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ABSTRACT

We develop a dynamic contracting theory of asset- and cash flow-based financing that demonstrates how firm, intermediary, and capital market characteristics shape firms' financing constraints. A firm with imperfect access to equity financing covers financing needs through costly sources—an intermediary and retained cash. The firm's financing capacity is endogenously determined by either the liquidation value of assets (asset-based) or the intermediary's going-concern valuation of the firm's cash flows (cash flow-based). We implement the optimal contract between the firm and intermediary with both unsecured and secured debt (credit lines) in an overlapping pecking order: the firm simultaneously finances cash flow shortfalls with unsecured debt and either cash reserves (if available) or secured debt (otherwise). Improved access to equity financing increases debt capacity, thus debt and equity are dynamic complements. When the firm does well, it repays its debt in full, while when in distress, repayment dynamics mirror U.S. bankruptcy procedures.

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What determines firms' financing constraints or borrowing capacity? Recent work by Lian and Ma (2021) highlights the importance of firm characteristics, specifically the liquidation value of assets or the going-concern value of cash flows, as key determinants of financing constraints. These constraints can be classified as *asset-based* and *cash flow-based* respectively. In this paper, we present a dynamic contracting theory of endogenous asset- and cash flow-based financing, shedding light on the impact of firm, capital market, and financial intermediary characteristics on financing capacity.

In our theory, a second-best holder of assets, an intermediary, provides interim financing to a liquidity constrained firm subject to endogenous financing capacity. If the intermediary's valuation of the firm's cash flows *plus* the option to raise new equity capital from first-best holders in the future is above the liquidation value of the firm's assets, then the intermediary provides financing against the going concern value of cash flows. Otherwise, it provides financing against the liquidation value of assets. The key implications are that (i) the optimal financing contract with the intermediary consists of both secured and unsecured debt, (ii) improved access to equity financing increases cash flow-based debt capacity making debt and equity *dynamic* complements, (iii) the firm uses secured debt when low on cash primarily for financing while using unsecured debt in all states primarily for hedging, and (iv) endogenous bankruptcy and bankruptcy resolution arise from the dynamics of the model.

In more detail, a firm owned by risk-neutral and penniless investors produces risky cash flows and has imperfect access to capital markets. The firm's current owners cannot inject cash into the firm. However, as in Hugonnier, Malamud, and Morellec (2015), the firm can raise external financing only at infrequent random times from newly arriving risk-neutral and competitive investors. Such intermittent access captures capital supply uncertainty or frictions that cause a delay between the firm's need for financing and its access to capital markets. Outside such access, the firm finances cash flow shortfalls with internal cash reserves and/or funds provided by an intermediary. However, both liquidity facilities are costly. First, cash held in the firm earns a return below the risk-free rate. Second, the intermediary requires compensation for any cash flow risk it bears. This compensation reflects the intermediary's own financial or regulatory constraints which limit its risk-bearing capacity.

We derive the optimal contract between the firm's investors and the intermediary that maximizes the value of the firm, subject to the intermediary's limited commitment which requires its continuation payoff to remain non-negative. Intuitively, in the optimal contract, the risk-neutral investors act as the firm's shareholders, and external financing takes the form of equity. The intermediary, in turn, provides debt-like bridge financing to the firm, covering financing needs between infrequent equity financing rounds. As such, the intermediary may represent a bank or non-bank lender, or a group or syndicate of different lenders.

Crucially, we can summarize the firm's state with a single variable that we term excess liquidity. This variable equals the firm's cash holdings less the risk-adjusted present value of future transfers to and from the intermediary. Under the optimal contract, the shareholders' value function solves an ordinary differential equation over excess liquidity with two free boundaries. The firm only pays dividends to its shareholders at the upper boundary. Because the firm has infrequent access to equity financing, negative cash flow shocks can induce financial distress, and the firm's shareholders become effectively risk-averse. Thus, they optimally share cash flow risk with the intermediary even though the intermediary requires compensation for bearing such risk. The intensity of risk-sharing between the intermediary and the shareholders decreases in excess liquidity. Further, the firm relaxes liquidity constraints by delaying some payouts to the intermediary until the next equity market access.

If the firm runs out of cash, which under the optimal contract happens when excess liquidity hits zero, the firm continues operations by relying purely on intermediary financing until it has equity market access. However, the firm faces an endogenous financing capacity vis-a-vis the intermediary, in that promised repayments to the intermediary cannot exceed this financing capacity. This financing capacity pins down the lower boundary on excess liquidity and coincides with the intermediary's valuation of the entire firm. Intuitively, repayment promises must be be backed by the firm as a collateral asset and so cannot exceed the intermediary's valuation of this collateral asset. When the intermediary values the firm above its liquidation value, financing capacity is cash flow-based, i.e., determined by the intermediary's going concern value of the firm's cash flows. Otherwise, it is asset-based, i.e., determined by the liquidation value of the firm's assets.

Once the firm exhausts its financing capacity, i.e., when excess liquidity approaches its lower boundary, the intermediary effectively takes ownership of the entire firm. It seizes the collateral backing its promised repayments and existing equity holders are wiped out. When financing capacity is cash flow-based, the intermediary keeps the firm alive until it can sell the firm to new risk-neutral equity investors. The intermediary never liquidates the firm under this kind of financing. Intuitively, if the intermediary's valuation of the firm — equal to the risk-adjusted expected value of receiving all interim cash flows plus the resale value — supports cash flow-based financing, then the intermediary prefers continuing to run the firm for however long it takes to locate new outside equity financing rather than liquidating. In contrast, when financing capacity is asset-based, the intermediary liquidates the firm once repayment promises to the intermediary reach the liquidation value of assets.

The optimal contract with the intermediary involves two key elements, risk-sharing and financing. To implement this optimal contract via standard securities, we first show that the sum of past transfers to and from the intermediary, compounded at an endogenous rate, is a sufficient statistic for the firm's state. This sum resembles a credit line balance, motivating an implementation via a risky unsecured and a risk-free secured credit line. The unsecured credit line implements risk-sharing, while the secured credit line implements financing against promises. For low excess liquidity, the firm is in financial distress and there are conflicting interests between shareholders and creditors when recapitalizing the firm with new equity.

The implementation of the optimal contract resolves financial distress in a manner that resembles U.S. bankruptcy procedures. Specifically, an appropriate debt covenant (e.g., balance sheet or earnings-based covenant) allocates control rights in distress to the creditors. When the firm raises new equity financing outside of distress, it fully repays credit lines while existing equity claims are diluted. When the firm is in distress, creditors force *Chapter 11* bankruptcy, facilitating continued operation. The firm emerges from bankruptcy when it finds new equity investors, repays the secured credit line in full, while partially defaulting on the unsecured credit line and wiping out the existing equity claims. It may also emerge from bankruptcy after a string of positive cash flow realizations. While in bankruptcy the firm may fully exhaust its financing capacity. It then optimally liquidates, effectively converting to *Chapter 7* bankruptcy, repays the secured credit line in full, while wiping out both the unsecured credit line and existing equity claims. In all cases, repayments respect the *absolute priority rule* (APR).

The implementation also sheds light on how different financing instruments, here debt, equity, and internal cash reserves, interact. First, debt and equity are *static* substitutes, but *dynamic* complements. When the firm raises new equity, it retires all debt; in this case, equity substitutes for debt. However, absent current access to equity financing, the prospect of *future* access increases debt capacity because it raises the likelihood that debt is repaid. Second, our implementation suggests an overlapping pecking order: when the firm holds (runs out of) cash, it finances cash flow shocks by drawing on cash reserves (the secured credit line) and the unsecured credit line. Third, unsecured and secured debt are complements. Unsecured debt allows the firm to share risk with the intermediary to stave off liquidation, ensuring repayment of secured creditors and raising secured debt capacity. Fourth, according to the definitions of cash flow- and asset-based debt in Lian and Ma (2021), the firm relies both on asset-based (secured by assets) and cash flow-based debt (unsecured or secured by blanket lien, i.e., senior unsecured).

We emphasize that the types of debt used by the firm and the drivers of financing capacity are related but distinct objects. Whether a firm's financing capacity is cash flow- or asset-based depends on which fundamentals drive the total amount of financing available to the firm: Financing capacity is asset-based (cash flow-based) if it increases in (is invariant to) the liquidation value of assets. Moreover, financing capacity coincides with the firm's secured debt capacity in our implementation. In contrast, whether a particular debt within the firm's capital structure is cash flow- or asset-based depends on the determinants of that debt's payoff in bankruptcy. In our implementation, a firm with cash flow-based financing capacity uses both cash flow- and asset-based debt. A fraction of the secured debt is backed by the liquidation value of assets, i.e., asset-based, whereas the secured debt balance in excess of the liquidation value of assets is cash flow-based (secured by a blanket lien). In addition, the firm always uses cash flow-based debt financing in the form of unsecured debt. While an increase in the liquidation value of asset does not raise total amount of financing available to this firm, i.e., its financing capacity, it does lead to more asset-based debt. A similar logic applies to a firm with asset-based financing capacity.

Our theory endogenizes not only financing constraints, but also their tightness, that is, a firm's utilization of intermediary financing relative to its capacity. We find that firms whose financing capacity is either very high or low use intermediary financing the least, and so face on average the least tight financing constraints. Thus, firms with large financing capacity do not rely much on the intermediary because their endogenous financing capacity reflects strong firm fundamentals that reduce the need for intermediary financing. Further, better firm fundamentals, higher intermediary risk-bearing capacity, or better access to equity financing — all associated with larger financing capacity — may increase utilization more than capacity of intermediary financing, thus tightening financing constraints.

Next, we consider that shareholders cannot commit to lowering their continuation value upon raising equity, limiting the extent to which equity can be diluted in distress resolution. Although the optimal contract can still be implemented via secured and unsecured debt, shareholders' limited commitment leads to the violation of absolute priority in bankruptcy and thus effectively implies weak creditor rights. We show that weaker creditor rights cause a shift from cash flow-based towards asset-based financing and from Chapter 11 (with reorganization) to Chapter 7 bankruptcy (with liquidation). This extension can inform international comparisons of financing arrangements and the underlying legal systems.

To capture lenders' monitoring (e.g., via covenants) or engagement in distress resolution often observed in practice, we introduce that the intermediary improves the performance of the firm through costly (monitoring) effort. We show that this monitoring effort is proportional to the extent of risk-sharing between firm and intermediary. This implies monitoring increases following negative cash flow shocks, and thus credit line drawdowns.

Finally, we give a brief overview of the main empirical implications of our theory; Section 6 provides a more detailed summary. First, better access to equity financing, e.g., due to more liquid private or public equity markets, improves access to cash flow-based financing, and expands financing capacity. Thus, our theory rationalizes why large public and private equity (PE) backed firms use more cash flow-based financing than private firms with limited access to equity financing (Lian and Ma, 2021; Haque, Jang, and Mayer, 2022). Second, intermediaries with higher risk-bearing capacity, for instance, non-bank lenders (Chernenko, Erel, and Prilmeier, 2022), provide more cash flow-based financing (Jang, 2022; Block, Jang, Kaplan, and Schulze, 2023). Consequently, a shock to the intermediary sector implies a shift from cash flow-based toward asset-based financing. Third, cash flow-based financing is more prevalent among firms with high profitability, low cash flow volatility, or low liquidation value (Kermani and Ma, 2023). Fourth, firms tend to use unsecured debt in all states and secured debt primarily when low on cash or in distress (Benmelech, Kumar, and Rajan, 2020a). Fifth, in financial distress, cash flow-based financing is associated with Chapter 11 bankruptcy and reorganization, while asset-based financing is associated with Chapter 7 bankruptcy and liquidation. Sixth, weak creditor protection reduces cash flow-based financing capacity, leading to more asset-based financing and liquidations. Seventh, cash flow-based financing is associated with high creditor monitoring in distress (Kermani and Ma, 2020). Eight, in an alternative application in which the intermediary represents a distress investor (e.g., PE or hedge fund) that acquires a (debt or equity) stake in the firm in distress, distress investment activity is hump-shaped with respect to firms' access to equity financing.

By providing a micro-foundation of financing constraints through the lens of dynamic contracting theory, our paper can provide guidance on reduced-form financing constraints, for instance, in dynamic macroeconomic models. While most papers in the macroeconomic literature focus on collateral, i.e., asset-based, constraints as key financing constraint (Kiyotaki and Moore, 1997; Bernanke, Gertler, and Gilchrist, 1999; Dávila and Korinek, 2018), Greenwald (2019), Drechsel and Kim (2022), and Drechsel (2023) introduce cash flow-based financing constraints in general equilibrium models. According to our theory, a firm's total (debt) financing B_t from intermediaries at time t is either constrained by the liquidation value of assets L_t or an increasing function of expected cash flows E_t, suggesting a reduced-form financing constraint $B_t \leq \max\{L_t, A_t E_t\}$. Although this constraint differs from the micro-founded one, it can capture its key qualitative aspects despite its simplicity. Our findings reveal that A_t should increase with intermediary risk-bearing capacity and a firm's access to equity financing as well as reflect firm characteristics; the exact functional form of A_t could be calibrated or structurally estimated which is left for future research.[1]

Related Literature. Our paper mainly relates to the literature on dynamic corporate liquidity management in continuous time, pioneered by Bolton, Chen, and Wang (2011) and Décamps, Mariotti, Rochet, and Villeneuve (2011). In a unified model of corporate investment, financing, and liquidity management, Bolton et al. (2011) demonstrate how

[1] Drechsel (2023); Greenwald (2019) use a cash flow-based borrowing constraints with constant $A_t \equiv A$.

liquidity management and firm financing interacts with a firm's investment decisions. Bolton, Wang, and Yang (2021) study dynamic liquidity management with short-term debt financing, thereby highlighting the interaction between the endogenous pricing of debt and the optimal the equity payout and issuance strategies. Further contributions in this literature include Gryglewicz (2011); Bolton, Chen, and Wang (2013); Décamps, Gryglewicz, Morellec, and Villeneuve (2016); Hugonnier and Morellec (2017); Malamud and Zucchi (2018), and, more recently, Abel and Panageas (2022); Dai, Giroud, Jiang, and Wang (2020); Bolton, Li, Wang, and Yang (2021). We add to this literature in two ways. First, methodologically, we introduce a second-best holder of the asset, an intermediary, who can however provide continuous financing, combining optimal long-term contracting with dynamic liquidity management. Second, while existing papers typically feature exogenous financing constraints or capital structure or both, we endogenize (i) the firm's capital structure, including the optimal use of secured and unsecured (or asset- and cash flow-based) debt, and (ii) financing constraints and capacity, thereby generating novel empirical implications.[2]

Our paper relates to the corporate finance literature on dynamic moral hazard without liquidity management. Bolton and Scharfstein (1990) show that financial constraints, in their case, early termination of a positive NPV project, may arise as dynamic solution to agency conflicts between investors and a firm's managers. Recent contributions include DeMarzo and Sannikov (2006); Biais, Mariotti, Plantin, and Rochet (2007); DeMarzo and Fishman (2007); Sannikov (2008); DeMarzo, Fishman, He, and Wang (2012); Malenko (2019). In this literature, financial constraints arise endogenously as part of the optimal contract in order to incentivize the agent. We contribute by solving for the optimal contract between principal (the firm's shareholders) and agent (the intermediary) when the principal faces liquidity constraints and therefore must both manage its liquidity and design the contract, inducing endogenous financing capacity.[3]

In particular, our work contributes to the dynamic contracting literature that studies optimal risk-sharing between a principal and an agent under limited commitment, such as Ai and Li (2015) and Ai, Kiku, and Li (2019), while the closest to our paper is Bolton, Wang, and Yang (2019). Our model differs as we introduce a deep-pocketed, costly intermediary that provides financing to the firm subject to endogenous constraints. Our work is complementary in that it highlights optimal financing from a costly intermediary in the presence of physical cash constraints and limited commitment. In contrast, their model is

[2]For instance, Bolton et al. (2011) consider a fully-equity financed firm with access to a credit line with an exogenous credit line limit. Their paper does not feature an exogenous financing capacity and does not distinguish between asset- and cash flow-based financing or secured vs. unsecured debt.

[3]The baseline model has no agency conflict. We also present a model extension in which the intermediary (the agent) exerts hidden effort, here monitoring, subject to a private cost, leading to a moral hazard issue.

driven by the connection between investment, firm scale, and the scale of the manager's outside option. Rampini and Viswanathan (2010) and Rampini, Sufi, and Viswanathan (2014) provide models in which limited enforcement constrains financing and creates a role for collateral. Rampini and Viswanathan (2020) applies their framework to distinguish between secured and unsecured debt. Abel (2018) develops a dynamic trade-off theory in which a (cash flow-based) borrowing constraint prevents shareholders from defaulting immediately.

The paper also is related to static contracting models, such as Holmstrom and Tirole (1997) and Holmström and Tirole (1998), in which external financing is constrained by the firm's "pledgeable income," akin to a cash flow-based financing constraint. The key differences between this classic literature and our model framework is that we consider dynamic market access and risk sharing. These elements allow us to differentiate between asset-based and cash flow-based financing, and their dynamic impact, such as bankruptcy and bankruptcy resolution arising from an optimal contract.

Paper overview. Section 1 sets up the model, while Section 2 solves it. Section 3 analyses the optimal contract and establishes the link to cash flow- and asset-based financing. Section 4 implements the optimal contract via secured and unsecured credit lines. Section 5 provides the weak creditor rights and monitoring extensions, as well as robustness considerations. Section 6 summarizes the empirical predictions of the model. Section 7 concludes.

1 Model Setup

Time $t \geq 0$ is continuous and infinite. We consider a firm whose assets produce cash flows X_t with stationary increments

$$dX_t = \mu dt + \sigma dZ_t, \tag{1}$$

where dZ_t is the increment of a standard Brownian Motion. The firm is owned by its current risk-neutral and penniless *investors*. Access to external financing from newly arriving risk neutral investors occurs only intermittently. An *intermediary* (distinct from the investors) is available to continuously provide (bridge) financing at a cost. Both the investors and the intermediary discount the future at the risk-free rate of $r > 0$. The intermediary and investors sign a long-term contract $\mathcal{C}$ at time $t = 0$. This contract, $\mathcal{C} = (Div, I, \Delta M)$, stipulates cumulative payouts Div_t to investors, money raised from new investors upon access to external financing ΔM_t, and cumulative transfers I_t to and from the intermediary. Cash flows dX_t are publicly observable, verifiable, and contractible.

As will become clear later, within the optimal contract, the investors act as the firm's shareholders, who receive dividend payouts $dDiv_t$, and external financing takes the form of private or public equity financing. The intermediary provides debt-like financing, and thus may represent a bank or non-bank lender. In anticipation of this result, we refer in the following to investors already as the firm's shareholders and to the value of their stake as the firm's equity value. However, we emphasize that we do not impose specific ex-ante restrictions on the investor-intermediary contract to take this form.

As in Hugonnier et al. (2015), the firm can only raise external (equity) financing from competitive and risk-neutral investors at Poisson times that arrive with constant intensity $\pi \geq 0$. Here, $d\Pi_t = 1$ means access to external financing or refinancing at time t, with $d\Pi_0 = 1$ to reflect access to equity financing at inception. Once outside investors provide financing, they become part of the current owners. This assumption reflects capital supply uncertainty or proxies for frictions that cause a delay between the firm's need for financing and its access to broader markets.[4] We assume no additional costs of refinancing upon $d\Pi_t$ for the main analysis. Online Appendix M extends our baseline by introducing a cost of refinancing. We denote capital infusions by new investors upon market access by $\Delta M_t d\Pi_t \geq 0$. The key assumption is that dividend payouts to existing shareholders must be non-negative. That is, $dDiv_t \geq 0$ at all times $t \geq 0$, including at refinancing times with $d\Pi_t = 1$. This reflects that the firm's existing shareholders are penniless and cannot inject cash into the firm. Alternatively, $dDiv_t \geq 0$ may capture existing shareholders' limited liability.

The firm's financial constraints and the fact that cash flow shocks can be negative imply that the firm has an incentive to accumulate cash M_t via retained earnings. The cash balance held within the firm accrues interest at the rate $(r - \lambda)$ where r is the common interest rate and $\lambda \in (0, r)$ represents a carrying cost of cash.[5] The dynamics of cash reserves M_t are

$$dM_t = dX_t + (r - \lambda) M_t dt - dDiv_t - dI_t + \Delta M_t d\Pi_t. \tag{2}$$

Absent access to equity financing, all cash flow realizations dX_t, payouts to investors $dDiv_t \geq 0$, and transfers to/from the intermediary $dI_t \gtrless 0$ flow through the cash balance M_t. Unlike investors, the intermediary can provide financing and inject cash into the firm at any time, so dI_t can be negative. However, this source of financing is costly, as we formalize below.

The cash balance of the firm at $t = 0^-$, i.e., before the contract is signed, to zero so that $M_{0^-} = 0$. We assume that the firm cannot borrow, except from the intermediary, so that cash holdings must remain non-negative, $M_t \geq 0$ for all $t \geq 0$. This constraint implies that if M_t reaches zero, the intermediary must either inject the necessary funds

[4] One may interpret the time it takes to arrange for financing as proxying for asymmetric information — outside investors take time to verify information. The intermediary as a specialist does not face such a delay.

[5] This assumption is standard (see, e.g., Décamps et al. 2011 and Bolton et al. 2011) and prevents the firm from saving itself out of the constraint. Assuming impatient investors leads to similar results.

or the firm must liquidate. Liquidation thus occurs at a stopping time $\tau \in [0, \infty]$, and $dDiv_t = dI_t = dX_t = 0$ for all $t > \tau$. We assume that the liquidation value of the firm's assets excluding cash is $L \in [0, \mu/r)$, so that the firm's total liquidation value is $M_\tau + L$ and liquidation is costly compared to first-best. In parallel with the current legal system, in liquidation, rights on the liquidation value L can be pre-assigned and cannot be reneged upon. We denote intermediary's payment of liquidation proceeds by $dI_\tau \in [0, L]$.[6]

1.1 Optimal Contracting Problem

We stipulate that, given a contract $\mathcal{C}$, the intermediary's continuation value (in certainty equivalent or dollar terms) is

$$Y_t = \mathbb{E}_t \left[\int_t^\tau e^{-r(s-t)} \left(dI_s - k_s ds \right) \right]. \tag{3}$$

We also refer to Y_t as the intermediary's (future) promised payments or as the intermediary's stake because it represents a portion of firm value promised to the intermediary. In (3), k_t is the intermediary's endogenous cost of providing financing to the firm. We will discuss k_t and specify its functional form once we characterize the dynamics of Y_t. As the intermediary may represent a bank or non-bank lender, we interpret k_t as a proxy for the intermediary's limited risk-bearing capacity stemming from regulatory constraints, under-diversification, or limited capital of its own. We micro-found the representation (3) and the cost of funds k_s in Online Appendix L, where we assume that the intermediary has CARA preferences, deep pockets, and a private savings technology.

We assume that the intermediary has an outside option, which we normalize to zero. It can always part from the contract and receive its outside option whenever it is privately optimal to do so and is thus subject to limited commitment, i.e., $Y_t \geq 0$ at any time $t \geq 0$.[7]

We denote the firm's equity value, i.e., shareholders' value function, at time t by P_t. Upon access to equity financing $d\Pi_t = 1$, the firm raises ΔM_t dollars from competitive risk-neutral investors at fair value by issuing ΔM_t dollars worth of new equity. Refinancing changes *total* equity value from $\lim_{s \uparrow t} P_s$ pre-refinancing to $P_t := \lim_{s \downarrow t} P_s$ post-refinancing, i.e., P_t equals its right-limit at time t, while *existing* shareholders' are diluted and their post-refinancing payoff is $P_t - \Delta M_t$. Recall that existing shareholders cannot inject cash into the firm, i.e., $dDiv_t \geq 0$ for all $t \geq 0$ including t such that $d\Pi_t = 1$. Thus, their payouts and payoff are

[6] The constraint $dI_\tau \in [0, L]$ reflects the limited commitment of shareholders and intermediary: neither can promise payments upon liquidation in excess of liquidation proceeds.

[7] The model's results remain qualitatively unchanged for a negative lower bound $\underline{Y}$, i.e., $Y_t \geq \underline{Y}$. For simplicity, we normalize $\underline{Y} = 0$. This constraint also prevents the firm from raising a large amount of cash upon refinancing and saving with intermediary to circumvent the internal cost of cash.

non-negative, i.e., $P_t - \Delta M_t \geq 0$. The maximum amount of cash the firm can raise upon market access equals the post-refinancing equity value, i.e., $\Delta M_t = P_t$. If the firm raises this maximal amount of financing, newly arriving investors buy the firm's entire equity at fair price P_t, while existing shareholders are fully diluted.

At time t and given contract $\mathcal{C}$, equity value is the expected discounted stream of future dividend payouts net of the costs of refinancing via dilution

$$P_t = \mathbb{E}_t \left[\int_t^{\tau} e^{-r(s-t)} (dDiv_s - \Delta M_s d\Pi_s) \right], \tag{4}$$

As $dDiv_t \geq 0$ and $\Delta M_t \leq P_t$, we have $P_t \geq 0$.[8] The optimal contract maximizes

$$P_{0^-} = \max_{\mathcal{C}} \mathbb{E} \left[\int_{0^-}^{\tau} e^{-rt} (dDiv_t - \Delta M_t d\Pi_t) \right] \quad \text{s.t.} \quad Y_t, M_t, dDiv_t \geq 0, \text{ and } \Delta M_t \in [0, P_t] \tag{5}$$

for all $t \geq 0$, where intermediary's stake Y_t is given by (3) with initial value $Y_{0^-} = 0$ and cash M_t follows (2) with initial balance $M_{0^-} = 0$, and equity value P_t is given by (4).

2 Model Solution

In this section, we solve the model and derive the optimal contract. We gain tractability by showing that a sufficient statistic for the state of the firm is the difference between the firm's cash holdings and the intermediary's future promised payouts, which we term excess liquidity. The following Lemma, proven in Online Appendix A, derives the dynamics of Y_t.

Lemma 1. *The intermediary's continuation payoff evolves according to*

$$dY_t + dI_t = (rY_t + k_t)\, dt + \beta_t \sigma dZ_t + \alpha_t (d\Pi_t - \pi dt), \tag{6}$$

where β_t captures the intermediary's exposure to Brownian cash flow shocks dZ_t and α_t captures the intermediary's exposure to the (compensated) market access process $(d\Pi_t - \pi dt)$.

We refer to equations (3) and (6) as the promise keeping constraints. It means that current transfers dI_t must be accompanied by a commensurate change in future promised transfers dY_t. Notice that promise-keeping requires that at the time of liquidation τ, the intermediary receives a lumpy payout of $dI_\tau = \lim_{t\uparrow\tau} Y_t$ dollars, where the left limit $\lim_{t\uparrow\tau} Y_t$ denotes the continuation payoff "just before" liquidation. In Online Appendix B, we impose

[8] To express P_t recursively, let $\tau_t = \inf\{s \geq t : d\Pi_s = 1\}$ denote the next time of refinancing after time t. Then $P_t = \mathbb{E}_t \left[\int_t^{\tau_t \wedge \tau} e^{-r(s-t)} dDiv_s + \mathbb{1}_{\{\tau_t < \tau\}} e^{-r(\tau_t - t)} \left(P_{\tau_t} - \Delta M_{\tau_t} \right) \right]$. $P_t \geq 0$ follows from $P_t - \Delta M_t \geq 0$.

standard regularity conditions on α_t and β_t which are needed in the formal proofs but do not play a role in the main text.

Intermediary cost of financing. Next, we specify k_t to coincide with the intermediary's cost of financing obtained under CARA(ρ) preferences:

$$k_t = \sigma^2 \cdot \underbrace{\frac{\rho r}{2}\beta_t^2}_{\equiv k_Z(\beta_t)} + \pi \cdot \underbrace{\left(\alpha_t - \frac{1 - e^{-\rho r \alpha_t}}{\rho r}\right)}_{\equiv k_\Pi(\alpha_t)}. \tag{7}$$

The intermediary charges a risk-premium $k_Z(\beta)$ for β exposure to dZ_t (scaled by σ^2), while charing $k_\Pi(\alpha)$ for α exposure to $(d\Pi_t - \pi dt)$ (scaled by π). Thus, using financing from the intermediary to absorb a fraction of cash flow shocks, i.e., $\beta_t > 0$, and delaying payouts to the intermediary to random future refinancing dates, i.e., $\alpha_t > 0$, are both costly.

