_{i,t+1} - R^E_{t+1}]$. On the one hand, investing in new blueprints acts as a hedge against the displacement of old blueprints by new ones. On the other hand, investing in new blueprints adds non-diversifiable risk to the portfolio whenever $\Delta < 1$. The tension between those two forces is illustrated in the two statements of the proposition: Fixing Δ and lowering the tradable fraction, υ, makes the hedging aspect of new blueprints more valuable to the point where their expected return becomes lower than investing in already traded equities (statement (i)). By contrast, fixing υ and reducing Δ makes the idiosyncratic-risk aspect more prominent; since this risk is non-diversifiable, it raises the required expected return $\mathrm{E}_t[R^N_{i,t+1}]$ above $\mathrm{E}_t[R^E_{t+1}]$ (statement (ii)).

Taken together, Propositions 3 and 4 imply that the signs of Cov $\left(\frac{M_{t+1}}{M_t}, \Gamma_{t+1}^N\right)$ and $\mathrm{E}_t[R_{i,t+1}^N - R_{t+1}^E]$ are not linked. This is easiest to illustrate with Figure 6. The x-axis depicts values of the tradable share, υ, while the y-axis depicts values of the arc width, Δ. The blue, upward-sloping curve AB separates positive from negative values of $\mathrm{E}_t[R_{i,t+1}^N - R_{t+1}^E]$ and illustrates the two statements (i) and (ii) of Proposition 4. Similarly, the red, downward-sloping curve CD separates positive from negative values of Cov $\left(\frac{M_{t+1}}{M_t}, \Gamma_{t+1}^N\right)$: By statement (i) of Proposition 3 (and continuity), the points close to G ($\upsilon = \Delta = 1$) entail a negative value of Cov $\left(\frac{M_{t+1}}{M_t}, \Gamma_{t+1}^N\right)$. By contrast, close to the origin (Point A), statement (ii) of Proposition 3 implies positive values of Cov $\left(\frac{M_{t+1}}{M_t}, \Gamma_{t+1}^N\right)$.

The most noteworthy result of Proposition 4 is part (ii). Combined with Proposition 3, it establishes that there is a small enough Δ such that both $\mathrm{E}_t[R_{i,t+1}^N - R_{t+1}^E]$ and Cov $\left(\frac{M_{t+1}}{M_t}, \Gamma_{t+1}^N\right)$ are positive (region $Z4$ in Figure 6): The small value of Δ implies that investing in new blueprints is too imperfect a hedge against high realizations of Γ_{t+1}^N.

One implication of Proposition 4 we wish to emphasize is that investments in private equity may offer a higher expected return than investments in traded equities simply because of their non-diversifiable risk. Interestingly, as a consequence, in our model the most popular method for computing the "risk-adjusted" performance of private equity investments (the Kaplan and Schoar (2005) PME) indicates "risk-adjusted outperformance," even though all expected returns are entirely driven by risk and there is no notion of "skill" or "alpha."

The next proposition proves this result for the special case $\gamma = 1$, i.e., the special case that is most favorable for the validity of PME calculations. Indeed, in that case usage of the PME is theoretically justified by the Sharpe-Lintner-Mossin CAPM: a PME above (below) 1 can be equivalently viewed as a CAPM alpha above (below) 0.[25]

Proposition 5 *For any $\Delta \in [0,1]$ and $\upsilon \in [0,1]$ it holds that $E_t\left[\frac{R_{i,t+1}^N}{R_{t+1}^E}\right] \geq 1$ if γ is equal (or sufficiently close) to 1.*

In our stylized model private-equity investments can be viewed as a cash outflow at time t ("commitment" in the language of the PE industry) and a cash inflow ("exit") at $t+1$. The numerator of $R_{i,t+1}^N$ is the value of exits and the denominator is the value of commitments, so that $\mathrm{E}_t\left[\frac{R_{i,t+1}^N}{R_{t+1}^E}\right]$ corresponds to the PME. The remarkable aspect of Proposition 5 is not merely that the PME is different from one. This is to be expected, since in our framework

[25] See Sorensen and Jagannathan (2015), Korteweg and Nagel (2016), and Korteweg et al. (2023).

the return on existing, traded equities, R_{t+1}^E, is not the return on the total wealth of the representative investor. The remarkable aspect is that for any $\Delta \in [0,1]$ and any $\upsilon \in [0,1]$ this misspecification leads to a value of the PME above one.

The argument is simple enough to sketch in a few lines and applies broadly to any unitary-risk-aversion model where the total return on wealth consists of two positive-net-supply assets, say asset 1 and 2, so that the total return on wealth is a weighted average, $R_w = a_1 R_1 + (1-a_2)R_2$, with weights a_1 and a_2 that are positive and sum to one, $a_1 + a_2 = 1$. The assumption of unit risk aversion implies that[26] $\mathrm{E}\left[\frac{R_1}{R_w}\right] = \mathrm{E}\left[\frac{R_2}{R_w}\right] = E\left[\frac{R_w}{R_w}\right] = 1$, and therefore we obtain

$$1 = \frac{1}{\mathrm{E}\left[\frac{R_w}{R_w}\right]} = \frac{1}{\mathrm{E}\left[\frac{R_1}{a_1 R_1 + a_2 R_2}\right]} \leq \mathrm{E}\left[\frac{a_1 R_1 + a_2 R_2}{R_1}\right] = a_1 + (1-a_1)\mathrm{E}\left[\frac{R_2}{R_1}\right], \tag{23}$$

where the inequality follows from Jensen's inequality applied to the convex function $x \mapsto x^{-1}$. Since $a_1 < 1$, inequality (23) is equivalent to $\mathrm{E}\left[\frac{R_2}{R_1}\right] \geq 1$.

In short, while it is true that $\mathrm{E}\left[\frac{R_2}{R_w}\right] = 1$, the expectation of the ratio $\mathrm{E}\left[\frac{R_2}{R_1}\right]$ is always larger than one. In our model, R_{t+1}^E is just one component of the total return on wealth of investors in location i. The other component is $R_{i,t+1}^N$. Therefore, inequality (23) applied to our model leads to $\mathrm{E}_t\left[\frac{R_{i,t+1}^N}{R_{t+1}^E}\right] \geq 1$.

Proposition 5 asserts that this bias of the PME towards values larger than one extends beyond the special case $\gamma = 1$. Indeed, when we calibrate the model (Section 6.2) we find that the PME is well above one also for larger values of risk aversion.

4.4 The participation arc Δ and the PE portfolio share

To gain some intuition on the size of PE investments compared to total assets, ϕ^N, we provide the following comparative-static results.

Proposition 6 *For γ sufficiently close to one, ϕ^N is an increasing function of the parameters $\eta, \upsilon,$ and Δ.*

Proposition 6 states that, as expected displacement of old by new blueprints increases (an increase in η), it becomes more attractive to invest in private equity. Similarly, improvements in the contracting technology (higher υ, and/or higher Δ) also raise the share invested in

[26]Specifically, the SDF is mR_w^{-1} for some constant m, but this constant m must equal 1. To see this, price the traded return on wealth R_w: $1 = \mathrm{E}\left[m\frac{R_w}{R_w}\right]$, implying $m = 1$.

private equity. Proposition 6 allows some broad conclusions about the joint determination of (a) aggregate growth, (b) inequality, (c) the size (that is, the portfolio share, ϕ^N) of the private-equity industry.

An acceleration of disruptive growth (an increase in η) raises aggregate growth (a direct implication of equation (2)), increases the portfolio share of the private-equity industry, and also boosts the wealth shares of newly-rich entrepreneurs.[27]

However, if the increase in the share of private equity investments results mostly from contractual improvements (an increase in υ or Δ), then aggregate output, Y_t, is unchanged. The increased share of private equity investments is only reflected in distributional outcomes (less displacement of the old rich by the new rich, or less dispersed returns among private equity investors.)

These observation may be helpful in providing an interpretation on the changing forces behind the growth in private equity investments in the last four decades. For example the rapid growth of the private-equity investment share in the mid-to-late nineties coincided with strong TFP growth, increased displacement of old rich by new rich, and large dispersion in the returns of PE investors.[28] An increase in disruptive growth (a rise in η) would be able to explain these observations jointly.

By contrast, while the private-equity investment share kept growing after 2000, TFP growth was moderate, the displacement of old by new rich decelerated (Gomez (2023)), and returns among private equity investors became less dispersed (Korteweg et al. (2023)). Taken together, these observations suggest that the driving force behind the continued growth of private equity during this period was not disruptive growth, but primarily improvements inside the private-equity industry.

5 Extensions and Discussion

For simplicity, the baseline model contains only two types of assets: claims to existing blueprints and claims to newly arriving blueprints. We next show how to embed additional positive supply assets into the framework with minimal modeling extensions.

[27]Recall that a newly-rich entrepreneurs retain a fraction of their equity, either because $\upsilon < 1$, or because they belong to the set of entrepreneurs who have no access to private equity (when $\Gamma^U \neq 0$.) A rise in η raises the value of new as opposed to old assets and thus raises the wealth share of new entrepreneurs.

[28]For the historical path of displacement activity, see Gomez (2023). For a plot of the cross-sectional dispersion of the IRR's obtained by public pension funds across time, see Korteweg et al. (2023).

5.1 Buyout funds

We have referred to the intermediaries that facilitate risk sharing between entrepreneurs and existing investors using the term "private-equity" funds, rather than the narrower "venture-capital" funds. The reason is that the crucial characteristics of private equity in our model are that (a) some of the gains from innovation accrue to arriving (rather than existing) cohorts of innovators and (b) the funds facilitate a partial trade between existing investors and arriving investors.

