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ABSTRACT

The use of distributional weights in economic analysis is receiving increasing attention in both research and policy circles. This paper examines the extent to which distributional weights affect economic analysis of public good provision. We make two contributions. First, we present a model with distributional weights that allows for marginal benefits and costs of the public good to differ across regions and individual characteristics, such as income or race. Samuelson's analysis of pure public goods is a special case, as are other cases in which marginal benefits and costs may differ by region when the distributional weights are unity. We show how the provision of a pure public good varies with distributional weights different from unity. Second, we analyze distributional weights in conjunction with the value of a statistical life (VSL). We compare the use of an average VSL with differentiated VSLs. We show when using an average VSL will increase or decrease optimal public goods provision relative to differentiated VSLs for given distributional weights. We also identify conditions under which a low-income group prefers using an average VSL to true VSLs. This depends on the fraction of the costs that the low-income group bears in the provision of the public good.

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1. Introduction

The analysis of inequality across groups and nations has become an important research area in economics (Piketty, 2022). Concerns about distributional issues, such as those related to race and income, are also becoming a political issue that national leaders are addressing (The White House, 2021). One of the ways suggested by economists for measuring the impact of policies on different groups is to introduce distributional weights (Finkelstein and Hendren, 2020). For example, with standard assumptions about how utility varies with income, a low-income group could have a higher weight than a high-income group. One might want to then employ such weights in doing a benefit-cost analysis. The U.S. Office of Management and Budget has suggested that benefit-cost analyses done for regulations and projects may want to assign distributional weights that are based on income (The White House, 2023).

This paper extends the theoretical literature on distributional weights with the goal of informing the practice of policy evaluation. It makes two contributions. First, we present a tractable model with distributional weights that allows for marginal benefits and costs to differ across regions and individual characteristics, such as income or race. The first order conditions for maximizing welfare are derived and compared with the case where distributional weights are unity, which is the typical case used in benefit-cost analysis. We show that Samuelson's analysis of pure public goods is a special case in our framework (Samuelson, 1954), as are other settings in which marginal benefits and costs may differ by region, but where distributional weights are unity (Muller and Mendelsohn, 2009). Our result demonstrates that whether distributional weights induce more or less public good provision depends on the marginal benefit function, the specification of the distributional weights, and the policy cost shares, but *not* the marginal cost function itself.

Second, we analyze the use of distributional weights in conjunction with the value of a statistical life (VSL). The VSL is a key statistic used in analyses of government policy that calculate the benefits of policies where mortality risks are affected (Robinson, 2007; USEPA, 1999; 2011; Viscusi, 2018). This value measures the tradeoff between fatality risk and income, and it has been shown to vary systematically with income levels (Viscusi, 2017). We compare the use of an average VSL, often used in the design of government policy, with the use of differentiated (in our case, income-specific) VSLs. We show how to determine when using an average VSL will increase or decrease optimal public goods provision relative to the case of income-specific VSLs for a given set of distributional weights. Much of the motivation for distributional analyses stems from concern about whether policies reflect the preferences of low-income groups. Given this concern and the frequent use of average VSLs in practice, we identify conditions under which a low-income group would prefer using an average VSL to true VSLs in a setting in which their mortality risks are valued less than the mortality risks faced by a high-income group. Our result depends critically on the fraction of the costs that the low-income group bears in the provision of the public good. This analysis of the VSL is also applicable to any willingness-to-pay based determinant of the benefits of public good provision, provided the willingness-to-pay varies with race, income, or other drivers of equity concerns. More generally, our analysis highlights the importance of considering costs as well as benefits in attempting to address equity concerns. We suggest that this may have implications for how decisionmakers and analysts frame equity issues, including concerns about environmental equity.

The theory we present builds on three key areas of research. First, we extend a model by Montgomery (1972), which used a general equilibrium framework to assess the cost-effectiveness of different policy instruments. Montgomery demonstrated that, in principle, pollution licenses could be cost-effective. His model includes a mapping between emission sources and environmental quality receptors, which we also use in developing our theory. In contrast to Montgomery, we introduce distributional weights and focus on social welfare maximization rather than cost-effectiveness.

A second literature that we build on highlights the importance of distributional weights. Adler (2016) summarizes many of the theoretical contributions in this area. Early research that discussed distributional weights includes Meade (1955), Dasgupta and Pearce (1972) and Dasgupta, Sen, and Marglin (1972). The UK Green Book explains how distributional weights could be used in analyzing government projects, programs, and regulations. It notes that, "in principle, each monetary cost and benefit should be weighted according to the relative prosperity of those receiving the benefit or bearing the cost." (HM Treasury, 2003, p. 92). In addition, there is a line of research that examines how tax policy affects the use of distributional weights (e.g., Harberger, 1978; Hylland and Zeckhauser, 1979; Christiansen, 1981).

A final literature we address attempts to link discussions of different VSLs with welfare. Closest to the spirit of our analysis of the VSL is Banzhaf (2011; 2023). Banzhaf compares the use of an average VSL with the true VSL using unitary distributional weights. He finds that the preferences for specific policies may shift depending on which set of weights are used. Hemel (2022) analyzes a regulation that would require new vehicles to have rearview cameras. He examines the welfare implications of using different VSLs for low and high-income groups, as well as the possibility of using different distributional weights. Sunstein (2023) also analyzes the welfare implications of using a VSL and argues that knowing the benefit and cost incidence is important. In contrast to these authors, we present a formal model of optimizing choice, we apply distributional weights to both benefits and costs, and we allow the distributional weights and the benefit and cost functions to vary. Our flexible modeling approach facilitates an analysis of how consumer choices would vary with different distributional weights and different VSL choices.

The paper proceeds as follows: Section 2 presents the theoretical framework and discusses how it relates to global and local public goods. Section 3 provides an application to the VSL, and considers when a low-income group would prefer using an average VSL to its true VSL. Section 4 concludes and identifies areas for future research.

2. Theoretical model

We first present a general model with distributional weights. These weights are allowed to vary by group. A group may refer to an income group, a racial group, or a regional group. For the purposes of the theory, it does not matter, so long as the groups cover all agents -- in particular, groups are mutually exclusive and exhaustive. Two special cases of the general model are derived in which distributional weights are unity; one in which marginal costs and marginal benefits can vary by region (Muller and Mendelsohn, 2009), and one in which there is a pure public good (Samuelson, 1954).