The specific form of k_t is micro-founded in Online Appendix L by the intermediary having CARA preferences with risk-aversion ρ and a private savings/borrowings technology at rate r.[9] We interpret $1/\rho$ as the intermediary's limited risk-bearing capacity due to regulatory or capital constraints. Further, we assume this risk-bearing capacity is constant for two reasons: First, the intermediary, which may represent a group of bank or non-bank lenders investing in many firms, is large relative to the firm. Thus, while the intermediary requires some compensation for bearing firm risk, its risk-bearing capacity is not significantly affected by the performance of one individual firm. Second, this assumption lends tractability to our model; otherwise, one would have to track the additional state variables that drive the intermediary's risk-bearing capacity such as net worth.[10]

Benchmarks. Finally, we introduce two benchmark valuations of the firm. First, consider the firm's value when $\pi \to \infty$ so that it has continuous access to new equity financing. In this case, the firm does not need to retain any cash as it can cover cash flow shortfalls by raising new financing at will. The value of the firm then is simply its first-best net present value (NPV)

$$NPV \equiv \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)} dX_s\right] = \frac{\mu}{r}. \tag{8}$$

Next, consider the *autarky* value of the firm to the intermediary. Suppose the intermediary owns the entire firm without access to outside financing and commits to continue opera-

[9] Our results remain qualitatively similar if we assume a simpler non micro-founded functional form for k_t, e.g., $k_t = \rho\sigma^2\beta_t$ with $\beta_t \geq 0$ and $k_\Pi(\alpha) = 0$. Under this specification, the intermediary exhibits risk-aversion only toward Brownian risk. A linear cost k_t thus allows one to aggregate individual, identical intermediaries cleanly into one representative intermediary. See Online Appendix N for details.

[10] We could assume that equity investors are risk-averse, in that they apply a stochastic discount factor. As long as it is not optimal to sell the entire firm to the intermediary, the model's dynamics remain similar.

tions indefinitely. The firm does not retain cash due to the firm's carry-cost-of-cash. The intermediary then must fully absorbs all shocks. Setting $dI_s = dX_s$, $\beta_s = 1$, and $\alpha_s = 0$ in equation (3), we calculate the autarky value of the firm to the intermediary under continued operation as

$$Y^A \equiv \frac{\mu - \sigma^2 k_Z(1)}{r} = \frac{\mu}{r} - \frac{\rho}{2}\sigma^2. \tag{9}$$

Then, if $Y^A < L$, the intermediary is unwilling to operate the firm in autarky, and instead immediately liquidates the firm for a value L. The net value of the firm is the sum of the shareholders' value function P_t and the intermediary's stake Y_t less the current cash-holdings M_t. For finite π and positive ρ, the net value satisfies

$$\max\left\{Y^A, L\right\} \leq P_t + Y_t - M_t < NPV. \tag{10}$$

2.1 Dynamic Program and HJB Equation

In principle, the dynamic optimization of the shareholders' value function depends on two state variables: the intermediary's continuation payoff Y_t, and the firm's cash holdings M_t. We heuristically show how to reduce the problem to a single state variable called *excess liquidity* which is the difference of cash and continuation value:

$$C_t \equiv M_t - Y_t. \tag{11}$$

Combining (2) and (6), excess liquidity C has the following law of motion:

$$dC_t = dM_t - dY_t = \mu_C dt + \sigma_C dZ_t + \left(C_t^* - C_t\right) d\Pi_t - dDiv_t, \tag{12}$$

where

$$\begin{aligned} \mu_{C,t} = \mu_C(C_t) =& \mu + (r - \lambda) C_t - \lambda Y_t - \sigma^2 k_Z(\beta_t) + \pi\left(\alpha_t - k_\Pi(\alpha_t)\right), \\ \sigma_{C,t} = \sigma_C(C_t) =& \sigma\left(1 - \beta_t\right), \\ C_t^* = C^*(C_t) =& \Delta M_t + C_t - \alpha_t, \end{aligned}$$

and C_t^* is the level of excess liquidity immediately after refinancing. As we show below, C_t becomes a sufficient state variable of the firm, so we can express drift and volatility of dC_t as well as C_t^* as functions of C_t, i.e., $\mu_{C,t} = \mu_C(C_t)$, $\sigma_{C,t} = \sigma_C(C_t)$, and $C_t^* = C^*(C_t)$.

Reduction of the state space. Let us rotate the state space and, instead of (M_t, Y_t), we work with (C_t, Y_t) as state variables (the formal argument is given in Online Appendix B.1).

Under the promise-keeping constraint (6), a transfer of cash dI_t between the balance M_t of the firm and the intermediary must be accompanied by a commensurate increase or decrease in the intermediary's stake Y_t. Thus, $C_t = (M_t - dI_t) - (Y_t - dI_t) = M_t - Y_t$ is invariant to transfers dI_t. Since dI_t can be positive or negative, we can use transfers to adjust Y_t freely without affecting excess liquidity C_t. Therefore, Y_t is effectively a control variable, albeit a constrained one: First, the intermediary's limited commitment requires $Y_t \geq 0$. Second, the definition of excess liquidity (11) and the physical constraint on cash $M_t \geq 0$ together imply that $Y_t \geq -C_t$. Combining,

$$Y_t \geq \max\{0, -C_t\}. \tag{13}$$

Excess liquidity C_t is negative if promised payments to the intermediary Y_t are larger than the cash balance M_t. Since Y_t is now a control variable, the only relevant state variable is C_t. Thus, the shareholders' value P_t only depends on C_t so that we can write $P_t = P(C_t)$. In what follows, we omit time subscripts unless necessary.

Refinancing limits. Refinancing via equity issuance moves excess liquidity from its pre-financing level C to the post-refinancing level C^*, with a prescribed increase of the intermediary's payoff $Y + I$ by α as per (6). Upon refinancing, the firm raises $\Delta M = C^* - C + \alpha$ dollars of cash to transition from C to C^* with an increase in $Y + I$ of α. Thus, refinancing in state C to C^* changes *existing* shareholders' payoff from $P(C)$ to $P(C) + J(C)$ by amount

$$J(C) \equiv P(C^*) - \Delta M - P(C) = [P(C^*) - C^*] - [P(C) - C] - \alpha, \tag{14}$$

while changing *total* equity by the amount $P(C^*) - P(C)$. Because existing shareholders cannot inject cash into the firm, their payouts as well as payoff are necessarily non-negative at all times. In particular, $P(C) + J(C) \geq 0$ at refinancing times. This is equivalent to $P(C^*) \geq \Delta M$, implying that the maximum dollar amount of new equity financing that can be raised equals the post-refinancing value $P(C^*)$. When $P(C^*) = \Delta M$, existing shareholders are fully diluted and newly arriving investors buy the firm's entire equity at fair price $P(C^*)$. Note that $P(C) + J(C) \geq 0$ implies the following constraint on α:

$$\alpha \leq [P(C^*) - C^*] + C. \tag{15}$$

We denote $\mathcal{S}(C^*, C) = \{x : x \leq [P(C^*) - C^*] + C\}$ the set of all admissible choices for α.[11]

The HJB Equation. We now conjecture and verify that P_t can be expressed as a function of excess liquidity only, $P_t = P(C_t)$, which in turn implies that total net value of the firm

[11] For robustness, Section 5.1 introduces the generalized constraint $J(C) \geq -\nu P(C), \nu \in [0, 1]$, which is tighter than (15) for $\nu < 1$. We show that qualitatively the results remain unchanged.

also reduces to a function of excess liquidity only, $P_t + Y_t - M_t = P(C_t) - C_t$. To solve the shareholders' dynamic problem (5), we first maximize equity value $P(C)$ for a given level of excess liquidity C and then determine the optimal level of initial excess liquidity C_0. We conjecture that the firm optimally makes dividend payouts to investors at an endogenous upper boundary $C = \overline{C}$, and that it either liquidates or receives sufficient financing to stay alive at some endogenous lower boundary $\underline{C}$. The intuition behind this payout policy is that the firm's investors are the firm's shareholders and can only receive positive dividend payouts. Thus, dividends are only paid out when the firm has performed sufficiently well in the past and accumulated enough liquidity. For $C \in (\underline{C}, \overline{C})$, $dDiv = 0$ and equity solves

$$rP(C) = \max_{\beta, Y \geq \max\{0, -C\}} \left\{ P'(C) \left[\mu + (r - \lambda) C - \lambda Y - \sigma^2 k_Z(\beta) \right] + P''(C) \frac{\sigma^2}{2} (1 - \beta)^2 \right\}$$
$$+ \pi \max_{C^*, \alpha \in \mathcal{S}(C^*, C)} \left\{ P'(C) [\alpha - k_\Pi(\alpha)] + [P(C^*) - C^*] - [P(C) - C] - \alpha \right\}. \quad (16)$$

Note that the right-hand-side of (16) only depends on the state variable C and control variables (α, β, Y, C^*) with constraints that depend on C and exogenous constants, confirming the conjecture that we can express equity value as a function of C alone, $P_t = P(C_t)$. Optimal control variables determined according to (16) are functions of C too.

Payout boundary. Payout boundary satisfies smooth pasting and super contact conditions,

$$P'(\overline{C}) = 1 \quad \text{and} \quad P''(\overline{C}) = 0. \quad (17)$$

For $C > \overline{C}$, the firm pays out $C - \overline{C} > 0$ dollars, and C drops immediately to $\overline{C}$, implying $P(C) = P(\overline{C}) + (C - \overline{C})$ for $C \geq \overline{C}$. For now, we assume that a well-behaved, non-negative, and twice continuously differentiable solution to (16) exists on the endogenous state space $(\underline{C}, \overline{C})$ subject to (17). In Online Appendix I, we establish existence of such a solution, and show that the dynamic contracting problem (5) has a unique solution.

The following Proposition summarizes our findings, formally proven in Online Appendix B.

Proposition 1. *Equity value under the optimal contract can be expressed as function of C only, $P_t = P(C_t)$, and solves the HJB equation (16) on the endogenous state space $(\underline{C}, \overline{C})$ subject to (17). Equity value is strictly concave on $(\underline{C}, \overline{C})$, so that $P''(C) < 0$ and $P'(C) > 1$ for $C < \overline{C}$. Optimal dividend payouts $dDiv$ cause C to reflect at $\overline{C}$ and are zero in the interior of the state space. The payout boundary is strictly positive, $\overline{C} > 0$.*

For $C < \overline{C}$, the value function is strictly concave, so $P''(C) < 0$ and $P'(C) > 1$. The concavity of equity value implies that the firm's shareholders are effectively risk-averse, since access to external equity financing is limited and hence negative cash flow shocks can trigger financial distress. To be able to withstand cash flow shocks absent access to equity financing, the firm accumulates cash and delays payouts until C reaches a strictly positive $\overline{C}$.[12]

2.2 Optimal State-Dependent Control Variables

We solve the optimization in the HJB equation (16) to characterize the four control variables:

Refinancing target C^*. First, the first order condition with respect to C^* yields $P'(C^*(C)) = 1$, so that refinancing occurs up until a point at which the internal value of cash is equalized with the value of paying it out. Recall that $P'(\overline{C}) = 1$ at the payout boundary $\overline{C}$ and $P'(C) > 1$ for $C < \overline{C}$, so that without loss of generality we set $C^* = C^*(C) = \overline{C}$.[13]

Intermediary stake Y. Second, since $P'(C) > 0$ and $\lambda > 0$, the optimal contract picks the lowest Y possible subject to constraint (13), so that

$$Y(C) = \max\{-C, 0\} \quad \text{and} \quad M(C) = \max\{C, 0\}. \tag{18}$$

As holding cash is costly, it is optimal to minimize cash holdings $M = C + Y$ given excess liquidity C. If $C > 0$, the firm holds cash $M = C > 0$, and the intermediary's stake Y is zero. If $C < 0$, the firm holds no cash, but the intermediary's stake $Y = -C$ is positive. Thus, by (12), changes in promised payments to the intermediary follow

$$dY = \begin{cases} 0 & \text{for} \quad C > 0 \\ [rY - \mu + \sigma^2 k_Z(\beta) - \pi(\alpha - k_\Pi(\alpha))]\, dt - (1-\beta)\sigma dZ - Y d\Pi & \text{for} \quad C < 0. \end{cases} \tag{19}$$

Combining this with the promise keeping constraint (6), we have that for any α, β the transfers to/from the intermediary before liquidation are given by

$$dI = \begin{cases} [\sigma^2 k_Z(\beta) - \pi(\alpha - k_\Pi(\alpha))]\, dt + \beta\sigma dZ + \alpha d\Pi & \text{for} \quad C > 0 \\ \mu dt + \sigma dZ + (\alpha + Y)\, d\Pi & \text{for} \quad C < 0. \end{cases} \tag{20}$$

Alternatively, we write (20) as $dI = \mu_I dt + \sigma_I dZ + \alpha_I d\Pi$ with drift $\mu_I = \mu_I(C)$, volatility $\sigma_I = \sigma_I(C)$, and $\alpha_I = \alpha_I(C)$. We note that upon liquidation at time τ (with $\tau = \infty$ possible), promise-keeping (6) requires that the intermediary receives a lumpy payout dI_τ equal to the value of promised payments Y_τ at or "just before" liquidation.

[12]To see that $\overline{C} > 0$, evaluate the ODE (16) at $\overline{C}$ (with $\alpha(\overline{C}) = \beta(\overline{C}) = 0$ as shown later) to obtain $P(\overline{C}) - \overline{C} = \frac{\mu}{r} - \frac{\lambda}{r}\left(\overline{C}\mathbb{1}_{\{\overline{C}\geq 0\}}\right)$. This must be strictly lower than the NPV $\frac{\mu}{r}$, implying $\overline{C} > 0$.

[13]Any $C^* > \overline{C}$ fulfills the first-order condition but leads to an immediate payout of $C^* - \overline{C} > 0$, causing C to drop to $\overline{C}$. Setting $C^* = \overline{C}$ minimizes these round-trip transactions.

Intermediary exposure to market access α. Third, setting the derivative with respect to α in (16) to zero when (15) is not binding yields

$$\alpha = \alpha_U(C) \equiv \frac{\ln P'(C)}{\rho r}, \tag{21}$$

while if (15) is binding we have $\alpha = \alpha_C(C) \equiv P(\overline{C}) - \overline{C} + C$. The optimal $\alpha = \alpha(C)$ is then characterized by

$$\alpha(C) = \min\{\alpha_U(C), \alpha_C(C)\}, \tag{22}$$

As $\alpha(C) > 0$ is a promised payout to the intermediary upon (stochastic) access to equity financing $d\Pi$, all else equal, it lowers the intermediary's required flow compensation, i.e., the drift of $dY + dI$, and thus increases the drift of dC. Intuitively, $\alpha(C) > 0$ shifts payouts to the intermediary from states in which the firm is financially constrained to states in which the firm is effectively unconstrained. As we show later in Lemma 2, the lower boundary satisfies $\underline{C} \geq -[P(\overline{C}) - \overline{C}]$, which implies $\alpha(C) \geq 0$.

Instantaneous risk-sharing β. Fourth, the first order condition with respect to β yields

$$\beta(C) = \frac{P''(C)}{P''(C) - \rho r P'(C)} \in [0, 1]. \tag{23}$$

Setting $\beta > 0$ transfers cash-flow risk to the intermediary, reducing the volatility of excess liquidity. However, setting $\beta > 0$ is costly due to the intermediary's required risk-premium $k_Z(\beta)$, also reducing the drift of excess liquidity.

2.3 The Lower Boundary for Excess Liquidity

In this section, we determine the lower boundary $\underline{C}$. First, note that total firm value $Y + P(C)$ is lower than the firm's first-best value (with cash reserves M), i.e., $Y + P(C) \leq \frac{\mu}{r} + M$. Due to $P(C) \geq 0$, we have $Y \leq \frac{\mu}{r} + M$ and $C \geq -\frac{\mu}{r}$, so C is bounded from below with probability one. Therefore, there must exist an endogenous lower boundary $\underline{C} \geq -\frac{\mu}{r}$ such that $C_t \geq \underline{C}$ at all times t. For $\underline{C}$ to be a lower bound for C with dynamics (12), it must be that either (i) the firm liquidates at $\underline{C}$, in which case we denote the lower boundary by C^L, or that (ii) $\underline{C}$ is either a reflection, inaccessible or an absorbing state (absent refinancing), in which case we denote the lower boundary by C^S and the firm is not liquidated at $\underline{C} = C^S$.

Liquidation. First, consider liquidation, i.e., case (i). As long as $L > 0$, the firm can always be kept alive until C reaches $-L$ without violating promise keeping vis-a-vis the intermediary. Thus, the firm need not liquidate the first time it runs out of cash at $C = M = 0$. Next, we establish that conditional on liquidation, it is optimal to liquidate at the lowest value C that does not violate promise keeping, leading to $\underline{C} = C^L = -L$, zero cash holdings at liquidation $M(C^L) = 0$, and $P(C^L) = 0$. The total liquidation value at any level of excess liquidity C is $M(C) + L$, which is divided between the intermediary and the investors, so that $P(C) + Y(C) = M(C) + L$ at the time of liquidation. Consider liquidation at a value $\hat{C} > -L$ with payouts $Y(\hat{C})$ to the intermediary and $\hat{P}(\hat{C}) = M(\hat{C}) + L - Y(\hat{C}) = L + \hat{C}$ to the investors. Compare this to the value function obtained under liquidation at $C^L = -L$, which satisfies $P(-L) = 0$, $P'(C) > 1$ for $C \in (-L, \overline{C})$, and $P'(C) = 1$ for $C \geq \overline{C}$. Then $P(\hat{C}) > \hat{C} + L = \hat{P}(\hat{C})$ and it is sub-optimal to liquidate at $\hat{C} > -L$. Intuitively, because liquidation is costly, it is optimal to delay it as long as possible without violating promise keeping. Thus, conditional on liquidation, the firm optimally liquidates at $C = -L$, with the intermediary receiving the full liquidation value of assets as a lump sum payout, $dI_\tau = L$.

Survival. Second, consider no liquidation, i.e., case (ii). For $\underline{C} = C^S$ to be a lower bound of C in the absence of liquidation, which we term "survival", it must not be crossed. This requires that as C approaches $\underline{C}$ the volatility of excess liquidity $\sigma_C(C)$ must vanish, requiring $\beta(C) = 1$, while its drift $\mu_C(C)$ and the shareholders' value function $P(C)$ both must stay non-negative. The intuition is that at $C = \underline{C}$, the intermediary keeps the firm alive by providing continuous financing and absorbing all cash flow risks through $\beta(\underline{C}) = 1$. However, it is optimal to delay setting $\beta = 1$ as long as possible due to the intermediary's cost of bearing risk. Thus, the lower boundary C^S is determined as the lowest level C at which $\mu_C(C) \geq 0$, $P(C) \geq 0$, and $\beta(C) = 1$ simultaneously hold. In Online Appendix C.2 we show that these inequalities hold with equality at $C = C^S$, as well as $\alpha(C^S) = \alpha_C(C^S)$, so

$$C^S = \frac{w\left(\frac{\pi}{r}\exp\left\{\rho r\left[\frac{\lambda}{r}\overline{C} + \frac{\pi}{\rho r^2} - \frac{\rho}{2}\sigma^2\right]\right\}\right) - \frac{\pi}{r}}{\rho r} - Y^A, \tag{24}$$

where $w(\cdot)$ is the primary branch of the Lambert-W function.

Combining, we have the following Lemma (formally proven in Online Appendix C):

Lemma 2. *The lower boundary and the associated value of equity are given by*

$$\underline{C} = \min\left\{C^S, C^L\right\} \quad \text{with} \quad P\left(\underline{C}\right) = 0, \tag{25}$$

where C^S is given in (24) and $C^L = -L$. When $\underline{C} = C^S$, the firm is never liquidated, and $\tau = \infty$. When $\underline{C} = C^L$, the firm defaults the first time C_t attains C^L, i.e., $\tau = \inf\{t \geq 0 : C_t = C^L\}$. The lower boundary satisfies $\underline{C} \in [-(P(\overline{C}) - \overline{C}), -L]$, where $P(\overline{C}) - \overline{C} \leq \frac{\mu}{r}$.

At the lower boundary $\underline{C}$, the shareholders' value function equals $P(\underline{C}) = 0$, the constraint (15) binds, and the intermediary's stake $Y(\underline{C}) = -\underline{C}$ coincides with total firm value. In other words, the intermediary owns the entire firm at $\underline{C}$, and $Y(\underline{C}) = -\underline{C}$ is the intermediary's valuation of the entire firm. On the one hand, when $\underline{C} = -L$, the intermediary finds it optimal to liquidate the firm at the lower boundary, so the intermediary's valuation of the firm equals its liquidation value L. On the other hand, when $\underline{C} = C^S$, the intermediary optimally provides all necessary financing to keep the firm alive and the firm is never liquidated, so the intermediary's valuation of the firm exceeds its liquidation value.[14]

To gain intuition about how the intermediary values the firm at $\underline{C} = C^S$, we consider an approximation in which we remove the limit on the intermediary's risk-bearing capacity for the uncertain refinancing time, i.e., $\pi k_\Pi(\alpha) = 0$:[15]

$$Y(C^S) = -C^S \approx \left\{ \frac{r}{r+\pi} Y^A + \frac{\pi}{r+\pi} \left[P\left(\overline{C}\right) - \overline{C} \right] \right\}. \tag{26}$$

The (approximate) intermediary's valuation of the entire firm $Y(C^S)$ is the weighted average of its autarky valuation Y^A and the gains from selling the firm to outside investors at the fair price $\left[P\left(\overline{C}\right) - \overline{C}\right]$ upon the next equity financing opportunity. Provided that the resale value $\left[P\left(\overline{C}\right) - \overline{C}\right]$ or the frequency of equity financing opportunities π are large enough, the intermediary might keep the firm alive, i.e., $\underline{C} < -L$, even when $Y^A < L$.

The following Proposition, formally proven in Online Appendix E, states that if the lower boundary is C^S, whilst conditionally absorbing, it is never attained, which implies the existence of a non-degenerate stationary distribution over C even when $\pi = 0$. We will use this stationary distribution at a later point in our analysis to simulate firm outcomes.

Proposition 2. *When $\underline{C} = C^S$, then the lower boundary is never attained and a non-degenerate stationary distribution of states C exists with support $(\underline{C}, \overline{C})$.*

2.4 Optimal Contract and Dynamics

We now characterize the optimal financing arrangement by summarizing our previous results.

Proposition 3. *Under the optimal contract, the shareholders' value function $P(C)$ solves on $(\underline{C}, \overline{C})$ the HJB equation (16) with boundary conditions (17) and $P(\underline{C}) = 0$ where $\underline{C}$ is given by (25). Dividend payouts cause C to reflect at the payout boundary $\overline{C} > 0$. Optimal controls are characterized by (18) (22), (23), and $C^* = \overline{C}$. The value function is concave,*

[14]The intermediary never liquidating the firm for $\underline{C} = C^S$ is a consequence of stationarity of dX_t and $d\Pi_t$.

[15]For a derivation, set $\beta(\underline{C}) = 1$, $\alpha(\underline{C}) = P(\overline{C}) - \overline{C} + \underline{C}$, $Y(\underline{C}) = -\underline{C}$, and $\pi k_\Pi(\alpha_t) = 0$ into the drift term of (12) to obtain $\mu_C(\underline{C}) = \mu - \sigma^2 k_Z(1) + (r+\pi)\underline{C} + \pi[P(\overline{C}) - \overline{C}]$. By Y^A from (9), we can solve $\mu_C(\underline{C}) = 0$ for the result. This approximation is precise for $\rho \to 0$ or $\pi \to 0$, since either limit implies $\pi k_\Pi(\alpha) \to 0$.

i.e., $P''(C) < 0$ for $C < \overline{C}$. Before liquidation at time τ, the associated transfers $dI = dI(C)$ and promises to the intermediary $dY = dY(C)$ are given in (20) and (19).

If $\underline{C} = C^L = -L$, the firm liquidates once C reaches the lower boundary $\underline{C}$ and $\beta(C) < 1$ for all $C \geq \underline{C}$. Then, the intermediary receives a lump-sump payment of $dI_\tau = L$, equal to the value of her promised payments at liquidation $Y(C^L) = L$. If $\underline{C} = C^S$, the firm never liquidates (i.e., $\tau = \infty$) and $\beta(\underline{C}) = 1$ while $\beta(C) < 1$ for all $C > \underline{C}$.

The formal proof is given in Online Appendix F. The firm's initial liquidity C_0 coincides with the payout boundary

$$\overline{C} \in \arg\max_{C_0} [P(C_0) - C_0], \tag{27}$$

which is strictly positive. The initial value of the original shareholders in (5) is $P_{0-} = P_0 - \Delta M_0 = P(\overline{C}) - \overline{C}$ as the firm raises $\Delta M_0 = \overline{C}$ from new investors who pay fair value and, since $Y_0 = Y(\overline{C}) = 0$ and $\alpha_0 = 0$, the intermediary does not provide initial financing.

How does the firm finance cash flow shortfalls under the optimal contract? When $C > 0$, the firm holds cash $M(C) = C > 0$ and covers a negative cash flow shock $dZ < 0$ partially out of the cash balance and through the intermediary, while the intermediary does not have a stake in the firm. Specifically, a fraction $1 - \beta(C)$ of a negative cash flow shock $dZ < 0$ is covered from the firm's cash and the fraction $\beta(C)$ is covered by the intermediary as part of the risk-sharing agreement; that is, the volatility of dI equals $\sigma\beta(C)$ in this case as (20) shows. In contrast, when $C \leq 0$, the firm's cash balance is exhausted and the intermediary provides financing in exchange for future promised payments. A negative cash flow shock of, say, one dollar ($\sigma dZ = -1$) is fully financed by the intermediary (so $dI = -1$), in that the volatility of payouts dI equals σ. The intermediary is only partially compensated via an increase of future promised payments by $dY = (1 - \beta(C))$ for covering this cash flow shock of one dollar, hence it bears fraction $\beta(C)$ of cash flow risk according to the risk-sharing agreement. Section 4 shows that there is a natural implementation of the intermediary financing contract via the combination of an unsecured, risky credit line, implementing risk-sharing, and a secured, risk-free credit line, implementing financing against promises.

Numerical illustration. To illustrate the contract dynamics, we present numerical examples based on the parameters given in Table 1. We follow Bolton et al. (2011) in setting r, μ, λ. We set the liquidation value to $L = 1$, which is about 33.3% of the firm's $NPV = \mu/r = 3$ in line with the liquidation values of nonfinancial firms reported in Kermani and Ma (2023). We take the intermediary's CARA coefficient $\rho = 6$, similar to He (2011), and normalize $\sigma = 1$, which also normalizes $Y^A = 0$. In the absence of refinancing opportunities, that is, $\pi = 0$, the firm is liquidated at $C = C^L = -L$. In the baseline, we pick $\pi = 0.5$ (that is, expected time until the next market access is $1/\pi = 2$ years) leading to $\underline{C} < -L$ and the firm

Parameter	Value	Interpretation
r	0.06	Common discount & interest rate
λ	0.01	Internal carry cost of cash
μ	0.18	Drift of cash flow process
σ	1	Volatility of cash flow process
ρ	6	Intermediary's CARA coefficient
π	0.5	Arrival rate of equity financing opportunities
NPV	3	First-best value of firm (μ/r)
Y^A	0	Autarky value of firm to intermediary
L	1	Liquidation value

Table 1: Baseline Parameter Values for all Figures.

does not liquidate; we also provide a plot for $\pi = 0.01$ in which case the firm is liquidated at $C = C^L = -L$.[16] The model's qualitative outcomes are robust to the choice of these parameters. Figure 1 illustrates the contract dynamics by plotting $\beta(C)$ and $\alpha(C), \alpha_U(C)$ against C in the state space $(\underline{C}, \overline{C})$, both for $\pi = 0.5$ (see Panels A and C) and $\pi = 0.01$ (see Panels B and D). The payout boundary $\overline{C}$ (lower boundary $\underline{C}$) is indicated as a vertical red (blue) line. In the right panels for $\pi = 0.01$, the firm liquidates at $C^L = -L$, with $\beta(\underline{C}) < 1$. In the left panels, $\pi = 0.5$ and $C^S < -L$, so the firm never liquidates and $\lim_{C \to \underline{C}} \beta(C) = 1$.

The upper row of Figure 1 (see Panels A and B) shows that the intensity of risk-sharing $\beta(C)$ decreases with C or, alternatively, increases with how financially constrained the firm is. Due to financial constraints, shareholders are effectively risk-averse, i.e., $P''(C) < 0$. It now becomes optimal to share risk with the intermediary through $\beta(C) > 0$, which is costly due to the intermediary's limited risk-bearing capacity. Because the effective risk aversion of shareholders decreases in C, while the intermediary's risk-bearing capacity $1/\rho$ is constant, $\beta(C)$ decreases with C.

