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MEASURING UPWARD MOBILITY

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ABSTRACT

We develop a measure of upward mobility that distills central features of the relative and absolute approaches to measuring mobility. The former is embodied in the Growth Progressivity axiom: transfers of instantaneous growth rates from relatively rich to poor individuals increases upward mobility. The latter is embodied in the Growth Alignment axiom: mobility increases with higher growth for all individuals. These axioms, along with standard auxiliary restrictions, identify a simple one-parameter family of upward mobility measures, linear in individual growth rates and exhibiting geometrically declining weights on baseline incomes. A serendipitous implication of our measure is that it does not rely on panel data, which greatly expands our analytical scope to data-poor settings.

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Abstract. We develop a measure of upward mobility that distills central features of the relative and absolute approaches to measuring mobility. The former is embodied in the *Growth Progressivity axiom*: transfers of instantaneous growth *rates* from relatively rich to poor individuals increases upward mobility. The latter is embodied in the *Growth Alignment axiom*: mobility increases with higher growth for all individuals. These axioms, along with standard auxiliary restrictions, identify a simple one-parameter family of upward mobility measures, linear in individual growth rates and exhibiting geometrically declining weights on baseline incomes. A serendipitous implication of our measure is that it does not rely on panel data, which greatly expands our analytical scope to data-poor settings.

JEL Classification Numbers: [Poverty and Inequality] I32, D63, O15

Keywords: income mobility, upward mobility, pro-poor growth

1. Introduction

Social mobility refers to the ease of transition between various socioeconomic categories. To the extent that those categories (such as income or wealth) are vertically ranked, mobility is linked to *directed* movement, upward or downward. Economic growth is related to mobility in this sense, without necessarily being identical to it. Additionally, we might say that mobility is higher if the relatively worse-off enjoy greater upward movement. Economic *equality* is connected to mobility in this sense, without necessarily being identical to it.

These observations connecting mobility, growth and equality lead to a view that mobility is related to *pro poor growth*, a concept that aggregates growth across individuals or groups, but weighted by their economic characteristics. (See Section 2 for relevant literature.) We follow this line of thinking and call our notion *upward* mobility. It is to be contrasted with the distinct concept of mobility as "pure movement," which allows mobility to increase with sheer volatility across categories, without imposing or accounting for any ranking on those categories.

We begin with mobility at an "instant of time," though our end goal is to build the discrete-time measure in equation (10) of Theorem 3. Our initial domain is a vector of individual observations, each consisting of a baseline income and an instantaneous growth rate of that income. We

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impose two key axioms: *Growth Progressivity* and *Growth Alignment*. The former states that a transfer of growth rates from richer to poorer individuals increases upward mobility. (The rates are instantaneous, so no income crossings occur when initial incomes are distinct, but of course crossings will be central to the main story.) By exploiting properties of multiaffine functions, we show that Growth Progressivity implies the linearity of upward mobility in individual growth rates, with weights that decline in baseline incomes (Theorem 1). This remarkable connection between Growth Progressivity and the *linearity* of the measure in growth rates is both of intrinsic interest and crucial to empirical implementation, as we shall see.

Growth Alignment states that mobility increases when all individuals experience higher growth. Along with some standard auxiliary restrictions, these two axioms further identify a simple one-parameter family of upward mobility measures: a weighted sum of instantaneous growth rates for each individual, with weights that geometrically decline in income and are indexed by a single parameter to mark the speed of that decline (Theorem 2).

Now we move to the domains that are typically available to the researcher: discrete income observations separated in time. Just what transpired in between we do not know. If the data form a panel, then we might know if any income crossings have occurred. Faced with such crossings, Growth Progressivity cannot be reasonably applied to growth between two points of time, simply based on initial incomes. Moreover, the problem is exacerbated by the fact that we don't know the exact path that connects the two observations. We might know that an income crossing has occurred, but there is no way to know *when* it occurred. Worse still, different interpolations of intermediate income trajectories may well yield different answers for mobility as a whole (with or without crossings), but those trajectories are not observed.

We deal with these issues by imposing two conditions. First, we ask that discrete upward mobility should be a functional defined on the path of all instantaneous upward mobilities as derived in Theorems 1 and 2; in short, the latter should form a "base" for the former. The base condition has the immediate implication that any such derived measure *does not require panel data for its implementation*; this argument is developed in Sections 4.1 and 4.2.

Our second condition imposes path independence. It asks that our discrete measure of mobility across two separated instants of time be independent of the precise intermediate paths that connect the initial and terminal points. This condition, while satisfying the demands of pragmatism (no intermediate data is available by definition), goes beyond pragmatism in that it is connected to the very meaning of upward mobility, which is a *directed* notion. Periods of upward mobility can be nullified by subsequent periods of downward mobility: the final outcome is all that matters for evaluation. We develop and defend this idea in Section 4.3. We note that such a condition eliminates entirely the notion of "mobility as pure movement." With mobility as pure movement, directions have no meaning; only movement does, so upward and downward movements do not cancel out and path independence fails. Theorem 3 combines the base condition, path independence and our earlier axioms to generate the main measure of mobility that we take to the data. This convenient discrete-time formula also has a welfarist interpretation as the annualized growth of Atkinson's equivalent income.

Finally, Section 6 represents an initial empirical exploration. Conceptually, our approach does not rely on panel data for its implementation. We therefore compare our measure to perhaps the most popular measure of directional mobility deployed empirically, which *does* rely on a panel. This is the share of families whose absolute fortune improved across generations. Chetty, Grusky, Hell, Hendren, Manduca and Narang (2017) is the leading study that uses this measure. They estimate the fraction of children who earn more than their parents for US birth cohorts from 1940 to 1984 and document a secular decline in this fraction. We implement our measure with the same data but do *not* use the transition copula of Chetty et al. (2017), which is estimated from a proprietary panel of tax records. Chetty et al. (2017) need the copula because their measure is panel-reliant. Ours is not, but tracks their measure very closely; see Figure 3.[2] With this empirical "proof of concept" in hand, we apply our measure of upward mobility to Brazil, India and France over 1970–2015, using *repeated cross-sectional data* from the World Inequality Lab. The empirical exercise that follows reveals new trends for these countries that may be of independent interest.

2. Related Literature

Different approaches have been taken to the measurement of mobility reflecting the variety of opinion on just what the term means (Fields 2010).

A large literature on mobility views mobility as relative and non-directional. It builds on the idea that transition probabilities or the copula of the intertemporal population distributions across (income or "status") categories are all we need to know to build a suitable measure, without distinguishing between gains and losses; see Prais (1955), Atkinson (1981), Bartholomew (1982), Conlisk (1974), Dardanoni (1993), Hart (1976), or Shorrocks (1978). This approach views mobility as the literal converse of immobility. In Shorrocks (1978), for instance, any increase in an off-diagonal entry of the transition matrix across incomes increases mobility. Other popular measures include the intergenerational income elasticity of progeny income with respect to parental income (Solon 1999, Jäntti and Jenkins 2015), and the rank-rank correlation or slope (Dahl and DeLeire 2008, Chetty, Hendren, Kline and Saez 2014a).

In the extreme form of this approach, mobility *is* movement. Good examples are the relative mobility measures of King (1983) and Chakravarty (1984) that measure the rerankings in the distribution, also called "exchange mobility" (see Dardanoni 1993 and Markandya 1982).

However, this view has been criticized for divesting itself of the ethical connotations that swirl around the concept of mobility (Fields and Ok 1999a; Jäntti and Jenkins 2015). We would all

[2] This exercise is closely in the spirit of Berman (2021), who also attempts to circumvent the panel structure, though entirely from an empirical perspective. See Section 6 for more discussion.

agree that gains are better than losses. When empirical studies emphasize a specific part of the transition matrix, say, the movement from bottom to upper ranks, they implicitly provide this welcome sense of direction (see, e.g., Bhattacharya and Mazumder 2011, Chetty, Hendren, Kline and Saez 2014a and Berman 2021). Contributions that embrace overall growth include Fields and Ok (1996) and Mitra and Ok (1998), but the main results in these papers use a measure of mobility that depends on the magnitude but not the direction of change in incomes. In this sense, these are also measures of "movement" rather than "upward mobility."

A different approach aims to measure mobility as an equalizer of aggregate income compared to either the initial or the per-period income (see Chakravarty, Dutta and Weymark 1985, Maasoumi and Zandvakili 1986, or Fields 2010). In these measures too, *overall* growth is typically normalized away, so the absolute aspect of overall income growth is removed.

Two well-known measures of mobility are both directional and absolute. Fields and Ok (1999b) propose "directional mobility measures" that sum individual income growth rates. More recently, the absolute mobility measure in Chetty, Grusky, Hell, Hendren, Manduca and Narang (2017) records the fraction of children who earn more than their parents. We discuss both measures in more detail in Section 5.4 below. For now, we note that both measures throw away information about who gains and who loses. In contrast, as already discussed, the relative mobility literature is sensitive to such matters.

In our twin emphases placed on both absolute and relative growth, we are closest to a literature on *pro-poor growth*. Chenery, Jolly, Ahluwalia, Bell and Duloy (1974) introduced an index of economic performance as a weighted sum of group growth rates. This led to a literature on pro-poor growth using a variety of weights decreasing with income (see Dardanoni 1993, Essama-Nssah 2005, and Ravallion and Chen 2003), or using growth incidence curves (Grimm 2007, Bourguignon 2011, Dhongde and Silber 2016, Palmisano and de Gaer 2016, Palmisano 2018). These authors observe that the difference between "anonymous" and "non-anonymous" versions of these curves correspond to pure exchange mobility. They also connected this literature on pro-poor growth with the literature on income convergence (see in particular O'Neill and Kerm 2008, Wodon and Yitzhaki 2005, Bourguignon 2011 and Dhongde and Silber 2016). Using non-anonymous growth incidence curves, indices of directional mobility have been proposed that place more weight on the growth rates of lower-ranked individuals (Jenkins and Kerm 2016, Palmisano and de Gaer 2016 and Berman and Bourguignon 2022).

Our analysis builds on these insights. As noted in the Introduction, we extract the notion of pro-poor growth as a partial order that any mobility index should satisfy, and show that (along with some other mild restrictions) that this axiom precipitates a measure of pro-poor growth that is linear in growth rates, with weights that decline geometrically in income. Our exercise therefore establishes the foundational principles of such measures. But more than that, it leads to a discrete-time measure which is shown at a conceptual level to apply to non-panel data.

3. Prelude: Instantaneous Upward Mobility

As an initial conceptual step, we develop a measure of “instantaneous" mobility that is:

1. *Growth-oriented*: it rewards growth, and punishes decay;

2. *Growth-progressive*: it rewards “growth transfers" from higher to lower incomes.

This forms the core of our analysis in Section 4, where we also argue that our measure is:

3. *Panel-independent*: it can be deployed on repeated cross-sections of data;

4. *Directional*: it does not value movement *per se*; downward moves negate upward moves.

3.1. Axiomatic Development. Each individual i is described by a pair $z_i = (y_i, g_i)$, where $y_i > 0$ is baseline income and g_i the instantaneous growth rate — positive or negative — of that income. Denote by $\mathbf{z} = \{z_i\}$ the collection of incomes and growth rates, including repetitions, for any finite population. An *instantaneous upward mobility index* is a continuous function $M(\mathbf{z})$, defined over all finite populations, and anonymous to permutations of indices within $\mathbf{z}$. Given anonymity, we order individuals so that $y_i \leq y_j$ for $j \geq i$, and if $y_i = y_j$, then $g_i \leq g_j$.

We place the following axioms on M, beginning with the declaration that zero growth for all individuals implies zero mobility:

1. **Zero Growth Anchoring.** If in two situations $\mathbf{z}$ and $\mathbf{z}'$, *every* individual has a zero growth rate, then $M(\mathbf{z}) = M(\mathbf{z}')$ and we normalize this common value to zero.

Our fundamental axiom that connects growth to mobility is

2. **Growth Progressivity.** For any $\mathbf{z}$ and ij-pair with $y_i < y_j$, and for $\epsilon > 0$, form $\mathbf{z}'$ by altering only g_i to $g_i + \epsilon$ and g_j to $g_j - \epsilon$. Then mobility goes up: $M(\mathbf{z}') > M(\mathbf{z})$.

This axiom imposes a substantive and defining restriction: it forces any instantaneous measure to reward higher growth to the relatively poor. Because growth rates are instantaneous, they do not result in an immediate “income crossing" of two individuals with distinct incomes. We will, of course, address income crossings in Section 4.