By focusing on these two characteristics, venture-capital and buyout funds can be viewed as performing similar economic functions. To understand this claim better, we next show how a re-interpretation of some of the model's mathematical structure would allow us to model the particular functions performed by buyout funds. Specifically, recall that in the model some of the new blueprints accrue to existing firms (a fraction $\Gamma_{t+1}^{E}/\Gamma_{t+1}$). Now, assume that each period a fraction of the existing firms lose their ability to receive new blueprints. Newly arriving entrepreneurs ("managers") can perform a "buyout," by purchasing those firms from existing investors and restoring their ability to receive an allocation of blueprints over the next period, at which point the firms are re-introduced into the public market. The assumption that the new entrepreneurs have the *unique* ability to restore the ability of existing firms to receive new blueprints is equivalent to assuming that these new blueprints are effectively their property, as in the baseline model. By making the same assumptions as in the baseline model on the random allotment of new blueprints to new entrepreneurs, there would again be room for buyout funds that would provide managers with the ability to diversify their risk etc.[29]

In summary, there is little difference between assuming that the new entrepreneurs out-

[29] The main difference with the baseline model is that buyout funds would pay the manager a price equal to the sum of (a) the present value of the existing blueprints of the firm and (b) the value of the new blueprints. The uniquely skilled manager would simply pass through the first component of this sum to old investors in order to purchase the firm from them. Therefore, the component of value that is shared between the private equity fund and the manager is the value of the new blueprints, exactly as in the baseline model. Assuming that the total sum, Γ_{t+1}^{E}, accruing to all existing firms is unchanged (with the understanding that it now only accrues to a subset of existing firms), this extended model leads to exactly the same consumption allocations and SDF as the baseline model. The only difference is an "accounting" difference: Because the price paid by investors to purchase a share of a private equity fund is the sum of existing and new assets, the return from investing in private equity (from the perspective of an investor) is a portfolio return from the claim to existing blueprints (without the ability to receive further blueprint allotments this period) and the return of investing in new blueprints. In that sense, an investment in a buyout fund is a weighted average of (a) the return of what we refer to in Section 5.3 as a return to a "value" stock, and (b) the idiosyncratic return in a portfolio of new blueprints, reflecting managers' ability to restore an existing firm's profitability.

right obtain the new blueprints and assuming that they obtain these blueprints indirectly by restoring old firms' ability to receive them.

5.2 Real assets

Equation (3) of the model states that the profit share $A_t \pi_t$ is a constant share of output. However, the profits of any individual blueprint are a declining fraction of output, akin to the value of each blueprint "depreciating" over time. Indeed, equation (3) implies that the change in the logarithm of the profits accruing to any fixed blueprint is $-\alpha \log(A_{t+1}/A_t) + \varepsilon_{t+1}$. In that sense, a positive shock to technological advancement is a negative depreciation shock for a fixed blueprint.[30]

One could view some assets (for example, land or commodities) as assets that are non-displaceable in the sense that they are factors of production that are just as useful to new or old firms. One simple way to capture the non-displaceability of such assets is to modify the baseline model by assuming that a portion $r_t^F = \zeta Y_t$ with $\zeta \in (0, 1-\alpha)$ of output is channeled as cashflow to the asset "land." (Appendix B shows how to modify the micro-foundations of the model to allow land and other non-displaceable factor of production.)

Once again, we construct an equilibrium where the price-to-rent ratio $P^F = \frac{P_t^F}{r_t^F}$ is constant, so that the return on land is given by $R_{t+1}^F = \frac{r_{t+1}^F + P_{t+1}^F}{P_t^F} = \frac{Y_{t+1}}{Y_t} \frac{1+P^F}{P^F}$. Repeating the arguments of Section 4.1, the wealth evolution of an individual investor, conditional on survival, is

$$\frac{W_{i,t+1}}{W_{i,t}} = \frac{1}{1-\lambda} \frac{W_{t+1}}{W_t} \left(\frac{\zeta \left(1 + P^F\right)}{\alpha(1-\alpha)\left(1 + P^E + P^N\right) + \zeta\left(1 + P^F\right)} + \frac{\alpha(1-\alpha)\left(1 + P^E\right)}{\alpha(1-\alpha)\left(1 + P^E + P^N\right) + \zeta\left(1 + P^F\right)} \left(\frac{1 + \eta \Gamma_{t+1}^E + \eta \upsilon \Gamma_{t+1}^N}{1 + \eta \Gamma_{t+1}} \right) X_{i,t+1} \right), \tag{24}$$

for some new constants P^E and P^N. Comparing (24) with (18), the only difference is that the wealth growth of an individual investor now gains a fraction $\frac{\zeta\left(1+P^F\right)}{\alpha(1-\alpha)(1+P^E+P^N)+\zeta(1+P^F)}$ of aggregate wealth growth. The reason is intuitive: Since land captures a constant fraction of total output, it actually benefits from higher values of Γ_{t+1}^N, since those are associated with higher output growth.

[30]This feature of the model is reminiscent of the role that a positive investment-specific shock ("IST" shock) plays for the value of existing capital.

Just as before, the return R^E_{t+1} is declining in Γ^N_{t+1}, while R^F_{t+1} is increasing in Γ^N_{t+1}. Using arguments similar to those we employed for Proposition 3(ii), one can show that, for sufficiently small Δ, υ, and ζ, we have $\mathrm{E}\left[R^F_{t+1}\right] < \mathrm{E}\left[R^E_{t+1}\right]$. Indeed, if Z_t were deterministic, then $\mathrm{E}\left[R^F_{t+1}\right] < 1 + r^f$, reflecting that land is a hedge against displacement risk.

In short, the model predicts that the investments in gold or land, which are not subject to displacement shocks, should underperform public equities in expectation. In turn, under the additional assumptions of Proposition 4(ii), the expected return of private equities should exceed the return of public equities. This ordering of expected returns across asset classes appears consistent with the data (e.g., Mauboussin and Callahan (2020), see footnote 2).

We conclude this section with a caveat: Implicitly, we assume that "land" is a factor of production that can be productively combined with any blueprints to produce profits. Therefore, the cashflows that accrue to land are proportional to the aggregate profits of all blueprints (or equivalently, aggregate output). If one assumed instead that a specific lot of land is useful only to a fixed set of blueprints, then it would be more meaningful to assume that the value of that lot is proportional to the profits of those specific blueprints. Accordingly, it would be subject to the same displacement risk as those blueprints.

5.3 Heterogeneity among existing equities and the value premium

In presenting the baseline model, we assumed that all existing firms have the same profit growth, $\frac{A_{t+1}\pi_{t+1}}{A_t\pi_t} = \frac{\pi_{t+1}}{\pi_t}\left(1+\eta\Gamma^E_{t+1}\right)$, which is the product of two components: (a) the profit growth of a fixed blueprint, $\frac{\pi_{t+1}}{\pi_t}$, and (b) the growth component $\left(1+\eta\Gamma^E_{t+1}\right)$, resulting from the allotment of new blueprints to existing firms.

It is straightforward to introduce within-existing-firm heterogeneity, by assuming that firms' profit growth is heterogeneously exposed to the arrival of new competitors. The simplest way to achieve this outcome, without changing any aggregate quantity, is to introduce a third, firm-specific component to a firm's profit growth. Specifically, we assume that firm j's profit growth is given by

$$\frac{A_{j,t+1}\pi_{t+1}}{A_{j,t}\pi_t} = \frac{\pi_{t+1}}{\pi_t}(1+\Gamma^E_{t+1})q_{j,t+1}, \tag{25}$$

where $q_{j,t+1}$ is a positive, random variable that is i.i.d. across firms and time and has a (cross-sectional) expectation equal to 1. For example, assume that $q_{j,t+1}$ is given by

$$q_{j,t+1} = e^{(\Gamma^N_{t+1}+\Gamma^U_{t+1})\sigma_\zeta \zeta_{j,t} - \frac{1}{2}(\Gamma^N_{t+1}+\Gamma^U_{t+1})^2\sigma^2_\zeta}, \tag{26}$$

with $\zeta_{j,t} \sim \mathcal{N}(0,1)$ a standard, normal, i.i.d., firm-specific shock, drawn *at the beginning of period* t. When $\sigma_\zeta = 0$, all existing firms experience identical profit growth, as in the baseline model. Furthermore, for any value of the parameter σ_ζ, the aggregate growth of existing firms' profits is the same as specified in Section 3, because integrating $q_{j,t+1}$ across j equals the cross-sectional expectation of $q_{j,t+1}$, which is equal to 1 for any realization $\Gamma^N_{t+1} + \Gamma^U_{t+1}$. In short, $q_{j,t+1}$ is purely a distributional shock across existing firms that does not impact any aggregate quantity (SDF, aggregate dividend growth of the market portfolio, consumption allocations, etc.).

However, the idiosyncratic component $q_{j,t+1}$ allows us to introduce cross-sectional differences in the expected returns of existing equities. The reason is that each firm draws the shock ζ_j at the beginning of period t, before trading commences. Investors don't know the magnitude of the displacement shocks $\Gamma^N_{t+1} + \Gamma^U_{t+1}$ until the end of the period. But they know that if $\zeta_j < \zeta_k$, then firm j's profits will experience slower growth than firm k's profits from t to $t+1$ for any realization of $\Gamma^N_{t+1} + \Gamma^U_{t+1}$. In particular, the across-all-firms decline in profits $\frac{\pi_{t+1}}{\pi_t}$ that occurs in response to a large realization of $\Gamma^N_{t+1} + \Gamma^U_{t+1}$ is partially offset for firms with a positive ζ_j and amplified for firms with a negative ζ_j. In that sense, it is helpful to think of the heterogeneous ζ_j at the beginning of the period as heterogeneous exposures ("betas") to the sum of the displacement shocks $\Gamma^N_{t+1} + \Gamma^U_{t+1}$.

We obtain the following.

Proposition 7 *Let $R_{i,t+1}$ and $R_{j,t+1}$ denote the returns of firms i and j. We have*

$$\frac{E_t\left[R_{i,t+1}\right]}{E_t\left[R_{j,t+1}\right]} = \frac{1 + Cov_t\left(\frac{M_{t+1}}{M_t} / E_t\left[\frac{M_{t+1}}{M_t}\right], \frac{A_{j,t+1}\pi_{t+1}}{A_{j,t}\pi_t} / E_t\left[\frac{A_{j,t+1}\pi_{t+1}}{A_{j,t}\pi_t}\right]\right)}{1 + Cov_t\left(\frac{M_{t+1}}{M_t} / E_t\left[\frac{M_{t+1}}{M_t}\right], \frac{A_{i,t+1}\pi_{t+1}}{A_{i,t}\pi_t} / E_t\left[\frac{A_{i,t+1}\pi_{t+1}}{A_{i,t}\pi_t}\right]\right)}. \tag{27}$$

We note that conditioning on the time-t information set implies that all moments in (27) are conditional on ζ_i and ζ_j.

The main implication of (27) is as follows. Take two firms i and j and assume that (a) $\zeta_i < \zeta_j$, so that firm i's profit growth between t and $t+1$ will be lower than firm j's for any realization of $\Gamma^N_{t+1} + \Gamma^U_{t+1}$, and (b) the stochastic discount factor assigns high marginal utility to large realization of Γ^N_{t+1} and Γ^U_{t+1}. These two assumptions imply that the covariance of the stochastic discount factor with firm i's profit growth, $\frac{A_{i,t+1}\pi_{t+1}}{A_t\pi_t}$ (normalized to have expectation one), is smaller than the respective covariance for firm j.[31] Accordingly, equation (27) implies that firm i has a higher expected return than firm j. Therefore, stocks with comparatively low profit growth ("value stocks") have comparatively high expected returns. In Section 6.2 we examine this "value premium" quantitatively.