The lower row of Figure 1 (see Panels C and D) shows that, provided (15) does not bind, $\alpha(C) = \alpha_U(C)$ smoothly decreases with C. Recall that α can be understood as a costly financing or risk-sharing instrument. Setting $\alpha(C) > 0$ essentially transforms flow payouts to the intermediary today (i.e., from states in which the firm is constrained) into a promised lumpy payout upon refinancing in the future (i.e., a state in which the firm is financially unconstrained). This transfer of promised payments to the future is relatively more beneficial when excess liquidity is lower and the firm is more constrained, but exposing the intermediary to shock $d\Pi$ through $\alpha > 0$ is costly. The optimal choice of α trades off relaxation of financial

[16] Our choice $\pi = 0.5$ follows Hugonnier et al. (2015) who assume an arrival rate of refinancing opportunities of 2 and incumbent shareholders' bargaining power of 0.25, resulting in an *effective* arrival rate of $0.25 \cdot 2 = 0.5$.

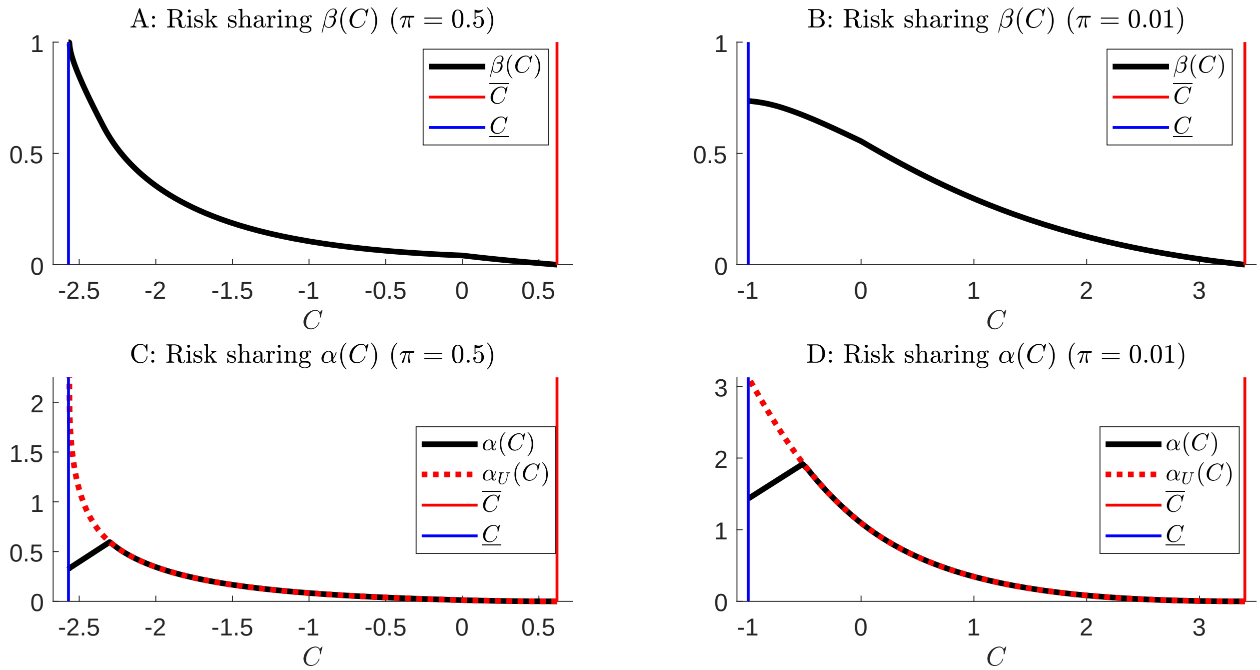

Figure 1: **Contract Dynamics.** This figure plots $\beta(C)$ and $\alpha(C)$ against C both for $\pi = 0.5$ (Panels A and C) and $\pi = 0.01$ (Panels B and D). The parameters follow Table 1.

constraints now versus larger payments to the intermediary in the future, so unconstrained $\alpha_U(C)$ decreases with financial slack C. However, limited liability limits what amount $\alpha(C)$ can be promised, as shown in the constraint (15). It is binding when C is low and close to $\underline{C}$, which we term financial distress. In this case, $\alpha(C)$ mechanically increases with C, as additional liquidity C relaxes the constraint (15). In summary, intermediary financing through the three instruments α, β, Y can be seen as a form of bridge financing that helps the firm cover financing needs and bridges the gap between equity financing rounds.

3 Analysis and Discussion

3.1 Cash Flow- vs. Asset-Based Financing Capacity

In the optimal contract, the intermediary provides financing against promised repayments, as long as the value of these promises Y does not exceed the firm's *financing capacity* defined as $\overline{Y} := Y(\underline{C})$. As formalized in the implementation of the optimal contract via secured and unsecured debt in Section 4, promises to the intermediary Y resemble a collateralized debt claim in the firm, with the firm as the collateral backing the claim. Financing capacity $\overline{Y} = -\underline{C} = \max\{L, -C^S\}$ is determined by either the liquidation value of assets L (asset-based) or the intermediary's going concern value of the firm's cash flows (cash flow-based).

When $\underline{C} = -L$, financing capacity is *asset-based*, andt the firm's financing from the intermediary is constrained by the liquidation value of its assets. Nonetheless, when $C < 0$, the firm taps into intermediary financing against future promised payments $Y(C) > 0$ and pledges the firm's assets to the intermediary so as not to default on the promises. Upon liquidation at $\underline{C} = -L$, the intermediary seizes the entire liquidation value L equal to its continuation payoff $Y(\underline{C}) = -L$.

When $\underline{C} = C^S$, financing capacity is *cash flow-based*, and the intermediary provides sufficient financing to prevent firm liquidation, for which it is compensated via future promised payments. The intermediary effectively obtains a stake in the firm which secures and collateralizes future promised repayments. Since this stake cannot exceed the value of the entire firm, the intermediary's valuation of the firm, including the value of future refinancing opportunities, constrains the amount of financing that the intermediary provides against promised repayments. Notably, cash flow-based financing capacity implies $\tau = \infty$ and so is associated with lower liquidation risk than asset-based financing capacity with $\tau < \infty$ (almost surely). Intuitively, for the firm to be able to obtain financing against future cash flows, the firm must not be liquidated beforehand so that these cash flows indeed realize.[17]

A cash flow-based financing capacity implies that cash flow-based financing becomes the *marginal* source of financing as C approaches $\underline{C}$, but need not preclude asset-based financing altogether. For example, suppose that financing capacity is cash flow-based and $\overline{Y} \geq Y > L$. Then, some part of the promises to the intermediary (i.e., $\min\{Y, L\}$ dollars) are backed by the asset liquidation value L, while the promises in excess of liquidation value (i.e., $\max\{Y - L, 0\}$ dollars) are cash flow-based. Indeed, as the implementation of the optimal contract from Section 4 illustrates, the firm generally relies on both asset- and cash flow-based credit line debt.

3.2 Determinants of Financing Capacity

When is the firm's financing capacity determined by the value of future cash flow or the liquidation value of assets? To this end, Figure 2 plots $\underline{C}$ and $\overline{C}$ against the expected time to refinancing $1/\pi$ (Panel A), cash flow drift μ (Panel B), cash flow volatility σ (Panel C), and intermediary CARA coefficient ρ (Panel D). Figure 2 shows that the firm's financing capacity $\overline{Y} = -\underline{C}$ increases with access to equity financing π, i.e., $\underline{C}$ increases with $1/\pi$, and the firm's cash flow rate μ, while it decreases with cash flow volatility σ and intermediary CARA coefficient ρ (recall that $1/\rho$ is the intermediary's risk-bearing capacity). Thus, financing capacity is cash flow-based (asset-based) for low (high) values of $1/\pi$, σ, ρ or when

[17] When the firm's financing capacity is asset-based, i.e., $\underline{C} = -L < C^S$, the firm could guarantee survival by using intermediary financing only up to C^S. However, this is sub-optimal as shown in Online Appendix C.

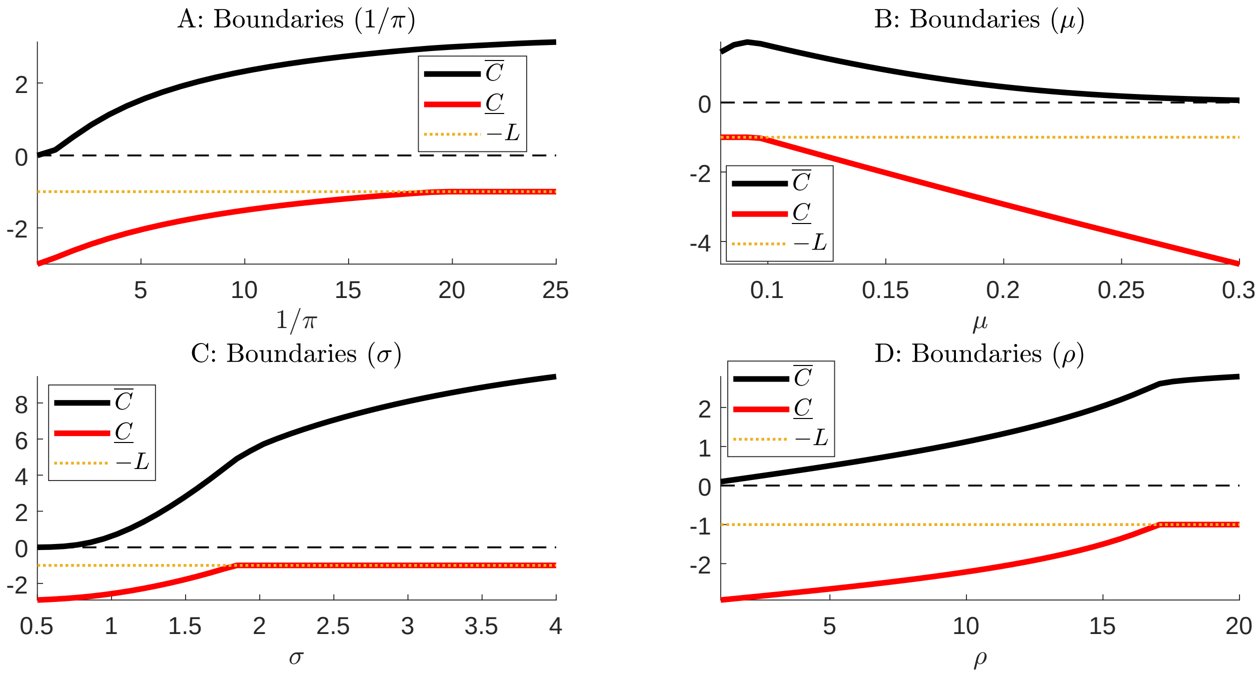

Figure 2: **Boundaries:** Comparative statics of the boundaries $\underline{C}, \overline{C}$ with respect to to the expected time until refinancing $1/\pi$ (top left panel), profitability μ (top right panel), cash flow volatility σ (bottom left panel), intermediary risk aversion ρ (bottom right panel). The parameters follow Table 1.

μ is high (low). Intuitively, intermediaries are more willing to provide financing against cash flow for firms with high profitability μ and less volatile cash flows σ. At the same time, low values of μ and large values of σ are associated with larger target cash holdings $\overline{C}$. [18] [19]

A key insight from the model is that a firm's financing capacity not only depends on firm characteristics but, as Figure 2 and the following Corollary show, also on intermediary characteristics, here the intermediary's risk-bearing capacity $1/\rho$, as well as market characteristics, here the firm's access to equity financing π.

Corollary 1. *When ρ is sufficiently large (small), then $\underline{C} = -L$ ($\underline{C} = C^S$) and financing capacity is asset-based (cash flow-based). Further, for $Y^A < L$, when $\pi \geq 0$ is sufficiently small (large) the financing capacity is asset-based (cash flow-based).*

To gain some intuition behind Corollary 1, consider the benchmark $\rho \to \infty$, so the intermediary has zero risk-bearing capacity, leading to $\beta(C) = \alpha(C) = 0$ and $\underline{C} = -L$. Note that the intermediary is still willing to provide financing against future promised payments as

[18] More volatile cash flows raise the risk of financial distress and the need for precautionary cash holdings (Décamps et al., 2011) which is exacerbated because an increase in σ also reduces financing capacity.

[19] $\overline{C}$ is non-monotone in μ because an increase in μ raises profitability and thus makes liquidation more costly while reducing the need for precautionary cash. The liquidation effect dominates for low μ because financing capacity is asset-based. Otherwise the precautionary saving effect dominates.

long as these promises are collateralized by the firm's liquidation value L. From (20) and (19), the intermediary provides asset-backed financing against the firm's liquidation value (which it seizes upon liquidation), with $dI = (dX + Y d\Pi)\mathbb{1}_{\{C<0\}}$ and $dY = (rY dt - dX - Y d\Pi)\mathbb{1}_{\{C<0\}}$ before the default time τ. Thus, although the intermediary covers all cash flow shocks when $C < 0$ (that is, the volatility of dI is σ), financing is risk-free because future promised payments grow at rate r and are fully backed by the liquidation value of assets, i.e., $dI + dY = rY dt$. Hence, the firm's financing capacity $\overline{Y}$ is asset-based for sufficiently large values of ρ. On the contrary, when $\rho \to 0$, so the intermediary has unlimited risk-bearing capacity, the intermediary values the firm at its NPV μ/r, so the firm's financing capacity $\overline{Y}$ is cash flow-based as $\mu/r > L$.

More generally, as Corollary 1 and Panel D of Figure 2 show, financing capacity decreases with ρ and is cash flow-based only for low values of ρ, while target cash holdings $\overline{C}$ increase with ρ. That is, a negative shock to intermediary risk-bearing capacity tightens financing constraints. Furthermore, intermediaries with higher risk-bearing capacity tend to provide more cash flow-based financing. This result is notable and perhaps counterintuitive, because cash flow-based financing is associated with no liquidation and thus seemingly risk-free, whereas the firm faces the risk of liquidation under asset-based financing. However, the reason is that cash flow-based financing requires the intermediary to have a high valuation for the firm which serves as collateral backing promised repayments. Since the firm's cash flows are risky and access to equity financing is uncertain, the intermediary's valuation of the firm as a going-concern is high only if the intermediary has sufficiently risk-bearing capacity.

In reality, different types of intermediaries may exhibit different risk-bearing capacities. For instance, because banks face regulatory capital constraints, they effectively have lower risk-bearing capacity and are likely to be characterized by lower $1/\rho$. On the other hand, direct lenders or private debt funds face less regulatory constraints than banks but still might be capital-constrained, so they are likely characterized by higher $1/\rho$, in line with their documented lending to riskier borrowers in practice (Chernenko et al., 2022).

To examine how capital market characteristics affect financing capacity, we first consider the limit case $\pi \to \infty$, leading to continuous and costless equity financing. Then, the financing capacity satisfies $\overline{Y} = \mu/r > L$ and thus is cash flow-based. Next, recall that financing capacity equals the intermediary's valuation of the entire firm, reflecting both the intermediary's autarky valuation as well as the resale option value. More frequent access to equity financing improves the intermediary's opportunities to exit by selling the firm to equity investors, hence boosting resale option value. As a result, it raises the intermediary's willingness to provide financing against promises and also the firm's financing capacity. Panel A shows that lower $1/\pi$, allowing the firm to raise new equity capital more frequently, increases financing capacity and reduces the reliance on costly precautionary cash holdings as captured by $\overline{C}$. An immediate consequence is that the better access to equity financing, the more likely the firm's financing capacity is cash flow-based rather than asset-based.

Finally, as formalized in Section 4, intermediary financing can be seen as (collateralized) debt financing. Thus, our results imply that debt-like intermediary financing and equity financing are *static* substitutes but *dynamic* complements. When the firm raises equity new financing, it repays the intermediary, in that C jumps to $\overline{C} \geq 0$ and Y to $Y(\overline{C}) = 0$; that is, equity financing substitutes for financing from the intermediary. On the other hand, the prospect of *future* access to equity financing improves financing capacity and thus the firm's access to financing against promises absent access to equity financing. The intuition is that the prospect of future access to equity financing ensures that the intermediary is repaid.

3.3 Tightness of Financing Constraints

In our model, not only is financing capacity endogenously determined, but also its utilization. To better connect to the empirical literature, we refer to the level of the firm's financing constraint $\overline{Y}$ as the firm's *financing capacity*. Further, we refer to the level of its usage of financing Y_t as *utilization* and its usage Y_t relative to its limit $\overline{Y}$ as the *tightness* of its financing constraint. As recent studies show, both the nature and tightness of financing constraints matter for corporate policies and investment (Chaney, Sraer, and Thesmar, 2012; Adler, 2020) as well as pin down firms' exposure to economic shocks that affect the availability of intermediary financing (e.g., financial crises) or the determinants of firms' financing constraints (e.g., aggregate profitability or uncertainty shocks like Covid-19).[20]

One key implication of our model is that firms with very high or low financing capacity $\overline{Y}$ tend to utilize intermediary financing the least and face on average the least tight financing constraints. This implication has several interesting aspects. First, better access to equity financing, higher profitability, or lower cash flow volatility — all of which imply a larger financing capacity $\overline{Y}$ — may increase the utilization of the financing capacity more than the capacity itself, thereby leading to tighter financing constraints. Second, firms with large financing capacity actually end up utilizing very little of it, precisely because their financing capacity reflects strong firm fundamentals (e.g., high profitability or low cash flow risk) that reduce the need for intermediary financing.

For analyzing the determinants of utilization of intermediary financing and constraint tightness, we focus on model parameters that induce survival $\underline{C} = C^S$ and thus admit a stationary distribution of the state variable C_t on the interval $(\underline{C}, \overline{C})$ as shown in Proposition 2.

[20]See, for instance, Ivashina, Laeven, and Moral-Benito (2020), Caglio, Darst, and Kalemli-Özcan (2021), Drechsel (2023), and Cloyne, Ferreira, Froemel, and Surico (2023).

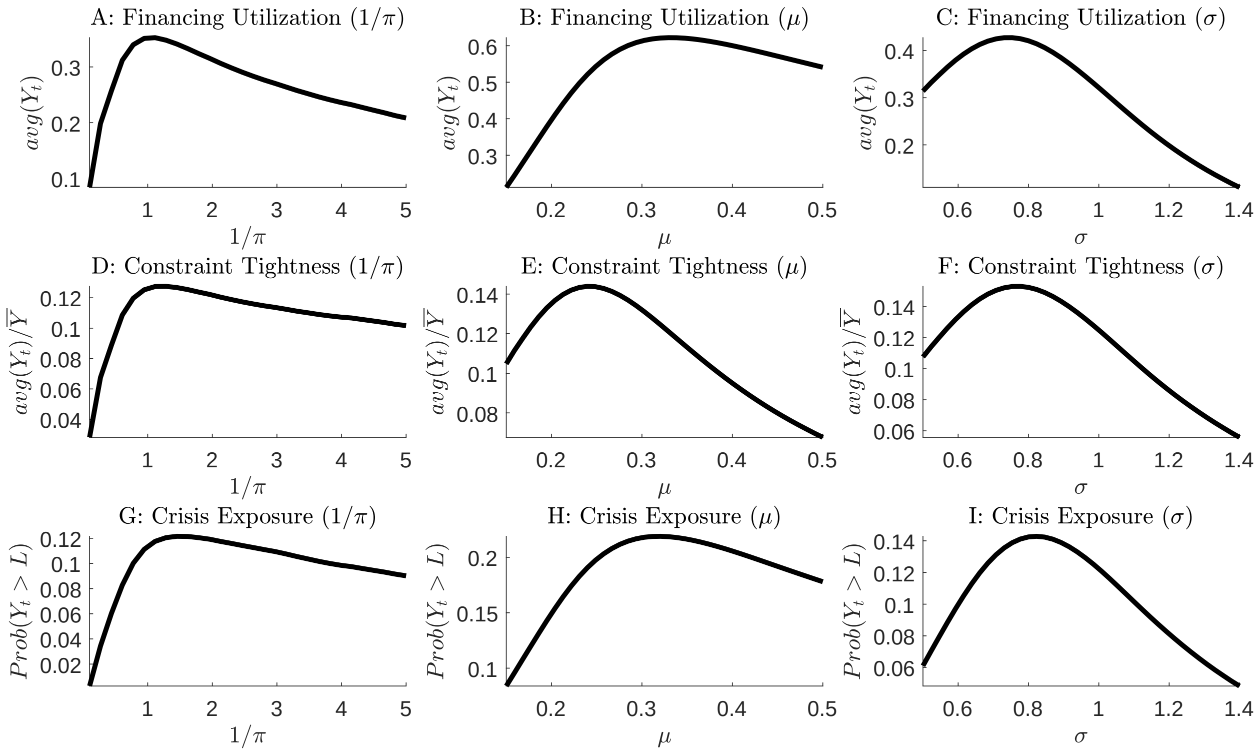

Figure 3: **Intermediary financing under survival.** This figure plots the average utilization of intermediary financing $avg(Y_t)$ (top row), the average tightness of financing constraints $avg(Y_t)/\overline{Y}$ (middle row), and crisis exposure $Prob(Y_t > L)$ (bottom row) against $1/\pi$ (left column), μ (middle column), and σ (right column). The parameters follow Table 1.

We then use the stationary distribution to calculate firms' average utilization of intermediary financing, i.e., $avg(Y_t)$, the average tightness of their financing constraints, i.e., $avg(Y_t)/\overline{Y}$, and the frequency that promises exceed liquidation value, i.e., $Prob(Y_t > L)$. We interpret $Prob(Y_t > L)$ as capturing the exposure to unforeseen economic shocks or financial crises. If there is a sudden unanticipated shock that sends intermediary risk-bearing capacity to zero ($1/\rho \to 0$) or freezes markets ($\pi \to 0$), cash flow-based intermediary financing evaporates. Thus, only asset-based intermediary financing is available and the financing capacity of all firms drops to L. Consequently, a fraction $Prob(Y_t > L)$ of the firms immediately default.

Figure 3 shows that $avg(Y_t)$, $avg(Y_t)/\overline{Y}$, and $Prob(Y_t > L)$ are all hump-shaped in $1/\pi$, μ, and σ. Firms with a relatively low financing capacity $\overline{Y}$ — characterized by high $1/\pi$, low μ, or high σ as seen in Figure 2 — do not rely much on intermediary financing, so their financing constraints on average are relatively loose. The intuition is that these types of firms accumulate a larger precautionary cash buffer $M(\overline{C}) = \overline{C}$ to rely less on intermediary financing. Surprisingly, even though these firms are characterized by a lower financing capacity, they face on average less tight financing constraints. Thus, an increase in μ or π, for example, can increase the utilization of intermediary financing more than the financing capacity, thereby tightening financing constraints. Finally, firms with intermediate financing capacity — characterized by intermediate levels of $1/\pi$, μ, or σ — rely on intermediary financing the most and so face the tightest financial constraints on average.

4 Implementing the Contract

Up to this point, the state variable for the contract was the net liquidity position of the firm $C = M - Y$, which contains the *forward* looking promises Y to the intermediary. While this is a natural variable for shareholders to consider when deciding on payout and financing policies, standard securities typically do not explicitly specify payouts contingent on the cash balance or similar measures of the firm's liquidity. For example, the relevant state variable for debt contracts and in particular credit lines is the borrowed amount or balance. We now provide an implementation for the optimal contract characterized by the *past* transfer process dI_t. Specifically, we first derive a variable that tracks past cumulative compounded transfers between the firm and the intermediary, reflecting an accrued principal balance that serves as a sufficient statistic for the contract. We then use this variable as the basis of our implementation of the contract in terms of credit lines.

4.1 Tracking the State of the Firm with Past Transfers

For any time $t > 0$, define the last refinancing time $\tau^{\Pi}(t) = \sup\{s \leq t : d\Pi_s = 1\}$. Next, define the cumulative net transfers received from the intermediary over $s \in (\tau^{\Pi}(t), t)$, with each transfer compounded at some rate $\int_s^t \hat{r}_u du$:

$$T_t \equiv -\int_{\tau^{\Pi}(t)}^{t} e^{\int_s^t \hat{r}_u du} dI_s. \tag{28}$$

The balance T_t quantifies the net amount of money that the intermediary has contributed to the firm over the time interval $(\tau^{\Pi}(t), t)$. To base the implementation of the optimal contract on T_t, it must be a sufficient statistic for C_t. Specifically, we are looking for a process $\hat{r}_t$ that makes T_t Markovian in C_t, with $T_t = T(C_t)$ and $T(\overline{C}) = 0$ because T_t is reset upon equity financing.

Proposition 4. *There exists a unique function $\hat{r}(C)$ with $\hat{r}(C_t) = \hat{r}_t$ which results in a unique (non-degenerate) Markovian process $T_t = T(C_t)$. Under the optimal $\beta(C)$, $T(C_t)$ satisfies*

$$T(C_t) = \alpha_U(C_t) + Y(C_t), \tag{29}$$

$T(\overline{C}) = 0$, *and* $T'(C) > 0$. *For any* $\alpha_t = \alpha(C_t)$ *differentiable almost everywhere, we have*

$$\hat{r}(C_t)T(C_t) = \frac{\lambda}{\rho r}\mathbb{1}_{\{C_t \geq 0\}} + rY(C_t) + \pi\left(\frac{e^{\rho r[\alpha_U(C_t)-\alpha(C_t)]} - 1}{\rho r}\right)\left(\frac{1-\alpha'(C_t)}{e^{\rho r \alpha_U(C_t)}}\right). \tag{30}$$

For a heuristic derivation of (29), notice from (28) that dT_t has volatility $-\sigma_I(C)$. At the same time, by Ito's Lemma, the volatility of $dT(C)$ is $T'(C)\sigma_C(C)$. Matching terms and using $\sigma_I(C)$ from (20) and $\sigma_C(C)$ from (12), we have that $T(C)$ solves the first-order ODE

$$-\sigma_I(C) = T'(C)\sigma_C(C) \quad \Longleftrightarrow \quad T'(C) = -\left(\frac{\left[\mathbb{1}_{\{C<0\}} + \mathbb{1}_{\{C \geq 0\}}\beta(C)\right]}{[1-\beta(C)]}\right) \tag{31}$$

with boundary condition $T(\overline{C}) = 0$. Solving the ODE yields (29). The formal proof is given in Online Appendix G.

With Proposition 4 in hand, we can change the state variable from C_t to $T_t = T(C_t)$. In general, $T(C_t)$ exhibits characteristics of a credit line balance, in that it records payments from the intermediary to and from the firm and compounds at "rate of return" $\hat{r}(C_t)$. The balance increases (decreases) after negative (positive) cash flow shocks in that the intermediary draws on (repays) the credit line. Moreover, the balance is retired upon equity financing through at least partial lump-sum repayment. In the following, we present an implementation that links the balance $T(C_t)$ to the balance of two separate credit lines.

4.2 Secured and Unsecured Credit Line Debt

Motivated by Proposition 4, we implement the optimal contract through a combination of secured and unsecured credit line debt, whose balances add up to the "joint balance" $T(C)$. The two credit lines implement the structure of intermediary financing and, as such, the control variables $Y(C), \alpha(C), \beta(C)$. In summary, the secured credit line implements $Y(C)$, while the unsecured credit implements contracted risk-sharing $\alpha(C)$ and $\beta(C)$.

Proposition 5. *The optimal contract can be implemented via two securities that respect absolute priority: (1) A secured, risk-free credit line with balance* $Y(C)$ *and (2) an unsecured, risky credit line with balance* $D(C) = \alpha_U(C)$ *for* $C \in (\underline{C}, \overline{C}]$. *At the dividend payout boundary, these balances are* $Y(\overline{C}) = D(\overline{C}) = 0$. *For* $C \in (\underline{C}, \overline{C})$, *the following holds:*

1. *The balance of the secured credit line* $Y(C)$ *grows with interest at rate* r *and rises and*

falls with transfers $dI^Y(C)$*:*

$$dY(C) = rY(C)dt - dI^Y(C) \tag{32}$$

$$dI^Y(C) = \left([\mu - \sigma^2 k_Z(\beta(C)) + \pi\,(\alpha - k_\Pi(\alpha))]dt + \sigma_C(C)dZ + Y(C)d\Pi\right)\mathbb{1}_{\{C\leq 0\}}. \tag{33}$$

Upon refinancing, the secured credit line is repaid in full.