In this formulation, a planner is assumed to maximize the difference between the weighted sum of benefits and the weighted sum of costs to different groups of consumers across all regions. We first consider the weighted benefits to groups and then the weighted costs to groups. There are assumed to be o groups, indexed by k, and m regions, index by j. Group k is assigned a distributional weight, w_k, which is assumed to be constant and positive.[1] In the standard benefit-cost evaluation, the distributional weights are set to one for all groups. Each group in each region has a benefit function $B_{jk}(q_j)$ where q_j represents the level of the local public good in region j (e.g., environmental quality), and B_{jk} represents the benefits to group k in region j. Marginal benefit functions are assumed to be positive and non-increasing ($B'_{jk} > 0, B''_j \leq 0$ for all groups and regions). The weighted benefits for a group k in region j are given by $w_k B_{jk}(q_j)$. Summing over all groups and regions gives aggregate weighted benefits of a given level of the public good in region j: $\sum_{j=1}^{m} \sum_{k=1}^{o} w_k B_{jk}(q_j)$.

To identify the weighted costs to different groups, we need to introduce the costs to firms and translate that into costs to groups of consumers in different regions. For concreteness, assume n firms produce emissions as a byproduct of their production. The cost function for emissions reductions for firm i is given by $C_i(r_i)$, where r_i represents emission reductions. The cost function could be associated with a multi-product or single product firm. The marginal cost of emission reductions is assumed to be greater than zero and non-decreasing in r_i, ($C_i{}' > 0, C_i{}'' \geq 0$ for all i). Firms are assumed to be profit maximizers in a perfectly competitive market. When a firm reduces emissions by r_i, the impact of its reduction on region j is given by $r_i h_{ij}$. h_{ij} is a transfer coefficient that defines the impact of one unit of a reduction in emissions from firm i on environmental quality in region j.

[1] The distributional weight needs to be greater than or equal to zero. One economic interpretation for this weight could be the marginal utility of income. Note that in the interest of simplicity we assume that the weights are constant. To the extent that weights vary with income, and income changes with the introduction of the public good, this would induce variation in the weights and could be taken into account.

A critical issue in the subsequent analysis is the economic incidence of costs: how the cost of emission reductions by firms is allocated across different consumer groups. This factor is often not given adequate attention in policy discussions as they pertain to distributional outcomes (Banzhaf, 2023; Cecot and Hahn, 2022; and Hemel, 2022). We define p_{jk} as the fraction of total costs borne by group k in region j. Furthermore, we assume these abatement costs are fully allocated, so that $\sum_j \sum_k p_{jk} = 1$.[2]

We can now derive the costs of a particular policy. The costs to group k in region j are $p_{jk} \sum_{i=1}^{n} C_i\,(r_i)$, which is the fraction of cost incurred by group k multiplied by the total costs, across polluting firms. In order to derive the total weighted cost, we aggregate over all regions and groups and multiply by the distributional weight for each group, yielding: $\sum_{j=1}^{m} \sum_{k=1}^{o} w_k p_{jk} \sum_{i=1}^{n} C_i\,(r_i)$.

Subtracting costs from benefits and simplifying yields the weighted sum of net benefits over regions and groups:

$$\sum_{j=1}^{m} \sum_{k=1}^{o} w_k \{B_{jk}(q_j) - p_{jk} \sum_{i=1}^{n} C_i\,(r_i)\} \quad (1)$$

The planner in this model chooses the level of emissions reduction, r_i, for each firm, to maximize this weighted sum. The local public good can be connected to the level of emission reduction by noting that $q_j = \sum_{i=1}^{n} r_i h_{ij}$. This says that the concentration level in region j is determined by the emission reductions by each firm i, multiplied by the corresponding transfer coefficient h_{ij}, and summed over all firms. Substitution into (1) yields the following maximization problem:

$$\underset{r_1,\dots,r_n}{\text{Maximize}} \sum_{j=1}^{m} \sum_{k=1}^{o} w_k \{B_{jk}\left(\sum_{i=1}^{n} r_i h_{ij}\right) - p_{jk} \sum_{i=1}^{n} C_i\,(r_i)\} \quad (2)$$

for $i = 1, \dots, n$.

Differentiating the preceding expression with respect to r_i yields:

$$\sum_{j=1}^{m} \sum_{k=1}^{o} w_k \{B'_{jk} h_{ij} - p_{jk} C'_i\} = 0 \quad (3)$$

for $i = 1, \dots, n$.

Equation (3) says the weighted sum of marginal net benefits must equal zero (from reducing a unit of emissions at the i th firm). Alternatively, the weighted sum of marginal benefits over regions and groups equals the weighted sum of marginal costs borne by different individuals (again for reducing a unit of emissions at the i th firm).[3]

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[2] In order to focus on the issue of cost allocation across groups, we do not consider the inefficiencies associated with raising revenue from various groups. See, for example, Mirrlees (1971) and more recently Hendren and Sprung-Keyser (2020).

[3] In what follows, we assume that the second order condition for an interior maximum are satisfied, which will be the case given our assumptions about marginal benefit and marginal cost curves.

2.1 Local public goods

Consider the local pubic goods case where environmental quality can differ across regions. We allow regional benefits and costs to have different distributional weights. This is a generalization of Muller and Mendelsohn (2009) and Mendelsohn (1986). In the context of our model, this is equivalent to assuming that each region corresponds to only one group. Assume for region j the distributional weight is w_j. Furthermore, there are only benefits for group k in region j (i.e., $B_{jk} = 0$ for $j \neq k$).

Equation (3) then becomes:

$$\sum_{j=1}^{m} w_j\{B'_{jj}h_{ij} - p_{jj}C_i{}'\} = 0. \quad (4)$$

for $i = 1, \ldots, n$.

Equation (4) is a special case of equation (3). The term in parenthesis is the unweighted net marginal benefit for group j (in region j) from a unit of emission reduction by firm i. The equation says that firm i chooses its emission reduction so that the weighted sum of marginal benefits equals a weighted sum of its own marginal costs ($C_i{}'$). The case that Muller and Mendelsohn (2009) explore assumes that the distributional weights are one for all j. Even in this case, the marginal cost of emissions reduction will, in general, *not* be equal across all n firms because the h_{ij} vector for a given firm i can differ across the n firms.

2.2 Pure public goods

In our framing, the pure public good case can be modeled as a single region with an arbitrary number of groups. Samuelson considered the case where $w_k = 1$ for all k (weights are unity across income classes) and $h_{ij} = 1$ for all i, j (the transport coefficients are set equal to 1).

