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MEASURING WELFARE BY MATCHING HOUSEHOLDS ACROSS TIME

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Measuring Welfare by Matching Households Across Time
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ABSTRACT

The money-metric utility function is an essential tool for calculating welfare-relevant growth and inflation. We show how to recover it from repeated cross-sectional data without making parametric assumptions about preferences. We do this by solving the following recursive problem. Given compensated demand, we construct money-metric utility by integration. Given money-metric utility, we construct compensated demand by matching households over time whose money-metric utility value is the same. We illustrate our method using household consumption survey data from the United Kingdom from 1974 to 2017 and find that real consumption calculated using official aggregate inflation statistics overstates money-metric utility for the poorest households by around half a percent per year and understates it by around a quarter of a percentage point per year for the richest households. We extend our method to allow for missing or mismeasured prices, assuming preferences are separable between goods with well-measured prices and the rest. We discuss how our results change if the price of some service sectors is mismeasured.

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1 Introduction

Money-metric utility functions convert incomes under different price systems into equivalent income under a common baseline price system, so that budget sets can be compared to one another. In other words, money-metric utility functions cardinalize preferences and have interpretable units. For this reason, they are the standard tool to measure growth and inflation in a theory-consistent way. Money metrics can be calculated by deflating nominal income using a weighted average of changes in prices, where the weights are compensated (or Hicksian) budget shares (see e.g. Hausman, 1981).

Since compensated budget shares are not directly observable, standard price deflators use uncompensated (or Marshallian) budget shares instead. This shortcut leads to the correct answer if preferences are homothetic, but fails when preferences are non-homothetic. This is because when preferences are non-homothetic, compensated and uncompensated budget shares are different, and using one in place of the other produces incorrect results.

In this paper, we show how to recover compensated budget shares, and money-metric utility, without estimating demand or making parametric assumptions about preferences. To do this, consider repeated cross-sections of households with identical preferences facing common prices.¹ To construct the money-metric utility function, in t_0 dollars, for a household with income I at time t , we must know the compensated demand of this household for every $s \in [t_0, t]$. This is revealed at each point in time s by the budget shares of another household with a different income level I' who is on the same indifference curve as the household with income I at t .

If we can find such households, then we can calculate the money-metric utility function by integration. That is, if we know how to match households over time, we can recover money-metric utility. Conversely, if we know money-metric utility, then we can match households through time, since households are on the same indifference curves if, and only if, their money-metric utility values coincide. The insight is that this is a fixed point problem in terms of observables that can be solved.

Our methodology endogenously identifies the set of households for which a money-metric value can be calculated reliably, without out-of-sample extrapolation. That is, our approach does not necessarily recover the money metric for all households in the sample because suitable matches may not exist. For example, if there is positive growth over

¹Even though we assume that households have common preferences that are unchanged over time (we relax this assumption in Section 3.4), our matching approach is based on revealed preference theory and is not based on interpersonal comparisons of utility. That is, we match households by their behavior (their compensated demand curves are the same) and do not need to postulate that two households are “equally well-off” if their utilities are the same.

time, then the richest household at any point in time is on an indifference curve that no other household was on in the previous periods. This means that for such a household, we cannot calculate compensated demand in the past and hence the money metric, unless we are prepared to extrapolate Engel curves out-of-sample.

Our method generalizes the standard practice of statistical agencies who weigh changes in prices over time using aggregate budget shares. Conventional price deflators like the CPI or the PCE recover money-metric utility under the assumptions of homothetic and stable preferences. However, when preferences are non-homothetic, we show that one must use the budget shares of a unique corresponding income level in the past for each income today instead of aggregate budget shares.²

Our paper also provides a contrast to the popular but ad hoc approach of constructing price indices by household-income group using the budget shares of some fixed percentile of the income distribution in each period. This method lacks a theoretical foundation if percentiles of the income distribution do not remain on the same indifference curve over time, and the shape of the indifference curve varies as a function of income.³

Our approach differs from alternatives that calculate compensated demand based on estimated elasticities of substitution, as it does not require the estimation of non-parametric elasticities of substitution.⁴ Intuitively, our method only recovers compensated demand evaluated at observed prices, whereas the elasticities of substitution determine how compensated demand will react to any price change, even those that have not been observed. As a result, our procedure can measure changes in welfare for observed changes in prices and income but is not suited for addressing counterfactual welfare questions, such as those explored by Baqaee and Burstein (2021).

The paper is organized as follows. In Section 2, we define money-metric utility and

²Chained-weighted indices measured by statistical agencies are generally uninterpretable when preferences are non-homothetic. However, under additional assumptions, chained indices do have meaningful interpretations. For example, Feenstra and Reinsdorf (2000) show that when the path of prices is linear in time, chained indices measure the cost-of-living price index for some intermediate utility level under AIDS preferences. Caves et al. (1982) establish a similar result for Tornqvist price indices, up to a second-order approximation. But these are not money metrics. In this paper, we focus on the money-metric utility function.

³National statistical agencies sometimes produce inflation statistics like this. For example, the UK's Office of National Statistics produces inflation indices by household expenditure groups (see <https://www.ons.gov.uk/economy/inflationandpriceindices/articles/inflationandthecostoflivingforhouseholdgroups/october2022>).

⁴In this respect, our approach resembles Oulton (2012), who demonstrates how to back out compensated budget shares by adjusting uncompensated budget shares using a Taylor series in income. He applies this methodology using the QAIDS demand system to estimate the cost-of-living index without needing to estimate price elasticities. Instead of relying on a Taylor series under a parametric functional form for demand, our approach purges income effects from substitution effects by matching households over time who are on the same indifference curve but face different prices.

its dual, the cost-of-living index, and explain their relationship to compensated demand. In Section 3, we demonstrate how to recover the cost-of-living index and money-metric utility given cross-sectional data when all prices are fully observed over time. We present two solution strategies, both of which exactly recover the money metric as long as the data is continuous in both the time series and the cross-section.

We apply our methodology to artificial data generated using a popular functional form for non-homothetic preferences and demonstrate that our procedure converges to the truth quickly as the number of households and the temporal frequency of observations increase. The errors are relatively small when our numerical examples are calibrated to match real-world data in terms of the frequency of observation and the rate at which prices and incomes change over time.

In Section 3, we also discuss how our results change when there is preference heterogeneity in either the time series or the cross-section. For instance, when sufficient data is available, we explain how to handle idiosyncratic taste shocks that are unrelated to income. Similarly, we describe how our method can be adapted to account for heterogeneity in preferences that depend on observable characteristics. Importantly, even if there are taste shocks, we demonstrate that our approach approximately recovers the true money metric as long as taste shocks are small and uncorrelated with price changes.

In Section 4, we illustrate our method by applying it to household expenditure survey data from the United Kingdom spanning from 1974 to 2017. We find that real consumption calculated by deflating income with aggregate chain-weighted inflation (as measured by official statistical agencies) overstates the money-metric utility for all households below the 60th percentile of the spending distribution in 2017 in our sample. In other words, for expenditures below the 60th percentile, the 1974 equivalent income is less than real consumption. The size of this gap is greatest for the poorest households, roughly 20 percentage points (0.5 percentage points per year on average), and gradually diminishes until it reaches zero for households close to the 60th percentile.

Conversely, real consumption calculated using aggregate inflation statistics understates the money-metric utility for households above the 60th percentile. For households in the 97th percentile of our sample, who spend around £81,000 per year, the size of this gap is 13 percentage points over the whole sample (0.25 percentage points per year on average).⁵ We are unable to compute the money metric for the richest households in 2017 (97th percentile and above). The reason is that for these households, there did not exist consumers in the past whose money metric utilities are high enough and whose observed

⁵These results are consistent with Blundell et al. (2003), which report a relatively greater rise on the cost of living for poorer households between 1975 and 1984 in the UK.

demand can be used in place of the compensated budget shares.

Whereas real consumption calculated using the aggregate inflation rate has large errors relative to our true estimated money metric, a decile-specific chained deflator produces smaller errors in our UK dataset. Of course, one needs to compute the true money metric first, before knowing whether or not the ad hoc approach is a good approximation. Furthermore, computing quantile-specific chained deflators requires more information than our method.

In Section 5, we extend our methodology to allow for missing prices. To do this, we require the restriction that the expenditure function be separable between observed and unobserved prices. Under this additional assumption, we show that money-metric utility can be recovered provided knowledge of the compensated elasticity of substitution between observed and unobserved goods. This generalizes the influential Feenstra (1994) approach to imputing missing prices beyond the homothetic CES case.

In particular, we show how to back out the change in the relative price of observed and unobserved goods using changes in the compensated budget share of the observed goods. For example, if the compensated budget share on observed goods is rising, and observed goods are net complements with unobserved goods, then this indicates that the relative price of unobserved goods is falling. This can then be used to calculate money-metric utility. Importantly, we also show that the elasticity of substitution between observed and unobserved goods, which is required to infer missing prices, can be identified without knowledge of those missing prices.

We provide an empirical illustration of this extension in Section 6. We assume that preferences are indirectly separable between goods and services, estimate elasticities of substitution between goods and services, and apply our methodology assuming that the prices for services are mismeasured. We find that the price of the compensated bundle of services has been rising faster than official data for rich but not for poor households. This implies that the money metric is overstated for rich but not poor households. We conclude in Section 7.

Related Literature. Our paper is closely related to Blundell et al. (2003) and Jaravel and Lashkari (2022), both of which develop non-parametric approaches to measuring welfare for non-homothetic preferences using cross-sectional household-level data. Although inspired by them, our approach, which integrates compensated demand curves by matching households, is distinct from theirs. We discuss these papers in turn.

Blundell et al. (2003) bound the money metric by using revealed choice arguments. For each income level at time t , Blundell et al. (2003) construct a bundle that is strictly

better and a bundle that is strictly worse in time $s \neq t$. The price of these two bundles then bound the true money-metric value. We exposit and implement an amended version of their methodology in Appendix D. We amend their algorithm for the lower-bound, since the version in their paper appears to have an error. Our approach has an advantage over Blundell et al. (2003) in that it provides a point estimate, rather than only bounds, for the money-metric utility. On the other hand, our approach requires the data to be smooth and observed continuously, whereas Blundell et al. (2003) need neither assumption. We show in Appendix D that our point estimates are always within their bounds for both artificial and real-world data.

Jaravel and Lashkari (2022) use a correction term to address non-homotheticity in household-level chain-weighted indices. Whereas our approach endogenously delineates a set of households for whom money-metric utility can be calculated, without relying on out-of-sample extrapolation, the Jaravel and Lashkari (2022) method aims to uncover the money metric for all households observed at any point in time. That is, unlike our methodology, their approach does not provide a boundary on the set of households whose money-metric values can be reliably computed. In Appendix E, we apply the Jaravel and Lashkari (2022) method to artificial examples and demonstrate that their algorithm can lead to large errors if the support of the cross-sectional distribution of utilities is not constant over time.

In contrast to both Jaravel and Lashkari (2022) and Blundell et al. (2003), we also extend our methodology to situations where some prices and expenditures are unobserved. Since our method can be extended to allow for unmeasured prices, our paper is also related to the literature that measures welfare allowing for incomplete information about prices. Most papers with non-homothetic preferences follow the approach of Costa (2001) and Hamilton (2001). These papers take advantage of horizontal shifts in Engel curves to identify money metric utility changes. The frontier in this literature is Atkin et al. (2020), who show how to identify welfare changes assuming that preferences are quasi-separable between the measured and unmeasured goods.

Our paper, instead, generalizes the Feenstra (1994) method beyond the homothetic CES case. One advantage of our approach is that we do not need to make strong parametric assumptions within the set of observed prices. This is in contrast to Atkin et al. (2020) who need to fully model the demand system for the subset of goods with observed prices. This advantage of our approach comes at the cost that we require a stronger form of separability between the observed and unobserved prices than Atkin et al. (2020). We discuss these issues in more detail in Section 5.

Our approach can also be contrasted with more parametric approaches where wel-

fare measures are computed using a fully-specified demand system (e.g. Deaton and Muellbauer 1980). Specific functional forms for non-homothetic preferences are used to understand phenomena as diverse as structural transformation (e.g. Boppart 2014, Comin et al. 2021, and Fan et al. 2022), international trade patterns (e.g. Matsuyama 2000, and Fajgelbaum et al. 2011), and savings behavior and inequality (e.g. Straub 2019). Our approach provides a non-parametric way to compute welfare measures from the data without relying on low-dimensional functional forms.

2 Money Metrics and the Cost of Living

We start by defining the objects of interest: money-metric utility and the closely related cost-of-living function. Consider a rational preference relation \succeq defined over consumption bundles c in \mathbb{R}^N . Suppose that these preferences can be represented by a utility function $\mathcal{U}(c)$ that maps consumption bundles to utility values. Given this utility function, we can define the *indirect utility function*

$$v(\mathbf{p}, I) = \max_c \{\mathcal{U}(c) : \mathbf{p} \cdot c \leq I\},$$

mapping a vector of prices \mathbf{p} and expenditures I to utility values. We interchangeably refer to I as income, but in the data, we measure I using expenditures. Define the *expenditure function* to be

$$e(\mathbf{p}, U) = \min_c \{\mathbf{p} \cdot c : \mathcal{U}(c) \geq U\}.$$

We assume that the expenditure function is absolutely continuous in all its arguments.

The expenditure and indirect utility functions are used to define money metrics and cost-of-living indices.

Definition 1 (Money Metric and Cost of Living). For a fixed reference vector of prices $\bar{\mathbf{p}}$, the *money-metric* function maps budget sets defined by (\mathbf{p}, I) to

$$e(\bar{\mathbf{p}}, v(\mathbf{p}, I)).$$

For a fixed reference budget set defined by $(\bar{\mathbf{p}}, \bar{I})$, the *cost-of-living index* maps prices, \mathbf{p} , to

$$e(\mathbf{p}, v(\bar{\mathbf{p}}, \bar{I})).$$

The money-metric function, $e(\bar{\mathbf{p}}, v(\cdot))$, converts the value of different budget sets (\mathbf{p}, I) into equivalent dollars under some baseline prices. It is itself an indirect utility func-

tion because a budget set (\mathbf{p}, I) is preferred to another budget set (\mathbf{p}', I') if, and only if, $e(\bar{\mathbf{p}}, v(\mathbf{p}, I)) > e(\bar{\mathbf{p}}, v(\mathbf{p}', I'))$. The cost-of-living function, $e(\cdot, v(\bar{\mathbf{p}}, \bar{I}))$, converts the value of some baseline budget constraint $(\bar{\mathbf{p}}, \bar{I})$ into equivalent income under different sets of prices.⁶

To summarize, the function $e(\mathbf{p}', v(\mathbf{p}, I))$, mapping $(\mathbf{p}', \mathbf{p}, I)$ into a scalar, is an object of paramount interest. The “money metric” is the cross-section of this function that holds \mathbf{p}' constant and the cost-of-living index is the cross-section that holds (\mathbf{p}, I) constant. The money metric is useful for ranking budget sets (i.e. measuring growth).⁷ The cost-of-living index is useful for converting a common utility level, attained by $v(\mathbf{p}, I)$, into equivalent income under different price systems (i.e. measuring the cost of maintaining a fixed standard of living).

Denote the compensated budget share for good i by $b_i(\mathbf{p}, U)$ where \mathbf{p} is a vector of prices and U is a utility level. The following lemma, which is a corollary of Lemma 1 from Baqaee and Burstein (2021) and follows from Shephard’s lemma, provides a characterization of both the cost-of-living index and the money metric using compensated budget shares.

Lemma 1 (Money Metric and Cost of Living). *The money metric of a budget set (\mathbf{p}, I) in terms of $\bar{\mathbf{p}}$ prices can be expressed as*

$$\log e(\bar{\mathbf{p}}, v(\mathbf{p}, I)) = \log I - \int_C \sum_{i \in N} b_i(\xi, v(\mathbf{p}, I)) d \log \xi_i, \quad (1)$$

where C is any smooth path connecting $\bar{\mathbf{p}}$ to \mathbf{p} .⁸ The cost of living for a budget set $(\bar{\mathbf{p}}, \bar{I})$ in terms of \mathbf{p} prices can be expressed as

$$\log e(\mathbf{p}, v(\bar{\mathbf{p}}, \bar{I})) = \log \bar{I} + \int_C \sum_{i \in N} b_i(\xi, v(\bar{\mathbf{p}}, \bar{I})) d \log \xi_i. \quad (2)$$

According to Lemma 1, both the money metric and the cost-of-living index can be expressed as integrals of compensated budget shares with respect to changes in prices. However, compensated demand curves are not directly observable, so operationalizing this result requires having a way to identify compensated budget shares. This is what we focus on in the next section.

⁶In index number theory, the cost-of-living index is also called the Konüs (1939) index.

⁷The equivalent and compensating variation are related to the money metric and the cost-of-living index. Specifically, to measure the change in welfare from some initial budget set (\mathbf{p}, I) to some other budget set (\mathbf{p}', I') , the equivalent variation is $e(\mathbf{p}, v(\mathbf{p}', I')) - I$ and the compensating variation is $I' - e(\mathbf{p}', v(\mathbf{p}, I))$.

⁸Formally, the path integral in (1) is defined by $\int_{t_0}^{t_1} \sum_{i \in N} b_i(\xi_t, v(\mathbf{p}, I)) \frac{d \log \xi_{it}}{dt} dt$ where $\{\xi_t : t \in [t_0, t_1]\}$ parameterizes the path C from $\bar{\mathbf{p}}$ and \mathbf{p} as a function of a scalar t . The integrals in (1) and (2) are both path independent and only depend on the end points.

3 Recovering the Money Metric by Matching Households

In this section, we discuss how Lemma 1 can be deployed to recover money-metric utility functions and cost-of-living indices if one has access to repeated cross-sectional data of consumers with common and stable preferences who all face common prices at each point in time but have different incomes. We start this section by introducing our main theoretical result. We then provide two solution methods, and test them with artificial discrete data to assess their accuracy. We end the section by discussing how our results are affected by taste shocks and mismeasurement.

3.1 Theoretical Result

Suppose we observe an absolutely continuous path of prices $\mathbf{p}_t \in \mathbb{R}^N$ at each point in time $t \in [t_0, T]$.⁹ We also observe vectors of budget shares $\mathbf{B}(I, t) \in \mathbb{R}^N$ for consumers with preferences \succeq and income levels $I \in [I_t, \bar{I}_t]$ at time t . Our aim is to recover the money-metric utility function using reference prices \mathbf{p}_{t_0} evaluated at budget set (\mathbf{p}_t, I) , which we denote by $u(I, t) \equiv e(\mathbf{p}_{t_0}, v(\mathbf{p}_t, I))$.

The function $u(I, t)$ converts the value of the budget constraint defined by prices \mathbf{p}_t and income I into income under base prices \mathbf{p}_{t_0} . By varying base prices, for fixed (\mathbf{p}_t, I) , we can also recover the cost-of-living index. Once we are equipped with $u(I, t)$, it is also straightforward to compute the money metric for other base prices.¹⁰

Denote the uncompensated budget share of good i by B_i^M (the superscript M stands for Marshallian). For every good i ,

$$B_i^M(\mathbf{p}_t, I) = B_i(I, t)$$

whenever $t \in [t_0, T]$ and $I \in [I_t, \bar{I}_t]$. For any cardinalization of the indirect utility function and its associated compensated demand curves, the following identity between compensated and uncompensated budget shares also holds:

$$b_i(\mathbf{p}_t, v(\mathbf{p}_t, I)) = B_i^M(\mathbf{p}_t, I).$$

⁹Prices of sectoral aggregates tend to be smooth over time. This is the level of aggregation typically considered in work that documents non-homotheticities (including our application in Section 4). If the underlying prices within these aggregates contain jumps (due to, e.g. temporary sales or entry and exit of goods) then this needs to be taken into account when constructing these sectoral price aggregates. Doing so is beyond the scope of this paper.