2. *The balance of the unsecured credit line* $D(C)$ *increases with a maintenance fee* $\frac{\lambda}{\rho r}\mathbb{1}_{\{C\geq 0\}}$*, and rises and falls with transfers* $dI^D(C)$*:*

$$dD(C) = \frac{\lambda}{\rho r}\mathbb{1}_{\{C\geq 0\}}dt - dI^D(C) - \Delta^\Pi(C)d\Pi \tag{34}$$

$$dI^D(C) = \left[\sigma^2 k_Z(\beta(C)) - \pi\,(\alpha - k_\Pi(\alpha))\right]dt + \sigma\beta(C)dZ + \alpha(C)d\Pi. \tag{35}$$

$$\Delta^\Pi(C) = D(C) - \alpha(C). \tag{36}$$

Upon refinancing, if $\alpha = \alpha_C(C)$*, the firm defaults on* $\Delta^\Pi(C) > 0$ *of the unsecured credit line, while existing equity claims are wiped out. Otherwise, the unsecured credit line is repaid in full and existing equity claims retain some value.*

Creditors have control rights over the firm's assets for all C *such that* $J(C) \leq 0$*. Finally,* $Y(\underline{C})$ *is the total secured debt capacity of the firm.*

The formal proof is given in Online Appendix H. First, observe that within the optimal contract, equity holders may commit to a value loss through dilution when raising new equity, i.e., $J(C) \in [-P(C), 0)$. Intuitively, the firm faces a debt-overhang problem when it raises equity in such states: raising equity financing is only possible if part of the existing debt is written off and shareholders are fully diluted. To implement refinancing with a value loss to the existing shareholders, we thus need to give creditors control rights for all states C with $J(C) < 0$. We can achieve this via bankruptcy or the threat of bankruptcy from debt covenant violation, and structure the covenant so that it is violated whenever $J(C) < 0$. It can be a balance sheet covenant, e.g., a maximum debt-to-asset ratio, or a financial covenant, e.g., an earnings-based covenant which is violated after a sufficient string of negative cash flow realizations.[21] For low C, Figure 1 shows that (15) binds, i.e., $J(C) = -P(C)$. In this case, the creditors enforce the covenant and force bankruptcy. If the opportunity to raise new equity arrives while the firm is in bankruptcy, the existing shareholders are wiped out as the proceeds from raising new equity are insufficient to pay all claims in full, a situation we term distress. The creditors then distribute the refinancing proceeds according to seniority.

[21] An earnings-based covenant, such as typically stipulate that a firm's total debt or interest expenses cannot exceed a multiple of EBITDA.

In contrast, for $J(C) \in (-P(C), 0)$, the covenant is violated and existing shareholders face the threat of bankruptcy. However, in these states refinancing proceeds are sufficient to repay the creditors in full. Thus, shareholders pay off the credit lines upon finding new equity financing, in the process lowering their own continuation value rather than facing bankruptcy and losing all value.

Bankruptcy. Given the preceding discussion, there are four cases to credit line repayment. First, credit lines are gradually repaid after positive cash flow realizations. Second, when the firm's liquidity reserves are sufficiently high, i.e., high C, credit lines are repaid in full while existing equity claims are partially diluted upon refinancing. Third, when the firm is in distress, i.e., for low C such that (15) is binding, its creditors force it to enter *Chapter 11* bankruptcy and the firm continues operations. If the firm finds new equity investors, the unsecured credit line is retired with a write-down, essentially a partial default, the secured credit line is repaid in full, and the existing equity claims are wiped out.[22] The firm then emerges from bankruptcy under the new ownership, completing the reorganization. The firm may also emerge from bankruptcy following a string of positive cash flow realizations without new equity infusions. Fourth, while in bankruptcy, the firm may hit its financing capacity, i.e., for $C = \underline{C}$, in which case it is optimally liquidated, akin to converting *Chapter 11* to *Chapter 7*. The secured credit line is repaid in full with the liquidation proceeds, while both the unsecured credit line and existing equity claims are wiped out.[23] In all cases, repayments respect the *absolute priority rule* (APR).[24] It follows that cash flow-based financing is associated with Chapter 11 bankruptcy and reorganization, while asset-based financing is associated with Chapter 7 bankruptcy and liquidation. Different from bargaining-based models of bankruptcy, such as Antill and Grenadier (2019), in our model, the current shareholders have full bargaining power vis-a-vis new shareholders and the intermediary. Thus, different bankruptcy resolutions are not a consequence of bargaining, but endogenously arise from the shareholders' commitment to the optimal contract. Consequently, bankruptcy and the different paths out of it are an ex-ante efficient outcome.

Secured Credit Line. The purpose of the secured credit line is to implement financing against future promised payouts, as characterized by $Y(C)$. Thus, the implementation of the optimal contract formalizes the interpretation of financing against promises as collateralized

[22]Indeed, restructuring of distressed and bankrupt firms in practice goes often along with partial default on existing debt claims (see, e.g., Ivashina, Iverson, and Smith (2016)) and write-down of equity claims; the restructuring process might induce a loss for some types of debt claims (e.g., unsecured debt) and equity, while benefiting others (e.g., secured debt).

[23]Under $\underline{C} = C^S$ the firm never exhausts its financing capacity, i.e., C_t never attains $\underline{C}$ (see Proposition 2).

[24]The reason is that the states in which there is partial default on the unsecured credit line are exactly the states in which (15) is binding, implying *existing* equity holders are wiped out while the firm survives.

debt financing. Crucially, the balance of the secured credit line $Y(C)$ is always fully backed by the firm as a collateral asset, and the limit of the secured credit line equals $Y(\underline{C}) = -\underline{C}$ and, as such, coincides with the firm's financing capacity. Therefore, the secured credit line limit is either determined by the liquidation of the firm's assets ("asset-based borrowing constraint") in that $Y(\underline{C}) = L$ or the firm's going concern value ("cash flow-based or earnings-based borrowing constraint") in that $Y(\underline{C}) = -C^S$. The secured credit line is risk-free and accrues interest at the risk-free rate r.

Unsecured Credit Line. The firm's unsecured, risky credit line implements contracted risk-sharing — that is, α and β. Notice that the interest rate is zero and the credit line only stipulates a maintenance fee which is waived when $C < 0$, i.e., intuitively when the firm undergoes financial distress. The utilization of this (zero-interest) credit line represents a wealth transfer from the intermediary to the firm for which the intermediary is compensated ex ante via the maintenance fee. Repayments and drawdowns on the unsecured credit line in response to Brownian cash flow shocks are proportional to $\beta(C)$: Upon a negative (positive) cash flow shock of \$1, the firm draws down (repays) the credit line by $\$\beta(C)$. As a result, the credit line and state-contingent drawdown/repayment as stipulated in Proposition 5 induce a wealth transfer from (to) the intermediary to (from) the firm upon a negative (positive) cash flow shock proportional to $\beta(C)$, thus implementing contracted risk-sharing of Brownian risk. Similarly, as can be seen from (35), the speed with which the intermediary draws on the credit line, that is, $-dI^D(C)$, increases with $\alpha(C)$, while the balance of the credit line is paid back by the amount $\alpha(C)$ upon refinancing $d\Pi = 1$. Thus, the higher $\alpha(C)$, the higher the transfer from the intermediary to the firm by means of this subsidized credit line.[25]

The interaction between secured and unsecured debt. Notably, secured and unsecured debt in our implementation exhibit a form of complementarity, in a sense that usage of one debt instrument stimulates usage of the other one. First, the unsecured credit line — which implements risk-sharing α and β — is necessary to prevent liquidation at $\underline{C}$ via $\beta(\underline{C}) = 1$ and thus for the secured credit line to have capacity beyond L (i.e., for financing capacity to be cash flow-based). Intuitively, unsecured debt allows the firm to offload risk to prevent liquidation, ensuring repayment of secured debt. Conversely, the unsecured credit line is used intensely for low C in that $\beta(\underline{C}) = 1$, if and only if the secured debt capacity exceeds L and, equivalently, financing capacity $\overline{Y}$ is cash flow-based.

Next, notice that for $C > 0$ only the unsecured credit line is used. For $C < 0$ however, both credit lines are used, giving rise to direct interactions. Secured and unsecured

[25] When intermediary risk-bearing capacity vanishes, i.e., $\rho \to \infty$, there is no more risk-sharing and the unsecured credit line vanishes, and intermediary financing solely occurs via the secured credit line, i.e., $\lim_{\rho\to\infty} D(C) = 0$ and $\lim_{\rho\to\infty} T(C) = Y(C)$.

credit lines absorb Brownian cash flow shocks in proportions $(1-\beta)$ and β, respectively. Mechanically, the two credit line therefore are substitutes in covering cash flow shortfalls when $C < 0$. Further, the secured creditors receive a flow payoff μ less the term $[\sigma^2 k_Z(\beta(C)) - \pi(\alpha - k_\Pi(\alpha))] > 0$, while the unsecured creditors receive a flow payoff equal to this term. Thus, the secured credit line is used to pay off the unsecured credit line, which, intuitively, ensures safety and seniority of the secured credit line.

Asset- and cash flow-based debt. Importantly, the types of debt used by the firm and the determinants of financing capacity are related but distinct objects. Whether a firm's financing capacity is cash flow- or asset-based depends on which fundamentals drive the total amount of financing available to the firm: Financing capacity is asset-based (cash flow-based) if it increases in (is invariant to) the liquidation value of assets. Moreover, financing capacity coincides with the firm's secured debt capacity in our implementation. Meanwhile, the firm utilizes several types of debt. According to the classification of asset- and cash flow-based debt in Lian and Ma (2021), the firm relies both on asset-based debt (secured by specific asset) and cash flow-based debt (unsecured or secured by a blanket lien). First, the secured debt may consist of both asset- and cash flow-based debt. The secured credit line balance below the liquidation value $\min\{L, Y\}$ is backed by the liquidation value of assets and thus asset-based debt. The secured credit line balance in excess of the liquidation value $\max\{Y - L, 0\}$ is then cash flow-based, i.e., secured by a blanket lien. While the firm always uses asset-based secured debt, it uses cash flow-based secured debt if and only if financing capacity is cash flow-based.[26] Second, the firm always relies on cash flow-based debt in the form of unsecured debt. Note that secured cash flow-based debt is senior to other forms of cash flow-based debt. Third, a firm with cash flow-based financing capacity uses both cash flow- and asset-based debt to achieve the optimal capital structure. An increase in the liquidation value of asset may lead to more asset-based debt, but does not raise total amount of financing available to this firm. A similar logic applies to asset-based financing capacity.

4.3 Analysis of the Implementation

Our implementation suggests an overlapping pecking order: First, the firm finances cash flow shortfalls with internal cash reserves and unsecured credit line debt. It relies on secured credit line debt only under financial distress when it has run out of cash, that is, for $C < 0$,

[26] In principle, we could "slice up" secured debt in asset- and cash flow-based secured debt in many ways, as both types of debt have the same return properties and the firm never defaults on its secured debt. For instance, we could stipulate for any $\omega \in [0, 1]$ that the balance $\min\{\omega L, Y\}$ is asset-based, while balance $\max\{Y - \omega L, 0\}$ is cash flow-based. If liquidation occurs, secured cash-flow based debt is senior to any unsecured debt, and receives any residual liquidation value after asset-based debt is repaid. As long as $L > 0$, the firm may always use some asset-based debt.

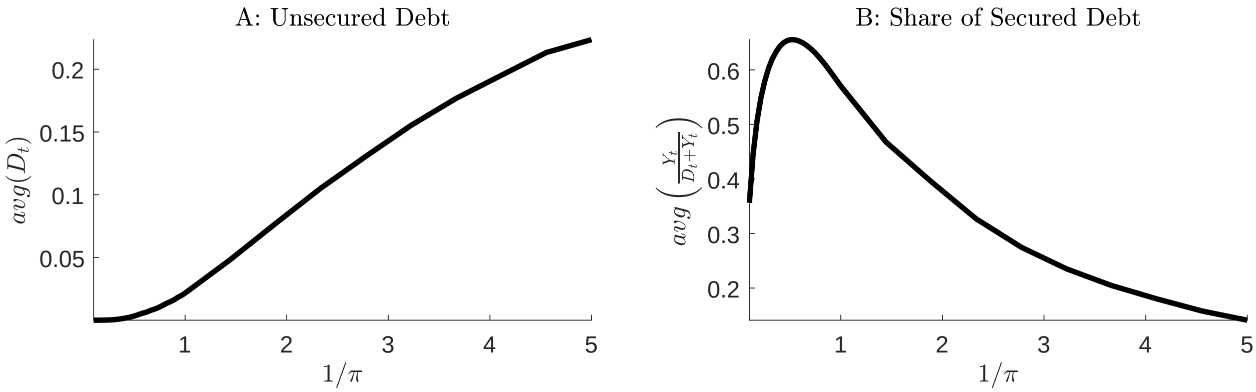

Figure 4: **Implementation in Steady State.** This figure plots the average unsecured debt $avg(D_t)$ and the average share of secured debt $avg\left(\frac{Y_t}{Y_t+D_t}\right)$ against $1/\pi$. The parameters follow Table 1 and throughout we consider parameter configurations under which $\tau = \infty$.

while the unsecured credit line is used in all states for risk-sharing purposes. That is, the firm first uses unsecured debt financing before it resorts to secured debt financing, consistent with the findings in Benmelech et al. (2020a); Benmelech, Kumar, and Rajan (2020b); Rauh and Sufi (2010).[27] At first glance, one might expect the opposite, that is, the firm first uses its available collateral to pledge for secured debt financing and, once collateral is exhausted, the firm "needs" to raise unsecured debt financing. However, this intuition does not apply in our context because secured and unsecured credit line debt serve different purposes. Unsecured debt is used to finance cash flow shortfalls in all states and even when the firm has cash $M(C) > 0$ to implement risk-sharing between the firm and the intermediary. Secured debt is purely a financing instrument that is only used when the firm runs out of cash.

Our theory sheds light on how intermediary characteristics (e.g., risk aversion ρ) or capital market characteristics (e.g., π) shape firms' use of secured and unsecured credit line financing. Notice that as $\rho \to \infty$, the firm only uses secured credit line debt, whereas for lower levels of ρ, the firm uses both unsecured and secured credit line debt. As such, our theory suggests that non-bank lenders (e.g., direct lenders or private debt funds) with larger risk-bearing capacity tend to provide more unsecured debt financing than traditional banks which tend to provide more secured debt financing. In practice, the two credit lines could also be provided by two different intermediaries. A bank with provides the relatively "standard" secured line and a private lender provides the unsecured credit line with "flexible" terms in distress (i.e., maintenance fee is waived).[28]

[27] Rauh and Sufi (2010) find that high-credit-quality firms, which may correspond in our model to the ones with high liquidity reserves, use more unsecured debt financing than low-credit-quality firms.

[28] Consistent with this idea, Block et al. (2023) or Jang (2022) document that private lenders indeed tend to provide more flexible debt terms for borrowers, especially in distress.

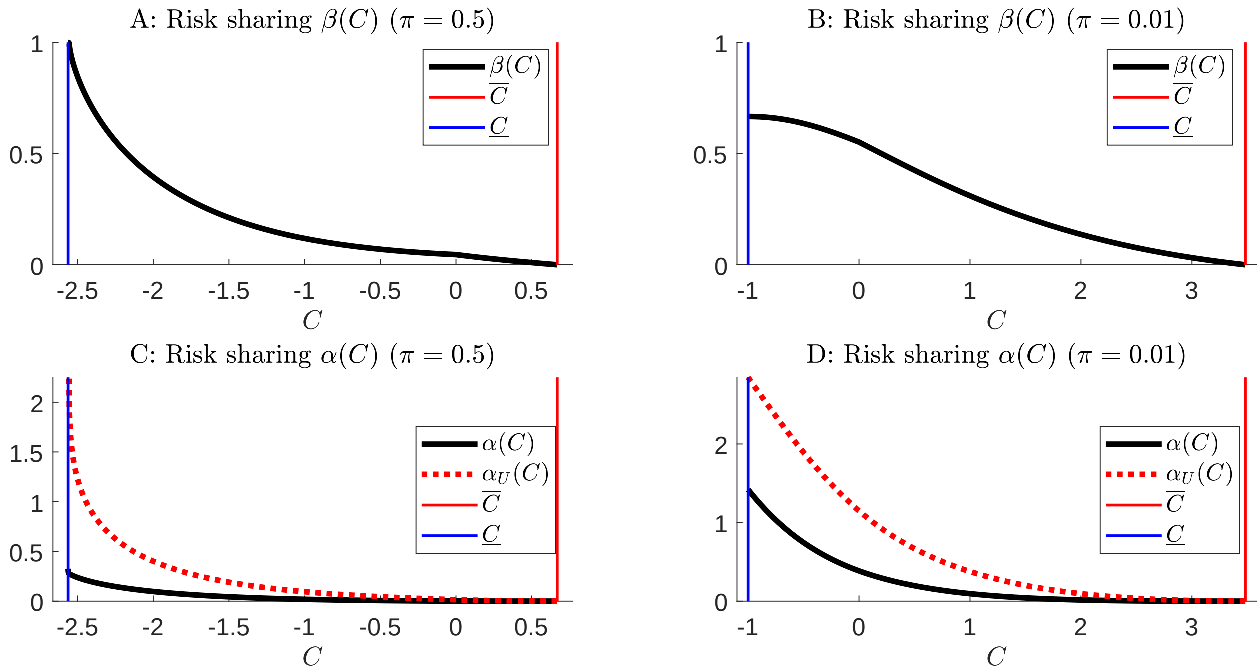

Figure 5: **Contract Dynamics under limited creditor protection.** This figure plots $\beta(C)$ and $\alpha(C)$ against C both for $\pi = 0.5$ (Panels A and C) and $\pi = 0$ (Panels B and D). The parameters follow Table 1 and $\nu = 0$ in refinancing constraint (37)

Interestingly, while $avg(Y_t)$ is hump-shaped in $1/\pi$ (see Figure 3), unsecured debt $avg(D_t)$ increases in $1/\pi$, as shown in Panel A of Figure 4 . Intuitively, when the firm's access to equity financing is limited, the firm relies more on unsecured debt provided by the intermediary to offload risk. At the same time, large $1/\pi$ limits the firm's financing capacity, thus curbing the use of secured credit line debt. The share of secured debt (Panel B) is hump-shaped in $1/\pi$. As such, better access to equity financing, i.e., lower $1/\pi$, actually can reduce the use of secured debt relative to unsecured debt and so the share of secured debt.

5 Further Results and Extensions

5.1 Refinancing and Weak Creditor Rights

Recall that by (15) the shareholders' continuation payoff must be positive at any point in time, which implied that the amount α that can be promised to the intermediary upon refinancing was limited by constraint (15). However, in states C such that $J(C) < 0$, existing shareholders are so heavily diluted upon refinancing that they are worse off than "just before" refinancing. While committing to such a refinancing policy is ex-ante optimal, shareholders would not want to raise equity financing at $d\Pi_t = 1$ if they had discretion at t.

For example, if shareholders can unobservably search for refinancing opportunities at zero costs, they would have no incentive to search when $J(C) < 0$.

For $\nu \in [0, 1]$, consider the following generalized constraint on α that nests our base case:

$$J(C) \geq -\nu P(C) \iff \alpha(C) \leq [P(\overline{C}) - \overline{C}] - [(1-\nu)P(C) - C]. \tag{37}$$

Here, $\nu = 1$ yields our baseline constraint (15) implying full commitment, while $\nu = 0$ yields a tighter constraint we term monotonicity, as refinancing cannot make current shareholders worse off via dilution. Thus, ν describes shareholders' strength of commitment, with higher ν implying higher commitment. The solution of this model variant is now akin to the baseline with the only difference that constraint (15) is replaced by constraint (37). The boundary conditions remain the same, specifically the expressions for the lower bound (25), as due to $P(\underline{C}) = 0$, we have $J(\underline{C}) = 0$ regardless of ν. For $\nu = 0$, (37) implies that optimal $\alpha(C)$ always decreases in C. The following proposition shows that weaker commitment, i.e., smaller ν, increases the payout boundary and lowers cash flow-based financing capacity.

Proposition 6. *The upper bound $\overline{C}$ and lower bound under survival C^S both decrease in ν.*

The proof is given in Online Appendix J. Notably, the solution, implementation, and the key results for $\nu < 1$ remain qualitatively similar to the baseline. Figure 5 illustrates the dynamics of the optimal contract under $\nu = 0$ using the same parameters as in Figure 1. The outcomes in Figure 5 are broadly similar to Figure 1, with the differences being that the constraint (37) now binds for all C and that $\underline{C}$ and $\overline{C}$ are larger than in the baseline case.

As Proposition 4 only relies on general $\alpha(C)$, our implementation from Proposition 5 via two separate credit lines applies after a change to $dD(C)$ and the bankruptcy rules.[29] The key difference to our baseline implementation is that for $0 \leq \nu < 1$, *absolute priority* may fail in bankruptcy – existing shareholders are not fully wiped out when unsecured creditors take a partial loss. We thus interpret ν as a proxy for creditor protection. Weaker creditor rights, i.e., lower ν, lead to a decrease in financing capacity, a shift from cash flow-based financing $\underline{C} = C^S$ toward asset-based financing $\underline{C} = -L$, and more liquidation. Thus, we would expect that firms in countries with weaker creditor rights rely more on asset-based financing, provide overall less credit, and self-insure more by holding higher cash balances.

[29] The interest rate on the unsecured credit line (30) changes as $\alpha_C(C)$ is redefined by (37), so that

$$dD(C) = \left[\frac{\lambda}{\rho r}\mathbb{1}_{\{C \geq 0\}} + \pi(1-\nu)\left(\frac{e^{\rho r[\alpha_U(C) - \alpha(C)]} - 1}{\rho r}\right)\right] dt - dI^D(C) - \Delta^{\Pi}(C)d\Pi. \tag{38}$$

Note that the term involving ν, i.e., the second term in $[\cdot]$, is always positive. Effectively, the unsecured interest rate is higher to compensate the unsecured creditors for the higher expected losses from bankruptcy.

5.2 Active Intermediaries and Monitoring

Financial intermediaries, such as banks or direct lenders, often monitor borrower firms to contain credit risk or improve operational performance, or actively engage in financial distress resolution. In this section, we extend our model to account for such intermediary actions by allowing the intermediary to affect cash flows via its effort.[30] Online Appendix K provides a detailed description and solution of this variant. Summarizing, we assume that cash flows evolve according to

$$dX_t = (\mu + a_t)dt + \sigma dZ_t, \tag{39}$$

where $a_t \geq 0$ is the intermediary's non-contractible and privately observable effort which entails a cost $\frac{\kappa a_t^2}{2}dt$ for a constant $\kappa > 0$.[31] The intermediary's effort boosts the firm's cash flows, which could capture the intermediary's active role in firm operations (or restructuring) or its role in disciplining management through monitoring. Such monitoring could also be related to covenants and to creditor actions taken after covenant violations. The intermediary's incentives to exert effort are determined by the incentive condition $a_t = \frac{\beta_t}{\kappa}$, and thus increase with the intermediary's exposure to cash flow shocks ("skin-in-the-game") β_t. That is, there is a moral hazard with regard to intermediary's monitoring effort.[32]

Panel A of Figure 6 illustrates the dynamics of effort a in two scenarios, (i) $\pi = 0.5$ in which case $\underline{C} = C^S$ and the firm is never liquidated and (ii) $\pi = 0$ in which case the firm is liquidated once $C = -L$. To facilitate graphical comparison across the two scenarios, we plot intermediary effort a against the firm's adjusted liquidity position $(C - \underline{C})/(\overline{C} - \underline{C}) \in [0, 1]$ both for $\pi = 0.5$ (solid black line) and for $\pi = 0$ (dotted red line). Intermediary effort decreases upon negative cash flow shocks and is highest in financial distress, that is, when C is low. Observe that in the event the intermediary provides debt financing, its monitoring effort could also be related to the enforcement of covenants, which are more likely to be breached following negative cash flow realizations. Notably, the intermediary exerts particularly high effort when $\beta(C)$ is close to one, which occurs at $C = \underline{C}$ in the case of cash flow-based financing. Thus, cash flow-based financing is associated with more intense monitoring under financial distress. This result also aligns with the observation that cash flow-based debt typically features earnings-based covenants which are violated and trigger monitoring (creditor actions) following negative cash flow shocks.

[30] Heitz, Martin, and Ufier (2022) provide empirical evidence that bank monitoring (e.g., via on-site inspections) and Nini, Smith, and Sufi (2012) that actions taken by creditors improve borrower performance, lending support to our modelling in (39) that a_t boosts firm performance.

[31] Note that an infinite cost of effort, i.e., $\kappa \to \infty$, implies $a_t = 0$, thus giving our baseline case.

[32] While this model extension relates to dynamic agency models with moral hazard over monitoring (Piskorski and Westerfield, 2016; Malenko, 2019; Gryglewicz and Mayer, 2021; Gryglewicz, Mayer, and Morellec, 2021), the key novelty is that it considers a financially constrained principal, here the firm's shareholders.

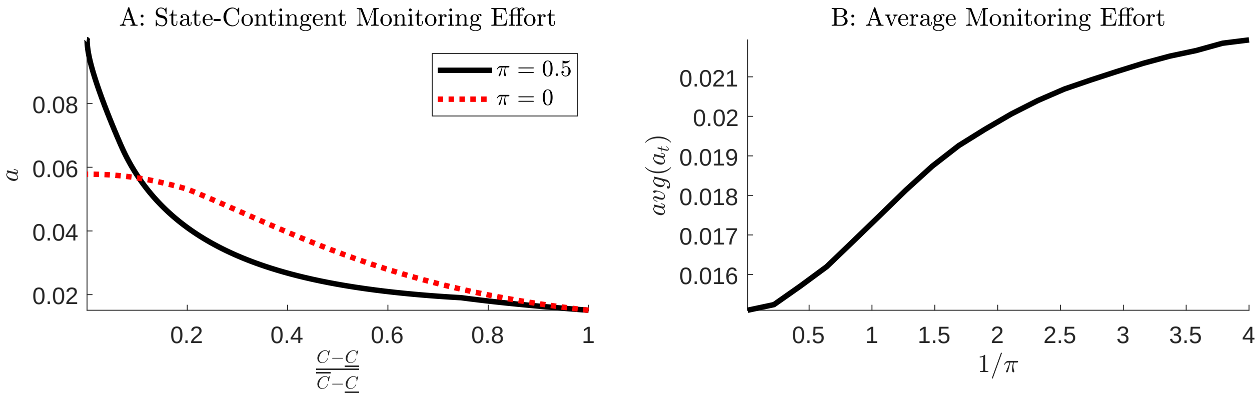

Figure 6: **Intermediary Incentives**. The left panel of this figure plots the intermediary's effort incentives $a = \beta/\kappa$ both for $\pi = 0.5$ (solid black line) and for $\pi = 0$ (dotted red line) in the transformed state space against $(C - \underline{C})/(\overline{C} - \underline{C})$ to ensure comparability across different parameterizations. The right panel plots average effort $avg(a_t)$ against $1/\pi$ under parameters that ensure $\underline{C} < -L$ (so that a stationary distribution exists). The parameters follow Table 1, and we use $\kappa = 10$ and $\sigma = 1.25$ (for illustrative purposes).

Panel B of Figure 6 plots the intermediary's average effort (in steady state focusing on parameters that admit a stationary density) against $1/\pi$, the expected time to refinancing. Interestingly, better access to equity markets (i.e., lower $1/\pi$) reduces the intermediary's incentives to monitor, so that $avg(a_t)$ increases with $1/\pi$. That is, that lenders monitor less when borrowers have better access to equity financing. As monitoring intensity may be related to covenants, the model predicts a less stringent covenant structure for firms with superior access to equity financing. On the contrary, when liquidity dries up and π is low (e.g., in a financial crisis), intermediaries, providing debt financing, exerts more effort to improve firm operations or, similarly, engage more in monitoring. Again, interpreting monitoring intensity as related to covenants, the model predicts stricter covenant structures (i.e., more monitoring) in crisis times.

Finally, as argued in Online Appendix K, cash flow-based financing capacity generally decreases with κ and so increases with the intermediary's monitoring ability. Intuitively, when the intermediary can add value to the firm through monitoring, it has a higher valuation for the firm and so is more willing to provide cash flow-based financing.

