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AN EQUILIBRIUM THEORY OF EXCESS VOLATILITY AND MEAN REVERSION IN STOCK MARKET PRICES

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ABSTRACT

Apparent mean reversion and excess volatility in stock market prices can be reconciled with the Efficient Market Hypothesis by specifying investor preferences that give rise to the demand for portfolio insurance. Therefore, several supposed macro anomalies can be shown to be consistent with a rational market in a simple and parsimonious model of the economy. Unlike other models that have derived equilibrium mean reversion in prices, the model in this paper does not require that the production side of the economy exhibit mean reversion. It also predicts that mean reversion and excess volatility will differ substantially across subperiods.

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1. Introduction

The two biggest challenges to the Efficient Market Hypothesis in recent years have been findings that stock market prices are excessively volatile compared to dividends [Shiller (1981)], and that aggregate stock price indices exhibit mean reversion [Fama and French (1988), Poterba and Summers (1988)]. Most researchers of these phenomena are careful to note that their findings are not necessarily inconsistent with efficient markets, and in particular, that they can be reconciled with the EMH via time-varying interest rates or risk premia. Nevertheless, it appears difficult at first glance to imagine credible models of the economy that would generate equilibrium behavior consistent with these phenomena, and the literature by and large leaves the impression that the most economical explanation of these "macro anomalies" lies in systematic overreaction of security prices to exogenous shocks. Indeed, the now common terminology "fads model" and "excess" volatility reveals the tentative inference drawn from these studies.

More recently, equilibrium models consistent with apparent mean reversion in stock market prices have been advanced by Cecchetti, Lam, and Mark (1988) and Brock and LeBaron (1989). Both of these papers use variants of Lucas' (1978) model, and both exploit consumption smoothing motives to generate stationarity in stock price distributions. In Cecchetti et al., the real sector is modeled by positing two states (boom and bust) for the macro economy with Markov transition probabilities. In low dividend periods, individuals desire to sell assets to maintain consumption levels. In aggregate, however, net demand cannot be negative. Instead, asset prices fall and expected returns rise. Thus, the driving force behind mean reversion in prices is the desire for consumption smoothing in the presence of stochastic, but (because of the two-state assumption) essentially mean-reverting, shocks to dividend growth. In Brock and LeBaron, a similar effect is achieved by considering i.i.d. shocks to an otherwise fixed production function. While they focus on liquidity constraints at the level of the firm, they also show that the tendency for high consumption to revert back to typical consumption will cause mean reversion in prices at an aggregate level. In both papers, there is a well-defined notion of "good times" and "bad times," and the economy tends on average toward normal times.

This paper also is an attempt to reconcile mean reversion and excess volatility with market rationality. Unlike the previous literature, however, the model focuses on risk aversion per se, and is unconcerned with consumption smoothing. Indeed, even if the real economy as measured by dividends or earnings follows a pure random walk, so that agents do not foresee changes in output, stock prices in this model still will seem to exhibit both mean reversion and excess volatility. This result therefore extends and complements the results in the Lucas-based models, and shows that even a very simple one-period model can generate meaningful mean reversion and excess volatility.

In contrast to the Lucas-based models, the model in this paper relies on the particular specification of utility functions to generate interesting results. However, not much structure is necessary to obtain these results. In particular, I show that a sufficient condition for both mean reversion and excess volatility is that the representative agent in the economy would be a demander of portfolio insurance if the risk-free rate and market price of risk were constant. This condition is straightforward and is consistent with the demand for portfolio insurance evident in the marketplace. The model used here is related to Black's (1989) model in that both rely on an inverse relationship between the market price of risk and asset prices to generate mean reversion. Black, however, posits this relationship a priori, and explores its consequences. Here, the derivation of the relationship is the central focus of the paper.

Section 2 of this paper lays out a formal model of the economy and shows how mean reversion and apparent excess volatility can arise in a rational market. Section 3 explores the potential magnitude of these effects. Section 4 concludes.

2. Equilibrium Macro Anomalies

Because the demand for portfolio insurance will play a central role in the analysis to follow, I will specify a utility function intimately related to such a demand. Consider, therefore, the family of utility functions of end-of-period wealth of the form

$$U(W) = \frac{1}{1-\gamma} (W-W_{min})^{1-\gamma} \tag{1}$$

where W_{min} is a floor on wealth that might correspond to a subsistence value and is the natural level at which wealth would be insured. The function in equation (1) is a member of the HARA family of utility functions. Perold (1986) shows that such a derived utility function is consistent with a more rigorously defined intertemporal utility of consumption function, U(C), where

$$U(C) = (C-C_{min})^{1-\gamma}$$

and C_{min} is subsistence consumption. Similarly, Constantinides (1988) derives a similar, though more complex, version of (1) where "subsistence" consumption is determined by habit formation. [See his equation (11).]

