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STRATEGYPROOFNESS-EXPOSING DESCRIPTIONS OF MATCHING MECHANISMS

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ABSTRACT

A menu description exposes strategyproofness by presenting a mechanism to player i in two steps. Step (1) uses others' reports to describe i's menu of potential outcomes. Step (2) uses i's report to select i's favorite outcome from her menu. We provide novel menu descriptions of the Deferred Acceptance (DA) and Top Trading Cycles (TTC) matching mechanisms. For TTC, our description additionally yields a proof of the strategyproofness of TTC's traditional description, in a way that we prove is impossible for DA.

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1 Introduction

Strategyproof mechanisms are often considered desirable. Under standard economic assumptions, these mechanisms eliminate the need for players to strategize, since straightforward play is a dominant strategy.[1] In practice, however, real participants in strategyproof mechanisms often play theoretically dominated strategies, raising the possibility that they do not perceive the mechanisms as strategyproof (see, e.g., Hakimov and Kübler 2021; Rees-Jones and Shorrer 2023).

In this paper, we posit that the way mechanisms are *described* can influence the extent to which participants perceive strategyproofness. In contrast to other recent works, which have sought to encourage straightforward play by implementing a given choice rule through different interactive mechanisms,[2] we propose changing only the (ex ante) description of a given static, direct-revelation mechanism. We propose a general outline—called *menu descriptions*—for describing mechanisms to one player at a time in a way that makes strategyproofness hold via an elementary, one-sentence argument. In this sense, menu descriptions expose strategyproofness.

Our focus is on matching, and particularly on two canonical mechanisms: Deferred Acceptance (henceforth DA) and Top Trading Cycles (henceforth TTC). These mechanisms are widely and successfully deployed. They are typically explained to participants using *outcome descriptions*—i.e., detailed and explicit algorithms for calculating the outcome matching. However, the traditional (outcome) descriptions of these mechanisms do not expose their strategyproofness, in the sense that proving this property from these descriptions requires technical mathematical arguments.

We present three main results. The first is a new menu description of DA. The second is a new menu description of TTC, which furthermore yields a new proof of the strategyproofness of the traditional description of TTC. The third is an impossibility result showing that such a proof via a menu description cannot work for the traditional description of DA.

As an initial illustration, consider the canonical Serial Dictatorship (henceforth SD) mechanism. When matching applicants to institutions,[3] the traditional descrip-

[1] We use the term "straightforward" to describe the strategy an agent would play under classic economic assumptions. While often referred to as the "truthtelling" strategy, we avoid this morally laden term, since deviations from this strategy should not be thought of as dishonesty.

[2] Prior works often consider interactive mechanisms designed to reduce non-straightforward play arising from behavioral factors, such as contingent-reasoning failures (Li, 2017; Pycia and Troyan, 2023) and loss aversion (Dreyfuss et al., 2022; Meisner and von Wangenheim, 2023).

[3] Throughout this paper, the only strategic players are the applicants. Institutions are non-

tion of SD is as follows: In some priority order, say $i = 1, \ldots, n$, applicant i is matched to her highest-ranked not-yet-matched institution. Strategyproofness is exposed by this description: Applicant i cannot influence the set of not-yet-matched institutions, and straightforward reporting guarantees i her favorite not-yet-matched institution. Our paper presents new descriptions of DA and TTC that make strategyproofness as evident as in the traditional description of SD.

In Section 2, we provide preliminaries. We study descriptions in terms of the classic notion of a *menu* (Hammond, 1979)—the set of all institutions an applicant might match to, given others' reports. In particular, a *menu description* for applicant i has the following two-step outline:

Step (1) uses only the reports of other applicants to describe i's menu.

Step (2) says that i's match is her highest-ranked institution from her menu.

For instance, the traditional description of SD is a menu description. In contrast with some other mechanisms' traditional descriptions, strategyproofness is exposed by any menu description in the same way as in SD: In Step (1), applicant i cannot influence her menu (in SD, the set of not-yet-matched institutions), and in Step (2), straightforward reporting guarantees i her favorite institution from her menu.

In Section 3, we present our first main result: A novel menu description of DA. Our description—which is summarized in Table 1—describes applicant i's menu as all institutions that prefer i to their outcome in "flipped-side-proposing" Deferred Acceptance without i. This directly conveys i's match in DA, while exposing strategyproofness for i. Prior to our work, it was not clear how to construct DA's menu, except via a trivial brute-force solution (applicable for any strategyproof mechanism)

Table 1: Two descriptions of DA (the applicant-optimal stable match)

Traditional Descr.:	Menu Description:
The applicants and institutions will be matched using the *applicant-proposing Deferred Acceptance* algorithm.	We will run *institution-proposing Deferred Acceptance* with all applicants *except you*, to obtain a hypothetical matching. Your menu consists of every institution that ranks you higher than its hypothetically matched applicant. You will be matched to the institution that you *ranked highest* out of your menu.

Note: In the menu description, others' hypothetical matches need not be their matches in DA.

strategic, and their preferences over the applicants are by convention called *priorities*.

involving running the traditional description many times to separately check whether or not each institution is on i's menu.

In Section 4, we present our second main result: A novel menu description of TTC, which furthermore yields a novel proof of the strategyproofness of TTC's traditional description. Our menu description is contained in an outcome description. We call any such description a *menu-in-outcome* description; such descriptions have the following three-step outline:

Step (1) uses only the reports of other applicants to describe i's menu.

Step (2) says that i's match is her highest-ranked institution from her menu.

Step (3) describes the rest of the matching (for all other applicants).

Our description of TTC, and our resulting proof of the strategyproofness of the traditional description, are as follows. TTC's traditional description works in terms of "eliminating trading cycles," and it is well known that these cycles can be eliminated in any order. Our menu-in-outcome description is a slight tweak of the traditional one: It differs only by changing the order in which cycles are eliminated (by eliminating the cycle involving applicant i as late as possible). Thus, for any applicant i, the match of i in the traditional description equals her match in our menu description—which (like all menu descriptions) is strategyproof. This proves that TTC's traditional description is strategyproof.[4]

In Section 5, we present our third main result. We ask: Like in our result for TTC, is there a menu description of DA that yields a proof of the strategyproofness of its traditional description through a slight tweak of it? We present an impossibility theorem showing that the answer is *no*.

To establish a formal notion of a "slight tweak," we consider three salient properties of DA's traditional description:

- It calculates the entire outcome matching; i.e., it is an outcome description.
- It looks at applicants' preferences in favorite-to-least-favorite order; we say that such descriptions are *applicant-proposing*.

[4] Following the first appearance of our paper, the survey article of Morrill and Roth (2024) adopted our proof of TTC's strategyproofness. Regarding the potential real-world adoption of TTC for public school choice, Morrill and Roth write:

> "Our experience [. . .] taught us that when we worked with school districts, we should help design not just a mechanism, but also the communication package that explained that mechanism [. . .]. Perhaps if we had already known of the proof of [Gonczarowski, Heffetz, and Thomas] we could have explained [TTC's strategyproofness] more clearly."

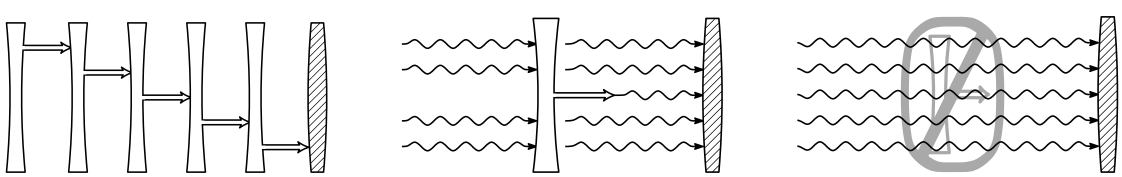
- It maintains bookkeeping by only tracking and iteratively modifying a single tentative matching. In particular, it uses only a small amount of bookkeeping per applicant; in computer science terms, such descriptions are *linear-memory*.

We formalize our impossibility theorem by treating the satisfaction of these three properties as a prerequisite for a description to qualify as a slight tweak of DA's traditional description. Indeed, not satisfying the first property means reaching a different final result than the traditional description, and not satisfying the second or third properties means having step-by-step calculations that are very different from those of the traditional one.[5]

Our third main result proves that, in contrast to (SD and) TTC, no slight tweak of DA's traditional description contains a menu description. Concretely, we prove that no menu-in-outcome description of DA is applicant-proposing and linear-memory. In fact, we prove a strong impossibility: Any applicant-proposing menu-in-outcome description of DA must use *quadratic* memory, an amount far greater than any description tracking only a single tentative matching. Thus, our approach above that exposes the strategyproofness of TTC's traditional description cannot work for DA.

Our results reveal a stark trichotomy among SD, TTC, and DA. The traditional description of SD already has each applicant choosing from her menu (see Figure 1(a) for an illustration), and thus exposes strategyproofness. For the traditional description of TTC, this is not the case, but for each applicant there is a slight tweak in which she chooses from her menu (see Figure 1(b)), which thus exposes strategyproofness.

Figure 1: Illustration of trichotomy for traditional descriptions of SD, TTC, and DA



(a) SD (b) TTC (c) DA

Notes: Each figure depicts an applicant-proposing and linear-memory outcome description, which progresses from left to right. The outcome matching is depicted as a shaded shape. Step (2) of a menu description (choice from a menu, which exposes strategyproofness for the chooser) is depicted as a white shape with an arrow. Other calculations are depicted as wavy arrows. The gray diagram in Panel (c) depicts the impossibility: DA cannot be described in finer detail as in Panels (a) and (b).

[5] For instance, our menu-in-outcome description of TTC (Section 4) is, like its traditional description, a linear-memory applicant-proposing outcome description. The same is not true for our menu description of DA (Section 3): It is *institution*-proposing, and is *not* an outcome description.

For the traditional description of DA, a comparable result is impossible, in the strong sense discussed above (see Figure 1(c)).

Table 2 summarizes our three main results discussed above. In Section 6, we review related work, including our empirical companion paper investigating our menu description of DA, and our work exploring menu descriptions in non-matching settings such as auctions and voting. We conclude in Section 7, where we also discuss potential practical concerns.

Table 2: Summary of main results

Main Result	Summary	Relevant Formal Properties
Positive result for **DA** (Sec. 3)	Description 1 is a novel description of DA (to one applicant at a time) that exposes strategyproofness.	Description 1 is a menu description of DA.
Positive result for **TTC** (Sec. 4)	Description 2 satisfies the above for TTC, and additionally yields a novel proof of the strategyproofness of the traditional description of TTC.	Description 2 of TTC is a linear-memory, applicant-proposing menu-in-outcome description.
Negative result for **DA** (Sec. 5)	Theorem 5.3 shows it is impossible to prove the strategyproofness of the traditional description of DA in the way we do for TTC (in Sec. 4).	Theorem 5.3 shows that for DA, any applicant-proposing menu-in-outcome description requires *quadratic* memory.

2 Preliminaries

2.1 Mechanisms

This paper studies (static, direct-revelation) matching mechanisms. This environment consists of n applicants $\{1, \ldots, n\}$ to be matched to institutions. Applicant i has a strict ordinal *preference* $\succ_i$ over institutions, also called her *type*. Let $\mathcal{T}_i$ denote the set of types of applicant i, and let A denote the set of matchings.[6] We focus on one-to-one matching for concreteness (though our results, particularly for DA, generalize

[6]Applicants may go unmatched, and their preference lists may be partial (indicating that they prefers to going unmatched over institutions not on her preference list). We also let $h_1 \succ_d h_2$ indicate that applicant d prefers h_1 to h_2; $h_1 \succeq_d h_2$ indicate $h_1 \succ_d h_2$ or $h_1 = h_2$; $\emptyset \succ_d h$ indicate that d does not rank h; $\mu(d)$ denote the match of d in μ; $\mu(d) = \emptyset$ denote d going unmatched; $\mathcal{T}_{-i}$ denote the set $\mathcal{T}_1 \times \ldots \times \mathcal{T}_{i-1} \times \mathcal{T}_{i+1} \ldots \mathcal{T}_n$, and for $\succ_i \in \mathcal{T}_i$ and $\succ_{-i} \in \mathcal{T}_{-i}$, we write $(\succ_i, \succ_{-i})$ for the naturally corresponding element of $\mathcal{T}_1 \times \ldots \times \mathcal{T}_n$.

substantially; see Section 3).

The applicants *report* their types to a mechanism, which determines the outcome matching. Formally, a *mechanism* is any mapping $f : \mathcal{T}_1 \times \ldots \times \mathcal{T}_n \to A$ from the reported types of all applicants to a matching. We focus on *strategyproof* mechanisms. This means that for every applicant i, every $\succ_i, \succ'_i \in \mathcal{T}_i$, and every $\succ_{-i} \in \mathcal{T}_{-i}$, we have $f_i(\succ_i, \succ_{-i}) \succeq_i f_i(\succ'_i, \succ_{-i})$, where $f_i(\succ_1, \ldots, \succ_n)$ denotes i's match in $f(\succ_1, \ldots, \succ_n)$.

We study the canonical strategyproof mechanisms SD, TTC, and DA. These mechanisms are defined with respect to *priorities* of the institutions over the applicants. Following much of the matching literature, we assume the institutions are non-strategic; hence, we treat the priorities as predetermined. SD uses a single priority order $\succ$ over all applicants; TTC and DA use a profile of priority orders $\{\succ_h\}_h$, one for each institution h.

Definition 2.1 (SD)**.** For a given priority order $\succ$, Serial Dictatorship (SD) is defined as follows. Given the reports, applicants are considered in order of highest-to-lowest priority, and each applicant is permanently matched to her highest-ranked not-yet-matched institution.

Definition 2.2 (TTC)**.** For a given profile of institutions' priorities $\{\succ_h\}_h$, Top Trading Cycles (TTC) is defined as follows. Given the reports, repeat the following until everyone is matched (or has exhausted their preference lists): Every remaining (i.e., currently unmatched) applicant "points" to her favorite remaining institution, and every remaining institution points to its highest priority remaining applicant. There must be some cycle in this directed graph (since there is only a finite number of vertices). Choose any such cycle and "eliminate" that cycle by permanently matching each applicant in the cycle to the institution she is pointing to (and removing all matched agents from consideration for later cycles).

Definition 2.3 (DA)**.** For a given profile of institutions' priorities $\{\succ_h\}_h$, Deferred Acceptance (DA) is defined as follows. Given the reports, repeat the following until every applicant is matched (or has exhausted her preference list): A currently unmatched applicant is chosen to "propose" to her favorite institution that has not yet "rejected" her. The institution then rejects every proposal except for its highest priority applicant who has proposed to it thus far. Rejected applicants become (currently) unmatched, while that highest priority applicant is tentatively matched to the institution. At the end, the tentative allocation becomes final.

Note that DA, by convention, refers to the above *applicant* proposing version of the mechanism. The sides can also be flipped, which results in the *institution* proposing variant of DA. When confusion might arise, we use APDA for applicant-proposing DA and IPDA for institution-proposing DA.

TTC and DA are the two canonical matching mechanisms that are priority-based and strategyproof. TTC is Pareto-efficient for the applicants, and DA is stable. Note that the outcomes of TTC is independent of the order in which cycles are eliminated (see Proposition C.8) and that DA is independent of the order of proposals (see Corollary C.4).

2.2 Descriptions

This paper studies ex ante descriptions of matching mechanisms, i.e., descriptions that are given before any concrete inputs are known. When matching markets are described in detail to participants, this is typically done by specifying a set of explicit, precise, step-by-step instructions for calculating the result, i.e., by specifying an algorithm.[7] Thus, we formally define a *description* to be any algorithm that uses as input the preferences of the applicants and the priorities of the institutions, and calculates some result (e.g., an outcome matching).[8]

For any mechanism f, an *outcome descriptions* of f is an algorithms that, using input $\succ_1, \ldots, \succ_n$ (and the priorities of institutions), outputs the outcome matching $f(\succ_1, \ldots, \succ_n)$. For instance, the descriptions in Definitions 2.1 through 2.3 are outcome descriptions for SD, TTC, and DA, respectively. We refer to each of these outcome descriptions as the *traditional description* of the corresponding mechanism.

Beyond outcome descriptions, we study two other description outlines: menu descriptions and menu-in-outcome descriptions; these are defined in Sections 2.3 and 2.4, respectively.

[7] Of course, the way a description/algorithm is actually conveyed to participants can vary. One common real-world approach to relaying matching algorithms is an illustrative video using an example (see, e.g., Figure 3 in Section 5.1 for such a video for DA). In our paper, we abstract over exactly how the algorithm is relayed.

[8] Algorithms can be defined in full mathematical detail in various ways; any such definition suffices for our purposes. In the Supplemental Appendix S, we present such a definition from first principles.