5.3 Alternative Application: Distress Investors

Because the intermediary effectively acquires a stake Y in the firm in distress, our theory also applies to the study of specialized distress investors. Distress investors — which can be PE or hedge funds — take (equity or debt) stakes in distressed firms and firms in Chapter 11 bankruptcy (see Ivashina et al. (2016) for empirical evidence), often with the goal of exiting their position and reselling the stake at a later point, for example, after distress is resolved. Note that at the lower boundary $\underline{C}$, the firm is not liquidated if and only if the intermediary stake $Y(\underline{C})$ exceeds the liquidation value of the firm L and, as such, is sufficiently large. The intermediary's willingness to acquire a stake in the firm in distress crucially depends on the firm's access to equity financing, allowing the intermediary to exit her position. Interestingly, the model generates a hump-shaped relationship between the average intermediary stake $avg(Y_t)$, a proxy for distress investment activity, and access to equity financing, which in the past may have improved with the rise of PE; see Figure 3. Intuitively, when $1/\pi$ is low, the intermediary can exit its positions quickly, reducing its stake on average. But, when $1/\pi$ is large and a successful exit is difficult, the intermediary is unwilling to acquire a stake in the firm because the option to resell the firm has little value.

As in Section 5.2, we can incorporate the costly intermediary effort to capture investors' engagement in restructuring, the model illustrates that distress investors exert substantial effort. The intermediary's effort is larger when the intermediary holds a larger stake Y. Very "high" intermediary effort in distress, i.e., $\beta(\underline{C} = 1$ and $a(\underline{C}) = 1/\kappa$, occurs if and only if $\underline{C} < -L$ and thus is associated with a low likelihood of liquidation (i.e., $\tau = \infty$) and with successful restructuring. As shown in Figure 6, average effort increases with $1/\pi$ and thus decreases with the firm's access to equity financing and the intermediary's opportunities to exit. That is, while good exit opportunities are necessary to entice the intermediary to take a stake in the firm under distress, they also undermine its incentives to exert effort.

6 Empirical Implications

Our theory rationalizes empirical evidence in Lian and Ma (2021) on asset- and cash flow-based borrowing constraints and produces several novel empirical predictions that relate the capital market and intermediary characteristics to these constraints. In this Section, we give an overview of the key empirical implications of our theory.

Firm Characteristics and Financing Constraints. Our results show that firms with higher profitability or lower cash flow risk have higher cash flow-based financing capacity and thus are associated with more cash flow-based and less asset-based financing. In addition, firms with lower liquidation value have lower asset-based financing capacity and thus tend to use more cash flow-based financing. These model implications are broadly consistent with Lian and Ma (2021) and Kermani and Ma (2023).

Capital Market Characteristics and Financing Constraints. According to the model, better access to equity financing, e.g., due to more liquid private or public equity markets, improves access to cash flow-based financing, and expands financing and debt capacity. This result rationalizes why large public firms use more cash flow-based debt than small private firms (Lian and Ma, 2021) that predominantly rely on asset-based debt (Gupta, Sapriza, and Yankov, 2021). It also explains why PE-owned firms use more cash flow-based debt than private firms without PE owner, as documented in Haque et al. (2022). Overall, our findings suggest that better access to equity financing relaxes financing constraints with other intermediaries, such as banks and private lenders, too, in accordance with the evidence in Ivashina and Kovner (2011) and Demiroglu and James (2010).

Financial Intermediary Characteristics and Financing Constraints. The model predicts that intermediaries with higher risk-bearing capacity tend to provide more cash flow-based financing and allow for larger debt capacity. Jang (2022), Block et al. (2023), and Chernenko et al. (2022) show that non-bank lenders with less regulatory or capital constraints are more willing to lend against cash flows than traditional banks. Moreover, higher (aggregate) intermediary risk-bearing capacity, which may reflect that the financial intermediary sector is well-capitalized, implies larger cash flow-based debt capacity and larger debt capacity for firms. Thus, an aggregate shock to intermediary risk-bearing capacity (e.g., financial crisis) causes a shift from cash flow-based toward asset-based financing.

Financing Instruments and Secured vs. Unsecured Debt. In our theory, the firm essentially has three financing instruments, namely, internal cash reserves, equity financing, and unsecured and secured debt financing provided by the intermediary. Our model can therefore shed light on how these financing instruments interact and to what extent they are used leading to the following predictions. First, debt and equity are *dynamic* complements. Thus, firms with better access to equity financing also have higher debt capacity and so may use more debt financing too. Second, our implementation reveals that, while unsecured debt is used in all states, the firm uses secured debt only in distress (Benmelech et al., 2020a,b; Rauh and Sufi, 2010). Third, unsecured and secured debt exhibit features of complements. Fourth, cash is not negative debt (Acharya, Almeida, and Campello, 2007): The firm uses cash reserves and credit line financing simultaneously to cover cash flow shortfalls. Fifth, the model predicts that lenders with high risk-bearing capacity provide more unsecured debt financing. That is, private or non-bank lenders with less regulatory or capital constraints and arguably higher risk-bearing capacity provide more unsecured debt as well as more cash flow-based debt. This is broadly in line with Block et al. (2023) or Jang (2022) which documents that non-bank lenders allow for more flexible loan terms. Sixth, better access to equity financing can reduce the share of secured debt of total debt.

Bankruptcy. Endogenous resolution of financial distress resembles U.S. bankruptcy procedures, namely Chapter 11, facilitating reorganization, and Chapter 7, leading to liquidation. In line with evidence in Ivashina et al. (2016), distress resolution via Chapter 11 features dilution of equity holders and junior unsecured debt holders, whereas senior secured debt is repaid in full. Moreover, the model predicts that cash flow-based financing is associated with Chapter 11 bankruptcy and reorganization and asset-based financing is associated with Chapter 7 bankruptcy and liquidation.

Legal Infrastructure and Creditor Protection. In practice, a country's legal infrastructure is key to debt enforcement and creditor protection. Notably, it also affects the practice of corporate borrowing and distress resolution via bankruptcy. Our results indicate that weak creditor protection — which limits the extent of dilution of equity in distress resolution — limits the availability and feasibility of cash flow-based financing and debt, leading to more asset-based financing. Overall, weak creditor protection reduces debt capacity and is associated with higher cash holdings. In addition, weak creditor protection implies a shift from Chapter 11 bankruptcy with reorganization toward Chapter 7 bankruptcy and liquidation for distress resolution. Related to these findings, Antill (2022) documents that so-called "363 sales," essentially reflecting weakened creditor protection, lead to liquidations inefficiently often, which harms creditors.

Tightness of Financing Constraints. As recent studies show (Chaney et al., 2012; Adler, 2020; Lian and Ma, 2021; Cloyne et al., 2023), both type and tightness of financing constraints matter for corporate policies and firms' exposure to shocks that affect the availability of intermediary financing (e.g., financial crises) or firm characteristics (e.g., shocks to profitability and cash flow risk like Covid-19). Overall, we find that firms whose financing capacity is either very high or low use intermediary financing the least, and therefore face on average the least tight financing constraints. Thus, firms with large financing capacity do not rely much on the intermediary because their endogenous financing capacity reflects strong firm fundamentals that reduce the need for intermediary financing. Further, better firm fundamentals, higher intermediary risk-bearing capacity, or better access to equity financing — all associated with larger financing capacity — may increase utilization more than capacity of intermediary financing, thus tightening financing constraints.

Monitoring. Consistent with evidence on bank monitoring in Gustafson, Ivanov, and Meisenzahl (2021) and creditor interventions in Nini et al. (2012), the intermediary's incentives to monitor the firm increase after negative cash flow shocks and credit line drawdowns. Interestingly, cash flow-based debt financing features high creditor monitoring in distress, in line with Kermani and Ma (2020). Further, an intermediary's monitoring ability increases financing and cash flow-based debt capacity. Thus, we expect that firms financed by lenders with more expertise and skill in monitoring — such as specialized banks (Paravisini, Rappoport, and Schnabl, 2017; Blickle, Parlatore, and Saunders, 2023) — have more cash flow-based debt. In line with evidence in Badoer, Emin, and James (2023) and Haque, Mayer, and Wang (2023) that loans to PE-backed firms tend to have less covenants and are monitored less, we show that monitoring is lower for firms with better access to equity financing. The model also predicts that, by improving access to cash flow-based financing, a better legal environment increases lenders' monitoring in distress. Interestingly, Jiang, Kundu, and Xu (2022) document such a positive link between legal environment, which, in their case, reduces loan renegotiation frictions, and bank monitoring.

Distress Investors. Distress investors, which can be PE or hedge funds, take (equity or debt) stakes in distressed and potentially bankrupt firms under Chapter 11. Our model predicts that the participation of distress investors is associated with successful restructuring of firms in Chapter 11 bankruptcy, consistent with evidence in Jiang, Li, and Wang (2012). In addition, our results suggest that overall distress investment activity is related to firms' access to equity financing, which determines distress investors' exit opportunities. In particular, distress investment activity is hump-shaped with respect to firms' access to equity financing: It is lowest for very liquid or illiquid equity markets.

7 Conclusion

We provide a dynamic theory of liquidity management and optimal long-term contracting with endogenous asset- and cash flow-based financing constraints. In the model, a firm with infrequent access to equity financing and a cost of holding cash bridges short-term financing needs via financing from an intermediary. Optimal financing from the intermediary takes the form of collateralized debt and is endogenously constrained by the firm's financing capacity which is either asset-based, i.e., determined by the asset liquidation value, or cash flow-based, i.e, determined by the going-concern value of cash flows. Further, we show that debt and equity are dynamic complements in that better equity market access increases debt capacity.

We then study the determinants of firms' financing constraints. Higher profitability, lower cash flow volatility, better access to equity financing, and lower intermediary capital constraints are all associated with larger financing capacity and more cash flow-based financing. Further, both type and tightness of financing constraints are endogenously determined. Surprisingly, we find that firms with very low or high financing capacity use intermediary financing the least and thus face on average the least tight financing constraints.

We implement the optimal contract with a combination of unsecured and secured credit lines, yielding an overlapping pecking order: The firm simultaneously finances cash flow shortfalls with the unsecured credit line and either cash reserves (if available) or the secured credit line (otherwise). In good times, credit lines are repaid in full and shareholders and creditors' interests are aligned. In bad times, shareholders and creditors' interests diverge, requiring resolution via bankruptcy or threat thereof. The firm enters Chapter 11 bankruptcy and continues operations when its liquidity reserves are low. It emerges from bankruptcy when it finds new equity investors, repays the secured credit line in full, partially defaults on the unsecured credit line, wipes out the existing equity claims. While in bankruptcy the firm may exhaust its financing capacity. It then optimally liquidates, effectively converting to Chapter 7, repays the secured credit line in full and wipes out both the unsecured credit line and existing equity claims. In all cases, repayments respect the absolute priority rule.

Going forward, our theory can hopefully be used to micro-found and incorporate endogenous financing constraints in dynamic macroeconomic models.

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Online Appendix

A Proof of Lemma 1

Recall from (3) that $Y_t = \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)}\left(dI_s - k_s ds\right)\right]$, and define

$$A_t = \mathbb{E}_t\left[\int_0^\infty e^{-rs}(dI_s - k_s ds)\right] = \int_0^t e^{-rs}(dI_s - k_s ds) + e^{-rt}Y_t. \tag{A.1}$$

By construction, $A = \{A_t\}$ is a martingale. By the martingale representation theorem, there exist stochastic processes $\alpha = \{\alpha_t\}$ and $\beta = \{\beta_t\}$ such that

$$e^{rt}dA_t = \beta_t\big(dX_t - \mu dt\big) + \alpha_t(d\Pi_t - \pi dt), \tag{A.2}$$

where $dZ_t = \frac{dX_t - \mu dt}{\sigma}$ is the increment of a standard Brownian Motion and $(d\Pi_t - \pi dt)$ is the increment of a compensated Poisson process (a martingale). We differentiate (A.1) with respect to time t to obtain an expression for dA_t, then plug this expression into (A.2) and solve (A.2) to get $dY_t = (rY_t + k_t)dt + \beta_t \sigma dZ_t + \alpha_t(d\Pi_t - \pi dt)$.

B Proof of Proposition 1

We prove Proposition 1 in several parts. First, we provide formal arguments for the reduction in state space and show that equity value can be expressed as function of C_t only and solves the HJB equation (16). Second we prove the concavity of the value function, assuming a well-behaved solution exists in the state space. Third, we provide the formal verification argument that under the optimal contract, the value function indeed solves (16). We impose the regularity condition that sensitivities are bounded, i.e., $|\alpha_t|, |\beta_t| \leq M$ for arbitrarily large $0 < M < \infty$ (see, e.g., Sannikov (2008)). This assumption is needed in the formal verification proof, but we pick M sufficiently large so that this constraint never binds in optimum. We can therefore ignore it in the follow-up analysis.

For convenience and to limit the number of distinct cases to deal with, we already conjecture that the lower boundary of the endogenous state space $(\underline{C}, \overline{C})$ satisfies $\underline{C} \geq -[P(\overline{C}) - \overline{C}]$. This conjecture will be verified in the proof of Lemma 2 in Online Appendix C. Further, we impose that dividend payouts must satisfy $dDiv_t \leq C_t - \underline{C}$. In Online Appendix D, we show that $dDiv_t > C_t - \underline{C}$ would lead to a violation of promise-keeping, i.e., an inconsistency with (6), giving rise to the constraint $dDiv_t \leq C_t - \underline{C}$ that applies under a full-commitment contract $\mathcal{C}$. It turns out that this constraint never binds.

Notably, all arguments in this proof are carried out under the assumption that a well-behaved, non-negative, and twice continuously differentiable solution $P(C)$ to (16) exists on the endogenous state space $(\underline{C}, \overline{C})$ (subject to $P'(\overline{C}) - 1 = P''(\overline{C}) = 0$). We formally establish existence of such a solution in Online Appendix I.

To proceed, we rewrite the law of motion of dC_t from (12) as $dC_t = \mu_C(C_t)dt + \sigma_C(C_t)dZ_t +$

$(C_t^* - C_t)\, d\Pi_t - dDiv_t$, where denote the drift and volatility of dC_t by

$$\mu_C(C_t) = \left[\mu + (r-\lambda)\, C_t - \lambda Y_t - \frac{\rho r}{2}\, (\beta_t \sigma)^2 + \pi \left(\frac{1 - e^{-\rho r \alpha_t}}{\rho r} \right)\right]; \sigma_C(C_t) = \sigma(1-\beta_t). \quad \text{(B.1)}$$

Note that $\pi\,(\alpha_t - k_\Pi(\alpha_t)) = \pi \left(\frac{1-e^{-\rho r \alpha_t}}{\rho r}\right)$ and $\frac{\rho r}{2}\,(\beta_t \sigma)^2 = \sigma^2 k_Z(\beta_t)$. As we verify, α_t and β_t will be functions of C_t in optimum, i.e., we can write $\beta_t = \beta(C_t)$ and $\alpha_t = \alpha(C_t)$.

B.1 Part I

The endogeneous state space is two dimensional, with two state variables (M, Y). The state space is contained in $\{(M, Y) \in \mathbb{R}^2 : M, Y \geq 0\}$, due to $Y \geq 0$ (intermediary limited commitment) and $M \geq 0$ (non-negativity constraint on cash). Take $C := M - Y$ and rotate the state space by considering (C, Y) rather than (M, Y). To respect $M \geq 0$, the state space must be a subset of $\{(C, Y) \in \mathbb{R}^2 : Y \geq -C\}$. Given $\mathcal{C}$, the time-t equity value is

$$P_t = P(C_t, Y_t) = \mathbb{E}_t \left[\int_t^\tau e^{-r(s-t)} (dDiv_s - \Delta M_s d\Pi_s) \Big| C_t = C, Y_t = Y \right].$$

By the dynamic programming principle, $P_t = P(C, Y)$ solves the Hamilton-Jacobi-Bellman (HJB) equation, i.e., the partial differential equation (PDE)

$$rP\,(C, Y))\, dt = \max_{dI, \Delta M, dDiv \geq 0} dDiv + \mathbb{E}[dP(C, Y) - \Delta M d\Pi]. \quad \text{(B.2)}$$

In the following, we prove that $P(C, Y)$ only depends on C, in that $\frac{\partial P(C,Y)}{\partial Y} = \frac{\partial^2 P(C,Y)}{\partial C \partial Y} = \frac{\partial^2 P(C,Y)}{\partial C^2} = 0$, and we can write with a slight abuse of notation $P(C, Y) = P(C)$, and denote $P'(C) = P_C(C, Y)$ as well as $P''(C) = P_{CC}(C, Y)$.

First, fix a state (C, Y) with $M = C + Y \geq 0$ as well as $Y \geq 0$. By (19) and (12), payouts to the intermediary dI do not change the level of C, but change the level of Y by amount $-dI$. It is always possible to stipulate (negative) payouts $dI = -\varepsilon < 0$ for $\varepsilon > 0$ to the intermediary, which by (19) moves the intermediary's continuation payoff from Y to $Y + \varepsilon$ and moves the firm's cash balance from $M = C + Y$ to $M + \varepsilon = C + Y + \varepsilon > 0$. In state (C, Y), payouts to the intermediary $dI = -\varepsilon < 0$ are possible but not necessarily optimal, so that $P(C, Y) \geq P(C, Y + \varepsilon)$. Likewise, in state $(C, Y + \varepsilon)$, positive payouts to the intermediary $dI = \varepsilon > 0$, which move intermediary's continuation payoff from $Y + \varepsilon$ to Y, are possible but not necessarily optimal, so $P(C, Y + \varepsilon) \geq P(C, Y - \varepsilon + \varepsilon) = P(C, Y)$. As a result, in the entire state space, we obtain $P(C, Y + \varepsilon) = P(C, Y)$ for $\varepsilon > 0$. Thus, above relationship implies $P_Y(C, Y) = 0$ and $P_{CY}(C, Y) = P_{YY}(C, Y) = 0$, whenever the respective derivatives exist. Thus, $P(C, Y)$ is constant in the Y-dimension, i.e., does not depend on the state of Y. Thus, we write from now on with slight abuse of notation $P(C, Y) = P(C)$.

Second, using $P_Y(C, Y) = P_{CY}(C, Y) = P_{YY}(C, Y) = 0$ and Ito's Lemma, we can expand the right-hand-side of (B.2) to obtain:

$$rP(C,Y)dt = \max_{dI,\Delta M, dDiv\geq 0}\Big\{dDiv + P'(C)\big[\mu_C(C)dt - dDiv\big] + \frac{P''(C)\sigma_C(C)^2dt}{2} + \pi[P(C^*) - \Delta M]dt\Big\}, \tag{B.3}$$

where the post-refinancing level C^* satisfies $C^* = \Delta M + C - \alpha$. As in related papers (e.g., Bolton et al. (2011)), dividend payouts are optimal only if $P'(C) \leq 1$, occur at a payout boundary $\overline{C}$, and follow a barrier strategy, that is, they cause C to reflect at $\overline{C}$. As such, we have $P(C) = P(\overline{C}) + C - \overline{C}$ and $P'(C) = 1$ for $C > \overline{C}$. The location of the payout boundary is determined by smooth pasting and super contact conditions, that is, $P'(\overline{C}) = 1$ and $P''(\overline{C}) = 0$. We verify the optimality of this dividend payout strategy in the verification argument in Part III. Due to the (downward) reflection of C at $\overline{C}$, the (endogenous) state space can be written as an interval $(\underline{C}, \overline{C})$, with the boundaries to be characterized later on.

Third, because payouts to the intermediary dI are always possible (when $M > 0$), do not change the level of C, but change the level of Y by amount $-dI$, controlling payouts dI implies controlling the level of Y. Therefore, Y becomes a control variable in the dynamic optimization, as its level can be freely adjusted via dI. More generally, the payout process dI is fully characterized by Y, α, and β, due to (6). Thus, instead of working with dI as control variable, we work with α, β, and Y. Once we have solved for the optimal contract and the optimal level of Y, we can back out the payout process dI (see (20)). Likewise, due to $C^* + \alpha - C = \Delta M$, controlling ΔM is equivalent to controlling C^*; in what follows, we work with C^* rather than ΔM as control variable.

For $C \in (\underline{C}, \overline{C})$ where $dDiv = 0$, the HJB equation (B.3) reduces to the ODE:

$$rP(C) = \max_{\alpha,\beta,Y,C^*}\Big\{P'(C)\left[\mu + (r-\lambda)C - \lambda Y - \sigma^2 \cdot k_Z(\beta) + \pi\left(\alpha - k_\Pi(\alpha)\right)\right] + P''(C)\frac{\sigma^2}{2}(1-\beta)^2 + \pi\left[P(C^*) - P(C) - (C^* - C + \alpha)\right]\Big\},$$

subject to $Y \geq \max\{-C, 0\}$ and $\alpha \in \mathcal{S}(C^*, C)$. The above ODE is equivalent to (16) after rewriting and spelling out the constraints on α and Y into the max operator.

We henceforth assume that a twice continuously differentiable and non-negative solution $P(C)$ to (16) exists on the endogenous state space $(\underline{C}, \overline{C})$ (subject to $P'(\overline{C}) - 1 = P''(\overline{C}) = 0$). We formally establish existence of such a solution in Online Appendix I. Next, we prove $\overline{C} > 0$. To see this, note that we can evaluate the ODE (16) at the payout boundary $\overline{C}$ to obtain $P(\overline{C}) - \overline{C} = \frac{\mu}{r} - \frac{\lambda}{r}\left(\overline{C}\mathbb{1}_{\{\overline{C}\geq 0\}}\right)$. This payoff must be strictly lower than the NPV of the firm, $\frac{\mu}{r}$, which implies $\overline{C} > 0$. Section 2.2 in the main text goes through the maximization of the HJB equation (16), and derives the optimal control variables as functions of excess liquidity C, that is, $Y = Y(C)$, $M = M(C)$, $\alpha = \alpha(C)$, $\beta = \beta(C)$, and $C^* = C^*(C)$.

B.2 Part II — Concavity of Value Function

For convenience, we already conjecture a lower boundary $\underline{C}$ satisfying $\underline{C} \geq -(P(\overline{C}) - \overline{C})$. Further, we can conjecture and verify that $P'(C) \geq 1$ on the state space.

We can solve the optimization in the HJB equation (16) (see also Section 2.2 in the main text). We have $\beta(C) = \frac{P''(C)}{P''(C) - \rho r P'(C)}$ if $P''(C) < 0$ and $\beta(C)$ if $P''(C) \geq 0$. As, by assumption, $P(C)$ is twice continuously differentiable, we have that $P''(C) > -\infty$ for any $C \in (\underline{C}, \overline{C})$, so that $\beta(C) \in [0, 1)$.

Recall that the jump in the value function upon refinancing $J(C)$ is defined in (14), and insert the optimal choice of the refinancing target $C^* = \overline{C}$ which is independent of C. When $J(C)$ is differentiable (which is the case when $\alpha(C)$ is differentiable), then $J'(C) = 1 - P'(C) - \alpha'(C)$. We now rewrite the HJB equation (16) as

$$rP(C) = \max_{\beta \in [0,1]} \left\{ P'(C)\mu_C(C) + \frac{P''(C)}{2}\sigma^2(1 - \beta(C))^2 + \pi \cdot J(C) \right\}, \tag{B.4}$$

under the optimal choice of α in (22), β, $Y = \max\{-C, 0\}$ and $C^* = \overline{C}$, and with $\mu_C(C)$ from (B.1).

When $P''(C)$ and $\alpha(C)$ are differentiable, we can use the envelope theorem and differentiate the HJB equation (B.4) under the optimal $\beta = \beta(C) \in [0, 1)$, satisfying $(1 - \beta(C))\sigma^2 > 0$, with respect to C and rearrange to obtain

$$P'''(C) = \frac{2}{(1 - \beta(C))^2\sigma^2} \left(P'(C)\lambda \mathbb{1}_{\{C \geq 0\}} - P''(C)\mu_C(C) - \pi \left(e^{-\rho r \alpha(C)} P'(C)\alpha'(C) + J'(C) \right) \right), \tag{B.5}$$

where $\mathbb{1}_{\{\cdot\}}$ denotes the indicator function which is equal to one if $\{\cdot\}$ is true and is equal to zero otherwise. The set of points at which either $P''(C)$ or $\alpha(C)$ is not differentiable is countable; therefore, for any C, the limits $\lim_{x \uparrow C} P'''(C), \lim_{x \downarrow C} P'''(C)$ and $\lim_{x \uparrow C} \alpha'(C), \lim_{x \downarrow C} \alpha'(C)$ exist and are well-defined.

Suppose that $\alpha(C)$ is differentiable, and recall $\alpha(C) = \min\{\alpha_C(C), \alpha_U(C)\}$. If $\alpha(C) = \alpha_C(C) = P(\overline{C}) - \overline{C} + C$, then $\alpha'(C) = 1$ and $J'(C) = -P'(C) < 0$. As such,

$$\pi \left(e^{-\rho r \alpha(C)} P'(C)\alpha'(C) + J'(C) \right) = \pi P'(C) \left(e^{-\rho r \alpha(C)} - 1 \right) \leq 0.$$

When $\alpha(C) = \alpha_U(C) = \frac{\ln P'(C)}{\rho r}$, then

$$\begin{aligned} \pi \left(e^{-\rho r \alpha(C)} P'(C)\alpha'(C) + J'(C) \right) &= \pi \left(e^{-\rho r \alpha(C)} P'(C)\alpha'(C) + 1 - P'(C) - \alpha'(C) \right) \qquad \text{(B.6)} \\ &= \pi \left(\alpha'(C) + 1 - P'(C) - \alpha'(C) \right) = \pi(1 - P'(C)) \leq 0. \end{aligned}$$

where it was used that $J'(C) = 1 - P'(C) - \alpha'(C)$ and $e^{-\rho r \alpha(C)} = 1/P'(C)$ as well as $P'(C) \geq 1$. Thus, altogether,

$$\pi \left(e^{-\rho r \alpha(C)} P'(C)\alpha'(C) + J'(C) \right) \leq 0, \tag{B.7}$$

provided $\alpha(C)$ is differentiable.

At the payout boundary, it therefore holds that $P'(\overline{C}) = 1$, and $P''(\overline{C}) = 0$ and

$$\lim_{C \uparrow \overline{C}} \left(e^{-\rho r \alpha(C)} P'(C) \alpha'(C) + J'(C) \right) = \lim_{C \uparrow \overline{C}} (1 - P'(C)) = 0. \tag{B.8}$$

As $\overline{C} > 0$, $P''(\overline{C}) = 0$, $P'(\overline{C}) = 1$, (B.5) and (B.8) imply $\lim_{C \uparrow \overline{C}} P'''(C) > 0$. Because $\lim_{C \uparrow \overline{C}} P'''(C) > 0$ for $\overline{C} > 0$, we obtain $P'''(C) > 0$, and therefore $P''(C) < 0$ and $P'(C) > 1$ and in a left-neighbourhood of $\overline{C}$. Suppose to the contrary that there exists $C' \in (\underline{C}, \overline{C})$, with $P''(C') \geq 0$. Define $\hat{C} = \sup\{C \in (\underline{C}, \overline{C}) : P''(C) \geq 0\}$; note that $\hat{C} < \overline{C}$ and $P''(C) < 0$ for $C \in (\hat{C}, \overline{C})$. By continuity of $P''(C)$, $P''(\hat{C}) = 0$, and $P'(\hat{C}) > 1$.

By (B.7), $\lim_{C \downarrow \hat{C}} \pi \left(e^{-\rho r \alpha(C)} P'(C) \alpha'(C) + J'(C) \right)$ is weakly negative. Hence, (B.5) implies $\lim_{C \downarrow \hat{C}} P'''(C) \geq 0$, where the inequality is strict if $\hat{C} \geq 0$. Consider $\hat{C} \geq 0$. Due to $\lim_{C \downarrow \hat{C}} P'''(C) > 0$, there exists $C' > \hat{C}$ with $P''(C') > 0$, contradiction the definition of $\hat{C}$.

Next, take $\hat{C} < 0 < \overline{C}$. Distinguish two cases. First, when $\pi J(\hat{C}) > 0 \geq -\pi P(\hat{C})$, then (15) does not bind in a neighbourhood of $\hat{C}$ and $\alpha(C) = \alpha_U(C)$. Thus, by (B.6):

$$\lim_{C \downarrow \hat{C}} \pi \left(e^{-\rho r \alpha(C)} P'(C) \alpha'(C) + J'(C) \right) = \pi (1 - P'(\hat{C})) < 0.$$

Then, (B.5) implies $\lim_{C \downarrow \hat{C}} P'''(C) > 0$. Thus, there exists $C' > \hat{C}$ so that $P''(C') > 0$, which contradicts the definition of $\hat{C}$.