Despite the superficial resemblance to the well-known Transfers Principle for Lorenz comparisons in inequality measurement (Fields and Fei 1978), Growth Progressivity is a distinct (and new) notion that involves transfers of growth *rates*, not incomes. In particular, it implies that not all aggregate growth is welcome on the grounds of upward mobility. Imagine, for instance, switching a growth rate of 5% (for a poor person) and 10% (for a rich person) between the two. Then Growth Progressivity states that mobility rises while aggregate growth is lower.

We now state an auxiliary restriction that gives M some cardinal meaning. For any pair of situations $\mathbf{z}$ and $\mathbf{z}'$, let $\mathbf{z} \oplus \mathbf{z}'$ denoted the merged or concatenated situation which simply takes the union of all income-growth pairs over both situations. For instance, $\mathbf{z} \oplus \mathbf{z}$ means that $\mathbf{z}$ has been duplicated. We place a restriction on "locally merged" situations that have identical sets of incomes and growth rates except for just one individual k. Specifically:

3. **Local Merge.** Suppose $\mathbf{z}$, $\mathbf{z}'$ and $\mathbf{z}''$ are identical except for possibly different growth rates for just one index k; specifically, $g'_k = g_k - \epsilon$ and $g''_k = g_k + \epsilon$ for some $\epsilon > 0$. Then $M(\mathbf{z}' \oplus \mathbf{z}'') \neq M(\mathbf{z} \oplus \mathbf{z})$ whenever $M(\mathbf{z}'') - M(\mathbf{z}) \neq M(\mathbf{z}) - M(\mathbf{z}')$.

This axiom is a cardinal restriction. It demands that if "average mobility" is altered by moving one person's growth rate up while her clone's growth rate is moved down, then mobility is also altered when both persons coexist and experience these same changes "at the same time."

Theorem 1. *Axioms 1–3 hold if and only if for every population of size $n \geq 1$, M can be written as*

$$M(\mathbf{z}) = \sum_{i=1}^{n} \phi_i(\mathbf{y}) g_i \tag{1}$$

for some collection $\{\phi_i\}$ with $\phi(\mathbf{y}) = \phi_j(\mathbf{y})$ when $y_i = y_j$, and $\phi_i(\mathbf{y}) > \phi_j(\mathbf{y})$ when $y_i > y_j$.

A complete proof of Theorem 1 is in the Appendix. We illustrate here the power of the Growth Progressivity axiom. It is essentially responsible for precipitating the additivity and linearity of the instantaneous measure in individual growth rates.[3] The first of two central steps in the proof is the assertion that $M(\mathbf{y}, \mathbf{g})$ is *multiaffine* in $\mathbf{g}$; i.e., for every k, $m(g_k) \equiv M(g_k | \mathbf{y}, \mathbf{g}_{-k})$ is affine in g_k:

$$m(g_k) = A g_k + B$$

for constants A and B that could depend on $(\mathbf{y}, \mathbf{g}_{-k})$. Because m is continuous, it is enough to show that for every $\epsilon > 0$,

$$m(g_k) = \frac{1}{2}[m(g_k - \epsilon) + m(g_k + \epsilon)]. \tag{2}$$

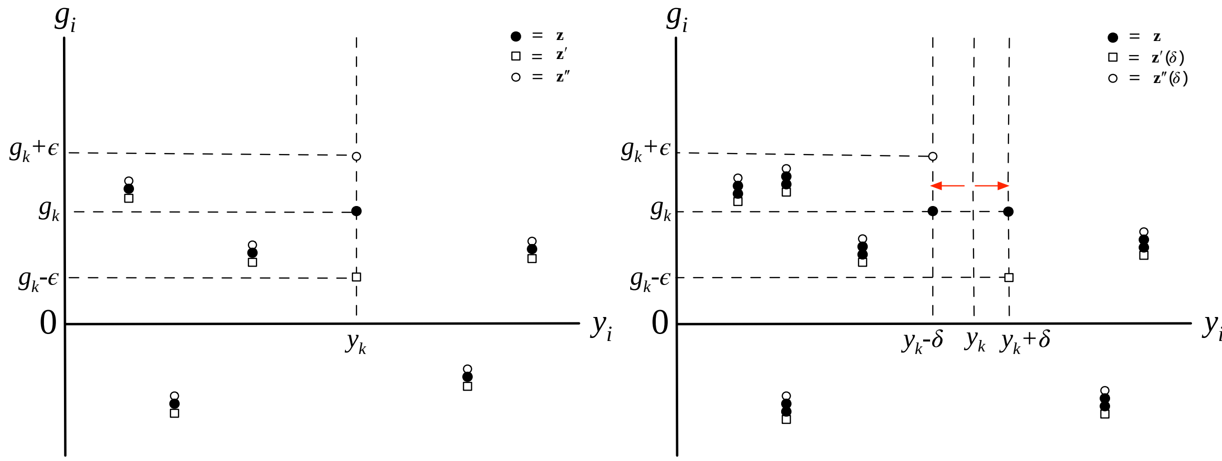
Suppose that the claim is false, so that (2) fails for some g_k and $\epsilon > 0$. Let $\mathbf{z} = (\mathbf{y}, \mathbf{g}_{-k}, g_k)$. The filled dots in Part (a) of Figure 1 depicts this situation. It also shows two other situations, $\mathbf{z}'$ (represented by the squares) and $\mathbf{z}''$ (represented by the hollow dots). The proximity of all the markings away from y_k is meant to imply that all these three situations are identical in incomes and growths, except at y_k, where z' exhibits a lower growth rate than $\mathbf{z}$ (by ϵ) and $\mathbf{z}''$ a higher growth rate (also by ϵ). Because (2) fails, we have

$$m(g_k + \epsilon) - m(g_k) \neq m(g_k) + m(g_k - \epsilon),$$

but using the definition of m, this just means that $M(\mathbf{z}'') - M(\mathbf{z}) \neq M(\mathbf{z}) - M(\mathbf{z}')$, or that

$$M(\mathbf{z}' \oplus \mathbf{z}'') \neq M(\mathbf{z} \oplus \mathbf{z}) \tag{3}$$

[3] The other axioms also play their part. Axiom 1 removes any intercept term that depends on incomes. Axiom 3 permits us to go across populations of varying size. Without it, Growth Progressivity would still limit the curvature of the measure in each g_i, but allow for some nonlinearity depending on cross-individual income gaps.



(a) Growth perturbation **(b)** Income perturbation

Figure 1. Illustration of proof that M is multiaffine. Part (a) shows two situations (filled and hollow dots) which are identical except for the growth rates at y_k, which are different (larger dots). Part (b) perturbs the incomes in both situation by generating incomes $y_k - \delta$ and $y_k + \delta$ instead of y_k. The perturbation δ is to be thought of as small.

by the Local Merge Axiom. Let us suppose that "<" holds in (3); the opposite inequality has a parallel argument. Part (b) of Figure 1 perturbs all three situations to separate the income y_k into $y_k - \delta$ and $y_k + \delta$, as shown, with the perturbed $\mathbf{z}'$ having $y_k + \delta$ and the perturbed $\mathbf{z}''$ having $y_k - \delta$. Denote these two perturbed situations by $\mathbf{z}'(\delta)$ and $\mathbf{z}''(\delta)$. This panel also perturbs the situation $\mathbf{z}$ in two ways: $\mathbf{z}^-(\delta)$ replaces y_k by $y_k - \delta$ while $\mathbf{z}^+(\delta)$ replaces y_k by $y_k + \delta$. Let $\mathbf{z}(\delta) \equiv \mathbf{z}^+(\delta) \oplus \mathbf{z}^-(\delta)$. Using the continuity of M and "<" in (3) and the fact that $\mathbf{z}'(\delta) \oplus \mathbf{z}''(\delta) \to \mathbf{z}' \oplus \mathbf{z}''$ and $\mathbf{z}(\delta) \to \mathbf{z} \oplus \mathbf{z}$, we must conclude that for $\delta > 0$ and small,

$$M(\mathbf{z}'(\delta) \oplus \mathbf{z}''(\delta)) < M(\mathbf{z}^+(\delta) \oplus \mathbf{z}^-(\delta)).$$

But this contradicts Growth Progressivity, for $\mathbf{z}'(\delta) \oplus \mathbf{z}''(\delta)$ can be achieved from $\mathbf{z}^+(\delta) \oplus \mathbf{z}^-(\delta)$ by transferring a positive growth rate of ϵ from $y_k + \delta$ to $y_k - \delta$.

A parallel argument applies when ">" holds in (3), by perturbing $\mathbf{z}''(\delta)$ to the higher income $y_k + \delta$ and $\mathbf{z}'(\delta)$ to the lower income $y_k - \delta$.

Because M is multiaffine, it can easily be expressible as follows: for every $\mathbf{y} \gg 0$, there is a collection of numbers $\phi_S(\mathbf{y})$ for every nonempty subset S of $\{1, \ldots, n\}$, such that

$$M(\mathbf{z}) = \sum_S \phi_S(\mathbf{y}) \left[\prod_{j \in S} g_j \right]. \tag{4}$$

where n is the population, the sum ranges over all nonempty[4] index subsets S of $\{1, \ldots n\}$, and the $\phi_S(\mathbf{y})$ are coefficients that depend on the income vector $\mathbf{y}$ (see, e.g., Gallier 1999, Chapter 4.5).

[4]The empty product can be excluded by the Zero Growth Anchoring Axiom.

The remainder of the proof argues that all nontrivial product terms *must have zero coefficients*. For if they do not, then for some $\mathbf{g}$, it is possible to obtain a contradiction by transferring growth rates from relatively poor to relatively rich and raising the mobility measure, thereby violating the Growth Progressivity Axiom. See Appendix for details. So the only terms that can have zero coefficients are the linear terms in (4). The fact that the smaller index terms among $\{\phi_i(\mathbf{y})\}$ have larger values than the larger-index terms is also an immediate consequence of Growth Progressivity. That outlines our proof of Theorem 1.

3.2. A One-Parameter Family for Instantaneous Upward Mobility. Expression (1) is almost exclusively a consequence of the Growth Progressivity Axiom, and illustrates the power of that Axiom. In contrast we have placed no restrictions on how instantaneous upward mobility changes with growth, or on the baseline incomes or weights apart from the monotonicity stated in Theorem 1. This is a deliberate strategy to illustrate just how far Growth Progressivity takes us. We now proceed to impose further axioms that bring the weights $\phi_i(\mathbf{y})$ into sharper focus.

4. **Income Neutrality.** Given $\mathbf{z}$, form $\mathbf{z}'$ by scaling all baseline incomes by the same positive constant. Then $M(\mathbf{z}) = M(\mathbf{z}')$.

5. **Growth Alignment.** For any $\mathbf{y}$, if $\mathbf{g} > \mathbf{g}'$, then $(\mathbf{y}, \mathbf{g}) \succ (\mathbf{y}, \mathbf{g}')$. And if $\mathbf{g} = (g, g, \ldots, g)$, then for every $\mathbf{y}$ and $\mathbf{y}'$, $(\mathbf{y}, \mathbf{g}) \sim (\mathbf{y}', \mathbf{g})$.

6. **Binary Growth Tradeoffs.** For any ij, any income pairs (y_i, y_j) and any two growth pairs (g_i, g_j) and (g_i', g_j'), the comparison of $\mathbf{z} = ((y_i, g_i), (y_j, g_j), (\mathbf{y}_{-ij}, \mathbf{g}_{-ij}))$ and $\mathbf{z}' = ((y_i, g_i'), (y_j, g_j'), (\mathbf{y}_{-ij}, \mathbf{g}_{-ij}))$ is insensitive to the value of $(\mathbf{y}_{-ij}, \mathbf{g}_{-ij})$.

Axiom 4 asserts that only *relative* baseline incomes matter. It is indispensable if one wants to implement any mobility measure across regions or countries. Axiom 5 (partially) aligns upward mobility with growth, in the sense that if *all* income levels grow faster, then mobility is deemed to be higher. We believe both these axioms to be innocuous, though see Section 3.3.

Axiom 6 declares that any tradeoffs across a pair of growth rates depends only on the characteristics of the pair in question. This is in the spirit of "independence of irrelevant alternatives." There are well-known misgivings about that axiom (see the critical assessment in Pearce 2021). One qualification that is particularly relevant here is the choice of variable over which this Axiom is to be applied. When the context is one of absolute upward mobility, individual growth rates are traded off across pairs of persons without regard to overall growth, so that Axiom 6 is reasonable. But if the context is one of *purely relative* upward mobility, then the Axiom more properly applies to the *excess* (positive or negative) of individual growth against overall growth. See the discussion in Section 3.3.