5.4 Labor income and pension funds

Aggregate labor income is a constant fraction of output in our paper. (In Appendix B, we micro-found this assumption by assuming a Cobb-Douglas production for the final good, and a production function that is linear in labor for the intermediate goods.) Since aggregate output is increasing in A_t, so is aggregate labor income.

In the baseline version of the model workers are hand-to-mouth consumers who do not participate in financial markets. It is straightforward to relax this assumption. In particular, if we make assumptions that ensure that the growth rate of the labor income of a *fixed* cohort of workers grows at the same rate as the profits of *existing* firms, then we can allow workers to participate in financial markets without substantially modifying the model.

At the end of Appendix B we micro-found such an extension, by assuming that a unit of labor in production, l_t, is a Cobb-Douglas aggregate of labor inputs provided by different cohorts of workers,

$$l_t = \prod_{s \leq t} (l_{t,s})^{\frac{a_{t,s}}{\sum_{s \leq t} a_{t,s}}}, \tag{28}$$

where $l_{t,s}$ is the labor input of workers born at time s and the weights $a_{t,s}$ are given by $a_{t,t} = \frac{\eta(\Gamma_t - \Gamma_t^E)}{1+\Gamma_t}$ and $a_{t,s} = a_{s,s} \prod_{s+1}^{t} \frac{1+\eta\Gamma_u^E}{1+\eta\Gamma_u}$. This specification implies that, although the aggregate wage bill grows at the rate of aggregate output, the fraction of wages accruing to a given worker cohort declines over time at the same rate as the profits of existing firms. One motivation for such a specification is skill obsolescence, in that the skills of a fixed worker cohort follow the fate of the technologies they encountered when they joined the workforce.

If one adopts equation (28), and allows workers access to financial markets on the same terms as entrepreneurs, then all our conclusions carry through without modification. While workers' and entrepreneurs' initial endowments may differ, the fact that their endowment

[31] Because $\frac{\pi_{t+1}}{\pi_t}$ declines with $\Gamma_{t+1}^N + \Gamma_{t+1}^U$, the typical covariance is negative. Accordingly, "smaller" means negative and larger in absolute value.

a^E	14.06	α	0.80
a^N	17.36	γ	9.00
a^U	1.23	Δ	0.50
b^E	0.26	υ	0.80
b^N	0.03	ρ	0.95
b^U	0.39	$1-\lambda$	0.90
$\mathrm{E}[\varepsilon]$	-0.23	β	0.85
$\sigma(\varepsilon)$	0.07		

Table 1: Parameters used for the calibration. One model period is five years.

growth rates are the same leads to the same SDF, portfolio choices, etc.[32] In particular, if one took the view that pension funds invest in bonds, existing stocks, and private equity on behalf of workers to maximize their welfare, then the workers' portfolios would be the same as the portfolios of the investors in our baseline model.

6 Calibration

Our approach to calibrating the model follows the asset pricing literature: We first fix some assumptions on preference parameters that are in line with the macro-asset pricing literature. We then choose technological parameters to match endowment-related first and second moments (aggregate consumption, the ratio of new-to-old stock market capitalization, the cross-sectional dispersion of new asset allocations, etc.) The combination of preference and endowment assumptions have implications for risk premiums (equity risk premium, value premium, etc.) as well as for the PME values of private-equity returns. The assumptions on the allocation of new blueprints along with the equilibrium asset prices have implications for inequality. In Section 6.1 we describe how we choose preference and technological parameters. In Section 6.2 we focus on the model implications for risk premia and in Section 6.3 we examine the model implications for inequality.

6.1 Parameter choice

We choose a period to be five years and thus compare five-year returns in the data to those in the model. The reason for this choice is that in our model t is the date of the capital

[32]This is a direct implication of the homogeneity properties of Epstein-Zin-Weil utilities.

outflow (when the capital is called on a private-equity investment) and $t+1$ is the date of "exit." Typically, there is a 5-7 year distance between these dates.[33]

For preference parameters we choose an annual discount rate of approximately 3% per year and an annual birth/death rate of approximately 2%, which are common choices in the literature. Taking into account that a period is five years we obtain $\beta = 0.97^5 \approx 0.85$ and $(1-\lambda) = 0.98^5 \approx 0.9$. For risk aversion we choose $\gamma = 9$, which is in the range of values that are commonly used in the asset pricing literature. We set $\upsilon = 0.8$, implying that entrepreneurs retain about 20% of the equity share in a typical private-equity deal.[34]

In terms of the parameters that control output, we set α to match the share of economic rents ("pure profits") in production. In particular, the value $\alpha = 0.8$ implies a profit share of $\alpha(1-\alpha) = 16\%$. This is in line with the estimates in the literature, especially in the more recent decades.[35]

We then choose the endowment parameters that control the stochastic properties of blueprint increments (Γ^E_{t+1}, Γ^N_{t+1}, and Γ^U_{t+1}) and the neutral process Z_{t+1}. The model has eight parameters that control the univariate distribution of these four random variables. Two of them, $\mathrm{E}[\varepsilon]$ and $\sigma(\varepsilon)$, are the mean and standard deviation of the normal, i.i.d., shock $\varepsilon_{t+1} = \Delta \log(Z_{t+1})$. The parameters a^l and b^l for $l \in \{E, N, U\}$ determine the distributions of Γ^l_{t+1}. (Without loss of generality, we may set $\eta = 1$.[36]) We choose these eight parameters with the goal of approximately matching eight moments in the data, as follows. We target (1) the mean and (2) standard deviation of aggregate consumption growth, (3) the mean and (4) standard deviation of the ratio of old-to-total stock market capitalization, $(1+\Gamma^E_{t+1}) \times (1+\Gamma^E_{t+1}+\Gamma^N_{t+1}+\Gamma^U_{t+1})^{-1}$, and (5) the mean ratio of VC-backed to non-VC-backed IPOs, $\frac{\Gamma^N_{t+1}}{\Gamma^U_{t+1}}$. Jointly with the other parameters in the model, the parameters a^l and b^l also determine (6) the standard deviation (and higher moments) of the logarithmic excess returns of private equity in the time-series, (7) the share of private equity in an investor's portfolio, and (8) the real interest rate.

[33] An additional benefit of the choice of five-year horizons is that it mitigates any measurement error in annually reported data of public-pension-fund alternative-investment returns.

[34] In a typical first round venture capital contract, the VC obtains about 30-40% of equity and the founders retain 20-30%. A fraction of 20-30% accrues to Angel investors and a fraction of about 20% remains in the option pool. Source: "https://www.entrepreneur.com/money-finance/business-dividing-equity-between-founders-and-investors/65028." By setting the entrepreneur's share to a low number, we wish to account for further dilution in future rounds of financing.

[35] See Barkai (2020) and Barkai and Panageas (2023).

[36] The scaling property of the gamma distribution implies that any other choice of η would be equivalent to multiplying each b^l by η.

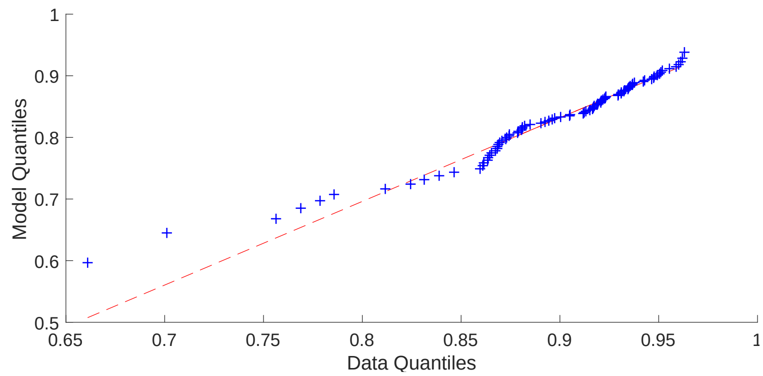

Figure 7: Quantile-quantile plot of the ratio of old-to-total stock market capitalization, $(1+\Gamma^E_{t+1})/(1+\Gamma_{t+1})$, in the model and the data. The source of the data on the ratio of old-to-total stock market capitalization is Gârleanu and Panageas (2023), which computes the ratio of the value of annual additions to the market portfolio as a fraction of the value of the market portfolio at an annual frequency. We compute rolling 5-year observations and compare them to the model.

We make one additional assumption to match the beta of private equity returns on public equity returns. Specifically, we assume that the shocks to ε_{t+1} and Γ^N_{t+1} are correlated. This allows us to capture the cyclical nature of private equity returns in the data. Since ε_{t+1} is normally distributed, whereas Γ^N_{t+1} is gamma distributed, we use a Gaussian copula to model the correlation between ε_{t+1} and Γ^N_{t+1}. We choose a correlation of $\rho = 0.95$ in the specification of the Gaussian copula to match a private-equity-return beta to the public-market-portfolio return around unity. This choice is motivated by the empirical literature,[37] and it helps illustrate that the above-unity PME values in our calibration are not the result of a beta that differs substantially from unity. Finally, we specify Δ to match the cross-sectional dispersion of private equity returns in the data.

6.2 Implications for displacement, cross-sectional dispersion, and asset prices

We next compare the model to the data. We start by illustrating that the model can match two aspects of the data with the chosen parameters, namely (a) the displacement of old firms by new firms in the market portfolio and (b) the cross-sectional dispersion of private-equity returns. In our model (a) controls the extent of intertemporal risk sharing, while (b) controls the extent of intratemporal risk sharing.

[37]Estimates of this "beta" vary a lot in the literature depending on the time-period, the methodology, etc. For a recent survey of this literature, see Section I of Boyer et al. (2023). In the literature one encounters values ranging from 0.75 all the way to numbers slightly above 2, but with most numbers being in the vicinity of one.