2.3 Menus and Menu Descriptions

The starting point of our framework for changing mechanism descriptions is the following characterization of strategyproofness in terms of applicants' *menus*.[9]

Definition 2.4 (Menu)**.** For any matching mechanism f, applicant i, and $\succ_{-i} \in \mathcal{T}_{-i}$, the *menu* $\mathcal{M}_{\succ_{-i}}$ of i with respect to $\succ_{-i}$ is the set of all institutions h for which there exists some $\succ_i \in \mathcal{T}_i$ such that $f_i(\succ_i, \succ_{-i}) = h$. That is,

$$\mathcal{M}_{\succ_{-i}} = \{\ f_i(\succ_i, \succ_{-i}) \mid \succ_i \in \mathcal{T}_i\ \}.$$

Theorem 2.5 (Hammond, 1979)**.** *A matching mechanism f is strategyproof if and only if each applicant i always receives her favorite institution from her menu. That is, for every $\succ_{-i} \in \mathcal{T}_{-i}$ and $\succ_i \in \mathcal{T}_i$, it holds that $f_i(\succ_i, \succ_{-i}) \succeq_i h$ for all $h \in \mathcal{M}_{\succ_{-i}}$.*

Proof. Suppose f is strategyproof and fix $\succ_{-i} \in \mathcal{T}_{-i}$. For every $\succ_i \in \mathcal{T}_i$, it holds by definition that for every $h = f_i(\succ'_i, \succ_{-i}) \in \mathcal{M}_{\succ_{-i}}$, we have $f_i(\succ_i, \succ_{-i}) \succeq_i h$. On the other hand, if applicant i always receives her favorite institution from her menu, then she always prefers reporting $\succ_i$ at least as much as any $\succ'_i$, so f is strategyproof. □

We use menus to describe mechanisms while exposing their strategyproofness:

Definition 2.6 (Menu Description)**.** A *menu description* of mechanism f for applicant i is a description with the following outline:

Step (1) uses only $\succ_{-i} \in \mathcal{T}_{-i}$ to calculate the menu $\mathcal{M}_{\succ_{-i}}$ of applicant i.

Step (2) uses $\succ_i \in \mathcal{T}_i$ to match applicant i to her favorite institution in $\mathcal{M}_{\succ_{-i}}$.

Formally, a menu description for i is thus an algorithm that initially receives only $\succ_{-i}$ as input and calculates $\mathcal{M}_{\succ_{-i}}$ as an intermediate result, then additionally receives $\succ_i$ as input and uses it to calculate i's favorite choice from $\mathcal{M}_{\succ_{-i}}$ as the final result.

[9] Definition 2.4 has been considered under many different names in many different contexts (e.g., *sets that decentralize the mechanism* in Hammond (1979); *option sets* in Barberà et al. (1991); *proper budget sets* in Leshno and Lo (2021); *feasible sets* in Katuščák and Kittsteiner (2025); and likely others). We follow the "economics and computation" literature (e.g., Hart and Nisan, 2017; Dobzinski, 2016; Babaioff et al., 2022) in calling these sets "menus." This notion is distinct from many other definitions of menus (e.g., those of Mackenzie and Zhou, 2022; Bó and Hakimov, 2023, and many others).

Note that while Step (1)—the calculation of $\mathcal{M}_{\succ_{-i}}$—varies between different mechanisms and menu descriptions, Step (2)—i's choice from $\mathcal{M}_{\succ_{-i}}$—is essentially the same across all menu descriptions.

The central premise of our paper is that menu descriptions are one way to expose strategyproofness. This is because the strategyproofness of any menu description can immediately be seen via a simple, one-sentence argument: First, applicant i's report cannot affect her menu, and second, straightforward reporting ("truthtelling") gets applicant i her favorite institution from the menu.[10]

2.4 Menu-in-Outcome Descriptions

Beyond menu descriptions, outcome descriptions that contain menu descriptions also play a key role in our results. We call these *menu-in-outcome descriptions*.

Definition 2.7 (Menu-in-Outcome Description)**.** A *menu-in-outcome description* of mechanism f for applicant i is an outcome description of f that contains a menu description for applicant i. Equivalently, it is a description with the following outline:

Step (1) uses only $\succ_{-i} \in \mathcal{T}_{-i}$ to calculate the menu $\mathcal{M}_{\succ_{-i}}$ of applicant i.

Step (2) uses $\succ_i \in \mathcal{T}_i$ to match applicant i to her favorite institution from $\mathcal{M}_{\succ_{-i}}$.

Step (3) uses both $\succ_i$ and $\succ_{-i}$ to calculate the full outcome matching $f(\succ_i, \succ_{-i})$.

Formally, a menu-in-outcome description for i is thus an algorithm that initially receives $\succ_{-i}$ as input and calculates $\mathcal{M}_{\succ_{-i}}$, then additionally receives $\succ_i$ as input and calculates i's favorite choice from $\mathcal{M}_{\succ_{-i}}$, and finally proceeds to calculate the entire outcome matching $f(\succ_i, \succ_{-i})$ as the final result.

For example, consider SD (Definition 2.1). This mechanism is easily seen to be strategyproof, directly from its traditional description (and even for many students encountering it for the first time). This is reflected by the fact that applicants are

[10] There is also a precise sense in which menu descriptions are the *only* ones for which the above argument for strategyproofness goes through. In particular, suppose a description calculates the match of applicant i in some mechanism f, and has the following outline:

Step (1) uses $\succ_{-i}$ to calculate a set S of institutions.

Step (2) uses $\succ_i$ to match i to her top-ranked institution in S.

Then, it is not hard to show that the set S must be i's menu.

matched in SD via menu descriptions. In particular, when applicants are prioritized $1 \succ 2 \succ \ldots \succ n$, the traditional description of SD can be divided into three steps for any applicant i:

(1) Each applicant $j < i$ is matched, in order, to her top-ranked remaining institution.

(2) Applicant i is matched to her top-ranked remaining institution.

(3) Each applicant $j > i$ is matched, in order, to her top-ranked remaining institution.

Steps (1) and (2) form menu description, but this menu description is contained within the (traditional) outcome description, and thus Steps (1) through (3) form a menu-in-outcome description.

2.5 Uses of Menu Descriptions

The positive results of our paper present new menu descriptions of DA and TTC. Before giving these results, we note that *every* strategyproof mechanism has a menu description, given by an argument from Hammond (1979).[11]

Example 2.8 (A "brute force" menu description)**.** Consider any strategyproof matching mechanism f, and let D be an outcome description of f. For each institution h, let $\{h\}$ denote the preference list that ranks only h as acceptable. Then, consider the following description for applicant i:

(1) Start with $M = \emptyset$. For each institution h separately, evaluate D on $(\{h\}, \succ_{-i})$; if this matches i to h, then add h to M.

(2) Match i to her highest-ranked institution in M.

By strategyproofness, h is included in M in Step (1) if and only if h is on the menu. Thus, the above provides a menu description of f.

We explore two uses of menu descriptions: as alternative standalone descriptions for participants, and as a tool to help illustrate the strategyproofness of a traditional description. A description such as Example 2.8 has drawbacks in both of these uses.

[11] This menu description was also identified by Katuščák and Kittsteiner (2025).

First, as a standalone description, Example 2.8 might—compared to the traditional descriptions presented in Section 2.1—be considered cumbersome or convoluted, since it repeats D many independent times. Given this, we look for more-direct new menu descriptions (such as our menu description of DA in Section 3).

Second, since there is no clear relation between the outcomes of Example 2.8 and those of the description D (absent prior knowledge that D is strategyproof), it is unclear how Example 2.8 might aid in conveying the strategyproofness of D. Given this, we look for menu descriptions that are closely related to the corresponding traditional description (such as our menu-in-outcome description of TTC in Section 4).

3 A Menu Description of DA

In this section, we present our first main result: A novel menu description of DA. This is Description 1 (which rephrases Table 1 in the introduction).

Description 1 A menu description of (*applicant*-proposing) DA for applicant i

(1) Run *institution*-proposing DA with applicant i removed from the market, to get a matching μ_{-i}. Let M be the set of institutions h such that $i \succ_h \mu_{-i}(h)$.

(2) Match i to i's highest-ranked institution in M.

Step (1) of Description 1 begins with a modified version of the traditional description of DA (from Definition 2.3). Then, it calculates i's menu as an (arguably) intuitive set of institutions: those that prefer i to their match in this modified version of DA. We speculate that many real market participants would find such a description understandable. (In fact, our empirical companion paper Gonczarowski et al. (2024) gives evidence that many lab participants can learn this description—see Section 6.)

Crucially, Description 1 uses the *institution*-proposing DA algorithm to describe DA (traditionally described via the *applicant*-proposing DA algorithm). To give intuition for why the proposing side is flipped, we show via an example that switching the proposing sides in Step (1) of Description 1 would not suffice.

Example 3.1. Consider a market with three applicants i, d_1, d_2 and two institutions h_1, h_2. Applicants have preferences $d_1 : h_1 \succ h_2$ and $d_2 : h_2 \succ h_1$, and institutions have priorities $h_1 : d_2 \succ i \succ d_1$ and $h_2 : d_1 \succ i \succ d_2$. Running *appli-*

cant-proposing DA on these preferences without i gives matching $\{(d_1, h_1), (d_2, h_2)\}$, and both h_1 and h_2 prefer i to their match. However, neither h_1 nor h_2 are on i's menu, since having i propose to any $h_i \in \{h_1, h_2\}$ (after running applicant-proposing DA without i) causes a "rejection cycle" that results in h_i rejecting i. Intuitively, institution-proposing DA fixes this issue by outputting a matching that has no potential "applicant-proposing rejection cycles."[12] Specifically, institution-proposing DA gives matching $\{(d_1, h_2), (d_2, h_1)\}$, corresponding to i's menu in this example being $\emptyset$.

Formally, the following theorem establishes the correctness of Description 1:

Theorem 3.2. *Description 1 is a menu description of DA.*

Proof. Fix institutions' priorities, an applicant i, and preferences $\succ_{-i}$ of applicants other than i. Let $\{h\}$ denote the preference list of i that reports only institution h as acceptable, and let $\emptyset$ denote the preference list of i that reports *no* institution as acceptable. For clarity, denote applicant-proposing DA by $APDA(\cdot) = DA(\cdot)$ and denote institution-proposing DA by $IPDA(\cdot)$.

Since $APDA$ is strategyproof (Theorem C.7), it suffices to prove that the set of institutions calculated in Step (1) of Description 1 is applicant i's menu. Now, for any institution h, we observe the following chain of equivalences:

$$
\begin{aligned}
& h \text{ is in the menu of } i \text{ in } APDA \text{ (with respect to } \succ_{-i}) \\
& \qquad \Longleftrightarrow \big(\text{By strategyproofness of } APDA\text{; Theorem C.7}\big) \\
& i \text{ is matched to } h \text{ by } APDA(\{h\}, \succ_{-i}) \\
& \qquad \Longleftrightarrow \big(\text{By the Lone Wolf / Rural Hospitals Theorem; Theorem C.6}\big) \\
& i \text{ is matched to } h \text{ by } IPDA(\{h\}, \succ_{-i}) \\
& \qquad \Longleftrightarrow \big(IPDA(\{h\}, \succ_{-i}) \text{ and } IPDA(\emptyset, \succ_{-i}) \text{ coincide until } h \text{ proposes to } i\big) \\
& h \text{ proposes to } i \text{ in } IPDA(\emptyset, \succ_{-i}) \\
& \qquad \Longleftrightarrow \big(IPDA(\emptyset, \succ_{-i}) \text{ and } IPDA(\succ_{-i}) \text{ produce the same matching;} \\
& \qquad\qquad \text{in } IPDA,\ h \text{ proposes in highest-to-lowest priority order}\big) \\
& i \text{ has higher priority at } h \text{ than } h\text{'s match in } IPDA(\succ_{-i}).
\end{aligned}
$$

□

[12] This intuition regarding "applicant-proposing rejection cycles" is related to the concept of an institution-improving rotation as in Gusfield and Irving (1989).

In addition to giving a arguably-appealing alternative description of DA, Theorem 3.2 provides a characterization of the menu in DA that is useful for reasoning about DA's properties. We briefly highlight two applications. First, one can immediately see from Description 1 that, if one applicant's priorities increase at some set of institutions, then (all other things being equal) the match of that applicant in DA can only improve (Balinski and Sönmez, 1999). Second, a short argument using Description 1, which we provide in Remark B.3, shows that in a market with $n+1$ applicants, n institutions, and uniformly random full length preference lists, applicants receive in DA roughly their $n/\log(n)$th choice in expectation—rather lower than in the case with n applicants, where they receive their $\log(n)$th choice—re-proving results from Ashlagi et al. (2017); Cai and Thomas (2022).

Description 1 generalizes to a broader class of stable matching markets. In fact, in Remark B.2, we observe that the same argument as in the above proof shows that a natural generalization of Description 1 characterizes the menu of DA in many-to-one markets, and even in a general class of markets with contracts, namely, those considered by Hatfield and Milgrom (2005).

Finally, we remark that Description 1 can facilitate a proof from first-principles of the strategyproofness of (traditionally described) DA (without relying on this fact as in the proof above). We give such a proof in Appendix B. While we view this proof as theoretically appealing, and perhaps useful for classroom instruction, we believe this approach remains far too mathematically involved to convey the strategyproofness of DA's traditional description to real-world participants. In contrast, if a clearinghouse directly adopts Description 1 as a way to describe participants' matches in explicit detail, then strategyproofness is exposed via a simple one-sentence argument.

4 A Menu-in-Outcome Description of TTC

In this section, we present our second main result: A novel menu description of TTC. In fact, we present a menu-in-outcome description that yields a novel proof that the traditional description of TTC is strategyproof. This is Description 2.

Description 2 modifies the traditional description of TTC (only) by delaying matching applicant i as long as possible.[13] This accurately describes the full out-

[13] This can also be thought of as running TTC, with a twist: During the first stage, applicant i does not point to any institutions. This stage lasts until no cycles exist, after which i points as

Description 2 A menu-in-outcome description of TTC for applicant i

(1) Using $\succ_{-i}$, iteratively eliminate as many cycles not involving applicant i as possible. Let M denote the set of remaining institutions.

(2) Using $\succ_i$, match i to her highest-ranked institution in M. Call this institution h.

(3) Using $(\succ_i, \succ_{-i})$, eliminate the cycle created when i points to h, then continue to eliminate cycles until all applicants match (or exhaust their preference lists).

---

come matching since, as is well known, TTC is independent of the order in which cycles are chosen to be eliminated and matched. Formally:

Theorem 4.1. *Description 2 is a menu-in-outcome description of of TTC.*

Proof. Recall that the traditional description of TTC is independent of the order in which cycles are eliminated (Proposition C.8). Now, consider modifying this traditional description by delaying matching cycles involving applicant i as long as possible, and consider the pointing graph of TTC once all remaining cycles involve applicant i. Observe that eliminating a cycle now requires matching i to her highest-ranked not-yet-matched institution; see Figure 2 for an illustration. Thus, Description 2 differs from the traditional description of TTC only in the order in which cycles are eliminated, and hence calculates the TTC outcome matching.

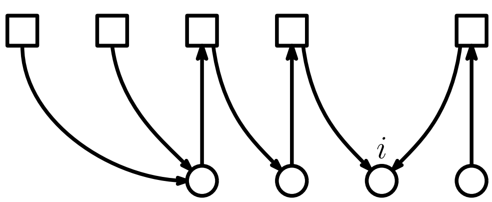


Figure 2: Menu calculation in Description 2

Notes: Circles represent applicants; squares represent institutions; each institution (resp. each applicant except i) points to her favorite remaining applicant (resp. institution). Cycles not involving i were already eliminated, so wherever i points will form a cycle.

By construction, Step (1) of Description 2 does not use $\succ_i$ to calculate the set M. Thus, since i is matched to her highest-ranked institution in M in Step (2), it follows that TTC is strategyproof. Moreover, since i can match to any institution in M (and only to institutions in M), it follows that M equals i's menu.[14] Hence, Description 2 is a menu-in-outcome description of TTC. □

normal (and immediately matches to the institution she points to).

[14] By the observation discussed in Footnote 10, the fact that M equals i's menu also follows from the fact that Description 2 calculates the outcome matching of TTC as i's top pick from M (which is independent of i's report).

In addition to constructing a new menu description of TTC, Theorem 4.1 yields an (arguably simple and intuitive) proof that the traditional description of TTC is strategyproof. In particular, Theorem 4.1 demonstrates—given (only) the fact that TTC is independent of the order in which cycles are eliminated—that in the traditional description of TTC, any applicant i is matched according to a menu description. Hence, TTC is strategyproof.

The above simple proof is enabled by two properties of Description 2. First, it contains a menu description. Second, it only slightly tweaks TTC's traditional, outcome description (in a way that clearly maintains the same result). Crucially, a description cannot satisfy these two properties without being an outcome description that contains a menu description, i.e., a menu-in-outcome description.

In sum, our description of TTC, and the simple argument for the strategyproofness it provides, give promising new ways to explain TTC's strategyproofness, both in the classroom and for real-world market participants.[15]

5 An Impossibility Result for Menu-in-Outcome Descriptions of DA

In this section, we present our third main result: an impossibility theorem showing that—in contrast to what our menu-in-outcome description of TTC (Section 4) achieves—no menu description of DA yields a simple proof of the strategyproofness of DA's traditional description.

Concretely, we show that no slight tweak of DA's traditional description contains a menu description. Here and throughout the paper, by "slight tweak," we mean that the same result is reached, and that the step-by-step calculation is similar enough for this fact to be evident. We formalize this notion of slight tweaks in Section 5.1. We then present our impossibility theorem in Section 5.2, showing that no such slight tweak of DA's traditional description contains a menu description (and hence showing that slight tweaks cannot expose the strategyproofness of DA's traditional description in a way analogous to TTC in Section 4).

[15] See Footnote 4.

5.1 Properties of Slight Tweaks of Traditional Descriptions

We now identify two salient properties of the step-by-step calculations used in the traditional description of DA (and of SD, and of TTC):

- First, the description only considers the preferences of each applicant once, in a specific, natural order—from favorite to least favorite. We call this property *applicant-proposing*.
- Second, the description requires only a small amount of bookkeeping, namely, that required to track a single tentative matching. We consider a flexible generalization of this property: that the description uses only a small (nearly constant, formalized below) amount of bookkeeping per applicant. Following standard terminology from computer science, we call this property *linear-memory*.