Theorem 2. *Axioms 1-6 hold if and only if for every population of size $n \geq 3$, M can be written as:*

$$M_\alpha(\mathbf{z}) = \frac{\sum_{i=1}^n y_i^{-\alpha} g_i}{\sum_{i=1}^n y_i^{-\alpha}}, \text{ for some } \alpha > 0. \tag{5}$$

In effect, four axioms characterize (5): Growth Progressivity, Income Neutrality, Growth Alignment, and Binary Growth Tradeoffs. Local Merge and Zero Growth Anchoring are automatically implied. The family of instantaneous measures characterized here will form the nucleus of our main analysis, to be developed in Section 4.

While the Appendix provides a self-contained proof of Theorem 2 that does not invoke Theorem 1, it is easy to see how the additional axioms precipitate it, given Theorem 1. When $n \geq 3$, Binary Growth Tradeoffs along with anonymity allow us to write

$$\phi_i(\mathbf{y}) = \psi(y_i)h(\mathbf{y})$$

for functions ψ and h. Growth Alignment implies that both functions are strictly positive-valued, and also permits us to normalize $M(\mathbf{z}) = g$ when all growth rates equal g. Using (1), that implies $\sum_i \psi(y_i)h(\mathbf{y}) = 1$. Substituting this information in (1), we have:

$$M(\mathbf{z}) = \frac{\sum_i \psi_i(y_i)g_i}{\sum_i \psi_i(y_i)}.$$

The Appendix shows that Income Neutrality must then precipitate the form $\psi(y) = y^{-\alpha}$, where $\alpha > 0$ by Growth Progressivity. That establishes (5), and Theorem 2.

3.3. The Instantaneous Relative Mobility Kernel. Summarizing the discussion, we see that our measure has both absolute and relative features, the former embodied in Growth Alignment and the latter in Growth Progressivity. Our preference is to retain both these aspects. But we might also want to "net out" aggregate growth and view what remains as a purely relative measure of mobility. We call this the *relative mobility kernel*, and we can define it by

$$K_\alpha(\mathbf{z}) = \frac{\sum_{i=1}^n y_i^{-\alpha} g_i}{\sum_{i=1}^n y_i^{-\alpha}} - g = \frac{\sum_{i=1}^n y_i^{-\alpha} e_i}{\sum_{i=1}^n y_i^{-\alpha}}, \tag{6}$$

where g is the overall rate of growth and $e_i \equiv g_i - g$ is the individual *excess* growth rate. Invoking (13), this can be rewritten as

$$K_\alpha(\mathbf{z}) = \sum_{i=1}^n \phi_i^*(\mathbf{y})g_i, \text{ where } \phi_i^*(\mathbf{y}) = \frac{y_i^{-\alpha}}{\sum_{j=1}^n y_i^{-\alpha}} - \frac{y_i}{\sum_{j=1}^n y_i}. \tag{7}$$

Notice that $\phi_i^*(\mathbf{y}) > \phi_j^*(\mathbf{y})$ whenever $y_i < y_j$, so the relative mobility kernel satisfies Growth Progressivity (it satisfies Local Merge as well). It is therefore one of the measures accommodated in the characterization of Theorem 1. But growth is not the variable of central interest in the relative mobility kernel. Rather, it is *excess growth* over and above the overall aggregate growth rate that's salient. From an axiomatic perspective, we must change our domain to pairs of the form $(\mathbf{y}, \mathbf{e})$: $\mathbf{y}$ is a vector of baseline incomes, and $\mathbf{e}$ is the vector of *excess growth rates* $e_i = g_i - g$. Axioms 2–5 apply without any change, in the sense that they can be shown to be equivalent across the two domains. But Axiom 6, on Binary Growth Tradeoffs, does change its meaning: the independence condition on binary tradeoffs across g_i and g_j is not the same as the independence condition on binary tradeoffs across e_i and e_j. Finally, because overall growth has no meaning, Axiom 7 is eliminated, while Axiom 1 is replaced by

1′. **Zero Excess Growth Anchoring.** If in two situations $\mathbf{z}$ and $\mathbf{z}'$, *every* individual has the same growth rate, then $M(\mathbf{z}) = M(\mathbf{z}')$; normalize this common value to zero.

These reconfigured axioms fully characterize the relative mobility kernel as in (6). We omit the proof, which follows that of Theorem 2. In Section 6, we apply the measure characterized in Theorem 2, but also use the relative mobility kernel to obtain further information.

4. Upward Mobility in Discrete Time

The "instantaneous measures" developed in Theorem 2 cannot be taken to the data as they stand. Three considerations bear on the development of an applicable measure. First, the data come in discrete form, with observations drawn at separated points in time. Second, income trajectories *do* cross. Third, and in unhappy contrast to the second point, we often do not have access to panel data. Frequently, incomes are arrayed on a cross-section under each observation, without any link across pairs of incomes in time. These core issues will now be taken up.

4.1. Instantaneous Mobility as a Base for Upward Mobility. Suppose that for a panel of two individuals or dynasties labeled 1 and 2, incomes are $\{y_1(0), y_2(0)\}$ at date 0 and $\{y_1(T), y_2(T)\}$ at some later date T. Suppose that $y_1(0) < y_2(0)$, so that 1 is initially poorer than 2, but that 1 eventually outperforms 2, with $y_1(T) > y_2(T)$. This situation is depicted in Figure 2a. Assume for now that income paths are continuous and right-differentiable everywhere (more on this later). Denote the full collection of paths by $\mathbf{y}(0, T)$. We are going to argue that this collective path should be examined in "pieces": for the segment in which 1 is poorer, Growth Progressivity acts in favor of 1, but as soon as a crossing occurs, Growth Progressivity must act in favor of 2. Of course, such switches could occur on multiple occasions, as in Figure 2b, for instance.

Specifically, we regard our measure of "instantaneous" upward mobility as providing a foundation for upward mobility measured over the entire period. That is, if μ measures "average" upward mobility over some interval $[s, t]$, we impose the *base condition* that

$$\mu(\mathbf{y}(s, t)) = \Psi(\{M(\mathbf{z}(\tau))\}_s^t), \tag{8}$$

where M is our choice of instantaneous measure, and $\mathbf{z}(\tau)$ is the collection of income and growth rate pairs for $\tau \in [s, t]$, induced by the right-hand derivatives of $\mathbf{y}(s, t)$. Condition (8) is analogous to the notion of non-paternalism in welfare economics — the idea that social welfare only depends on individual utilities (and not *how* those utilities were generated). Similarly, overall mobility under the base condition depends only on the collection of instantaneous mobilities at every date, and not on other details about how those measures were generated. This condition, while eminently reasonable to us, has a very strong implication.

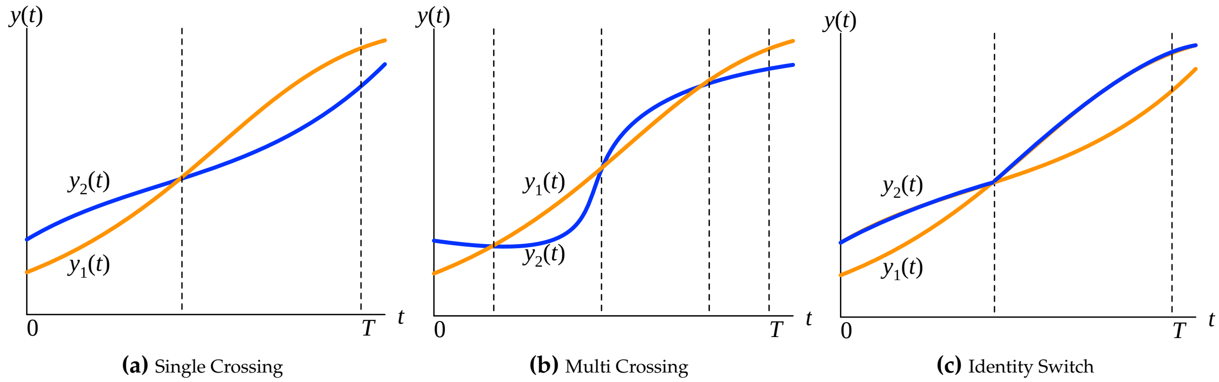


Figure 2. Possible Paths for Discrete Data.

4.2. Freedom from Panel Data. Compare the paths in Figures 2a and 2c. The latter replicates the same paths as in panel (a), but flips the identities of persons 1 and 2 at the moment of crossing. Thus, instead of person 1 proceeding from $y_1(0)$ to $y_1(T)$, she follows a path that goes up, touches the itinerary of person 2, and then follows person 2's original path, as indicated by the change in line color. Under our instantaneous upward mobility measure, this change makes no difference to the collection $\{M(\mathbf{z}(t))\}$, and consequently, via equation (8), it makes no difference to the measurement of discrete upward mobility. Because the link of identity between a starting and terminal income is thereby fully broken, *it follows that our measure does not need panel data for its implementation.* The base condition therefore provides a foundation for using non-panel data, which can be of significant value in information-poor settings.

This assertion questions the need for estimating a cupola for transitions, which is typically accomplished with much difficulty, as the data is often proprietary; see Chetty et al. (2017) and Acciari et al. (forthcoming). Such exercises are near-impossible to conduct in the majority of countries. Chetty et al. (2017) use their cupola to estimate the fraction of children who fare better than their parents. This leads to a measure that we will discuss in Section 5.4, and revisit for empirical comparison in Section 6. This measure is certainly of intrinsic interest and captures important aspects of upward mobility. That said, we are going to argue that our proposed mobility measure both has conceptual advantages (Section 5.4) *and* does not require access to a panel. Moreover, in situations where panel data is available, as in the Chetty et al. (2017) study, it appears to *empirically* track the panel measure very closely; see the discussion in Section 6.1.

4.3. Path Independence. We now introduce an additional concept. Return to the variety of paths described in Figure 2. The statistician will have access to the date $\mathbf{y}(0)$ and $\mathbf{y}(T)$, but not to the exact time path of incomes that connect these starting and terminal observations. It is natural, at least from this pragmatic perspective, to demand a discrete measure of mobility that is independent of intermediate trajectories, except for their coincident starting and ending points. But the issue goes beyond pragmatism and is connected to the very meaning of upward mobility, which is a *directed* notion. If there is a period of “positive" upward mobility, followed by a period of “negative upward" — or downward — mobility, then the overall upward mobility stemming from these two sub-periods should net out the latter from the former.

Thus say that an upward mobility measure $\mu(\mathbf{y}(s,t))$ is *path-independent* if any two paths with the same initial and final income vectors $\mathbf{y}(s)$ and $\mathbf{y}(t)$ have the same mobility; that is, $\mu(\mathbf{y}(s,t)) = M^{\Delta}(\mathbf{y}(s), \mathbf{y}(t))$ for some function M^{Δ} that depends only on starting and ending incomes.

Path independence eliminates the idea of mobility as pure movement, leaving only its directed component. For instance, the overall upward mobility over two identical configurations $\mathbf{y}(0) = \mathbf{y}(T)$ would be the same as the overall upward mobility that would obtain from a stationary path connecting the two identical income vectors, which is presumably zero. In particular, if some individuals became rich “in between" and then lost their temporary gains to the poor, that would generate the same overall upward mobility as well — zero again. Any upward phase in mobility is exactly nullified by a downward phase.

With exchange mobility, or mobility as pure movement, such path independence cannot obtain. Directions have no meaning; only movement does, so upward and downward movements do not cancel out. Perhaps such movements indeed matter from a welfare perspective: for instance, two individuals might prefer to constantly switch incomes rather than be allocated to one income or the other for their lifetimes. Risk-preferences might generate such examples (Gottschalk and Spolaore 2002). We are agnostic on the matter. The above arguments have more to do with just *what* it is that we should measure the mobility of. Our answer would be: fix the state variable of interest — current incomes, consumption, lifetime incomes or continuation values — *before* applying the notion of path independence, or indeed, any notion of mobility.