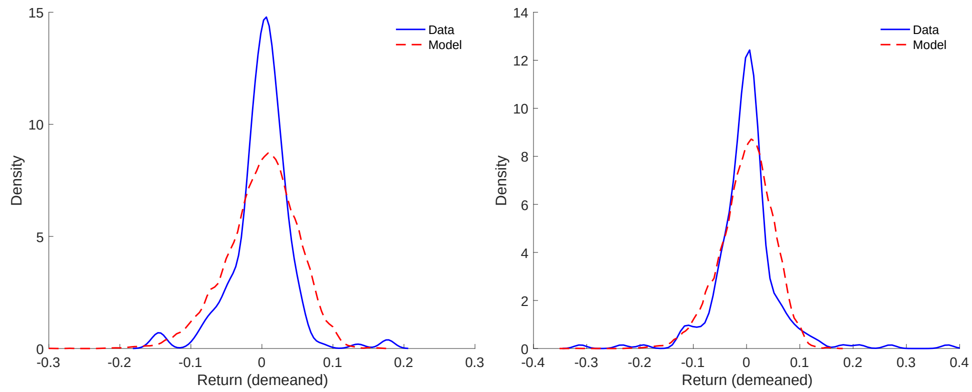

Figure 8: Dispersion of private-equity returns in the model and the data. The left plot uses 5-year averages of the (log) returns on alternatives reported in public pension plans' Comprehensive Annual Financial Reports, 2001–2015. The right plot uses internal rates of returns on the private equity investments obtained by public pension plans on their investments for all PE investments made between 1995 and 2009, ensuring that all funds are closed and the IRR is not affected by unrealized exits. Source: Korteweg et al. (2023).

Over a five-year period, the old-to-total stock market capitalization ratio has a mean of about 0.89 in the data (0.82 in the model) and a standard deviation of 0.074 in the data (0.087 in the model).[38] Figure 7 provides a quantile-quantile plot of this ratio in the data and the model, which shows that the model matches the shape of the distribution of this ratio reasonably well.

Figure 8 provides a visual illustration of the cross-sectional dispersion of private equity returns across investors in the model and the data. The data is from US public pension plans. As mentioned earlier, public pension plans are an attractive source of information, since they are subject to FOIA requests, and therefore they produce annual reports detailing their performance in various asset classes. Under the extended assumptions of Section 5.4, pension plans can be viewed as one of the marginal investors in our framework, so that we can use the dispersion of their private equity returns to discipline the respective dispersion in the model. Specifically, over every (non-overlapping) five-year period we depict the distribution of a pension plan's five-year average of log-private equity returns minus the cross-sectional mean of this quantity (across pension plans) over that same period. We contrast this cross-sectional dispersion in the model and the data. In the left plot we use the data provided by the Comprehensive Annual Financial Reports (CAFRs) of defined-benefit, public pension plans in the US between 2000-2018, resulting in three non-overlapping five-year periods

[38]The source of the data on the ratio of old-to-total stock market capitalization is Gârleanu and Panageas (2023), which computes the ratio of the value of annual additions to the market portfolio as a fraction of the value of the market portfolio.

	Model	Data
Consumption growth mean	0.021	0.020
Consumption growth std. dev.	0.039	0.037
Mean ratio of PE-backed to non-PE-backed IPOs	1.065	1.174
Std. dev. of log-PE return, $\sigma(\log(R_{i,t}^N))$	0.186	0.141
Market-beta of private equity, $\beta_{R^E}(R_i^N)$	0.853	1.000
Portfolio share of PE, ϕ_N	0.074	0.080
Mean of real log risk-free rate, $\log(1+r^f)$	0.017	0.024
Log eq. prem., $\mathrm{E}[\log(R_t^E)] - \log(1+r^f)$ (model: unlevered, data: levered)	0.028	0.040
Std. dev. of pub. equity, $\sigma(\log(R_t^E))$ (model: unlevered, data: levered)	0.049	0.148
Std. dev. of real log-dividend growth (model: unlevered, data: levered)	0.049	0.107
PME, $\mathrm{E}\left[R_{i,t}^N / R_t^E\right]$	1.084	1.130

Table 2: Model and data — various moments. The source of the data for real returns on public equity, the interest rate, real dividend growth, and real consumption growth is the long historical sample provided by R. Shiller's website. To be consistent with the model, we compute non-overlapping five-year log returns and annualize the moments. We do the same for consumption and dividend growth. The data for returns are the 29 non-overlapping five-year intervals covering 1871–2016. (For consumption the data starts in 1889). For the rest of the data sources, we refer to the text.

(2001-2015) for the 138 plans in the sample. To ensure that this cross-sectional dispersion is not just a result of measurement error in CAFRs, the right plot uses deal-level internal rates of return (IRR) of the private equity investments made by each public pension plan.[39] Specifically, we compute a five-year average of IRRs for all the private equity investments that each pension fund made over this period, and then subtract the cross-sectional mean (across pension funds) of this quantity over that same period. Irrespective of which data set we use, the figure shows that our model matches the large cross-sectional dispersion of returns quite well.

Having discussed how the model matches the old-to-new capitalization ratio and the cross-sectional dispersion of private-equity returns, Table 2 reports some additional, predominantly asset-pricing, moments for the model (left column) and the data (right column). The first two rows confirm that the model roughly matches the first two moments of aggregate consumption growth. The combination of a unit IES and the fact that old investors' consumption growth is below aggregate consumption growth allows the model to match both the aggregate-consumption growth moments and the low risk-free rate that we observe in reality. The mean ratio of PE-backed to non-PE-backed IPOs mainly disciplines the split between the

[39]Both data sets are described in Korteweg et al. (2023).

Portfolio	2-1	3-1	4-1	5-1	6-1	7-1	8-1	9-1	10-1
Ex. returns, data	-0.01	0.01	0.03	0.03	0.05	0.06	0.06	0.08	0.10
Ex. returns, model	0.02	0.02	0.03	0.03	0.04	0.04	0.05	0.05	0.06

Table 3: Value premium. The table reports the return to each of the top nine decile portfolios ranked by the earnings-to-price ratio (E/P) in excess of the return on the first of the decile portfolios.

shocks Γ^N and Γ^U in the model. In terms of data sources, the data for the ratio of PE-backed to non-PE-backed IPOs is from Jay Ritter's website,[40] and the standard deviation of private equity returns (in the time series) is from CAFR reports. Finally, the table reports a private equity share in the data equal to 8%. The data counterpart for this number has been constantly growing from levels around 5% in 2001 to a number around 13% in 2023,[41] with the 8% number in the data column being characteristic of the early 2010s.

Turning to the equity premium for existing equities, the model matches about 70% of its empirical counterpart. (The logarithmic equity premium is 2.8% in the model and 4% in the data.) In this context, it is useful to note that in the real world the leverage ratio is approximately 1.7; accordingly, the Modigliani-Miller formula implies that the levered equity premium should be about 1.7 times larger than the un-levered one. The same observation about leverage applies to the model-implied dividend growth, which is about 50% of its empirical counterpart. However, because the model abstracts from mechanisms that could make the price-dividend ratio time-varying, it cannot account for the discrepancy between the volatility of returns and the volatility of dividends in the data. By now there are several approaches in asset pricing to account for this "excess" volatility (heterogeneous preferences, beliefs, time-varying risk aversion, etc.), but such extensions would be just a distraction for this paper.[42]

In terms of private equity returns, the model-implied PME is well above one (1.08), and close to the mean PME value observed in the data (the mean PME is 1.13, the median 1.06.[43]) The fact that the PME is well above one shows that the conclusion of Proposition 4

[40] "Inital Public Offerings:Updated Statistics", Page 18, reports that 54% percent of IPOs between 1980-2023 were backed by PE (either buyout or VC).

[41] The source of the data is the website of Equable Institute, in particular, https://equable.org/wp-content/uploads/2023/07/Equable-Institute_State-of-Pensions-2023_Final.pdf.

[42] Gârleanu and Panageas (2023) illustrates that by simply adding time-varying, expected displacement, one can generate time-varying discount rates even in economies where the aggregate endowment is deterministic.

[43] These numbers are from the 14,528 observations reported in the first table of Boyer et al. (2023).

Min Rank	350	300	250	200	150	100	50
Data	20.5	17.2	13.0	16.0	10.7	11.1	6.8
Model	11.3	11.5	11.2	12.1	12.1	13.2	14.5

Table 4: Proportion of Forbes 400 represented by new entrants. Every year from 1999 to 2013, we compute the proportion of new entrants over the previous five years classified by Forbes as "self-made" and who have not appeared in the list previously. We do this separately for each rank range 351–400, 301–350, and so on. We report the average of these proportions, for each rank range.

holds even for values of risk aversion quite far from one. This quantitatively non-trivial PME in a model calibrated to match several of the features of PE investments (beta with respect to the market, cross-sectional dispersion of returns, the share of private equity in an investor's portfolio, etc.) shows that the theoretical problems with the PME of Section 4.3 also appear quantitatively relevant. To be clear, in our calibrated model the return of PE investments exceeds that of public equities by 1.7% per annum. However, this is not "outperformance" due to skill, it is just compensation for the risk of these investments.

Finally, we confirm that under our parameterization we obtain not only a positive risk premium for new blueprints, but simultaneously a value premium. We use the model of Section 5.3 with $\sigma_\zeta = 0.25$. Every period we sort stocks into ten bins based on their beginning-of-period realization of the shock ζ_j. Because there is a monotone relation between the stock's ζ_j and its price-earnings ratio, it is immaterial if we form the portfolios based on ζ_j or the stock's price-earnings ratio. We compute the average expected returns to portfolios consisting of all stocks in each of the ten bins, and we report, in Table 3, the spread between each of the top nine decile portfolio-returns and the bottom-decile portfolio return. For comparison, we also report the corresponding numbers from Fama and French (1992) (Table IV). The data explains about 60% of the value premium. Specifically, the difference in (levered) returns between the 10th and the 1st portfolio is 10% in the data, whereas the respective difference between the model-implied (unlevered) returns is 6%.

6.3 Wealth distribution

Compared to the majority of models that study the wealth distribution in the presence of heterogeneous returns, while keeping the population of agents fixed, this model highlights the role of entry of new agents. In particular, the key feature of the model is that some investors (especially the non-PE backed entrepreneurs who do not sell any of their equity) experience a rapid accession to wealth upon entry, followed by less volatile but lower average returns thereafter.

This is an empirically attractive feature of the model, as we illustrated with Figure 2. We tabulate the data plotted in that figure, as well as the model counterpart, in Table 4. Using data from the Forbes 400, we find that approximately 20% of the individuals joining the Forbes 400 list over a five-year period are new entrants — people who were not in any of the previous 400 lists — and who displace previous members of the Forbes 400 from the ranks of the 400 richest. The remarkable feature of the data to us, which we already highlighted, is that the new entrants do not predominantly populate the lower ranks of the Forbes 400 list. Only 20.5% of the entrants find themselves in the 350–400 range, while almost 20% populate the ranks 150 and above. With its very dispersed distribution of wealth in the early stage of an entrepreneur's life, the model can match this remarkable phenomenon in the data, as the second row of the table shows.[44]

Table 5 compares the stationary distribution of the model to the data. We simulate the model, normalize the log wealth of the 100th richest individual to zero, and then report the log wealth difference between the 150th to the 100th individual, the 200th to the 100th individual, etc.[45] The table shows that the model leads to a more dispersed wealth distribution than what we observe in the data. For instance, in the data the 250th richest individual has approximately 49.6% ($e^{-0.7}$) of the wealth of the 100th richest individual, while in the model that ratio is about 27.2% ($e^{-1.3}$). We note that such real-world forces as taxes and split inheritances, if modeled, would reduce the dispersion of the stationary wealth distribution.