It is easy to show that the relative risk aversion, A, of an investor with utility function (1) is

$$A = \gamma \frac{W}{W-W_{min}} \qquad (2)$$

As W becomes infinite, preferences asymptote to the familiar constant relative risk aversion (CRRA) formulation. As W approaches W_{min}, however, agents become absolutely risk-averse.

Merton (1971) demonstrates that in an economy with one riskless asset paying r_f and one risky asset (the market) with expected return r_M and variance σ_M^2, the optimal allocation, x, to the risky asset will be

$$x = \frac{r_M - r_f}{A\sigma_M^2} \qquad (3)$$

Therefore, the dollar demand by the representative agent for holdings of the risky asset is

$$xW = \frac{r_M - r_f}{\gamma\sigma_M^2}(W - W_{min}) \qquad (4)$$

Demand is formally identical to the CRRA case except that wealth is replaced by the surplus of wealth over W_{min}.

Call the value of the risky asset P, and the net supply of the risk-free asset F, so that W = P + F. The model allows F = 0. Then the market clearing condition for the risky asset is obtained by equating asset demand from (4) to the value of the risky asset.

$$\frac{r_M - r_f}{\gamma\sigma_M^2} (P + F - W_{min}) = P \qquad (5)$$

Now consider an exogenous stock to aggregate profitability which lowers the value of P. For fixed values of r_M and σ_M^2, the fall in the left-hand side of (5) will exceed that of the right-hand side if and only if $W_{min} > F$, which certainly would be the case if risk-free borrowing were an inside asset.[1] When $W_{min} > F$, the exogenous reduction in the value of holdings of the risky asset leads to an even larger reduction in demand for the risky asset. After the shock, there will be a desire to sell off shares to restore portfolio balance. But this response constitutes a generalized version of portfolio insurance, whereby investors follow a rule that shifts the portfolio from risky to riskless assets as the risky asset falls in value (Perold and Sharpe, 1988).

Of course, there cannot be a net demand for portfolio insurance in the face of fixed market parameters, since not everyone in equilibrium can simultaneously desire to buy or sell shares. Leland (1980) takes the investment opportunity set as given and examines the conditions determining which heterogeneous investors will be suppliers or demanders of portfolio insurance. Here, we impose more homogeneity on preferences consistent with a more rigorously defined intertemporal utility of consumption function, U(C), where

$$U(C) = (C-C_{min})^{1-\gamma}$$

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Demand is formally identical to the CRRA case except that wealth is and allow the market parameter r_M to adjust to maintain equilibrium. As agents sell off shares in response to the exogenous shock, prices fall further and r_M increases. The market clearing condition, equation (5), is restored when r_M increases by enough to equate quantity demanded to the market value of the risky asset.[2]

Therefore, the same condition that gives rise to a demand for portfolio insurance, $W_{min} > F$, also will result in both excess volatility and mean reversion in the price of the risky asset. Consider first excess volatility. The shock to corporate profitability, or for concreteness dividends, lowers prices directly through the present value relationship. Then there is a secondary, or "multiplier effect" as decreased demand for shares lowers prices even further. As a result, P will be more volatile than dividends. Now consider mean reversion. The equilibrium expected market return will increase after the market price falls, thereby leading to the appearance of mean reversion in prices. We turn to the potential magnitude of these effects in the next section.

As a digression, however, note that it is easy to stretch the model a bit to rationalize the Lo and MacKinley (1987) result that at short horizons, the market exhibits positive serial correlation. If portfolio adjustments occur with a lag after the market is shocked, then the price effect of the change in equilibrium r_M, which reinforces the exogenous shock to prices, will be spread out over a brief period, leading to positive short-term serial correlation in aggregate market returns. Over longer horizons, however, after the market equilibrates to the new expected return, serial correlation in returns will appear negative.