¹⁰Suppose we wish to obtain $\tilde{u}(I, t) = e(\mathbf{p}_s, v(\mathbf{p}_t, I))$ for some $s \in [t_0, T]$. The solution is $\tilde{u}(I, t) = I'$ where I' satisfies $u(I', s) = u(I, t)$. By construction, $v(\mathbf{p}_t, I) = v(\mathbf{p}_s, I')$, hence $\tilde{u}(I, t) = e(\mathbf{p}_s, v(\mathbf{p}_t, I)) = e(\mathbf{p}_s, v(\mathbf{p}_s, I')) = I'$.

Using the money-metric cardinalization of indirect utility, and slightly abusing notation, we can combine the previous two identities to obtain:¹¹

$$b_i(\mathbf{p}_t, u(I, t)) = B_i(I, t).$$

Using this identity, Lemma 1 can be rewritten as the following recursive integral equation.

Proposition 1 (Money metric as Solution to Integral Equation). *For $t \in [t_0, T]$, the money metric $u(I, t) \equiv e(\mathbf{p}_{t_0}, v(\mathbf{p}_t, I))$ is a fixed point of the following integral equation*

$$\log u(I, t) = \log I - \int_{t_0}^t \sum_i B_i(u^{-1}(u(I, t), s), s) \frac{d \log p_{is}}{ds} ds \quad (3)$$

with boundary condition $u(I, t_0) = I$. Here, $u^{-1}(\cdot, s)$ is the inverse of u with respect to its first argument (income) given its second argument (time) is equal to s . That is, $u^{-1}(u(I, t), s)$ is a level of nominal income I^* in s such that $u(I^*, s) = u(I, t)$.

Since the money metric exists, the integral equation (3) has a solution. Proposition A.1 in Appendix A shows that the solution to this integral equation is unique because the operator defined by (3) is a contraction mapping.

Proposition 1 follows immediately from Lemma 1 once we recognize that in the integral equation above, $B_i(u^{-1}(\cdot, s), s) : \mathbb{R}_+ \rightarrow [0, 1]$ maps utility values to the budget share of good i at time s . That is, it is the compensated budget share of i .

To better understand (3), observe the simplification that occurs when preferences are homothetic. In this case, budget shares do not depend on income levels, only on time. Therefore, when preferences are homothetic, (3) simplifies to

$$\log u(I, t) = \log I - \int_{t_0}^t \sum_i B_i(s) \frac{d \log p_{is}}{ds} ds, \quad (4)$$

which eliminates the need to find a fixed point. This equation, called a Divisia (1926) index, justifies the standard chain-weighting practices adopted in the national accounts for calculating price and quantity indices.

If we can solve (3), then we can compute the compensated budget shares $b(\mathbf{p}_s, \bar{u})$ for a utility level \bar{u} at time t under prices \mathbf{p}_s at time s by using the budget shares of a different household on the same indifference curve at time s . That is, we “match” households with

¹¹Our “abuse of notation” is that we do not index compensated budget shares by the utility cardinalization. This is to simplify notation, since we are primarily interested in compensated budget shares only under the money-metric cardinalization.

income I^* at time s to households with income I at time t if $u(I^*, s) = u(I, t)$. The budget shares of this “matched” household, $B(I^*, s)$, are equal to the compensated budget shares $b(p_s, \bar{u})$.

Proposition 1 provides a way to recover the money metric and cost-of-living functions without needing direct knowledge of the potentially very high-dimensional demand system $B_i^M(p_t, I)$. Recall that the number of cross-price elasticities scales in the square of the number goods, and generically depends on both income and relative prices. Proposition 1 obviates the need to undertake this onerous estimation exercise by using the demand from other households and time periods in place of a counterfactual model of compensated demand.

The integral equation in Proposition 1 may initially appear abstract, but its underlying intuition is quite simple. In the next section, we clarify its intuition and provide step-by-step methods for solving it in practice.

3.2 Two Solution Methods

The money metric is a fixed point of (3), which is a system of nonlinear equations, albeit an infinite-dimensional one. We provide two solution methods. The first is a simple iterative procedure that converges to the desired solution as we approach the continuous-time limit. The second is a recursive solution that is equivalent to the iterative one in the continuous time limit but has better properties when the data is discrete.

Iterative Solution. For some interval of time $[t_0, T]$, suppose we have data on a grid of points $\{t_0, \dots, t_M\}$ where $t_n < t_{n+1}$, with $t_M = T$. Use the following iterative procedure for each $n \in \{1, \dots, M\}$ starting with $n = 1$:

$$\log u(I, t_n) \approx \log I - \sum_{m=0}^{n-1} B(I_m^*, t_m) \cdot \Delta \log p_{t_m}, \quad (5)$$

where I_m^* satisfies

$$u(I_m^*, t_m) = u(I, t_{n-1}), \quad (6)$$

with the boundary condition $u(I, t_0) = I$. If we cannot find I_m^* satisfying (6) for all $m \leq n - 1$, then $u(I, t_n)$ cannot be calculated for that value of I (without out-of-sample extrapolation).

These equations converge to the exact solution $u(I, t)$ as we approach the continuous

time limit since (3) has a unique solution and the limit is unique. To see this, note that the summation in (5) approximates the integral in (3) using a Riemann sum and becomes exact in the continuous-time limit because the Riemann sum becomes an integral and $u(I, t_{n-1})$, in (6), converges to $u(I, t_n)$.¹²

This procedure endogenously delineates those values of (I, t) for which $u(I, t)$ can be computed, and it does not require an assumption of full constant support over time on either the set of observed incomes or unobserved utilities. Furthermore, if data is continuous (in both time and income), then the result is an exact solution to the money metric that requires no estimation or interpolation.

The iterative procedure that we describe is useful for building intuition. However, one can also find a fixed point by solving the system of equations directly. This gives a recursive variation on the iterative procedure described above. The two approaches are equivalent in the continuous time limit.

Recursive Solution. Apply the iterative solution in (5) and (6) and call the resulting money metric $u_0(I, t)$. For each $i \geq 1$, and each $n \in \{1, \dots, M\}$, starting with $n = 1$, define

$$\log u_{i+1}(I, t_n) \approx \log I - \sum_{m=0}^{n-1} B(I_m^*, t_m) \cdot \Delta \log p_{t_m}, \quad (7)$$

where I_m^* satisfies

$$u_{i+1}(I_m^*, t_m) = u_i(I, t_n). \quad (8)$$

If we cannot find I_m^* satisfying (8) for all $m \leq n - 1$, then $u(I, t_n)$ cannot be calculated for that value of I (without out-of-sample extrapolation). Continue until $u_{i+1}(I, t) = u_i(I, t)$ for all feasible values of I and t . Then set $u(I, t) = u_i(I, t)$.

Once the recursive solution converges, it solves a fixed point problem. The difference between the iterative and recursive solution is that we replace $u(I, t_{n-1})$ on the right hand side of equation (6) with $u(I, t_n)$ in (8). Proposition A.1 in Appendix A shows that the continuous time version of this recursive procedure is a contraction mapping and must

¹²In practice, we do not observe total expenditures continuously in $[L_t, \bar{I}_t]$. We only observe I over a discrete set of grid points. Therefore, we use some numerical refinements. First, we use interpolation to deduce $B(I, t)$ and $u(I, t)$ where needed (we do not extrapolate). Second, we always normalize the interpolated budget shares to ensure they add up to one for every income level and time period. Finally, because time is discrete, to approximate any integrals, we use the trapezoid rule rather than the left-Riemann sum. For example, we use $(B(I_m^*, t_m) + B(I_{m+1}^*, t_{m+1}))/2$ in place of $B(I_m^*, t_m)$ in (5). These refinements also apply to the recursive solution method.

necessarily converge to the unique solution (which is the money metric).

Using artificial examples with discrete data, we show that the recursive solution has smaller errors than the iterative solution. However, both methods work well. For our empirical results using UK data, in Section 4, the results are almost unchanged between the iterative and recursive methods. Since the iterative procedure is simpler and faster to compute, we only show results for the iterative method for our empirical results.

To give more intuition, it helps to explicitly spell out the first few steps of the iterative procedure. Start with the boundary condition $u(I, t_0) = I$ since t_0 -equivalent income at t_0 is just initial income. Abusing notation, let $b_i(u, t)$ be the compensated budget share of good i at prices p_t for utility value u . For period t_1 , compute

$$\log u(I, t_1) \approx \log I - \mathbf{b}(u(I, t_0), t_0) \cdot \Delta \log p_{t_0} = \log I - \mathbf{B}(I, t_0) \cdot \Delta \log p_{t_0}$$

where the last equation uses the boundary condition, which implies $\mathbf{b}(u(I, t_0), t_0) = \mathbf{B}(I, t_0)$. For values of I outside of $[L_0, \bar{I}_0]$, we cannot compute $u(I, t_1)$.¹³

With $u(I, t_1)$ in hand, construct compensated budget shares for period t_1 :

$$\mathbf{b}(u(I, t_1), t_1) = \mathbf{B}(I, t_1).$$

That is, to each budget share $B_i(I, t_1)$, assign a utility value based on $u(I, t_1)$. Intuitively, we know budget shares as a function of income at t_1 , and we know utility as a function of income at t_1 . Since utility is monotone in income, this means that we can associate with each $B_i(I, t_1)$ a utility value, which is precisely the compensated budget share. We now have compensated budget shares $\mathbf{b}(u, t_0)$ and $\mathbf{b}(u, t_1)$.

Next, calculate

$$\log u(I, t_2) \approx \log I - \mathbf{b}(u(I, t_1), t_1) \cdot \Delta \log p_{t_1} - \mathbf{b}(u(I, t_1), t_0) \cdot \Delta \log p_{t_0},$$

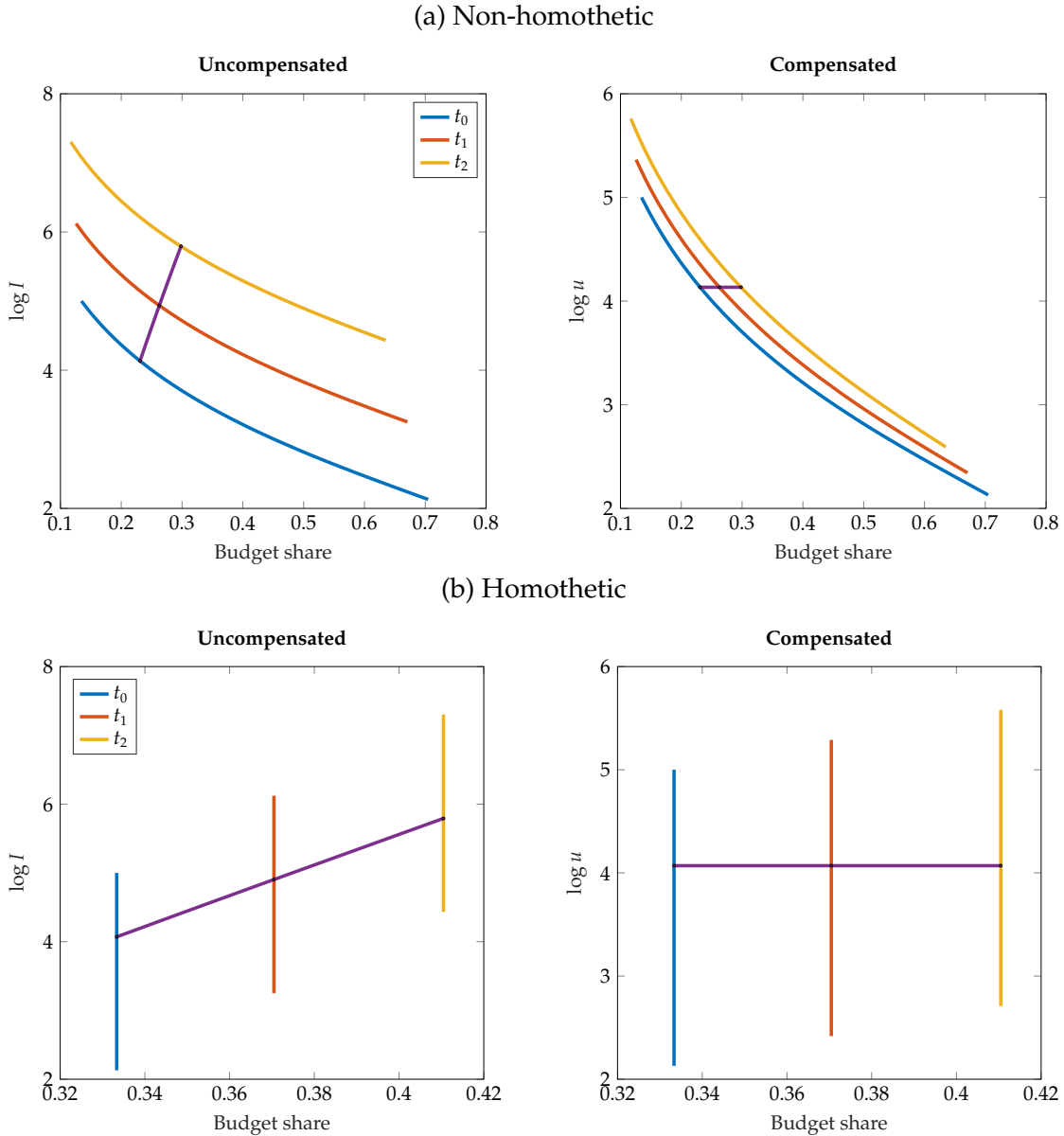
and using $u(I, t_2)$, construct compensated budget shares for period t_2 :

$$\mathbf{b}(u(I, t_2), t_2) = \mathbf{B}(I, t_2).$$

That is, for each budget share $B_i(I, t_2)$ in t_2 , assign a utility value based on $u(I, t_2)$. Note that we can only calculate $u(I, t_2)$ for those I 's for which $I_{t_1}^* = u^{-1}(u(I, t_2), t_1)$ and $I_{t_0}^* = u^{-1}(u(I, t_2), t_0)$ are observed. Continue this iterative process until t_M .

¹³This requirement is not very binding if the support of the income distribution is wide or if it moves slowly from period to period (the latter condition is satisfied if the data is smooth and the interval between each period is relatively short).

Figure 1: Budget share for some good against nominal income and money-metric utility in different periods.



To see this procedure graphically, consider the left panel of Figure 1a showing the budget share on some good against nominal income for three different points in time. The fact that the lines are downward sloping means that higher incomes are associated with lower budget shares on the good. In this example, incomes grow over time, so the range of nominal income levels shifts up over time.

In the data we observe budget shares as a function of income over time (uncompensated budget shares), but to construct the money metric we require budget shares as a

function of utility (compensated budget shares). The right panel of Figure 1a displays the compensated budget shares for the same good. The purple line in the right panel of Figure 1a shows for each period the compensated budget share for the good evaluated at some fixed utility level \bar{u} . The change in budget shares, holding utility constant, are pure substitution effects over time due to changes in relative prices. As implied by Lemma 1, multiplying the compensated budget shares by log price changes and summing over time gives the money-metric utility for the household with utility \bar{u} at time t_2 .

But, we cannot directly observe the figure on the right. How do we infer compensated budget shares? The purple line in the left panel of Figure 1a plots, for each period s , the income that gives the utility of \bar{u} , that is $u^{-1}(\bar{u}, s)$, and the associated budget share for the good, $B_i(u^{-1}(\bar{u}, s), s)$. In other words, we can infer compensated budget shares for \bar{u} by using the observed budget share along the purple line in the left panel. Then we can construct the mapping between income and utility at each point (the purple line) by iteratively applying the summation in (5).

To understand why Proposition 1 is unnecessary when preferences are homothetic, Figure 1b plots the same information as Figure 1a but for homothetic preferences. Since there are no income effects, budget shares at a point in time do not vary with household income or utility. That is, uncompensated and compensated budget shares coincide. Therefore, we can construct the money metric using a price index based on uncompensated budget shares by good.

3.3 Example with Artificial Discrete Data.

To illustrate how our method fares when faced with discrete, rather than continuous, data we consider a simple artificial example. Suppose the expenditure function is nonhomothetic CES¹⁴

$$e(\mathbf{p}, U) = \left(\sum_i \omega_i (U^{\varepsilon_i} p_i)^{1-\gamma} \right)^{\frac{1}{1-\gamma}}. \quad (9)$$

The money-metric function for t_0 reference prices is

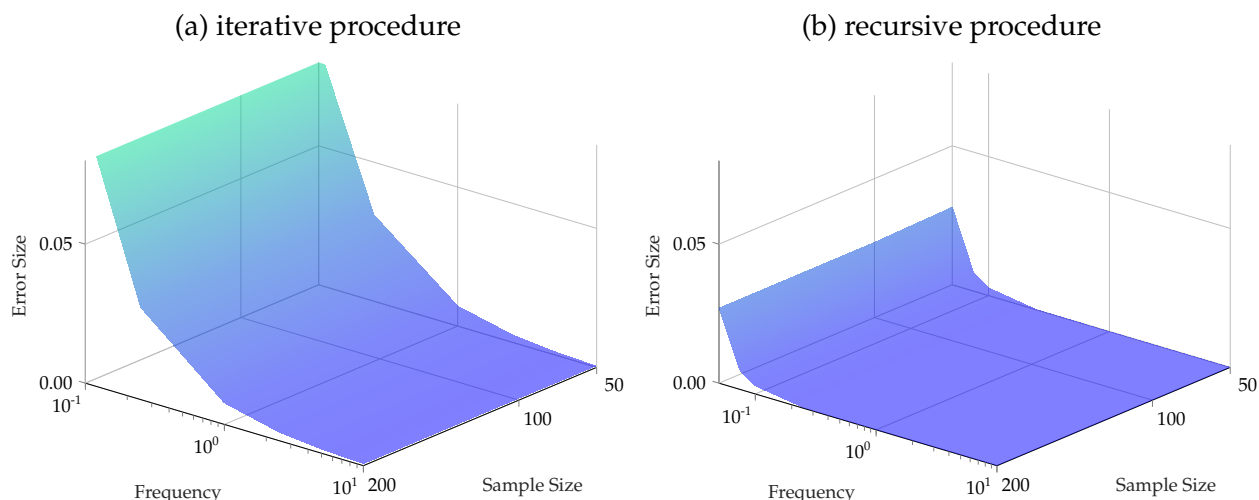
$$u(I, t) = \left(\sum_i \omega_i (V^{\varepsilon_i} p_{i,t_0})^{1-\gamma} \right)^{\frac{1}{1-\gamma}},$$

¹⁴See Hanoch (1975), Comin et al. (2021), and Matsuyama (2019) for more information on these preferences.

where V is the solution to $I = e(p_t, V)$.¹⁵ We evaluate the accuracy of our algorithm by comparing this exact expression for $u(I, T)$ with the results of our numerical procedure applied to artificial data generated using these preferences.