Before formally defining these two properties, Figure 3 illustrates how they are used to describe DA in one of its most celebrated practical applications: matching medical doctors to residencies in the US National Resident Matching Program (NRMP). The figure shows a screenshot of a video from the NRMP that relays the traditional description of DA by applying it to a small example. The explanation in

Figure 3: An illustration of the traditional description of DA through an example

Note: Screenshot taken from https://youtu.be/kVTwXNawpbk (NMS, 2020), a video produced by National Matching Services (the company providing matching software to the NRMP).

the video is aided by two visual elements: sequentially crossing off institutions from applicants' lists as the description progresses, and keeping track of a "current tentative matching" illustrated by the yellow-highlighted names. In order for these two simple visual elements to illustrate the description, it is necessary that the description is applicant-proposing and linear-memory. First, the applicant-proposing property is necessary for the video to sequentially cross off institutions from applicants' lists as the description progresses. Second, the linear-memory property is necessary for the yellow highlighting in the video to capture all required bookkeeping.

Definition 5.1 (Applicant-proposing and Linear-memory Descriptions)**.**[16]

- A description is *applicant-proposing* if it reads applicants' preferences by querying a single applicant at a time, such that the j^{th} query to applicant d returns the j^{th} institution on d's preference list. (The priorities of the institutions, on the other hand, can be accessed by the description in any way.)

  Formally, an applicant-proposing description is thus a procedure that maintains some internal *state* that is iteratively updated while querying applicants' preference lists (one applicant at a time), with the following property. For any possible inputs and for any applicant d, suppose the algorithm queries d's preference list sequentially in states $s_1, s_2, \ldots, s_k$ as it runs, and for each $j = 1, \ldots, k$, let s'_j denote the updated state that the algorithm reaches immediately after querying d's preferences in s_j. Then, s'_j depends only on (s_j and) the j^{th} institution on d's preference list (which is considered to be the "empty institution" if d's list contains fewer than j institutions).

- The *memory requirement* of a description is the number of bits required to represent the state of the description. Intuitively, this is the amount of extra bookkeeping or "scratch paper" required by the description. Formally, it is the logarithm in base 2 of the number of possible internal states of the algorithm.

  In a matching environment with n applicants and n institutions, we say a description is *linear-memory* if its memory requirement is at most $\widetilde{\mathcal{O}}(n)$.[17]

[16] As discussed in Section 2.2, we formally define descriptions to be algorithms. For completeness, the Supplemental Appendix S gives a self-contained mathematical definition of algorithms sufficient for our purposes.

[17] The standard computer-science notation $\widetilde{\mathcal{O}}(n)$ means $\mathcal{O}(n \log^{\alpha} n)$ for some constant α. That is, for large enough n, memory is upper-bounded by $cn \log^{\alpha} n$ for some constants c, α that do not depend on n. Using $\widetilde{\mathcal{O}}(n)$ memory means using only nearly constant bookkeeping per applicant.

We note that linear memory is the smallest possible memory requirement for outcome descriptions (as well as for menu descriptions) of matching mechanisms. Indeed, $\widetilde{\mathcal{O}}(n)$ is exactly (up to the precise logarithmic factors) the number of bits of memory required to describe a single matching (or a single applicant's menu).[18]

The applicant-proposing and linear-memory properties capture salient properties of the traditional descriptions of many matching mechanisms. As discussed above, this includes DA, but also includes SD, and TTC.[19] In particular:

- In the traditional description of SD, the linear memory stores a set S of not-yet-matched institutions. The applicant-proposing property enables the description to choose an applicant's highest-ranked institution in S by reading the applicant's preference list until the first institution in S is found.

- In the traditional description of TTC, the linear memory stores the set S of not-yet-matched institutions, and a pointing graph in which some applicants point to their top-ranked institution in S. The applicant-proposing property enables the description to update an applicant's pointing edge by reading her list further down to the highest-ranked institution remaining in S.

Even beyond permitting these diverse traditional descriptions, the applicant-proposing and linear-memory properties are quite flexible. The linear-memory requirement allows for arbitrary calculations or data structures, so long as a small amount of bookkeeping per-applicant is used. Additionally, applicant-proposing linear-memory descriptions permit many variations to the order in which applicant preferences are used by the description; for instance, the description could query and remember one institution from *each* applicant's preference list—or could query and remember one applicant's *entire* preference list.[20]

Given the above, any description retaining sufficiently similar step-by-step calculations to the traditional description of DA (or SD or TTC) must, at the very

[18]To see this formally, note that there are $n! = 2^{O(n \log n)}$ distinct matchings involving n applicants and n institutions (and exactly 2^n possible menus). Intuitively, this means that the number of letters it takes to write down a single matching with n applicants and n institutions (or a single menu, i.e., a subset of the n institutions) is roughly proportional to n.

[19]These properties are additionally satisfied by the popular non-strategyproof Boston mechanism (see, e.g., Abdulkadiroğlu et al., 2011).

[20]We also note that if no memory requirement is considered, then *every* algorithm can be implemented as an applicant-proposing description, by reading all applicants' preference lists and storing them fully in the bookkeeping, and then finally running any algorithm on these preference lists. See also the discussion regarding Figure 4 below.

least, maintain the applicant-proposing and linear-memory properties. Slight tweaks of the traditional description of DA should retain similar step-by-step calculations, and should also calculate the same result as the traditional description, that is, be *outcome descriptions*.

Overall, we thus take the view that all slight tweaks of the traditional description of DA should share three formal properties: applicant-proposing, linear-memory, and being an outcome description.[21] For an example for TTC, our menu-in-outcome description is a slight tweak of the traditional description; as a consequence, we have:

Proposition 5.2. *Description 2 is an applicant-proposing linear-memory menu-in-outcome descriptions of TTC.*

5.2 Impossibility Theorem

We now present our main impossibility result. Using the properties discussed in Section 5.1—applicant-proposing, linear-memory, and being an outcome description—we prove that no slight tweak of the traditional description of DA contains a menu description.

Theorem 5.3. *DA has no applicant-proposing, linear-memory, menu-in-outcome description. In fact, with n applicants and n institutions, any applicant-proposing menu-in-outcome description of DA requires $\Omega(n^2)$ memory.*[22]

We prove Theorem 5.3 in Appendix A below. The theorem shows a precise sense in which slight tweaks of DA's traditional description cannot expose its strategyproofness via a menu description. This is in sharp contrast to TTC, which (in the same sense) has a slight tweak that exposes strategyproofness as shown in Section 4, and in contrast to SD, whose traditional description already exposes strategyproofness.

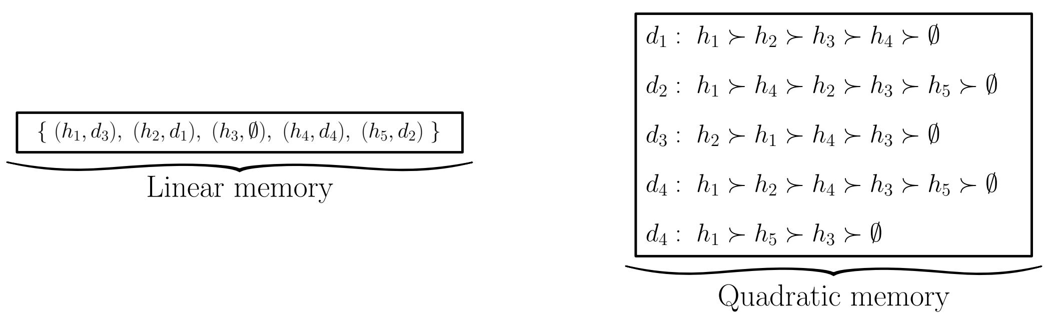
Theorem 5.3 is a strong impossibility result. Namely, we show that applicant-proposing menu-in-outcome description of DA require *quadratic* memory—$\Omega(n^2)$ bits. This nearly matches the memory requirement of simply memorizing all applicants'

[21]Note that we do not view *every* description satisfying these properties as a slight tweak of a traditional one. Instead, we take (only) the stance that these are necessary conditions that all slight tweaks satisfy.

[22]The standard computer-science notation $\Omega(n^2)$ means that, for large enough n, memory is lower-bounded by cn^2 for some constant c that does not depend on n.

preferences—$\widetilde{\mathcal{O}}(n^2)$ bits.[23] If an applicant-proposing description memorizes all applicants' preferences, then it can calculate *any* desired result (formally, by querying each applicant's entire preference list in order, with a separate state of the algorithm's memory for each possible preference profile, and returning a separate desired result for each such state). This shows that quadratic memory is the highest possible amount of memory that an algorithm might require. Thus, where applicant-proposing menu-in-outcome description of (SD and) TTC use memory as *low* as possible (linear, see Section 5.1), for DA the memory requirement is as *high* as possible (quadratic). See Figure 4 for an illustration of the qualitative gap between these two memory requirements.

Figure 4: Linear versus quadratic memory



Theorem 5.3 is also tight in the following sense. The theorem shows that descriptions of DA cannot simultaneously satisfy four criteria: being an outcome description, containing a menu description, being applicant-proposing, and using linear-memory. The impossibility holds only when all four of these criteria are assumed. We establish this as follows. First, DA's traditional description is an applicant-proposing, linear-memory outcome description. Second, DA has an applicant-proposing *quadratic memory* menu-in-outcome description, since (as discussed above) quadratic-memory is as high as possible. Third and fourth, we show in the Supplemental Appendix T that DA has an applicant-proposing linear-memory menu description, and a linear-memory menu-in-outcome description that is not applicant-proposing.[24] Hence, Theorem 5.3 captures the complexity of DA in our framework very precisely.

[23] To see this formally, observe that there are $(n!)^n = 2^{O(n^2 \log(n))}$ possible preference profiles for all applicants. Intuitively, this means that the number of letters it takes to write down n applicants' preferences over all n institutions is roughly proportional to n^2.

[24] Appendix T.2 constructs an applicant-proposing linear-memory menu description of DA, and

All told, our results establish a stark trichotomy—mentioned in the introduction—between SD, TTC, and DA. The traditional description of SD is already a menu description, simultaneously for all applicants, exposing its strategyproofness easily.[25] The traditional description of TTC does not expose strategyproofness; however, once this description is slightly tweaked and specialized to each individual applicant, strategyproofness is exposed easily.[26] For DA, in contrast with both other mechanisms, no small tweak of the traditional description suffices to expose strategyproofness using a menu, in the robust and strong sense provided by Theorem 5.3.

6 Related work

Our paper is most directly inspired by the contemporary "strategic simplicity" program in mechanism design theory, which largely considers different dynamic implementations of mechanisms. A cornerstone of this literature is Li (2017), which introduces obviously strategyproof (OSP) mechanisms as a way to expose strategyproofness. Unfortunately, TTC (Li, 2017) and DA (Ashlagi and Gonczarowski, 2018) do not have OSP mechanisms (except in rare special cases of institutions' priorities; see Troyan, 2019; Mandal and Roy, 2021; Thomas, 2021).[27]

In contrast to the above literature, we consider different ex ante descriptions of (static, direct-revelation) mechanisms. Breitmoser and Schweighofer-Kodritsch (2022) provide empirical evidence that framing a static auction as an OSP (ascending-clock) auction can be effective towards conveying strategyproofness. Since DA and TTC do not have OSP implementations, they cannot be framed in this way. Nonetheless, by relaying the match of only a single applicant at a time, menu descriptions frame the mechanism in a way that is OSP for that applicant (and in fact *strongly* OSP; Pycia and Troyan 2023).

The experimental paper of Katuščák and Kittsteiner (2025) also suggests describ-

Appendix T.1 constructs an *institution*-proposing linear-memory menu-in-outcome description of DA.

[25]SD has an (S)OSP implementation (Li, 2017; Pycia and Troyan, 2023) for a similar reason.

[26]One can show that if a mechanism is not OSP-implementable—as is the case for TTC (Li, 2017)—then any description of the mechanism *must* be specialized to a given applicant i in order to contain a menu description for i. In Remark B.4 we give a short direct proof that TTC's order requires such specialization.

[27]A different line of work also considers notions of strategic simplicity that are weaker than strategyproofness (Börgers and Li, 2019; Fernandez, 2020; Troyan and Morrill, 2020; Chen and Möller, 2024; Mennle and Seuken, 2021).

ing matching mechanisms to participants via menu descriptions, but does not investigate any menu description beyond that of Example 2.8, which essentially calculates the menu by iterating over all possible reports and running the traditional mechanism description each time.

We are not aware of any prior characterizations of the menu in DA. Our characterization builds on a large literature developing techniques for reasoning about stable matchings.[28] The menu in DA is different than other commonly considered definitions in the theory of stable matching, such as applicant i's set of stable partners (Gale and Shapley, 1962) or her budget set of institutions h where she is above h's cutoff (Segal, 2007; Azevedo and Leshno, 2016; Luflade, 2017; Azevedo and Budish, 2019; Immorlica et al., 2020). In particular, in finite matching markets, these other commonly-considered sets depend on applicant i's report, and hence do not equal i's menu. We provide explicit examples and more discussion in Remark B.5 and Remark B.6.

Proposition 2 in Leshno and Lo (2021) characterizes the menu in TTC in a different way from our Description 2. Their characterization does not give a menu-in-outcome description for TTC, and hence cannot be used in the same way as Description 2 to derive a simple proof of the strategyproofness of TTC's traditional description.

Our paper is also loosely inspired by the literature within computer science studying menus. These works largely focus on single-player selling mechanisms (e.g., Hart and Nisan, 2019; Daskalakis et al., 2017; Babaioff et al., 2022; Saxena et al., 2018; Gonczarowski, 2018).[29] Papers considering menus in multi-player mechanisms include Dobzinski (2016) and Dobzinski et al. (2022), who use menus as a tool for bounding communication complexity. We do not know of any prior algorithmic work on menus of matching mechanisms, nor of any prior work that analyzes different ways to describe multi-player mechanisms in terms of menus.

The present paper is part of our broader research agenda. In an earlier work-

[28]In particular, our proof of Theorem 3.2 in Appendix B analyzes DA by incrementally modifying preference lists. Similar techniques appear in Gale and Sotomayor (1985); Teo et al. (2001); Immorlica and Mahdian (2005); Hatfield and Milgrom (2005); Gonczarowski (2014); Ashlagi et al. (2017); Cai and Thomas (2022), for example. Our proof of Theorem 3.2 in Section 3 uses the strategyproofness of DA; to our knowledge, this is a fairly novel technique. Certain other properties of DA (e.g., in Blum et al., 1997; Adachi, 2000) and of unit-demand auctions (e.g., in Gul and Stacchetti, 2000; Alaei et al., 2016), despite not being studied with relation to menus, bear some technical similarity to the menu calculation in Description 1. However, the proofs seem unrelated.

[29]Brânzei and Procaccia (2015); Golowich and Li (2022) study the computational complexity of checking whether a mechanism, given its extensive- or normal-form representation, is strategyproof.

ing paper version of the present article,[30] we consider more general environments, study a basic extension of our theory for auctions, and conduct an experiment for a second-price auction and median voting. The theoretical computer science paper Gonczarowski and Thomas (2024) investigates a number of complexity questions related to our three main results.

Most relevantly, the empirical companion paper Gonczarowski, Heffetz, Ishai, and Thomas (2024) investigates participants' responses to different descriptions of DA, including the traditional one and Description 1 (our menu description). We find evidence that, while Description 1 is more complex for participants to understand than the traditional one, many participants can understand Description 1 and calculate its outcomes. Interestingly, while levels of strategyproofness-understanding are similar under both descriptions of DA, we see very high levels of strategyproofness-understanding under a less-complex, stripped-down menu description that omits the details of how the menu is calculated. This stripped-down menu description—which relays *only* strategyproofness—yields levels of strategyproofness-understanding well above a zero-information treatment benchmark, and even higher than a description relaying strategyproofness that is inspired by textbook definitions of strategyproofness. For real-world descriptions of DA, this may suggest complementing Description 1 with a stripped-down summary focusing on the properties important for strategyproofness.[31]

7 Discussion

Strategyproofness has long been proposed as a way to make mechanisms fair by leveling the playing field for players who do not strategize well (Pathak and Sönmez, 2008). We warmly embrace this agenda. However, we observe that if participants do not all *understand* that the mechanism is strategyproof, then disparities may remain. Menu descriptions may improve this understanding. They relay ex ante how participants' matches will be calculated while ensuring that strategyproofness follows via a simple argument, offering an alternative to status-quo tactics such as appeals

[30] For this earlier version, see https://arxiv.org/abs/2209.13148v2.

[31] Katuščák and Kittsteiner (2025) show the promise of a description (in their case, the description in Example 2.8 for TTC) complemented with a stripped-down summary.

to authority, asserting that the mechanism is strategyproof.[32]

While menu descriptions expose strategyproofness, they may obscure other properties of the mechanism. For example, since Description 1 (our menu description of DA) relays each applicant's match separately, it is unclear why this description always produces a feasible (one-to-one) matching.[33] In contrast, in DA's traditional description, feasibility of the outcome matching is clear, but strategyproofness is not exposed. Description 2 (our menu-in-outcome description of TTC) might be used to simultaneously expose strategyproofness and make feasibility clear. Future empirical work may present TTC to lab participants using our Description 2—or use this description to explain the strategyproofness of TTC's traditional description (as advocated by Morrill and Roth, 2024 for real-world participants)—and measure participants' understanding of both strategyproofness and feasibility.