Rewriting (3) with these assumptions, and allowing region 1 to be the region of interest yields $\sum_{k=1}^{o} \{B'_{1k} - p_{1k}C_i{}'\} = 0$ for $i = 1, \ldots, n$. In the remainder of our discussion of the pure public good case, we drop the region subscript because there is only one region. Doing so simplifies the notation. This yields $\sum_{k=1}^{o} \{B'_k - p_k C_i{}'\} = 0$, and simplification yields:

$$\sum_{k=1}^{o} B'_k = C'_i = C' \text{ for } i = 1, \ldots, n. \quad (5)$$

We will refer to this as the "unit weight" case which assumes that distributional weights are unity.[4] Equation (5) is the familiar first order condition for a pure public good derived by Samuelson. The vertical summation of marginal benefit curves must equal the

[4] The weights only need to be constant to be constant and positive to get the same result here in terms of public good provision.

marginal cost curve when the optimal quantity of the public good is selected. Equation (5) can be derived by first noting that $\sum_{k=1}^{o} p_k = 1$, which says all groups cover the full costs of emission reductions from each firm. In addition, the marginal costs of each firm will be equal across all firms and equal to a point on the industry marginal cost schedule defined as $C'(\sum r_i)$. To see this last point, note that the term on the left-hand side of equation (5) is the same for each i because this case focuses on a pure public good. That implies $C_1' = C_2' = \cdots = C_n{}'$ when evaluated at the optimal r_i for each i. But that implies by construction $C'(\sum r_i) = C'(q) = C_i'(r_i)$.[5] That follows from the horizontal aggregation across the marginal cost functions for all firms,.

The generalization with arbitrary welfare weights is given by:

$$\sum_{k=1}^{o} w_k\{B_k' - p_k C_i{}'\} = 0 \text{ for } i = 1, \dots, n. \quad (6)$$

We will refer to this as the “general” case because weights can vary, whereas in the original Samuelson formulation, the weights are (implicitly) restricted to unity. Equation (6) says that a weighted sum of marginal benefit curves equals a weighted sum of marginal cost curve curves. It reduces to the Samuelson condition for a pure public good when all the weights are constant and equal.[6] However, in general the optimal provision of the public good will not be the same when different distributional weights are used across groups (compared to when the weights are the same). We will use equation (6) to explore this difference and to highlight some of the welfare implications of using different values of the VSL for different groups. One such example is allowing the VSL to vary across income groups.

2.2.1 Distributional weights and provision of pure public goods

We wish to understand how provision of the public good (e.g., abatement) will vary when distributional weights vary for different consumer groups.

It will be useful to solve for $C_i{}'$ and compare the case when weights are 1 versus the case of arbitrary weights. We can solve equation (6) for C_i' to obtain:[7]

$$C'(\sum r_i) = C_i'(r_i) = \frac{\sum_{k=1}^{o} w_k B_k'}{\sum_{l=1}^{o} w_l p_l}. \quad (7)$$

Equation (7) says that the C_i' are equal for all i, which means that the marginal cost of reducing emissions is equal across all sources, and hence equal to C'. This point is worth noting. Efficiency requires that firms have the same marginal cost of emissions reductions

[5] In the pure public good case, $\sum r_i = q$. That is the emission reductions from each firm are treated as homogeneous and sum to the total level of abatement, q, also called the public good.
[6] In this case the constant for the distributional weight can be divided on both sides of equation (6) and equation (6) becomes equation (5).
[7] We introduce the “l” subscript to note that the groups comprising the sum in the denominator may be different from those in the sum in the numerator.

even in the case where distributional weights differ from unity. Note, however, that the level of the marginal cost in (7) will likely differ from that in (5).

Equation (7) also says that whether abatement goes up or down when incorporating nonunitary weights depends on whether a *weighted average* of the sum of the marginal benefit curves goes up or down compared to the unitary case. This direction of change does *not* depend on the marginal cost function. Rather it depends on how the marginal benefit functions adjusted by their weights compare (for the case of unit weights and the general case) and it depends on the cost shares.[8] To see this, consider that the weights for the kth group in the general case are $\frac{w_k}{\sum_{l=1}^{o} w_l p_l}$, and thus both the distributional weights *and* the cost shares are important considerations in determining optimality.[9]

Another way of interpreting the equilibrium condition represented by equation (7) is to note that $C' \sum_{l=1}^{o} w_l p_l = \sum_{k=1}^{o} w_k B_k'$. Viewed in this way the equilibrium is determined by the intersection of a weighted marginal cost curve, where the weight is $\sum_{l=1}^{o} w_l p_l$, and a weighted sum of marginal benefit curves, where the weight on the k th curve is w_k . Both interpretations of the equilibrium can be illustrated graphically.[10]

Leveraging the equilibrium condition in (7), we wish to compare the optimal provision of the pure public good when distributional weights are arbitrary with optimal provision when distributional weights are set to one. Let q_w^* be the optimal abatement with arbitrary distributional weights, w_k; and q^* be the optimal abatement with unit weights.

Suppose

$$\frac{\sum_{k=1}^{o} w_k B_k'(q_w^*)}{\sum_{l=1}^{o} w_l p_l} > \frac{\sum_{k=1}^{o} B_k'(q_w^*)}{\sum_{l=1}^{o} p_l} = \sum_{k=1}^{o} B_k'(q_w^*)$$

where the equality follows because $\sum_{k=1}^{o} p_k = 1$. The above expression says the weighted marginal benefit at q_w^* exceeds the unit weight marginal benefit at q_w^*. Reducing q from q_w^* will decrease C' (if the marginal cost curve is upward sloping) or increase the sum of marginal benefits (if that curve is downward sloping). Thus, q must be reduced from q_w^* to satisfy the unit weights first order condition, and it follows that $q_w^* > q^*$.

This yields the following result:

Proposition 1: Assume the marginal cost curve is upward sloping and/or the sum of marginal benefits curve for all groups is downward sloping.[11] Then:

[8] The marginal benefit function in the unit case is the vertical summation of the marginal benefit functions across groups, following Samuelson. In the weighted case, the vertical summation must be weighted in accord with the weights on the right-hand side of equation (7).
[9] This makes sense because the cost share is weighted by the distributional weight. See equation (6).
[10] Figure 1 in the next section illustrates this graphical approach using an application to the VSL.
[11] We also assume a unique interior equilibrium exists, which will be satisfied for reasonable choices of the parameters, given the assumed shapes of the marginal benefit and marginal cost functions.

$$q_w^* \,(>)(=)(<) q^* \; iff \; \frac{\sum_{k=1}^{o} w_k B_k'(q_w^*)}{\sum_{l=1}^{o} w_l p_l} (>)(=)(<) \sum_{k=1}^{o} B_k'(q_w^*) \tag{8}$$

Proof: See appendix.

This proposition shows that the distributional weights and the cost shares are both crucial for determining whether abatement goes up or down with the introduction of non-unit distributional weights. Further, this comparison does not depend on the marginal cost function.

3. Model application to the VSL

In this section we demonstrate the policy relevance of the machinery we developed in the previous section by applying it to the VSL.[12] The following analysis of the VSL is, in fact, applicable to any willingness-to-pay based determinant of the benefits of public good provision, provided the willingness to pay varies with race, income, or other drivers of equity concerns. Examples include ecosystem services, endangered species preservation, or willingness-to-pay based measures of morbidity risk reductions.