3. Numerical Solutions

To quantify the potential for mean reversion and excess volatility, we need to specify the stock valuation process. One of the simplest specifications is that dividends, D, follow a lognormal random walk with trend. Suppose, therefore, that

$$D_t = D_{t-1} \exp(u + g - \sigma^2/2)$$

where u is normally distributed as $N(0, \sigma^2)$ and g is the trend growth rate of dividends. Given this specification, the value of the risky asset is simply

$$P = \frac{D}{r_M - g} \tag{6}$$

Recall that as D is shocked, r_M will change as well, leading to a secondary impact on P. The elasticity of P with respect to D is, from equation (6)

$$\frac{dP/P}{dD/D} = 1 - P \frac{dr_M}{dD} \tag{7}$$

Equation (7) is the excess volatility relationship. Because $dr_M/dD < 0$, the proportional change in the market price when D is shocked will exceed the proportional change in dividends, and indeed, in a single factor model with shocks only to D, the relative volatilities will be

$$\sigma_M^2 = \sigma_D^2 \left(1 - P \frac{dr_M}{dD}\right)^2 \tag{8}$$

where σ_M^2 is the variance of the market price, P.

Equations (5), (6), and (8) allow for numerical solution of the market equilibrium. For chosen values of W_{min}, D, γ, σ_D^2, g, and F, it is possible to solve these equations for σ_M^2, P, and r_M.

The solution algorithm proceeds as follows. At a given value of D and an initial guess for σ_M^2, calculate from (5) and (6) the equilibrium rate r_M, and corresponding value of P. Calculate r_M for a slightly higher value of D and evaluate dr_M/dD. Use (8) to update the guess for σ_M^2. Using this new guess, repeat the process. Iterate until the the guess for σ_M^2 and the resultant values of P and dr_M/dD are consistent with equation (8).

Figure 1 graphs the equilibrium expected market return as a function of the current dividend levelfor parameters as follows: W_{min}=100; F=90; r_f=.025; g=.02; σ_D^2=.03. At a dividend level of 12, for example, the market rate is about .095, so that the market price is 12/(.095-.02) = 160, meaning that floor wealth, 100, is about 40% of total wealth (90+160). As D becomes large, the equilibrium market return asymptotes to

$$r_M = \gamma\sigma_D^2 + r_f \tag{9}$$

which is .08 using the chosen parameters.

Equation (9) is the standard solution for market equilibrium in a CRRA economy with no inside asset, and no floor on subsistence wealth. As D and, correspondingly, P become large, both floor wealth and the value of the riskless asset become relatively trivial in comparison to W, while risk aversion asymptotes to γ. Therefore, (9) becomes the solution to (3) with A = γ, x = 1, and $\sigma_M^2 = \sigma_D^2$. At the other extreme, as D falls, the equilibrium value of r_M becomes unbounded at a positive value of D, slightly below 6.8 in Figure 1. Values of D less than the asymptote value do not allow for market equilibrium. At these points the economy is so poor and correspondingly risk averse that no promised return can induce enough demand to absorb the supply of risky shares. Figures 2 and 3 illustrate the problem. In Figure 2, as r_M increases, stock demand initially increases (a substitution effect), but ultimately must fall because increases in r_M reduce P, and thereby overall wealth, which eventually dominates the substitution effect. Indeed, by the time r_M is high enough to drive P down to $W_{min} - F$, demand will fall to zero. Note that while there are two intersections of the demand and supply curves, only the equilibrium on the left is stable. If D is too low given the values of γ and σ_D^2, as in Figure 3, there will be no intersection between the demand for and the value of the risky asset.

Returning to Figure 1, it is apparent that equilibrium "mean reversion" can vary considerably across subperiods. When wealth is high relative to W_{min}, the equilibrium expected return is nearly constant, and the aggregate market should obey a random walk if the driving variable D is a random walk. In periods that include severe recessions or depressions, however, wealth-induced changes in risk aversion will be correspondingly severe and can lead to equilibrium returns that vary substantially and inversely with the level of stock prices. This implication is consistent with the finding of Kim, Nelson, and Startz (1988) that mean reversion in the postwar period is not significantly different from zero, but is significant in periods that include the Great Depression.

Figure 4 presents the ratio of σ_M^2 to σ_D^2 as a function of D. The relationship is similar to that observed for equilibrium r_M. At low wealth levels, small changes in wealth result in relatively large changes in r_M and, consequently, to a larger multiplier effect on stock prices. Hence, the "excess volatility" ratio is greater in this region. At higher wealth levels, the volatility ratio asymptotes to 1.0. Again, the excess volatility ratio can be arbitrarily high, and should be expected to vary across subperiods.