For illustration, we set $\gamma = 0.25$, $\varepsilon_1 = 0.2$, $\varepsilon_2 = 1$, $\varepsilon_3 = 1.65$, which are values taken from Comin et al. (2021). We generate repeated cross-sectional data on income and budget shares over 3 goods for a finite number of households facing a common price vector over forty years. The distribution of income in the first period is lognormal (parameterized to match the distribution of household expenditures in the 1974 UK household survey, described in the next section). The share parameters are calibrated so that the budget share of each good for the median household in the first period are uniform. All incomes and prices grow exponentially, at different rates, over the sample period. Figure A.1 in Appendix B plots the paths of prices and incomes in our numerical example.

Figure 2: Maximum error as function of frequency of observation and sample size



Notes: Throughout, we hold the path of price and income changes constant. Our baseline calibration is annual frequency corresponding to a value of $10^0 = 1$ observations per year on the x -axis. If we observe the data once every decade, then the frequency is $1/10$, and if we observe the data every month, then the frequency is 12. The left panel uses the iterative and the right panel the recursive solution method in Section 3.2.

We apply both the iterative and recursive solution methods. We use linear interpolation to evaluate budget shares for I between two observed income levels. To assess the accuracy of our procedure, we use the infinity norm — that is, the maximum absolute value of the log difference between the true money-metric function and our estimate in the final period.

¹⁵As shown in Baqaee and Burstein (2021), $u(I, t)$ can be expressed in terms of observable budget shares and elasticities as $u(I, t) = I \times \left(\sum_i B_i(I, t) \left(\frac{p_{i,t_0}}{p_{i,t}} \right)^{1-\gamma} \right)^{\frac{1}{1-\gamma}}$.

The error is very small. For example, with 100 households and annual data, the maximum error in the final period is 0.0078 for the iterative procedure and 2×10^{-5} for the recursive procedure. So, the error is less than 1% of income for the iterative procedure and around 1/1000th of 1% for the recursive procedure. Figure 2 shows how this error varies as we vary the number of households and the frequency of observations. As expected, the error converges to zero as we approach the continuous-time limit. The error also falls as the number of households in the sample increases.¹⁶

3.4 Tastes Shocks and Mismeasured Expenditures

In practice, data is imperfect and noisy. There are two potential sources of error: (1) expenditures by good may be mismeasured or affected by variations in tastes; (2) prices may be missing or mismeasured. In this section, we focus on mismeasured expenditures due to measurement error and or taste shocks. We address missing or mismeasured prices in Section 5, where we impose stronger assumptions on preferences.

If there are arbitrary unobservable shocks to preferences or measurement error, then our methodology cannot be used reliably. However, there are certain tractable cases with shocks where we can still apply our methodology. In this section, we discuss these cases. We begin by considering the straightforward scenario where preferences vary as a function of observable characteristics – for instance, households with children have distinct tastes compared to those without.¹⁷

Proposition 2 (Tastes Vary by Observed Characteristics). *If there are differences in preferences that are functions of observable characteristics, then split the sample by characteristic and apply Proposition 1 to each subsample separately.*¹⁸

Next, we consider the more difficult case where observed expenditures depend on unobservable taste shocks or measurement error. Suppose that observed budget shares are

$$\tilde{B}(I, t|\kappa) = B(I, t) + \kappa\epsilon(I, t),$$

where $B(I, t)$ are the true expenditures — that is, the expenditures generated by the preferences that we wish to construct a money metric for — but we can only see $\tilde{B}(I, t|\kappa)$. The

¹⁶In Appendix E, we apply the Jaravel and Lashkari (2022) algorithm to this example and find similarly small errors. However, in the same appendix, we provide other examples where the Jaravel and Lashkari (2022) approach yields large errors or diverges.

¹⁷This assumption is similar to that considered in Section 2.3 of Jaravel and Lashkari (2022).

¹⁸Similarly, if we observe two groups of households that face different prices at a point in time (e.g. households living in different locations), then we can apply our method to each sample separately.

functions $\epsilon(I, t)$ are unobserved errors in the expenditures either due to mismeasurement and or taste shocks.¹⁹ The scalar $\kappa \geq 0$ controls the importance of these errors.

Define $\tilde{u}(I, t|\kappa)$ to be the solution to the integral equation

$$\log \tilde{u}(I, t|\kappa) = \log I - \int_{t_0}^t \sum_i \tilde{B}_i(\tilde{u}^{-1}(\tilde{u}(I, t|\kappa), s|\kappa), s|\kappa) \frac{d \log p_{is}}{ds} ds. \quad (10)$$

Proposition 1 assumes that $\kappa = 0$. That is, $\tilde{u}(I, t|0) = u(I, t)$.

When there is idiosyncratic (mean-zero) noise at the level of individual households, averaging over households ensures that $\kappa = 0$ as long as the law of large numbers holds. In such situations, we can apply Proposition 1 without concerns about taste shocks and recover the money metric for preferences in the absence of the idiosyncratic noise. However, if the errors do not average out, they could potentially impact the results. To analyze the extent of this influence, we derive a first-order approximation of $\tilde{u}(I, t|\kappa)$ with respect to the error term κ . The general form of this first-order approximation can be found in Lemma A.1 in the appendix. Within the main text, we highlight two tractable and salient special cases.

Proposition 3 (Taste Shocks Uncorrelated with Price Shocks). *Suppose that for all I and $s \leq t$, we have $\text{Cov}(\epsilon(I, s), d \log \mathbf{p}/ds) = 0$. Then, to a first-order approximation around $\kappa \approx 0$,*

$$\tilde{u}(I, t|\kappa) \approx u(I, t),$$

where the remainder term is order κ^2 .

In words, if the shocks are uncorrelated with price changes, then the money metric we construct by solving the wrong integral equation is, to a first-order approximation, correct. This approximation assumes that κ is small, but does not require that t be close to t_0 .²⁰

Next, we consider how taste shocks that are correlated with price changes affect our results.

Proposition 4 (Engel Curve Slopes Uncorrelated with Price Shocks). *Suppose that for all I and $s \leq t$, the slope of Engel curves is uncorrelated with price changes $\text{Cov}(\partial \mathbf{B}(I, s)/\partial I, d \log \mathbf{p}/ds) =$*

¹⁹See Baqaee and Burstein (2021) for a detailed analysis of how welfare should be defined when preferences are subject to taste shocks.

²⁰This result bears a superficial resemblance to previous results, for example by Baqaee and Burstein (2021), that Divisia indices approximately measure welfare correctly when taste shocks are uncorrelated with price changes. However, Proposition 3 is different since it is characterizing the solution to an integral equation, and not the Divisia index. Importantly, the results about the Divisia index require that t be close to the base year t_0 . On the other hand, in Proposition 3, t can be far from t_0 .

0. Then, to a first-order approximation around $\kappa \approx 0$,

$$\tilde{u}(I, t|\kappa) - u(I, t) \approx -\kappa \int_{t_0}^t \text{Cov}(\epsilon(u(I, t), s), d \log \mathbf{p}/ds),$$

where the remainder term is order κ^2 .

If the slope of Engel curves is uncorrelated with price shocks, then the money metric we construct is biased according to how the taste/measurement shocks $\epsilon(I, t)$ covary with price shocks. That is, although our methodology will have errors, the sign and magnitude of these errors can be linked to the underlying shocks in a straightforward way. In particular, if the mismeasured expenditures are biased upwards for goods whose relative price rose, then the constructed money metric will be biased downwards.

We need the requirement that the slope of Engel curves be uncorrelated with price shocks because otherwise, as we solve the integral equation forward, errors in prior values of $\tilde{u}(I, s|\kappa)$, for $s \leq t$, contaminate the matching process in a systematic way and induce additional biases in $\tilde{u}(I, t|\kappa)$.

4 Empirical Illustration

In this section, we apply our algorithm to long-run cross-sectional household data. Our goal is to compare welfare as measured by the money metric with real consumption. We define *real consumption* consistently with how it is constructed by statistical agencies in the national accounts: nominal expenditures deflated by a chain-weighted price index that reflects observed (either aggregate or decile-specific) budget shares.²¹ When preferences are homothetic, then real consumption for every household coincides with money-metric utility.

We use the *Family Expenditure Survey and Living Costs and Food Survey Derived Variables* for the UK (see Oldfield et al., 2020), which is a repeated cross-section of UK household expenditures over different sub-categories of goods and services from 1974 to 2017.²² The

²¹The analog to real consumption in our theoretical model is $\log RC(I, t) = \log I - \int_{t_0}^t \sum_{i=1}^N \bar{B}_i(t) \frac{d \log p_{is}}{ds} ds$, where $\bar{B}_i(t)$ is some average budget share of good i in period t . If we use the average budget shares, then the price deflator is common for all households. Alternatively, we can group households by quantiles of the spending distribution and use average budget shares by quantile. We compare our results with both aggregate and decile-specific price deflators.

²²Aggregate nominal consumption growth in our sample is lower than that in the UK national accounts. According to the UK Office for National Statistics, this difference is due to differences in sample coverage. While these sample coverage issues affect aggregate nominal growth rates, they do not affect our results, which are at the household-level.

UK Family Expenditure Survey was also used in Blundell et al. (2003) and Blundell et al. (2008) to estimate Engel curves, test for deviations from revealed preference theory, and compute bounds for a true cost of living index.

Following the practice of the Office of National Statistics (ONS), we measure prices using the retail price index (RPI) in the period 1974-1998 and the consumer price index (CPI) in the period 1998-2017. To concord the RPI, CPI, and household expenditure data, we assemble 17 aggregate product categories that can be used consistently over the entire period of analysis.²³ Between 1974 and 2017 prices rose relatively less for product categories that are disproportionately consumed by richer households, such as leisure goods and services. Even though we consider product categories that are more aggregated than the official data, our data tracks the official inflation figures from the ONS fairly well.²⁴

We pool all households in our sample and assume that they have the same stable preference relation over the 17 categories of goods and services for which we have price data. To investigate the validity of this assumption, we can split the sample by observable characteristics (following Proposition 2). We provide examples using marital status and age in Appendix B. This added flexibility comes at the expense of shrinking the boundaries over which the money metric can be computed, since households with different characteristics (e.g. married and unmarried households) cannot be matched to one another through time. We do not find marked differences in the money-metric function by age or marital status, so these results are relegated to the appendix.

4.1 Mapping Data to the Model

Our procedure requires expenditures I and budget shares $B(I, t)$ at time t across all goods. To deal with idiosyncratic noise, we fit a smooth curve to the budget share of each good i at time t as a function of I . We use these curves as $B(I, t)$. More precisely, we estimate the true $B_i(I, t)$ function for some good i by fitting the following curve for each t using

²³See Appendix C for details about our concordance table. We also calculate our results using more disaggregated spending categories, using only CPI data, from 2001 to 2017. Figure A.4 compares these results to what we get if we instead use the more aggregated 17 spending categories instead for the same time period. The gaps relative to the chain-weighted inflation index are qualitatively similar but moderately larger when we use more disaggregated spending categories. Unfortunately, the more disaggregated data is not available for the full sample, so we use the more aggregated data for our benchmark. In principle, one should apply our methodology to the most disaggregated spending categories possible in order to minimize aggregation bias.

²⁴See Figure A.6 and Table A.1 for comparisons of our data with aggregate inflation and inflation by decile of expenditures as reported by the ONS.

ordinary least squares

$$B_{iht} = \alpha_{0it} + \alpha_{1it} \log I_{ht} + \alpha_{2it} (\log I_{ht})^2 + \varepsilon_{iht},$$

where i is the good, h is the household, and t is the time period. The estimated regression line gives us $B(I, t)$. Importantly, we only evaluate the estimated $B(I, t)$ in-sample to avoid out-of-sample extrapolation errors. As mentioned before this potentially limits the set of values for which we can construct the money metric, but ensures that our estimates are more reliable. Our results are virtually unchanged if we estimate the Engel curves non-parametrically (i.e. using locally weighted scatterplot smoothing, LOWESS) instead of quadratic functions (see Figure A.3 in Appendix B).

Since this regression is the only source of sampling uncertainty in our exercise, we calculate standard errors for our estimates of the money metric by bootstrapping this regression. To do this, we redraw repeated samples with replacement. Although the Engel curves are estimated with considerable uncertainty, the standard errors for the money metric are fairly tight. This is due to the law of large numbers, since the money metric combines many Engel curve estimates. For this reason, and to make the figures less cluttered, when we present our results, we do not report the bootstrapped standard errors.

We calculate money-metric utility using 1974 base prices by applying our procedure sequentially from 1974 to 2017 to the UK data.²⁵ Computing $u(I, t)$ requires that for each time $s < t$, we can estimate the compensated budget share $b(p_s, u(I, t))$. That is, for each expenditure level I at time t , we must be able to find consumers at $s < t$ who were on the same indifference curve as the one delivered by I at time t .

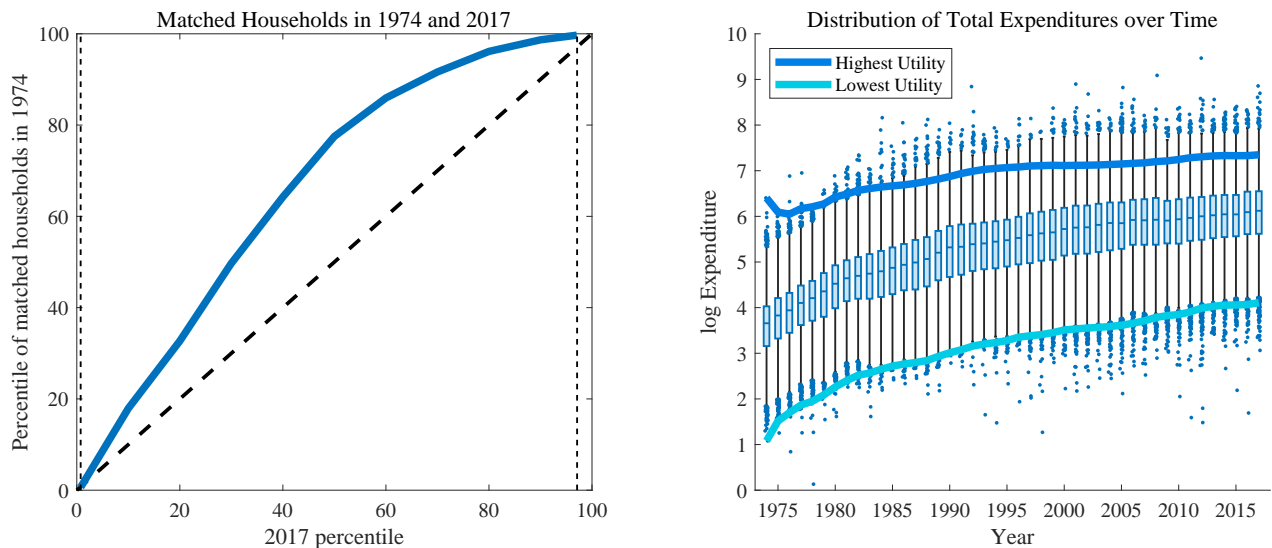
The left panel of Figure 3 illustrates how households in 2017 are matched with households in 1974 in order to estimate $b(p_{1974}, u(I, 2017))$. For example, households in the 50th percentile of expenditures in 2017 are matched with households in the 78th percentile of expenditures in 1974. The dashed diagonal line is the 45-degree line and is what we would get if we matched households by percentile of the distribution. This is how price deflators by spending group are typically calculated by statistical agencies (we compare our results with such a measure below).

Our methodology naturally implies that we can only compute $u(I, t)$ if $u(I, t)$ is lower than the upper-bound and higher than the lower-bound of utility levels at all past times $s < t$. Otherwise, we cannot carry out the inversion in (6). The right panel of Figure 3 plots

²⁵Given the money metric at some base prices, we can easily obtain the money metric at any other base prices in $t_m \in [t_0, T]$, as explained in Footnote 10.

the distribution of log expenditures in our data and the solid lines show the sample of households for which we can calculate $u(I, t)$. Our algorithm can recover the money metric up to about the 97th percentile of households in 2017. For the richest households, we are unable to compute $u(I, t)$ because there are no households in our sample that were on the same indifference curve in the past. Nevertheless, our algorithm covers a significant range of households. Our sample coverage is high because the distribution of expenditures is highly fat-tailed, which means that in 1974, there are households who are on the same indifference curve as the richest 97th percentile of households in 2017.

Figure 3: Results of matching process



Notes: The figure on the left shows, for each expenditure percentile in 2017, the expenditure percentile in 1974 of the matched household that is on the same indifference curve as the 2017 household. The dashed diagonal line is the 45 degree line. The vertical dotted lines are the boundaries for households that can be matched. The figure on the right shows the sample distribution of (weekly) log expenditures from 1974 to 2017. The upper and lower blue boxes represent the 75th and 25th percentiles, respectively. The solid lines indicate the upper and lower bounds of the sample for whom the compensated budget share can be computed as a function of time. The lower and upper bounds in 2017 represent the 0.8th and 97th percentile, respectively, of the spending distribution.

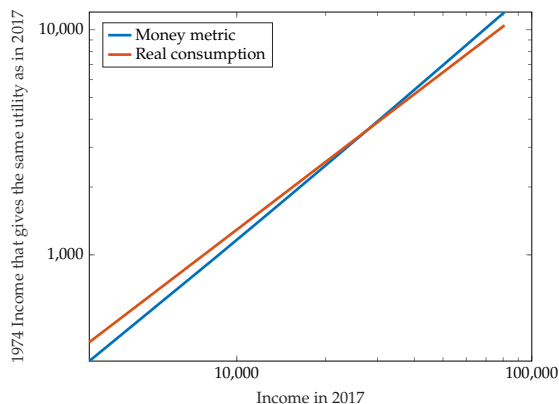
4.2 Results

The blue line in the left panel of Figure 4 plots the expenditure function $e(p_{1974}, v(p_{2017}, I))$ for different values of I . This expresses different income levels in 2017 (x-axis) in terms of 1974 pounds (y-axis). We can also use this figure to convert different income levels in

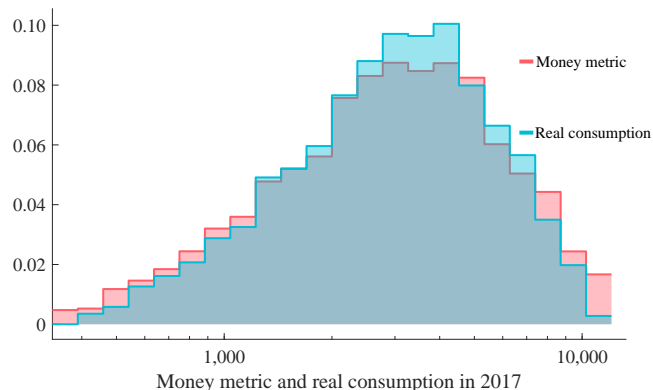
1974 (y-axis) in terms of 2017 pounds (x-axis).²⁶

For comparison, the red line shows the equivalent incomes in 1974 if all households faced the same effective inflation rate, as given by the chain-weighted aggregate inflation rate. When the red line is above the blue line, this means that real consumption based on chain-weighted aggregate inflation is higher than equivalent income using the money metric for households in the sample. Hence, the money metric is higher than real consumption for richer households and lower for poorer households, and the size of the gap is largest for the poorest households. That is, the poorest households are not as well-off as implied by using an aggregate price deflator calculated as in the official statistics. Conversely, the gap reverses around the 60th percentile of the distribution and then widens suggesting that the richest households are better off in 1974 pounds than what is implied by official statistics. Accordingly, the histograms in the right panel of Figure 4 show that inequality across households is larger based on money metric values than based on real consumption.