In this paper and its experimental companion (Gonczarowski, Heffetz, Ishai, and Thomas, 2024), we suggest that some principled alternative framings of mechanisms (namely, menu descriptions) might better convey their properties (namely, strategyproofness), and we analyze such framings theoretically and empirically. We view this general premise—of reasoning about different descriptions (of the same mechanism) that expose different properties—as being of potential broader use. Future theoretical work might consider other properties one may wish to expose (e.g., fairness or optimality) and study opportunities and tradeoffs for exposing these properties using different descriptions in a variety of mechanisms and settings.

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[32]One common prior approach taken by clearinghouses is to encourage straightforward reporting without explaining strategyproofness. For example, Dreyfuss et al. (2022) notes that an informative video by the National Resident Matching Program (NRMP) was formerly introduced with the text:

> Research on the algorithm was the basis for awarding the 2012 Nobel Prize in Economic Sciences. To make the matching algorithm work best for you, create your rank order list in order of your true preferences, not how you think you will match.

[33]While traditional mechanism descriptions require participants to trust the description (as noted in, e.g., Akbarpour and Li, 2020), the fact that menu descriptions obscure feasibility may influence some participants' levels of trust. While our work focuses on understanding, trust may be an interesting direction for future theoretical or empirical work.

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Appendix

A Proof of Main Impossibility Theorem

In this appendix, we prove Theorem 5.3.

Theorem 5.3 considers applicant-proposing menu-in-outcome descriptions of DA. Recall that such descriptions must—while reading applicants' preferences only once in favorite-to-least-favorite order—calculate i's menu using $\succ_{-i}$, and then proceed to calculate the full matching using $(\succ_i, \succ_{-i})$. The theorem states that such descriptions require quadratic memory.

To prove the theorem, we construct a large set of applicant preference profiles that, intuitively speaking, has two properties: (A) to calculate i's menu given preferences in this set, essentially the full preference list of every applicant other than i must be read in its entirety, and (B) to calculate the final matching, essentially all this information must be remembered in its entirety. These properties ensure that the description must store the entire preference profile in its memory. There are many preference profiles in our construction, which implies the description has a high memory requirement.

Proof of Theorem 5.3. Fix an applicant i and let D be any applicant-proposing menu-in-outcome description of DA for i.

We now describe a set $\mathcal{S} \subseteq \mathcal{T}_{-i}$ of possible inputs to DA, illustrated in Figure A.1, which allows us to establish property (B) discussed above (intuitively, by allowing i's possible reports to affect the outcome matching in a different way for each different $\succ_{-i} \in \mathcal{S}$). For simplicity, let the number n of non-i applicants and institutions be a multiple of 4. Other than i, there are applicants and institutions d_j, d'_j, h_j, h'_j for each $j \in \{1, \ldots, n/2\}$. There are $n/2$ total "cycles" containing two applicants and two institutions each. Cycle j has applicants d_j and d'_j and institutions h_j and h'_j. The cycles are divided into two classes, "top" cycles (for $j \in \{1, \ldots, n/4\}$) and "bottom" cycles (for $j \in \{n/4 + 1, \ldots, n/2\}$).

The institutions' priorities are fixed, and defined as follows:

For top cycles ($j \le n/4$):

$$h_j : \quad d'_j \succ i \succ d_j$$
$$h'_j : \quad d_j \succ d'_j.$$

For bottom cycles ($j > n/4$):

$$h_j : \quad d'_j \succ d_1 \succ d_2 \succ \ldots \succ d_{n/4} \succ d_j$$
$$h'_j : \quad d_j \succ d'_j.$$

For the top cycle applicants (d_j with $j \leq n/4$), the preferences vary (in a way we will specify momentarily). Other applicants' preferences are fixed, as follows:

$$\text{For bottom cycles } (j > n/4): \qquad d_j : \quad h_j \succ h'_j.$$
$$\text{For all cycles } (j \in \{1, \ldots, n/2\}): \qquad d'_j : \quad h'_j \succ h_j.$$

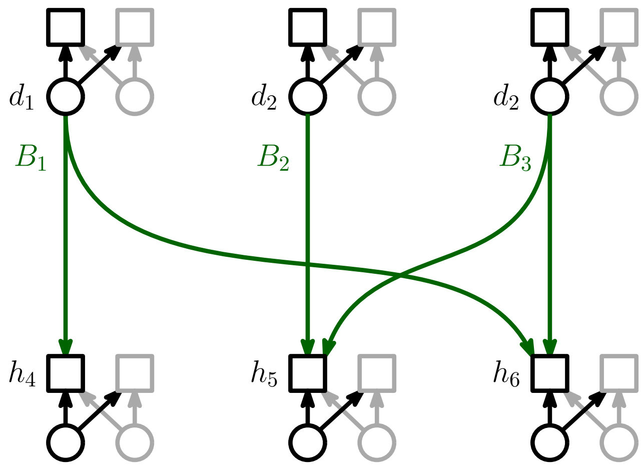

Figure A.1: Illustration of the construction used to prove Theorem 5.3

Notes: Dark nodes represent d_j or h_j for some j, and grey nodes represent d'_j or h'_j. The green arrows directed outwards from a top cycle d_j represent the sets B_j.

Let $\mathcal{S}$ denote the set of preference profiles where we additionally have:

$$\text{For top cycles } (j \leq n/4): \qquad d_j : \qquad h_j \succ B_j \succ h'_j,$$

where B_j is an arbitrary subset of $\{h_k \mid k > n/4\}$, ranked in any fixed order (say, increasing order of j). Any such collection of $\left(B_j\right)_{j=1}^{n/4}$ defines a distinct preference profile in $\mathcal{S}$. Note that $|\mathcal{S}| = 2^{(n/4)^2}$. See Figure A.1 for an illustration.

We additionally define a set of inputs $\mathcal{S}' \supseteq \mathcal{S}$, which allow us to establish property (A) discussed above (intuitively, by making i's menu depend on the final institution ranked above $\emptyset$ on other applicants' lists). Specifically, let $\mathcal{S}'$ denote the set containing every element of $\mathcal{S}$, and additionally any top cycle applicant d_j ($j \leq n/4$) may or may not truncate the final institution h'_i off her list, marking it as unacceptable. In other words, in addition to the sets $\left(B_j\right)_{j=1}^{n/4}$, an element of $\mathcal{S}'$ is defined by bits $(c_j)_{j=1}^{n/4}$, such that, for each top cycle j ($j \leq n/4$):[1]

$$\text{When } c_j = 0: \qquad d_j: \quad h_j \succ B_j \succ h'_j.$$
$$\text{When } c_j = 1: \qquad d_j: \quad h_j \succ B_j.$$

We now proceed to prove the two crucial properties of DA, and the description D, when run on this family of preference profiles. The following lemmas formalize properties (A) and (B) discussed above, showing (respectively) that D must essentially read all of the preferences in $\succ_{-i} \in \mathcal{S}'$ in order to calculate i's menu, and (before knowing $\succ_i$) must remember essentially all of this information in order to calculate the outcome matching DA($\succ_i, \succ_{-i}$).

Lemma A.1. *Consider a preference profile in $\mathcal{S}'$. For each top cycle j (with $j \leq n/4$), we have that h_j is in applicant i's menu in DA if and only if d_j does not rank h'_j (i.e., $c_j = 1$). Hence, to correctly calculate i's menu, description D must read the entire preference list of each such d_j (up to the position of h'_j).*

To prove this lemma, consider the execution of the APDA algorithm when i submits a list containing only h_j. First, d_j is rejected, then she proposes to every institution $h_k \in B_j$. This "rotates" the bottom cycle containing h_k; in more detail, h_k will accept the proposal from d_j, then d_k will propose to h'_k, then d'_k with propose to h_k, and d_j will be rejected from h_k. This will occur for every $h_k \in B_j$, so d_j will not match to any h_k with $k \in \{n/4 + 1, \ldots, n/2\}$.

Finally, after getting rejected from each institution in B_j, applicant d_j may or may not propose to h'_j, depending on the bit c_j. If she does not, then d_* remains matched to h_j and in this case h_j is on i's menu. If she does, then h'_j will reject d'_j, who will propose to h_j, which will reject i. So i will go unmatched, and thus in this case h_j is not on i's menu.

The final sentence of the lemma then follows from the fact that D is an applicant-proposing and must calculate i's menu. This proves Lemma A.1.

Lemma A.2. *Each distinct preference profile $\succ_{-i} \in \mathcal{S}$ induces a distinct function $DA(\cdot, \succ_{-i}) : \mathcal{T}_i \to A$ from applicant i's report to outcome matchings. Hence, to*

[1] This collection of preferences can also be constructed with full preference lists by adding some unmatched institution $h_\emptyset$ to represent truncating d_i's list.

correctly calculate the outcome matching, the description D must—across all states where it finishes calculating i's menu—have at least one state for each element of $\mathcal{S}$.

To prove this lemma, consider two distinct preference profiles in $\mathcal{S}$, one profile $\succ_{-i}$ corresponding to $\left(B_j\right)_{j=1}^{n/4}$, and the other profile $\succ'_{-i}$ corresponding to $\left(B'_j\right)_{j=1}^{n/4}$. Without loss of generality, there is some j and k such that $h_k \in B_j \setminus B'_j$. Suppose now that i's report $\succ_i$ lists only h_j. Then, consider execution of the APDA algorithm under $(\succ_i, \succ_{-i})$ and under $(\succ_i, \succ'_{-i})$. Under $\succ_{-i}$, the bottom tier cycle containing h_k will be "rotated," i.e. since $h_k \in B_j$, the sequence of rejections will cause h_k to match to d'_k. However, this is not the case under $\succ'_{-i}$, since $h_k \notin B_j$. Thus, $\mathrm{DA}(\cdot, \succ_{-i}) \neq \mathrm{DA}(\cdot, \succ'_{-i})$.

We now prove the second sentence of the lemma. As argued in Lemma A.1, D must have read all top cycle applicants' preferences in order to calculate i's menu. Moreover, since D is a menu-in-outcome description, it must do so before learning $\succ_i$. Hence, to calculate the outcome matching correctly at the end, D must remember the entirety of $\left(B_j\right)_{j=1}^{n/4}$. This proves Lemma A.2.

We now prove Theorem 5.3. Together, Lemma A.1 and Lemma A.2 show that when D has just calculated the menu of applicant i, the description must be in a distinct state for each distinct $\succ_{-i} \in \mathcal{S}$. There is one such $\succ_{-i}$ for each possible way of assigning the sets $B_j \subseteq \{h_k \mid k > n/4\}$ for all $j \in \{1, \ldots, n/4\}$. For each such j, there are $2^{n/4}$ ways to assign B_j, and hence there are $\left(2^{n/4}\right)^{n/4} = 2^{(n/4)^2} = 2^{\Omega(n^2)}$ possible ways to set this collection $\left(B_j\right)_{j=1}^{n/4}$. Thus, the description requires at least this many states, and thus requires memory $\Omega(n^2)$. This finishes the proof. □

B Additional Proofs and Remarks

In this appendix, we provide additional supplemental proofs and remarks omitted from the main text.

We start by proving Theorem 3.2, which shows that Description 1 is a menu description of DA, without assuming the strategyproofness of DA. This provide an alternative, potentially-instructive approach for proving DA's strategyproofness.

Alternative proof of Theorem 3.2. We show that, for any applicant i, Description 1 correctly calculates i's match in DA. To this end, fix the priorities of institutions and preferences $\succ = (\succ_i, \succ_{-i})$ of all applicants. Let $h_* = APDA_i(\succ)$ denote the match of

i according to applicant-proposing DA. Our goal is to show that h_* is the outcome of Description 1. Hence, we must show h_* is the $\succ_i$-favorite institution in the set containing (1) the "outside option" of going unmatched, and (2) all institutions h such that h prefers i to $IPDA_h(\succ_{-i})$ (the match of h according to institution-proposing DA in the market without i).

Let $\emptyset$ denote the empty preference report of i (i.e., the report marking all institutions as unacceptable). Observe that $IPDA(\succ_{-i})$ and $IPDA(\emptyset, \succ_{-i})$ match applicants (other than i) in exactly the same way, and furthermore, the institutions h that prefer i to $IPDA_h(\succ_{-i})$ are exactly those that propose to i during the calculation of $IPDA(\emptyset, \succ_{-i})$. Therefore, it suffices to prove:

(I) If $h_* \neq \emptyset$, then then h_* proposes to i during $IPDA(\emptyset, \succ_{-i})$.

(II) Applicant i gets no proposal in $IPDA(\emptyset, \succ_{-i})$ that is $\succ_i$-preferred to h_*.

We start by proving (I). Assume that $h_* \neq \emptyset$. Let $\{h_*\}$ denote the preference list of i ranking only h_* (i.e., marking all other institutions as unacceptable). Observe that $APDA(\succ_i, \succ_{-i})$ is also stable under preferences $(\{h_*\}, \succ_{-i})$. Thus, by the Lone Wolf / Rural Hospitals Theorem (Theorem C.6), since i is matched in $APDA(\succ_i, \succ_{-i})$, she must be matched in $IPDA(\{h_*\}, \succ_{-i})$ as well. Thus, $IPDA(\{h_*\}, \succ_{-i}) = h_*$. Since $IPDA(\{h_*\}, \succ_{-i})$ and $IPDA(\emptyset, \succ_{-i})$ coincide until h_* proposes to i, we conclude that h_* must propose to i during $IPDA(\emptyset, \succ_{-i})$, proving (I).

We now prove (II). Let T denote i's preference list, truncated at and below h^*, i.e., the report listing only institutions that i strictly prefers to h^*. Observe that i must go unmatched in $APDA(T, \succ_{-i})$, since every proposal by i before h_* was rejected in $APDA(\succ_i, \succ_{-i})$. Hence, by the Lone Wolf / Rural Hospitals Theorem (Theorem C.6), i goes unmatched in $IPDA(T, \succ_{-i})$. Now, since i goes unmatched in $IPDA(T, \succ_{-i})$, we see that i does not receive any proposal that is $\succ_i$-preferred to h_* in $IPDA(\emptyset, \succ_{-i})$, proving (II).

We have shown that the outcome calculated at the end of Step (2) of Description 1 is i's outcome in DA. Moreover, observe that the set calculated in Step (1) of is independent of i's report. Hence, DA is strategyproof (by the same proof outline that applies to every menu description). Moreover, as observed in Footnote 10, the menu is the *only* set M of institutions that is independent of i's report such that i always receives her favorite institution in M. Hence, the set in Step (1) must be i's menu, and Description 1 is a menu description of DA. □

Remark B.1. As noted in Section 3, Theorem 3.2 extends to many-to-one markets with substitutable priorities. To quickly see why this extension holds in the special case in which institutions have responsive preferences (i.e., the special case in which each institution has a master preference order and a capacity), fix a many-to-one market, and following a standard approach, consider a one-to-one market where each institution from the original market is split into "independent copies." That is, the number of copies of each institution equals the capacity of the institution, each "copied" institution has the same preference list as the original institution, and each applicant ranks all the copies of the institution (in any order) in the same way she ranked the original institution. Ignoring the artificial difference between copies of the same institution, the run of applicant-proposing DA is equivalent under these two markets. Thus, an applicant's menu is equivalent under both markets, and so by Theorem 3.2, a menu description for the many-to-one market can be given through institution-proposing DA under the corresponding one-to-one market, which in turn is equivalent to institution-proposing DA under the original market (where at each step, each institution proposes to a number of applicants up to its capacity). The only change in Description 1 in this case would be replacing the condition $i \succ_h \mu_{-i}(h)$ with $\exists d' \in \mu_{-i}(h) : i \succ_h d'$.

Remark B.2. As additionally noted in Section 3, Theorem 3.2 also extends to many-to-one markets with contracts in which the institutions have substitutable preferences that satisfy the law of aggregate demand (the conditions under which Hatfield and Milgrom (2005) prove that the strategyproofness of applicant-proposing DA and the rural hospitals theorem hold). Description A.1 gives a menu description of DA in this environment, which generalizes Description 1 as follows: (1) Description A.1 uses the generalized Gale–Shapley algorithm of Hatfield and Milgrom (2005) starting from $(\emptyset, X)$ (where X is the set of all possible contracts) to calculate the institution-optimal stable outcome without i to get a matching μ_{-i}. (2) A given contract $c = (i, h, c)$ (i.e., an $(\text{applicant}, \text{institution}, \text{term})$ tuple) is on i's menu if and only if h would choose (i, h, c) if given a choice from the set containing (i, c) and its matches in μ_{-i} (in the notation of Hatfield and Milgrom (2005), $c \in C_h(\mu_{-i}(h) \cup \{c\})$). Under this modification, each step of the proof of Theorem 3.2 in Section 3 holds by a completely analogous argument for this market.

Description A.1 A menu description for applicant i of the applicant-optimal stable matching in a many-to-one market with contracts

(1) Calculate the institution-optimal stable matching with applicant i removed from the market using the generalized Gale–Shapley algorithm of Hatfield and Milgrom (2005). Call the resulting matching μ_{-i}. Let M be the set of contracts $c = (i, h, t)$ involving applicant i such that $c \in C_h(\mu_{-i}(h) \cup \{c\})$.

(2) Match i to i's highest-ranked contract in M.

Remark B.3. In this remark, we show how Theorem 3.2, which characterizes the menu in DA in terms of Description 1, can be used to prove results from Ashlagi et al. (2017) via arguments similar to Cai and Thomas (2022). Consider a randomized market with $n + 1$ applicants and n institutions, where such that each applicant/institution draws a full-length preference list uniformly at random, and let μ be the result of (applicant-optimal) DA with these preferences. We prove that the expected rank each applicant receives on their preference list (formally, the expectation of $|\{h : h \succeq_d \mu(d)\}|$ for any d) is at least $(1 - \epsilon)n/\log(n)$ for any $\epsilon > 0$ and large enough n.