4. Conclusion

Apparent mean reversion and excess volatility in stock market prices are consistent with a rational market equilibrium in an economy that would be characterized by a net demand for portfolio insurance if the market price of risk were fixed. The model developed here therefore shows that the Efficient Market Hypothesis is broadly consistent with a range of recent "macro anomalies." Moreover, the model sheds some light on the different degrees of mean reversion measured in different subperiods, as well as on the shorter-term positive serial correlation in stock market prices.

One obvious question is whether an equilibrium model like the one in this paper can be empirically tested against a fads model. An empirical implication of this model is that periods of high mean reversion and highe excess volatility ought to coincide and ought to occur following severe market declines. A model of market irrationality that holds that fads are essentially overreactions to exogenous shocks which the market eventually corrects also would predict a coincidence of mean reversion and excess volatility, but not necessarily following bear markets. If, however, market mispricing is independent of fundamentals, perhaps deriving from noise traders for example, then it is possible that episodes of excess volatility and mean reversion would occur independently, leading to another difference in the empirical implications of the two models.

Footnotes

1. If W_{min} exceeds F then the market equilibrium [equation (5)] requires that $(r_M - r_f)/\gamma\sigma_M^2$ exceeds 1, implying in turn that the LHS has a greater sensitivity to P than the RHS.

2. It is possible that an equilibrium might not exist. As r_M rises, demand as a <u>fraction</u> of wealth increases, but wealth falls because the price of the risky asset will fall as its discount rate increases. It is possible that there is no market clearing value of r_M. This issue is explored more fully in the next section.