7 Conclusion

This paper presents a unified, risk-based asset-pricing theory of expected returns across conventional and alternative asset classes. The model makes two core assumptions: (a) young firms engage in creative destruction and displace old firms, and (b) the gains to innovation

[44]In comparing the model's implications to the data we use the discrete interval construction of Section 3.3. We partition the interval $[0, 1)$ into $K = 1000$ subintervals of equal length and allocate the 1000 i.i.d.-drawn gamma-increments to the innovators in that sub-interval. We record the wealth of the representative innovator inside each subinterval, track its dynamics according to the model, and compute order statistics. For large enough K, the order statistics of the discrete approximation on $[0, 1)$ converge to the order statistics of the gamma process on $[0, 1)$, which do not depend on K. Therefore, the choice of K is largely immaterial for our results (as long as it is large.)

[45]We choose the 100th individual as the base for comparison because the log wealths of the very richest several individuals are estimated with error even if we simulate the model for a very long time.

Rank	400	350	300	250	200	150
1982	-1.2	-0.9	-0.9	-0.7	-0.5	-0.2
1992	-1.1	-1.0	-0.8	-0.7	-0.5	-0.3
2002	-1.2	-1.0	-0.8	-0.7	-0.5	-0.3
2012	-1.3	-1.1	-0.9	-0.7	-0.5	-0.3
Model	-2.4	-2.0	-1.7	-1.3	-0.9	-0.5

Table 5: Log-wealth of the richest individuals. Data is from Forbes 400. The log wealth of the 100th richest individual is normalized to zero.

are extremely skewed; only a small number of arriving firms will end up valuable. The random and extremely skewed allotment of new innovations is a source of risk for arriving entrepreneurs, and also for old investors whose shares in existing firms lose value in response to large creative-destruction shocks. The financial sector plays a key role in the model as a device to facilitate risk sharing across arriving entrepreneurs and existing investors ("inter-cohort risk sharing") and within cohorts of existing investors ("intra-cohort risk sharing").

Investing in every single arriving new venture is infeasible. In addition, entrepreneurs are forced to retain a stake in their corporation. These two assumptions, along with the extremely skewed distribution of the gains from innovation, imply that (a) both inter- and intra-cohort risk sharing are imperfect and there is dispersion across investor returns in their private-equity investments, and (b) a set of measure zero of entrepreneurs become ultra rich.

Using this model we show several results: (a) It is possible to reconcile some seemingly contradictory patterns in the returns to private and public equities. In particular, the value premium, which requires that public-equity growth options command a low risk premium, is consistent with the seemingly high risk premium commanded by growth-option-intensive venture capital investments. (b) The "public market equivalent" approach to risk-adjusting the returns of private equity has an expected value higher than one. This result holds even though all expected returns in our model reflect exclusively compensation for risk. Thus, PME values above unity are not necessarily an indication of outperformance. (c) The wealth dispersion among the ultra rich forms at an early stage. The cohort of new entrants into the distribution of the ultra rich have a wealth dispersion not too dissimilar with the existing rich. This suggests that entry and displacement are important forces to explain the dynamics of the wealth distribution of the ultra rich. (d) Land and natural resources are attractive as displacement-risk hedges, since they are useful to all firms, young and old.

Therefore, the model can account for a ranking of expected returns, whereby private equity investments command the highest risk compensation followed by public equities, followed by land and natural resources. (e) The model makes joint predictions for asset returns, aggregate growth, inequality, and the relative size of the private equity industry as a fraction of total investments. Taken together, these predictions allow us to better understand the type of economic forces that are likely to be driving broad economic trends. For instance, increased creative destruction leads to strong growth, increased displacement of old rich by young rich, strong (realized) returns to investing in new ventures, and a larger share of investments in private equity, consistent with the events of the mid-to-late nineties. By contrast, a growing share of private equity investments, but with unchanged GDP growth, smaller displacement of old by young rich, less dispersed PE returns, and comparatively weaker returns to investing in new ventures (consistent with the post-2000 experience), may be just the result of a more developed private-equity industry that facilitates a larger quantity of transfers.

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Selective List of Mathematical Symbols

A_t	Total number of blueprints	p. 12
$M_{i,t+1}/M_{i,t}$	SDF of an individual investor	p. 22
M_{t+1}/M_t	Average SDF, $M_{t+1}/M_t = \mathrm{E}_t[M_{i,t+1}/M_{i,t} \mid \Gamma^E_{t+1}, \Gamma^N_{t+1}, \Gamma^U_{t+1}]$	p. 22
R^E_{t+1}	Return on market portfolio	p. 18
$R^N_{i,t+1}$	PE return to agent in location i	p. 18
$V_{t,s}$	Value function	p. 12
$W_{i,t}$	Wealth of agent in location i	p. 18
$X_{i,t+1}$	Idiosyncratic component of individual wealth growth	p. 21
Y_t	Aggregate output	p. 12
Z_t	Neutral productivity factor	p. 12
Γ^E_t	Number of new blueprints distributed to existing firms	p. 13
Γ^N_t	Number of new blueprints distributed to entrepreneurs	p. 13
Γ^U_t	Number of new blueprints completely untradable	p. 13
Γ_t	$\Gamma_t = \Gamma^E_t + \Gamma^N_t + \Gamma^U_t$	p. 13
Π^E_t	Market value of a blueprint	p. 18
$\Pi^N_{j,t}$	Price at which entrepreneur j sells share; $\Pi^N_{j,t} = \Pi^N_t \; \forall j$ in equilibrium	p. 17
$H_{i,t+1}$	Auxiliary variable: $H_{i,t+1} = \left(1 + \eta \Gamma^E_{t+1}\right)^{-1} \times \frac{\eta \upsilon}{\Delta} \int_{i-\Delta/2}^{i+\Delta/2} d\Gamma^N_{i,t+1}$	p. 20
α	Quantifies effect of new blueprints on existing blueprint profits	p. 12
β	Time discount rate	p. 12
η	Scaling factor for Γ^E_t, Γ^N_t, and Γ^U_t	p. 13
γ	Risk aversion	p. 12
P^N	Normalized value of PE share	p. 20
$\hat{\beta}$	Effective time discount rate	p. 12
λ	Death probability and entry rate	p. 12
ϕ^B	Portfolio allocation to risk-free asset	p. 18
ϕ^E	Portfolio allocation to market asset	p. 18
ϕ^N	Portfolio allocation to PE	p. 18
π_t	Profit accruing to a single blueprint	p. 12
P^E	Normalized value of a blueprint, $P^E = \frac{\Pi^E_t}{\pi_t}$	p. 20
P^N	Normalized value of a PE share, $P^N = \frac{P_t}{A_t \pi_t}$	p. 20
θ	Fraction of agents who are entrepreneurs	p. 13
υ	Fraction of own firm an entrepreneur sells	p. 16
ε_{t+1}	Log growth in Z_t	p. 12
$\zeta_{j,t}$	Idiosyncratic shock determining exposure of $q_{j,t+1}$ to $\Gamma^N_{t+1} + \Gamma^U_{t+1}$	p. 31
a^l	Shape parameter of Γ^l	p. 13
b^l	Rate parameter of Γ^l	p. 13
c	Consumption-to-wealth ratio	p. 20
$c_{t,s}$	Consumption	p. 12
$dL_{j,t+1}$	Proportion of Γ^N_{t+1} accruing to location j: $dL_{j,t+1} = d\Gamma^N_{j,t+1}/\Gamma^N_{t+1}$	p. 21
$q_{j,t+1}$	Firm j profit growth normalized by market profit growth	p. 31
r^f_t	Interest rate	p. 18

A Proofs

Proof of Proposition 1. We start by positing that, with $R^f = 1 + r^f$, the portfolio choice of all investors obeys the pair of Euler equations

$$\mathrm{E}_t\left[\left(\phi^B R^f + \phi^E R^E_{t+1} + \phi^N R^N_{i,t+1}\right)^{-\gamma}\left(R^E_{t+1} - R^N_{i,t+1}\right)\right] = 0 \tag{29}$$

$$\mathrm{E}_t\left[\left(\phi^B R^f + \phi^E R^E_{t+1} + \phi^N R^N_{i,t+1}\right)^{-\gamma}\left(R^E_{t+1} - R^f\right)\right] = 0. \tag{30}$$

Using the definitions of ϕ^E and ϕ^N and imposing $\phi^B = 0$ and market clearing in the stock markets implies $\phi^E = \frac{P^E}{P^E+P^N}$ and $\phi^N = 1 - \phi^E$. Accordingly, using (10) and (11),

$$\phi^B R^f + \phi^E R^E_{t+1} + \phi^N R^N_{i,t+1} = \frac{P^E}{P^E + P^N} R^E_{t+1}\left(1 + H_{i,t+1}\right). \tag{31}$$

Using (31) and (11) inside (29) and noting that $\frac{P^E}{P^N} = \frac{\phi^E}{1-\phi^E}$ leads to (13).

Having determined ϕ^E, it is straightforward to determine P^E and P^N. To start, we note that with unitary IES $c = 1 - \hat{\beta}$. Aggregating (7) across agents and imposing asset market clearing implies

$$W_t = A_t \pi_t \left(P^E + P^N + 1\right). \tag{32}$$

Using goods market clearing, with aggregate consumption equal to $A_t\pi_t$, along with $c = 1-\hat{\beta}$ we obtain

$$1 + P^E + P^N = \frac{1}{1-\hat{\beta}}. \tag{33}$$

Combining (33) with $\phi^E = \frac{P^E}{P^E+P^N}$ results in (14)–(15).

Using (31) together with (30) for portfolio choice, yields (16).