Fix an applicant d_*, and consider calculating d_*'s menu using Description 1 in this market. This is equivalent to considering IPDA in a market where d_* rejects all proposals, and setting d_*'s menu to consist of all proposals she receives. By the principle of deferred decisions, this run of IPDA can be constructed by letting each institution h proposes to a uniformly random applicant (among those h has not yet proposed to) each time she proposes. Observe that this run of IPDA will terminate as soon as each of the n applicants other than d_* receives a proposal. Thus (much like the standard case of n applicants and n institutions in APDA Wilson (1972)), the total number of proposals made in this run of IPDA is stochastically dominated by a coupon collector random variable. Thus, intuitively, the total number of proposals will be $n \log(n)$, and $\log(n)$ of these will go to d_* in expectation, and d_*'s top choice out of these $\log(n)$ proposals will be their $n/\log(n)$th ranked choice overall.

Formally, let Y denote the number of proposals d_* receives, and let $\overline{Y}$ denote the same quantity in a market where each institution makes each proposal completely uniformly at random (without regard to prior proposals); it follows that Y is stochastically dominated by $\overline{Y}$. Let $\overline{Z_i}$ denote the total number of proposals between the

$(i-1)$th and ith distinct applicant in $\mathcal{D} \setminus \{d_*\}$ receiving a proposal (in the market with repeated proposals). The expected value of Z_i is exactly $(n+1)/(n+1-i)$, and each of these Z_i proposals (except for the final one) has a $1/i$ probability of going to d_*. Thus, we have

$$\mathbb{E}[Y] \le \mathbb{E}\left[\overline{Y}\right] = \sum_{i=1}^{n} \frac{1}{i}\left(\frac{n+1}{n+1-i} - 1\right) = \sum_{i=1}^{n} \frac{1}{i}\left(\frac{i}{n+1-i}\right) = H_n \le \log(n) + 1.$$

Now, let $R = |\{h : h \succeq_d h_*\}|$, where h_* is d_*'s top-ranked proposal received (i.e., d_*'s match in APDA). One can show that, conditioned on $Y = y$, we have the expected value of R exactly equal to $(n+1)/(y+1)$ (see for example (Cai and Thomas, 2022, Claim A.1)). Thus, by Jensen's inequality, we have

$$\mathbb{E}[R] = \mathop{\mathbb{E}}_{y \sim Y}\left[\frac{n+1}{y+1}\right] \ge \frac{n+1}{\mathbb{E}[Y]+1} \ge \frac{n+1}{\log(n)+2} \ge \big(1-\epsilon\big)\frac{n}{\log(n)}$$

for any $\epsilon > 0$ and large enough n, as desired.

Remark B.4. We now formally show that, unlike SD, a description of TTC *must* be specialized to individual applicants in order to contain a menu description for them. Concretely, we show that any outcome description of TTC cannot contain a menu description for *two* applicants (where, in contrast, our Description 2 contains a menu description for exactly one student).

To do this, it suffices to construct an instance containing two applicants d_1 and d_2 such that each of their menus depends on the other. For example, consider an instance where $h_i : d_i \succ d_{3-i}$ and $d_i : h_{3-i} \succ h_i$ for $i \in \{1, 2\}$. Under this instance, for each $i \in \{1, 2\}$, institution h_{3-i} is on d_i's menu, but if applicant d_{3-i} changed her preference list, this would no longer be true. Hence, a description cannot calculate either applicant's menu before the description queries the other applicant's type.

Remark B.5. We now show that in (finite-market) DA, budget sets and menus are different sets; moreover, we show that neither set includes the other. For a fixed profile of preferences and priorities, denote an applicant i's budget set $B(i) = \{h | i \succeq_h \mu(i)\}$, where μ is the outcome of DA. Let $M(i) = \mathcal{M}_{\succ_{-i}}$ denote i's menu.

Now, consider the market with institutions h_1, h_2, h_3, and h_4, and applicants

d_1, d_2, d_3, and d_4. Let the preferences and priorities be as follows:

$$h_1 : d_1 \succ d_2 \succ \dots \qquad d_1 : h_1 \succ \dots$$
$$h_2 : d_4 \succ d_3 \succ d_2 \succ d_1 \succ \dots \qquad d_2 : h_1 \succ h_2 \succ h_4 \succ \dots$$
$$h_3 : d_3 \succ \dots \qquad d_3 : h_3 \succ \dots$$
$$h_4 : d_2 \succ d_4 \succ \dots \qquad d_4 : h_4 \succ h_2 \succ \dots$$

Then, one can check that DA pairs h_i to d_i for each $i = 1, \dots, 4$, and that $h_2 \in B(d_3) \setminus M(d_3)$, and also $h_2 \in M(d_1) \setminus B(d_1)$. Thus, neither the menu nor the budget set contain the other. Moreover, the relationship between the two sets does not seem to be restricted in a straightforward way based on priorities and the outcome of DA: despite the fact that $d_3 \succ_{h_2} d_2$, we have $h_2 \notin M(d_3)$; despite $d_1 \prec_{h_2} d_2$, we have $h_2 \in M(d_1)$.

Remark B.6. We now show that in DA, an applicant's set of stable partners is a (possibly strict) subset of her menu. For a given profile of preferences and priorities, let $S(i)$ denote the set of stable partners of applicant i, and let $M(i)$ denote her menu. We begin by showing that $M(i) \neq S(i)$. Consider any instance with two institutions h_1, h_2 which both rank i above all other applicants. Both h_1 and h_2 must be in i's menu. However, if i ranks h_1 above all other institutions, then h_1 is i's unique stable partner; thus $h_2 \in M(i) \setminus S(i)$.

We now show that $S(i) \subseteq M(i)$. Suppose the profile of preferences and priorities is P. Consider any $h \in S(i)$, and let μ be a stable matching with $\mu(i) = h$. Then, let $\widetilde{P}$ denote modifying P by having i submit a list which ranks only h. Then, observe that μ is also stable under $\widetilde{P}$. Thus, by the Rural Hospital Theorem (Theorem C.6), i and h must be matched in every stable matching under $\widetilde{P}$, in particular, in $DA(\widetilde{P})$. Thus, $h \in M(i)$, and $S(i) \subseteq M(i)$.

C Proofs of Known Results

In this appendix, we recall classically-known lemmas on DA and TTC that are needed for our paper. We also provide full proofs, making all the arguments in this paper self-contained.

C.1 Known Results for DA and Stable Matchings

We now provide properties of DA and stable matchings. Let D denote the set of applicants, and H the set of institutions. Recall that a matching μ is *stable* if $\mu(a) \succ_a \emptyset$ for all $a \in D \cup H$, and moreover there is no pair $d \in D, h \in H$ such that $h \succ_d \mu(h)$ and $d \succ_h \mu(d)$.

Lemma C.1 (Gale and Shapley, 1962)**.** *The outcome of DA is a stable matching.*

Proof. Consider running the traditional description of DA (Definition 2.3) on some profile of preferences (and priorities), and let the output matching be μ. Consider a pair $d \in D$, $h \in H$ which is unmatched in μ. Suppose for contradiction $h \succ_d \mu(d)$ and $d \succ_h \mu(h)$. In the DA algorithm, d would propose to h before $\mu(d)$. However, it's easy to observe from the traditional description of DA that once an institution is proposed to, they remain matched and can only increase their priority for their match. This contradicts the fact that h was eventually matched to $\mu(h)$. □

Note that Lemma C.1 also proves that at least one stable matching always exists. Next, we show that DA (i.e., the matching output by the APDA algorithm) is (simultaniously) the best stable matching for all applicants.

Lemma C.2 (Gale and Shapley, 1962)**.** *If an applicant $d \in D$ is ever rejected by an institution $h \in H$ during some run of the APDA algorithm, then no stable matching can pair d to h.*

Proof. Let μ be any matching, not necessarily stable. We will show that if h rejects $\mu(h)$ at any step of DA, then μ is not stable.

Consider the first time during in the run of APDA where such a rejection occurred. In particular, let h reject $d \stackrel{\text{def}}{=} \mu(h)$ in favor of $\widetilde{d} \neq d$ (either because $\widetilde{d}$ proposed to h, or because $\widetilde{d}$ was already matched to h and d proposed). We have $\widetilde{d} \succ_h d$. We have $\mu(\widetilde{d}) \neq h$, simply because μ is a matching. Because this is the *first* time an applicant has been rejected by her match in μ, $\widetilde{d}$ has not yet proposed to $\mu(\widetilde{d})$. This means $h \succ_{\widetilde{d}} \mu(\widetilde{d})$, and μ is not stable.

Thus, no institution can ever reject a stable partner in APDA. □

The following corollaries are immediate:

Corollary C.3 (Gale and Shapley, 1962)**.** *In the outcome of DA, every applicant is matched to her favorite stable partner.*

Corollary C.4 (Dubins and Freedman, 1981)**.** *The matching output by the traditional DA algorithm is independent of the order in which applicants are selected to propose.*

A phenomenon dual to Corollary C.3 occurs for the institutions:

Lemma C.5 (McVitie and Wilson, 1971)**.** *In the outcome of DA, every $h \in H$ is paired to her least-favorite stable partner.*

Proof. Let $d \in D$ and $h \in H$ be paired by applicant-proposing deferred acceptance. Let μ be any stable matching which does not pair d and h. We must have $h \succ_d \mu(d)$, because h is the d's favorite stable partner. If $d \succ_h \mu(h)$, then μ is not stable. Thus, we must in fact have $\mu(h) \succ_h d$. □

Finally, we show that the set of matched agents must be the same in each stable matching.

Theorem C.6 (Lone Wolf / Rural Hospitals Theorem, Roth, 1986)**.** *The set of unmatched agents is the same in every stable matching.*

Proof. Consider any stable matching μ in which applicants D^μ and institutions H^μ are matched, and let D^0 and H^0 be matched in DA. By Corollary C.3, we know that for all $d \in D^\mu$, the match of d can only improve in DA; in particular, d is still matched in DA, and thus $D^\mu \subseteq D^0$. Similarly, Lemma C.5 implies that each agent in H^0 is matched in every stable outcome, so $H^0 \subseteq H^\mu$. But then, since the matching is one-to-one, we have $|D^0| = |H^0|$ as well as $|D^0| \geq |D^\mu| = |H^\mu| \geq |H^0|$, so the same number of agents (on each side) are matched in μ and in DA. Thus, $D^0 = D^\mu$ and $H^0 = H^\mu$. □

Additionally, DA is strategyproof. This follows from our Theorem 3.2. While our proof of Theorem 3.2 in Section 3 relies on DA's strategyproofness, our proof of Theorem 3.2 in Appendix B only uses properties proven above in this appendix. Hence, these arguments show that DA is strategyproof from first-principles.

Theorem C.7 (Roth, 1982; Dubins and Freedman, 1981)**.** *DA is strategyproof for the applicants.*

C.2 Known Result for TTC

We now prove that TTC is independent of the order in which the steps are chosen in the traditional description (analogous to Corollary C.4 for DA). This will follow from the observation that cycles in the pointing graph of the traditional description of TTC must always be disjoint, since the pointing graph has out-degree 1. See also Carroll (2014); Morrill and Roth (2024) for similar contemporary proofs.

Proposition C.8 (Follows from Shapley and Scarf, 1974; Roth and Postlewaite, 1977)**.** *The TTC algorithm is independent of the order in which cycles are chosen and eliminated.*

Proof. Fix a profile of priorities and preferences. Define the *elimination graph* G as follows. The vertices of G are the set of all partial matchings between applicants and institutions. There is an edge $\mu_1 \to \mu_2$ in G whenever μ_2 differs from μ_1 by the elimination of exactly one cycle, as defined in Definition 2.2, under the given preferences and priorities. Formally, this is defined as follows. Fix μ_1, and consider the *pointing graph* $B = B_{\mu_1}$ given μ_1 to be the bipartite graph formed by applicants and institutions who are unmatched in μ_1, where each agent points to her top-ranked agent on the other side who is unmatched in μ_1 (if any such agents on the other side remain). Then, we have an edge $\mu_1 \to \mu_2$ whenever there exists a cycle in B such that, if μ_1 is modified such that every applicant in the cycle is matched to the institution she points to, then the resulting matching is μ_2. When $\mu_1 \to \mu_2$ in G, and the cycle C in B_{μ_1} represents the difference between μ_2 and μ_1, we say that C is *available* in μ_1.

Now, define a *elimination sequence* T to be any sequence $T = \mu_1 \to \mu_2 \to \ldots \to \mu_k$ of adjacent edges in G, such that μ_1 is the empty matching which pairs no agents, and T is of maximal possible length. Observe that the outcome of TTC is defined to be the final matching μ_k of an elimination sequence.

We make the following observations regarding any elimination sequence $T = \mu_1 \to \ldots \to \mu_k$:

- For any fixed pointing graph B_{μ_i}, all of the cycles C in B_{μ_i} are disjoint. This follows because the pointing graph has out-degree 1.
- If C is available in some μ_x, then there exists a $z > x$ such that C is available in every subsequent μ_y for $x \le y < z$. This follows from the previous observation,

since for each $\mu_y \to \mu_{y+1}$ with $x \leq y < z$ with y increasing inductively, the vertices in the cycle C are not changed as we switch from μ_y to μ_{y+1}, unless the cycle C itself is eliminated. Thus, in particular, μ_z differs from μ_{z-1} by the elimination of C.

- Suppose that in T, cycle C_1 is available in some μ_x, but $C_2 \neq C_1$ eliminated in μ_x to get μ_{x+1}. Then, there exists another elimination sequence $T' = \mu_1 \to \mu_x \to \mu'_{x+1} \to \ldots \to \mu'_k$ which agrees with T up until μ_x, but C_1 is eliminated at μ_x to get μ'_{x+1}, and which ends in the same final matching $\mu'_k = \mu_k$. To show this, we construct T' as follows. After eliminating C_1 at μ_x to get μ'_{x+1}, follow the same order of eliminating cycles as in T until cycle C_1 is eliminated in T—i.e., go from μ'_{y+1} to μ'_{y+2} via the same cycle used to go from μ_y to μ_{y+1}, for each $y \geq x$ such that C_1 is not eliminated in $\mu_y \to \mu_{y+1}$ in T. (All such cycles must be available as needed in T', since before C_1 was eliminated in T, none of these cycles could have involved agents in C_1 in any way.) At some point, C_1 must be eliminated in T, say in $\mu_z \to \mu_{z+1}$. After this point, the elimination sequence T' will from that point onward agree with T, i.e., $\mu_w = \mu'_w$ for $w \geq z + 1$.

Now, suppose for contradiction that there are two elimination orderings T_1 and T_2 which produce different final matchings, and additionally suppose among all such pairs, the index $j > 1$ where T_1 and T_2 first disagree is *as large as possible*. Then, at index j, two cycles C_1 and C_2 are eliminated in T_1 and T_2, respectively. Then, by final observation listed above, we can consider the elimination sequence T'_2 that disagrees with T_1 at least one step later than j (by eliminated C_1), but has the same final matching as T_2. This contradicts the assumption that j was as large as possible.

This proves that all elimination sequences must produce the same final matching, which is the outcome of TTC. This proves the result. □

Supplemental Material

S Mathematical Model of Algorithms

In this appendix, we define from first-principles a mathematical model of descriptions of mechanisms which can express all our results.

We introduce the notion of an *extensive-form description*. For generality, we state this definition in terms of a general mechanism design environment with players $1, \ldots, n$, type spaces $\mathcal{T}_1, \ldots, \mathcal{T}_n$, and outcome space A. At a technical level, an extensive-form description is similar to an extensive-form mechanism, except that different branches may "merge," i.e., the underlying game tree is actually a directed acyclic graph (DAG).[1] Note, however, that the interpretation is different from that of an extensive-form mechanism: Rather than modeling an interactive process where the players may act multiple times, an extensive-form description spells out the steps used to calculate some result by iteratively querying the directly-reported types of the players.

We formally define three types of extensive-form descriptions, corresponding to our three description outlines: outcome descriptions, menu descriptions, and menu-in-outcome descriptions.

Definition S.1 (Extensive-Form Descriptions)**.**

- An *extensive-form description* in some environment is defined by a directed graph on some set of vertices V.[2] There is a (single) root vertex $s \in V$, and the vertices of V are organized into *layers* $j = 1, \ldots, L$ such that each edge goes between layer j and $j + 1$ for some j. For a vertex v, let $S(v)$ denote the edges outgoing from v. Each vertex v with out-degree at least 2 is associated with some player i, whom the vertex is said to *query*, and some *transition function* $\ell_v : \mathcal{T}_i \to S(v)$ from types of player i to edges outgoing from v. (It will be convenient to also allow vertices with out-degree 1, which are not associated with any player.) For each type profile $(t_1, \ldots, t_n)$, the *evaluation path* on

[1] Alternatively, extensive-form descriptions can be viewed as finite automata where state transitions are given by querying the types of players.

[2] Formally, a directed graph G on vertices V is some set of ordered pairs $G \subseteq V \times V$. An element $(v, w) \in G$ is called an *edge* from v to w. A *source* (resp., *sink*) vertex is any v where there exists no vertex w with an edge from w to v (resp., from v to w).

$(t_1, \ldots, t_n) \in \mathcal{T}_1 \times \ldots \times \mathcal{T}_n$ is defined as follows: Start in the root vertex s, and whenever reaching any non-terminal vertex v that queries a player i and has transition function ℓ_v, follow the edge $\ell_v(t_i)$.