4.4. Discrete Upward Mobility.

Theorem 3. *Axioms 1–3, the base condition (8), and path independence together imply that over any collection $\mathbf{y}(s,t)$ of continuous and right-differentiable income trajectories on $[s,t]$, $\mu(\mathbf{y}(s,t))$ must have a representation of the form:*

$$\mu(\mathbf{y}(s,t)) = M^{\Delta}(\mathbf{y}(s), \mathbf{y}(t)) = \frac{1}{t-s} \int_s^t \left[\sum_{i=1}^{n} \phi_i(\mathbf{y}(\tau)) g_i(\tau) \right] d\tau, \tag{9}$$

where $\{\phi_i(\mathbf{y})\}$ has the properties described in Theorem 1. Under the additional conditions for Theorem 2, $M^{\Delta}(\mathbf{y}(s), \mathbf{y}(t))$ has the more specific form

$$M^{\Delta}_{\alpha}(\mathbf{y}(s), \mathbf{y}(t)) = \frac{1}{t-s} \ln \left[\frac{\sum_{i=1}^{n} y_i^{-\alpha}(t)}{\sum_{i=1}^{n} y_i^{-\alpha}(s)} \right]^{-\frac{1}{\alpha}} \text{ for some } \alpha > 0. \tag{10}$$

While we relegate some routine details of the proof to the Appendix, the main lines of the argument are worth mentioning here. By Theorem 1 — and therefore Axioms 1–3 — instantaneous upward mobility at any date is given by $M(\mathbf{z}) = \sum_{i=1}^{n} \phi_i(\mathbf{y}) g_i$ for some collection

$\{\phi_i(\mathbf{y})\}$ of functions. Therefore, by the base condition,

$$\mu(\mathbf{y}(s,t)) = \Psi\left(\left\{\sum_{i=1}^{n} \phi_i(\mathbf{y}(\tau))g_i(\tau)\right\}_s^t\right) = \Psi\left(\left\{\sum_{i=1}^{n} \frac{\phi_i(\mathbf{y}(\tau))}{y_i(\tau)}\dot{y}_i(\tau)\right\}_s^t\right),$$

where the dot over the y_i stands for the right-hand derivative of income in time. The only way in which path-independence can be satisfied is for Ψ to have a linear representation in these derivatives. That leads to the integral expression in (9).

Under the conditions that characterize Theorem 2, ϕ_i assumes the particular form:

$$\phi_i(\mathbf{y}) = \frac{y_i^{-\alpha}}{\sum_{j=1}^{n} y_j^{-\alpha}}$$

for some $\alpha > 0$. Substituting this expression into (9) and integrating (see Appendix for the details), we obtain (10), which is the form in which we wish to take our mobility measure to the data. Notice that (9) and (10) both divide by the normalization term $t - s$, and so pick out "average mobility" over the period, expressible as, say, an annual percentage rate.

The trajectories chosen to connect initial and final income vectors are continuous and right-differentiable in time. This is done so that growth rates are defined and we can apply our instantaneous measure. But discontinuous events might well occur. For our two-person example at the start of this section, a possible income trajectory connecting the outcomes is one in which incomes never grow, except at just one instant of time (a sudden loss of job, or a promotion, or a change in identity along a dynasty from parent to child) when they jump to their terminal values. Instantaneous mobility would be zero at every date except at the jump, where a mass point appears in the integral in (9), formally requiring the use of a Stieltjes integral. But we can approximate the integral of such trajectories (with piece-wise jumps) arbitrarily closely by differentiable functions that all generate the same answer via (9), and so there is no difficulty at all in sticking to our original assumption continuous, right-differentiable income paths.

4.5. Relative Upward Mobility. Equation (9) of Theorem 3 applies fully to the discrete counterpart of the instantaneous relative mobility kernel introduced in Section 3.3. Impose the base condition and path independence on relative upward mobility. Then Theorem 3 asks us to integrate the relative mobility kernel described in Section 3.3 over an income trajectory to generate a corresponding *discrete* relative mobility kernel. That measure is independent of the particular choice of that trajectory, for the same reason that the discrete mobility measure and the overall growth rate both are. In the context of our main measure described in (10), we have:

$$\begin{aligned} K_\alpha^\Delta(\mathbf{y}(s), \mathbf{y}(t)) &= M_\alpha^\Delta(\mathbf{y}(s), \mathbf{y}(t)) - \frac{1}{t-s}\Big[\ln(\bar{y}(t)) - \ln(\bar{y}(s))\Big] \\ &= \frac{1}{t-s}\left\{\ln\left[\sum_i \left(\frac{y_i(t)}{\bar{y}(t)}\right)^{-\alpha}\right]^{-\frac{1}{\alpha}} - \ln\left[\sum_i \left(\frac{y_i(s)}{\bar{y}(s)}\right)^{-\alpha}\right]^{-\frac{1}{\alpha}}\right\}, \end{aligned} \tag{11}$$

where $\bar{y}$ denotes per-capita income.

5. Discussion

5.1. Upward Mobility and Growth. Our measure connects upward mobility to *pro-poor growth* (Chenery, Jolly, Ahluwalia, Bell and Duloy 1974, Dardanoni 1993, Ravallion and Chen 2003, Essama-Nssah 2005, Jenkins and Van Kerm 2006; Jenkins and Kerm 2011, Palmisano and de Gaer 2016 and Berman and Bourguignon 2022). It evaluates economic performance as a weighted sum of individual growth rates. In particular, Theorem 2 declares that the weights must be powers of the inverse of an individual's initial income. For instance, $\alpha = 0.5$ doubles the weight on someone earning \$40, 000 relative to someone earning \$160, 000. If α is close to 0 then our upward mobility measure (1) converges to the sum of the log growth rates of individual income, as in Fields and Ok (1999b),[5] and as $\alpha \to \infty$ the measure becomes Rawlsian.

We allow several individuals to have the same baseline incomes, so there is no population weighting in equation (10). Typically, data on income-specific growth rates are provided by income categories (say m of them), and are therefore be available as $\{y_i(\tau), n_i(\tau)\}_{i=1}^m$, where $n_i(\tau)$ is the population share in each category at date τ. Our mobility measure would then be rewritten as

$$M_\alpha^\Delta(\mathbf{y}(s), \mathbf{y}(t)) = \frac{1}{t-s} \ln \left[\frac{\sum_{j=1}^m n_j(t) y_j^{-\alpha}(t)}{\sum_{j=1}^m n_j(s) y_j^{-\alpha}(s)} \right]^{-\frac{1}{\alpha}}. \tag{12}$$

Using (12), the distinction between upward mobility and aggregate growth becomes very sharp. The former tolerates a sacrifice of aggregate growth provided the relatively poor grow faster. The logarithm of aggregate growth over the same period is given by

$$\text{Log Growth} = M_{-1}^\Delta(\mathbf{y}(s), \mathbf{y}(t)) = \frac{1}{t-s} \ln \left[\frac{\sum_{j=1}^m n_j(t) y_j(t)}{\sum_{j=1}^m n_j(s) y_j} \right], \tag{13}$$

which is formally in the class of our measures as shown, but is emphatically excluded because the implicit value of α under the growth measure is -1. Indeed, (13) is "fully" separated from our class of upward mobility measures, and does not even sit on its boundary of that class as $\alpha \to 0$. Nevertheless, the exercise shows that our measure can be viewed as a "growth rate equivalent." Indeed, when all growth rates are the same, our measure *is* the (logarithm of) growth. Otherwise, when growth is uneven, it corrects for the progressivity of that growth.

5.2. Upward Mobility and Inequality. Upward mobility rewards greater equalization of "terminal" incomes. But it is only interested in the equalization implicit in the *change* of incomes, as a consequence of rewarding (positive) differential growth for the relatively poor. That comes from Growth Progressivity.

[5] This can be seen by applying L'Hospital's Rule to $-\ln\left(\frac{\sum_i y_i(t)^{-\alpha}}{\sum_i y_i(s)^{-\alpha}}\right)/\alpha(t-s)$, as $\alpha \to 0$.

But upward mobility is not a measure of equality. For instance, (10) is insensitive to inequality in baseline incomes. For instance, as already noted, if all incomes grow at the same rate, our upward mobility measure returns the same answer irrespective of the initial distribution of income. It also values growth all around relative to zero-growth situations, even if that growth is disequalizing.

5.3. Upward Mobility as Change in Welfare. The representation in (9) implies that our discrete measure is "time-additive" in the sense that for every triple of time periods $r < s < t$ and jointly ordered income vectors $(\mathbf{y}(r), \mathbf{y}(s), \mathbf{y}(t))$,

$$(r-s)M^{\Delta}(\mathbf{y}(r), \mathbf{y}(s)) + (t-s)M^{\Delta}(\mathbf{y}(s), \mathbf{y}(t)) = (t-r)M^{\Delta}(\mathbf{y}(r), \mathbf{y}(t)). \tag{14}$$

Fix dates 0 and 1, an arbitrary vector $\mathbf{x}$, and define a function $W(\mathbf{y})$ by

$$W(\mathbf{y}) \equiv \exp\left[M^{\Delta}(\mathbf{x}(0), \mathbf{y}(1))\right] \tag{15}$$

where $\mathbf{x} = \mathbf{x}(0)$ and $\mathbf{y} = \mathbf{y}(1)$. Observe that for any $s < t$ and trajectory $\mathbf{y}(s,t)$,

$$\ln W(\mathbf{y}(t)) - \ln W(\mathbf{y}(s)) = tM^{\Delta}(\mathbf{x}(0), \mathbf{y}(t)) - sM^{\Delta}(\mathbf{x}(0), \mathbf{y}(s)) = (t-s)M^{\Delta}(\mathbf{y}(s), \mathbf{y}(t)), \tag{16}$$

where the first equality follows from a simple change of variables applied to (9),[6] and the second from (14). Equation (16) has the interpretation that mobility over the period s to t can be viewed as the "average percentage change" in some time-stationary welfare function W over that period. Its time separability is a consequence of our elimination of exchange mobility once path-independence is imposed; see Section 4.3.[7]

Applying the same definition (15) to our focal case of equation (10), we see that W can be taken to be proportional to *Atkinson's equivalent income* from the class

$$W_{\alpha}(\mathbf{y}) = \left(\frac{1}{n}\sum_{i} y_i^{-\alpha}\right)^{-\frac{1}{\alpha}}, \text{ for } \alpha > 0, \tag{17}$$

which is the equally held income that yields the same (Atkinson) welfare as the original distribution. Our measure $M_{\alpha}^{\Delta}(\mathbf{y}(s), \mathbf{y}(t))$ can then be written as the *average growth rate of Atkinson's equivalent income* over $[s, t]$.

A similar interpretation applies to the relative mobility index. The expression in (11) can be rewritten as

$$K_{\alpha}^{\Delta}(\mathbf{y}(s,t)) = \frac{1}{t-s}\Big[\ln\left(W_{\alpha}(\mathbf{y}(t)\right) - \ln\left(W_{\alpha}(\mathbf{y}(s)G)\right)\Big],$$

where $G = \bar{y}(t)/\bar{y}(s)$ is the overall growth factor. That is, K_{α}^{Δ} can be seen as a net adjustment in welfare experienced in from going from the initial distribution to final distribution, relative to going from the initial distribution to a hypothetical distribution $\mathbf{y}(s)G$, obtained by giving

[6] It suffices to show that $tM^{\Delta}(\mathbf{x}(0), \mathbf{y}(t)) = M^{\Delta}(\mathbf{x}(0), \mathbf{y}(1))$ for every $t > 0$. For any t, use the change of variables $s = \tau/t$ in (9) to establish this.

[7] Markandya (1982) and Atkinson and Bourguignon (1982) also note the link between exchange mobility and the non-separability of the associated welfare function

everyone the average growth rate experienced by the economy. This view of relative mobility as a net change in social welfare over and above balanced growth underlies the ethical measures of mobility of Chakravarty et al. (1985), though in our case it has emerged endogenously from more primitive axioms.

Our interpretation is restricted to coefficients of inequality aversion that exceed one, or to welfare curvatures that exceed the curvature of the logarithmic function. Redistributing growth *rates* from rich to poor, and not just absolute changes in income, increases mobility. So the change-in-welfare interpretation applies to a subclass of the Atkinson welfare functions. In any case, our intention is to make this welfare connection somewhat lightly, as we do not view the measurement of mobility as fully connected to welfare, but only as a component of it. For instance, our discussion at the very end of Section 4.3 underlines the fact that the researcher must first choose the state variable of direct welfare interest (such as current or lifetime income or consumption) before applying a mobility measure.