The final step of the proof is to show that equations (29) and (30) characterize the portfolio choice of all investors. Clearly, this is true for all investors born at times $s < t$, since the probability of death is age independent and all investors have access to the same return distributions. Equations (29) and (30) also describe the portfolio choices of investors arriving at the beginning of $t+1$, who are simply endowed with the blueprints Γ^U_{t+1}. When $\upsilon < 1$, the Euler equations of investors born at time t, who could receive some additional allocation of blueprints at time $t+1$, is in principle different. Specifically, equations (29) and (30) become

$$\lim_{h\to 0} \mathrm{E}_t \left[\left(\phi^B R^f + \phi^E R^E_{t+1} + \phi^N R^N_{i,t+1} + K \frac{1}{h} \int_{i-\frac{h}{2}}^{i+\frac{h}{2}} d\Gamma^N_{j,t+1} \right)^{-\gamma} \left(R^E_{t+1} - R^N_{i,t+1} \right) \right] = 0 \qquad (34)$$

$$\lim_{h\to 0} \mathrm{E}_t \left[\left(\phi^B R^f + \phi^E R^E_{t+1} + \phi^N R^N_{i,t+1} + K \frac{1}{h} \int_{i-\frac{h}{2}}^{i+\frac{h}{2}} d\Gamma^N_{j,t+1} \right)^{-\gamma} \left(R^E_{t+1} - R^f \right) \right] = 0. \qquad (35)$$

where $K = \frac{(1-\upsilon)\lambda}{P^N}$ is constant.

Lemma 2, which we state and prove below, ensures that for any $\varepsilon > 0$, the probability of the event $\frac{1}{h}\int_{i-\frac{h}{2}}^{i+\frac{h}{2}} d\Gamma^N_{j,t+1} > \varepsilon$ can be made less than ε by limiting h to be small enough. Additionally, on the complementary, ε-probability event, the kernel in (34)–(35) is bounded above by the kernel in (29)–(30), namely $\left(\phi^B R^f + \phi^E R^E_{t+1} + \phi^N R^N_{i,t+1}\right)^{-\gamma}$. Since ε can be chosen to be arbitrarily small, any solution to the set of Euler equations (29)–(30) also solves (34)–(35). ■

Proof of Proposition 2. Equation (17) follows from (32) and (4). To arrive at equation (18), we note that equation (33) and $c = 1 - \hat{\beta}$ imply

$$1 - c = \frac{P^E + P^N}{1 + P^E + P^N}. \qquad (36)$$

■

Proof of Proposition 3. Throughout this proof, we will make use of the following lemma, which we prove after the proof of the proposition.

Lemma 1 *Let X_i, $i = 1, \ldots, n$, $n \geq 1$, Y_i, $i = 1, \ldots, p$, $p \geq 0$, and Z_i, $i = 1, \ldots, q$, $q \geq 0$, be independent (one-dimensional) non-trivial random variables and functions $f : \mathbb{R}^{n+p} \to \mathbb{R}$ and $g : \mathbb{R}^{n+q} \to \mathbb{R}$ non-decreasing in the first n arguments and strictly increasing in at least one of the first n arguments. Then*

$$Cov(f(X_1, \ldots, X_n, Y_1, \ldots, Y_p), g(X_1, \ldots, X_n, Z_1, \ldots, Z_q)) > 0. \qquad (37)$$

To prove statement (i) of the proposition, note that when when $\upsilon = 1$ and $\Delta = 1$,

$\frac{M_{t+1}}{M_t} \propto \left(1+\eta\Gamma^E_{t+1}+\eta\Gamma^N_{t+1}+\eta\Gamma^U_{t+1}\right)^{-\gamma(1-\alpha)}$, which is decreasing in $\Gamma^N_{t+1}, \Gamma^E_{t+1}$, and Γ^U_{t+1}. Applying Lemma 1 with $f(\Gamma^E_{t+1}, \Gamma^N_{t+1}, \Gamma^U_{t+1}) = -\left(1+\eta\Gamma^E_{t+1}+\eta\Gamma^N_{t+1}+\eta\Gamma^U_{t+1}\right)^{-\gamma(1-\alpha)}$ and $g(\Gamma^E_{t+1}, \Gamma^N_{t+1}, \Gamma^U_{t+1}) = \Gamma^N_{t+1}$ leads to statement (i).

To prove statement (ii), note first that, for $\upsilon = 0$,

$$\frac{M_{t+1}}{M_t} \propto \left(R^E_{t+1}\right)^{-\gamma} \propto (1+\eta\Gamma^E_{t+1})^{-\gamma}\left(1+\eta\Gamma^E_{t+1}+\eta\Gamma^N_{t+1}+\eta\Gamma^U_{t+1}\right)^{\alpha\gamma}, \tag{38}$$

which is increasing in Γ^N_{t+1}, Γ^U_{t+1}, and $-\Gamma^E_{t+1}$. Using continuity around $\upsilon = 0$, along with Lemma 1, proves statement (ii) for small enough υ.

Finally, for the case of small Δ, we are going to use the following result, which we formalize as a lemma, and prove after the end of the proof to the proposition.

Lemma 2 *The random variable $\frac{1}{\Delta}\int_0^\Delta dL_{j,t+1}$ tends to zero in probability as Δ tends to zero. That is, for every $\varepsilon > 0$*

$$\lim_{\Delta\to 0} Prob\left(\frac{1}{\Delta}\int_0^\Delta dL_{j,t+1} < \epsilon\right) = 1. \tag{39}$$

Note now that, for any $\upsilon > 0$ and $\Delta > 0$,

$$\frac{M_{i,t+1}}{M_{i,t}} \propto \left(1+\eta\Gamma^E_{t+1}+\eta\Gamma^N_{t+1}+\eta\Gamma^U_{t+1}\right)^{\alpha\gamma}\left(1+\eta\Gamma^E_{t+1}+\frac{\eta\upsilon}{\Delta}\Gamma^N_{t+1}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_{j,t+1}\right)^{-\gamma}. \tag{40}$$

Partial differentiation of the right-hand side of (40) with respect to Γ^E_{t+1} shows that if $\frac{1}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_{j,t+1} < 1$ then $\frac{M_{i,t+1}}{M_{i,t}}$ is decreasing in Γ^E_{t+1}. Similarly, if $\frac{1}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_{j,t+1} < 1$ then $\frac{M_{i,t+1}}{M_{i,t}}$ is increasing in Γ^N_{t+1}. $\frac{M_{i,t+1}}{M_{i,t}}$ is always increasing in Γ^U_{t+1}.

Now define $Q_{i,t+1} = \frac{1}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_{j,t+1}$ and also define $f_{i,t+1} = f(\Gamma^E_{t+1}, \Gamma^N_{t+1}, \Gamma^U_{t+1}, Q_{i,t+1})$ to be equal to the right-hand side of (40). Using that $Q_{i,t+1}$ is independent of Γ^E_{t+1} and Γ^N_{t+1}, the covariance of $\frac{M_{i,t+1}}{M_{i,t}}$ and Γ^N_{t+1} is proportional to

$$\mathrm{cov}\left(f_{i,t+1}, \Gamma^N_{t+1}\right) = \mathrm{E}\left[\mathrm{cov}\left(f_{i,t+1}, \Gamma^N_{t+1}|Q_{i,t+1}\right)\right] + \mathrm{cov}\left(\mathrm{E}\left(f_{i,t+1}|Q_{i,t+1}\right), \mathrm{E}\left(\Gamma^N_{t+1}|Q_{i,t+1}\right)\right). \tag{41}$$

By the independence of $Q_{i,t+1}$ and Γ^N_{t+1}, the second term on the RHS in (41) is zero, since

$\mathrm{E}\left[\Gamma^N_{t+1}|Q_{i,t+1}\right] = \mathrm{E}\left[\Gamma^N_{t+1}\right]$. Therefore we may write

$$\begin{aligned}\operatorname{cov}\left(f_{i,t+1}, \Gamma^N_{t+1}\right) = \mathrm{E}\big[&1_{Q_{i,t+1}<1}\operatorname{cov}\left(f_{i,t+1}, \Gamma^N_{t+1}|Q_{i,t+1}\right) + \\ &1_{Q_{i,t+1}\geq 1}\operatorname{cov}\left(f_{i,t+1}, \Gamma^N_{t+1}|Q_{i,t+1}\right)\big].\end{aligned} \tag{42}$$

The first term inside the square brackets of (42) is non-negative. For the second term we have that

$$\left|\mathrm{E}\left[1_{Q_{i,t+1}\geq 1}\operatorname{cov}\left(f_{i,t+1}, \Gamma^N_{t+1}|Q_{i,t+1}\right)\right]\right| \leq \Pr\left(Q_{i,t+1} \geq 1\right) \times \sigma\left(\Gamma^N_{t+1}\right) \times \sup_{Q_{i,t+1}\geq 1} \sigma\left(f_{i,t+1}|Q_{i,t+1}\right), \tag{43}$$

where we used the Cauchy-Schwarz inequality, and due to the independence of Γ^N_{t+1} and $Q_{i,t+1}$ we have $\sigma\left(\Gamma^N_{t+1}|Q_{i,t+1}\right) = \sigma\left(\Gamma^N_{t+1}\right)$. By Lemma 2, the term $\Pr\left(Q_{i,t+1} \geq 1\right)$ approaches zero as $\Delta \to 0$. Also, $\sup_{Q_{i,t+1}\geq 1} \sigma\left(f_{i,t+1}|Q_{i,t+1}\right)$ is finite because for values $Q_{i,t+1} \geq 1$ we have

$$\sigma^2\left(f_{i,t+1}|Q_{i,t+1}\right) \leq \mathrm{E}_t\left[f^2_{i,t+1}|Q_{i,t+1}\right] \leq \mathrm{E}_t\left[f^2_{i,t+1}|Q_{i,t+1} = 1\right] < \infty, \tag{44}$$

where the second inequality follows from the fact that $\left(1 + \eta\Gamma^E_{t+1} + \eta\upsilon\Gamma^N_{t+1}Q_{i,t+1}\right)^{-2\gamma}$ is decreasing in $Q_{i,t+1}$. Using (44) and taking the limit as $\Delta \to 0$ on the right-hand side of (43) shows that the second term on the right-hand side of (42) converges to zero. This implies that $\operatorname{cov}\left(f_{i,t+1}, \Gamma^N_{t+1}\right) \geq 0$.