Second, suppose $\pi J(\hat{C}) \leq 0$. Because $\hat{C} \in (\underline{C}, \overline{C})$, $P'(\hat{C}) > 1$, and $P(C) \geq 0$ for all $C \in (\underline{C}, \overline{C})$, it follows that $P(\hat{C}) > 0$ which — by (B.4) — implies that $\mu_C(\hat{C}) > 0$. By definition of $\hat{C}$ and because $\hat{C} < \overline{C}$, there must exist $\epsilon > 0$ such that $P''(C) < 0$, $\mu_C(C) > 0$, and $P'''(C)$ exists with $P'''(C) < 0$ for $C \in (\hat{C}, \hat{C} + \epsilon)$. But, using $P''(C) < 0$ and $\mu_C(C) > 0$ as well as (B.7), we obtain from (B.5) that $P'''(C) > 0$ for $C \in (\hat{C}, \hat{C} + \epsilon)$, a contradiction.

Either way, it follows that $P''(C) < 0$ for all $C \in (\underline{C}, \overline{C})$, which concludes the proof.

B.3 Part III — Verification Argument

Let $\mathcal{C}$ be the contract which implements the controls according to the optimization in the HJB equation (16) and under which equity value $P_t = P(C_t)$ solves (16) on $(\underline{C}, \overline{C})$ with dividend payout boundary $\overline{C}$ subject to $P'(\overline{C}) - 1 = P''(\overline{C}) = 0$. And, consider any other contract $\hat{\mathcal{C}}$ that respects the intermediary's and the investors' limited commitment and $dDiv_t \leq C_t - \underline{C}$. We show that contract $\mathcal{C}$ yields higher payoff at $t = 0$ than any other admissible contract $\hat{\mathcal{C}}$.

For $t \leq \tau$ (possibly $\tau = \infty$), define $\mathcal{L}P(C_t) = P'(C_t)\mu_C(C_t) + \frac{\sigma_C(C_t)^2 P''(C_t)}{2}$, with $\mu_C(C_t)$ and $\sigma_C(C_t)$ from (B.1), and

$$G_t^P = \int_0^t e^{-rs}(dDiv_s - \Delta M_s d\Pi_s) + e^{-rt} P(C_t) \mathbb{I}_{\{t < \tau\}},$$

where $\Delta M_t = C_t^* - C_t + \alpha_t$. Note that G_t^P is the equity value when the contract $\hat{\mathcal{C}}$ is followed up to time t and after time t the contract $\mathcal{C}$ is followed. By Itô's Lemma for $t < \tau$:

$$\begin{aligned} e^{rt}dG_t^P &= \{-rP(C_t) + \mathcal{L}P(C_t)\}\,dt + (P(C_t^*) - P(C_t))d\Pi_t \\ &\qquad - \Delta M_t d\Pi_t + \big(1 - P'(C_t)\big)d\hat{D}iv_t + \sigma_{Ct}P'(C_t)dZ_t \\ &\equiv \mu_t^G dt + \big(1 - P'(C_t)\big)dDiv_t + \sigma_{Ct}P'(C_t)dZ_t + (P(C_t^*) - P(C_t) - \Delta M_t)(d\Pi_t - \pi dt). \end{aligned}$$

We define μ_t^G as

$$\mu_t^G = -rP(C_t) + \mathcal{L}P(C_t) + \pi(P(C_t^*) - P(C_t) - \Delta M_t).$$

Next, note that we can rewrite the HJB equation (16) as

$$rP(C_t) = \max_{\alpha_t, \beta_t, Y_t, C^*} \{\mathcal{L}P(C_t) + \pi\,[P(C_t^*) - P(C_t) - \Delta M_t]\}\,, \tag{B.9}$$

subject to all relevant constraints, where we use time subscripts in this part of the proof.

As a result, the HJB equation (B.9) implies that the drift term of $e^{rt}dG_t^P$ — that is, $\mu_t^G = -rP(C_t) + \mathcal{L}P(C_t) + \pi(P(C_t^*) - P(C_t) - \Delta M_t)$ — is zero under the controls obtained via the optimization in the HJB equation (16). Moreover, any other strategy $\hat{\mathcal{C}}$ and choice of $(\alpha_t, \beta_t, Y_t, C_t^*)$ makes this term (weakly) negative, so that $\mu_t^G \leq 0$ for $t < \tau$. Because of $dDiv_t \geq 0$, $P'(C_t) > 1$ for $C_t < \overline{C}$, and $P'(C_t) = 1$ for $C_t \geq \overline{C}$, the term $\big(1 - P'(C_t)\big)$ is (weakly) negative under any dividend payout policy $dDiv_t$ and zero under the dividend payout policy $dDiv_t$ that causes C_t to reflect at $\overline{C}$.

Next, our regularity conditions ensure that α_t, β_t are bounded. Thus, $\sigma_{Ct} = \sigma(1 - \beta_t)$ is bounded too. In addition, $P'(C)$ and $P(C)$ are bounded over $(\underline{C}, \overline{C})$, as $P(C)$ is twice continuously differentiable on the same interval. Thus, $\mathbb{E}\left[\int_0^t e^{-rs}\sigma_{Cs}P'(C_s)dZ_s\right] = 0$. for all $t \leq \tau$. As α_t and $P(C_t)$ are bounded, we obtain likewise $\mathbb{E}\left[\int_0^t e^{-rs}(P(C_s^*) - P(C_s) - \Delta M_s)(d\Pi_s - \pi ds)\right] = 0$. Therefore, the process $\{G_t^P\}$ follows a supermartingale (i.e., decreases in expectation) up to τ. By the optional stopping theorem, the process $\{G_{t\wedge\tau}^P\}$ follows a supermartingale too, so $\mathbb{E}[G_{t\wedge\tau}^P] \leq G_0^P$. Taking the limit $t \to \infty$ yields

$$P(C_0) - C_0 = G_0^P \geq \lim_{t\to\infty} \mathbb{E}[G_{t\wedge\tau}^P] = \mathbb{E}[G_\tau^P] = \mathbb{E}\left[\int_0^\tau e^{-rs}\Big(dDiv_s - \Delta M_s d\Pi_s\Big)\right].$$

At inception, the firm is penniless and has access to equity financing, i.e., $d\Pi_0 = 1$ and $\Delta M_0 = C_0$, so ex-ante payoff under $\mathcal{C}$ is $P(C_0) - C_0$. Because contract $\mathcal{C}$ yields equity value $G_0^P = P(C_0) - C_0$, it maximizes the equity value over all admissible contracts that respect the intermediary's and the investors' limited commitment.

C Proof of Lemma 2

The proof is split in three parts. Part I presents auxiliary results and conditions that the lower boundary satisfies in the survival scenario. Part II derives (24). Part III demonstrates that the firm optimally never liquidates and $\underline{C} = C^S$ if and only if $C^S \leq -L$; otherwise, liquidation occurs in finite time and $\underline{C} = -L$. In either scenario, $P(\underline{C}) = 0$.

C.1 Part I — Auxiliary Results

Lemma 3. *Suppose the firm never liquidates (i.e., $\tau = \infty$) and $P(C)$ solves the HJB equation (16). Define $C^S = \inf\{C \geq -\frac{\mu}{r} : \mu_C(C) \geq 0, P(C) \geq 0, \sigma_C(C) = 0\}$. Then, $\mu_C(C^S) = \sigma_C(C^S) = 0$, $\beta(C^S) = 1$, $P(C^S) = 0$, and $\alpha(C^S) = P(\overline{C}) - \overline{C} + C^S$.*

Proof of Lemma 3. Rewrite the HJB equation (16) under the optimal controls $\alpha(C)$ (see (22)), $\beta(C)$ (see (23)), $Y(C) = \max\{-C, 0\}$, $J(C)$ (see (14)), and $C^* = \overline{C}$ as

$$rP(C) = P'(C)\mu_C(C) + \frac{P''(C)(\sigma_C(C))^2}{2} + \pi J(C). \tag{C.1}$$

Holding $\alpha = \alpha(C)$ fixed, $J(C)$ decreases with C, as $P'(C) \geq 1$.

Note that $\sigma_C(C) = \sigma(1 - \beta(C)) = 0$ is equivalent to $\beta(C) = 1$. As $\mu_C(C)$ increases with C, decreases with $\beta(C)$, and increases with $\alpha(C)$, as $J(C)$ decreases with $\alpha(C)$, as the right hand side of (15) (with $C^* = \overline{C}$) increases with C, and as equity value $P(C)$ is characterized by (C.1), it follows that $\sigma_C(C^S) = 0 \iff \beta(C^S) = 1$, $J(C^S) = 0$, $\mu_C(C^S) = 0$, and $P(C^S) = 0$ (so $J(C^S) = -P(C^S)$ and (15) is tight at $C = C^S$). In more detail, if it were $\mu_C(C^S) > 0$, there would exist $C' < C^S$ such that the contract could implement $\sigma_C(C') = 0$ and $\mu_C(C') \geq 0$ with the same choice of β (i.e., $\beta(C') = \beta(C^S) = 1$) and α such that $J(C') \geq 0 \geq -P(C')$ holds and $P(C') \geq 0$ due to (C.1), contradicting the definition of C^S.

Likewise, if it were $J(C^S) > -P(C^S)$, then there would exist $C' < C^S$, such that the contract can stipulate $J(C') \geq -P(C')$, $\alpha(C') \geq \alpha(C^S) \geq 0$, $\beta(C') = 1$, and $\mu_C(C') \geq 0$ as $\mu_C(C)$ increases with α which leads to $P(C') \geq 0$, contradicting the definition of C^S. Finally, $P(C^S) > 0$ while $\beta(C^S) = 1$ would imply $\mu_C(C^S) > 0$ or $J(C^S) > 0$, again leading to a contradiction. Thus, $\sigma_C(C^S) = 0$, $J(C^S) = 0$, $\mu_C(C^S) = 0$, and $P(C^S) = 0$.

Next, we show that $P(C^S) = 0$ implies $\beta(C^S) = 1$, $J(C^S) = 0$, and $\mu_C(C^S) = 0$ under the optimal controls from (16). According to the optimization in (16), the optimal choice of $\alpha(C^S)$ and $\beta(C^S)$ induces $P(C^S) = 0$. Setting $\beta(C^S) = 1$ and $\alpha(C^S)$ such that $J(C^S) = 0$ implies, by definition of C^S, $\mu_C(C^S) = \sigma_C(C^S) = J(C^S) = 0$ and therefore $rP(C^S) = P'(C^S)\mu_C(C^S) + \frac{P''(C^S)(\sigma_C(C^S))^2}{2} + \pi J(C^S) = 0$. Thus, setting $\beta(C^S) = 1$ and $\alpha(C^S)$ such that $J(C^S) = 0$ is optimal and consistent with the optimization in (16). □

C.2 Part II — Derivation of (24)

As shown in Lemma 3, we have $\mu_C(\underline{C}) = P(\underline{C}) = 0$ and $\beta(\underline{C}) = 1$ for $\underline{C} = C^S$. To derive an expression for C^S, one first uses (12) to calculate the drift of excess liquidity under the optimal choice of Y derived in the previous section (that is, $Y(C) = \max\{-C, 0\}$):

$$\mu_C(C) = \mu + \left(r - \lambda \mathbb{1}_{\{C \geq 0\}}\right) C - \frac{\rho r}{2}\sigma^2 \beta(C)^2 + \pi \left(\frac{1 - e^{-\rho r \alpha(C)}}{\rho r} \right), \tag{C.2}$$

where $\mathbb{1}_{\{.\}}$ denotes the indicator function, i.e., it is 1 if $\{\cdot\}$ is true and 0 otherwise. The HJB equation (16) evaluated under the optimal controls $\alpha(C)$ and $\beta(C)$ as well as $C^* = \overline{C}$ can be rewritten as in (C.1). Due to $\mu_C(\underline{C}) = P(\underline{C}) = \sigma_C(\underline{C}) = 0$, we have by means of (C.1) that $J(\underline{C}) = 0$ and therefore $\alpha(\underline{C}) = P(\overline{C}) - \left[\overline{C} - \underline{C}\right] = \frac{\mu}{r} - \frac{\lambda}{r}\overline{C} + \underline{C}$. The last equality uses that at the payout boundary $\overline{C} > 0$, the HJB equation (16) implies $P(\overline{C}) = \frac{\mu}{r} + \overline{C} - \frac{\lambda \overline{C}}{r}$, due to $\beta(\overline{C}) = \alpha(\overline{C}) = P'(\overline{C}) - 1 = P''(\overline{C}) = 0$.

Substituting in for the optimal policies, and using $\alpha(\underline{C})$ from above in $\mu_C(\underline{C}) = 0$ while using that $\sigma_C(\underline{C}) = 0 \iff \beta(\underline{C}) = 1$, we have

$$0 = \mu_C(\underline{C}) = \mu + r\underline{C} - \frac{\rho r}{2}\sigma^2 + \frac{\pi}{\rho r}\left(1 - e^{-\rho r\left[\frac{\mu}{r} - \frac{\lambda}{r}\overline{C} + \underline{C}\right]}\right). \tag{C.3}$$

We use the following Lemma to solve for $\underline{C}$:

Lemma 4. *The solution to*

$$0 = a + x + e^{(b + c \cdot x)} \tag{C.4}$$

is given by

$$x = -\frac{w\left(c \cdot \exp\{b - a \cdot c\}\right) + a \cdot c}{c}, \tag{C.5}$$

where $w(\cdot)$ is the primary branch of the Lambert-w function.

Proof. Define $z \equiv c \cdot \exp\{b - ac\}$. Plugging in proposed solution (C.5) into (C.4), we have

$$\begin{aligned} 0 &= a + \left(-\frac{w\left(c \cdot \exp\{b - ac\}\right)}{c} - a\right) + \exp\{b - w\left(c \cdot \exp\{b - ac\}\right) - ac\} \\ &= -\frac{w\left(c \cdot \exp\{b - ac\}\right)}{c} + \exp\{b - ac\}\exp\{-w\left(c \cdot \exp\{b - ac\}\right)\} \\ &= -w\left(c \cdot \exp\{b - ac\}\right) + c \cdot \exp\{b - ac\}\exp\{-w\left(c \cdot \exp\{b - ac\}\right)\} \\ &= -w(z) + z\exp(-w(z)) \end{aligned}$$

where we multiplied through by $c \neq 0$ in the second-to-last line. The last line equals zero by definition of the Lambert-w function $w(z)e^{w(z)} = z \iff w(z) = z \cdot e^{-w(z)}$. □

Next, we rewrite (C.3) as

$$0 = \frac{-\rho r}{\pi}\left(\mu - \frac{\rho r}{2}\sigma^2 + \frac{\pi}{\rho r}\right) - \frac{\rho r^2}{\pi}\underline{C} + e^{-\rho(\mu - \lambda\overline{C}) - \frac{\pi}{r}\frac{\rho r^2}{\pi}\underline{C}}. \tag{C.6}$$

Define $a \equiv \frac{-\rho r}{\pi}\left(\mu - \frac{\rho r}{2}\sigma^2 + \frac{\pi}{\rho r}\right)$, $b \equiv -\rho(\mu - \lambda\overline{C})$, $c \equiv \frac{\pi}{r}$, and $x \equiv -\frac{\rho r^2}{\pi}\underline{C}$. We now apply the above lemma to solve (C.6) for x and to thus obtain (24), that is,

$$\underline{C} = C^S = \frac{w\left(\frac{\pi}{r}\exp\left\{\rho r\left[\frac{\lambda}{r}\overline{C} + \frac{\pi}{\rho r^2} - \frac{\rho}{2}\sigma^2\right]\right\}\right) - \frac{\pi}{r}}{\rho r} - Y^A, \tag{C.7}$$

where $w(\cdot)$ is the Lambert function (i.e., $w(z)$ is the principal-branch solution to $we^w = z$). Note that when $\pi = 0$, then $C^S = -Y^A$, where Y^A is the autarky value defined in (9).

Finally, we show that $-\underline{C} \leq P(\overline{C}) - \overline{C}$ and as such $\alpha(\underline{C}) \geq 0$ as well as $\alpha(C) = \min\{\alpha_C(C), \alpha_U(C)\} \geq 0$ for all C. For this sake, note that $P(\overline{C}) - \overline{C}$ must weakly exceed Y^A (see, for instance, (10)). When $\pi = 0$, then $-\underline{C} = Y^A$ and the claim follows. Next, consider $\pi > 0$. Suppose to the contrary $-\underline{C} > P(\overline{C}) - \overline{C}$, i.e., $\alpha(\underline{C}) < 0$. Then, $\pi\left(\frac{1-e^{-\rho r \alpha(\underline{C})}}{\rho r}\right) < 0$ and, by (C.3), we obtain $-\underline{C} < Y^A \leq P(\overline{C}) - \overline{C}$, a contradiction.

C.3 Part III

We determine when liquidation or survival (i.e., $\tau = \infty$) scenario applies. We distinguish between i) $C^S < -L$ and ii) $C^S > -L$. The knife-edge case $C^S = -L$ follows analogously.

Suppose that $C^S < -L \leq 0$. We conjecture and verify that the survival scenario prevails (so that $\tau = \infty$). Suppose that $P(C)$ solves (16) subject to $P'(\overline{C}) - 1 = P''(\overline{C}) = P(C^S) = 0$. Previous results imply that $P'(C) > 1$ on $(C^S, \overline{C})$. Note that $Y(C) = \max\{0, -C\}$ implies $Y(C) \leq -C^S$. Due to $P'(C) > 1$ for $C < \overline{C}$ and $P(C^S) = 0$, it follows that $P(C) > C - C^S$. If in state $C \geq C^S$ the firm is liquidated and all cash holdings $M(C)$ are paid out (to shareholders and intermediary), total firm value "just before" liquidation is the sum of cash balance $M(C)$ and liquidation value L, which is split between intermediary and shareholders. Thus, shareholders would obtain upon liquidation in state $C > C^S$, $M(C) + L - Y(C)$, while the intermediary receives $Y(C)$ (dollars). Note that

$$M(C) + L = C + Y(C) + L \leq C - C^S + Y(C) < P(C) + Y(C),$$

where the first inequality used that $L \leq -C^S$, and the second that $P(C) > C - C^S$. As a result, $P(C) > M(C) + L - Y(C)$, and liquidation in state $C > C^S$ is not optimal for shareholders. As, in addition, liquidation in state $C = C^S$ would violate $Y_\tau = -C^S \leq L$, the survival scenario prevails in optimum, i.e., $\tau = \infty$.

Suppose that $C^S > -L$. It follows that $Y(C) = \max\{0, -C\} = 0$ for $C \geq C^S$, and $M(C) = \max\{C, 0\}$. Conditional on survival, i.e., no liquidation at the lower boundary $\underline{C} = C^S$ and $\tau = \infty$, the boundary condition $P(C^S) = 0$ applies for the hypotehtical value function. However, survival cannot be optimal for shareholders. Liquidating the firm at $C = C^S$ and paying out $M(C^S) = \max\{0, C^S\} \geq 0$ dollars as dividends yields value $L + \max\{0, C^S\} > 0$ for shareholders. As such, the liquidation scenario prevails.

It remains to show that liquidation occurs the first time C falls to $-L$ so that $\underline{C} = -L \leq 0$. To start with, note that liquidation at $C < -L \leq 0$ with $M(C) = 0$ is not possible because at the time of liquidation, $Y(C) \leq L$ must hold to ensure promise-keeping, and $C < -L$ would imply $Y(C) = -C > L$. Thus, liquidation can only occur in states $C \geq -L$. Next, suppose the firm is liquidated at $C = -L$, so the intermediary receives a payout of

$Y(C) = -C = L$ and $P(-L) = 0$ at liquidation. With $P(C)$ the value function solving (16) subject to $P(-L) = P'(\overline{C}) - 1 = P''(\overline{C}) = 0$, we have that $P'(C) > 1$. This implies for $C < \overline{C}$ that $P(C) > C + L$ for $C > -L$. Consider $C > -L \iff -C < L$. If the firm were liquidated in state $C > -L$, then shareholders and intermediary would jointly receive

$$M(C) + L = C + Y(C) + L < P(C) + Y(C),$$

where the inequality uses $P(C) > C + L$. Thus, liquidation at $C > -L$ is not optimal. As liquidation must occur for $C \geq -L$, it follows that optimal liquidation occurs at $C = \underline{C} = -L$, i.e., $\tau = \inf\{t \geq 0 : C_t = -L\}$. Taken together, $\underline{C} = \min\left\{C^S, -L\right\}$ and $P\left(\underline{C}\right) = 0$.

D Dividend Payouts and Promise-Keeping

Lemma 5. *In a full-commitment contract $\mathcal{C}$, dividends must satisfy $dDiv_t \leq C_t - \underline{C}$.*

Proof. Due to $P(C) \geq 0$, $Y(C) \leq M(C) + \frac{\mu}{r}$ and $C \geq -\frac{\mu}{r}$. Suppose to the contrary that at time t, the firm pays $dDiv_t > C_t - \underline{C}$ and so causes C_t to drop to value $C' < \underline{C} = \min\{-L, C^S\} \leq 0$. After the dividend payout, we have $Y = Y' > L$. By definition of C^S and Lemma 3, we have for any $C < C^S$ that $\mu_C(C) < 0$ or $\sigma_C(C) \neq 0$ in case the firm does not liquidate. Liquidation in any state $C < -L$ violates promise-keeping (as we have $Y > L$). Suppose the firm liquidates in state $C'' < \underline{C}$. Then, with strictly positive probability, there exists time $T > t$ such that C_T reaches C'', leading to liquidation and a violation of promise-keeping. Next, consider that the firm does not liquidate in any state $C < -L$. As a result, with strictly positive probability, there exists time $T > t$ such that $C_T < -\frac{\mu}{r}$, a contradiction. Thus, Y_t must exhibit a discrete downward jump upon the payout $dDiv_t = C_t - \underline{C}$ ensuring $C' \geq \underline{C}$, which is inconsistent with the dynamics (6) and violates promise-keeping. Thus, dividend payouts satisfy $dDiv_t \leq C_t - \underline{C}$. □

E Proof of Proposition 2

Part I derives a sufficient condition for the stationary distribution to exist. In particular, under this condition, the lower boundary $\underline{C} = C^S$, while conditionally absorbing, is not attainable. Parts II and III show that this condition is satisfied.

E.1 Part I

In the interior of the state space for $C \in (\underline{C}, \overline{C})$ when $\sigma_C(C)$ is twice differentiable, the stationary density $g(C)$ — provided its existence — satisfies the Kolmogorov forward (Fokker Planck) equation: $\pi \cdot g(C) = -\frac{\partial}{\partial C}\left[\mu_C(C)g(C)\right] + \frac{1}{2}\frac{\partial^2}{\partial C^2}\left[\sigma_C(C)^2 g(C)\right]$. We can integrate over C to obtain

$$\pi G(C) = G(\underline{C}) - \mu_C(C)g(C) + \frac{1}{2}\frac{\partial}{\partial C}\left[\sigma_C(C)^2 g(C)\right]. \tag{E.1}$$

with stationary distribution function $G(C) = \int_{\underline{C}}^{x} g(x)dx$. When $\underline{C}$ is not accessible, (E.1) is solved subject to $G(\overline{C}) = 1$ and $G(\underline{C}) = 0$. Define the scaled stationary density $\hat{g}(C) =$

$\sigma_C(C)^2 g(C)$, so that $\hat{g}'(C) = 2\pi G(C) + \mu_C(C) g(C) = 2\pi G(C) + 2\hat{g}(C)\left(\frac{\mu_C(C)}{\sigma_C(C)^2}\right)$. Equivalently, the log scaled density has derivative $\frac{d \ln \hat{g}(C)}{dC} = \frac{\hat{g}'(C)}{\hat{g}(C)} = \frac{2\pi G(C)}{\hat{g}(C)} + 2\left(\frac{\mu_C(C)}{\sigma_C(C)^2}\right)$. The boundary $\underline{C}$ is absorbing conditionally on no jumps. A non-degenerate stationary density, with the absorbing boundary at $\underline{C}$, exists if the boundary condition $\hat{g}(\underline{C}) = 0$ can be satisfied together with $\hat{g}(\hat{C}) > 0$ for $\hat{C} > \underline{C}$; in this case, the boundary $\underline{C}$ is never reached or inaccessible. For this to happen, we need that

$$\ln \hat{g}(C) = \ln \hat{g}(\hat{C}) - 2\int_C^{\hat{C}} \frac{2\pi G(c)}{\hat{g}(c)} dc - 2\int_C^{\hat{C}} \frac{\mu_C(c)}{\sigma_C(c)^2} dc$$

tends to $-\infty$, as $C \to \underline{C}$; see Brunnermeier and Sannikov (2014) for an analogous argument in a similar context. A sufficient condition is Feller's test for explosions, i.e.,

$$\lim_{C \to \underline{C}} \int_C^{\hat{C}} \frac{\mu_C(c)}{\sigma_C(c)^2} dc = +\infty. \tag{E.2}$$

In the following two parts, we show that (E.2) is met, which then implies that $\underline{C}$ is never reached and a stationary distribution of states exists.

E.2 Part II

Define $\Gamma(C) = -P''(C)/P'(C)$ and rewrite

$$\beta(C) = 1 - \frac{\rho r}{\Gamma(C) + \rho r} \iff \sigma_C(C) = \sigma(1 - \beta(C)) = \frac{\sigma \rho r}{\Gamma(C) + \rho r}. \tag{E.3}$$

Next, we focus on states $C < 0$ and in which $P''(C)$ is differentiable. For $C < 0$, the drift of C becomes

$$\begin{aligned} \mu_C(C) &= \mu + rC - \frac{\sigma^2 \rho r}{2}\left[1 - \frac{2\sigma_C(C)}{\sigma} + \left(\frac{\sigma_C(C)}{\sigma}\right)^2\right] + \pi\left[\alpha(C) - k_\Pi(\alpha(C))\right] \\ &= r(C - \underline{C}) + \pi\big[\alpha(C) - k_\Pi(\alpha(C)) - \alpha(\underline{C}) + k_\Pi(\alpha(\underline{C}))\big] + \frac{\rho r \sigma_C(C)}{2}\left(\frac{2}{\sigma} - \sigma_C(C)\right), \end{aligned} \tag{E.4}$$

where in the second line we subtracted $\mu_C(\underline{C}) = \mu + r\underline{C} - \frac{\sigma^2 \rho r}{2} + \pi\left[\alpha(\underline{C}) - k_\Pi(\alpha(\underline{C}))\right] = 0$. For C sufficiently close to $\underline{C}$ all of these three terms are positive, since $\sigma_C(C)$ tends to zero as C approaches $\underline{C}$. Therefore, a sufficient condition for (E.2) is

$$\lim_{C \to \underline{C}} \int_C^{\hat{C}} \frac{(c - \underline{C})}{\sigma_C(c)^2} dc = +\infty. \tag{E.5}$$

Using (E.3), a sufficient condition for (E.5), and thus for (E.2), is

$$\lim_{C \to \underline{C}} \int_C^{\hat{C}} (c - \underline{C}) \Gamma(c)^2 dc = \lim_{C \to \underline{C}} \int_C^{\hat{C}} \frac{(c - \underline{C})}{1/(\Gamma(c))^2} dc = +\infty.$$

In Part III, we will show that there exists constant $\mathcal{K} > 0$ such that $1/\Gamma(C) < \mathcal{K}(C - \underline{C})$ in a neighbourhood $(\underline{C}, \underline{C} + \varepsilon)$ with $\varepsilon > 0$ of $\underline{C}$. Then, we can pick $\hat{C} \in (\underline{C}, \underline{C} + \varepsilon)$ and calculate

$$\lim_{C \to \underline{C}} \int_C^{\hat{C}} \frac{(c - \underline{C})}{1/(\Gamma(c))^2} dc \geq \lim_{C \to \underline{C}} \int_C^{\hat{C}} \frac{(c - \underline{C})}{\mathcal{K}^2 (c - \underline{C})^2} dc = \frac{1}{\mathcal{K}^2} \lim_{C \to \underline{C}} \int_C^{\hat{C}} \frac{1}{c - \underline{C}} dc$$
$$= \frac{1}{\mathcal{K}^2} \lim_{C \to \underline{C}} \left[\ln(\hat{C} - \underline{C}) - \ln(C - \underline{C}) \right] = +\infty.$$

E.3 Part III

To show $1/\Gamma(C) < \mathcal{K}(C - \underline{C})$ in a neighbourhood of $\underline{C}$, we assume sufficient differentiability and conduct a Taylor expansion around $\underline{C}$ (noting that $1/\Gamma(\underline{C}) = 0$):

$$\frac{1}{\Gamma(C)} = -\frac{\Gamma'(C)}{\Gamma(C)^2}(C - \underline{C}) + o((C - \underline{C})^2). \tag{E.6}$$

Calculate $\Gamma'(C) = \frac{-P'''(C)P'(C) + P''(C)^2}{P'(C)^2} = \frac{-P'''(C)}{P'(C)} + \Gamma(C)^2$. It boils down to show that $\frac{\Gamma'(C)}{\Gamma(C)^2}$ is bounded in a neighbourhood of $\underline{C}$.