Figure 4: Comparison of money metric with chain-weighted real consumption.



(a) Real consumption using aggregate chain-weighted inflation between 1974 to 2017 (annualized pounds, log scale) and the money metric $e(\mathbf{p}_{1974}, v(\mathbf{p}_{2017}, I_{2017}))$. This figure converts income in 1974 into equivalent income in 2017 and vice versa.



(b) Histogram (using household weights) of money metric $e(\mathbf{p}_{1974}, v(\mathbf{p}_{2017}, I_{2017}))$ and real consumption using aggregate chain-weighted inflation (annualized pounds, log scale). The distributions are truncated at the upper and lower bounds of Figure 3.

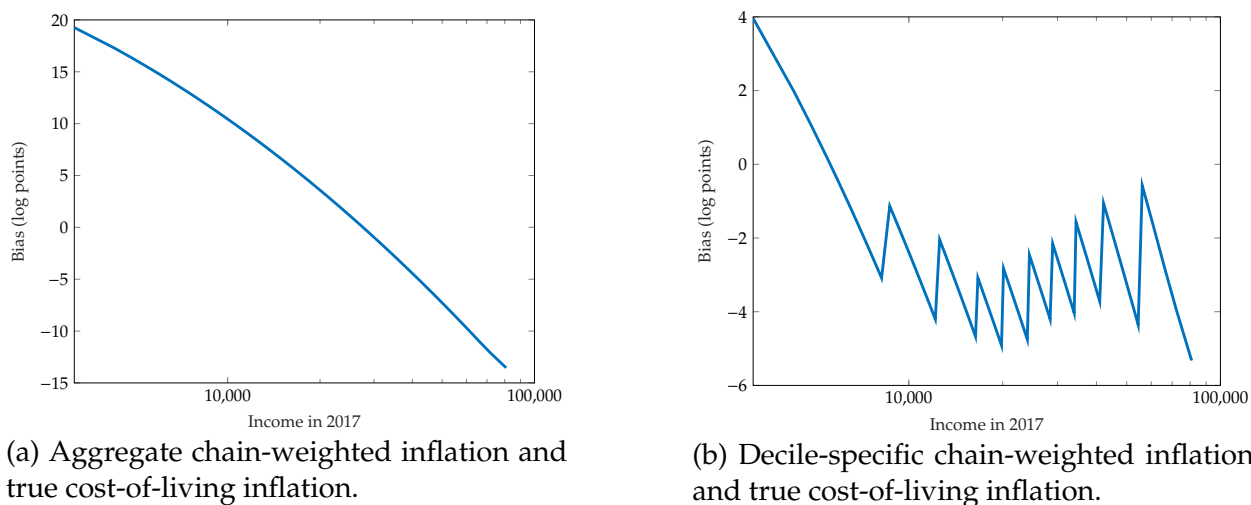
The left panel of Figure 5 displays the log difference between the red and blue lines in Figure 4. As expected, the difference is positive for poor households, meaning that real consumption calculated using aggregate inflation is upward biased, and negative for rich households, meaning that real consumption is downward biased. The size of the bias is

²⁶That is, pick an I' in the y-axis, and find the associated I in the x-axis. Then, since $v(\mathbf{p}_{1974}, I') = v(\mathbf{p}_{2017}, I)$, it must be that $e(\mathbf{p}_{2017}, v(\mathbf{p}_{1974}, I')) = e(\mathbf{p}_{2017}, v(\mathbf{p}_{2017}, I)) = I$.

20 log points for the poorest households. This means that over the 44 year sample, annual inflation rates calculated as in the official statistics understate the true welfare-relevant inflation (i.e. the deflator implied by the money metric and cost-of-living functions in Lemma 1) for these households by around 0.5 percentage points per year. On the other hand, for the richest households, the official inflation rate overstates the true inflation by around 0.25 percentage points per year on average.

The right panel of Figure 5 shows the errors between the true inflation rate and chain-weighted decile-specific inflation. The errors are much smaller, but not zero. We stress that this does not guarantee that quantile-specific chained deflators always approximate the true money metric well. We expect that in contexts where growth is more rapid, the differences can be larger. Importantly, the data requirements for constructing the money metric, following our method, are slightly less demanding than the ones required for constructing quantile-specific chained deflators.²⁷

Figure 5: Log difference between chain-weighted inflation and true cost-of-living inflation



Notes: Results are reported in log points (i.e. 100 times the log difference). The sample is from 1974 to 2017.

5 Extension with Partially Observed Prices

In this section, we extend our methodology to allow for the possibility of missing or unreliably measured price changes. This may occur because the infrastructure for collecting

²⁷Whereas quantile-specific chained deflators require a representative sampling of the entire distribution of households, our methodology can recover the money metric for a subsample of observed households even if that subsample does not sample incomes at the same frequency as the population, as explained in Appendix C. Otherwise, the data requirements of the two methodologies are the same.

comprehensive price data is absent, as in developing country contexts, or because changes in some prices are inherently difficult to measure, for example those of services and new goods. The results in this section generalize Feenstra (1994) beyond the homothetic CES case.

To compute welfare without data on some prices, we impose the following assumption about preferences throughout this section.

Assumption 1 (Separability). Partition the set of goods into X and Y . Suppose that preferences are *separable* in the sense that the expenditure function can be written as

$$e(\mathbf{p}, U) = e(e^X(\mathbf{p}^X, U), e^Y(\mathbf{p}^Y, U), U), \quad (11)$$

where U is utils, \mathbf{p}^X and \mathbf{p}^Y are vectors of prices in X and Y , and e^X and e^Y are non-decreasing in and homogeneous of degree one in prices.

We assume that prices and budget shares of goods in X are observed, but prices and budget shares in Y are unobserved. Assumption 1 does not restrict cross-price elasticities for goods within X or Y but does restrict cross-price effects between X and Y . CES aggregators, used by Feenstra (1994), are separable in every partition of their arguments, so our separability assumption is much weaker. Separability can be tested using the Leontief-Sono conditions, see Blackorby et al. (1998).²⁸

Denote the compensated budget share of X goods by

$$b_X = \sum_{i \in X} b_i(\mathbf{p}, U) = b_X(e^X(\mathbf{p}^X, U)/e^Y(\mathbf{p}^Y, U), U),$$

where the second equality uses Assumption 1 and the fact that e is homogenous of degree one in prices. Hence, the budget share on X goods is pinned down, for a fixed U , by a single scalar, $e^X(\mathbf{p}^X, U)/e^Y(\mathbf{p}^Y, U)$, which we can interpret as the relative price of the X and Y bundles. Raising all prices in X by the same amount, holding utility constant, changes the share of spending on X by

$$\sum_{i \in X} \frac{\partial \log b_X}{\partial \log p_i} = (1 - b_X)(1 - \sigma(\mathbf{p}, U)),$$

for some scalar-valued function $\sigma(\mathbf{p}, U)$. We can think of $\sigma(\mathbf{p}, U)$ as the (compensated)

²⁸The Leontief-Sono conditions, which are necessary and sufficient for separability, imply that, for each $i, j \in X$ and $k \in Y$, we must have $\partial \log(b_i/b_j)/\partial \log p_k = 0$, where b_i and b_j are both compensated budget shares. The same must hold if we swap X and Y .

elasticity of substitution between X and Y goods.²⁹

We provide an example of separable non-homothetic preferences below.

Example 1 (Indirect Addilog). Suppose that the expenditure function is implicitly defined by

$$U = \frac{\omega_X}{\sigma_X - 1} \left(\frac{e(\mathbf{p}, U)}{e^X(\mathbf{p}^X, U)} \right)^{\sigma_X - 1} + \frac{\omega_Y}{\sigma_Y - 1} \left(\frac{e(\mathbf{p}, U)}{e^Y(\mathbf{p}^Y, U)} \right)^{\sigma_Y - 1}. \quad (12)$$

For this demand system, the compensated elasticity of substitution between X and Y is

$$\sigma(\mathbf{p}, U) = b_X(\mathbf{p}, U)\sigma_Y + (1 - b_X(\mathbf{p}, U))\sigma_X,$$

which varies both as a function of utility and as a function of prices. CES is the special case where $\sigma_X = \sigma_Y$.

In general, $\sigma(\mathbf{p}, U)$ may depend on the entire vector of prices, some of which are unobserved. The following assumption ensures that σ can always be expressed as a function of b_X and U , rather than as a function of all prices.

Assumption 2 (Monotone Budget Share). Suppose that $b_X(\mathbf{p}, U)$ is strictly monotone in the price of some $i \in X$.

For the example in (12), Assumption 2 requires that either $\sigma_X, \sigma_Y > 1$ or $\sigma_X, \sigma_Y < 1$. This assumption is different to the monotonicity assumption of budget shares in utility required in Atkin et al. (2020). They require that some budget share be strictly monotone in income (since they infer money-metric utility by inverting budget shares). Assumption 2 allows every budget share to be non-monotone (or even invariant) in income, but it requires that the compensated budget share of X be monotone in at least one price (unitary price elasticities for all goods is not allowed).

Lemma 2. *Assumption 2 implies that we can write $\sigma(\mathbf{p}, U)$ as $\sigma(b_X(\mathbf{p}, U), U)$. By abusing notation, we denote this function by $\sigma(b_X, U)$.*

Lemma 2 allows us to express the elasticity of substitution σ as a function of two scalars: utility and the compensated budget share of X goods.

²⁹This elasticity of substitution is disciplined by the curvature of the upper-nest of the expenditure function $\sigma(\mathbf{p}, U) = 1 - \frac{1}{(1-b_X)b_X} \frac{\partial^2 \log e}{(\partial \log e^X)^2}$.

Recovering money metric using $\sigma(b_X, U)$. Denote the *relative* uncompensated and compensated budget share on $i \in X$ by

$$B_{Xi}(I, t) = \frac{B_i(I, t)}{B_X(I, t)}, \quad \text{and} \quad b_{Xi}(\mathbf{p}, U) = \frac{b_i(\mathbf{p}, U)}{b_X(\mathbf{p}, U)}.$$

The following proposition extends Proposition 1 to account for unmeasured prices.

Proposition 5 (Money metric with Missing Prices). *Under Assumptions 1 and 2, the money metric $u(I, t)$ solves the following integral equation*

$$\log u(I, t) = \log I - \int_{t_0}^t \sum_{i \in X} b_{Xi}(\mathbf{p}_s, u(I, t)) \frac{d \log p_{is}}{ds} ds - \int_{t_0}^t \frac{d \log b_X(\mathbf{p}_s, u(I, t))/ds}{\sigma(b_X(\mathbf{p}_s, u(I, t)), u(I, t)) - 1} ds, \quad (13)$$

where

$$b_{Xi}(\mathbf{p}_s, u(I, t)) = B_{Xi}(u^{-1}(u(I, t), s), s), \quad b_X(\mathbf{p}_s, u(I, t)) = B_X(u^{-1}(u(I, t), s), s).$$

If we know the shape of the function $\sigma(b_X, u)$, Proposition 5 can be used to obtain the money metric utility function using similar procedures to the ones in Section 3.2. Proposition 5 is a consequence of Proposition 1. To derive it, we use changes in the compensated budget share of X goods, $d \log b_X(\mathbf{p}_s, u(I, t))/ds$, to infer the compensated-budget-share-weighted changes in prices for the unobserved goods $\sum_{i \in Y} b_i(\mathbf{p}_s, u(I, t)) d \log p_{is}/ds$ given the elasticity of substitution $\sigma(b_X, u(I, t))$. Plugging this into (3) yields Proposition 5.

Compared to Proposition 1, the fixed point in Proposition 5 has some additional terms. First, the compensated elasticity of substitution $\sigma(b_X, u(I, t))$ on the right-hand side depends on $u(I, t)$, and since $u(I, t)$ depends on the compensated elasticity of substitution, there is a fixed point in this term. Second, the changes in the budget share of X goods, $d \log b_X(\mathbf{p}_s, u(I, t))/ds$, are compensated. To compute these changes, we must use the money-metric utility function, $u(I, t)$, to match households on the same indifference curve through time and use changes in the budget shares of matched households over time. Hence, there is also a fixed point in this term.

To better understand Proposition 5, it helps to consider the homothetic special case.

Example 2 (Homothetic preferences). Suppose that preferences are homothetic. In this case, Proposition 5 simplifies to

$$\log u(I, t) = \log I - \int_{t_0}^t \sum_{i \in X} B_{Xi}(\mathbf{p}_s) \frac{d \log p_{is}}{ds} ds - \int_{t_0}^t \frac{d \log B_X/ds}{\sigma(B_X(s)) - 1}. \quad (14)$$

When preferences are homothetic, there is no longer a fixed point problem since budget shares and elasticities of substitution do not depend on utility. If we also assume that the upper-nest expenditure function is CES, then $\sigma(b_X(p_s))$ is a constant and we get

$$\log u(I, t) = \log I - \int_{t_0}^t \sum_{i \in X} B_{Xi}(\mathbf{p}_s) \frac{d \log p_{is}}{ds} ds - \frac{\log B_X(t) - \log B_X(t_0)}{\sigma - 1}. \quad (15)$$

Equation (15) is a version of the popular Feenstra (1994) formula.³⁰ This formula is commonly used in the macroeconomics and trade literatures for adjusting price indices to account for missing price changes (typically those of new goods). Relative to this CES case, Proposition 5 allows the elasticity of substitution to vary as a function of prices, allows for non-homotheticities, and does not impose parametric assumptions on preferences among the X goods and among the Y goods.

Relative to the homothetic special case in (14), the additional complication in (13) is that changes in the budget share of X and the elasticity of substitution must both be compensated. To see the issue, restate (13) using uncompensated budget shares as

$$\log u(I, t) = \log I - \int_{t_0}^t \sum_{i \in X} B_{Xi}(I_s^*, s) \frac{d \log p_{is}}{ds} ds - \int_{t_0}^t \frac{d \log B_X(I_s^*, s)/ds}{\sigma(B_X(I_s^*, s), u(I, t)) - 1} ds, \quad (16)$$

where I_s^* is implicitly defined by $u(I_s^*, s) = u(I, t)$.

With more structure on the demand system, this expression can be further simplified. For example, suppose that the expenditure function in (11) can be written as

$$e(\mathbf{p}, U) = \left(\omega_X U^{\xi_X} e^X(\mathbf{p}^X, U)^{1-\gamma(U)} + \omega_Y U^{\xi_Y} e^Y(\mathbf{p}^Y, U)^{1-\gamma(U)} \right)^{\frac{1}{1-\gamma(U)}} \quad (17)$$

for any level of utility U . In this case, $\sigma(b_X, u)$ varies as a function of utility but not as a function of relative prices, as in Fally (2022). With this restriction, equation (16) simplifies to

$$\log u(I, t) = \log I - \int_{t_0}^t \sum_{i \in X} B_{Xi}(I_s^*, s) \frac{d \log p_{is}}{ds} ds - \frac{\log B_X(I, t) - \log B_X(I_{t_0}^*, t_0)}{\sigma(u(I, t)) - 1} ds.$$

Of course, if the elasticity of substitution σ is also constant as a function of utility, then the denominator becomes just σ .

³⁰The only (relatively inconsequential) difference between (15) and Feenstra (1994) is the assumption that e^X and e^Y also be homothetic CES aggregators.

Recovering $\sigma(b_X, U)$ without prices in Y . We now show that $\sigma(b_X(p_s, u), u)$, the unknown term required to apply Proposition 5, can be expressed non-parametrically in terms of elasticities that are estimable using only data on prices in X . This is an important result as it demonstrates that, in general, recovering $\sigma(b_X(p_s, u), u)$ does not require data on unobserved prices. Denote by $\epsilon_X(I, s)$ the uncompensated elasticity of the budget share of X with respect to the price of the X bundle. That is, let $\epsilon_X(I, s)$ be the scalar that satisfies the following equation for each level of income I at each time s :

$$\sum_{k \in X} \frac{\partial \log B_X(I, s)}{\partial \log p_k} d \log p_k = \epsilon_X(I, s) \sum_{k \in X} B_{Xk}(I, s) d \log p_k.$$

Proposition 6 shows that the compensated elasticity of substitution between X and Y can be deduced given knowledge of $\epsilon_X(I, s)$ and income elasticities.

Proposition 6 (Identifying Substitution Elasticity of X and Y). *Suppose Assumptions 1 and 2 hold. Let $\eta_i(I, t) - 1 = \partial \log B_i(I, t) / \partial \log I$ be the income elasticity of demand for each $i \in X$ at time t . Then, we have*

$$\sigma(p_s, u(I, t)) = 1 - \frac{\epsilon_X(I_s^*, s) + B_X(I_s^*, s) \sum_{i \in X} (\eta_i(I_s^*, s) - 1) B_{Xi}(I_s^*, s)}{1 - B_X(I_s^*, s)},$$

where I_s^* is defined by $u(I_s^*, s) = u(I, t)$.

Proposition 6 shows that if we know income elasticities for all goods in X and can estimate the uncompensated elasticity of B_X with respect to prices in X , ϵ_X , then we can recover the relevant elasticity of substitution and apply Proposition 5. Estimating the income elasticities, η_i for $i \in X$, is relatively straightforward since we simply need to fit a curve that relates the budget share of i to income in each period. Estimating the price elasticity ϵ_X is more challenging, but we only require a single elasticity per income group and period. That is, the number of elasticities that needs to be estimated does not depend on the number of goods.

With more structure on the demand system, then even less information is required. We provide one example below.

Example 3 (Generalized non-homothetic CES). Consider the case where the expenditure function takes the form in (17). According to Proposition 6, the function $\sigma(\cdot)$ is determined by the following expression

$$\sigma(I) = 1 - \frac{\epsilon_X(I, t_0) + B_X(I, t_0) \sum_{i \in X} (\eta_i(I, t_0) - 1) B_{Xi}(I, t_0)}{(1 - B_X(I, t_0))}, \quad (18)$$

Since σ is not a function of relative prices, Proposition 6 needs to be applied only in the initial period, t_0 , to recover the shape of the σ function.³¹ If σ also does not vary with utility, as in the example in Section 3.3, then equation (18) can still be used but only needs to be applied for one income group.

Relation to previous literature. When price data is unavailable or unreliable, a large strand of the literature relies on Feenstra (1994), which our method generalizes. A different strand, building on Hamilton (2001) and Costa (2001), estimates changes in welfare by inverting Engel curves. This procedure requires that relative budget shares be strictly monotone in income (i.e. homothetic preferences are ruled out). Atkin et al. (2020) provide a recent micro-founded treatment of this idea. To apply their method, one needs to estimate a compensated demand sub-system for the set of goods where prices are measured, a task that can suffer from a curse of dimensionality if the number of goods with observed prices is large. In their applications, they either rely on first-order approximations or use a CES sub-system to keep the estimation challenges manageable.

In contrast, we make stronger assumptions about preferences (separability rather than quasi-separability). In exchange, we do not require that budget shares be strictly monotone in income. More importantly, without making further assumptions, our approach only requires a single uncompensated price elasticity as a function of income in each period (rather than a compensated system). Given estimates of this elasticity, we can non-parametrically and non-linearly back out the elasticity of substitution between the measured and unmeasured goods and use this to non-linearly solve for welfare changes.

6 Empirical Illustration with Partially Observed Prices

As an illustration, we apply Proposition 5 to the UK data that we used in Section 4. Since service prices are difficult to measure, as a test case, we partition the consumption bundle into a subset of luxury services and the rest. That is, we assume that prices for recreation & culture, education, accommodation, leisure goods & services, restaurants and drinking establishments are not reliably observed. These are the Y goods, which in our data account for roughly 30% percent of spending. We assume that prices for all other categories of spending are measured accurately. These other categories are the X goods. We impose Assumptions 1 and 2.