This section presents a simplified version of the pure public good model to explore one issue: how the government's use of an average VSL affects overall welfare and the welfare of specific groups. At the outset, we note that use of an average VSL (prevalent in government policy analyses) with true welfare weights will typically reduce social welfare relative to the use of true welfare weights with a true VSL. This is because the true welfare weights and true distributional weights are used in the maximization problem solved by the planner.[13]

Of perhaps greater interest is that use of an average VSL may be *preferred* by a group that we refer to as low-income (this group is defined below), even when use of the average VSL is not preferred by society. This result depends on the share of the costs for the public good paid by the low-income group. The intuition is that the application of the average VSL to the mortality risk incurred by the low-income group attributes a higher value to such risks than their true VSL. Provided costs borne by the low-income group are sufficiently small, this group enjoys higher net benefits. We underscore the importance of economic cost incidence in driving this result.

Because government agencies often use a VSL that does not differ by income, it is instructive to consider how this might affect welfare. To model this, we consider two population groups: group 1 is the low-income group and group 2 is the high-income group. Each group has a constant marginal benefit curve which incorporate different VSLs. For group 1, the marginal benefit curve is given by $B_1' = n_1 c$, where n_1 represents the number of people in group 1 and $c > 0$ represents the per capita marginal benefits for that group

[12] Note that we choose to focus here on the VSL because of its importance in federal benefit-cost analyses and because the VSL varies systematically with income. Our analysis in this section is sufficiently general to be applied to any such parameter.

[13] This result on reducing welfare will be true when $w_1 \neq w_2$, and both weights are positive.

associated with each unit of the public good. For group 2, the marginal benefit curve is given by $B_2' = n_2 d$, where n_2 represents the number of people in group 2 and $d > c$ represents the per capita marginal benefits associated for that group associated with each unit of the public good. The high-income group thus has higher per capita marginal benefits than the low-income group for each unit of the public good that is provided. The motivation for this specification lies in the positive VSL-income elasticity (Viscusi, 2018). The marginal cost curve is given by a linear upward sloping curve through the origin, which implies $C'(q) = bq$ and $b > 0$. We assume for simplicity there is a single firm.[14] The cost share for group 1 is given by $n_1 p_1 = p$ and for group 2 is given by $n_2 p_2 = 1 - p$, so that costs are fully allocated. In this case p_1 and p_2 can be interpreted as the per capita cost share for a member of group 1 and group 2, respectively.

We can link this formally to the use of VSLs by assuming there is a constant relationship between provision of the public good, q, and the number of statistical lives saved in each group.[15] The benefits for a unit reduction in emissions can be expressed as the product of some constant and the specific VSL for that group. As above, the distributional weights are assumed to be positive. In this context, the social planner maximizes social welfare as shown below:[16]

$$\underset{q}{\text{Maximize}}\; w_1\left(n_1 c q - p\frac{bq^2}{2}\right) + w_2\left(n_2 d q - (1-p)\frac{bq^2}{2}\right)$$

We begin by considering the first order conditions for a maximum. Substitution of the linear functional forms and the true marginal benefit values (c, d) into equation (6) yields:

$$w_1(n_1 c - pbq) + w_2(n_2 d - (1-p)bq) = 0$$

This first order condition is a linear equation in q. We define the solution to this problem as q_t^*, where "t" denotes using true values for the VSL for both groups. Solving for q_t^* yields:

$$q_t^* = \frac{w_1 n_1 c + w_2 n_2 d}{w_1 p\, b + w_2 (1-p) b} \qquad (9)$$

Note that q_t^* is a function of both the weighted sum of marginal benefits and costs.

We next define the average per capita marginal benefit as $\frac{c+d}{2}$, which is a constant proportion of the average VSL. If we substitute the average VSL for the true per capita VSL for both groups, then the first order conditions for a maximum become:

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[14] As noted above, this assumption is easily relaxed to allow for multiple firms. See the discussion around equation (7).

[15] The benefits functions also allow for a formulation in which the VSLs differ by group, and the risk per unit of the public good differs by group.

[16] This formulation assumes total costs are $C(q) = \frac{bq^2}{2}, B_1(q) = cq$ and $B_2(q) = dq$. This follows from integrating the marginal benefit functions and the marginal cost function from 0 to q.

$$w_1\left(n_1\left(\frac{c+d}{2}\right)-pbq\right)+w_2\left(n_2\left(\frac{c+d}{2}\right)-(1-p)bq\right)=0$$

The equation using the average VSL is also a linear equation in q. We define the solution to this problem as q_a^*, where the "a" denotes using average values for the VSL for both groups. Solving for q_a^* yields:

$$q_a^*=\frac{(w_1n_1+w_2n_2)(\frac{c+d}{2})}{w_1p\,b+w_2(1-p)b} \quad (10)$$

The denominators in equation (9) and equation (10) are the same and positive, but the numerators will, in general be, different. Under the special case when the distributional weights multiplied by population are equal (*i.e.*, $w_1n_1=w_2n_2$), then $q_t^*=q_a^*$. To see this result let $\bar{w}=w_1n_1=w_2n_2$. Then the numerator in (9) is $\bar{w}c+\bar{w}d$ and the numerator in (10) is $\bar{w}\left(\frac{c+d}{2}\right)+\bar{w}\left(\frac{c+d}{2}\right)$. These terms are clearly equal.[17] If, however, weighted marginal benefits are higher (lower) using the true value versus the average, then $q_t^*>(<)q_a^*$.

In the more general case where $n_1\neq n_2$, we have the following proposition:

<u>Proposition 2</u>: $q_t^*(<)(=)(>)q_a^*$ iff $w_1n_1(>)(=)(<)w_2n_2$.[18]

<u>Proof</u>: See appendix.

This proposition says that for the case of linear marginal benefit and marginal cost curves, the distributional weights and the size of the groups determine how the optimal public good provision changes with the introduction of an average VSL. Note that these results do not depend on the fraction of costs borne by group 1. As can be seen from the expression for q_t^* and q_a^*, these quantities are affected by changes in p, but the relationship between these two values is not (that is, whether one is greater than the other or they are equal).

In summary, this model provides a transparent way of determining the welfare implications of substituting an average VSL for a true VSL for low-income and high-income groups. We have assumed, in this particular analysis, that the distributional weights are true. However, the model also enables analyses comparing the case with distributional weights set to unity and an average VSL with distributional weights set to their true value with true VSLs.

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[17]In the example where $n_1=n_2=1$, the term for weighed marginal benefits is $\frac{w_1c+w_2d}{w_1p+w_2(1-p)}$ in the true case and $\frac{(w_1+w_2)(\frac{c+d}{2})}{w_1p+w_2(1-p)}$ in the average VSL case (see equation (7) which is the first order condition). Viewing the numerator of those two terms reveals they will be equal if and only if $w_1=w_2$. In this case (7) requires they have the same optimal level of the public good.

[18] We use the assumptions on the linear marginal benefit function and cost functions defined above.