5.4. Our Axioms and Alternative Measures. We illustrate our central axiom by using it to evaluate some alternative mobility measures. It will be easiest to do this by going back to the instantaneous upward mobility kernel. For our first comparison, it will be useful to consider an extended family of measures, which includes our measures described in (5):

$$M_\alpha^h(\mathbf{z}) = \frac{\sum_{i=1}^n y_i^{-\alpha} h(g_i)}{\sum_{i=1}^n y_i^{-\alpha}} \tag{18}$$

for some possibly non-linear function h. Now, a special measure from this class can be found by setting $\alpha = 0$ and h equal to the indicator function $I(g) = 0$ for $g < 0$, and $I(g) = 1$ for $g \geq 0$. For this setting, the general measure in (18) is easily seen to reduce to

$$M_0^I(\mathbf{z}) = \text{Population share under } \mathbf{z} \text{ for whom the future improves on the present.}$$

This measure is used in Chetty et al. (2017) and Berman (2021), and we've already encountered it in the context of our discussion on panels. Our points here are different. By nesting the measure within (18), which also includes our characterized family, we uncover two essential contrasts with our measure. First, the growth experiences of the poor are treated on par with those of the rich: M_0^I counts only the unweighted share of those families whose absolute fortunes improved ($\alpha = 0$). Second, h is a step function, while in our case it is the identity function. Both these differences can be traced back to a failure of M_0^I to satisfy the Growth Progressivity axiom.[8] The former difference is easily fixed by injecting some weighting to create a modified measure in which poorer families that improve receive larger weight than richer families that also improve. This modified measure M_α^I could, of course, approximate M_0^I as closely as we please. But the second difference cannot be easily fixed. To see this, consider the following example.

[8]Strictly speaking, the Growth Alignment axiom is also not satisfied, as the measure only seeks to know if future prospects improved or deteriorated without asking by how much, but we consider this a minor issue which can easily be rectified by any strictly increasing approximation h of I.

Suppose that there are two income groups of equal size, at incomes \$19,000 and \$20,000. This is a decaying society: assume that their children earn \$18,700 and \$19,700 respectively. Call this situation $\mathbf{z}$; then $M_0^I(\mathbf{z}) = 0$; no one earns more than their parents. Now suppose that we alter the situation so that the children of the first group earn still lower income; say \$18.000. Give that extra loss to the children of the second group, so that they now earn \$20,400. Call this situation $\mathbf{z}'$. We believe it would be hard to argue that upward mobility is higher under $\mathbf{z}'$ compared to $\mathbf{z}$, but $M_0^I(\mathbf{z}') = 0.5 > M_0^I(\mathbf{z})$.

Or consider a growing society with two equally-sized groups at incomes \$10,000 and \$20,000, and with growth rates equal to 1% for each group. Then we see that $M_0^I(10000, 20000, 1, 1) = 1$: all individuals improve. If we transfer 2 percentage points of growth from the rich group to the poor, Growth Progressivity states that upward mobility must go up. But upward mobility as measured by M_0^I declines: $M_0^I(10000, 20000, 1, 1) = 1 > M_0^I(10000, 20000, 3, -1) = 0.5$. This sort of example can be constructed for *any* nonlinear h function. Indeed, that is why the linearity of mobility in individual growth rates is implied by Theorem 1.

To this one might respond that the fault lies in our axioms and not the measure M_0^I. We disagree. The new situation in the second example has poorer families actually *catching up* with their richer counterparts. Upward mobility rewards — and in our opinion *should* reward — this narrowing of inequalities.[9] It is the fact that M_0^I actually falls instead that is problematic. In this case it comes from a psychological anchor built into the zero-improvement threshold. Cross that threshold, and policymakers are presumably delighted. Fail to cross it, and they are not. This knife-edge preoccupation with the zero threshold is, we feel, not warranted, especially in a world where granular income data is increasingly available, and indeed, already available to some of the authors who have used M_0^I.[10]

Fields and Ok (1999b) also provide an axiomatic derivation for a mobility measure that (a) rewards growth and (b) is sensitive to inequality. Without going into detail about the setting or the axioms, we simply record the measure that they obtain:[11]

$$M^{FO}(\mathbf{z}) = \frac{1}{n}\sum_{i=1}^{n} \ln g_i = \frac{1}{n}\sum_{i=1}^{n}\left[\ln(y_i') - \ln(y_i)\right],$$

where y_i and y_i' are initial and final incomes over two periods. While Fields and Ok use different axioms in a different setting, we can use our axioms to evaluate this measure. It rises with higher growth rates for all individuals, so that Growth Alignment is satisfied. Income Neutrality and

[9] This parallel between pro-poor growth and convergence has been emphasized by O'Neill and Kerm (2008), Wodon and Yitzhaki (2005), Bourguignon (2011) and Dhongde and Silber (2016), among others.

[10] Our points are echoed in a different context in the critique of the head count measure of poverty, in which a disequalizing transfer from the poor to the less poor could result in a fall in the head count, which is an unsatisfactory property. See Sen (1976) and Foster, Greer and Thorbecke (1984).

[11] They call this a "directional measure" to emphasize that "upward" changes in income are preferred to downward changes; as already discussed, their other measures do not have this property.

Binary Growth Tradeoffs are easily seen to be satisfied as well. We must conclude, therefore, that their measure cannot satisfy Growth Progressivity.[12]

The following example shows this directly. Suppose that there are just two individuals, with incomes 100 and 200, and suppose that both grow at 10%. Compare this with another situation in which the poor person grows at 15%, while the rich person grows at 5%. Growth is now pro-poor, and our mobility measure goes up. The measure M^{FO}, however, comes *down*. That is not to say that the measure we propose is necessarily "better" — though our support for Growth Progressivity suggests that we believe it is — but to observe that the researcher has an explicit axiomatic (and therefore intuitive) basis on which to compare the measures.

6. Upward Mobility in the Data

A central feature of our measure, discussed in Section 4.4, is that it does not rely on panel data for its implementation. In this section, we apply our measure of upward mobility to the United States, Brazil, India and France using repeated cross-sectional data from the World Inequality Lab (World Inequality Database 2021, see Appendix B.1 for more on the data). The empirical exercise that follows demonstrates the applicability of our measure, and also contributes to a growing literature comparing upward income mobility across regions (among others Ayala and Sastre 2002, Fields and Ok 1999a, Jenkins and Kerm 2011, Chetty et al. 2014a) with the added advantage that we are using a measure of mobility with explicit conceptual foundations.

6.1. An Initial Comparison with Existing Empirical Studies. Perhaps the most popular measure of directional mobility (deployed empirically) is the share of families whose absolute fortune has improved across generations.[13] In Section 5.4, we discussed how our measure of upward mobility differs from this "absolute mobility" measure. In this section, we are interested in comparing how these measures behave empirically. As pointed out by Deutscher and Mazumder (2020), in practice the trends exhibited by various mobility measures do tend to be similar, differing mainly depending on whether they are directional or not. This is quite apart from the conceptual considerations highlighted in this paper.

In well-known work, Chetty et al. (2017) estimated this absolute mobility measure — the fraction of children who earn more than their parents — for US birth cohorts from 1940 to 1984 and documented its decline. They combined the estimated copula of the parent-child income distribution from a unique panel of tax records (Chetty et al. 2014b), with estimates of the marginal income distributions by generation using the CPS and decennial Census data. Chetty et al. (2017)'s estimates of absolute mobility are plotted in both panels of Figure 3.

[12] We are being a bit cavalier here, as the domain of their measure is set in a discrete-time context; see Section 4 below. But it is easy to rewrite their measure for the continuous case.

[13] This measure is more often used in the context of parents and children to measure inter-generational mobility but a similar measure can also be used to measure absolute intra-generational mobility over time.

Their exercise exploits the panel structure of the data, of course. Yet Berman (2021) shows that *in practice*, these estimates of absolute mobility depend largely on the *marginal* income distributions, and relatively little on the estimated copula. From this finding, he concludes that it is possible to approximate Chetty et al. (2017)'s measure of absolute mobility on non-panel data by using available copulas estimated for other countries or periods. The World Inequality Database 2021 provides yearly percentile distributions of income for the adult US population:

$$\mathbf{y}^c(\tau) \equiv \{y_1^c(\tau), y_2^c(\tau), ...y_{100}^c(\tau)\} \text{ for } c = \text{US and year } \tau \in [1940, 1984].$$

Using these, Berman (2021) estimates the mean and variance of each marginal distribution which, under a log-normality restriction, suffices to characterize the entire marginals at 30-year intervals with starting year ranging from 1945 to 1985. Applying his empirical approximation, he then obtains estimates of absolute income mobility. Figure 3(a) plots our replication of these estimates using Berman's approach.[14]

Next, using the same data, we calculate our measure of upward mobility $M_\alpha^\Delta(\mathbf{y}(t), \mathbf{y}(t+30))$ over the same 30-year intervals for $\alpha = 0.5$, *which needs no approximation for non-panels*, and add it to Figure 3(a). And finally, Figure 3(a) also contains estimates of annual growth (measured as the difference in natural logarithm of the per capita income[15]) averaged over thirty-year intervals: one series from the dataset in Chetty et al. (2017) and the other from World Inequality Database (2021). All estimates of mobility and growth are tagged by their *starting* year.

We see that despite the difference in data and approaches, all three measures capture the overall large decline in mobility in the generations that followed World War II. It appears that the panel-dependent Chetty *et al.* estimates, the (empirically motivated) panel-independent Berman variant, and our (conceptually) panel-free measure move closely with one another. Some differences do arise, and they appear to stem largely from differences in the growth patterns recorded by the two datasets. Figure 3(b) compares our measure and that of Chetty et al. (2017), using their dataset rather than World Inequality Database 2021.[16] Figure 3b plots upward mobility computed on the decile data and reproduces Chetty et al. (2017)'s absolute mobility as shown in Panel (a). This exercise confirms even more strongly that the two measures are very closely aligned in their ordinal movements.

Figure 6 in Appendix B.2 shows that very similar patterns of decline in upward mobility are observed for various values of the pro-poorness factor α ranging from 0.1 to 5, as well as $\alpha \simeq 0$, which corresponds to Fields and Ok (1999b). At the same time, increasing α predictably puts more weight on growth at the lowest quantiles. See Appendix B.2 for more discussion.

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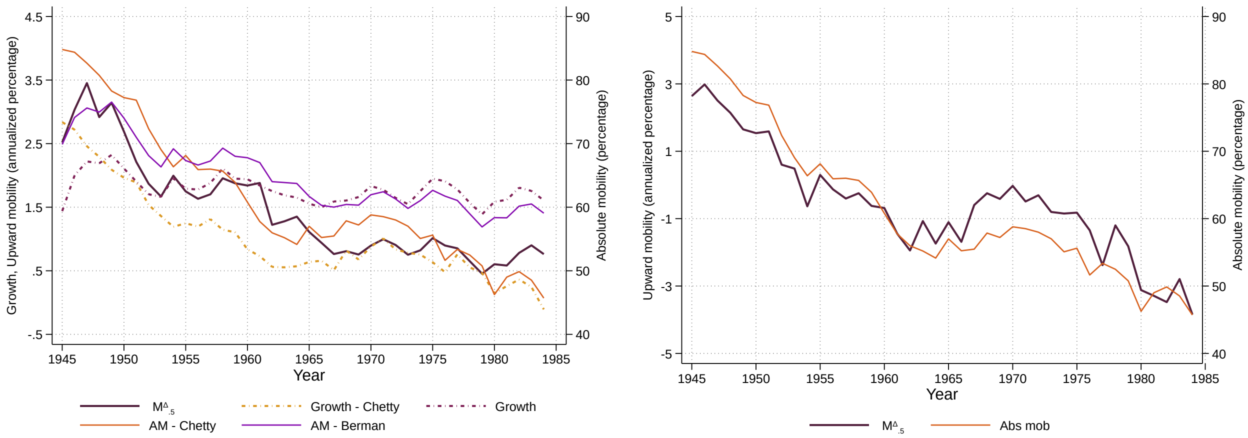
[14] We thank Yonathan Berman for sharing his code with us. Our estimates and Berman (2021) differ slightly due to updates to the WID database.

[15] That is, annual log growth equals $[\ln(\bar{y}(t+30)) - \ln(\bar{y}(t))]/30$.

[16] Chetty et al. (2017)'s sample has some negative and zero income entries among the poorest percentiles, while we know that all individuals must receive something. Our measure tend to be sensitive to the imputation assumptions for these low values, especially for higher values of α. Our favorite solution to this issue is to measure upward mobility on decile data. See the appendix for a discussion on the topic.



(a) WID data

(b) Chetty *et al.* data

Figure 3. Mobility Trends in the United States. This figure displays trends in mobility over thirty-year intervals for the United States, indexed by starting years. Panel (a) builds on percentile income data from World Inequality Database (2021). We use $M^{\Delta}_{0.5}$ for the upward mobility measure. We also display the Chetty et al. (2017) measure and replicate Berman (2021) using the World Inequality Database (2021) data. These values of absolute mobility are recorded on the right vertical axes. Upward mobility and growth rates are displayed on the left vertical axes. Panel (b) displays $M^{\Delta}_{0.5}$ applied to the decile distribution from Chetty et al. (2017) along with the Chetty et al. (2017) measure. Sources: Berman (2021), Chetty et al. (2017) and World Inequality Database (2021).