³¹In writing (18), we assume that $\epsilon_X(I, t_0)$ and $\eta_i(I, t_0)$ are known at t_0 . This is without loss of generality since Proposition 5 can be applied with time running forward $t > t_0$ and backward $t < t_0$. Furthermore, once we apply Proposition 5 to obtain the money metric with t_0 reference prices, we can easily obtain the

Table 1: Elasticity of budget share of X with respect to price index of X

	(1)	(2)	(3)	(4)
	OLS	IV	OLS	IV
$\sum_{i \in X} B_{Xi}(h, t) \Delta \log p_{it}$	0.144** (0.069)	0.073*** (0.019)	0.146** (0.069)	0.061*** (0.021)
$\sum_{i \in X} B_{Xi}(h, t) \Delta \log p_{it} \times \mathbf{1}(h \geq \text{median})$			0.005 (0.007)	0.025 (0.039)
F-stat		403,945		177,760
Quantile FE	Y	Y	Y	Y
Year FE	Y	N	Y	N
Obs	41,000	41,000	41,000	41,000

Notes: Columns (2) and (4) use the log difference in world oil prices as an instrument. All lags are two-year differences (results are similar for annual and triennial differences). The sample years are 1974-2017. Standard errors are clustered at the household quantile level (we have 1000 quantiles). Two and three stars indicate statistical significance at the 5% and 1% level.

To apply Proposition 5, we must estimate the elasticity of substitution $\sigma(b_X, u)$ between X and Y . To do this, we group households into a thousand groups by quantiles of the spending distribution. We run the following regression

$$\Delta \log B_X(h, t) = \epsilon_X \sum_{i \in X} B_{Xi}(h, t) \Delta \log p_{it} + \text{controls} + \text{error},$$

where t is time, h is the quantile of the spending distribution, and ϵ_X measures the uncompensated elasticity of the budget share of X goods with respect to the price of X goods. To check for heterogeneity, we allow this elasticity to depend on whether quantile h is above or below the median.

We estimate this regression by OLS. Given that we include year fixed effects, identification comes from variation across households in the price change of the X bundle. We also instrument the right-hand side variable using world oil prices (in which case we cannot include year fixed effects). The identification strategy requires that oil price shocks exogenously move the price of goods versus services. We view our exercises as a proof of concept rather than a full-fledged elasticity estimation.

The results of this regression are reported in Table 1. The first two columns assume that ϵ_X does not vary as a function of expenditures, and the last two columns allow

money metric at $t_s \in [t_0, T]$ base prices, as described in Section 3.1.

for the possibility that ϵ_X varies as a function of expenditures. Since the second row is insignificant with small coefficients, we assume ϵ_X does not vary by quantile. We also assume that ϵ_X does not vary as a function of time (we check for subsample stability by re-running the regression on the first and second half of the time period).

The OLS and IV point estimates are $\epsilon_X = 0.14$ and $\epsilon_X = 0.07$, though with overlapping confidence intervals. For concreteness, we take $\epsilon_X = 0.14$ and apply Proposition 6 to recover an estimate of the compensated elasticity of substitution σ for each value of I and in each time period.³² The results, for the 25th, 50th, and 75th percentile of the expenditures distribution are plotted in Figure 6. The estimated elasticity is below one, so X and Y are complements, and increasing in income. Richer households are more willing to substitute between X and Y goods than poorer households.

Figure 7 uses these estimates of the compensated elasticity and computes the money metric. The resulting money metric is plotted against the results from Section 4 when we assumed that all prices are perfectly observed. For low-income households, the two money metrics are quite similar and both are below real consumption (computed using an aggregate chain-weighted price deflator assuming that all prices are observed). However, for households with high incomes, the money metric calculated using Proposition 5 is lower than the one calculated using Proposition 1. The fact that the blue line is lower than the yellow line for rich households suggests that, for these households, prices in Y have risen more than the official price data suggest.

Figure 8 shows the percent difference between the money metric with observed prices and the money metric with unobserved prices for different deciles of expenditures as well as the breakdown of the difference into two terms. The first is the difference between overall inflation and inflation for X goods implied by the two methods:

$$\frac{\int_{1974}^{2017} \sum_{i=1}^N b_i(\mathbf{p}_s, u)(d \log p_{is}/ds)ds - \int_{1974}^{2017} \sum_{i \in X} b_{Xi}(\mathbf{p}_s, u)(d \log p_{is}/ds)ds}{\int_{1974}^{2017} \sum_{i=1}^N b_i(\mathbf{p}_s, u)(d \log p_{is}/ds)ds}.$$

These are the blue bar graphs in Figure 8. The remainder is the adjustment due to changes

³²See Figure A.5 for results using the IV point estimates instead. For illustration, Figure A.5 also shows how the results change if we instead calibrate the compensated elasticity of substitution between X and Y goods to be constant in both the time series and the cross-section and equal to $1/2$. When we use the IV point estimates, the results are qualitatively similar, but the adjustment to the money-metric values for rich households is larger than in Figure 7 because the implied elasticity of substitution σ is closer to one for richer households.

Figure 6: Compensated elasticity of substitution between X and Y .

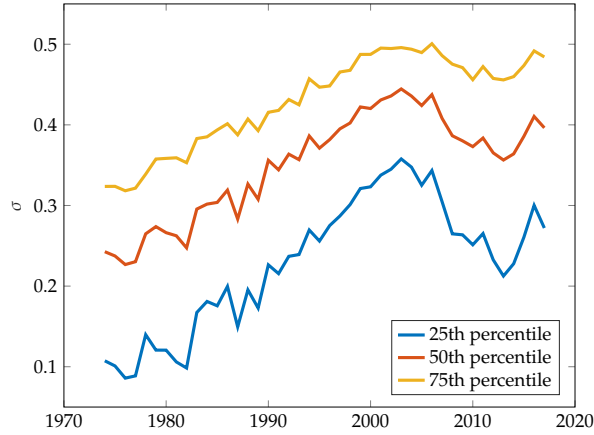


Figure 7: Money metric $e(p_{1974}, v(p_{2017}, I))$ and real consumption.

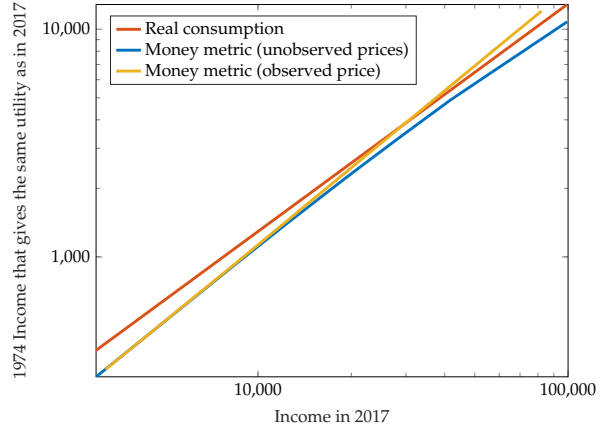
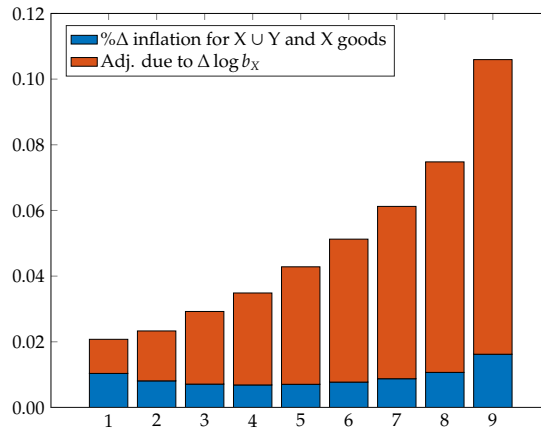


Figure 8: Log difference in estimated money metrics under observed and unobserved prices by decile of the expenditures distribution.



in the budget share of X goods, similar to the Feenstra (1994) adjustment:

$$\frac{\int_{1974}^{2017} \frac{1}{\sigma(p_s, u) - 1} (d \log b_X(p_s, u) / ds) ds}{\int_{1974}^{2017} \sum_{i=1}^N b_i(p_s, u) (d \log p_{is} / ds) ds} \quad (19)$$

These are the orange bar graphs in the right panel of Figure 8. This decomposition shows that inflation among X goods has tended to be higher than among all goods by roughly the same amount (around 1 percentage point) for all deciles. However, the change in compensated expenditures on X goods has been very different. Compensated expenditures on X goods have been falling much more quickly for rich households than poor.

Figure 9: Compensated changes in the budget share B_X for different I percentiles in 2017.

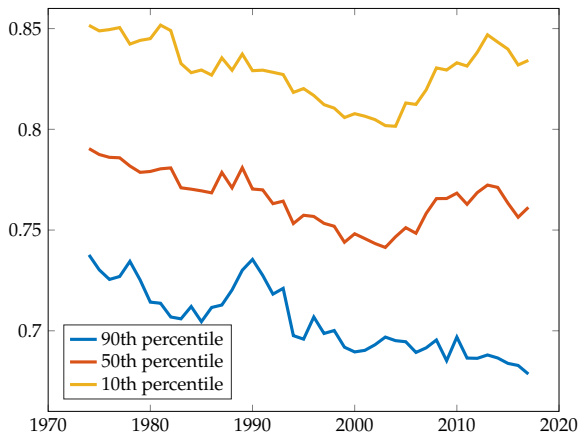
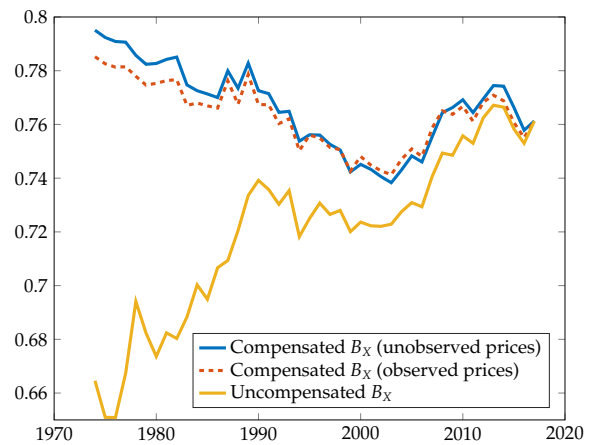


Figure 10: Compensated and uncompensated changes in the budget share B_X for the median household in 2017.



To better understand this, we investigate how compensated expenditures on X goods have changed over time. Figure 9 shows the compensated budget share on X goods for households at three different points in the distribution: the 10th, 50th and 90th percentiles in 2017. For poor households, there was almost no change on expenditures on X goods. This explains why the adjustment term in (19) is small for these households. For the median household, there was a modest decrease in the share of spending on X goods. Since X and Y are complements, this indicates that the relative price of Y goods rose relative to X goods for these households. Finally, for the richest households, there was a fairly dramatic reduction in their spending on X goods from around 74% to around 63%. This suggests that for these households, the relative price of Y goods rose fairly rapidly compared to X goods. This explains why the adjustment term, (19), for these households is large and negative. Furthermore, since the elasticity of substitution for rich households is closer to one, the implied difference in the relative price of X and Y goods is larger. This explains why the money metric according to Proposition 5 (the blue line in Figure 7) has a flatter slope than the money metric calculated according to Proposition 1 (the yellow line in Figure 7).

These difference in compensated expenditures are not mirrored in uncompensated expenditures. Figure 10 compares the compensated and uncompensated changes in expenditures for the median household. Whereas, for the median household, the compensated expenditures on X goods declined somewhat over time, the uncompensated expenditures on X goods increased very strongly. Intuitively, a household in 1974 with nominal expenditures equal to the median of the expenditure distribution in 2017 is actually fairly rich.

Such a household spends relatively less on goods (X) and relatively more on services (Y). As we roll time forward, such a household is effectively becoming poorer, due to inflation, and this causes the expenditures on the X goods to rise due to income effects. That is, the income effect overwhelms the substitution effect.

7 Conclusion

In this paper, we propose a straightforward approach to construct money-metric representations of utility — an essential input to measuring welfare-relevant growth — using repeated cross-sectional data. Our method does not require any estimation when the data on prices is comprehensive, aside from cross-sectional interpolation of how budget shares vary with income. If the data on prices is incomplete, the method can still be used, but stronger assumptions on preferences and knowledge of one uncompensated elasticity is required.

Whether prices are fully or partially observed, the unifying idea in both cases is that money-metric utility can be calculated using observed demand of matched households in the cross-sectional distribution over time. Doing so involves solving a simple fixed point equation in terms of observable variables.

Despite its advantages, our approach does not allow for preferences to vary in arbitrary and unobserved ways in the cross-section or the time-series, and requires that all consumers face common prices that evolve smoothly. Relaxing these assumptions is an interesting avenue for future work.

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Online Appendix

A Proofs and Additional Results

A.1 Proofs

Proof of Lemma 1. By definition,

$$\begin{aligned}\log e(\mathbf{p}, v(\bar{\mathbf{p}}, \bar{I})) &= \log e(\bar{\mathbf{p}}, v(\bar{\mathbf{p}}, \bar{I})) + \log e(\mathbf{p}, v(\bar{\mathbf{p}}, \bar{I})) - \log e(\bar{\mathbf{p}}, v(\bar{\mathbf{p}}, \bar{I})) \\ &= \log \bar{I} + \log e(\mathbf{p}, v(\bar{\mathbf{p}}, \bar{I})) - \log e(\bar{\mathbf{p}}, v(\bar{\mathbf{p}}, \bar{I})).\end{aligned}$$

Rewrite

$$\log e(\mathbf{p}, v(\bar{\mathbf{p}}, \bar{I})) - \log e(\bar{\mathbf{p}}, v(\bar{\mathbf{p}}, \bar{I})) = \int_{t_0}^{t_1} \sum_{i \in N} \frac{\partial \log e(\xi_t, v(\bar{\mathbf{p}}, \bar{I}))}{\partial \log \xi_{it}} \frac{\partial \log \xi_{it}}{dt} dt,$$

where $\{\xi_t : t \in [t_0, t_1]\}$ is a smooth path connecting $\bar{\mathbf{p}}$ and \mathbf{p} as a function of a scalar t . Finally, use Shephard's lemma to express the price elasticity of the expenditure function in terms of budget shares, and obtain (2). To obtain (1), switch \mathbf{p} and $\bar{\mathbf{p}}$ as well as I and \bar{I} . ■

Proof of Proposition 1. This follows immediately from the definition of $u^{-1}(\cdot, s)$ which maps incomes at t_0 to equivalent income at time s . Hence, for some amount of t_0 income, say $u(I, t)$, the equivalent income at time s is $u^{-1}(u(I, t), s)$. The uncompensated budget share $B(u^{-1}(u(I, t), s), s)$ is just $b(u(I, t), s)$. ■

Proof of Proposition 2. Suppose that preferences \succeq_x vary by some observable characteristic x . For example, x could be marital status. In this case, we can split our sample by x and apply Proposition 1 to each subsample separately resulting in $u(I, t|x)$ — money metrics for different levels of expenditures I , at different points in time t , for different values of the characteristic x . ■

To prove Proposition 3 and Proposition 4, we make use of the following lemma.

Lemma A.1. Define $\tilde{u}(I, t|\kappa)$ to be the solution to the integral equation (10). Then

$$\frac{\partial \log u(I, t)}{\partial \kappa} = \frac{-\int_{t_0}^t \text{Cov}(\epsilon(u(I, t), s), \frac{d \log p}{ds}) + \int_{t_0}^t \frac{\partial u(I^*(I, t, s), s)}{\partial \kappa} \text{Cov}_b(\frac{\partial \log b(u(I, t), s)}{\partial \log u(I, t)}, \frac{d \log p}{ds})}{[1 + \int_{t_0}^t \text{Cov}_b(\frac{\partial \log b(u(I, t), s)}{\partial \log u(I, t)}, \frac{d \log p}{ds})]}.$$

Proof of Lemma A.1. Define the integral equation

$$\log u(I, t|\kappa) = \log I - \int_{t_0}^t \sum_i B_i(\Gamma^*(I, t, s|\kappa), s) + \kappa \epsilon_i(\Gamma^*(I, t, s|\kappa), s) \frac{d \log p_i}{ds} ds$$

where

$$u(\Gamma^*(I, t, s|\kappa), s|\kappa) = u(I, t|\kappa).$$

Now differentiate this with respect to κ :

$$\frac{1}{u(I, t|\kappa)} \frac{\partial u(I, t|\kappa)}{\partial \kappa} = - \int_{t_0}^t \sum_i \left[\frac{\partial B_i}{\partial \Gamma^*} \frac{\partial \Gamma^*}{\partial \kappa} + \epsilon_i(\Gamma^*(I, t, s|\kappa), s) + \kappa \frac{\partial \epsilon_i}{\partial \Gamma^*} \frac{\partial \Gamma^*}{\partial \kappa} \right] \frac{d \log p_i}{ds} ds$$

where

$$\frac{\partial \Gamma^*(I, t, s|\kappa)}{\partial \kappa} = \frac{\frac{\partial u(I, t|\kappa)}{\partial \kappa} - \frac{\partial u(\Gamma^*(I, t, s|\kappa), s|\kappa)}{\partial \kappa}}{\frac{\partial u(\Gamma^*(I, t, s|\kappa), s|\kappa)}{\partial I}}$$

At $\kappa = 0$, this is

$$\frac{\partial \Gamma^*(I, t, s|\kappa)}{\partial \kappa} = \frac{\frac{\partial u(I, t)}{\partial \kappa} - \frac{\partial u(\Gamma^*(I, t, s), s)}{\partial \kappa}}{\frac{\partial u(\Gamma^*(I, t, s), s)}{\partial I}}$$

At $\kappa = 0$, we have

$$\begin{aligned} \frac{1}{u(I, t)} \frac{\partial u(I, t)}{\partial \kappa} &= - \int_{t_0}^t \sum_i \left[\frac{\partial B_i}{\partial \Gamma^*} \frac{\partial \Gamma^*}{\partial \kappa} \right] \frac{d \log p_i}{ds} ds - \int_{t_0}^t \sum_i \epsilon_i(\Gamma^*(I, t, s), s) \frac{d \log p_i}{ds} ds \\ &= - \int_{t_0}^t \sum_i \left[\frac{\partial B_i(\Gamma^*(I, t, s), s)}{\partial \Gamma^*(I, t, s)} \frac{\frac{\partial u(I, t)}{\partial \kappa} - \frac{\partial u(\Gamma^*(I, t, s), s)}{\partial \kappa}}{\frac{\partial u(\Gamma^*(I, t, s), s)}{\partial I}} \right] \frac{d \log p_i}{ds} ds \\ &\quad - \int_{t_0}^t \sum_i \epsilon_i(\Gamma^*(I, t, s), s) \frac{d \log p_i}{ds} ds. \end{aligned}$$