3.1 Low-income group preferences and the average VSL

In this subsection, we raise the possibility that certain population groups may prefer policy calibrated to the population average VSL rather than their true VSL. We do so both because environmental policymakers often employ the average VSL in policy design and evaluation and because this exercise highlights the importance of economic cost incidence in determining welfare outcomes. In particular, we wish to show that there may be situations in which the low-group group prefers an average VSL to the true VSL. This situation can arise, for example, when a low-income group shoulders a sufficiently low fraction of the costs. In this case, that group may prefer an average VSL that results in a higher level of abatement to using a true VSL that results in a lower level of abatement. In contrast, with a sufficiently high cost share, the high-income group may prefer a lower level of abatement. We present this example below along with a graphical illustration.

We begin by defining the optimal level of the public good from group 1's point of view, (q^*). We then construct an example in which $q_t^* < q_a^* < q^*$. If this is the case, we will show that group 1 prefers q_a^* to q_t^*, that society prefers $q_t^* < q_a^*$, and that group two prefers q_t^* to q_a^*. For simplicity, assume the low-income group and the high-income group each consist of one agent so that $n_1 = n_2 = 1$.[19]

Group 1's maximization problem is:

$$\underset{q}{\text{Maximize}}\; w_1(B_1(q) - pC(q)).$$

Substituting the functional forms for the benefit and cost functions and differentiating yields $c - pbq = 0$ for the first order condition, which yields $q^* = \frac{c}{pb}$. From (10), we have $q_a^* = \frac{(w_1+w_2)[\frac{1}{2}(c+d)]}{w_1 p\, b + w_2(1-p)b}$. We wish to see if there is a $q_a^* < q^*$. From the formula for q^* and q_a^*, it follows that

$$q_a^* < q^* \text{ iff } \frac{(w_1+w_2)[\frac{1}{2}(c+d)]}{w_1 p + w_2(1-p)} < \frac{c}{p}. \tag{11}$$

The key insight is that for p "sufficiently" small in (11), the right-hand side goes to infinity, while the left-hand side remains finite. The numerator stays constant and the denominator tends to w_2 as $p \to 0$. This shows that for p sufficiently small, we have $q_a^* < q^*$. To establish $q_t^* < q_a^*$, we assume $w_1 > w_2$ and $n_1 = n_2 = 1$, and apply Proposition 2.[20]

[19] The example can easily be extended to an arbitrary number of members in each group, but we assume one member in each group to avoid notational clutter.

[20] More generally, for $q_t^* < q_a^*$ to hold, we require conditions related to both the weights and the population group sizes. First, we require that $w_1 > w_2$ which, if the weights reflect group-specific marginal utility of income, should hold for concave utility functions. Second, if $n_1 >= n_2$, then $q_t^* < q_a^*$. This is likely to hold writ large in societies with right-skewed income distributions. Finally, the only case in which $q_t^* > q_a^*$ occurs if $n_2/n_1 > w_1/w_2$.

The intuition for this result is straightforward. Think of the limiting case in which p is zero, so that group 1 does not pay for the public good. In this case, group 1 prefers an infinite amount of the public good. Thus, it is possible to move group 1's preferred level of the public good beyond an arbitrary q_a^*, for p sufficiently small. Stated another way, as the public good becomes cheaper from the standpoint of group 1, it wishes to have more of it.

We are now in a position to demonstrate the three claims noted above. First, under the conditions laid out above, group 1 prefers q_a^* to q_t^* because q_a^* is closer in distance to q^*, and group 1's objective function is a parabola that achieves its maximum at q^*. Second, society, or the social planner, prefers q_t^* to q_a^* for the same reason—that is, society's objective function is a parabola that achieves its maximum at q_t^* and $q_t^* < q_a^*$ by construction. Finally, at q_a^*, we know that group 1's welfare is higher than at q_t^*. Because total social welfare is lower at q_a^* than at q_t^*, this implies group 2's welfare is lower at q_a^* than at q_t^*. This demonstrates the third claim.

A graphical representation of this result is shown in Figure 1. Figure 1a shows the optimum values for the average and true levels of abatement when group 1 and group 2 are both included. The weighted marginal benefit function for the average VSL case is given by MB_a and the weighted marginal benefit function for the true VSL case is given by MB_t (with $MB_t < MB_a$). The weighted marginal cost curve is given by MC_w. The weights are given by the first order condition for a pure public good (Equation (6)) for the linear functions defined in this VSL example.[21]

The optimum for the true value case, q_t^*, occurs at the intersection of MB_t and MC_w; and the optimum for the average VSL value case, q_a^*, occurs at the intersection of MB_a and MC. Note that $q_t^* < q_a^*$. The social loss in moving from q_t^* to q_a^* is given by triangle A.[22]

Figure 1b shows why group 1 is better off with the optimum, q_a^*, which uses the average VSL, rather than the true optimum, q_t^*, which uses the true VSL The optimum value for group 1, q^* is defined by the intersection of its weighted marginal cost curve MC_1, and its weighted marginal benefit curve MB_1. Note that that $q_t^* < q_a^* < q^*$. The increase in weighted welfare for group 1 in moving from q_t^* to q_a^* is given by trapezoid B.[23]

This framework for comparing an average VSL with a true VSL (or willingness to pay) can be extended in a number of different ways. To illustrate, we consider three extensions. First, we consider the case of a pure public good in which the low-income group and the high-income group face different risk reductions per unit of abatement. This may stem

[21] The specific values for the objective function are $w_1 = 2$, $w_2 = .5$, $b = 1$, $c = 1$, $d = 2$, $n_1 = 1$, $n_2 = 1$, and $p = .15$.

[22] This social loss is measured in weighted dollars because we are using the distributional weights.

[23] Sunstein (2023) explores the case when a low-income group could be better off, but does not use a formal model. He argues that costs are important, which this model supports. He also argues that a subsidy would likely make this group better off. His subsidy appears to be an in-kind transfer used to provide more abatement. If the subsidy reduced the price of the public good for the low-income group by reducing p instead, then Proposition 2 would apply.

from differential baseline risks that interact with pollution exposure (Spiller et al., 2021). Suppose, for the sake of illustration, that the risk reduction per person for the low-income group from a unit reduction in abatement were higher than the risk reduction per person for the high-income group. In this case, we can no longer assume that the marginal benefit per person for a unit of abatement was higher for the high-income group than the low-income group (i.e., $d > c$). If, in fact, it were the case that $d < c$, then the signs in Proposition 2 would be reversed, but the machinery we have developed would still apply.

A second extension is to consider the case in which the transfer coefficients are different for the low-income group and the high-income group. This could arise if the two groups live in different neighborhoods. This is the case of local public goods. The analysis in this case is very similar to the first case presented in the preceding paragraph, except risk now varies by income class per unit of emission reduction because of the location of the groups rather than differences in health status. The same type of analysis would apply and would be dependent on whether the marginal benefit per person per unit of emission reduction would be higher for the low-income group than the high-income group.