Finally, for part (iii) we use again Lemma 1, predicated on the strict monotonicity of $M_{i,t+1}$ in Γ^U_{t+1} and the independence of Γ^U_{t+1} from the other random variables. ■

Proof of Lemma 1. For simplicity of notation, we omit the arguments to the functions f and g. Also, we denote by $\mathbf{Y}$ and $\mathbf{Z}$ the vectors consisting of all of the random variables Y_i and Z_i, respectively. Let's first note that

$$\operatorname{Cov}(f, g) = \mathrm{E}\left[\operatorname{Cov}(f, g|\mathbf{Y}, \mathbf{Z})\right] + \operatorname{Cov}\left(\mathrm{E}\left[f|\mathbf{Y}, \mathbf{Z}\right], \mathrm{E}\left[g|\mathbf{Y}, \mathbf{Z}\right]\right) \tag{45}$$

and observe that (i) for every realization of $(\mathbf{Y}, \mathbf{Z})$ f and g are non-decreasing and further, strictly increasing in at least one argument — let that be X_1 without loss of generality; and

(ii) $\mathrm{E}\left[f|\mathbf{Y},\mathbf{Z}\right] = \mathrm{E}\left[f|\mathbf{Y}\right]$ is a function of $\mathbf{Y}$, and thus indepent of $\mathrm{E}\left[g|\mathbf{Y},\mathbf{Z}\right] = \mathrm{E}\left[g|\mathbf{Z}\right]$, which is a function of $\mathbf{Z}$, so that the second term on the right-hand side of (45) is zero.

We therefore only need to prove the statement for $p = q = 0$. We have

$$E[fg] = \mathrm{E}[\mathrm{E}[fg|X_n]] > \mathrm{E}[\mathrm{E}[f|X_n]\mathrm{E}[g|X_n]], \tag{46}$$

where we are making use of the induction hypothesis that the result holds for the $n-1$ variables X_1,...,X_{n-1}. (Note that, trivially, for every realization of X_n both f and g continue to be strictly increasing in X_1.) It also holds that $\mathrm{E}[f|X_n]$ and $\mathrm{E}[g|X_n]$ are non-decreasing functions of X_n, however, so that we can apply the result with $n = 1$ (the "classical" result) to infer

$$\mathrm{E}[\mathrm{E}[f|X_n]\mathrm{E}[g|X_n]] \geq \mathrm{E}[\mathrm{E}[f|X_n]]\mathrm{E}[\mathrm{E}[g|X_n]] = E[f]\mathrm{E}[g]. \tag{47}$$

The desired conclusion follows by combining the two inequalities. The only remaining detail to fill in is to observe that, for $n = 1$ again, if f and g are strictly increasing then $\mathrm{Cov}(f,g) > 0$, providing the initial step for the induction argument. ■

Proof of Lemma 2. The distribution of $\int_0^\Delta dL_{j,t+1}$ is beta with parameters $a^N\Delta$ and $a^N(1-\Delta)$. We wish to estimate

$$Pr\left(\int_0^\Delta dL_{j,t+1} < \varepsilon\Delta\right) = \frac{\Gamma(a^N)}{\Gamma(a^N\Delta)\Gamma(a^N(1-\Delta))}\int_0^{\varepsilon\Delta} x^{a^N\Delta-1}(1-x)^{a^N(1-\Delta)-1}dx, \tag{48}$$

where $\Gamma(\cdot)$ is the Gamma function. The right-hand side of the (48) has the same limit as $\Delta \to 0$ as

$$\lim_{\Delta\to 0}\frac{1}{\Gamma(a^N\Delta)}\int_0^{\varepsilon\Delta} x^{a^N\Delta-1}dx = \lim_{y\to 0}\frac{1}{\Gamma(y)}\int_0^{\frac{y\varepsilon}{a^N}} x^{y-1}dx = \lim_{y\to 0}\frac{(\varepsilon/a^N)^y y^y}{y\Gamma(y)} = 1, \tag{49}$$

since both the numerator and the denominator tend to 1. ■

Proof of Proposition 4. For the proof of this proposition we simplify matters by taking $\eta = 1$. (This is without loss of generality, after adjusting the rate parameter of the Gamma distribution by η.) We also economize notation by dropping time subscripts from the random variables. Let us first note that, given that $\mathrm{E}[M_i R^E] = \mathrm{E}[M_i R_i^N]$,

$$\frac{\mathrm{E}[R_i^N]}{\mathrm{E}[R^E]} > 1 \quad \Leftrightarrow \quad \frac{\mathrm{E}[M_i R_i^N]}{\mathrm{E}[M_i R^E]} < \frac{\mathrm{E}[R_i^N]}{\mathrm{E}[R^E]}. \tag{50}$$

This equivalence implies that proving the statements of the proposition amounts to proving that $\frac{\mathrm{E}[M_i d_1 R_i^N]}{\mathrm{E}[M_i d_2 R^E]} < \frac{\mathrm{E}[d_1 R_i^N]}{\mathrm{E}[d_2 R^E]}$ for any arbitrary positive scalars d_1 and d_2. This fact allows us to replace each of R_i^N and R^E with an arbitrary (scalar) multiple of itself when proving the second inequality of (50). Next note that

$$R^E \propto \xi^E \equiv e^{\varepsilon}(1 + \Gamma^E + \Gamma^N + \Gamma^U)^{-\alpha}(1 + \Gamma^E) \tag{51}$$

$$R_i^N \propto \xi_i^N \equiv e^{\varepsilon}(1 + \Gamma^E + \Gamma^N + \Gamma^U)^{-\alpha}\Gamma^N \frac{1}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_j \tag{52}$$

$$M_i \propto \zeta_i \equiv e^{\varepsilon}(1 + \Gamma^E + \Gamma^N + \Gamma^U)^{\alpha\gamma}\left(1 + \Gamma^E + \Gamma^N \frac{\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_j\right)^{-\gamma}, \tag{53}$$

and therefore we replace (50) with

$$\frac{\mathrm{E}[R_i^N]}{\mathrm{E}[R^E]} > 1 \quad \Leftrightarrow \quad \frac{\mathrm{E}[\zeta_i \xi_i^N]}{\mathrm{E}[\zeta_i \xi^E]} < \frac{\mathrm{E}[\xi_i^N]}{\mathrm{E}[\xi^E]}. \tag{54}$$

(i) Taking the limit as $\upsilon \to 0$ eliminates the last term inside the second parenthesis on the right-hand side of (53). In words, ζ_i does not depend on L. Because e^{ϵ} and $\frac{1}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_j$ are independent of all other random variables, the inequality $\frac{\mathrm{E}[\zeta_i \xi_i^N]}{\mathrm{E}[\zeta_i \xi^E]} < \frac{\mathrm{E}[\xi_i^N]}{\mathrm{E}[\xi^E]}$ simplifies to

$$\frac{\mathrm{E}\left[(1 + \Gamma^E + \Gamma^N + \Gamma^U)^{\alpha(\gamma-1)}(1 + \Gamma^E)^{-\gamma}\Gamma^N\right]}{\mathrm{E}\left[(1 + \Gamma^E + \Gamma^N + \Gamma^U)^{\alpha(\gamma-1)}(1 + \Gamma^E)^{1-\gamma}\right]} > \frac{\mathrm{E}\left[(1 + \Gamma^E + \Gamma^N + \Gamma^U)^{-\alpha}\Gamma^N\right]}{\mathrm{E}\left[(1 + \Gamma^E + \Gamma^N + \Gamma^U)^{-\alpha}(1 + \Gamma^E)\right]}, \tag{55}$$

which is the same as

$$\frac{\mathrm{E}\left[(\xi^E)^{1-\gamma}\frac{\Gamma^N}{1+\Gamma^E}\right]}{\mathrm{E}\left[(\xi^E)^{1-\gamma}\right]} > \frac{\mathrm{E}\left[\xi^E \frac{\Gamma^N}{1+\Gamma^E}\right]}{\mathrm{E}\left[\xi^E\right]}. \tag{56}$$

We note that ξ^E increases with Γ^E and decreases with Γ^N, precisely the opposite pattern from $\frac{\Gamma^N}{1+\Gamma^E}$. We next use Lemma 1 with $X_1 \equiv \Gamma^N$, $X_2 \equiv (1 + \Gamma^E)^{-1}$, and $X_3 \equiv \Gamma^U$, and treat $-\xi^E$ as an increasing function in each X_i, as is $\Gamma^N/(1 + \Gamma^E)$. We then apply the same logic to $(\xi^E)^{1-\gamma}$. Consequently, we have

$$\operatorname{Cov}\left(\xi^E, \frac{\Gamma^N}{1+\Gamma^E}\right) < 0 < \operatorname{Cov}\left((\xi^E)^{1-\gamma}, \frac{\Gamma^N}{1+\Gamma^E}\right). \tag{57}$$

Inequality (56) now follows using the definition of covariance.

(ii) For this part we rely on the fact that $\frac{1}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_j$ tends to 0 in probability as $\Delta \to 0$ (Lemma 2). We next prove that this property implies that $\mathrm{E}[\zeta_i \xi_i^N] \to 0$ as $\Delta \to 0$, where $\zeta_i \xi_i^N$ is given by

$$\begin{aligned}(1+\Gamma^E+&\Gamma^N+\Gamma^U)^{\alpha(\gamma-1)}\left(1+\Gamma^E+\Gamma^N\frac{\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_j\right)^{-\gamma}\Gamma^N\frac{\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_j \\ &\leq (1+\Gamma^E+\Gamma^N+\Gamma^U)^{\alpha(\gamma-1)}\left(1+\Gamma^N\frac{\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_j\right)^{-1}\Gamma^N\frac{\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_j.\end{aligned} \tag{58}$$

To show this result, define the random variables X and Y as

$$X = (1+\Gamma^E+\Gamma^N+\Gamma^U)^{\alpha(\gamma-1)} \tag{59}$$

$$Y = \Gamma^N\frac{\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_j \tag{60}$$

and note that Y tends to zero in probability as Δ goes to 0.

For an arbitrary ε, choose Δ such that $Y < \varepsilon$ with probability $1-\varepsilon$. We have

$$\begin{aligned}\mathrm{E}\left[X\frac{Y}{1+Y}\right] &= \mathrm{E}\left[X1_{X<\varepsilon^{-\frac{1}{2}}}\frac{Y}{1+Y}1_{Y<\varepsilon}\right] + \mathrm{E}\left[X1_{X<\varepsilon^{-\frac{1}{2}}}\frac{Y}{1+Y}1_{Y\geq\varepsilon}\right] + \\ &\quad \mathrm{E}\left[X1_{X\geq\varepsilon^{-\frac{1}{2}}}\frac{Y}{1+Y}\right] \\ &\leq \mathrm{E}\left[\varepsilon^{-\frac{1}{2}}1_{X<\varepsilon^{-\frac{1}{2}}}\varepsilon 1_{Y<\varepsilon}\right] + \mathrm{E}\left[\varepsilon^{-\frac{1}{2}}1_{X<\varepsilon^{-\frac{1}{2}}}1_{Y\geq\varepsilon}\right] + \mathrm{E}\left[X1_{X\geq\varepsilon^{-\frac{1}{2}}}\right] \\ &\leq \varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} + \mathrm{E}\left[X1_{X\geq\varepsilon^{-\frac{1}{2}}}\right],\end{aligned} \tag{61}$$

where the very last term tends to zero as $\varepsilon \to 0$ since X has finite mean.