Simplifying further gives

$$\begin{aligned} \frac{\partial \log u(I, t)}{\partial \kappa} &= - \frac{\partial u(I, t)}{\partial \kappa} \int_{t_0}^t \sum_i \left[\frac{\partial B_i}{\partial \Gamma^*} \frac{1}{\frac{\partial u(\Gamma^*(I, t, s), s)}{\partial I}} \right] \frac{d \log p_i}{ds} ds \\ &\quad + \int_{t_0}^t \sum_i \left[\frac{\partial B_i}{\partial \Gamma^*} \frac{\frac{\partial u(\Gamma^*(I, t, s), s)}{\partial \kappa}}{\frac{\partial u(\Gamma^*(I, t, s), s)}{\partial I}} \right] \frac{d \log p_i}{ds} ds - \int_{t_0}^t \sum_i \epsilon_i(\Gamma^*(I, t, s), s) \frac{d \log p_i}{ds} ds \\ \frac{\partial \log u(I, t)}{\partial \kappa} &= \frac{\int_{t_0}^t \sum_i \left[\frac{\partial B_i}{\partial \Gamma^*} \frac{\frac{\partial u(\Gamma^*(I, t, s), s)}{\partial \kappa}}{\frac{\partial u(\Gamma^*(I, t, s), s)}{\partial I}} \right] \frac{d \log p_i}{ds} ds - \int_{t_0}^t \sum_i \epsilon_i(\Gamma^*(I, t, s), s) \frac{d \log p_i}{ds} ds}{\left[1 + u(I, t) \int_{t_0}^t \sum_i \left[\frac{\partial B_i}{\partial \Gamma^*} \frac{1}{\frac{\partial u(\Gamma^*(I, t, s), s)}{\partial I}} \right] \frac{d \log p_i}{ds} ds \right]}. \end{aligned}$$

We know that

$$B_i(I^*(I, t, s), s) = b_i(u(I, t), s)$$

Hence

$$\frac{\partial B_i(I^*(I, t, s), s)}{\partial I^*} \frac{\partial I^*}{\partial u(I, t)} = \frac{\partial b_i(u(I, t), s)}{\partial u(I, t)}$$

Therefore, we can write

$$\begin{aligned} \frac{\partial \log u(I, t)}{\partial \kappa} &= \frac{\int_{t_0}^t \sum_i \left[\frac{\partial B_i(I^*(I, t, s), s)}{\partial (I^*(I, t, s))} \left[\frac{\partial u(I^*(I, t, s), s)}{\partial I} \right]^{-1} \frac{\partial u(I^*(I, t, s), s)}{\partial \kappa} \right] \frac{d \log p_i}{ds} ds - \int_{t_0}^t \sum_i \epsilon_i(I^*(I, t, s), s) \frac{d \log p_i}{ds} ds}{\left[1 + \int_{t_0}^t \sum_i \left[\frac{\partial b_i(u(I, t), s)}{\partial \log u(I, t)} \right] \frac{d \log p_i}{ds} ds \right]} \\ &= \frac{\int_{t_0}^t \sum_i \left[\frac{\partial B_i(I^*(I, t, s), s)}{\partial (I^*(I, t, s))} \left[\frac{\partial I^*(I, t, s)}{\partial u} \right] \frac{\partial u(I^*(I, t, s), s)}{\partial \kappa} \right] \frac{d \log p_i}{ds} ds - \int_{t_0}^t \sum_i \epsilon_i(I^*(I, t, s), s) \frac{d \log p_i}{ds} ds}{\left[1 + \int_{t_0}^t \sum_i \left[\frac{\partial b_i(u(I, t), s)}{\partial \log u(I, t)} \right] \frac{d \log p_i}{ds} ds \right]} \\ &= \frac{\int_{t_0}^t \sum_i \left[\frac{\partial b_i(u(I, t), s)}{\partial u(I, t)} \frac{\partial u(I^*(I, t, s), s)}{\partial \kappa} \right] \frac{d \log p_i}{ds} ds - \int_{t_0}^t \sum_i \epsilon_i(I^*(I, t, s), s) \frac{d \log p_i}{ds} ds}{\left[1 + \int_{t_0}^t \sum_i \left[\frac{\partial b_i(u(I, t), s)}{\partial \log u(I, t)} \right] \frac{d \log p_i}{ds} ds \right]} \\ \frac{\partial \log u(I, t)}{\partial \kappa} &= \frac{\int_{t_0}^t \sum_i \left[\frac{\partial b_i(u(I, t), s)}{\partial u(I, t)} \frac{\partial u(I^*(I, t, s), s)}{\partial \kappa} \right] \frac{d \log p_i}{ds} ds - \int_{t_0}^t \sum_i \epsilon_i(I^*(I, t, s), s) \frac{d \log p_i}{ds} ds}{\left[1 + \int_{t_0}^t \sum_i \left[\frac{\partial b_i(u(I, t), s)}{\partial \log u(I, t)} \right] \frac{d \log p_i}{ds} ds \right]} \end{aligned}$$

The adding up constraint ensures that

$$\sum_i \epsilon_i(I^*(I, t, s | \kappa), s) = 0.$$

Hence,

$$\begin{aligned} \frac{\partial \log u(I, t)}{\partial \kappa} &= \frac{- \int_{t_0}^t \text{Cov}(\epsilon(u(I, t), s), \frac{d \log p}{ds}) + \int_{t_0}^t \sum_i \left[\frac{\partial b_i(u(I, t), s)}{\partial u(I, t)} \frac{\partial u(I^*(I, t, s), s)}{\partial \kappa} \right] \frac{d \log p_i}{ds} ds}{\left[1 + \int_{t_0}^t \text{Cov}_b\left(\frac{\partial \log b(u(I, t), s)}{\partial \log u(I, t)}, \frac{d \log p}{ds}\right) \right]} \\ &= \frac{- \int_{t_0}^t \text{Cov}(\epsilon(u(I, t), s), \frac{d \log p}{ds}) + \int_{t_0}^t \frac{\partial u(I^*(I, t, s), s)}{\partial \kappa} \sum_i \left[\frac{\partial b_i(u(I, t), s)}{\partial u(I, t)} \right] \frac{d \log p_i}{ds} ds}{\left[1 + \int_{t_0}^t \text{Cov}_b\left(\frac{\partial \log b(u(I, t), s)}{\partial \log u(I, t)}, \frac{d \log p}{ds}\right) \right]} \\ &= \frac{- \int_{t_0}^t \text{Cov}(\epsilon(u(I, t), s), \frac{d \log p}{ds}) + \int_{t_0}^t \frac{\partial u(I^*(I, t, s), s)}{\partial \kappa} \text{Cov}_b\left(\frac{\partial \log b(u(I, t), s)}{\partial \log u(I, t)}, \frac{d \log p}{ds}\right)}{\left[1 + \int_{t_0}^t \text{Cov}_b\left(\frac{\partial \log b(u(I, t), s)}{\partial \log u(I, t)}, \frac{d \log p}{ds}\right) \right]} \end{aligned}$$

■

Proof of Proposition 3. Assume that for all I and s , we have

$$\text{Cov}(\epsilon(I, s), \frac{d \log p}{ds}) = 0.$$

Assume that for all $s < t$, we have

$$\frac{\partial \log u(I, s)}{\partial \kappa} = 0$$

Then, using Lemma A.1, we know that

$$\frac{\partial \log u(I, t)}{\partial \kappa} = \frac{\int_{t_0}^t \sum_i \frac{\partial u(I^*(I, t, s), s)}{\partial \kappa} \left[\frac{\partial b_i(u(I, t), s)}{\partial u(I, t)} \right] \frac{d \log p_i}{ds} ds}{\left[1 + \int_{t_0}^t \text{Cov}_b \left(\frac{\partial \log b(u(I, t), s)}{\partial \log u(I, t)}, \frac{d \log p}{ds} \right) \right]}.$$

This is equal to zero if $\frac{\partial u(I^*(I, t, s), s)}{\partial \kappa}$ is equal to zero for every $s \leq t$. We also know that

$$\frac{\partial \log u(I, t_0)}{\partial \kappa} = 0.$$

Hence

$$\frac{\partial \log u(I, t)}{\partial \kappa} = 0$$

by transfinite induction. ■

Proof of Proposition 4. If, for every s and I , we have

$$\text{Cov}_b \left(\frac{\partial \log B(I, s)}{\partial \log I}, \frac{d \log p}{ds} \right) = 0,$$

then we know that, for every s , we have

$$\text{Cov}_b \left(\frac{\partial \log b(u(I, t), s)}{\partial \log u(I, t)}, \frac{d \log p}{ds} \right) = 0.$$

Substituting this into Lemma A.1 yields

$$\frac{\partial \log u(I, t)}{\partial \kappa} = - \int_{t_0}^t \text{Cov}(\epsilon(u(I, t), s), \frac{d \log p}{ds}).$$

■

Proof of Lemma 2. Start by assuming that b_X is increasing in p_k for some $k \in X$. Then, we have that $\frac{\partial \log b_X}{\partial \log p_k} = (1 - b_X)(1 - \sigma)b_{Xk} > 0$. That is, $\sigma(p, u) < 1$. This implies that

$b_X(e^X(\mathbf{p}^X, u)/e^Y(\mathbf{p}^Y, u), u)$ is increasing in its first argument. In other words, we can write $e^X(\mathbf{p}^X, u)/e^Y(\mathbf{p}^Y, u) = f(b_X, u)$. Hence, we can write $\sigma(\mathbf{p}, u) = \sigma(e^X(\mathbf{p}^X, u)/e^Y(\mathbf{p}^Y, u), u) = \sigma(f(b_X, u), u)$ as needed. A symmetric argument applies when b_X is decreasing in p_k for some $k \in X$. ■

Proof of Proposition 5. By Euler's theorem of homogeneous functions, we know that

$$\frac{\partial \log e}{\partial \log e^X} + \frac{\partial \log e}{\partial \log e^Y} = 1.$$

Differentiating this identity with respect to e^X and e^Y yields the following equations

$$\frac{\partial^2 \log e}{(\partial \log e^X)^2} = -\frac{\partial^2 \log e}{\partial \log e^X \partial \log e^Y} = \frac{\partial^2 \log e}{(\partial \log e^Y)^2}.$$

Next, we know that

$$b_X = \sum_{i \in X} b_i = \sum_{i \in X} \frac{\partial \log e}{\partial \log e^X} \frac{\partial \log e^X}{\partial \log p_i} = \frac{\partial \log e}{\partial \log e^X} \sum_{i \in X} \frac{\partial \log e^X}{\partial \log p_i} = \frac{\partial \log e}{\partial \log e^X}.$$

Hence, fixing utility, the total derivative of b_X with respect to prices is

$$\begin{aligned} b_X d \log b_X &= \frac{\partial^2 \log e}{(\partial \log e^X)^2} \sum_{i \in X} \frac{\partial \log e^X}{\partial \log p_i} d \log p_i + \frac{\partial^2 \log e}{\partial \log e^Y \partial \log e^X} \sum_{i \in Y} \frac{\partial \log e^Y}{\partial \log p_i} d \log p_i \\ &= \frac{\partial^2 \log e}{(\partial \log e^X)^2} \left[\sum_{i \in X} \frac{\partial \log e^X}{\partial \log p_i} d \log p_i - \sum_{i \in Y} \frac{\partial \log e^Y}{\partial \log p_i} d \log p_i \right] \\ &= \frac{\partial^2 \log e}{(\partial \log e^X)^2} \left[\sum_{i \in X} b_{X_i} d \log p_i - \sum_{i \in Y} b_{Y_i} d \log p_i \right] \end{aligned}$$

Using the fact that

$$\sigma(\mathbf{p}, u) = 1 - \frac{1}{(1 - b_X)b_X} \frac{\partial^2 \log e}{(\partial \log e^X)^2},$$

we can rewrite this as

$$d \log b_X = (1 - b_X)(1 - \sigma) \left[\sum_{i \in X} b_{X_i} d \log p_i - \sum_{i \in Y} b_{Y_i} d \log p_i \right],$$

where we suppress the fact that σ is a function of prices and utility. Rearranging this gives

$$-\frac{d \log b_X}{1 - \sigma} + (1 - b_X) \sum_{i \in X} b_{Xi} d \log p_i + b_X \sum_{i \in X} b_{Xi} d \log p_i = \sum_{i \in X} b_i d \log p_i + \sum_{i \in Y} b_i d \log p_i,$$

or

$$-\frac{d \log b_X}{1 - \sigma} + \sum_{i \in X} b_{Xi} d \log p_i = \sum_{i \in X} b_i d \log p_i + \sum_{i \in Y} b_i d \log p_i.$$

Plug this back into Proposition 1 to get the desired result. It is important to note however that $d \log b_X$ in the expression above is the compensated change in the budget share of X . ■

Proof of Proposition 6. Consider a perturbation to p_k for $k \in X$ holding fixed utils:

$$\begin{aligned} \frac{\partial \log b_X}{\partial \log p_k} &= \frac{1}{b_X} \frac{\partial}{\partial \log p_k} \left[\sum_{i \in X} \frac{\partial \log e}{\partial \log e^X} \frac{\partial \log e^X}{\partial \log p_i} \right] \\ &= \frac{1}{b_X} \frac{\partial}{\partial \log p_k} \left[\sum_{i \in X} \frac{\partial \log e}{\partial \log e^X} b_{Xi} \right] \\ &= \frac{1}{b_X} \left[\sum_{i \in X} \frac{\partial}{\partial \log p_k} \frac{\partial \log e}{\partial \log e^X} b_{Xi} + \sum_{i \in X} \frac{\partial \log e}{\partial \log e^X} \frac{\partial b_{Xi}}{\partial \log p_k} \right] \\ &= \frac{1}{b_X} \left[\sum_{i \in X} \frac{\partial^2 \log e}{(\partial \log e^X)^2} b_{Xk} b_{Xi} + \sum_{i \in X} \frac{\partial \log e}{\partial \log e^X} \frac{\partial b_{Xi}}{\partial \log p_k} \right] \\ &= \frac{1}{b_X} \left[\sum_{i \in X} \frac{\partial^2 \log e}{(\partial \log e^X)^2} b_{Xk} b_{Xi} + \frac{\partial \log e}{\partial \log e^X} \frac{\partial \sum_{i \in X} b_{Xi}}{\partial \log p_k} \right] \\ &= \frac{1}{b_X} \frac{\partial^2 \log e}{(\partial \log e^X)^2} b_{Xk}, \end{aligned}$$

where the last line uses the fact that $\frac{\partial \sum_{i \in X} b_{Xi}}{\partial \log p_k} = 0$. Using the following relationship

$$\frac{\partial^2 \log e}{(\partial \log e^X)^2} = b_X \frac{\partial \log b_X}{\partial \log e^X} = b_X (1 - b_X) (1 - \sigma(\mathbf{p}, u)),$$

the compensated change in expenditures on X in response to a change in the price of $k \in X$ is given by

$$\frac{\partial \log b_X}{\partial \log p_k} = (1 - b_X) (1 - \sigma(\mathbf{p}, u)) b_{Xk}.$$

The following identity links the uncompensated and compensated budget share of X

goods:

$$B_X(p, e(p, u)) = b_X(p, u).$$

Differentiating both sides of this identity with respect to the price of some good $k \in X$ yields

$$\begin{aligned} \frac{\partial \log B_X}{\partial \log p_k} &= \frac{\partial \log b_X}{\partial \log p_k} - \frac{\partial \log B_X}{\partial \log I} \frac{\partial \log e}{\partial \log p_k}, \\ &= \frac{\partial \log b_X}{\partial \log p_k} - \frac{\partial \log B_X}{\partial \log I} b_k, \\ &= \frac{\partial \log b_X}{\partial \log p_k} - \sum_{i \in X} b_{Xi} \frac{\partial \log b_i}{\partial \log I} b_k, \\ &= (1 - b_X)(1 - \sigma) b_{Xk} - b_X b_{Xk} \sum_{i \in X} b_{Xi} (\eta_i - 1), \end{aligned}$$

where we use the fact that $\partial \log e / \partial \log p_k = b_k$. Summing over all $k \in X$, we get

$$\sum_{k \in X} \frac{\partial \log B_X}{\partial \log p_k} d \log p_k = \left[(1 - b_X)(1 - \sigma) - b_X \sum_{i \in X} b_{Xi} (\eta_i - 1) \right] \left(\sum_{k \in X} b_{Xk} d \log p_k \right).$$

In other words,

$$\sum_{k \in X} \frac{\partial \log B_X}{\partial \log p_k} d \log p_k = \epsilon_X d \log p_X,$$

where $d \log p_X = \sum_{k \in X} b_{Xk} d \log p_k$ and $\epsilon_X = (1 - b_X)(1 - \sigma(p, u)) - b_X \sum_{i \in X} (\eta_i - 1) b_{Xi}$. Rearranging this for $\sigma(p, u)$ yields the desired result

$$\sigma(p, u) = 1 - \frac{\epsilon_X + b_X \sum_{i \in X} (\eta_i - 1) b_{Xi}}{1 - b_X}.$$

That is,

$$\sigma(p, u(I, t)) = 1 - \frac{\epsilon_X(I, t) + B_X(I, t) \sum_{i \in X} (\eta_i(I, t) - 1) B_{Xi}(I, t)}{1 - B_X(I, t)}.$$

■

A.2 Additional Results

Proposition A.1 (Existence and Uniqueness). *Consider the integral equation*

$$u(I, t) = \log I - \int_{t_0}^t \sum_i b_i(s, u(I, t)) \frac{d \log p_i}{ds} ds.$$

Suppose that $b_i, \partial b_i / \partial u$, and p_i are smooth functions. Then the integral equation has a unique solution in some closed interval $[t_0, t_0 + h]$ where $h > 0$. Furthermore, the iterations defined by

$$u_{n+1}(I, t) = \log I - \int_{t_0}^t \sum_i b_i(s, u_n(I, t)) \frac{d \log p_i}{ds} ds$$

produces a sequence that converges uniformly to this solution on $[t_0, t_0 + h]$.

Before showing the proof, we note that local uniqueness implies global uniqueness. Suppose there exist two solutions to the integral equation $u(I, t)$ and $v(I, t)$. Pick the largest s such that $u(I, s) = v(I, s)$ for some s . Such an s must exist since $u(I, t_0) = v(I, t_0) = I$. We then apply Proposition A.1 starting at s , and conclude that $u(I, t + h) = v(I, s + h)$ for some $h > 0$. By transfinite induction, $u(I, t) = v(I, t)$ for all t and for every I .

Proof. To prove uniqueness, we use the contraction mapping theorem. We begin by showing that there exists a sufficiently small compact set, around the boundary condition, over which the integral equation is a continuous self-map. We then show that this self-map is a contraction mapping if the compact set is sufficiently small. This shows local uniqueness inside that set. Using the argument above, we can extend this to global uniqueness.

Part (i): To begin, adopt the infinity norm, and define the operator:

$$T(v(I, t)) = \log I - \int_{t_0}^t \sum_i b_i(s, v(I, t)) \frac{d \log p_i}{ds} ds.$$

Choose h_1 and α_1 such that

$$R_1 = \{(t, y) : |t - t_0| \leq h_1, |y - I| \leq \alpha_1\}.$$

It follows that $b_i, \partial b_i / \partial u$, and p_i all attain their supremum on R_1 . It follows that there exist $M > 0$ and $L > 0$ such that

$$\forall (t, y) \in R_1, \sum_i \left| b_i \frac{d \log p_i}{ds} \right| \leq M \text{ and } \left| \frac{\partial b_i}{\partial u} \frac{d \log p_i}{ds} \right| \leq L.$$

Let g be a continuous function on R_1 satisfying $g(t, I) \leq \alpha_1$ for all $(t, I) \in R_1$. Then

$$\begin{aligned} |T(g(I, t)) - \log I| &= \left| \int_{t_0}^t \sum_i b_i(s, g(I, t)) \frac{d \log p_i}{ds} ds \right| \\ &\leq \int_{t_0}^t \sum_i \left| b_i(s, g(I, t)) \frac{d \log p_i}{ds} \right| ds \\ &\leq M|t - t_0|. \end{aligned}$$

Choose h such that $0 < h < \min\{h_1, \frac{\alpha_1}{M}, \frac{1}{L}\}$. Hence

$$|T(g(I, t)) - \log I| \leq \alpha_1.$$

Hence, for the set

$$S = \{g \in C([t_0, t_0 + h]) : \|g - \log I\| \leq \alpha_1\},$$

the operator T is a self-map of continuous functions satisfying $g(t, I) \leq \alpha_1$ over R_1 .