Finally, consider a case in which we have two local public goods, one for the low-income group and one for the high-income group, both of which are provided separately by different firms (so the costs and benefits are separate). In this case we make the following two observations. First, if both groups are paying the full costs, each will always weakly prefer the true value of the VSL to the average value. The reason is that costs are fully internalized in these cases, so a group cannot do better, and may do worse, if it does not use its true benefit function (or VSL in this case). Second, if one group, say the low-income group, is not paying its full cost, then it may prefer an average VSL to a true VSL. The reasoning is similar to the reasoning provided in Proposition 2. As the price the group faces for the public good declines (e.g., measured in terms of its cost share), the group will want more of the public good to increase its welfare. Fundamentally, this is because the costs to that group are not fully internalized.

3.2 Modelling framework strengths and limitations

The modeling framework developed here is quite flexible, and allows for many formulations that allow operationalization of various concepts of equity. For example, an important issue for policy makers is how to address "environmental equity." A full treatment of this issue is beyond the scope of this paper, but we wish to make two points. Recent efforts by the federal government to address environmental justice considerations focus exclusively on benefits (The White House, 2021). However, our work demonstrates the importance of considering costs borne by the low-income group as well as benefits. First, as the previous example illustrates consideration of costs has fundamental implications for group welfare related to the use of an average or true VSL. Second, the model demonstrates that not considering cost could result in corner solutions where the optimum allocation including costs is not achieved. For example, without considering costs, the optimum quantity demanded by group one in the example would be infinite. Our joint inclusion of distributional weights *and* cost shares brings these two insights to light in a manner that facilitates operationalizing a broader definition of equity in economic analysis.

The model also has some limitations as well. We consider two here. The first relates to the fact that it is developed in a partial setting, and a second relates to the inclusion of uncertainty. One limitation of the model is that it relies on a partial equilibrium framework that does not consider tradeoffs with other goods and leisure. In principle, this could be added to our model, but would add complexity. A second limitation is that prices are assumed to be given, and do not change with the level of the public good that is provided. While a general equilibrium framing may be more realistic for some public goods problems, we decided to opt for model simplicity using a partial equilibrium setting that is applied in many real-world applications.

A second issue relates to the inclusion of uncertainty. The basic problem is that the decision maker may have very limited information over the key parameters in the model. Uncertainty in benefits and costs has been analyzed by several scholars (Weitzman, 1974), and we do not have much to add to that discussion because we are not considering different policy instruments. Uncertainty in other key parameters, such as the distributional weights and the cost allocation shares, could be quite important. Under plausible independence assumptions, one can show that uncertainty over the weights and the cost shares would still lead to using a framework in which expected net benefits are maximized. The basic insight in this case would be a familiar one. As uncertainty in parameter estimates increase, the value of obtaining better (or perfect) information would tend to increase.

4. Conclusion

This paper examines the use of distributional weights in economic analysis. We develop a formal model that allows us to derive some comparative statics and welfare results. We show how our model is a generalization of work by scholars on pure public goods and local public goods. We also show how the provision of a pure public good varies with the introduction of distributional weights different from unity.

Our model also sheds light on an important policy issue: the welfare impacts of using a single average VSL versus separate values that may differ by income. We show how to determine when using an average VSL will increase or decrease optimal public goods provision relative to the case of differentiated VSLs for given distributional weights. We also identify conditions under which a low-income group would prefer using an average VSL to true VSLs in which their mortality risks are valued less than the mortality risks faced by a high-income group.

Our analysis highlights the importance of considering costs as well as benefits in attempting to address equity concerns. It also underscores the need for rigorous empirical analysis to determine the incidence of costs when government make decisions about the provision of public goods.

References

Adler, Matthew D. 2016. "Benefit–cost analysis and distributional weights: an overview." *Review of Environmental Economics and Policy*. 10: 264-285.

Banzhaf, H. Spencer. 2023. "Distribution and Disputation: Net Benefits, Equity, and Public Decision Making." Available at SSRN: https://ssrn.com/abstract=4449328.

Banzhaf, H. Spencer. 2011. "Regulatory impact analysis of environmental justice effects." *J. Land Use & Envtl. L.* 27(1): 1-30.

Cecot, Caroline, and Robert W. Hahn. 2022 "Incorporating equity and justice concerns in regulation." *Regulation & Governance*. https://doi.org/10.1111/rego.12508.

Christiansen, Vidar. "Evaluation of public projects under optimal taxation." *The Review of Economic Studies* 48, no. 3 (1981): 447-457.

Dasgupta, Ajit K., and D. W. Pearce. 1972. *Cost-benefit analysis: Theory and practice.* New York: Harper and Row.

Dasgupta, Partha, Amartya Kumar Sen, and Stephen A. Marglin. 1972. *Guidelines for project evaluation.* United Nations. New York, NY, USA.

Finkelstein, Amy, and Nathaniel Hendren. 2020. "Welfare analysis meets causal inference." *Journal of Economic Perspectives* 34 (4): 146-167.

Harberger, Arnold C. 1978. "On the Use of Distributional Weights in Social Cost-Benefit Analysis." *Journal of Political Economy.* 86(2) Part 2: Research in Taxation: S87 – SS120.

Hemel, Daniel. 2022. "Regulation and Redistribution with Lives in the Balance." *U. Chi. L. Rev.* 89: 649-734.

Hendren, Nathaniel, and Ben Sprung-Keyser. 2020. "A unified welfare analysis of government policies." *The Quarterly Journal of Economics* 135(3): 1209-1318.

HM Treasury, The Green Book: Central Government Guidance on Appraisal and Evaluation (Stationery Office, 2003).

Hylland, Aanund, and Richard Zeckhauser. 1979. "Distributional Objectives Should Affect Taxes but Not Program Choice or Design, *Scandinavian Journal of Economics*, 81: 264-284.

Meade, J. E. 1955. *Trade and welfare: Mathematical supplement.* London: Oxford University Press.

Mendelsohn, Robert. 1986. "Regulating heterogeneous emissions." *Journal of Environmental Economics and Management* 13(4): 301-312.

Montgomery, W. David. 1972. "Markets in licenses and efficient pollution control programs." *Journal of Economic Theory* 5(3): 395-418.

Muller, Nicholas Z., and Robert Mendelsohn. 2009. "Efficient pollution regulation: getting the prices right." *American Economic Review* 99(5): 1714-1739.

Piketty, Thomas. 2022. *A brief history of equality*. Harvard University Press. Cambridge, MA, USA.

Robinson, Lisa A. 2007. "Policy Monitor: How US Government Agencies Value Mortality Risk Reductions." *Review of Environmental Economics and Policy.* 1(2): 283-299.