- An *extensive-form outcome description* of a mechanism f is an extensive-form description in which each terminal vertex is labeled by an outcome, such that for each type profile $(t_1, \ldots, t_n) \in \mathcal{T}_1 \times \mathcal{T}_n$, the terminal vertex reached by following the evaluation path on $t \in T$ is labeled by the outcome $f(t_1, \ldots, t_n)$.

- An *extensive-form menu description* of a social choice function f for player i is an extensive-form description with $k+1$ layers, such that (a) each vertex preceding layer k queries some player other than i, (b) each vertex v in layer k queries player i and is labeled by some set $M(v) \subseteq A_i$, such that if v is on the evaluation path on a type profile $(t_1, \ldots, t_n) \in \mathcal{T}_1 \times \mathcal{T}_n$, then $M(v) = \mathcal{M}_{t_{-i}}$ is the menu of player i with respect to t_{-i} in f, and (c) each (terminal) vertex v in the final layer $k+1$ is labeled by an outcome for player i,[3] such that if v is reached by following the evaluation path on a type profile $(t_1, \ldots, t_n)$, then v is labeled by i's outcome in $f(t_1, \ldots, t_n)$.

- An *extensive-form menu-in-outcome description* of f for player i is an extensive-form outcome description such that, for some k, the first $k+1$ layers are an extensive-form menu description.

For a concrete example of an extensive-form description, we consider a menu description of a second price auction.[4] In this mechanism, a bidder's menu consists of two options: winning the item and playing the highest bid placed by any other bidder, or winning nothing and paying nothing. Thus, a menu description can be given as follows:

(1) Your "price to win" the item will be set to the highest bid placed by any other player.

[3]Formally, in a general mechanism design environment, an *outcome of player i* (or, an *i-outcome*) is a maximal set E of outcomes such that all possible types of player i in $\mathcal{T}_i$ view each outcome in E as equally desirable.

[4]While we have not formally defined menus or menu descriptions in non-matching environments, they naturally generalize by considering the menu of i induced by reports t_{-i} to be the set of i's outcomes consistent with t_{-i}.

(2) If your bid is higher than this "price to win," then you will win the item and pay this price. Otherwise, you will win nothing and pay nothing.

An extensive-form description can formalize this menu description by querying the other bidders one-by-one, while keeping track of only the highest bid placed by any of them. Figure S.1 provides an illustration.

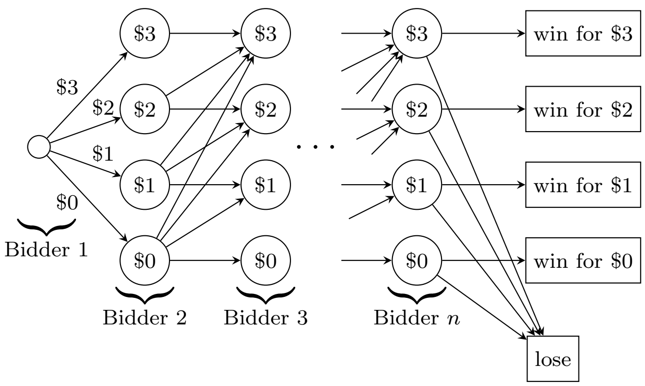

Figure S.1: An extensive-form menu description for bidder n in a second-price auction

Note: The second-to-last layer is labeled with bidder n's menu, abbreviated in the figure by the price she must pay to win the item.

More broadly, any precise algorithm taking players types as inputs induces an extensive-form description in a natural way: the vertices in layer j are the possible states of the algorithm after querying the types of different players altogether j times. In particular, our positive results (Description 1 and Description 2) correspond to extensive-form descriptions. The definitions of all our simplicity desiderata (Definition 5.1 and Definition T.11 below) also extend naturally to extensive-form descriptions. Moreover, the proofs of our impossibility theorems (Theorem 5.3 and Proposition T.12 below) hold, mutatis mutandis, for the relevant class of extensive form descriptions.

In addition to providing a self-contained mathematical language for expressing our results, the definition of an extensive-form description allows us to clarify some ways in which our impossibility results are strong. Namely, while algorithms are often required to work for any number of players, our impossibility results hold even if one can use a separate extensive-form description for each number of players n, and regardless of the computational complexity of such a description. Relatedly, our impossibility results follow from direct combinatorial arguments and do not depend on any complexity-theoretic conjectures such as $P \neq NP$.

T On Additional Descriptions of DA

In this appendix, we present additional findings regarding descriptions of DA. We examine a broad classification of mechanism descriptions. While we uncover additional descriptions of DA, we find that all such uncovered descriptions (beyond the traditional one and Description 1) are unintuitive and convoluted algorithms that are impractical for real-world use.

To motivate our search for additional descriptions of DA, consider the outline of menu-in-outcome descriptions, which provided our highly-useful Description 2 for TTC. Our description in Description 2 satisfies applicant-proposing and linear-memory, that may be regarded as certain formal simplicity properties. Our main impossibility theorem (Theorem 5.3) shows that applicant-proposing menu-in-outcome description of DA must, in some formal sense, be complex; formally, they cannot be linear-memory. However, this theorem does not give any impossibility result for menu-in-outcome descriptions of DA which—like our menu description of DA, Description 1—are *institution*-proposing.[5] Given this, one might still hope for a useful institution-proposing menu-in-outcome description of DA, which might yield an alternative outcome description of DA together with a simple proof of its strategyproofness.

Perhaps surprisingly, in Appendix T.3 we construct a new institution-proposing menu-in-outcome description of DA which is, in fact, linear memory. Unfortunately, this description is *exceedingly* unintuitive and convoluted. Indeed, as one can see from the details in Appendix T.3, this description is a highly technical algorithm that requires careful bookkeeping to maintain its linear-memory. Thus, in contrast to DA's traditional description and our Description 1, this algorithm is impractical for describing DA to real-world participants.

Motivated by the intricacies of the description we uncover in Appendix T.3, in Appendix T.4, we additionally use an established formal simplicity property to demonstrate a sense in which institution-proposing menu-in-outcome descriptions of DA *must be* convoluted and impractical. Our linear-memory property used in Theorem 5.3 does not suffice for this purpose (since our convoluted descriptions in Appendix T satisfy this flexible property). Instead, we use the *pick-an-object* simplicity

[5] We use the term institution-proposing to mean the definition perfectly analogous to applicant-proposing (Definition 5.1), in which sides of the market are interchanged.

desideratum of Bó and Hakimov (2023). We prove that institution-proposing menu-in-outcome descriptions for DA cannot be pick-an-object.[6] Briefly and informally, this means that all such descriptions must learn the match of some applicant d when making queries which seem unrelated to d, showing a precise sense in which such descriptions cannot be simple. Combined with our main impossibility result (Section 5), this shows that one-side-proposing menu-in-outcome descriptions of DA cannot (in appropriate senses) be simple.

More broadly, in pursuit of potentially useful descriptions of DA, we consider a broad classification of matching mechanism descriptions. We consider applicant-proposing descriptions (like traditional ones), and institution-proposing descriptions (like Description 1). We consider our three description outlines: menu descriptions, outcome descriptions, and menu-in-outcome descriptions. Altogether, this gives six classes of one-side-proposing descriptions with one of these outlines. In this appendix, we construct linear-memory descriptions of DA of *every* class that is not ruled out by our main impossibility result Theorem 5.3. Unfortunately, all of the additional descriptions are (like our institution-proposing menu-in-outcome description) exceedingly unintuitive and convoluted algorithms. See Table S.1 for an overview of all our descriptions and results for DA.

Table S.1: Classification of descriptions of DA

	Menu Description	Outcome Description	Menu-in-Outcome Description
Applicant proposing	Unintuitive, convoluted algorithm in Appendix T.2.	Traditional DA algorithm.	**Impossible** (without quadratic memory) by Theorem 5.3.
Institution proposing	Description 1 in Section 3.	Unintuitive, convoluted algorithm in Appendix T.1.	Unintuitive, convoluted (e.g., not pick-an-object) algorithm in Appendix T.3.

Notes: We consider descriptions which either read preferences in an applicant-proposing manner or read priorities in an institution-proposing manner. We consider three description outlines: menu descriptions (conveying strategyproofness), outcome descriptions (conveying the fully matching), or menu-in-outcome descriptions (conveying both).

All told, our results exhaustively consider all classes of descriptions of DA that

[6] In Appendix T.4, we demonstrate more generally that for DA, institution-proposing outcome descriptions—and thus menu-in-outcome descriptions as a special case—cannot be pick-an-object.

are one-side-proposing and fit one of our three description outlines. Within this classification, we find two simple and practical descriptions of DA: the traditional one, and our menu description. This suggests that within our framework, simple descriptions of DA face a trade-off between conveying strategyproofness and conveying the full outcome matching.

The organization of this appendix is as follows. We present an institution-proposing outcome description of DA, adapted from Ashlagi et al. (2017), in Appendix T.1. We present our applicant-proposing menu description in Appendix T.2. We present our institution-proposing menu-in-outcome description in Appendix T.3. We present our supplemental impossibility theorem for DA in Appendix T.4.

T.1 Institution-proposing outcome description of DA

First, we construct an *institution*-proposing linear-memory outcome description of DA. Interestingly, essentially this same algorithm was used as a lemma by Ashlagi et al. (2017) (henceforth, AKL).[7] For notational convenience, throughout the rest of this appendix, we refer to the priorities of institutions as "preferences." We also denote the set of applicants by $\mathcal{D}$, the set of institutions by $\mathcal{H}$, and (when relevant) we describe the menu to applicant d_*.

Theorem T.1 (Adapted from Ashlagi et al., 2017)**.** *Description S.1 computes the applicant-optimal stable outcome. Moreover, Description S.1 is an institution-proposing and $\widetilde{\mathcal{O}}(n)$-memory description.*

Proof. AKL refer to the sides of the market as "men" and "women", and define "Algorithm 2 (MOSM to WOSM)", a men-proposing algorithm for the women-optimal stable matching. Description S.1 follows the exact same order of proposals as this algorithm from AKL. The only difference apart from rewriting the algorithm in a more "pseudocode" fashion is that Description S.1 performs bookkeeping in a slightly different way—Algorithm 2 from AKL maintains *two* matchings, and their list V keeps track of only women along a rejection chain; our list V keeps track of both applicants

[7]For context, Ashlagi et al. (2017) needs such an algorithm to analyze (for a random matching market) the expected "gap" between the applicant and institution optimal stable matching. Their algorithm builds on the work of Immorlica and Mahdian (2005), and is also conceptually similar to algorithms for constructing the "rotation poset" in a stable matching instance Gusfield and Irving (1989) (see also Cai and Thomas (2019)).

and institutions along the rejection chain (and can thus keep track of the "difference between" the two matchings which AKL tracks).

Moreover, the algorithm is institution-proposing, by construction. Furthermore, as it runs it stores only a single matching μ, a set $\mathcal{D}_{\text{term}} \subseteq \mathcal{D}$, and the "rejection chain" V (which can contain each applicant $d \in \mathcal{D}$ *at most once*). Thus, it uses memory $\widetilde{O}(n)$. □

T.2 Applicant-proposing menu description of DA

In this section, we construct an applicant-proposing linear-memory menu description of DA. On an intuitive level, the algorithm works as per the "brute-force" menu description in Example 2.8, but avoiding the need to "restart many times" by using the various properties of DA and by careful bookkeeping (to intuitively "simulate all of the separate runs of the brute-force description on top of each other").

On a formal level, we describe the algorithm as a variant of Description S.1. The proof constructing this algorithm uses a bijection between one applicant's menu in DA under some preferences, and some data concerning the *institution*-optimal stable matching under a related set of preferences. Our applicant-proposing menu description is then phrased as a variation of Description S.1, which (reversing the roles of applicants and institutions from the presentation in Description S.1) is able to compute the institution-optimal matching using an applicant-proposing algorithm.

Fix an applicant d_* and set P that contains (1) the preferences of all applicants $\mathcal{D} \setminus \{d_*\}$ *other than* d_* over $\mathcal{H}$ and (2) the preferences of all institutions $\mathcal{H}$ over all applicants $\mathcal{D}$ (including d_*). We now define the "related set of preferences" mentioned above. Define the *augmented preference list* P' as follows: For each $h_i \in \mathcal{H}$, we create two additional applicants $d_i^{\text{try}}, d_i^{\text{fail}}$ and two additional institutions $h_i^{\text{try}}, h_i^{\text{fail}}$. The entire preference lists of these additional agents in P' are as follows: for each $h_i \in \mathcal{H}$:

$$d_i^{\text{try}} : h_i^{\text{try}} \succ h_i \succ h_i^{\text{fail}} \qquad\qquad d_i^{\text{fail}} : h_i^{\text{fail}} \succ h_i^{\text{try}}$$
$$h_i^{\text{try}} : d_i^{\text{fail}} \succ d_i^{\text{try}} \qquad\qquad h_i^{\text{fail}} : d_i^{\text{try}} \succ d_i^{\text{fail}}$$

We need to modify the preference lists of the pre-existing institutions as well. But this modification is simple: for each $h_i \in \mathcal{H}$, replace d_* with d_i^{try}. The institution-optimal

Description S.1 An institution-proposing outcome description of DA

Input: Preferences of all applicants $\mathcal{D}$ and institutions $\mathcal{H}$
Output: The result of applicant-proposing deferred acceptance

▷ *We start from the institution-optimal outcome, and slowly "improve the match for the applicants"* ◁
Let μ be the result of institution-proposing DA
Let $\mathcal{D}_{\text{term}}$ be all applicants unmatched in μ ▷ *$\mathcal{D}_{\text{term}}$ is all applicants at their optimal stable partner*
while $\mathcal{D}_{\text{term}} \neq \mathcal{D}$ **do**
  Pick any $\widehat{d} \in \mathcal{D} \setminus \mathcal{D}_{\text{term}}$, and set $d = \widehat{d}$
  Let $h = \mu(d)$ and set $V = [(d, h)]$
  while $V \neq []$ **do**
    Let $d \leftarrow$ NextAcceptingApplicant(μ, h)
    if $d = \emptyset$ or $d \in \mathcal{D}_{\text{term}}$ **then**
      ▷ *In this case, all the applicants in V have reached their optimal stable partner.* ◁
      Add every applicant which currently appears in V to $\mathcal{D}_{\text{term}}$
      Set $V = []$
    else if $d \neq \emptyset$ and d does not already appear in V **then** ▷ *Record this in the rejection chain*
      Add $(d, \mu(d))$ to the end of V
      Set $h \leftarrow \mu(d)$ ▷ *The next proposing institution will be the "old match" of d.*
    else if $d \neq \emptyset$ and d appears in V **then**
      ▷ *A new "rejection rotation" should be written to μ* ◁
      WriteRotation(μ, V, d, h) ▷ *Updates the value of μ, V, and (possibly) h*
Return μ

function NextAcceptingApplicant(μ, h)
  repeat
    Query h's preference list to get their next choice d
  until $d = \emptyset$ or $h \succ_d \mu(d)$
  Return d

procedure WriteRotation(μ, V, d, h)
  Let $T = (d_1, h_1), \ldots, (d_k, h_k)$ be the suffix of V starting with the first occurrence of $d = d_1$
  Update μ such that $\mu(h_i) = d_{i+1}$ (for each $i = 1, \ldots, k$, with indices taken mod k)
  ▷ *Now we fix V and h to reflect the new μ* ◁
  Update V by removing T from the end of V
  if $V \neq \emptyset$ **then**
    Let (d_0, h_0) denote the final entry remaining in V
    ▷ *The next proposing institution will either h_k or h_0, depending on which d_1 prefers* ◁
    if $h_k \succ_{d_1} h_0$ **then**
      Set $h \leftarrow h_0$
    else if $h_0 \succ_{d_1} h_k$ **then**
      Add (d_1, h_k) to the end of V
      Set $h \leftarrow h_k$

matching for this augmented set of preferences P' will encode the menu, as we need.[8]

Proposition T.2. *An institution $h_i \in \mathcal{H}$ is on d_*'s menu in APDA with preferences P if and only if in the institution-optimal stable matching with the augmented preferences P', we have h_i^{try} matched to d_i^{try}.*

Proof. For both directions of this proof, we use the following lemma, which is a special case of the main technical lemma in Cai and Thomas (2022):

Lemma T.3. *In P', each h_i^{try} has a unique stable partner if and only if, when h_i^{try} rejects d_i^{try} (i.e. if h_i^{try} submitted a list containing only d_i^{fail}, and all other preferences remained the same), h_i^{try} goes unmatched (say, in the applicant-optimal matching).*

Note that each h_i^{try} is matched to d_i^{try} in the applicant-optimal matching with preferences P' (and the matching among all original applicants and institutions is the same as μ_{app}).

($\Leftarrow$) By the lemma, if h_i^{try} is matched to d_i^{try} in the institution-optimal matching under P', then h_i^{try} must go unmatched when h_i^{try} rejects d_i^{try}. But, after h_i^{try}, we know d_i^{try} will propose to h_i, and some rejection chain may be started. Because d_i^{try}'s very next choice is h_i^{fail} (and proposing there would lead directly to h_i^{try} receiving a proposal from d_i^{fail}), the *only* way for h_i^{try} to remain unmatched is if d_i^{try} remains matched to h_i. But because (relative to all the original applicants) d_i^{try} is in the same place as d_* on h_i's preference list, the resulting set of rejections in P' will be precisely the same as those resulting from d_* submitting a preference list in P which contains only h_i. In particular, d_* would remain matched at h_i in P if they submitted such a list. Thus, h_i is on d_*'s menu.