Finally, notice that these measures do not merely track overall growth — and this will become even more apparent in the exercise in the next section. There is a good reason for this. Figure 5 in Appendix B.2 clearly shows how starting in the early 1950s, the upper income quintile has experienced higher than average 30-year growth while the bottom two quintiles of the distribution have seen their real growth almost vanish. These trends are reflected in mobility.

6.2. Upward Mobility in Brazil, India and France. Encouraged by the comparison above, and additionally given our theoretically-grounded permission to dispense with panel data limitations, we now study upward mobility in settings where panel data are not available. This can bring developing countries into focus.

Specifically, we apply our measures to study ten-year upward mobility in Brazil, India and France using decile data from the World Inequality Database (2021).[17] We apply our discrete measure (10) for the benchmark value of $\alpha = 0.5$ to measure upward mobility $M^{\Delta}_{0.5}(\mathbf{y}(t), \mathbf{y}(t+10))$ over ten-year intervals, for each of our three countries and for all t ranging from 1980 to 2010.[18] (Figure 9 in Appendix B.3 shows robustness to different values of α.) For this exercise, we also bring on board our measure of relative upward mobility, which nets out growth. Recall

[17]Panel data do exist for France (European Community Household Panel ECHP) and India (India Human Development Survey IHDS), though not for all years.

[18]The first year for which the data are available for all three countries is 1980.

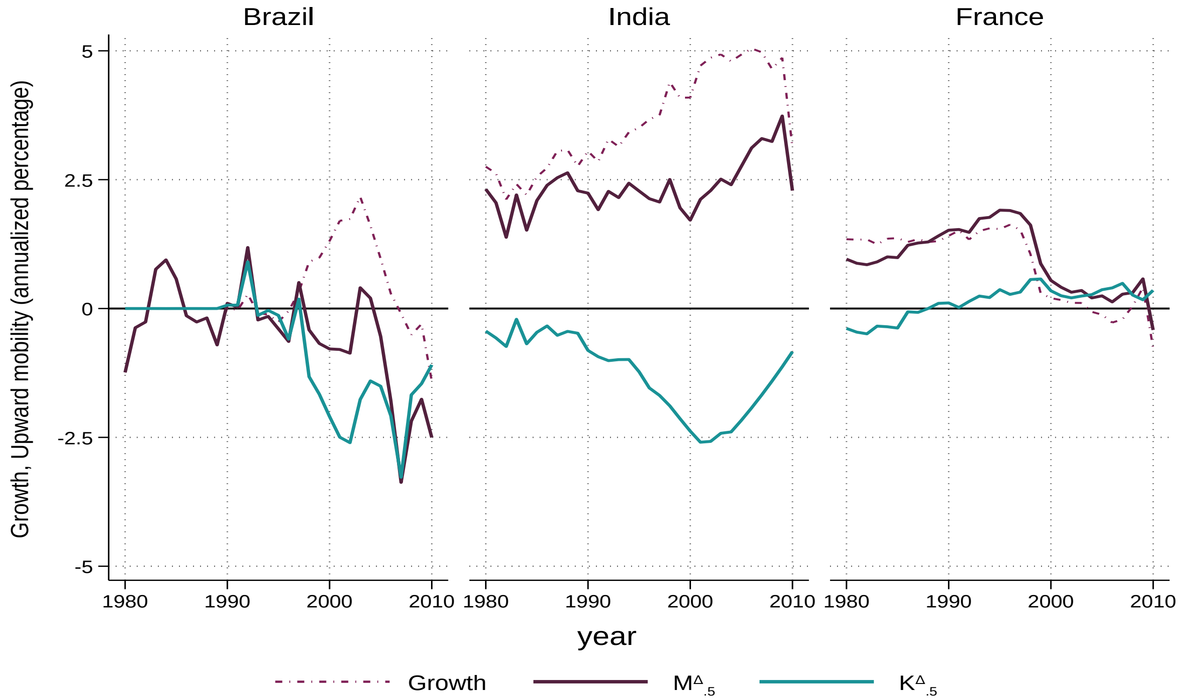


Figure 4. Upward Mobility in Brazil, India and France. These diagrams show growth (approximated by the difference in the natural logarithm of the per capita income), upward mobility and the relative mobility kernel for $\alpha = 0.5$. The numbers capture annualized averages over ten-year intervals, and are indexed by starting years.

that upward mobility is akin to an equivalent growth rate and hence it can take positive or negative values and can be expressed as an annual percentage, just as growth rates are. The relative mobility kernel shows the departure of the measured upward mobility from the per capita income growth (measured as the difference in the natural logarithm of the per capita income).

Figure 4 plots these measures, with — it is fair to say — striking effect.

After the debt crisis of 1980, Brazil entered a long decade of stagnation. In Figure 4, we see that ten-year upward mobility fluctuated between −1% and 1% over the period, but also that upward mobility and growth co-moved closely. The relative mobility kernel is therefore null over the period. Figure 8 in the Appendix confirms that all quintiles experienced the same ten-year growth throughout the 1980s. By the mid-1990s, however, Brazil had been transformed by trade liberalization, a series of privatizations and several pro-business policies. Growth reappeared between 1997 and 2007, but the quintiles diverged significantly in their growth experiences. The second to the fourth quintile did sustain positive growth, but incomes of the lowest quintile essentially decayed for most of these years except for the mid 90s. Finally, income growth in the top quintile significantly surpassed those in the other quintiles between

1999 and 2003. This is mirrored in a dramatic drop-off in upward mobility even as growth rose, with an attendant and even more severe plunge in the relative mobility kernel. Indeed, upward mobility is negative between 1993 and 2003. The implementations of strong social programs in 2003 may have helped to partially reverse the trend then. In 2007, Brazil's growth was negative and upward mobility was at its lowest at -3.35%.

The Indian story is equally dramatic, albeit on a different growth scale. Unlike Brazil, the overall period was one of steady economic growth. Following market deregulation in the early 1990s, India's per-capita growth rate experienced a steady acceleration, from 2.75% to an impressive 5% in 2005. But upward mobility, already short of growth after 1980, increasingly departs from it after 1990. This is reflected in the relative mobility kernel which trends sharply downward into the 2000s (note: our mobility estimates are indexed by *starting* years), though a later recovery is visible. The overall picture is consistent with a post-1990s reform regime that is unambiguously pro-business. Separately, Figure 8 in Appendix B.3 makes it abundantly clear that this acceleration of growth is purely concentrated in the top quintile. Our findings are in line with the inequality estimates of Chancel and Piketty (2019), who showed that the share of income of the top 1% rose from 6% in 1980 to over 22% in 2005. In the late 2000s, India suffered from the severe contraction in global trade when the financial meltdown morphed into a worldwide economic downturn. This shock particularly affected the top quintile (Figure 8), which explains the upward trend in the relative upward mobility kernel.

Finally, France paints a very different picture. Despite growth stagnating at about 1.5 % until 1997, upward mobility has risen to about 2% in 1995. Figure 8 in Appendix B.3 reveals how the growth has been systematically higher among the lowest quintiles. As a result, upward mobility *exceeded* income growth from 1988 onwards and the relative mobility kernel displays positive values. In fact, even though the great recession made resulted in negative growth rates, we see that upward mobility remained positive at around 0.5% between 2000 and 2009, which is in striking contrast to the experiences of India and Brazil.

6.3. Summary. These vignettes are not a substitute for a detailed study of mobility trends. But they serve as proof of concept for a new measure of upward mobility. We have argued that our measure is *conceptually* free from the need for panel data. If this argument is found convincing, it has significant implications for the *empirical* study of mobility in many countries for which panel data are unavailable. Indeed, for many countries in which such data are available, as in the case of India and Brazil, they are present in very limited settings. The current measure therefore greatly expands the scope for analyzing mobility in these societies.

At the same time, we do not rest our case on the theory alone. We have shown that empirically, our measure moves very closely with leading studies in the field which uses a different measure that *does* rely on a panel for its application. This empirical conformity gives us additional confidence that our analysis elsewhere can be viewed as useful.

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Appendix

A. Proofs

Proof of Theorem 1. Certainly, (1), along with the restrictions on $\{\phi_i\}$, satisfies Axioms 1–3. We therefore establish the converse.

Step 1. Suppose that $\mathbf{z}^*$ has two indices i and j with $y_i = y_j$. For any $\epsilon > 0$, define $\mathbf{z}^{**}$ identical to the old situation except that $g_i' = g_i - \epsilon$ and $g_j' = g_j + \epsilon$. Then $M(\mathbf{z}^*) = M(\mathbf{z}^{**})$.

Proof. For $\delta > 0$ but small, define $\mathbf{z}^*(\delta)$ and $\mathbf{z}^{**}(\delta)$ as follows: each has the same set of incomes and the same growth rates for every individual as $\mathbf{z}^*$ and $\mathbf{z}^{**}$ respectively, except that y_i and y_j are replaced by $y_i - \delta$ and $y_j + \delta$. By Growth Progressivity, $M(\mathbf{z}^*(\delta)) > M(\mathbf{z}^{**}(\delta))$ for every $\delta > 0$. Passing to the limit as $\delta \to 0$ and using the continuity of M, we have:

$$\lim_{\delta \to 0} M(\mathbf{z}^*(\delta)) = M(\mathbf{z}^*) \text{ and } \lim_{\delta \to 0} M(\mathbf{z}^{**}(\delta)) = M(\mathbf{z}^{**}).$$

We must therefore conclude that

$$M(\mathbf{z}^*) \geq M(\mathbf{z}^{**}). \tag{19}$$

Next, define a new situation $\mathbf{z}'^{**}(\delta)$ which is exactly like $\mathbf{z}^{**}(\delta)$ except that the growth rates are flipped: income $y_i - \delta$ now has the growth rate $g_i + \epsilon$, while income $y_j + \delta$ has the growth rate $g_j - \epsilon$. Applying Growth Progressivity again, we now have $M(\mathbf{z}^*(\delta)) < M(\mathbf{z}'^{**}(\delta))$ for every $\delta > 0$. Passing to the limit as $\delta \to 0$ just as we did before, we must now conclude that

$$M(\mathbf{z}^*) \leq M(\mathbf{z}^{**}). \tag{20}$$

Combining (19) and (20), we obtain Step 1.

Step 2. $M(\mathbf{y}, \mathbf{g})$ is multiaffine in $\mathbf{g}$; i.e., for every k, $M(\mathbf{y}, \mathbf{g}_{-k}, g_k)$ is affine in g_k:

$$M(\mathbf{y}, \mathbf{g}_{-k}, g_k) = A(\mathbf{y}, \mathbf{g}_{-k})g_k + B(\mathbf{y}, \mathbf{g}_{-k}) \tag{21}$$

for two functions A and B.

Proof. Because M is continuous, it is enough to show that for every $\epsilon > 0$,

$$M(\mathbf{y}, \mathbf{g}_{-k}, g_k) = \frac{1}{2}[M(\mathbf{y}, \mathbf{g}_{-k}, g_k - \epsilon) + M(\mathbf{y}, \mathbf{g}_{-k}, g_k + \epsilon)]. \tag{22}$$

Suppose that (22) fails for some g_k and $\epsilon > 0$. Let $\mathbf{z} = (\mathbf{y}, \mathbf{g}_{-k}, g_k)$. Define two situations $\mathbf{z}'$ and $\mathbf{z}''$, both identical to $\mathbf{z}$ for all income-growth pairs other than at y_k, where under $\mathbf{z}'$, $g_k' = g_k - \epsilon$, whereas under $\mathbf{z}''$ it is $g_k'' = g_k + \epsilon$. Because (22) fails, it is easy to see that

$$M(\mathbf{z}'') - M(\mathbf{z}) \neq M(\mathbf{z}) - M(\mathbf{z}'),$$

or that

$$M(\mathbf{z}' \oplus \mathbf{z}'') \neq M(\mathbf{z} \oplus \mathbf{z}) \tag{23}$$

by the Local Merge Axiom. But that contradicts Step 1 with $\mathbf{z}^* = \mathbf{z} \oplus \mathbf{z}$ and $\mathbf{z}^{**} = \mathbf{z}' \oplus \mathbf{z}''$.