On the other hand, $\mathrm{E}[\zeta_i\xi^E]$ and $\mathrm{E}[\xi_i^N]$ are bounded below away from 0 (since $\frac{1}{\Delta}\mathrm{E}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_j = 1$ for all $\Delta > 0$), implying that $\mathrm{E}[R_i^N] > \mathrm{E}[R^E]$. ■

Proof of Proposition 5. The proof for the log case ($\gamma = 1$) is contained in the text. As long as $R^N_{i,t+1}$ and R^E_{t+1} are not fully correlated, the expectation $\mathrm{E}\left[\frac{R^N_{i,t+1}}{R^E_{t+1}}\right]$ is strictly larger than 1 when $\gamma = 1$, and therefore by continuity also for γ close enough to 1. ■

Proof of Proposition 6. It follows from (13) that

$$\phi^N = \frac{\mathrm{E}_t\left[\left(R^E_{t+1}\right)^{1-\gamma}\left(1+H_{i,t+1}\right)^{-\gamma} H_{i,t+1}\right]}{\mathrm{E}_t\left[\left(R^E_{t+1}\right)^{1-\gamma}\left(1+H_{i,t+1}\right)^{1-\gamma}\right]}, \tag{62}$$

which specializes to

$$\phi^N = \mathrm{E}_t\left[\left(1+H_{i,t+1}\right)^{-1} H_{i,t+1}\right] \tag{63}$$

in the logarithmic utility case. The term inside the expectation is an increasing and concave function of $H_{i,t+1}$.

Since $H_{i,t+1}$ increases strictly in η and in υ, so does ϕ^N. Further, an increase in Δ results in a dominating random variable $H_{i,t+1}$ in the sense of second-order stochastic dominance, and therefore a higher expectation in (63).

Having established strict monotonicity in the logarithmic-utility case, it follows by continuity that, for any compact set of parameters, there exists an open set for γ containing the value 1 on which the proposition holds. ■

Proof of Proposition 7. The price of a firm i is given by

$$\Pi^E_{i,t} = E_t\left[\frac{M_{t+1}}{M_t}\left(A_{i,t+1}\pi_{t+1} + \Pi^E_{i,t+1}\right)\right].$$

Using the assumption that the shock ζ_i is drawn in an i.i.d. fashion across periods and that the price-to-dividend ratio, $\frac{\Pi^E_{i,t}}{A_{i,t}\pi_t}$, is only a function of i, we obtain

$$\frac{\Pi^E_{i,t}}{A_{i,t}\pi_t} = E_t\left[\frac{M_{t+1}}{M_t}\frac{A_{i,t+1}\pi_{t+1}}{A_{i,t}\pi_t}\right]\left(1 + E_i\frac{\Pi^E_{i,t+1}}{A_{i,t+1}\pi_{t+1}}\right), \tag{64}$$

where E_i denotes averaging across i. The i.i.d. draws of ζ_i across periods imply that $E_t\left(R_{i,t+1}\right) = E_t\left[\left(\frac{A_{i,t+1}\pi_{t+1}}{A_{i,t}\pi_t}\right)\right] \times \left(\frac{\Pi^E_{i,t}}{A_{i,t}\pi_t}\right)^{-1} \times \left(1 + E_i\frac{\Pi^E_{i,t+1}}{A_{i,t+1}\pi_{t+1}}\right)$ and therefore equation (64) can be written as

$$1 = \mathrm{E}_t \left[\frac{M_{t+1}}{M_t} \frac{\frac{A_{i,t+1}\pi_{t+1}}{A_{i,t}\pi_t}}{E\left(\frac{A_{i,t+1}\pi_{t+1}}{A_{i,t}\pi_t}\right)} \right] \mathrm{E}_t \left(R_{i,t+1} \right),$$

which, using $(1+r^f)^{-1} = \mathrm{E}_t \left[\frac{M_{t+1}}{M_t} \right]$, gives

$$\begin{aligned} \frac{1+r^f}{\mathrm{E}_t \left[R_{i,t+1} \right]} &= \mathrm{E}_t \left[\frac{\frac{M_{t+1}}{M_t}}{\mathrm{E}_t \left[\frac{M_{t+1}}{M_t} \right]} \frac{\frac{A_{i,t+1}\pi_{t+1}}{A_{i,t}\pi_t}}{\mathrm{E}_t \left(\frac{A_{i,t+1}\pi_{t+1}}{A_{i,t}\pi_t} \right)} \right] \\ &= 1 + \mathrm{Cov}_t \left(\frac{M_{t+1}}{M_t} / \mathrm{E}_t \left[\frac{M_{t+1}}{M_t} \right], \frac{A_{i,t+1}\pi_{t+1}}{A_{i,t}\pi_t} / \mathrm{E}_t \left[\frac{A_{i,t+1}\pi_{t+1}}{A_{i,t}\pi_t} \right] \right). \end{aligned} \tag{65}$$

Since this equation applies for any i, it implies equation (27). ■

B Micro-Foundations for the Endowment Assumptions

In the text, we simply postulated aggregate output, the profit share and the labor share exogenously. Here we show how to micro-found (2), and (3) in a production economy.

Specifically, assume that there is a continuum of intermediate-good firms that own non-perishable blueprints. Each blueprint allows the production of an intermediate good. The final good is produced by a representative, competitive firm, which purchases x_{jt} units of each intermediate good j and produces Y_t units of the final good. The production function is given by

$$Y_t = Z_t (L_t^F)^{1-\alpha} \int_0^{A_t} x_{jt}^{\alpha} dj$$

where Z_t is neutral productivity,[46] A_t is the number of blueprints, L_t^F is the amount of labor used in the production of final goods, and x_{jt} is the input of the intermediate input j. The intermediate good j is supplied by firms engaging in monopolistic competition, and the production of one unit of the intermediate good j requires one unit of labor. Labor is supplied inelastically and the total measure of workers is $1-\theta$.

[46]By "neutral" we mean productivity that does not lead to disruption of the profit shares of existing firms, as we explain below.

The final-goods firm profit function is $Y_t - \int p_{jt}x_{jt} - w_t L_t^F$, where w_t is the prevailing wage and p_{jt} are the prices of each intermediate good. Maximizing over w_t leads to $w_t L_t^F = (1-\alpha) Y_t$, and maximizing over x_{jt} leads to the demand function for intermediate goods, $p_{jt} = Z_t \left(L_t^F\right)^{1-\alpha} \alpha x_{jt}^{\alpha-1}$. Aggregating over j and using the definition of Y_t implies that $\int_0^{A_t} p_{jt}x_{jt}dj = \alpha Y_t$.

In turn, the maximization problem of the intermediate-goods firm is $\max_{x_{jt}} p_{jt}(x_{jt}) x_{jt} - w_t x_{jt}$, where we have used the assumption that one unit of x_{jt} requires one unit of labor. The first-order condition is the familiar pricing rule $p_{jt} = \frac{w_t}{\alpha}$. As a result, aggregate profits are equal to $\int_0^{A_t} (p_{jt} - w_t) x_{jt}dj = (1-\alpha) \int_0^{A_t} p_{jt}x_{jt}dj = \alpha(1-\alpha) Y_t$. (Note that the final goods firms make zero profits.)

Since all firms face the same wage, w_t, the pricing rule $Z_t \left(L_t^F\right)^{1-\alpha} \alpha x_{jt}^{\alpha-1} = p_{jt} = \frac{w_t}{\alpha}$ implies that $x_{jt} = x_t$ is identical across j. Producing one unit of j takes one unit of labor and therefore $x_t = \frac{1}{A_t} L_t^I$, where L_t^I is the total labor employed by intermediate goods firms. Accordingly, $Y_t = Z_t \left(L_t^F\right)^{1-\alpha} \int_0^{A_t} x_t^{\alpha} dj = Z_t \left(L_t^F\right)^{1-\alpha} \left(L_t^I\right)^{\alpha} A_t^{1-\alpha}$. In turn, the first order conditions $L_t^F = \frac{(1-\alpha)Y_t}{w_t}$ (for the final goods firm) and $L_t^I = \alpha^2 \frac{Y_t}{w_t}$ for the aggregate labor demand of intermediate goods firms imply that $\frac{L_t^F}{L_t^I}$ is constant. Because $L_t^I + L_t^F = L$, it follows that both $L_t^I = L^I$ and $L_t^F = L^F$ are constants, and therefore (2) follows. (Note that we can always L so that $\left(L_t^F\right)^{1-\alpha} L_t^I = 1$)

Finally, because $x_{jt} = x_t$ and $p_{jt} = p_t$, it follows that the profits per intermediate good firm are equal to each other and given by $\pi_t = \frac{\int_0^{A_t}(p_{jt}-w_t)x_{jt}dj}{A_t} = \alpha(1-\alpha)Y_t$, which is (3).

In Section 5.2 we introduce a non-displaceable factor. In the context of the micro-foundation of our model, the introduction of such a production factor (say "land") is straightforward. To be as explicit as possible that land is not tied to any intermediate good, we assume that land is useful only in the production of the final good. Land is owned by existing agents and rented out to final-good producing firms, so that aggregate output is given by

$$Y_t = Z_t F_t^{\zeta} \left(L_t^F\right)^{1-\alpha-\zeta} \int_0^{A_t} x_{j,t}^{\alpha} dj, \tag{66}$$

where $\zeta \in (0, 1-\alpha)$ is the share of output that accrues to land. Total land is fixed and normalized to one ($F_t = 1$). Given the Cobb-Douglas structure of (66), it follows that the rental rate of land is $r_t^F = \zeta Y_t$.

In Section 5.4 we consider an extension where labor is a composite good, so that workers of different cohorts obtain different wages $w_{t,s}$. The Cobb-Douglas specification in (28) implies that the wage bill of cohort s is $w_{t,s}l_{t,s} = \frac{a_{t,s}}{\sum_{s\leq t} a_{t,s}}(1-\alpha(1-\alpha))Y_t$. In words, this means that the wage bill of cohort s is a fraction $\frac{a_{t,s}}{\sum_{s\leq t} a_{t,s}}$ of aggregate labor income $(1-\alpha(1-\alpha))Y_t$. Because the endowments (at birth) of a worker-cohort born at time s and an entrepreneur-cohort born at time s exhibit the same growth rates, the consumption growth rates of workers and entrepreneurs are aligned.