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Figure 1

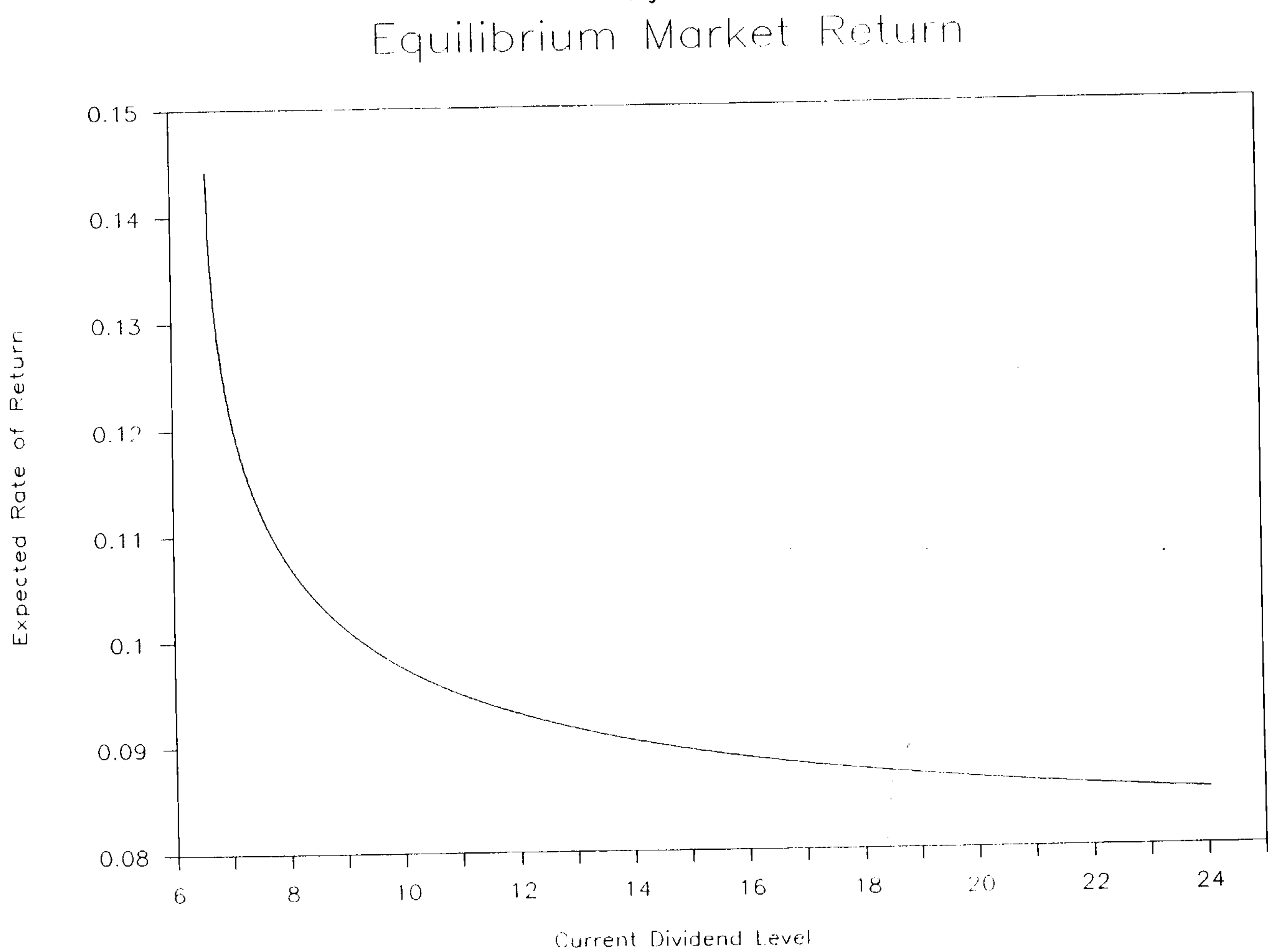

Equilibrium Market Return
Expected Rate of Return
0.15
0.14
0.13
0.12
0.11
0.1
0.09
0.08
6
8
10
12
14
16
18
20
22
24
Current Dividend Level