A well-known consequence of multiaffine real-valued functions (see, e.g., Gallier 1999, Chapter 4.5) is that M has the following representation: for every $\mathbf{y} \gg 0$, there is a collection of numbers $\phi_S(\mathbf{y})$ for every nonempty subset S of $\{1, \ldots, n\}$, such that

$$M(\mathbf{z}) = \sum_S \phi_S(\mathbf{y}) \left[\prod_{j \in S} g_j \right]. \tag{24}$$

(The empty product can be excluded from (24), by the Zero Growth Anchoring Axiom.)

Step 3. $\phi_S(\mathbf{y}) = 0$ for any S with $|S| \geq 2$.

Proof. Suppose the assertion is false. Then there are indices i and j with $i < j$ and $\{ij\} \subset S$ for some S with $\phi_S(\mathbf{y}) \neq 0$. Fix any numbers $\{\bar{g}_k\}$, for $k \neq i, j$, such that

$$\alpha \equiv \sum_{T: i,j \in T} \phi_T(\mathbf{y}) \left[\prod_{k \neq i,j} \bar{g}_k \right] \neq 0. \tag{25}$$

Also define

$$\beta \equiv \sum_{T: i \in T, j \notin T} \phi_T(\mathbf{y}) \left[\prod_{k \in T-i} \bar{g}_k \right], \quad \gamma \equiv \sum_{T: i \notin T, j \in T} \phi_T(\mathbf{y}) \left[\prod_{k \in T-j} \bar{g}_k \right], \text{ and } \delta \equiv \sum_{T: i \notin T, j \notin T} \phi_T(\mathbf{y}) \left[\prod_{k \in T} \bar{g}_k \right], \tag{26}$$

where the numbers are to be interpreted as zero in case any of the above products are empty. Fix some real number $G > 0$, and consider any growth vector $\mathbf{g}$ such that $g_k = \bar{g}_k$ for all $k \neq i, j$, and such that $g_i + g_j = G$. (We place more restrictions on g_i, g_j and G below.) Then, combining (25) and (26), it is easy to see that

$$M(\mathbf{y}, \mathbf{g}) = \alpha g_i g_j + \beta g_i + \gamma g_j + \delta = \alpha g_i (G - g_i) + \beta g_i + \gamma (G - g_i) + \delta.$$

Differentiating with respect to g_i and noting that $g_j = G - g_i$, we see that

$$\frac{\partial M(\mathbf{y}, \mathbf{g})}{\partial g_i} - \frac{\partial M(\mathbf{y}, \mathbf{g})}{\partial g_j} = \alpha G - 2\alpha g_i + \beta - \gamma.$$

In what follows, recall by (25) that $\alpha \neq 0$. Now we consider the following cases. First, if $y_i = y_j$, we know from Step 1 that the above derivative should be zero, but that clearly cannot hold for arbitrary values of G and g_i, both of which we are absolutely free to choose. Second, if $y_i < y_j$, we know from Growth Progressivity that the above derivative should be positive. Again note that we are free to choose G and g_i. If $\alpha > 0$, choose $G > 0$ and large and g_i smaller than G but close to it; then the above derivative must be negative, a contradiction. Finally, if $\alpha < 0$, again choose $G > 0$ and large, but choose g_i to be small; then the above derivative must be negative, a contradiction.

It follows that $\alpha = 0$, which means that $\phi_S(\mathbf{y}) \neq 0$ *only* for sets S that are singletons. But that establishes (1). We complete the proof by noting that by anonymity, it must be that

$\phi_i(\mathbf{y})g_i = \phi_j(\mathbf{y})g_j$ when $(y_i, g_i) = (y_j, g_j)$, which implies that $\phi_i(\mathbf{y}) = \phi_j(\mathbf{y})$ whenever $y_i = y_j$. And if $y_i < y_j$, Growth Progressivity implies that $\phi_i(\mathbf{y}) > \phi_j(\mathbf{y})$. □

Proof of Theorem 2. The representation (5) clearly satisfies Axioms 1–6. We prove the converse in Steps. (As discussed in the main text, we will not need to invoke Axiom 3 or Local Merge, while Axiom 1 — Zero Growth Anchoring — is obviously implied by the stronger Growth Alignment axiom.)

Step 1. Under $n \geq 3$, Growth Alignment, Binary Growth Tradeoffs, and the anonymity and continuity of M, Theorem 2 in Chatterjee ⓡ Ray ⓡ Sen (2021) applies, and implies that M can be written as

$$M(\mathbf{z}) = f\left(\sum_{i=1}^{n} h(y_i, g_i), \mathbf{y}\right), \tag{27}$$

where h is strictly increasing in its second argument for each y_i, and $x \mapsto f(x, \mathbf{y})$ is strictly increasing over x in the range of $\sum_i h(y_i, g_i)$ as we range over all $(\mathbf{y}, \mathbf{g})$.[19]

Step 2. By Growth Alignment, whenever $g_1 = \cdots = g_n = g$, then

$$f\left(\sum_{i=1}^{n} h(y_i, g), \mathbf{y}\right) = \lambda(g) \text{ for every income vector } \mathbf{y} \tag{28}$$

for some increasing, continuous λ, where we can normalize $\lambda(0) = 0$.

Step 3. For any $\mathbf{z} = (\mathbf{y}, \mathbf{g})$, define $g = g(\mathbf{z})$ by the equality

$$\sum_{i=1}^{n} h(y_i, g) \equiv \sum_{i=1}^{n} h(y_i, g_i). \tag{29}$$

Such a g is always well-defined for every $\mathbf{z}$. To see this, observe that h is increasing in its second argument by Step 1, so

$$\sum_{i=1}^{n} h(y_i, \min_j g_j) \leq \sum_{i=1}^{n} h(y_i, g_i) \leq \sum_{i=1}^{n} h(y_i, \max_j g_j).$$

So there is $g \in [\min_j g_j, \max_j g_j]$ such that (29) holds.[20] Because h is strictly increasing in its second argument, this value of g is uniquely pinned down.

Combining (28) in Step 2 with (29) in Step 3, we conclude that for all $\mathbf{z}$,

$$M(\mathbf{z}) = f\left(\sum_{i=1}^{n} h(y_i, g(\mathbf{z})), \mathbf{y}\right) = \lambda(g(\mathbf{z})), \tag{30}$$

[19] Chatterjee ⓡ Ray ⓡ Sen (2021) build on Debreu (1960), Gorman (1968) and Wakker (1988).

[20] This step does *not* require the continuity of h in its second argument, only the fact that it is increasing.

STEP 4. We claim that $h(y, g)$ is affine in g;[21] i.e., there exist $\psi(y)$ and $\mu(y)$ such that

$$h(y, g) = \psi(y)g + \mu(y). \tag{31}$$

Suppose not; then there is $y > 0$, $g \in \mathbb{R}$ and $\epsilon > 0$ such that

$$h(y, g + \epsilon) - h(y, g) \neq h(y, g) - h(y, g - \epsilon). \tag{32}$$

Consider any pair $\mathbf{z}^*$ and $\mathbf{z}^{**}$ such that $\mathbf{y} = \mathbf{y}'$, and for two indices i and j, $y_i = y'_i = y_j = y'_j = y$, while $g_i = g_j = g$, $g'_i = g_i - \epsilon$, $g'_j = g_j + \epsilon$, and $\mathbf{g}_{-ij} = \mathbf{g}'_{-ij}$. Observe that

$$\sum_{i=1}^{n} h(y_i, g_i) - \sum_{i=1}^{n} h(y'_i, g'_i) = [h(y, g + \epsilon) - h(y, g)] - [h(y, g) - h(y, g - \epsilon)] \neq 0,$$

and so, because f is strictly increasing in its first argument, it must be that

$$M(\mathbf{z}^*) - M(\mathbf{z}^{**}) = f\left(\sum_{i=1}^{n} h(y_i, g_i), \mathbf{y}\right) - f\left(\sum_{i=1}^{n} h(y'_i, g'_i), \mathbf{y}\right) \neq 0. \tag{33}$$

But this contradicts Step 1 in the proof of Theorem 1 (that Step only uses Growth Progressivity).

STEP 5. Applying (31) to (29), we obtain

$$\left[\sum_{i=1}^{n} \psi(y_i)\right] \lambda(g(\mathbf{z})) = \sum_{i=1}^{n} \psi(y_i) g_i + \sum_{i=1}^{n} \mu(y_i).$$

Because $h(y, g)$ is strictly increasing in g, we have $\psi(y) > 0$ (and therefore $\sum_{i=1}^{n} \psi(y_i) > 0$ as well). So the above equality and (30) allow us to conclude that

$$M(\mathbf{z}) = \lambda(g(\mathbf{z})) = \frac{\sum_{i=1}^{n} \psi(y_i) g_i}{\sum_{i=1}^{n} \psi(y_i)} + \frac{\sum_{i=1}^{n} \mu(y_i)}{\sum_{i=1}^{n} \psi(y_i)}. \tag{34}$$

By applying (34) to the case in which all growth rates are 0, and applying Growth Alignment, we must conclude that the second term on the right hand side is a constant, independent of $\mathbf{y}$. We can therefore normalize it to 0, and so $M(\mathbf{z})$ can be written as

$$M(\mathbf{z}) = \lambda(g(\mathbf{z})) = \frac{\sum_{i=1}^{n} \psi(y_i) g_i}{\sum_{i=1}^{n} \psi(y_i)}, \tag{35}$$

where by Growth Alignment and Growth Progressivity, $\psi(y)$ is a positive-valued, continuous function decreasing in y.

STEP 6. We claim, finally, that $\psi(y)$ is proportional to $y^{-\alpha}$ for some $\alpha > 0$. To prove this, we first show that for every strictly positive (y_1, y_2, λ),

$$\frac{\psi(y_1)}{\psi(y_2)} = \frac{\psi(\lambda y_1)}{\psi(\lambda y_2)}. \tag{36}$$

[21] Unlike in the proof of Theorem 1, Local Merge will not required here.

Suppose that this is false for some (y_1, y_2, λ). Without loss, suppose that $y_1 < y_2$ and that ">" holds in (36).[22] Pick values g_1, g_1', g_2, g_2' such that $g_1 > g_1'$ and $g_2' > g_2$, and such that

$$\frac{\psi(y_1)}{\psi(y_2)} > \frac{g_2' - g_2}{g_1 - g_1'} > \frac{\psi(\lambda y_1)}{\psi(\lambda y_2)}. \tag{37}$$

Now consider two situations $\mathbf{z}$ and $\mathbf{z}'$. Under $\mathbf{z}$, the values for persons 1 and 2 are (y_1, g_1) and (y_2, g_2), while under $\mathbf{z}'$, the corresponding values are (y_1, g_1') and (y_2, g_2'). Otherwise, the two situations are identical. Manipulating the left inequality in (37), we must conclude that

$$\psi(y_1)g_1 + \psi(y_2)g_2 > \psi(y_1)g_1' + \psi(y_2)g_2',$$

and consequently, that $M(\mathbf{z}) > M(\mathbf{z}')$. Now scale every income in $\mathbf{y}$ and $\mathbf{y}'$ by the common factor λ in (36) and call the new situations $\mathbf{z}_\lambda = (\mathbf{y}_\lambda, \mathbf{g}_\lambda)$ and $\mathbf{z}_\lambda' = (\mathbf{y}_\lambda', \mathbf{g}_\lambda')$. Manipulating the right inequality in (37), we must conclude that

$$\psi(\lambda y_1)g_1 + \psi(\lambda y_2)g_2 < \psi(\lambda y_1)g_1' + \psi(\lambda y_2)g_2',$$

so that now we have $M(\mathbf{z}_\lambda') > M(\mathbf{z}_\lambda)$. But this reversal contradicts Income Neutrality. Therefore (36) must be true.

By defining $y = y_1$, $y' = y_2/y_1$, and $\lambda = 1/y$, we see from (36) that ψ satisfies the fundamental Cauchy equation

$$\psi(y)\psi(y') = \psi(yy')\psi(1) \tag{38}$$

for every $(y, y') \gg 0$. The class of solutions to (38) (that also satisfy continuity and $\psi(y) > 0$ for $y > 0$) must be proportional to $\psi(p) = p^{-\alpha}$ for some constant α (see Aczél 1966, p.41, Theorem 3). Invoking Growth Progressivity, it is obvious that α must be positive.