($\Rightarrow$) Suppose h_i^{try} is matched to d_i^{fail} in the institution optimal matching under P'. Again, h_i^{try} must receive a proposal from d_i^{fail} when h_i^{try} rejects d_i^{try}. But this can only happen if d_i^{try} is rejected by h_i (then proposes to h_i^{fail}). But because the preferences

[8]For the reader familiar with the rotation poset of stable matchings (Gusfield and Irving, 1989), the intuition for this construction is the following: having h_i^{try} reject applicant d_i^{try} corresponds to d_* "trying" to get $h_i \in \mathcal{H}$, i.e., "trying to see if h_i is on their menu." If d_* would be rejected by h_i after proposing, either immediately or after some "rejection rotation," then so will d_i^{try} (because they serve the same role as d_* at h_i). So if a rotation swapping h_i^{try} and h_i^{fail} exists (e.g., in the institution optimal matching) then h_i is *not* on d_*'s menu. On the other hand, if d_* could actually permanently match to h_i, then d_i^{try} proposing to h_i will result in a rejection chain that ends at some other applicant (either exhausting their preference list or proposing to an institution in $\mathcal{H}_{\text{term}}$), which does not result in finding a rotation (or writing a new set of matches as we "work towards the institution-optimal match"). Thus, if h_i^{try} and h_i^{fail} do not swap their matches in the institution-optimal stable outcome, then h_i *is* on d_*'s menu.

of the original applicants in P' exactly corresponds to those in P, we know that d_* would get rejected by h_i if they proposed to them in μ_{app} under P. But then h_i cannot be on d_*'s menu. □

With this lemma in hand, we can now show that there is an applicant-proposing linear-memory menu description of (applicant-optimal) DA. This description is given in Description S.2.

Description S.2 An applicant-proposing menu description of DA

Input: An applicant d_* and preferences of all applicants $\mathcal{D} \setminus \{d_*\}$ and institutions $\mathcal{H}$
Output: The menu of d_* in applicant-optimal DA given these preferences

1: Simulate the flipped-side version of Description S.1 (such that applicants propose) on preferences P' to get a matching μ
2: **Return** the set of all institutions h_i such that h_i^{try} is matched to d_i^{try} in μ

Theorem T.4. *There is an applicant-proposing, $\widetilde{O}(n)$ memory menu description of (applicant-optimal) DA.*

Proof. The algorithm proceeds by simulating a run of Description S.1 on preferences P' (interchanging the role of applicants and institutions, so that applicants are proposing). This is easy to do while still maintaining the applicant-proposing and $\widetilde{O}(n)$ memory. In particular, P' adds only $O(n)$ applicants and institutions, with each d_i^{try} and d_i^{fail} making a predictable set of proposals. Moreover, the modification made to the preferences lists of the institutions $h \in \mathcal{H}$ is immaterial—when such institutions receive a proposal from d_i^{try}, the algorithm can just query their lists for d_*. □

T.3 Institution-proposing menu-in-outcome description of DA

In this section, we construct an institution-proposing linear-memory menu-in-outcome description of DA.[9] Throughout this section, let $P|_{d_i:L}$ denote altering preferences P by having d_i submit list L.

[9]For some technical intuition on why such a description might exist, consider the construction used in Theorem 5.3, and consider a menu-in-outcome description for applicant i executed on these preferences. To find the menu in this construction with an applicant-proposing algorithm, all of the "top tier rotations" must be "rotated", but to find the correct final matching after learning t_i, some arbitrary subset of the rotations must be "unrolled" (leaving only the subset of rotations which t_i

Unlike our applicant-proposing menu description of DA from Appendix T.2, our institution-proposing menu-in-outcome description cannot be "reduced to" another algorithm such as Description S.1. However, the algorithm is indeed a modified version of Description S.1 that "embeds" our simple institution-proposing menu algorithm Description 1 (i.e., IPDA where an applicant d_* submits an empty preference list) as the "first phase." The key difficulty the algorithm must overcome is being able to "undo one of the rejections" made in the embedded run of Description 1. Namely, the algorithm must match d_* to her top choice from her menu, and "undo" all the rejections caused by d_* rejecting her choice.[10] To facilitate this, the description has d_* reject institutions that propose to d_* "as slowly as possible," and maintains a delicate $\widetilde{O}(n)$-bit data structure that allows it to undo one of d_*'s rejections.[11] The way this data structure works is involved, but one simple feature that illustrates how and why it works is the following: *exactly one* rejection from d_* will be undone, so if some event is caused by *more than one* (independent) rejection from d_*, then this event will be caused regardless of what d_* picks from the menu.

We present our algorithm in Description S.3. For notational convenience, we define a related set of preferences P_{hold} as follows: For each $h_i \in \mathcal{H}$, add a "copy of d_*" called d_i^{hold} to P_{hold}. The only acceptable institution for d_i^{hold} is h_i, and if d_* is on

actually proposes to). Theorem 5.3 shows that all of this information must thus be remembered in full. Now consider a run of Description S.1 on these preferences (or on a modified form of these preferences where institutions' preference lists determine which top tier rotations propose to bottom tier rotations). Some subset of top-tier institutions will propose to applicant i. To continue on with a run of Description S.1, it suffices to undo *exactly one* of these proposals. So, if two or more top-tier rotations trigger a bottom-tier rotation, then we can be certain that the bottom-tier rotation will be rotated, and we only have to remember which bottom-tier rotations are triggered by exactly one top-tier rotation (which takes $\widetilde{O}(n)$ bits).

[10] Description S.1 is independent of the order in which proposals are made. Moreover, one can even show that d_* receives proposals from all h on her menu in Description S.1. However, this does not suffice to construct our menu-in-outcome description simply by changing the order of Description S.1. The main reason is this: in Description S.1, the preferences of d_* are already known, so d_* can reject low-ranked proposals without remembering the effect that accepting their proposal might have on the matching. While the "unrolling" approach of Description S.3 is inspired by the way Description S.1 effectively "unrolls rejection chains" (by storing rejections in a list V and only writing these rejections to μ when it is sure they will not be "unrolled"), the bookkeeping of Description S.3 is far more complicated (in particular, the description maintains a DAG Δ instead of a list V).

[11] Interestingly, this "rolled back state" is *not* the result of institution-proposing DA on preferences $(P, d_i : \{h_j\})$, where h_j is d_i's favorite institution on her menu. Instead, it is a "partial state" of Description S.1 (when run on these preferences), which (informally) may perform additional "applicant-improving rotations" on top of the result, and thus we can continue running Description S.1 until we find the applicant-optimal outcome.

h_i's list, replace d_* with d_i^{hold} on h_i's list. Given what we know from Section 3, the proof that this algorithm calculates the menu is actually fairly simple:

Lemma T.5. *The set $\mathcal{H}_{\text{menu}}$ output by Description S.3 is the menu of d_* in (applicant-proposing) DA.*

Proof. Ignoring all bookkeeping, Phase 1 of this algorithm corresponds to a run of $IPDA(P|_{d_*:\emptyset})$. The only thing changed is the order in which d_* performs rejections, but DA is invariant under the order in which rejections are performed. Moreover, $\mathcal{H}_{\text{menu}}$ consists of exactly all institutions who propose to d during this process, i.e. d_*'s menu (according to Section 3). □

The correctness of the matching, on the other hand, requires an involved proof. The main difficulty surrounds the "unroll DAG" Δ, which must be able to "undo some of the rejections" caused by d_* rejecting different h. We start by giving some invariants of the state maintained by the algorithm (namely, the values of Δ, μ, P, and h):

Lemma T.6. *At any point outside of the execution of* AdjustUnrollDAG*:*

(1) P contains all nodes in Δ of the form (d, h) (where h is the "currently proposing" $h \in \mathcal{H}$).

(2) All of the nodes in P have out-degree 0.

(3) The out-degree of every node in Δ is at most 1.

(4) Every source node in Δ is of the form (d_*, h_i) for some $h_i \in \mathcal{H}_{\text{menu}}$.

(5) For every edge (d_0, h_0) to (d_1, h_1) in Δ, we have $\mu(d_1) = h_0$.

(6) For each $d \in \mathcal{D} \setminus \{d_*\}$, there is at most one node in Δ of the form (d, h_i) for some h_i.

Each of these properties holds trivially at the beginning of the algorithm, and it is straightforward to verify that each structural property is maintained each time AdjustUnrollDag runs.

We now begin to model the properties that Δ needs to maintain as the algorithm runs.

Description S.3 An institution-proposing menu-in-outcome description of DA

Phase 1 input: An applicant d_* and preferences of applicants $\mathcal{D} \setminus \{d_*\}$ and institutions $\mathcal{H}$
Phase 1 output: The menu $\mathcal{H}_{\text{menu}}$ presented to d_* in (applicant-proposing) DA
Phase 2 input: The preference list of applicant d_*
Phase 2 output: The result of (applicant-proposing) DA

▷ ***Phase 1:*** ◁
Simulate a run of $IPDA(P_{\text{hold}})$ and call the result μ'
Let $\mathcal{H}_*$ be all those institutions $h_i \in \mathcal{H}$ matched to d_i^{hold} in μ' ▷ *These institutions "currently sit at d_*"*
Let μ be μ', ignoring all matches of the form (d_i^{hold}, h)
Let $\mathcal{H}_{\text{menu}}$ be a copy of $\mathcal{H}_*$ ▷ *We will grow $\mathcal{H}_{\text{menu}}$*
Let Δ be an empty graph ▷ *The "unroll DAG". After Phase 1, we'll "unroll a chain of rejections"*
while $\mathcal{H}_* \neq \emptyset$ **do**
    Pick some $h \in \mathcal{H}_*$ and remove h from $\mathcal{H}_*$
    Add (d_*, h) to Δ as a source node
    Set $P = \{(d_*, h)\}$ ▷ *This set stores the "predecessors of the next rejection"*
    while $h \neq \emptyset$ **do**
        Let $d \leftarrow$ NextInterestedApplicant(μ, Δ, h)
        AdjustUnrollDag(μ, Δ, P, d, h) ▷ *Updates each of these values*
Return $\mathcal{H}_{\text{menu}}$
▷ ***Phase 2:*** *We now additionally have access to d_*'s preferences* ◁
Permanently match d_* to their top pick h_{pick} from $\mathcal{H}_{\text{menu}}$
$(\mu, \mathcal{D}_{\text{term}}) \leftarrow$ UnrollOneChain$(\mu, \Delta, h_{\text{pick}})$
Continue running the Description S.1 until its end, using this μ and $\mathcal{D}_{\text{term}}$, starting from Description 4
Return the matching resulting from Description S.1

function NextInterestedApplicant(μ, Δ, h)
    repeat
        Query h's preference list to get their next choice d
    until $d \in \{\emptyset, d_*\}$ OR (d is in Δ, paired with h' in Δ, and $h \succ_d h'$) OR (d is not in Δ and $h \succ_d \mu(d)$)
    Return d

procedure UnrollOneChain$(\mu, \Delta, h_{\text{pick}})$
    Let $(d_0, h_0), (d_1, h_1), \ldots, (d_k, h_k)$ be the (unique) longest chain in Δ starting from $(d_0, h_0) = (d^*, h_{\text{pick}})$
    Set $\mu(d_i) = h_i$ for $i = 0, \ldots, k$
    Set $\mathcal{D}_{\text{term}} = \{d_*, d_1, \ldots, d_k\}$
    return $(\mu, \mathcal{D}_{\text{term}})$

procedure ADJUSTUNROLLDAG(μ, Δ, P, d, h)
  if $d = \emptyset$ **then**
    Set $h = \emptyset$ ▷ *Continue and pick a new h*
  else if $d = d^*$ **then** ▷ *h proposes to d_*, so we've found a new h in the menu*
    Add h to $\mathcal{H}_{\text{menu}}$
    Add (d_*, h) to Δ
    Add (d_*, h) to the set P ▷ *h still proposes; the next rejection will have multiple predecessors*
  else if d does not already appear in Δ **then** ▷ *Here $h \succ_d \mu(d)$*
    Add $(d, \mu(d))$ to Δ ▷ *Record this in the rejection DAG*
    Add an edge from each $p \in P$ to $(d, \mu(d))$ in Δ, and set $P = \{(d, \mu(d))\}$
    Set $h' \leftarrow \mu(d)$, then $\mu(d) \leftarrow h$, then $h \leftarrow h'$
    ▷ *The next proposing institution will be the "old match" of d.* ◁
  else if d appears in Δ **then**
    ADJUSTUNROLLDAGCOLLISION(μ, Δ, P, d, h) ▷ *Updates each of these values*

procedure ADJUSTUNROLLDAGCOLLISION(μ, Δ, P, d, h)
  Let $p_1 = (d_1, h_1)$ be the pair where $d = d_1$ appears in Δ ▷ *We know $h \succ_{d_1} h_1$*
  Let P_1 be the set of all predecessors of p_1 in Δ

  ▷ *First, we drop all rejections from Δ which we are now sure we won't have to unroll* ◁
  Let $(d_1, h_1), \ldots, (d_k, h_k)$ be the (unique) longest possible chain in Δ starting from (d_1, h_1) such that *each node (d_j, h_j) for $j > 1$ has exactly one predecessor*
  Remove each (d_i, h_i) from Δ, for $i = 1, \ldots, k$, and remove all edges pointing to these nodes

  ▷ *Now, we adjust the nodes to correctly handle d_1 (which might have to "unroll to h_{min}")* ◁
  Let $h_{\min}$ be the institution among $\{\mu(d_1), h\}$ which d_1 prefers least
  Let $p_{\text{new}} = (d_1, h_{\min})$; add p_{new} to Δ
  if $h_{\min} = h$ **then** ▷ *We replace p_1 with p_{new}*
    Add an edge from each $p \in P_1$ to p_{new}
    Add p_{new} to P ▷ *h is still going to propose next*
  else ▷ *Here $h_{min} = \mu(d_1)$; we add p_{new} below the predecessors P*
    Add an edge from every $p \in P$ to p_{new}
    Set $P = P_1 \cup \{p_{\text{new}}\}$
    Set $h' \leftarrow \mu(d_1)$, then $\mu(d) \leftarrow h$, then $h \leftarrow h'$ ▷ *d_1's old match will propose next*

Definition T.7. At some point during the run of any institution-proposing algorithm with preferences Q, define the *truncated revealed preferences* $\overline{Q}$ as exactly those institution preferences which have been queried so far, and assuming that all further queries to all institutions will return $\emptyset$ (that is, assume that all institution preference lists end right after those preferences learned so far).

For some set of preferences Q we say the revealed truncated preferences $\overline{Q}$ and the pair $(\mu', \mathcal{D}'_{\text{term}})$ is a *partial AKL state* for preferences Q if there exists some execution order of Description S.1 and a point along that execution path such that the truncated revealed preferences are $\overline{Q}$, and μ and $\mathcal{D}_{\text{term}}$ in Description S.1 take the values μ' and $\mathcal{D}'_{\text{term}}$

Let Q be a set of preferences which does not include preference of d_*, and let $\overline{Q}$ a truncated revealed preferences of Q. Call a pair (μ, Δ) *unroll-correct for Q at $\overline{Q}$* if 1) μ is the result of $IPDA(\overline{Q})$, and moreover, for every $h \in \mathcal{H}_{\text{menu}}$, the revealed preferences $\overline{Q}$ and pair UnrollOneChain(μ, Δ, h) is a valid partial AKL state of preferences $(\overline{Q}, d_* : \{h\})$.

The following is the main technical lemma we need, which inducts on the total number of proposals made in the algorithm, and shows that (μ, Δ) remain correct every time the algorithm changes their value:

Lemma T.8. *Consider any moment where we query some institution's preferences list withing* NextInterestedApplicant *in Description S.3. Let h be the just-queried institution, let d be the returned applicant, and suppose that the truncated revealed preferences before that query are $\overline{Q}$, and fix the current values of μ and Δ. Suppose that (μ, Δ) are unroll-correct for Q at $\overline{Q}$.*

Now let $\overline{Q}'$ be the revealed preferences after adding d to h's list, and let μ' and Δ' be the updated version of these values after Description S.3 processes this proposal (formally, if NextInterestedApplicant *returns d, fix μ' and Δ' to the values of μ and Δ after the algorithm finishes running* AdjustUnrollDag*; if* NextInterestedApplicant *does not return d, set $\mu' = \mu$ and $\Delta' = \Delta$). Then (μ', Δ') are unroll-correct for Q at $\overline{Q}'$.*

Proof. First, observe that if h's next choice is $\emptyset$, then the claim is trivially true, because $\overline{Q} = Q$ (and AdjustUnrollDAG does not change μ or Δ). Now suppose h's next choice is $d \neq \emptyset$, but is not returned by NextInterestedApplicant. This means that: 1) $d \neq d_*$, 2) $\mu(d) \succ_d h$, and 3) either d does not appear in Δ, or d

does appear in Δ, in which case d matched to some h' such that $h' \succ_d h$. Because (μ, Δ) are unroll-correct for Q at $\overline{Q}$, and because Lemma T.6 says that d can appear at most once in Δ, the only possible match which d could be unrolled to at truncated revealed preferences $\overline{Q}$ is h' (formally, if the true complete preferences were $\overline{Q}$, then for all $h_* \in \mathcal{H}_{\text{menu}}$, the partial AKL state under preferences $(\overline{Q}, d : \{h_*\})$ to which we we would unroll would match d to either $\mu(d)$ or h'). But d would not reject $\mu(d)$ in favor of h, nor would she reject h' in favor of h. Thus, (for all choices of $h_* \in \mathcal{H}_{\text{menu}}$) we know h will always be rejected by d, and (μ, Δ) are already unroll-correct for Q at $\overline{Q}'$.