Next, recall (B.5), which implies for $C < 0$:

$$P'''(C) = \frac{2}{\sigma_C(C)^2} \left(-P''(C)\mu_C(C) - \pi \left[e^{-\rho r \alpha(C)} P'(C)\alpha'(C) + J'(C) \right] \right)$$
$$= \frac{2\left[\Gamma(C) + \rho r\right]^2}{(\sigma \rho r)^2} \left(-P''(C)\mu_C(C) - \pi \left[e^{-\rho r \alpha(C)} P'(C)\alpha'(C) + J'(C) \right] \right)$$

For C close to $\underline{C}$, the constraint (15) binds, so that $\pi \left[e^{-\rho r \alpha(C)} P'(C)\alpha'(C) + J'(C) \right] = \pi P'(C) \left(e^{-\rho r \alpha(C)} - 1 \right)$. Thus,

$$\frac{P'''(C)}{P'(C)} = \frac{2\left[\Gamma(C) + \rho r\right]^2}{(\sigma \rho r)^2} \left[\Gamma(C)\mu_C(C) - \pi \left(e^{-\rho r \alpha(C)} - 1 \right) \right]$$

and

$$\frac{P'''(C)}{P'(C)\Gamma(C)^2} = \frac{2}{(\sigma \rho r)^2} \left[\Gamma(C)\mu_C(C) - \pi \left(e^{-\rho r \alpha(C)} - 1 \right) \right] + o\left(\frac{1}{\Gamma(C)} \right)$$

The term $\pi \left(e^{-\rho r \alpha(C)} - 1 \right)$ is clearly bounded, and so are the remainder terms of order $1/\Gamma(C)$, since $\lim_{C \to \underline{C}} \Gamma(C) = +\infty$.

Suppose now that $\Gamma(C)\mu_C(C)$ is bounded in a neighbourhood of $\underline{C}$. Then, $\frac{P'''(C)}{P'(C)\Gamma(C)^2}$ and therefore also $-\frac{\Gamma'(C)}{\Gamma(C)^2} = \frac{P'''(C)P'(C)}{P''(C)^2} - 1$ are bounded in a neighbourhood of $\underline{C}$. Thus, there exists $\mathcal{K} > 0$ such that $1/\Gamma(C) < \mathcal{K}(C - \underline{C})$ in a neighbourhood of $\underline{C}$, as desired.

Next, consider that $\Gamma(C)\mu_C(C)$ is *not* bounded in a neighbourhood of $\underline{C}$. In particular,

$\lim_{C \to \underline{C}} \Gamma(C)\mu_C(C) = +\infty$. Using (E.4), we obtain for $C < 0$ close to $\underline{C}$:

$$\begin{aligned}\Gamma(C)\mu_C(C) = r\Gamma(C)(C - \underline{C}) + \pi\Gamma(C)\big[\alpha(C) - k_\Pi(\alpha(C)) - \alpha(\underline{C}) + k_\Pi(\alpha(\underline{C}))\big] \\ + \frac{(\rho r)^2\sigma\Gamma(C)}{2(\Gamma(C) + \rho r)}\left(\frac{2}{\sigma} - \sigma_C(C)\right) \\ \leq r\Gamma(C)(C - \underline{C}) + \pi\Gamma(C)\big[\alpha(C) - k_\Pi(\alpha(C)) - \alpha(\underline{C}) + k_\Pi(\alpha(\underline{C}))\big] + \rho^2 r^2.\end{aligned}$$

Next, note that because of (15) — which binds close to $\underline{C}$ — the derivative with respect to C of the term $\pi\big[\alpha(C) - k_\Pi(\alpha(C)) - \alpha(\underline{C}) + k_\Pi(\alpha(\underline{C}))\big]$ is bounded, whenever it exists. As such, there exists constant $\mathcal{K}_1 \in (0, \infty)$ such that $\pi\big[\alpha(C) - k_\Pi(\alpha(C)) - \alpha(\underline{C}) + k_\Pi(\alpha(\underline{C}))\big] < \mathcal{K}_1(C - \underline{C})$ in a neighbourhood of $\underline{C}$. Accordingly, in a neighbourhood of $\underline{C}$, we have $\Gamma(C)\mu_C(C) \leq \mathcal{K}_2\Gamma(C)(C - \underline{C})$ for an appropriate constant $\mathcal{K}_2 \in (0, \infty)$. Due to $\lim_{C \to \underline{C}} \Gamma(C)\mu_C(C) = +\infty$, we have $\lim_{C \to \underline{C}} \Gamma(C)(C - \underline{C}) = +\infty$. In particular, there exists constant $\mathcal{K} > 0$ such that $\Gamma(C)(C - \underline{C}) > \frac{1}{\mathcal{K}}$, i.e., $\frac{1}{\Gamma(C)} < \mathcal{K}(C - \underline{C})$, in a neighbourhood of $\underline{C}$, concluding the proof.

F Proof of Proposition 3

The claims of Proposition 3 follow from the previous results presented in the main text and proven in the Appendix. The optimal control variables — that is, $Y = Y(C)$, $M = M(C)$, $\alpha = \alpha(C)$, $\beta = \beta(C)$, and $C^* = C^*(C)$ — are derived in the main text in Section 2.2 by going through the optimization in the HJB equation (16).

Online Appendix I establishes existence of a (twice differentiable) non-negative solution $P(C)$ to the system (16) subject to $P'(\overline{C}) - 1 = P''(\overline{C}) = P(\underline{C}) = 0$ with $\underline{C}$ from (25). Given this existence result, it follows $\beta(C) < 1$ for $C \in (\underline{C}, \overline{C})$. To see this, recall that the optimal choice of $\beta(C)$ is determined according to the optimization in (16) and therefore satisfies (23), with $\beta(C) \in [0, 1]$ due to concavity on $(\underline{C}, \overline{C})$. It follows that $\beta(C) \to 1$ only if $P''(C) \to -\infty$, as $P'(C) \geq 1$. However, because the value function $P(C)$ is twice continuously differentiable on $(\underline{C}, \overline{C})$, there cannot exist $C' \in (\underline{C}, \overline{C})$ such that $\lim_{C \to C'} P''(C) = -\infty$. As such, there cannot exist $C' \in (\underline{C}, \overline{C})$ such that $\lim_{C \to C'} \beta(C) = 1$. Thus, $\beta(C) < 1$ for $C > \underline{C}$.

That is, while it is always possible to set $\beta(C') = 1$ for $C' > C^S$ to ensure $\mu_C(C') > 0$ and $\sigma_C(C') = 1$ and $C_t \geq C'$ at all times t (with certainty), this is not optimal.[33] So, all states C within $(\underline{C}, \overline{C})$ are reached with positive probability.

G Proof of Proposition 4

G.1 Part I

We are looking for functions $\hat{r}(C)$ such that T_t can be represented as a Markovian function $T(C_t)$. Doing so, we show that there exists (unique) $\hat{r}(C)$ such that T_t is Markovian and

[33] Note that the stipulation of the boundary condition(s) $P(C^S) = 0$ and $\beta(C^S) = 1$ to solve the HJB equation (16) is not an optimality result but a consequence of the requirement C must be bounded from below under incentive compatible contracts and survival; the stipulation of the boundary condition $P(C^S) = 0$ does not per-se preclude $\beta(C') = 1$ for $C' > C^S$.

depends on C_t only, in that we can write $T_t = T(C_t)$. We also solve for $T(C_t)$, thereby showing its uniqueness. First, we differentiate T_t for $t > \tau^{\Pi}(t)$ in (28) w.r.t. t to get

$$dT_t = \hat{r}(C_t)T_t dt - dI_t - T_{t^-}d\Pi_t = \hat{r}(C_t)T_t dt - \mu_I(C_t)dt + \sigma_I(C_t)dZ_t + \alpha_I(C_t)d\Pi_t - T_{t^-}d\Pi_t, \tag{G.1}$$

where dI_t follows (20) and $\mu_I(C_t)$, $\sigma_I(C_t)$, $\alpha_I(C_t)$ are implicitly defined in (20). When $d\Pi_t = 1$, then T_t is reset to zero, where $T_{t^-} := \lim_{s\uparrow t} T_s$ denotes the left-limit of T_t.

Applying Ito's Lemma to $T(C_t)$ for $C_t \in (\underline{C}, \overline{C})$ we get:

$$\begin{aligned} dT(C_t) = &\left(T'(C_t)\mu_C(C_t) + \frac{T''(C_t)\sigma_C(C_t)^2}{2}\right)dt + T'(C_t)\left[\sigma_C(C_t)dZ_t - dDiv_t\right] \\ &+ \left[T(\overline{C}) - T(C_t)\right]d\Pi_t. \end{aligned} \tag{G.2}$$

By conjecture $T_t = T(C_t)$, so $dT_t = dT(C_t)$ for $C_t \in (\underline{C}, \overline{C})$. We now match the terms in (G.1) and (G.2). First, matching the exposure to Poisson shocks $d\Pi_t$ in (G.1) and (G.2), we obtain $T(\overline{C}) = 0$. Second, matching the exposure to Brownian shocks, we obtain $T'(C)\sigma_C(C) = -\sigma_I(C)$. Plugging in $\sigma_I(C)$ from (20) — that is, $\sigma_I(C) = \sigma$ for $C < 0$ and $\sigma_I(C) = \beta(C)\sigma$ for $C \geq 0$ — and $\sigma_C(C) = \sigma(1 - \beta(C))$, we have the first-order ODE for $C \in (\underline{C}, \overline{C})$

$$T'(C) = -\left(\frac{\left[\mathbb{1}_{\{C<0\}} + \mathbb{1}_{\{C\geq 0\}}\beta(C)\right]}{[1 - \beta(C)]}\right) \tag{G.3}$$

with boundary condition $T(\overline{C}) = 0$. As $\beta(C) < 1$ for $C \in (\underline{C}, \overline{C})$, the right-hand-side of (G.3) is well-defined on $C \in (\underline{C}, \overline{C})$, with potentially degenerate boundary behavior as $C \to \underline{C}$. Plugging in the optimal $\beta(C)$ from (23) into (G.3) and imposing the boundary condition, we can solve (G.3) for $T(C) = \frac{\ln P'(C)}{\rho r} + \max\{-C, 0\}$, i.e., (29).

Finally, we denote the drift term of $dT(C_t)$ by $\mu_T(C_t) = T'(C_t)\mu_C(C_t) + \frac{T''(C_t)\sigma_C(C_t)^2}{2}$. By matching drift terms, we obtain $\hat{r}(C) = \mu_T(C) + \mu_T(I)$.

G.2 Part II

The next part of the proof shows that for $(\underline{C}, \overline{C})$:

$$\mu_T(C) + \mu_I(C) = \frac{\lambda}{\rho r}\mathbb{1}_{\{C\geq 0\}} + rY(C). \tag{G.4}$$

We assume that that $\alpha(C)$ and $\beta(C)$ are differentiable in state C. We establish (G.4) for points C at which $\alpha(C)$ and $\beta(C)$ are differentiable. Because the set of points at which $\alpha(C)$ or $\beta(C)$ are not differentiable is countable, (G.4) then holds for all C in $(\underline{C}, \overline{C})$

Part II.A — Auxiliary Result: Simplified HJB under optimal $\beta(C)$

Assume that $\alpha(C)$ and $\beta(C)$ are differentiable. Note that $1 - \beta(C) = 1 - \frac{P''(C)}{P''(C) - \rho r P'(C)} = \frac{-\rho r P'(C)}{P''(C) - \rho r P'(C)}$. Thus, $\frac{P''(C)}{-\rho r P'(C)} = \frac{\beta(C)}{1-\beta(C)}$ and $1 - \beta(C) = \frac{-\rho r P'(C)}{P''(C)}\beta(C)$ and $\beta(C) = \frac{P''(C)}{-\rho r P'(C)}(1-$

$\beta(C))$. As a result, we can calculate

$$-\frac{\rho r}{2}\beta(C)^2P'(C)+\frac{P''(C)}{2}((1-\beta(C))^2=-\frac{\rho r}{2}\beta(C)^2P'(C)+\frac{(\rho rP'(C))^2}{2P''(C)}(\beta(C))^2 \quad \text{(G.5)}$$
$$=-\frac{\rho r}{2}\beta(C)^2P'(C)\left(1-\frac{\rho rP'(C)}{P''(C)}\right)=-\frac{\rho r}{2}\beta(C)P'(C),$$

where the last equality uses that $\frac{1}{\beta(C)}=\frac{P''(C)-\rho rP'(C)}{P''(C)}=1-\frac{\rho rP'(C)}{P''(C)}$. We can insert relation (G.5) as well as $C^*=\overline{C}$ into (16) to obtain

$$rP(C)=P'(C)\left[\mu+\left(r-\lambda\mathbb{1}_{\{C\geq 0\}}\right)C-\frac{\rho r}{2}\beta(C)\sigma^2+\pi\left(\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right)\right]$$
$$+\pi\left[P\left(\overline{C}\right)-P(C)-\left(\overline{C}-C+\alpha(C)\right)\right]. \quad \text{(G.6)}$$

Part II.B — Auxiliary Result: Slope of optimal risk-sharing $\beta'(C)$

Consider $C\neq 0$ and that $\alpha(C)$ and $\beta(C)$ are differentiable. We differentiate both sides of the ODE (G.6) with respect to C:

$$rP'(C)=P''(C)\left[\mu+\left(r-\lambda\mathbb{1}_{\{C\geq 0\}}\right)C-\frac{\rho r}{2}\beta(C)\sigma^2+\pi\left(\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right)\right]$$
$$+P'(C)\left[\left(r-\lambda\mathbb{1}_{\{C\geq 0\}}\right)-\frac{\rho r}{2}\beta'(C)\sigma^2+\pi\alpha'(C)e^{-\rho r\alpha(C)}\right]+\pi\left[1-\alpha'(C)-P'(C)\right].$$

Rearranging, we have

$$\left[\pi\left(1-\alpha'(C)e^{-\rho r\alpha(C)}\right)+\lambda\mathbb{1}_{\{C\geq 0\}}+\frac{\rho r}{2}\beta'(C)\sigma^2\right]P'(C) \quad \text{(G.7)}$$
$$=P''(C)\left[\mu+\left(r-\lambda\mathbb{1}_{\{C\geq 0\}}\right)C-\frac{\rho r}{2}\beta(C)\sigma^2+\pi\left(\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right)\right]+\pi\left[1-\alpha'(C)\right].$$

Dividing through by $\rho rP'(C)$ and solving for $\frac{\sigma^2}{2}\beta'(C)$, we have

$$\frac{\sigma^2}{2}\beta'(C)=\frac{P''(C)}{\rho rP'(C)}\left[\mu+\left(r-\lambda\mathbb{1}_{\{C\geq 0\}}\right)C-\frac{\rho r}{2}\beta(C)\sigma^2+\pi\left(\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right)\right]$$
$$+\frac{\pi}{\rho r}\frac{\left[1-\alpha'(C)\right]}{P'(C)}-\frac{\pi}{\rho r}\left[1-\alpha'(C)e^{-\rho r\alpha(C)}\right]-\frac{\lambda\mathbb{1}_{\{C\geq 0\}}}{\rho r}$$
$$=-\frac{\beta(C)}{1-\beta(C)}\left[\mu+\left(r-\lambda\mathbb{1}_{\{C\geq 0\}}\right)C-\frac{\rho r}{2}\beta(C)\sigma^2+\pi\left(\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right)\right]$$
$$+\frac{\pi}{\rho r}\frac{\left[1-\alpha'(C)\right]}{P'(C)}-\frac{\pi}{\rho r}\left[1-\alpha'(C)e^{-\rho r\alpha(C)}\right]-\frac{\lambda\mathbb{1}_{\{C\geq 0\}}}{\rho r},$$

where the second equality uses $\frac{1}{\rho r}\frac{P''(C)}{P'(C)} = -\frac{\beta(C)}{1-\beta(C)}$. Using the expression for the drift of excess liquidity $\mu_C(C)$ in (B.1), we get

$$\frac{\sigma^2}{2}\beta'(C) = -\frac{\beta(C)}{1-\beta(C)}\mu_C(C) + \frac{\rho r}{2}\beta^2(C)\sigma^2 + \frac{\pi}{\rho r}\alpha'(C)\left[e^{-\rho r\alpha(C)} - \frac{1}{P'(C)}\right] - \frac{\pi}{\rho r}\left(1-\frac{1}{P'(C)}\right) - \frac{\lambda \mathbb{1}_{\{C\geq 0\}}}{\rho r} \tag{G.8}$$

Part II.C Derivation of (G.4)

Consider $C \neq 0$ and that $\alpha(C)$ and $\beta(C)$ are differentiable. We can calculate

$$\begin{aligned}
\mu_T(C) =& T'(C)\mu_C(C) + T''(C)\frac{\sigma_C^2(C)}{2} = T'(C)\mu_C(C) - \frac{\beta'(C)}{[1-\beta(C)]^2}\frac{\sigma^2[1-\beta(C)]^2}{2} \\
=& T'(C)\mu_C(C) - \frac{\sigma^2}{2}\beta'(C) \\
=& -\left[\frac{\rho r}{2}\beta^2(C)\sigma^2 + \frac{\pi}{\rho r}\alpha'(C)\left[e^{-\rho r\alpha(C)} - \frac{1}{P'(C)}\right] - \frac{\pi}{\rho r}\left(1-\frac{1}{P'(C)}\right) - \frac{\lambda\mathbb{1}_{\{C\geq 0\}}}{\rho r}\right] \\
=& \mu_C \mathbb{1}_{\{C\geq 0\}}(C) - \left[\mu + \left(r - \lambda\mathbb{1}_{\{C\geq 0\}}\right)C - \frac{\lambda\mathbb{1}_{\{C\geq 0\}}}{\rho r}\right] \\
& - \frac{\pi}{\rho r}\left\{\alpha'(C)\left[e^{-\rho r\alpha(C)} - \frac{1}{P'(C)}\right] - \left(1-\frac{1}{P'(C)}\right) + \left(1-e^{-\rho r\alpha(C)}\right)\right\},
\end{aligned}$$

where the second equality uses $T''(C) = -\frac{\beta'(C)}{[1-\beta(C)]^2}$, and the fourth equality uses (G.8) to substitute in for $\frac{\sigma^2}{2}\beta'(C)$. We can simplify to obtain

$$\mu_T(C) = \left[\frac{\lambda}{\rho r} + \mu_C(C)\right]\mathbb{1}_{\{C\geq 0\}} - \left[\mu + \left(r-\lambda\mathbb{1}_{\{C\geq 0\}}\right)C\right] - \frac{\pi}{\rho r}\left[\frac{1}{P'(C)} - e^{-\rho r\alpha(C)}\right][1-\alpha'(C)].$$

Utilizing $\mu_I(C) = \mu + [(r-\lambda)C - \mu_C]\mathbb{1}_{\{C\geq 0\}}$, we have

$$\mu_T(C) + \mu_I(C) = \frac{\lambda}{\rho r}\mathbb{1}_{\{C\geq 0\}} + rY(C) - \frac{\pi}{\rho r}\left[\frac{1}{P'(C)} - e^{-\rho r\alpha(C)}\right][1-\alpha'(C)]. \tag{G.9}$$

When $\alpha(C) = \alpha_U(C) = \frac{\ln P'(C)}{\rho r}$, we have $e^{-\rho r\alpha(C)} = 1/P'(C)$ and $\left[\frac{1}{P'(C)} - e^{-\rho r\alpha(C)}\right] = 0$. When $\alpha(C) = \alpha_C(C) = P(\overline{C}) - \overline{C} + C$, then $\alpha'(C) = 1$. Either way, under the optimal contract, $\frac{\pi}{\rho r}\left[\frac{1}{P'(C)} - e^{-\rho r\alpha(C)}\right][1-\alpha'(C)] = 0$ and (G.9) simplifies to (G.4).

H Proof of Proposition 5

By construction, balances of the credit lines add up to $T(C)$, i.e., $T(C) = D(C)+Y(C)$. The sum of interest payments on the secured credit line, $rY(C)$, plus maintenance $\frac{\lambda}{\rho r}\mathbb{1}_{\{C\geq 0\}}$ equals

$\hat{r}(C)T(C)$. We verify $dI^D(C) + dI^Y(C) = dI(C)$. When $C > 0$, then $dI^Y(C)$ and it can be readily seen from (20) and the expression in Proposition 5 that $dI^D(C) = dI(C)$, as desired. When $C \leq 0$, we can calculate $dI^D(C) + dI^Y(C) = \mu dt + \sigma dZ + (Y(C) + \alpha(C))d\Pi = dI(C)$.

I Existence of Solution

We consider $\varepsilon > 0$ and impose the constraint $\beta_t \in [0, 1-\varepsilon]$ to deal with the problem of vanishing volatility $\sigma_C(C)$. We also impose the constraint $\alpha \geq 0$ which never binds in optimum in the interior of the state space. We establish that for any ε, a solution to the system considered exists. Lastly, we take the limit $\varepsilon \to 0$ to show that there exists a solution $(P(C), \overline{C}, \underline{C})$ to the system (16) with (25) and $P'(\overline{C}) - 1 = P''(\overline{C}) = P(\underline{C}) = 0$.

The proof proceeds in five parts. Part I establishes existence and uniqueness to an auxiliary ODE system, with exogenous boundaries $\overline{C}_1 \geq 0$ and boundary $\underline{C}_1$. Part II shows that this solution is concave, and strictly concave for $\overline{C}_1 > 0$. Using these results, Part III and VI show that there exists a solution to the system (16) subject to $P(\overline{C}) - 1 = P''(\overline{C}) = P(\underline{C}) = 0$ and with (25). Part V concludes, and argues that a unique solution to the dynamic contracting problem (5) exists.

I.1 Part I

For any exogenous $\overline{C}_1 \geq 0 > \underline{C}_1$, consider the auxiliary function $P_{\varepsilon,\overline{C}_1}(C)$ solving the HJB

$$\begin{aligned} rP_{\varepsilon,\overline{C}_1}(C) = & \\ \max_{\beta\in[0,1-\varepsilon], Y\geq \max\{0,-C\}} & \left\{ P'_{\varepsilon,\overline{C}_1}(C)\left[\mu + (r-\lambda)C - \lambda Y - \sigma^2 k_Z(\beta)\right] + P''_{\varepsilon,\overline{C}_1}(C)\frac{\sigma^2}{2}(1-\beta)^2 \right\} \\ & + \pi \max_{\alpha} \left\{ P'_{\varepsilon,\overline{C}_1}(C)\left[\alpha - k_\Pi(\alpha)\right] + \left[P_{\varepsilon,\overline{C}_1}\left(\overline{C}_1\right) - P_{\varepsilon,\overline{C}_1}(C) - \left(\overline{C}_1 - C + \alpha\right)\right] \right\}, \end{aligned} \tag{I.1}$$

with α satisfying $0 \leq \alpha \leq P_{\varepsilon,\overline{C}_1}(\overline{C}_1) - \overline{C}_1 + C$ and subject to the boundary conditions $P'_{\varepsilon,\overline{C}_1}(\overline{C}_1) - 1 = P''_{\varepsilon,\overline{C}_1}(\overline{C}_1) = 0$. In other words, we are solving a second-order ODE which is pinned down by two boundary conditions at $\overline{C}_1$. Note that above HJB equation becomes akin to (16) upon setting $\varepsilon = 0$. We can then calculate

$$Y(C) = \max\{-C, 0\},\ \alpha(C) = \min\left\{ \max\left\{0, \frac{\ln P'_{\varepsilon,\overline{C}_1}(C)}{\rho r}\right\}, P_{\varepsilon,\overline{C}_1}(\overline{C}_1) - \overline{C}_1 + C \right\}. \tag{I.2}$$

We now prove uniqueness and existence of a solution to (I.1) adopting the argument in Sannikov (2008). Inserting optimal $Y = Y(C)$ and $\alpha = \alpha(C)$, we can rewrite (I.1):

$$\begin{aligned} P''_{\varepsilon,\overline{C}_1}(C) = \min_{\beta\in[0,1-\varepsilon]} \frac{2}{\sigma^2(1-\beta)^2} & \Big[rP_{\varepsilon,\overline{C}_1}(C) - P'_{\varepsilon,\overline{C}_1}(C)\left[\mu + (r-\lambda)C - \lambda Y(C) - \sigma^2 k_Z(\beta)\right] \\ & - \pi \left\{ P'_{\varepsilon,\overline{C}_1}(C)\left[\alpha(C) - k_\Pi(\alpha(C))\right] + \left[P_{\varepsilon,\overline{C}_1}\left(\overline{C}_1\right) - P_{\varepsilon,\overline{C}_1}(C) - \left(\overline{C}_1 - C + \alpha(C)\right)\right] \right\} \Big], \end{aligned} \tag{I.3}$$

where due to the constraint $\beta \in [0, 1-\varepsilon]$: $\frac{2}{\sigma^2} \geq \frac{2}{\sigma^2(1-\beta)^2} \geq \frac{2}{\sigma^2\varepsilon^2} > 0$. Next, we can rewrite (I.3) as $P''_{\varepsilon,\overline{C}_1}(C) = \min_{\beta\in[0,1-\varepsilon]} \mathcal{H}_\beta(C, P_{\varepsilon,\overline{C}_1}(C), P'_{\varepsilon,\overline{C}_1}(C))$, where $\mathcal{H}_\beta(C, P_{\varepsilon,\overline{C}_1}(C), P'_{\varepsilon,\overline{C}_1}(C))$ is implicitly defined via (I.3). Note that for all $\beta \in [0, 1-\varepsilon]$, functions $H_\beta(\cdot)$ are Lipschitz-continuous in the arguments. It follows the solution to (I.3) with $P'_{\varepsilon,\overline{C}_1}(\overline{C}_1) - 1 = P''_{\varepsilon,\overline{C}_1}(\overline{C}) = 0$ exists on $(\underline{C}_1, \overline{C}_1)$ and is unique and continuous in boundary conditions and $\varepsilon, \overline{C}_1$.

I.2 Part II — Concavity of $P_{\varepsilon,\overline{C}_1}(C)$

We restrict now attention to $\overline{C}_1 > 0$ and $\underline{C}_1 \geq -\frac{\mu}{r} + \frac{\lambda \overline{C}_1}{r}$. We show that, in this case, $P_{\varepsilon,\overline{C}_1}(C)$ is strictly concave on $(\underline{C}_1, \overline{C}_1)$, satisfying $P'_{\varepsilon,\overline{C}_1}(C) > 1$. We already conjecture (and then verify) that $P'_{\varepsilon,\overline{C}_1}(C) > 1$. By continuity, it then follows that the solution is at least weakly concave for any $\overline{C}_1 \geq 0$. To begin with note that at $P_{\varepsilon,\overline{C}_1}(\overline{C}_1) - \overline{C}_1 = \frac{\mu}{r} + \frac{\lambda \overline{C}_1}{r}$. Thus, for $C > \underline{C}_1$, we have $\alpha(C) > 0$ by (I.2).

Define the jump in the value function upon refinancing as

$$J(C) \equiv P_{\varepsilon,\overline{C}_1}\left(\overline{C}_1\right) - P_{\varepsilon,\overline{C}_1}(C) - \left(\overline{C}_1 - C + \alpha(C)\right). \tag{I.4}$$

When $P''_{\varepsilon,\overline{C}_1}(C)$ and $\alpha(C)$ are differentiable, we can use the envelope theorem and differentiate the HJB equation (B.4) under the optimal $\beta = \beta(C) \in [0, 1-\varepsilon]$ with respect to C and rearrange to obtain

$$\begin{aligned} P'''_{\varepsilon,\overline{C}_1}(C) =& \frac{2}{(1-\beta(C))^2\sigma^2}\Bigg(P'_{\varepsilon,\overline{C}_1}(C)\lambda \mathbb{1}_{\{C\geq 0\}} \\ & - P''_{\varepsilon,\overline{C}_1}(C)\mu_C(C) - \pi\left(e^{-\rho r\alpha(C)} P'_{\varepsilon,\overline{C}_1}(C)\alpha'(C) + J'(C)\right)\Bigg), \end{aligned} \tag{I.5}$$

where $\mathbb{1}_{\{\cdot\}}$ denotes the indicator function which is equal to one if $\{\cdot\}$ is true and is equal to zero otherwise. The set of points at which either $P''_{\varepsilon,\overline{C}_1}(C)$ or $\alpha(C)$ is not differentiable is countable; therefore, for any C, the limits $\lim_{x\uparrow C} P'''_{\varepsilon,\overline{C}_1}(C)$, $\lim_{x\downarrow C} P'''_{\varepsilon,\overline{C}_1}(C)$, and $\lim_{x\uparrow C} \alpha'(C)$, $\lim_{x\downarrow C} \alpha'(C)$ exist and are well-defined.