Part (ii): Now we show that T is a contraction mapping.

$$\begin{aligned} |T(v(I, t)) - T(u(I, t))| &= \left| \int_{t_0}^t \sum_i [b_i(s, v(I, t)) - b_i(s, u(I, t))] \frac{d \log p_i}{ds} ds \right| \\ &\leq \int_{t_0}^t \sum_i \left| [b_i(s, v(I, t)) - b_i(s, u(I, t))] \frac{d \log p_i}{ds} \right| ds. \end{aligned}$$

By the mean value theorem, there exists $\tilde{u}(I, t) \in [v(I, t), u(I, t)]$ such that

$$\begin{aligned} |T(v(I, t)) - T(u(I, t))| &\leq \int_{t_0}^t \sum_i \left| \frac{\partial b_i(s, \tilde{u}(I, t))}{\partial u} (u(I, t) - v(I, t)) \frac{d \log p_i}{ds} \right| ds \\ &\leq \int_{t_0}^t \sum_i L |u(I, t) - v(I, t)| ds \\ &\leq \sum_i L |u(I, t) - v(I, t)| |t - t_0| \\ &= \kappa |u(I, t) - v(I, t)| \end{aligned}$$

where $\kappa = \sum_i L |t - t_0|$. This holds if we choose $h < 1/LN$, so we have $\sum_i L |t - t_0| < hNL < 1$. Hence, T is a contraction mapping and we can apply the contraction mapping theorem. ■

B Additional Figures

Figure A.1: Exogenous price and income paths for the artificial example in Figure 2.

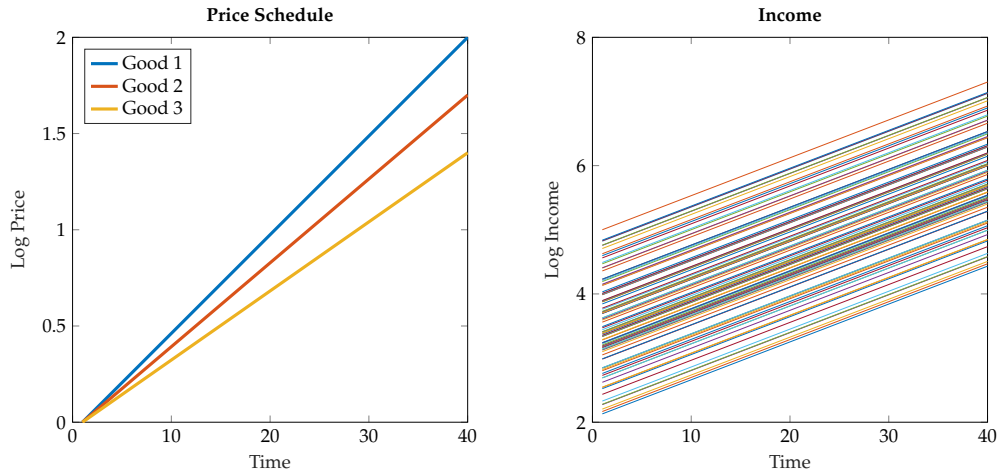


Figure A.2: Money metric $e(p_{1974}, v(p_{2017}, I_{2017}))$ by household characteristic (annualized pounds, log scale) for the UK data in Section 4.

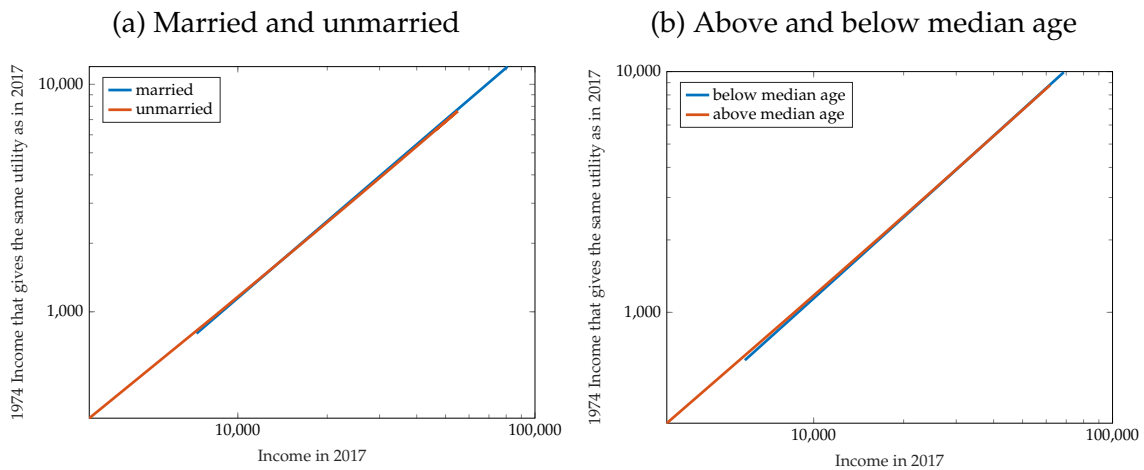
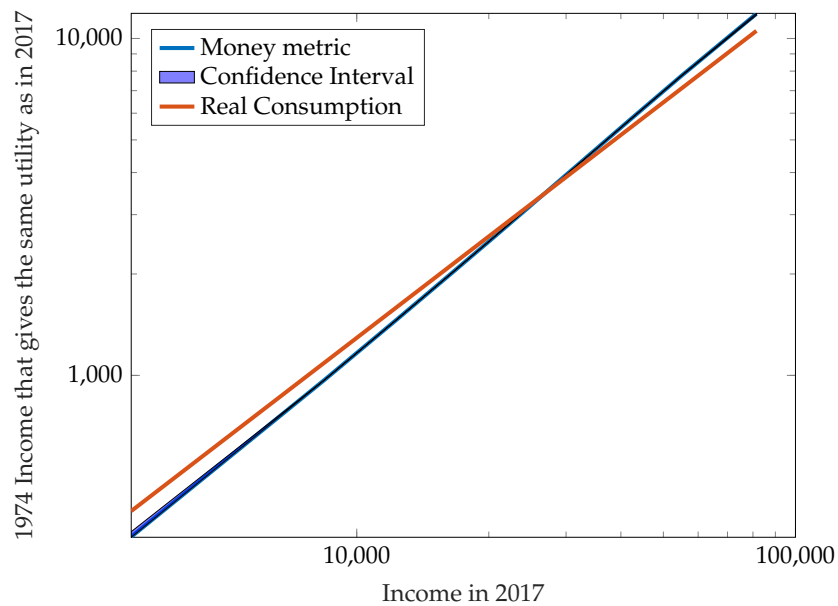


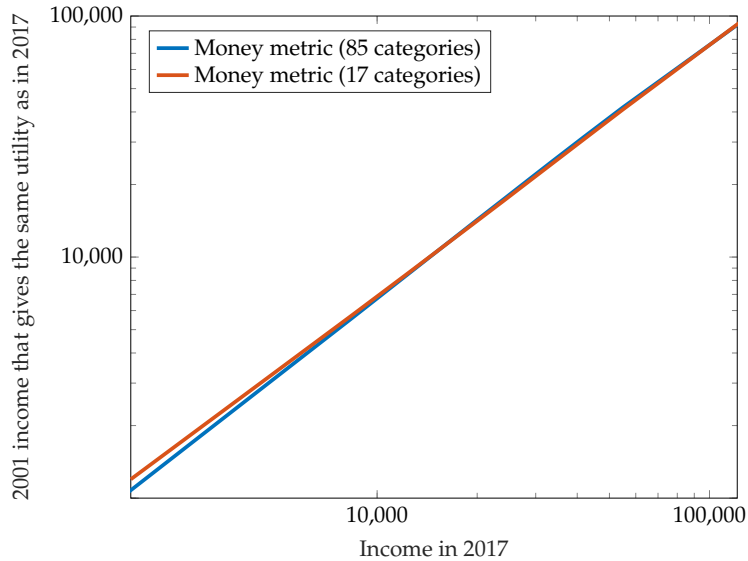
Figure A.3: Money metric $e(p_{1974}, v(p_{2017}, I))$ and real consumption as a function of I in 2017 using LOWESS



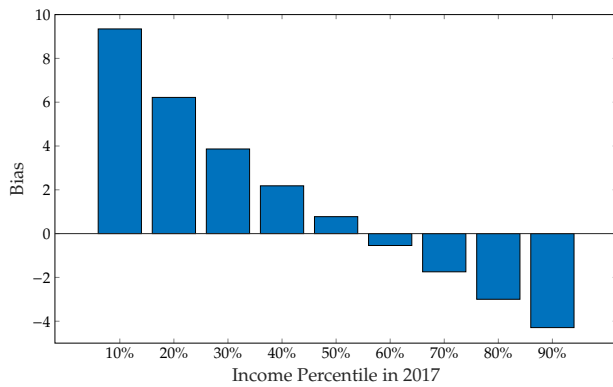
Notes: This figure is calculated using the recursive solution method rather than the iterative one. The 95% confidence intervals are bootstrapped using 500 draws with replacement.

Figure A.4: Results using more disaggregated spending categories

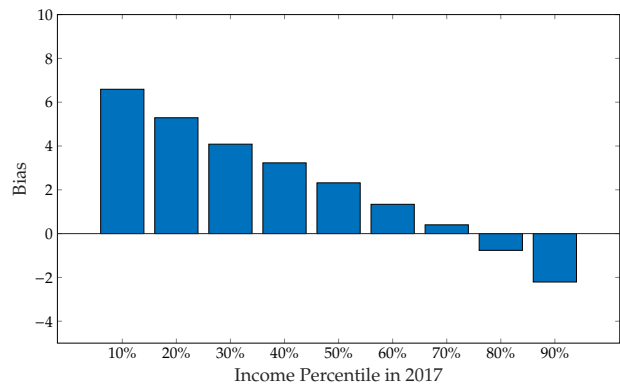
(a) Comparison of $e(p_{2001}, v(p_{2017}, I))$ computed using 17 and 85 spending categories.



(b) Log difference between chain-weighted inflation and true cost-of-living inflation using 85 spending categories.



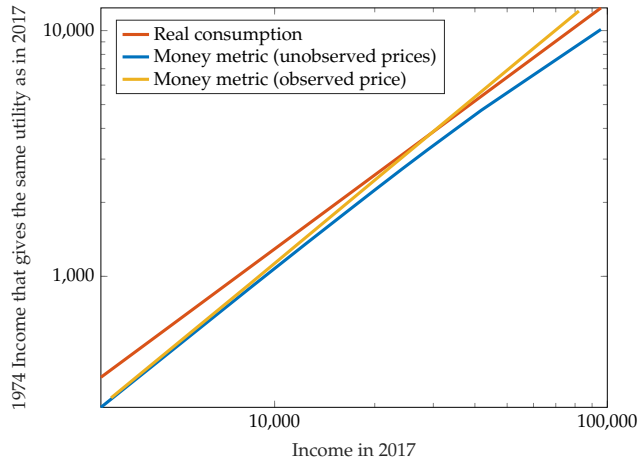
(c) Log difference between chain-weighted inflation and true cost-of-living inflation using 17 spending categories.



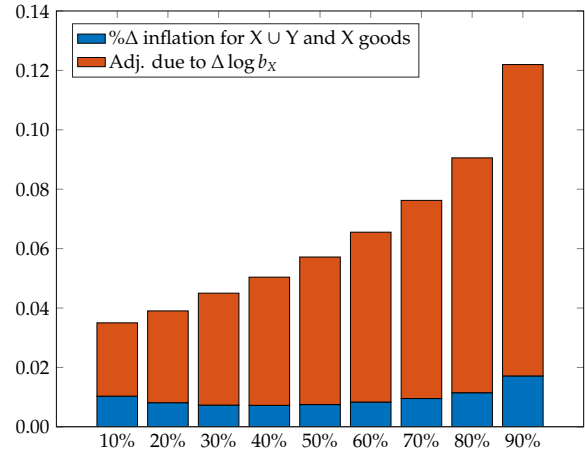
Notes: Figure A.4 uses the restricted sample from 2001 – 2017 using CPI price data.

Figure A.5: Replication of Section 5 using a constant σ and the IV estimates.

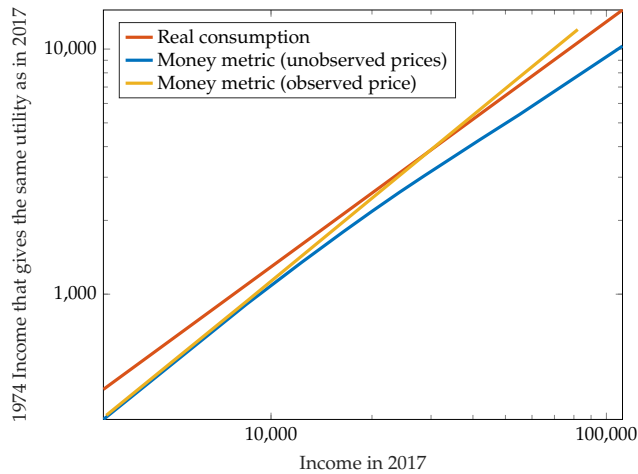
(a) Money metric $e(p_{1974}, v(p_{2017}, I))$ and real consumption as a function of I in 2017 assuming $\sigma = 0.5$.



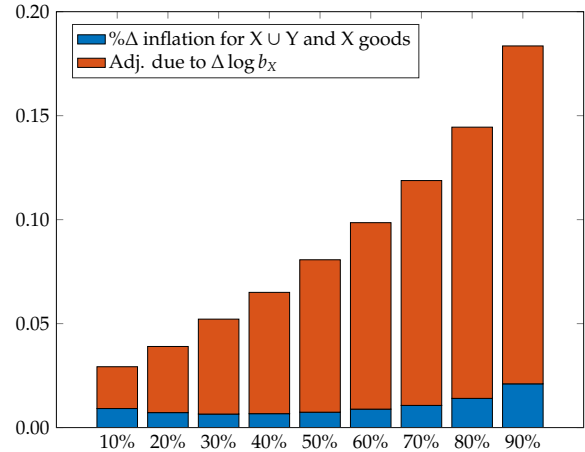
(b) Percent difference in money-metric values with observed and unobserved prices for different percentiles of the I distribution assuming $\sigma = 0.5$



(c) Money metric $e(p_{1974}, v(p_{2017}, I))$ and real consumption as a function of I in 2017 using IV estimates.



(d) Percent difference in money-metric values with observed and unobserved prices for different percentiles of the I distribution using IV estimates.



C Additional details of the UK data used in Section 4

We use two different datasets. One is a household-level expenditure survey and the other is data on prices of different categories of goods. The first data set is *Family Expenditure Survey and Living Costs and Food Survey Derived Variables*, which is a dataset of annual household expenditures with demographic information compiled from various household surveys conducted in the UK. Each sample includes about 5,000-7,000 households. The spending categories in the survey correspond to RPI (Retail Price Index) categories. We have continuous data from 1974 to 2017. Starting in 1995, the data are split into separate files for adults and children, so we merge them into households by adding up their expenditures.

Our algorithm does not require a representative sampling of the entire distribution of households, and can recover the money metric for a subsample of observed households, even if that subsample does not sample incomes at the same frequency as the population. The expenditure survey samples from the entire income distribution except for top earners and some pensioners. In order to correct for possible nonresponse bias, household weights are provided since 1997.³³ We use these weights to calculate the chained aggregate price index, which we use to calculate real consumption as in the official statistics. However, our approach for the money metric does not use household weights.

For the prices, we use the underlying data for the consumer price index (CPI) and the retail price index (RPI). To construct the consumption deflator in the national accounts, the Office of National Statistics switched from the Retail Price Index (RPI) to the Consumer Price Index (CPI).³⁴ By comparing the RPI and CPI with the consumption deflator provided by the Office of National Statistics, we identify the switching point as 1998 and do the same for our price data.

Because the CPI and RPI consider different baskets of goods and services, we merged various sub-categories to obtain a consistent set of categories over time. For example, “alcohol” in the RPI includes some items served outdoors, which is included in “restaurants” in the CPI. In this case, we merged “Catering and Alcohol” in the RPI and matched it with “Restaurant and Alcohol” in the CPI. We end up with 17 categories that are available for the entire period for both RPI and CPI. Table A.2 summarizes how we integrated the CPI

³³Prior to 1997, benefit unit weights are provided instead of household weights. Since a benefit unit is a single person or a couple with any dependent children, there can be more than one benefit unit weight in a household. For example, if a couple with their children and the father’s parents live together, then two benefit unit weights are recorded. In this case, we use the simple average as the household weight.

³⁴https://webarchive.nationalarchives.gov.uk/ukgwa/20151014001957mp_/http://www.ons.gov.uk/ons/guide-method/user-guidance/prices/cpi-and-rpi/mini-triennial-review-of-the-consumer-prices-index-and-retail-prices-index.pdf.

and RPI baskets.

Figure A.6: Comparison of aggregate annual inflation reported by the UK Office of National Statistics and aggregate inflation calculated in our dataset following the same methodology.

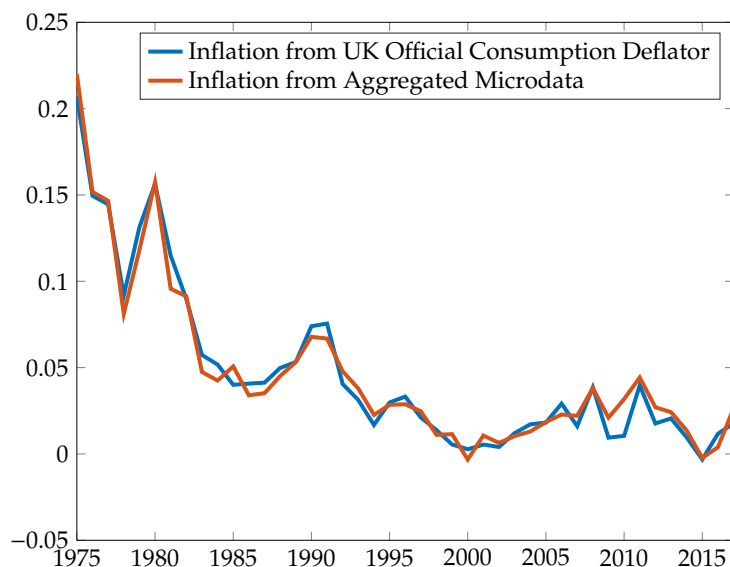


Table A.1: Comparison of ONS and our microdata.

	Decile								Difference
	2	3	4	5	6	7	8	9	D9-D2
ONS	2.8%	2.7%	2.6%	2.5%	2.4%	2.4%	2.3%	2.3%	-0.5%
Microdata	2.6%	2.6%	2.5%	2.4%	2.4%	2.3%	2.2%	2.1%	-0.5%

Notes: We report average annual inflation 2005-2017, in percentages. The ONS data is from Table 9 of “Data tables for the CPI consistent inflation rate estimates for UK household groups” Release date: 15th February 2023. We do not compare the 1st and 10th decile since those deciles are sensitive to how the tails of the distribution are treated. The last column is the difference between the ninth and second deciles.