Samuelson, Paul A. 1954. "The pure theory of public expenditure." *The Review of Economics and Statistics.* 36(4): 387-389.

Spiller, Elisheba, Jeremy Proville, Ananya Roy, Nicholas Z. Muller. 2021 "Mortality Risk from $PM_{2.5}$: A Comparison of Modeling Approaches to Identify Disparities Across Ethnic Groups in Policy Outcomes." *Environmental Health Perspectives*. 129(12): 1 – 12.

Sunstein, Cass R. 2023. "Inequality and the Value of a Statistical Life." *Journal of Benefit-Cost Analysis* 14(1): 1-7.

United States Environmental Protection Agency. 1999. *The Benefits and Costs of the Clean Air Act: 1990--2010.* EPA Report to Congress. EPA 410-R-99-001, Office of Air and Radiation, Office of Policy, Washington, D.C.

United States Environmental Protection Agency. 2011. *The Benefits and Costs of the Clean Air Act: 1990--2020.* Final Report. Office of Air and Radiation, Office of Policy, Washington, D.C.

Weitzman, Martin L. 1974. "Prices vs. quantities." *The Review of Economic Studies.* 41(4): 477-491.

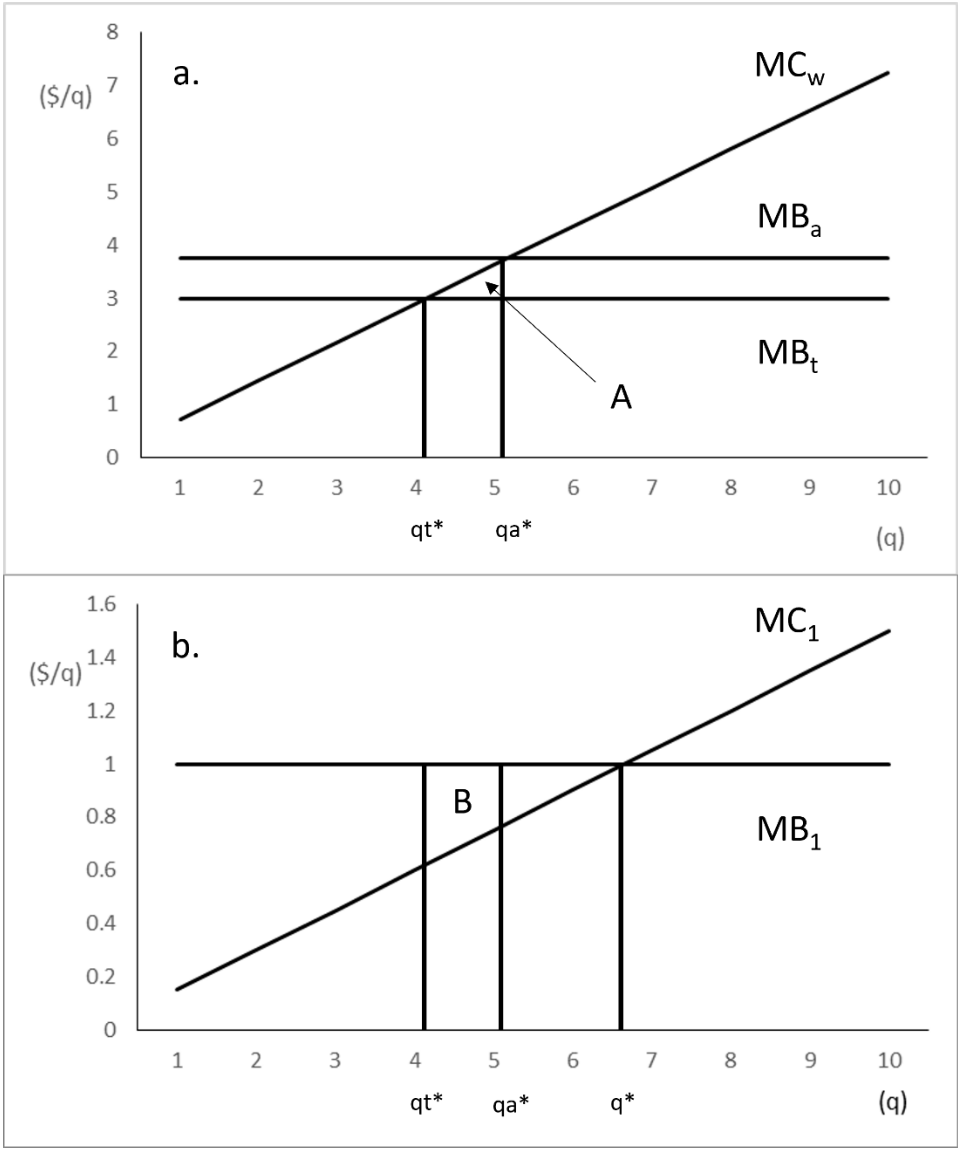
The White House. "Executive order on advancing racial equity and support for underserved communities through the federal government." (2021).

The White House. “Circular A-4.” (2023), April 6.

Viscusi, W. Kip. 2017. “Income Elasticities and Global Values of a Statistical Life.” *Journal of Benefit-Cost Analysis.* 8(2): 226-250.

Viscusi, W. Kip. 2018. "Pricing lives." In *Pricing Lives*. Princeton University Press, Princeton, NJ, USA.

Figure 1: Social welfare loss and low-income group gain with average VSL



Notes: MC_w = weighted marginal cost; MB_a = weighted marginal benefit using average VSL; MB_t = weighted marginal benefit using true VSL; MC_1 = low-income group marginal cost; MB_1 = low-income group marginal benefit; A = social welfare loss associated with q_a^* rather than q_t^*; and B = welfare gain to low-income group associated with q_a^* rather than q_t^*. q refers to quantity of the public good.

Appendix

This appendix contains the proofs of Proposition 1 and Proposition 2.

Proposition 1: Assume the industry marginal curve is upward sloping and/or the sum of marginal benefits curve for all groups is downward sloping.