Figure 2

Market Equilibrium

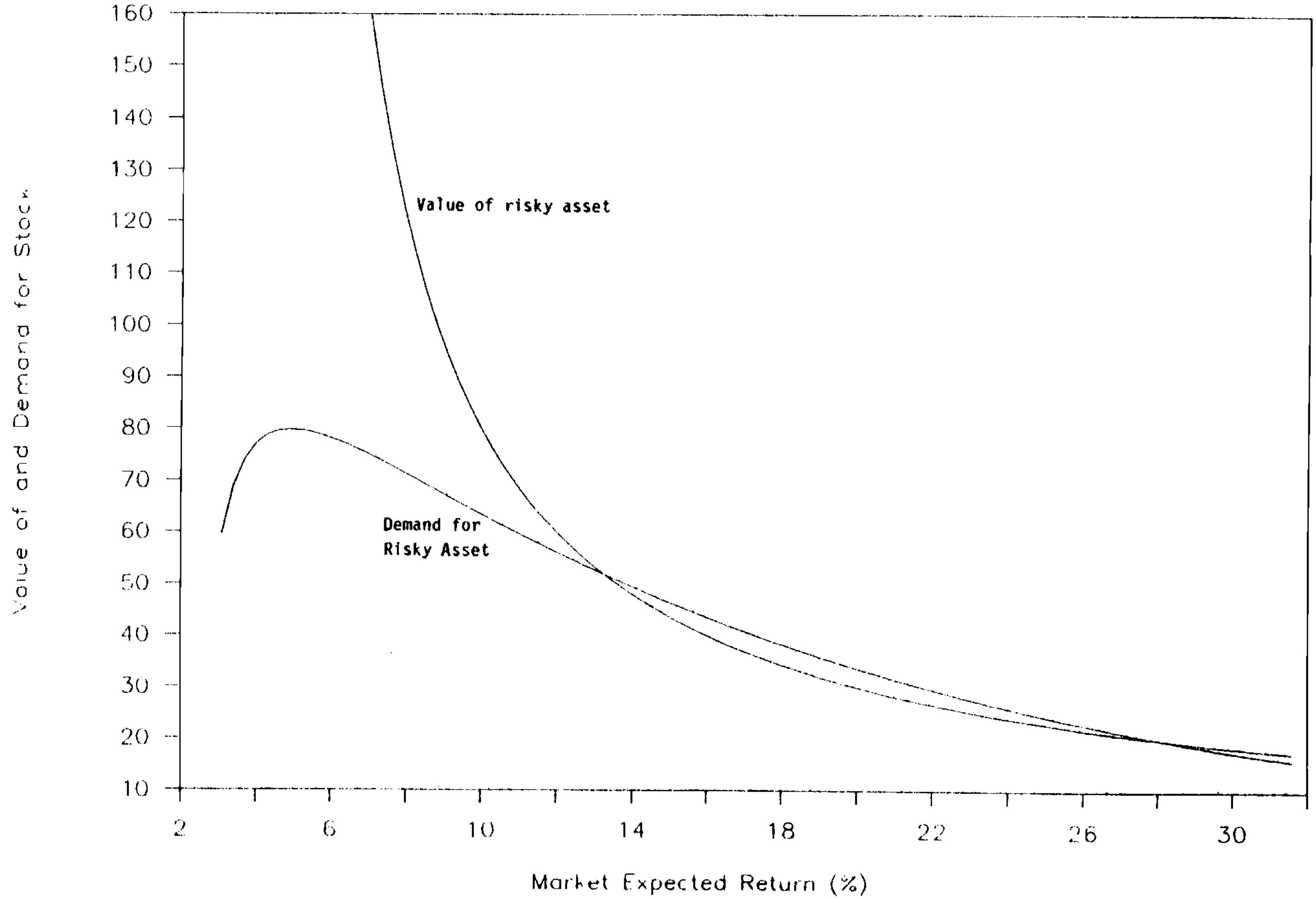
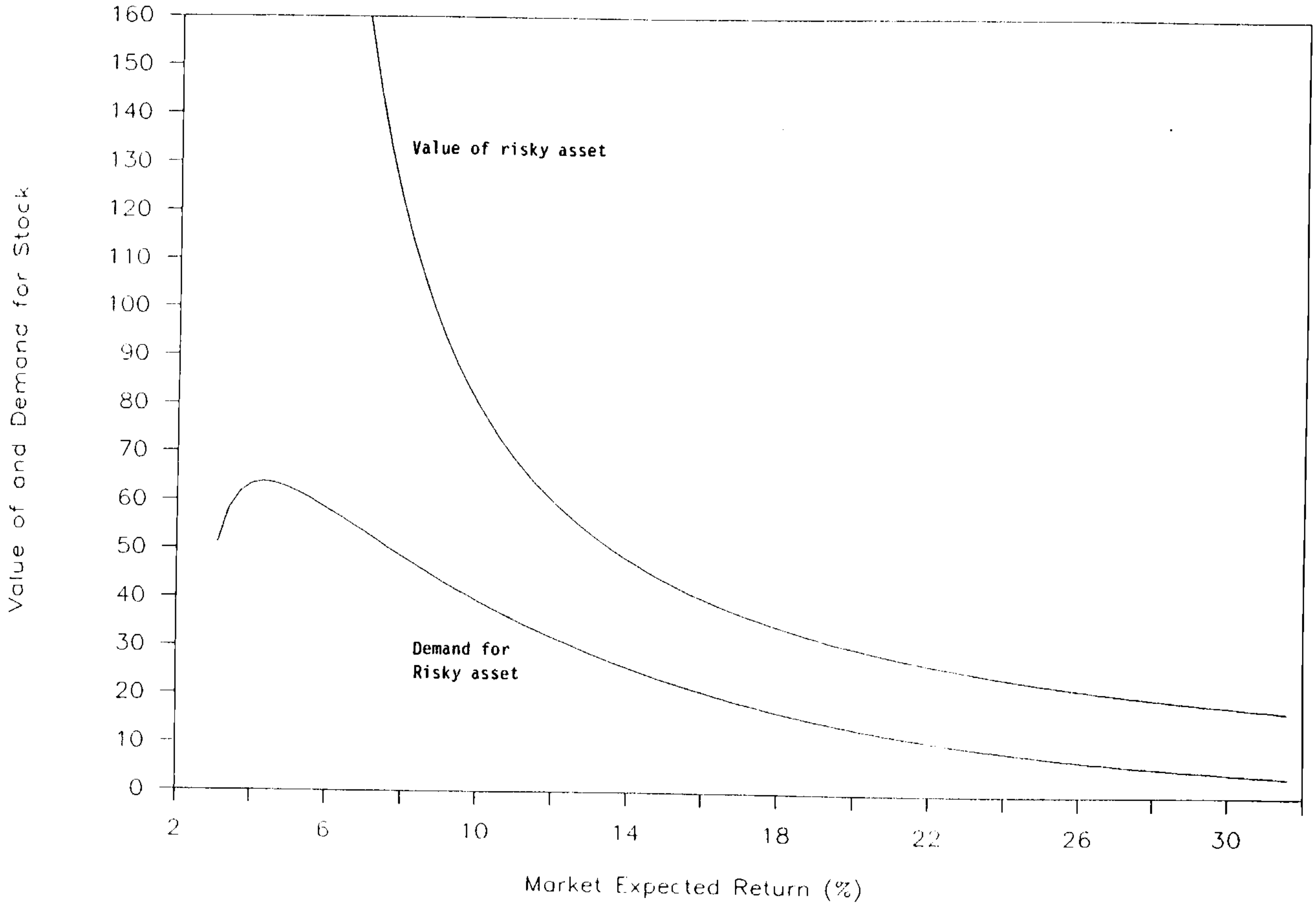
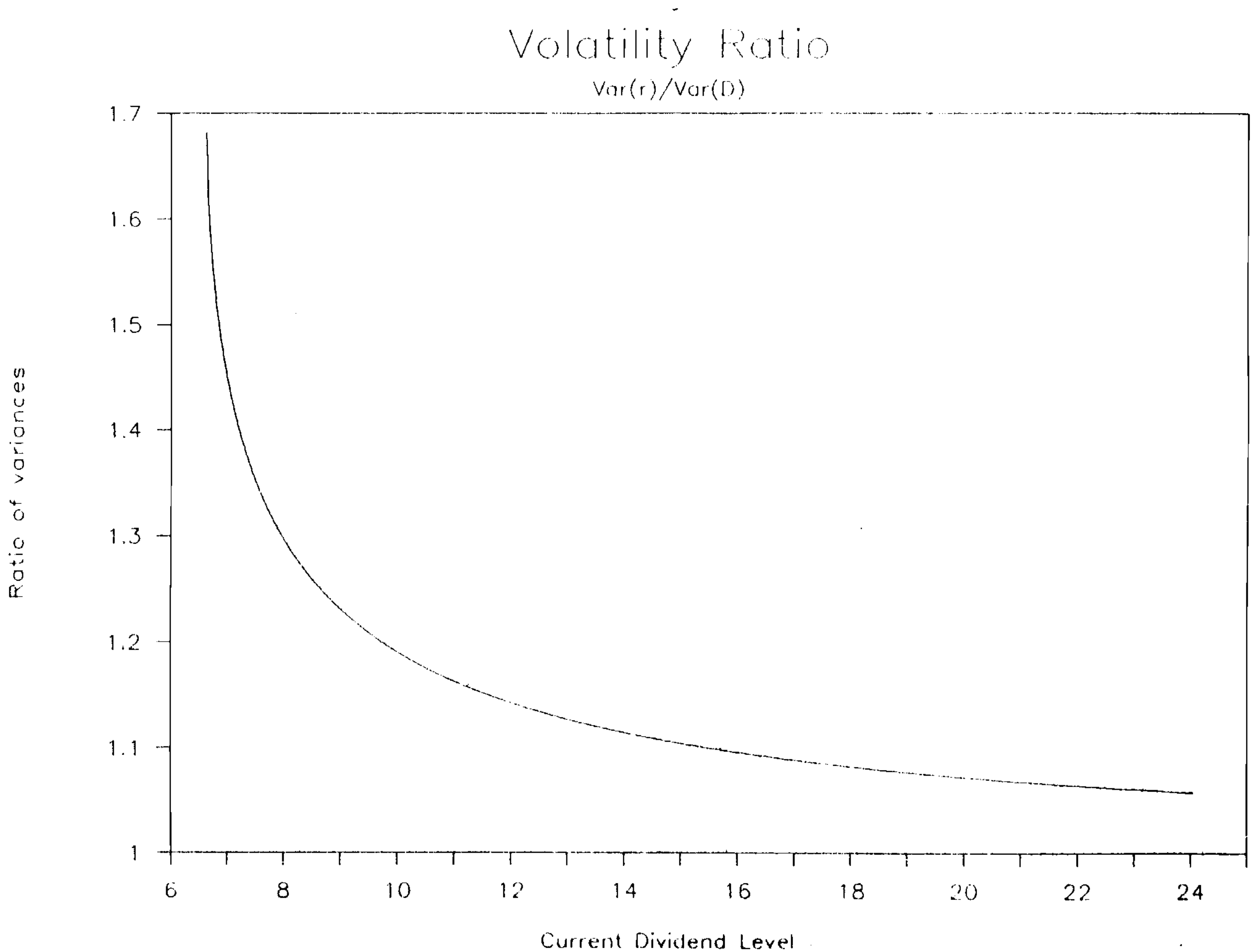


Figure 3

Market Equilibrium




Volatility Ratio
Var(r)/Var(D)
Ratio of variances
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1
6
8
10
12
14
16
18
20
22
24
Current Dividend Level