Figure A.6 shows that our aggregated microdata closely matches the official consumption price deflator series for the UK. Table A.1 compares average chain-weighted inflation by expenditure decile reported by the ONS to similar statistics calculated using our microdata. We do not compare the 1st and 10th decile since those deciles are sensitive to how the tails of the distribution are treated. Once again, our microdata matches the official rates reasonably closely.

Integrated Categories	RPI	CPI
Bread & Cereals	Bread, Cereals and Biscuits	Bread & cereals
Meat & Fish	Meat, Fish, Beef, Lamb and Pork	Meat & fish
	Poultry and Other meat	-
Milk & Eggs	Butter, Cheese and Eggs	Milk, cheese & eggs
	Fresh milk and Milk products	-
Oils & fats	Oils & fats	Oils & fats
Fruit	Fruit	Fruit
Vegetable	Potatoes and Other vegetables	Vegetables including potatoes & other tubers
Other food	Sweets & Chocolates	Food Products
	Other Foods	Sugar, jam, honey, syrups, chocolate & confectionery
Non-Alcoholic Beverages	Tea and Soft drinks	Non-Alcoholic Beverages
	Coffee & other hot drinks	
Tobacco	Cigarettes & tobacco	Tobacco
Catering	Catering	Catering services
	Alcoholic drink	Alcoholic beverage
Household & Fuel	Housing except mortgage interest	Housing, water and fuels
	Fuel & light	
	(-)Dwelling insurance & ground rent	
Clothing	Clothing & footwear	Clothing & footwear
Household Goods	Household goods	Furniture and household equipment & routine repair of house
	domestic services	
Postage & Telecom	Postage	Communication
	Telephones & Telemessages	
Personal Goods	Personal goods & services	Health
	Fees & subscriptions	Miscellaneous goods and service
	Dwelling insurance & ground rent	-
Transport	Motoring expenditure	Transport
	Fares & other travel costs	-
Leisure Goods & Service	Leisure goods	Recreation & culture
	Leisure services	Education
	-	Accommodation service

Table A.2: RPI and CPI Correspondence Table

D Comparison with Blundell et al. (2003)

In this appendix, we exposit and apply the welfare bounds in Blundell et al. (2003) to artificial and real data. We start by discussing how we implement their methodology, since, due to an inconsistency in their equations, we do not exactly implement their procedure.

D.1 Description of Bounding Algorithm

To bound the cost-of-living, Blundell et al. (2003) provide an algorithm for an upper-bound and a lower-bound. Following the notation in their paper, let $q_t(I)$ be bundle of goods consumed by a household with income I in period t . Blundell et al. (2003) assume that $q_t(I)$ is an injective function (each I maps to a unique bundle of quantities in each period).

Algorithm A (Upper-bound). To recover an upper-bound for $e(p_s, v(p_t, I_t))$, start by defining $q^* = q_t(I_t)$ and let T be the set of periods for which we have data.

- (1) Set $i = 0$ and $F^{(i)} = \{q_s^i = q_s(p_s \cdot q^*)\}_{s \in T}$.
- (2) Set $F^{(i+1)} = \{q_s^{i+1} = q_s(\min_{q \in F^{(i)}} p_s \cdot q)\}_{s \in T}$.
- (3) If $F^{(i+1)} = F^{(i)}$, then set $Q_B(q^*) = F^{(i)}$ and stop. Else set $i = i + 1$ and go to step (2).

We have that $e(p_s, v(p_t, I_t)) \leq \min_q \{p_s \cdot q : q \in Q_B(q^*)\}$.

Intuitively, the cost of living in period s associated with q^* , $e(p_s, v(p_t, I_t))$, is weakly less than $p_s \cdot q^*$. Hence, for every s , we must have that $q_s^0 = q_s(p_s \cdot q^*)$ is weakly preferred to q^* . This collection of bundles, $\{q_s^0\}_{s \in T}$, all of which are preferred to q^* , is $F^{(0)}$ defined in step (1). In step (2), we search across all of these bundles to find the cheapest one in each period s . We update each q_s^i to be the bundle that households with that level of income actually picked in each period (which is still better than q^*). We continue this indefinitely until this procedure converges, at which point we have our upper-bound.

As mentioned in the text, the lower-bound algorithm provided by Blundell et al. (2003) is not correct. We provide an amended version below.

Amended Algorithm B (Lower-bound). To recover a lower-bound for $e(p_s, v(p_t, I_t))$, start by defining $q^* = q_t(I_t)$ and let T be the set of periods for which we have data.

- (1) Set $i = 0$, and let $F^{(i)} = \{I_s^i : p_t \cdot q_s(I_s^i) = I_t\}_{s \in T}$.

(2) Set $F^{(i+1)} = \{\max_{I_k \in F^{(i)}} \{I_s^{i+1} : I_k = p_k \cdot q_s(I_s^{i+1})\}\}_{s \in T}$.

(3) If $F^{(i+1)} = F^{(i)}$, then set $Q_W(q^*) = \{q_s(I_s^i)\}_{s \in T}$ and stop. Else set $i = i + 1$ and go to step (2).

We have that $\max_{q_s \in Q_W(q^*)} p_s \cdot q_s \leq e(p_s, v(p_t, I_t))$.

Intuitively, in step (1), for each period s , we find the income level I_s^0 such that $p_t \cdot q_s(I_s^0) = I_t$. The bundle $q_s(I_s^0)$ was affordable at t but was not purchased. Hence, the true cost-of-living in period s must be greater than I_s^0 . The collection of income levels constructed in this step is $F^{(0)}$ and all are less than the true cost-of-living. In step (2), for each period s , we search over I_k^i and find the maximum level of income I_s^{i+1} such that $I_k^i = p_k \cdot q_s(I_s^{i+1})$ is satisfied. The new I_s^{i+1} is weakly greater than I_s^i but we still know that I_s^{i+1} is less than the true cost-of-living. We continue this indefinitely until this procedure converges, at which point we have our lower-bound.

D.2 Results with UK Data

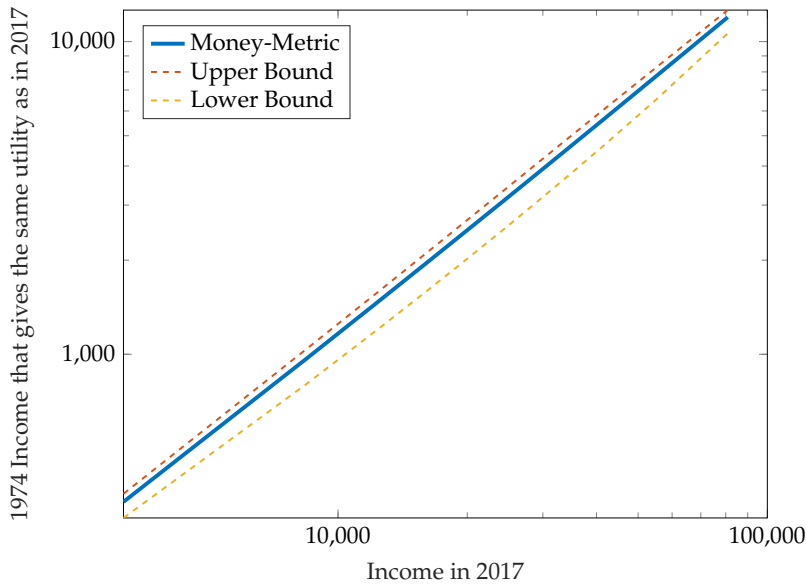


Figure A.7: Upper- and lower-bound using the amended Blundell et al. (2003) algorithm for the UK data in Section 4. Our algorithm produced the blue line.

E Comparison with Jaravel & Lashkari (2022)

In this appendix, we apply the first-order and second-order algorithms described in Jaravel and Lashkari (2022) (JL) to some artificial examples and compare the performance with our method.³⁵ We start with the example in Section 3.3, where both methods perform well. We then provide others examples where the errors in their methodology are very large. These examples are selected to contrast the mathematical properties of our two methodologies when the support of the cross-sectional distribution of utilities changes over time.

We compute the errors for each method relative to the truth for the entire range over which each method produces estimates. We do this because identifying the set of households over which the money metric can be reliably estimated (without extrapolation) is a contribution of our methodology. The JL method purports to estimate the money metric for all households in the sample and does not provide a way to know if they are performing out-of-sample extrapolations, so we calculate the error accordingly.

Table A.3 shows that both methodologies perform very well for the simple example in Section 3.3, even though the support of the cross-section distribution of utilities is not constant over time. However, if we change parameter values, then the two methods can perform very differently.

³⁵By setting the base year in the Jaravel and Lashkari (2022) algorithm to t_0 , their definition of *real consumption* (which differs from our definition of real consumption) matches our money metric. Our method only requires repeated cross-sections. However, the second-order JL method requires a panel to construct household-specific inflation indexes. Therefore, to apply their method we create panels by the most disaggregated income quantile possible (i.e. if we have N households per period, then we form panels based on income N-quantiles). Finally, for the polynomial fitting stage of the Jaravel and Lashkari (2022) method, we use Matlab's polyfit function because it gives lower errors than a naive OLS regression.

Table A.3: Comparison of errors for simple example in Section 3.3

Jaravel and Lashkari (2022) method:				
K	Infinity Norm		Root Mean Square Error	
	First Order	Second Order	First Order	Second Order
1	0.03	0.03	8.7×10^{-3}	8.3×10^{-3}
2	0.02	0.02	1.6×10^{-3}	1.4×10^{-3}
4	0.01	0.01	1.2×10^{-3}	1.0×10^{-3}
6	7.8×10^{-4}	4.0×10^{-4}	6.4×10^{-4}	1.5×10^{-4}
8	3.6×10^{-3}	4.4×10^{-3}	6.6×10^{-4}	2.5×10^{-4}
12	1.1×10^{-3}	7.0×10^{-4}	6.4×10^{-4}	1.5×10^{-4}

Baqae, Burstein, Koike-Mori method:				
	Infinity Norm		Root Mean Square Error	
	Iterative	Recursive	Iterative	Recursive
	7.9×10^{-3}	1.5×10^{-4}	5.0×10^{-3}	7.7×10^{-6}

Notes: The Jaravel and Lashkari (2022) methodology is applied to the artificial example in Section 3.3. We report two different norms (infinity norm and root mean square error) with respect to the absolute value of the log difference between the true money metric and the estimate in the final period. The first column is their “first-order” algorithm and the second column is their “second-order” algorithm. The parameter K is the order of the polynomial used. The sample has 1000 households and annual data.

One example is provided in Table A.4. Our method, which tracks the boundary of overlapping support, does not produce any numbers for this example because there is no overlap in the support of the utility distribution between t_0 and T . However, the Jaravel and Lashkari (2022) algorithm does produce estimates and they are very inaccurate. Importantly, the Jaravel and Lashkari (2022) methodology does not provide a way to know whether their estimates are reliable (like in Table A.3) or unreliable like in (Table A.4). On the other hand, our methodology does not produce estimates that are not guaranteed to be reliable (given our assumptions).

Table A.4: Errors in Jaravel and Lashkari (2022) method with different parameters

K	Infinity Norm		Root Mean Square Error	
	First Order	Second Order	First Order	Second Order
1	0.27	0.26	0.25	0.23
2	0.47	0.41	0.44	0.37
4	0.38	0.35	0.34	0.31
6	1.5×10^{104}	Not converged	4.6×10^{102}	Not converged
8	1.08	1.09	0.97	0.99
12	Polyfit error	Polyfit error	Polyfit error	Polyfit error

Notes: This table shows the accuracy of JL algorithm for each different K (polynomial degree). We report two different norms with respect to the absolute value of the log difference between the true money metric and the estimate in the final period. The expenditure function is $e(p, U) = \left(\sum_i \omega_i U^{\varepsilon_i(1-\gamma)} p_i^{1-\gamma} \right)^{1/(1-\gamma)}$ where $(\gamma, \varepsilon_1, \varepsilon_2, \varepsilon_3) = (5, 0.3, 1, 2)$ and ω is all 1. There are 1000 households uniformly distributed in the income distribution over $[1, 1.1]$. Average nominal income is the numeraire and the income distribution does not change over time. There are 40 periods and the price of the three goods rise (relative to income) at a constant rate from $(1, 1, 1)$ to $(2, 3, 4)$. Doubling the number of households and frequency of observation does not appreciably change the results in this table. If Matlab fails to find a unique polynomial due to (numerical) multi-collinearity, we report write "Polyfit error." Although we do not report the numbers, the errors in these cases are large.

In Table A.4, there is no overlapping support, so our method produces no estimates. In the next example, the distribution of money metric values in the final period is, by construction, a subset of the one in the initial period. This means that our method produces estimates for every household in the sample. That is, we compare the performance of our method to JL for the same set of households (since all households in the final period are in a region of overlapping support). The results are reported in Table A.5

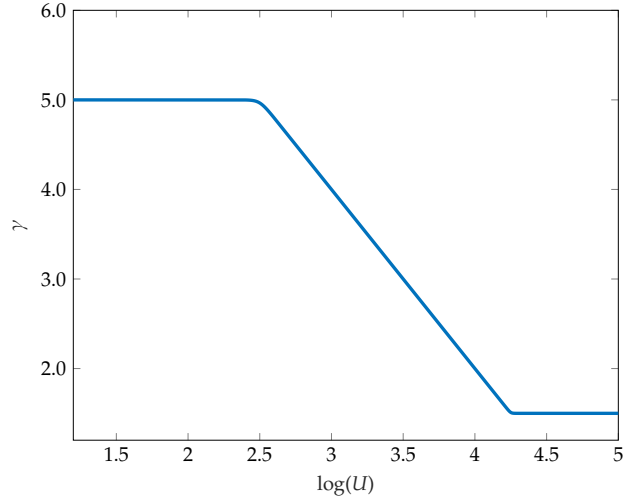
Table A.5: Comparison of errors for non-homothetic CES example with different parameters

Jaravel and Lashkari (2022) method:				
K	Infinity Norm		Root Mean Square Error	
	First Order	Second Order	First Order	Second Order
1	0.15	0.15	0.08	0.07
2	0.17	0.12	0.05	0.05
4	1.6×10^{16}	Not converged	6.1×10^{14}	Not converged
6	7.9×10^4	Not converged	2.6×10^3	Not converged
8	NaN	Polyfit error	NaN	Polyfit error
12	Polyfit error	Polyfit error	Polyfit error	Polyfit error

Baqae, Burstein, Koike-Mori method:				
	Infinity Norm		Root Mean Square Error	
	Iterative	Recursive	Iterative	Recursive
	1.4×10^{-3}	2.0×10^{-6}	1.2×10^{-3}	8.5×10^{-7}

Notes: Accuracy of Jaravel and Lashkari (2022) algorithm for different K (polynomial degree) and our method. The errors are computed for the same set of households. We report two different norms with respect to the absolute value of the log difference between the true money metric and the estimate in the final period. The expenditure function is $e(p, U) = \left(\sum_i \omega_i U^{(1-\gamma)\varepsilon_i} p_i^{1-\gamma} \right)^{1/(1-\gamma)}$ where $(\gamma, \varepsilon_1, \varepsilon_2, \varepsilon_3) = (5, 1.6, 2, 3.3)$ and $\omega = (1, 1, 1)$. There are 1000 households equally distributed in the income distribution and 100 periods. The initial income distribution is $[0.8, 1.4]$. Between period 1 and 50, the income distribution uniformly and linearly changes to $[0.003, 34.4]$. Between period 51 and 75, the income distribution uniformly and linearly changes to $[0.5, 8.2]$. Between period 76 and 100, the income distribution uniformly and linearly changes to $[2.7, 3.1]$. The price vector changes from $(1, 1, 1)$ to $(2, 3, 4)$. If the second-order algorithm does not converge, we write "Not converged." If the estimated values diverge, we write "NaN." If Matlab fails to find a unique polynomial (due to numerical multi-collinearity), we write "Polyfit error." Although we do not report the numbers, the errors in these cases are large. Results are similar for higher order polynomials, if we double the number of households, or double the frequency of observations.

Figure A.8: The elasticity of substitution as a function of utility for the example in Table A.6



Our final example uses a more nonlinear demand system. Let preferences be defined by

$$e(p, U) = \left(\sum_i \omega_i (U^{\varepsilon_i} p_i)^{1-\gamma(U)} \right)^{1/(1-\gamma(U))},$$

where we allow the elasticity of substitution γ to depend on utility, as in Fally (2022). To keep the preferences well behaved, we constrain the elasticity of substitution to be between a lower- and upper-bound value. For example, the most straightforward way to do this is to set

$$\gamma(U) = \max \left\{ \min \left\{ \underline{\gamma}, \gamma_0 - \eta \log U \right\}, \bar{\gamma} \right\}. \quad (20)$$

The Jaravel and Lashkari (2022) propositions require smoothness, so we instead use the following functional form

$$\gamma(U) = \left(\bar{\gamma}^{\chi_1-1} + \left(\left[\underline{\gamma}^{\frac{\chi_2-1}{\chi_2}} + (\gamma_0 - \eta \log(U))^{\frac{\chi_2-1}{\chi_2}} \right]^{\frac{\chi_2}{\chi_2-1}} \right)^{\chi_1-1} \right)^{\frac{1}{\chi_1-1}}, \quad (21)$$

where we set $\chi_1 = 100$ and $\chi_2 = 0.01$. This function is plotted in Figure A.8 and smoothly approximates the maximum and minimum functions. In practice, the errors are similarly large whether we use (20) or (21).

We simulate artificial data using this demand system and report the results in Table A.6. The Jaravel and Lashkari (2022) methodology has substantially larger errors and does not seem to converge as we increase the number of parameters in the polynomial

approximation. Our methodology, in contrast, produces very small errors.

Table A.6: Comparison of Jaravel and Lashkari (2022) and Baqae, Burstein, Koike-Mori errors for more complex example

Jaravel and Lashkari (2022) method:				
K	Infinity Norm		Root Mean Square Error	
	First Order	Second Order	First Order	Second Order
1	0.17	0.17	0.11	0.10
2	0.25	0.25	0.16	0.15
4	14	Not converged	0.53	Not converged
6	1.3×10^{205}	Not converged	4.1×10^{203}	Not converged
8	2.2×10^{73}	Polyfit error	7.0×10^{71}	Polyfit error
12	Polyfit error	Polyfit error	Polyfit error	Polyfit error

Baqae, Burstein, Koike-Mori method:			
Infinity Norm		Root Mean Square Error	
Iterative	Recursive	Iterative	Recursive
7.6×10^{-3}	1.7×10^{-5}	4.6×10^{-3}	1.1×10^{-5}

Notes: The parameter K indicates the polynomial degree. The artificial data are generated with $\gamma_0 = 10$, $\underline{\gamma} = 1.5$ $\bar{\gamma} = 5$ and $\eta = 2$. The income distribution starts as a uniform distribution between $[2, 50]$ and grows uniformly by a factor of 14 over 40 periods. The price vector changes from $(1, 1, 1)$ to $(7, 5, 3)$. If the second-order algorithm does not converge, we write “Not converged.” If Matlab fails to find a unique polynomial (due to numerical multi-collinearity), we write “Polyfit error.” Although we do not report the numbers, the errors in these cases are large.