Then:

$q_w^* \,(>)(=)(<) q^* \; iff \;\; \frac{\sum_{k=1}^{o} w_k B_k'(q_w^*)}{\sum_{l=1}^{o} w_l p_l} (>)(=)(<) \sum_{k=1}^{o} B_k'(q_w^*)$ (12)

Proof: Consider the case of equality. Assume that $q_w^* = q^*$. From equation (5), $\sum_{k=1}^{o} \; B_k'(q^*) = C'(q^*)$, and from equation (7), $C'(q_w^*) = \frac{\sum_{k=1}^{o} w_k B_k'(q_w^*)}{\sum_{l=1}^{o} w_l p_l}$. Because $q_w^* = q^*$, it follows that $\frac{\sum_{k=1}^{o} w_k B_k'(q_w^*)}{\sum_{l=1}^{o} w_l p_l} = \sum_{k=1}^{o} B_k'(q_w^*)$. Now, assume that $\frac{\sum_{k=1}^{o} w_k B_k'(q_w^*)}{\sum_{l=1}^{o} w_l p_l} = \sum_{k=1}^{o} B_k'(q_w^*)$. We wish to show that this implies $q_w^* = q^*$. q_w^* is optimal in the general case. But it is also optimal in the unit weight case because $C'(q_w^*) = \frac{\sum_{k=1}^{o} w_k B_k'(q_w^*)}{\sum_{l=1}^{o} w_l p_l} = \sum_{k=1}^{o} \; B_k'(q_w^*)$. Thus, $q_w^* = q^*$. This shows $q_w^* \; = q^*$ iff

$$\frac{\sum_{k=1}^{o} \; w_k B_k'(q_w^*)}{\sum_{l=1}^{o} w_l p_l} = \sum_{k=1}^{o} B_k'(q_w^*).$$

Consider the greater-than case, and assume that $q_w^* > q^*$. We wish to show that this implies that $\frac{\sum_{k=1}^{o} w_k B_k'(q_w^*)}{\sum_{l=1}^{o} w_l p_l} > \sum_{k=1}^{o} B_k'(q_w^*)$. We use a proof by contradiction. That is, assume that $\frac{\sum_{k=1}^{o} w_k B_k'(q_w^*)}{\sum_{l=1}^{o} w_l p_l} \leq \sum_{k=1}^{o} B_k'(q_w^*)$, and arrive at a contradiction. Consider the cases of equal-to and less-than separately. We know that if $\frac{\sum_{k=1}^{o} w_k B_k'(q_w^*)}{\sum_{l=1}^{o} w_l p_l} = \sum_{k=1}^{o} B_k'(q_w^*)$ then $q_w^* = q^*$, which violates the assumption that $q_w^* > q^*$. Now, consider the case where we assume $\frac{\sum_{k=1}^{o} w_k B_k'(q_w^*)}{\sum_{l=1}^{o} w_l p_l} < \sum_{k=1}^{o} B_k'(q_w^*)$. Given q_w^* is optimal for the general case, $C'(q_w^*) = \frac{\sum_{k=1}^{o} w_k B_k'(q_w^*)}{\sum_{l=1}^{o} w_l p_l}$. But this implies under this assumption that $C'(q_w^*) < \sum_{k=1}^{o} B_k'(q_w^*)$. This inequality implies that q must be increased from q_w^* to satisfy the unit weight case first order condition, which means that $q^* > q_w^*$. But this contradicts the assumption that $q_w^* > q^*$. This shows that $q_w^* > q^*$ implies $\frac{\sum_{k=1}^{o} w_k B_k'(q_w^*)}{\sum_{l=1}^{o} w_l p_l} > \sum_{k=1}^{o} B_k'(q_w^*)$.

Now, consider the greater-than case and assume that $\frac{\sum_{k=1}^{o} w_k B_k'(q_w^*)}{\sum_{l=1}^{o} w_l p_l} > \sum_{k=1}^{o} B_k'(q_w^*)$. We wish to show this implies $q_w^* > q^*$. We use a proof by contradiction. That is, assume that $q_w^* \leq q^*$ and arrive at a contradiction. Consider the cases of equal-to and less-than separately. We know that if $q_w^* = q^*$, $\frac{\sum_{k=1}^{o} w_k B_k'(q_w^*)}{\sum_{l=1}^{o} w_l p_l} = \sum_{k=1}^{o} B_k'(q_w^*)$ which violates the assumption that $\frac{\sum_{k=1}^{o} w_k B_k'(q_w^*)}{\sum_{l=1}^{o} w_l p_l} > \sum_{k=1}^{o} B_k'(q_w^*)$.

Consider the case where $q_w^* < q^*$. Given q_w^* is optimal for the general case, $C'(q_w^*) = \frac{\sum_{k=1}^{o} w_k B_k'(q_w^*)}{\sum_{l=1}^{o} w_l p_l}$ which implies, under the assumption that $q_w^* < q^*$, that $C'(q_w^*) < \sum_{k=1}^{o} B_k'(q_w^*)$,[24] which in turn implies that $\frac{\sum_{k=1}^{o} w_k B_k'(q_w^*)}{\sum_{l=1}^{o} w_l p_l} < \sum_{k=1}^{o} B_k'(q_w^*)$. This contradicts our assumption. This shows that $\frac{\sum_{k=1}^{o} w_k B_k'(q_w^*)}{\sum_{l=1}^{o} w_l p_l} > \sum_{k=1}^{o} B_k'(q_w^*)$ implies $q_w^* > q^*$. Taking the results on the less-than case together, we have shown that
$q_w^* > q^*$ iff $\frac{\sum_{k=1}^{o} w_k B_k'(q_w^*)}{\sum_{l=1}^{o} w_l p_l} > \sum_{k=1}^{o} B_k'(q_w^*)$.

The same proof holds, *mutatis mutandis*, for the less-than case.

Proposition 2: $q_t^*(<)(=)(>)q_a^*$ iff $w_1 n_1 (>)(=)(<) w_2 n_2$, with the assumption on the linear marginal benefit function and cost functions defined above.

Proof: Because the denominator is positive and equal, we form the difference between the numerators in equation (9) and equation (10). This difference is $(<)(=)(>)0$ iff $(q_t^* - q_a^*)\ (<)(=)(>)0$.

This yields: $(q_t^* - q_a^*)\ (<)(=)(>)0$

iff

$$[(w_1 n_1 c + w_2 n_2 d) - (w_1 n_1 + w_2 n_2)(\frac{c+d}{2})]\ (<)(=)(>)0$$

iff

$$(w_1 n_1 c + w_2 n_2 d) - (w_1 n_1 + w_2 n_2)(\frac{c+d}{2})\ (<)(=)(>)0$$

iff

$$w_1 n_1 (c - (\frac{c+d}{2})) + w_2 n_2 (d - (\frac{c+d}{2}))\ (<)(=)(>)0$$

iff

$$.5[w_1 n_1 (c - d) + w_2 n_2 (d - c)]\ (<)(=)(>)0$$

iff

[24] This follows directly from the assumption that the marginal benefit function is non-decreasing and the marginal cost function is increasing.

$w_1 n_1(c - d) + w_2 n_2(d - c)\ (<)(=)(>) 0$

Letting $k = d - c$, and substituting yields

iff
$k\ (w_2 n_2 - w_1 n_1)\ (<)(=)(>)\ 0$

iff
$w_2 n_2 - w_1 n_1\ (<)(=)(>)\ 0$

iff $w_1 n_1 (>)(=)(<) w_2 n_2$

Each of the if-and-only-if steps represent standard algebraic operations. This proves Proposition 2.