Combining Step 6 with (35), we obtain

$$M(\mathbf{y}, \mathbf{g}) = \frac{\sum_{i=1}^n y_i^{-\alpha} g_i}{\sum_{i=1}^n y_i^{-\alpha}},$$

thereby completing the proof. □

Proof of Theorem 3. We have already made the argument leading up to (9) in the main text. This proof contains the details needed to establish (10). To this end, consider any collection of right-differentiable income trajectories $\mathbf{y}(s,t)$ that connect $\mathbf{y}(s)$ and $\mathbf{y}(t)$. That generates an implied trajectory of instantaneous growth rates $\mathbf{g}(\tau) \equiv \{g_1(\tau), g_2(\tau), \ldots, g_n(\tau)\}$ for every $\tau \in [s,t]$ with the properties that for every $i = 1, \ldots, n$,

$$\frac{d \ln y_i(\tau)}{d\tau} = g_i(\tau) \text{ and } \int_s^t g_i(\tau) ds = \ln y_i(t) - \ln y_i(s).$$

[22] Asking for $y_1 < y_2$ is without loss. Suppose, however, that "<" holds in (36). Then simply rename λy_1 to y_1', λy_2 to y_2' and set $\lambda' = 1/\lambda$. Then the assertion in the main text holds: $y_1' < y_2'$ and ">" holds in (36).

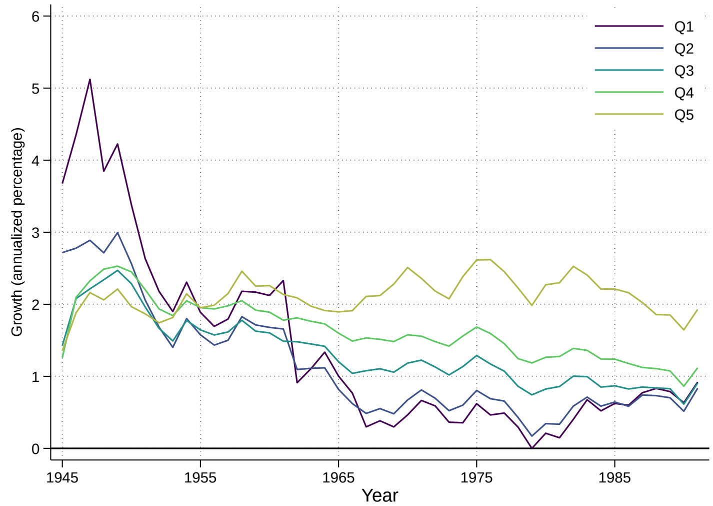

Figure 5. Growth Incidence Curves for the United States. These diagrams show the annualized growth rate of each income quintile for a 30-year interval, and are indexed by starting years.

Using all this information in (9) along withe specific functional form for $\{\phi_i(\mathbf{y})\}$, we have:

$$\begin{aligned}
M_\alpha^\Delta(\mathbf{y}(s,t)) &= \frac{1}{t-s}\int_s^t \frac{\sum_i y_i(\tau)^{-\alpha} g_i(\tau)}{\sum_i y_i(\tau)^{-\alpha}} d\tau \\
&= \frac{1}{t-s}\int_s^t \left[\frac{\sum_i y_i(s)^{-\alpha} \exp\left\{-\alpha \int_s^\tau g_i(x)dx\right\} g_i(\tau)}{\sum_i y_i(s)^{-\alpha} \exp\left\{-\alpha \int_s^\tau g_i(x)dx\right\}}\right] d\tau \\
&= -\frac{1}{\alpha(t-s)}\left[\ln\left(\sum_i y_i(s)^{-\alpha} \exp\left\{-\alpha \int_s^\tau g_i(x)dx\right\}\right)\right]_{\tau=s}^{\tau=t} \\
&= -\frac{1}{\alpha(t-s)} \ln\left[\frac{\sum_i y_i(s)^{-\alpha} \exp\left\{-\alpha \int_s^t g_i(x)dx\right\}}{\sum_i y_i(s)^{-\alpha}}\right] \\
&= \frac{1}{t-s} \ln\left[\frac{\sum_i y_i(t)^{-\alpha}}{\sum_i y_i(s)^{-\alpha}}\right]^{-\frac{1}{\alpha}},
\end{aligned}$$

which clearly does not depend on the particular collection $\{\mathbf{y}(\tau)\}$ of posited trajectories. □

B. Upward Mobility using the WID

B.1. Data. We use data from the The World Inequality Database (World Inequality Database 2021) that combines fiscal, survey and national accounts data in a systematic manner. In countries with a small informal economy and where high-quality tax microdata is available, the tax data is the main source. Income surveys and imputation methods are used to make minor adjustments in order to account for non-filers and certain tax-exempt incomes. In contrast,

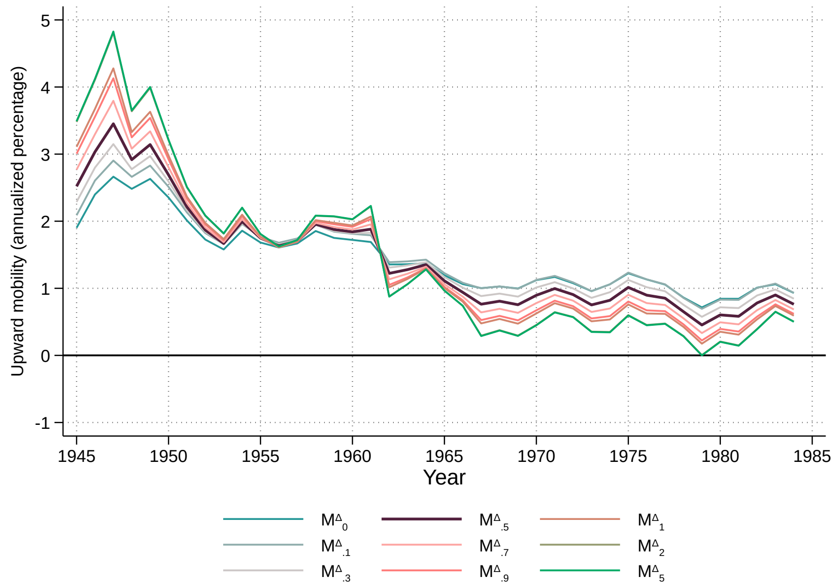

Figure 6. Upward Mobility in the US over 30-year intervals. This figure displays trends in upward mobility for different values of the pro-poorness factor α, indexed by starting years.

income surveys are main sources for most emerging economies, and the tax datasets are only used to correct the top of the income distribution. Income surveys are mainly coming from the World Bank (via PovcalNet). A detailed description of the methodology is available on the WID website.

The income data are pre-tax total incomes, computed using the equal-split assumption (that is, if the tax unit has more than one income-contributing individual contributing, the assumption is that everyone contributes in equal part to the total income of that tax unit). All incomes are expressed in PPP and in real terms, with a base year of 2021.

B.2. Upward Mobility in the United States: 30 Year Intervals.

Growth Incidence Curves. The upward mobility measures do not merely track overall growth, and there is a good reason for this. Figure 5 shows how starting in the early 1950s, the upper income quintile has experienced higher than average 30-year growth while the bottom two quintiles of the distribution have seen their real growth almost vanish.

Sensitivity to α. Chetty et al. (2017)'s sample has negative and zero income entries among the poorest percentiles. As observed in Section 5.1, our measure converges to the continuous growth rate of the lowest percentile as $\alpha \to \infty$. Negative or zero values are therefore problematic, especially for large values of α, and our measure could be sensitive to imputation assumptions. We therefore measure upward mobility on the Chetty et al. (2017) data aggregated into deciles. Figure 6 plots 30-year upward mobility in the US for a range of values of the pro-poorness

(a) Absolute and Upward Mobility in the US

(b) Different values of α

Figure 7. Chetty *et al* data censored at $100. This figure displays trends in mobility over thirty-year intervals for the United States, indexed by starting years. Panel (a) builds on data from Chetty et al. (2017), censored $100 to remove negative and zero incomes, and displays $M^{\Delta}_{0.5}$ along with the Chetty et al. (2017) measure. Panel (b) display upward mobility for a range of values of α.

factor α ranging from 0 (Fields and Ok 1999b) to 5. We see that the exact value of α does not affect the pattern to a large degree.

Censoring Low Incomes in Chetty et al. (2017)'s Sample. Aggregating the data into deciles is our preferred approach to deal with low income values. Alternatively, we could "censor" Chetty et al. (2017)'s sample and set all income values at a given minimum; e.g., $100. However, the level of upward mobility can be sensitive to these imputations, and we avoid them.

For very large values of α, disproportionate weight is put on the lowest percentile which makes the measure more variable and more susceptible to measurement errors.

B.3. Brazil, India and France.

Growth Incidence Curves Figure 8 shows the ten-year growth rates by quintile in Brazil, India and France.

Sensitivity to α As we did for the US data, we explore the robustness to different values of α. Figure 9 plots the 10-year upward mobility in Brazil, India and France for values of the pro-poor factor α ranging from 0 to 5. We see that the measure is not very sensitive to the exact value of α for India where similar growth was experienced by the bottom four quintiles (see Figure 8). In Brazil and France where the growth patterns among the lower quintiles are more differentiated, increasing α affects the upward mobility in a predictable manner: it lowers it in Brazil where the bottom quintile fared relatively poorly and increases it in France when the bottom quintiles outperform the others.

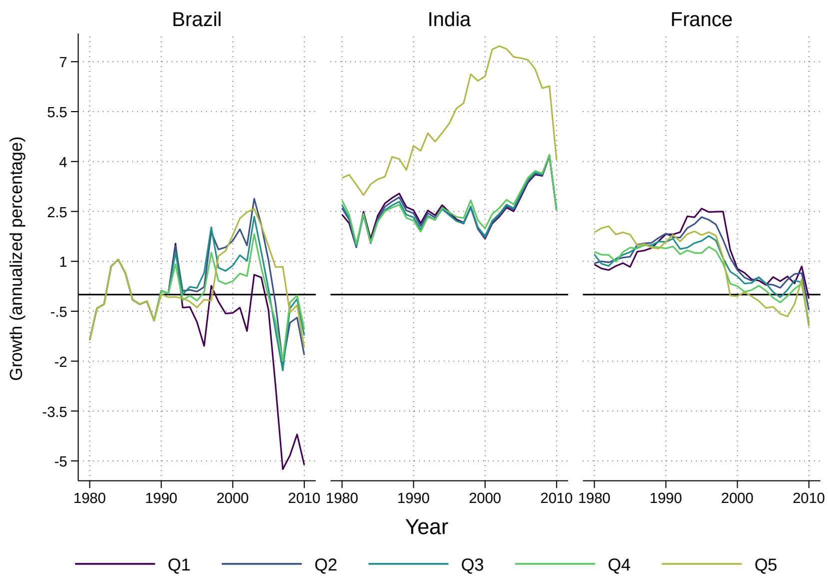

Figure 8. Growth Incidence Curves. These diagrams show the annualized growth rates for each quintile over ten-year intervals, and are indexed by starting years.

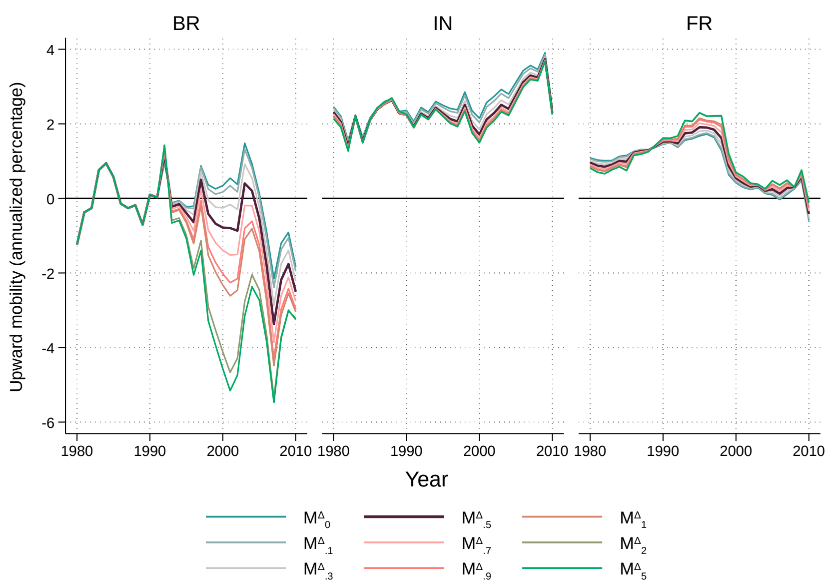

Figure 9. Upward Mobility over 10-Year Intervals. This figure displays trends in upward mobility for different values of the pro-poorness factor α, indexed by starting years.