Now, consider a case where h's next proposal $d \neq \emptyset$ is returned by NextInterestedApplicant. There are a number of ways in which AdjustUnrollDAG may change Δ. We go through these cases.

First, suppose $d = d_*$. In this case, the menu of d_* in $\overline{Q}'$ contains exactly one more institution than the menu in $\overline{Q}$, namely, institution h. Moreover, for any $h_* \in \mathcal{H}_{\text{menu}} \setminus \{h\}$, the same partial AKL state is valid under both preferences $(\overline{Q}, d : \{h_*\})$ and $(\overline{Q}', d : \{h_*\})$ (the only difference in $(\overline{Q}', d : \{h_*\})$ is a single additional proposal from h to d_*, which is rejected; the correct value of $\mathcal{D}_{\text{term}}$ is unchanged). For $h_* = h$, the current matching μ, modified to match h to d_*, is a valid partial AKL state for $(\overline{Q}', d : \{h\})$, and this is exactly the result of UnrollOneChain (with $\mathcal{D}_{\text{term}} = \{d_*\}$, which is correct for preferences $(\overline{Q}', d : \{h\})$). Thus, (using also the fact from Lemma T.6 that P contains all nodes in Δ involving h), each possible result of UnrollOneChain is a correct partial AKL state for each $(\overline{Q}', d : \{h_*\})$, so (μ', Δ') is unroll-correct for Q at $\overline{Q}'$.

Now suppose $d \notin \{\emptyset, d_*\}$ is returned from AdjustUnrollDAG, and d does not already appear in Δ. In this case, $h \succ_d \mu(d)$, and for every $h_* \in \mathcal{H}_{\text{menu}}$, the unrolled state when preferences $(\overline{Q}, d : \{h_*\})$ will pair d to $\mu(d)$. Under preferences $(\overline{Q}', d : \emptyset)$, a single additional proposal will be made on top of the proposals of $(\overline{Q}, d : \emptyset)$, namely, h will propose to d and d will reject $\mu(d)$. However, *if h_* is such that h is "unrolled"* (formally, if h_* is such that UnrollOneChain(μ, Δ, h_*) changes the partner of h) then h cannot propose to d in $(\overline{Q}, d : \emptyset)$ (because all pairs in Δ can only "unroll" h to partners before $\mu(h)$ on h's list), nor in $(\overline{Q}', d : \emptyset)$ (because $\overline{Q}'$ only adds a partner to h's list after $\mu(h)$). Thus, for all h_* such that h is unrolled, the pair $(d, \mu(d))$ should be unrolled as well. On the other hand, for all h_* such that h is not unrolled, h will propose to d (matched to d'), so d will match to h in the unrolled-to state. This is exactly how μ' and Δ' specify unrolling should go, as needed.

(Hardest case: AdjustUnrollDagCollision.**)** We now proceed to the hardest case, where $d \notin \{\emptyset, d_*\}$ is returned from AdjustUnrollDag, and d already appears in Δ. In this case, AdjustUnrollDagCollision modifies Δ. Define p_1, P_1, and $h_{\min}$, following the notation of AdjustUnrollDagCollision. Now consider any $h_* \in \mathcal{H}_{\text{menu}}$ under preferences $\overline{Q}$. There are several cases of how h_* may interact with the nodes changed AdjustUnrollDagCollision, so we look at these cases and prove correctness. There are two important considerations which we must prove correct: first, we consider the way that AdjustUnrollDagCollision removes nodes from Δ (starting on Description 19), and second, we consider the way that it creates a new node to handle d (starting on Description 22).

(First part of AdjustUnrollDagCollision.**)** We first consider the way AdjustUnrollDagCollision removes nodes from Δ. There are several subcases based on h_*. First, suppose UnrollOneChain(μ, Δ, h_*) does not contain p_1. Then, because AdjustUnrollDagCollision only drops p_1 and nodes only descended through p_1, the chain unrolled by UnrollOneChain(μ', Δ', h_*) is unchanged until h. (We will prove below that the behavior when this chain reaches h is correct.) Thus, the initial part of this unrolled chain remains correct for Q at $\overline{Q}'$.

On the other hand, suppose that UnrollOneChain(μ, Δ, h_*) contains p_1. There are two sub-cases based on Δ. First, suppose that there exists a pair $p \in P$ in Δ such that p is a descendent of p_1 (i.e. there exists a $p = (d_x, h) \in P$ and a path from p_1 to p in Δ). In this case, under preferences $\overline{Q}$, UnrollOneChain(μ, Δ, h_*) would unroll to each pair in the path starting at h_*, which includes p_1 and all nodes on the path from p_1 to p. Under Δ', however, *none of the nodes from p_1 to p* will be unrolled in this case. The reason is this: in Description S.1, the path from p_1 to p, including the proposal of h to d_1, form an "improvement rotation" when the true preferences are $\overline{Q}'$. Formally, under preferences $(\overline{Q}', d_* : \{h_*\})$, if d_1 rejected h_1, the rejections would follow exactly as in the path in Δ between p_1 and p, and finally h would propose to d_1. Description S.1 would then call WriteRotation, and the value of μ would be updated for each d on this path. So deleting these nodes is correct in this subcase.[12]

For the second subcase, suppose that there is *no* path between p_1 and any $p \in P$ in Δ. In this case, there must be some source $(d_*, \overline{h})$ in Δ which is an ancestor of

[12]This is the core reason why Description S.3 cannot "unroll" to $IPDA(Q, d : \{h_i\})$—instead, it unrolls to a "partial state of AKL".

some $p \in P$, and such that the path from $(d_*, \overline{h})$ to p does not contain any descendent of p_1. (This follows because each $p \in P$ must have at least one source as an ancestor, and no ancestor of any $p \in P$ can be descendent of p_1.) To complete the proof in this subcase, it suffices to show that at preferences $(\overline{Q}', d_* : \{h_*\})$, we "do not need to unroll" the path in Δ starting at h_* after p_1 (formally, we want to show that if you unroll from μ' the path in Δ from h_* to just before p_1 (including the new node added by the lines starting on Description 22), then this is a partial AKL state of Q at $\overline{Q}'$). The key observation is this: in contrast to preferences $(\overline{Q}, d_* : \{h_*\})$, where pair p_1 is "unrolled", under preferences $(\overline{Q}', d_* : \{h_*\})$, we know *$h$ will propose to d_1 anyway*, because d_* will certainly reject $\overline{h}$ (and trigger a rejection chain leading from $(d_*, \overline{h})$ to h proposing to d_1).

(Second part of AdjustUnrollDagCollision.) We now consider the second major task of AdjustUnrollDagCollision, namely, creating a new node to handle d. The analysis will follow in the same way regardless of how the first part of AdjustUnrollDagCollision executed (i.e., regardless of whether there exists a path between p_1 and P). The analysis has several cases. First, suppose (d_*, h_*) is not an ancestor of any node in $P_1 \cup P$ in Δ. This will hold in Δ' as well, so neither UnrollOneChain(μ, Δ, h_*) nor will UnrollOneChain(μ', Δ', h_*) will not change the match of d. Instead, the match of d under UnrollOneChain(μ', Δ', h_*) will be $\mu'(d)$, which is a correct partial AKL state under $(\overline{Q}', d_* : \{h_*\})$, as desired.

Second, suppose h_* is such that (d_*, h_*) is an ancestor of some node in P_1 in Δ. There are two subcases. If $h_{\min} = h$, then we have $\mu(d_1) = \mu'(d_1)$, but when UnrollOneChain(μ', Δ', h_*) is run, we unroll d_1 to h. Correspondingly, in IPDA with preferences $(\overline{Q}', d_* : \{h_*\})$, we know d_1 will not receive a proposal from $\mu(d_1)$ (as this match is unrolled in $\overline{Q}$) but d_1 will receive a proposal from h (as this additional proposal happens in $\overline{Q}'$ but not in $\overline{Q}$, regardless of whether this happens due to a "rejection rotation" of AKL, or simply due to two rejection chains causing this proposal, as discussed above), which d_1 prefers to the unrolled-to match under preferences $\overline{Q}$. Thus, under preferences $(\overline{Q}', d_* : \{h_*\})$, we know d_1 will match to $h_{\min} = h$ in a valid partial AKL-state. So (μ', Δ') is correct for $\overline{Q}'$ in this subcase. If, on the other hand, $h_{\min} = \mu(d_1)$, then in Δ', UnrollOneChain(μ', Δ', h_*) will not contain the new node p_{new}. However, $\mu'(d_1) = h$, and we know d would receive a proposal from h $(\overline{Q}', d_* : \{h_*\})$, and would accept this proposal. So (μ', Δ') is correct for $\overline{Q}'$ in this subcase.

Third and finally, suppose h_* is such that (d_*, h_*) is an ancestor of some node in P in Δ. The logic is similar to the previous paragraph, simply reversed. Specifically, there are two subcases. If $h_{\min} = h$, then when preferences are $(\overline{Q}', d_* : \{h_*\})$, then d_1 will no longer receive a proposal from h, but will still receive a proposal from $\mu(d_1)$. So d_1 should remain matched to $\mu(d_1)$ during $\textsc{UnrollOneChain}(\mu', \Delta', h_*)$, and (μ', Δ') is correct for $\overline{Q}'$ in this subcase. If $h_{\min} = \mu(d_1)$, then $\mu'(d_1) = h$, and in Δ', $\textsc{UnrollOneChain}(\mu', \Delta', h_*)$ will contain the new node p_{new}, which unrolls d_1 to their old match $\mu(d_1)$. This is correct, because in $\overline{Q}$, according to Δ, we know h will be unrolled to some previous match, and correspondingly, in preferences $(\overline{Q}', d_* : \{h_*\})$, we know d_1 will never receive a proposal from h. So (μ', Δ') is correct for $\overline{Q}'$ in this subcase.

Thus, for all cases, (μ', Δ') are unroll-correct for Q at $\overline{Q}'$, as required. □

To begin to wrap up, we bound the computational resources of the algorithm:

Lemma T.9. *Description S.3 is institution-proposing and uses memory $\widetilde{O}(n)$.*

Proof. The institution-proposing property holds by construction. To bound the memory, the only thing that we need to consider on top of AKL is the "unroll DAG" Δ. This memory requirement is small, because there are at most $O(n)$ nodes of the form (d_*, h) for different $h \in \mathcal{H}$, and by Lemma T.6, a given applicant $d \in \mathcal{D} \setminus \{d_*\}$ can appear *at most once* in Δ. So the memory requirement is $\widetilde{O}(n)$. □

We can now prove our main result:

Theorem T.10. *Description S.3 is an institution-proposing, $\widetilde{O}(n)$ memory menu-in-outcome description for DA.*

Proof. We know Description S.3 correctly computes the menu, and that it is institution-proposing and $\widetilde{O}(n)$ memory. So we just need to show that it correctly computes the final matching. To do this, it suffices to show that at the end of Phase 1 of Description S.3, (μ, Δ) is unroll-correct for Q at the truncated revealed preferences $\overline{Q}$ (for then, by definition, running Description S.1 after $\textsc{UnrollOneChain}$ will correctly compute the final matching).

To see this, first note that an empty graph is unroll-correct for the truncated revealed preference after running $IPDA(P_{\text{hold}})$, as no further proposals beyond d_* can be made in these truncated preferences. Second, each time we pick an $h \in \mathcal{H}_*$ on

Description 8, a single (d_*, h) added to Δ (with no edges) is unroll-correct for Q at $\overline{Q}'$, by construction. Finally, by Lemma T.8, every other query to any institution's preference list keeps (μ, Δ) unroll-correct after the new query. So by induction, (μ, Δ) is unroll-correct at the end of Phase 1, as desired. □

T.4 Supplemental Impossibility Result for DA

In this appendix, we give a supplemental impossibility result for descriptions of DA. We prove that institution-proposing outcome descriptions—and hence institution-proposing menu-in-outcome descriptions as a special case—cannot satisfy the *pick-an-object* simplicity condition of Bó and Hakimov (2023).

Bó and Hakimov (2023) introduce the pick-an-object condition in the context of interactive mechanisms, where (informally speaking) agents are iteratively asked to pick their favorite objects from some set, and whenever the mechanism terminates, every agent is matched to their most recently picked object. For example, in a dynamic mechanism implementing DA, applicants can be iteratively asked to pick their favorite institution from the set of all institutions they have not yet proposed to. We consider the pick-an-object condition within the context of one-side-proposing outcome descriptions. In this context, the condition requires that when the description terminates, every agent on the proposing side must be matched to whichever agent they proposed to most recently.

Like the linear-memory condition we use in Section 5, the pick-an-object condition captures one feature of matching mechanism descriptions used to explain these mechanisms in practice. Indeed, the description in Figure 3 on page 16 that is used by the NRMP is pick-an-object, since the yellow highlighting in that figure tracks the most recent proposal of each applicant and, at the end of the description, relays the outcome matching. However, where linear-memory is a fairly permissive desideratum concerning the amount of bookkeeping used, pick-an-object is a more restrictive desideratum concerning the manner in which the bookkeeping is updated and used. Thus, we do not interpret our pick-an-object impossibility result as strongly as our linear-memory impossibility result, e.g., we do not argue that all small tweaks of the traditional description of DA should be pick-an-object. Nevertheless, our pick-an-object impossibility result is quite useful: It shows a potentially-desirable class of descriptions cannot satisfy an established and intuitive simplicity condition, and gives a specific barrier that hypothetical more-practical alternatives to our unintuitive and convoluted description in Appendix T would have to circumvent.

We now formally define pick-an-object, adapting the definition from Bó and Hakimov (2023) to focus on institution-proposing outcome descriptions.

Definition T.11 (Pick-an-Object)**.** An institution-proposing outcome description is *pick-an-object* if, whenever the description terminates and calculates some outcome matching μ, it satisfies the following. For every institution h, let d_h be the most recently queried applicant from h's preference list, i.e., if the description made j queries to h, then d is the j^{th} applicant on h's preference list. Then, $\mu(h) = d_h$ for every institution h.

Observe that the traditional descriptions of SD, TTC, and DA are applicant-proposing outcome descriptions that are pick-an-object (according to a definition perfectly analogous to Definition T.11, but interchanging the roles of the applicants and institutions). DA (the applicant-optimal stable matching mechanism) has a non-trivial institution-proposing outcome description as well (Appendix T.1). However, as we now show, such a description cannot be pick-an-object, giving a sense in which they cannot be simple. Formally:

Proposition T.12. *No institution-proposing outcome description of DA is pick-an-object.*

Proof. Assume for contradiction that D is an institution-proposing outcome description of DA which is pick-an-object. Consider a market with institutions h_1, h_2 and applicants d_1, d_2, d_3. We first define preferences of three applicants as follows:

$$d_1 : h_2 \succ h_1$$
$$d_2 : h_1 \succ h_2$$
$$d_3 : \text{(any complete preference list)}$$

Next, we consider two possible preference lists for each of h_1, h_2:

$$\succ_1 : d_1 \succ d_2 \succ d_3 \qquad\qquad \succ_2 : d_2 \succ d_1 \succ d_3$$
$$\succ_1' : d_1 \succ d_3 \succ d_2 \qquad\qquad \succ_2' : d_2 \succ d_3 \succ d_1$$

One can check that DA (the applicant-optimal stable matching) produces outcome matching μ_1 that assigns d_1 to h_2 and d_2 to h_1 when the priorities are $(\succ_1, \succ_2)$; on any other profile of priorities among those defined above, DA has as outcome the matching μ_2 that assigns d_1 to h_1 and d_2 to h_2. Thus, our description D can know the outcome on these inputs only when it has read the second-highest-priority spot of *both* h_1 and h_2. However, intuitively, this means that our institution-proposing description D of DA cannot be pick-an-object, because the highest-priority applicant for both h_1 and h_2 must be read before we can know whether these institutions are assigned to these applicants.

Formally, consider the execution of D when institutions have priorities $(\succ_1, \succ_2)$. Consider the final time during this execution when D learns the difference between $\succ_j$ and $\succ'_j$ for some $j \in \{1, 2\}$; i.e., the latest possible state s during the execution of the description with priority profile $Q = (\succ_1, \succ_2)$ where the execution diverges from that of some priority profile in $\big\{(\succ'_1, \succ_2), (\succ_1, \succ'_2)\big\}$. (Note that the description must learn this difference in order to calculate DA.) By the symmetry in the defined preferences, it is without loss of generality to suppose that in state s, the description queries the preferences of applicant 1, and thus has one successor state consistent with Q and another consistent with $Q' = (\succ'_1, \succ_2)$. However, since D is institution-proposing, this means that in state s, the description has already read d_1 off the priority list of h_1 (and is proceeding to read either d_2 or d_3 next). Since D is pick-an-object, this means that h_1 cannot match to d_1 in any the final outcome matching of any execution of D consistent with s. But this is a contradiction, since h_1 must match to d_1 in DA when the priority profile is $(\succ'_1, \succ_2)$. This finishes the proof. □

Proposition T.12 directly implies that DA has no institution-proposing menu-in-outcome description satisfying the pick-an-object condition (since such a description is, in particular, an outcome description). Combined with our robust main impossibility result (Section 5), this establishes precise impossibilities for simple one-side-proposing menu-in-outcome descriptions of DA: Such applicant-proposing descriptions cannot be linear-memory, and such institution-proposing descriptions cannot be pick-an-object.