Suppose that $\alpha(C)$ is differentiable, and recall (I.2). If $\alpha(C) = P_{\varepsilon,\overline{C}_1}(\overline{C}_1) - \overline{C}_1 + C > 0$, then $\alpha'(C) = 1$ and $J'(C) = -P'_{\varepsilon,\overline{C}_1}(C) < 0$. As such,

$$\pi\left(e^{-\rho r\alpha(C)} P'_{\varepsilon,\overline{C}_1}(C)\alpha'(C) + J'(C)\right) = \pi P'_{\varepsilon,\overline{C}_1}(C)\left(e^{-\rho r\alpha(C)} - 1\right) \leq 0.$$

When $\alpha(C) = \frac{\ln P'_{\varepsilon,\overline{C}_1}(C)}{\rho r}$, then

$$\begin{aligned} \pi\left(e^{-\rho r\alpha(C)} P'_{\varepsilon,\overline{C}_1}(C)\alpha'(C) + J'(C)\right) &= \pi\left(e^{-\rho r\alpha(C)} P'_{\varepsilon,\overline{C}_1}(C)\alpha'(C) + 1 - P'_{\varepsilon,\overline{C}_1}(C) - \alpha'(C)\right) \\ &= \pi\left(\alpha'(C) + 1 - P'_{\varepsilon,\overline{C}_1}(C) - \alpha'(C)\right) = 1 - P'_{\varepsilon,\overline{C}_1}(C) \leq 0. \end{aligned}$$

where it was used that $J'(C) = 1 - P'_{\varepsilon,\overline{C}_1}(C) - \alpha'(C)$ and $e^{-\rho r\alpha(C)} = 1/P'_{\varepsilon,\overline{C}_1}(C)$ and $P'_{\varepsilon,\overline{C}_1}(C) \geq 1$. Thus, $\pi\left(e^{-\rho r\alpha(C)}P'_{\varepsilon,\overline{C}_1}(C)\alpha'(C) + J'(C)\right) \leq 0$, if $\alpha(C)$ is differentiable.

At the payout boundary, it therefore holds that $P'_{\varepsilon,\overline{C}_1}(\overline{C}_1) = 1$, and $P''_{\varepsilon,\overline{C}_1}(\overline{C}_1) = 0$ and

$$\lim_{C\uparrow\overline{C}_1}\left(e^{-\rho r\alpha(C)}P'_{\varepsilon,\overline{C}_1}(C)\alpha'(C) + J'(C)\right) = \lim_{C\uparrow\overline{C}}(1 - P'_{\varepsilon,\overline{C}_1}(C)) = 0. \tag{I.6}$$

Recall $\overline{C}_1 > 0$. Then, $P''_{\varepsilon,\overline{C}_1}(\overline{C}_1) = 0$, $P'_{\varepsilon,\overline{C}_1}(\overline{C}_1) = 1$, we get $\lim_{C\uparrow\overline{C}_1} P'''_{\varepsilon,\overline{C}_1}(C) > 0$. Because $\lim_{C\uparrow\overline{C}_1} P'''_{\varepsilon,\overline{C}_1}(C) > 0$ for $\overline{C}_1 > 0$, continuity implies $P'''(C) > 0$, $P'(C) > 1$, and $P''(C) < 0$ in a left-neighbourhood of $\overline{C}_1$.

Suppose to the contrary that there exists $C' \in (\underline{C}_1, \overline{C}_1)$ with $P''_{\varepsilon,\overline{C}_1}(C) \geq 0$. Define $\hat{C} = \sup\{C \in (\underline{C}_1, \overline{C}_1) : P''_{\varepsilon,\overline{C}_1}(C) \geq 0\}$ and suppose to the contrary that $\hat{C} < \overline{C}$. As $P''_{\varepsilon,\overline{C}_1}(C) < 0$ in a neighbourhood of $\overline{C}_1$, it follows that $\hat{C} < \overline{C}_1$ and, by continuity of $P''_{\varepsilon,\overline{C}_1}(C)$, that $P''_{\varepsilon,\overline{C}_1}(\hat{C}) = 0$. Since $P''_{\varepsilon,\overline{C}_1}(C) < 0$ for $C \in (\hat{C}, \overline{C}_1)$, it follows that $P'_{\varepsilon,\overline{C}_1}(\hat{C}) > 1$. Note that the term $\lim_{C\downarrow\hat{C}} \pi\left(e^{-\rho r\alpha(C)}P'_{\varepsilon,\overline{C}_1}(C)\alpha'(C) + J'(C)\right)$ is weakly negative. As such, $\lim_{C\downarrow\hat{C}} P'''_{\varepsilon,\overline{C}_1}(C) \geq 0$, where the inequality is strict if $\hat{C} > 0$.

Consider now $\hat{C} \geq 0$. Due to $\lim_{C\downarrow\hat{C}} P'''_{\varepsilon,\overline{C}_1}(C) > 0$, there exists $C' > \hat{C}$ so that $P''_{\varepsilon,\overline{C}_1}(C') > 0$, which contradicts the definition of $\hat{C}$. Next, suppose that $\hat{C} < 0$. If $\lim_{C\downarrow\hat{C}} P'''_{\varepsilon,\overline{C}_1}(C) > 0$, we achieve a contradiction similar to above. Consider $\lim_{C\downarrow\hat{C}} P'''_{\varepsilon,\overline{C}_1}(C) = 0$. Due to $P''_{\varepsilon,\overline{C}_1}(\hat{C}) = 0$, there exists then $\delta > 0$ such that the (unique) solution to (I.3) on $(\hat{C}, \hat{C} + \delta)$ satisfies $P'''_{\varepsilon,\overline{C}_1}(C) \geq 0$. Thus, there exists $C' > \hat{C}$ with $P''_{\varepsilon,\overline{C}_1}(C') = 0$, a contradiction.

Either way, it follows that $P''_{\varepsilon,\overline{C}_1}(C) < 0$ for all $C \in (\underline{C}_1, \overline{C}_1)$, which concludes the proof.

I.3 Part III — $P_{\varepsilon,\overline{C}_1}(C)$ has a Root

It follows that $P_{\varepsilon,\overline{C}_1}(C)$ is strictly concave, so that $P'_{\varepsilon,\overline{C}_1}(C) \geq 1$. Take $\overline{C}_1 \geq 0$ and $\underline{C}_1 = -\frac{\mu}{r} + \lambda\overline{C}_1$. Then, $P_{\varepsilon,\overline{C}_1}(C)$ is strictly concave on $(\underline{C}_1, \overline{C}_1)$ with $P'_{\varepsilon,\overline{C}_1}(C) \geq 1$.

Using the boundary conditions at $\overline{C}_1$, i.e., $P'_{\varepsilon,\overline{C}_1}(C) - 1 = P''_{\varepsilon,\overline{C}_1}(C) = 0$, and strict concavity on $(\underline{C}_1, \overline{C}_1)$, we bound the function $P_{\varepsilon,\overline{C}_1}(C)$ via a linear function $\overline{P}(C; \overline{C}_1)$:

$$P_{\varepsilon,\overline{C}_1}(C) \leq \overline{P}(C; \overline{C}_1) \equiv \frac{\mu + (r-\lambda)\overline{C}_1}{r} + C - \overline{C}_1 = \frac{\mu - \lambda\overline{C}_1}{r} + C.$$

By construction, $P(\overline{C}_1; \overline{C}_1) = P_{\varepsilon,\overline{C}_1}\left(\overline{C}_1\right)$ and $\overline{P}(C; \overline{C}_1) = 0$ for $C = \hat{C}(\overline{C}_1) = -\frac{\mu}{r} + \frac{\lambda\overline{C}_1}{r}$. Thus, there exists a unique $\hat{C}_\varepsilon(\overline{C}_1) \geq \hat{C}(\overline{C}_1) = -\frac{\mu}{r} + \frac{\lambda\overline{C}_1}{r}$ at which $P_{\varepsilon,\overline{C}_1}(\hat{C}_\varepsilon(\overline{C}_1)) = 0$. The function $\hat{C}_\varepsilon(\overline{C}_1)$ is continuous in $\overline{C}_1$ and ε. For $\overline{C}_1$ sufficiently large, $\hat{C}_\varepsilon(\overline{C}_1) \geq \hat{C}(\overline{C}_1) \geq 0$.

Next, consider $\overline{C}_1 = 0$. Thus, by (I.5), we obtain after using (I.6) and $P''_{\varepsilon,\overline{C}_1}(0) = 0$ that $\lim_{C\uparrow 0} P'''_{\varepsilon,\overline{C}_1}(C) = 0$. Therefore, the solution to (I.1) satisfies $P_{\varepsilon,\overline{C}_1} = \frac{\mu}{r} + C$ on $(-\frac{\mu}{r}, 0)$, so that $\hat{C}_\varepsilon(0) = \frac{-\mu}{r}$. Take

$$\underline{C}_\varepsilon(\overline{C}_1) = \min\left\{\frac{w\left(\frac{\pi}{r}\exp\left\{\rho r\left[\frac{\lambda}{r}\overline{C}_1 + \frac{\pi}{\rho r^2} - \frac{\rho}{2}\sigma^2\right]\right\}\right) - \frac{\pi}{r}}{\rho r} - Y^A, -L\right\}.$$

Observe that by Lemma 2, we have $\underline{C}_\varepsilon(\overline{C}_1) \in [-[P_{\varepsilon,\overline{C}_1}(\overline{C}_1) - \overline{C}_1], -L]$, with $P_{\varepsilon,\overline{C}_1}(\overline{C}_1) - \overline{C}_1 = \frac{\mu}{r} - \frac{\lambda \overline{C}_1 \mathbb{I}_{\{\overline{C}_1>0\}}}{r}$. Further, $\underline{C}_\varepsilon(\overline{C}_1)$ increases with $\overline{C}_1$ and is continuous in $\overline{C}_1 \geq 0$.

The image of the continuous function $\hat{C}_\varepsilon : [0,\infty) \to \mathbb{R}$ contains $[-\frac{\mu}{r}, 0]$, i.e., $[-\frac{\mu}{r}, 0] \subseteq \hat{C}_\varepsilon([0,\infty])$. Since, in addition, for $\overline{C}_1$ sufficiently large, $\hat{C}(\overline{C}_1) \geq 0$, it must be that there exists $\overline{C}_\varepsilon \geq 0$ such that $\underline{C}_\varepsilon = \underline{C}_\varepsilon(\overline{C}_\varepsilon) = \hat{C}(\overline{C}_\varepsilon)$. If there exist multiple values C' with $\underline{C}_\varepsilon(C') = \hat{C}(C')$, we define $\overline{C}_\varepsilon$ as the lowest intersection point.

Thus, the function $P_{\varepsilon,\overline{C}_\epsilon}(C)$ solves (I.1) in $(\underline{C}_\varepsilon, \overline{C}_\varepsilon)$ subject to $P_{\varepsilon,\overline{C}_\varepsilon}(\underline{C}_\varepsilon) = P'_{\varepsilon,\overline{C}_\varepsilon}(\overline{C}_\varepsilon) - 1 = P''_{\varepsilon,\overline{C}_\varepsilon}(\overline{C}_\varepsilon) = 0$. As $\underline{C}_\varepsilon \geq \frac{\mu}{r} - \frac{\lambda \overline{C}_\varepsilon}{r}$, it follows from Part II that $P_{\varepsilon,\overline{C}_\varepsilon}$ is concave on $(\underline{C}_\varepsilon, \overline{C}_\varepsilon)$, and strictly so if $\overline{C}_\varepsilon > 0$.

I.4 Part IV

Take $\overline{C}_\varepsilon$ and $\underline{C}_\varepsilon = \underline{C}(\overline{C}_\varepsilon)$ with $\underline{C}(\cdot)$ from (25) as well as the function $P_{\varepsilon,\overline{C}_\varepsilon}(C)$ solving (I.1) on $(\underline{C}_\varepsilon, \overline{C}_\varepsilon)$ subject to $P_{\varepsilon,\overline{C}_\varepsilon}(\underline{C}_\varepsilon) = P'_{\varepsilon,\overline{C}_\varepsilon}(\overline{C}_\varepsilon) - 1 = P''_{\varepsilon,\overline{C}_\varepsilon}(\overline{C}_\varepsilon) = 0$. The functions $P_{\varepsilon,\overline{C}_\varepsilon}$ are strictly concave and increasing on $(\underline{C}_\varepsilon, \overline{C}_\varepsilon)$.

We can take the limit $\varepsilon \to 0$ to obtain $\overline{C} := \lim_{\varepsilon\to 0} \overline{C}_\varepsilon$ and $\underline{C} := \lim_{\varepsilon\to 0} \underline{C}_\varepsilon$. As $\underline{C}_\varepsilon \in [-\frac{\mu}{r}, -L]$, $\underline{C}$ is well-defined with $\underline{C} \in [-\frac{\mu}{r}, -L]$. Define $P(C) := \lim_{\varepsilon\to 0} P_{\varepsilon,\overline{C}_\varepsilon}(C)$ on $(\underline{C}, \overline{C})$ as well as $P'(C) := \lim_{\varepsilon\to 0} P'_{\varepsilon,\overline{C}_\varepsilon}(C)$ and $P''(C) := \lim_{\varepsilon\to 0} P''_{\varepsilon,\overline{C}_\varepsilon}(C)$. We show now that for all $C \in (\underline{C}, \overline{C})$, these limits are finite (and exist).

For a given value $C \in (\underline{C}, \overline{C})$, there exist $\delta > 0$ and $\bar{\varepsilon} > 0$ such that for all $0 < \varepsilon < \bar{\varepsilon}$, we have $C > \underline{C}_\varepsilon + \delta$ and, due to $P'_{\varepsilon,\overline{C}_\varepsilon}(\cdot) \geq 1$, consequently $P_{\varepsilon,\overline{C}_\varepsilon}(C) > \delta$. Therefore $P(C) = \lim_{\varepsilon\to 0} P_{\varepsilon,\overline{C}_\varepsilon}(C) > 0$. As $P_{\varepsilon,\overline{C}_\varepsilon}(C)$ is concave and increasing, it follows that, if $P(C)$ is not finite, then $P(C) = -\infty$. However, due to $P(C) > 0$, $P(C)$ must be finite.

If $P'(C)$ is not finite, then $P'(C) = +\infty$. Suppose $P'(C) = +\infty$. Pick $C' \in (\underline{C}, C)$. Then, there is ε_1 such that for all $\varepsilon < \varepsilon_1$ we have $C', C \in (\underline{C}_\varepsilon, \overline{C}_\varepsilon)$ and $P'_{\varepsilon,\overline{C}_\varepsilon}(C'') \geq P'_{\varepsilon,\overline{C}_\varepsilon}(C)$ for $C'' \in [C', C]$, due to concavity. As a consequence, $P_{\varepsilon,\overline{C}_\varepsilon}(C') \leq P_{\varepsilon,\overline{C}_\varepsilon}(C) - (C - C')P'_{\varepsilon,\overline{C}_\varepsilon}(C)$. Owing to $P'(C) = +\infty$, for any $R > 0$, there exists $0 < \bar{\varepsilon} \leq \varepsilon_1$ such that $P'_{\varepsilon,\overline{C}_\varepsilon}(C) > R$ for all $\varepsilon < \bar{\varepsilon}$. That is, $P_{\varepsilon,\overline{C}_\varepsilon}(C') < P_{\varepsilon,\overline{C}_\varepsilon}(C) - (C - C')R$ for all $\varepsilon < \bar{\varepsilon}$. Pick $R > P(C)/(C - C')$ and let $\varepsilon \to 0$ to obtain $P(C') < 0$, a contradiction. Thus, $P'(C)$ must be finite.

If $P''(C)$ is not finite, then $P''(C) = -\infty$. Suppose to the contrary $P''(C) = -\infty$. We pick C as the largest value on $(\underline{C}, \overline{C})$ with $P''(C) = -\infty$; thus, $P''(c) > -\infty$ for $c \in (C, \overline{C})$. Then, it follows that $P'''(C) = +\infty$. Pick $C' \in (\underline{C}, C)$ such that $P'''(\cdot) > 0$ on (C', C). Note that, as we have shown, $P'(C') \geq 1$ is finite. Then, there is ε_1 such that for all $\varepsilon < \varepsilon_1$ we have $C', C \in (\underline{C}_\varepsilon, \overline{C}_\varepsilon)$, and $P'''_{\varepsilon,\overline{C}_\varepsilon}(\cdot) \geq 0$ on (C', C). As such, we have $P''_{\varepsilon,\overline{C}_\varepsilon}(C'') \leq P''_{\varepsilon,\overline{C}_\varepsilon}(C)$ for $C'' \in [C', C]$. Because $P''(C) = -\infty$, for any $R > 0$ there exists $0 < \bar{\varepsilon} < \varepsilon_1$ such that $P''_{\varepsilon,\overline{C}_\varepsilon}(C) < -R$ for $\varepsilon < \bar{\varepsilon}$. Thus, $P'_{\varepsilon,\overline{C}_\varepsilon}(C) \leq P'_{\varepsilon,\overline{C}_\varepsilon}(C') + (C - C')P''_{\varepsilon,\overline{C}_\varepsilon}(C)$. Take $R > P'(C')/(C - \underline{C})$, and take the limit $\varepsilon \to 0$ to obtain $P'(C) = \lim_{\varepsilon\to 0} P'_{\varepsilon,\overline{C}_\varepsilon}(C) \leq 0$, a contradiction. Thus, $P''(C)$ is finite.

Having established that the limit expressions $P(C), P'(C), P''(C)$ are finite for any $C \in (\underline{C}, \overline{C})$, we take the limit ε in (I.1) on $(\underline{C}, \overline{C}) = \lim_{\varepsilon\to 0}(\underline{C}_\varepsilon, \overline{C}_\varepsilon)$.We obtain that $P(C) = \lim_{\varepsilon\to 0} P_{\varepsilon,\overline{C}_\varepsilon}(C)$ solves on $(\underline{C}, \overline{C})$ the HJB equation:

$$
\begin{aligned}
&rP(C) = \lim_{\varepsilon\to 0} rP_{\varepsilon,\overline{C}_\varepsilon}(C) = \\
&\lim_{\varepsilon\to 0} \max_{\beta\in[0,1-\varepsilon], Y\geq \max\{0,-C\}} \left\{ P'_{\varepsilon,\overline{C}_\varepsilon}(C)\left[\mu + (r-\lambda)C - \lambda Y - \sigma^2 k_Z(\beta)\right] + P''_{\varepsilon,\overline{C}_\varepsilon}(C)\frac{\sigma^2}{2}(1-\beta)^2 \right\} \\
&+ \lim_{\varepsilon\to 0} \pi \max_{\alpha\in\mathcal{S}(C,\overline{C})} \left\{ P'_{\varepsilon,\overline{C}_\varepsilon}(C)[\alpha - k_\Pi(\alpha)] + \left[P_{\varepsilon,\overline{C}_\varepsilon}(\overline{C}_\varepsilon) - P_{\varepsilon,\overline{C}_\varepsilon}(C) - (\overline{C}_\varepsilon - C + \alpha)\right]\right\} \\
&= \max_{\beta, Y\geq\max\{0,-C\}} \left\{ P'(C)\left[\mu + (r-\lambda)C - \lambda Y - \sigma^2 k_Z(\beta)\right] + P''(C)\frac{\sigma^2}{2}(1-\beta)^2 \right\} \\
&+ \pi \max_\alpha \left\{ P'(C)[\alpha - k_\Pi(\alpha)] + \left[P(\overline{C}) - P(C) - (\overline{C} - C + \alpha)\right]\right\}.
\end{aligned}
$$

Thus, $(P(C), \overline{C}, \underline{C})$ is a solution to the system (16) subject to $P(\overline{C}) - 1 = P''(\overline{C}) = P(\underline{C}) = 0$ and with (25). Given $\overline{C}$, there exists — by construction — no other function that solves (16) subject to $P(\overline{C}) - 1 = P''(\overline{C}) = P(\underline{C}) = 0$ with $\underline{C} = \underline{C}(\overline{C})$ from (25).

I.5 Part V — Uniqueness of Solution to (5)

We were able to establish existence of a solution to the ODE system subject to endogenous boundary conditions. Moreover, by construction, the function $P(C)$ that solves (16) subject to $P'(\overline{C}) - 1 = P''(\overline{C}) = P(\underline{C}) = 0$ is unique, but there could exist multiple boundary values consistent with such a solution; nevertheless, there exists a unique solution to the optimization problem (5). If there exist multiple solutions $\mathcal{S}_i := (P_i(C), \overline{C}_i, \underline{C}_i)$ to the system (16) subject to $P(\overline{C}_i) - 1 = P''_i(\overline{C}_i) = P(\underline{C}_i) = 0$ and with $\underline{C}_i = \underline{C}(\overline{C}_i)$ from (25), we can index these solutions by $i \in \mathcal{I}$ and denote their set by $\Omega = \{(P_i(C), \overline{C}_i, \underline{C}_i) : i \in \mathcal{I}\}$.

Notice that at time $t = 0$, shareholders payoff under solution i is $P(\overline{C}_i) - \overline{C}_i = \frac{\mu}{r} - \frac{\lambda \overline{C}_i}{r}$, because $\overline{C}_i > 0$, so that ex-ante payoff decreases with $\overline{C}_i$. As a result, the optimal contract is obtained by picking the solution with the lowest upper and lower boundary. Formally, take $i^* = \arg\min_{i\in\mathcal{I}} \overline{C}_i$, and set $\overline{C} := \overline{C}_{i^*}$ and $\underline{C} = \underline{C}_{i^*}$ as well as $P(C) = P_{i^*}(C)$. Then, under the optimal contract, shareholders' value function $P(C)$ is the unique solution to (16) subject to $P(\overline{C}) - 1 = P''(\overline{C}) = P(\underline{C}) = 0$ and with (25). Consequently, a unique solution to shareholders' optimization — that is, the optimal contracting problem in (5) — exists.

J Proof of Proposition 6

Proof. As the HJB (16) and boundary conditions remain unchanged besides the new constraint (37), Footnote 12 applies, $P(\overline{C}) - \overline{C} = \frac{\mu}{r} - \frac{\lambda}{r}\overline{C}$. Notice that ν affects the value function only via the constraint (37), so $P(\overline{C}) - \overline{C}$ must increase with ν, as a larger ν relaxes the constraint (37). Thus, $\frac{\partial[P(\overline{C})-\overline{C}]}{\partial\nu} \geq 0 \iff \frac{\partial\overline{C}}{\partial\nu} \leq 0$, with strict inequality if the constraint is binding on a set with non-zero measure of the state space $(\underline{C}, \overline{C})$. Under survival, $\underline{C} = C^S$, the constraint is binding in a neighbourhood of $\underline{C}$ regardless of ν, by (24), $\frac{\partial C^S}{\partial\nu} < 0$. □

K Active Intermediaries and Monitoring

We now solve the model variant with endogenous intermediary effort a_t. For simplicity, we do not distinguish between actual effort levels and effort levels anticipated by outside investors, and simply write a_t for the optimal effort.

K.1 State Variables

In this model variant, the intermediary's continuation value reads

$$Y_t = \mathbb{E}_t\left[\int_t^{\infty} e^{-r(s-t)}\left(dI_s - k_s ds - \frac{\kappa a_s^2}{2}ds\right)\right]. \tag{K.1}$$

By the martingale representation theorem, there exists processes α and β such that

$$dY_t = \left[rY_t + k_t - \frac{\kappa a_t^2}{2}\right]dt + \beta_t(dX_t - \mu - a_t)dt + \alpha_t(d\Pi_t - \pi dt). \tag{K.2}$$

The intermediary chooses at each time t its effort $a_t \geq 0$ to solve $\max_{a_t}\left(\beta_t a_t - \frac{\kappa a_t^2}{2}\right)$, leading to $\beta_t = a_t\kappa$. As in the baseline, we set k_t according to (7). Next, note that the firm's cash balance M_t evolves according to (2) Excess liquidity has then the law of motion $dC_t = dM_t - dY_t$ with

$$\begin{aligned} dC_t = &\left[\mu + a_t + (r-\lambda)C_t - \lambda Y_t - \sigma^2 k_Z(\beta) - \frac{\kappa a_t^2}{2} + \pi\left(\alpha_t - k_\Pi(\alpha_t)\right)\right]dt \\ &+ \sigma\left(1-\beta_t\right)dZ_t + \left(C_t^* - C_t\right)d\Pi_t - dDiv_t, \end{aligned} \tag{K.3}$$

where we define the post-refinancing level of excess liquidity as $C_t^* \equiv \Delta M_t + C_t - \alpha_t$.

Finally, note that in autarky $\beta_t = 1$, so $a_t = 1/\kappa$. As such, $Y^A = \frac{\mu}{r} - \frac{\rho\sigma^2}{2} + \frac{1}{2\kappa r}$.

K.2 HJB Equation and Optimization

As in the baseline, C_t is the only state variable, and dividend payouts $dDiv_t$ cause C_t to reflect at the endogenous upper boundary $\overline{C}$ satisfying smooth pasting and super contact conditions $P'(\overline{C}) - 1 = P''(\overline{C}) = 0$. In the interior of the state space (i.e., for $C \in (\underline{C}, \overline{C})$ the value function $P(C)$ solves the following HJB equation:

$$rP(C) = \max_{\beta,Y}\left\{P'(C)\left[\mu + \frac{2\beta-\beta^2}{2\kappa} + (r-\lambda)C - \lambda Y - \frac{\rho 2\beta^2}{2}\right] + P''(C)\frac{\sigma^2}{2}(1-\beta)^2\right\}$$
$$+\pi \max_{C^*,\alpha\in\mathcal{S}(C^*,C)}\left\{P'(C)\left(\frac{1-e^{-\rho r\alpha}}{\rho r}\right) + [P(C^*) - P(C) - (C^* - C + \alpha)]\right\}, \tag{K.4}$$

where we already inserted the incentive condition $a_t = \beta_t/\kappa$. One can show that the value function $P(C)$ is strictly concave on $(\underline{C}, \overline{C})$ (omitted here), with $P'(C) > 1$.

The controls Y, C^*, and α are determined analogously to the baseline. That is, we have $Y(C) = \max\{-C, 0\}$ and $M(C) = \max\{C, 0\}$ which is (18). The firm chooses refinancing target $C^* = \overline{C}$. The optimal choice of $\alpha = \alpha(C)$ is characterized in (22), so $\alpha(C) = \max\{\alpha_U(C), \alpha_C(C)\}$.

Next, the first order condition in (K.4) with respect to β reads $\frac{1-\beta}{\kappa} - \rho r\sigma^2 P'(C)\beta - P''(C)\sigma^2(1-\beta) = 0$, which can be solved for

$$\beta = \beta(C) = \frac{1 - \kappa\sigma^2 P''(C)}{1 + \rho r\kappa\sigma^2 P'(C) - \kappa\sigma^2 P''(C)}. \tag{K.5}$$

Notice that the expression for $\beta(C)$ in (K.5) becomes (23) in the limit $\kappa \to \infty$.

Finally, notice that at the payout boundary $\overline{C}$, we have $P''(\overline{C})$ and $P'(\overline{C}) = 1$ so that $\beta^B := \beta(\overline{C}) = \frac{1}{1+\rho r\kappa\sigma^2}$ and $a^B := a(\overline{C}) = \frac{1}{\kappa+\rho r\kappa^2\sigma^2}$, so $a(\overline{C})$ is smaller than the autarky effort $1/\kappa$. As a result, the value function at the payout boundary satisfies

$$P(\overline{C}) - \overline{C} = \frac{1}{r}\left(\mu - \lambda\overline{C} + a^B - \frac{\kappa(a^B)^2 - \rho r\sigma^2(\beta^B)^2}{2}\right). \tag{K.6}$$

K.3 Lower Boundary $\underline{C}$

We determine the lower boundary in the state space $\underline{C}$ using arguments analogous to the ones from the main text. To begin with, suppose that $\underline{C} = C^S < -L$ in which case the firm is never liquidated. The following conditions are satisfied. First, it must be that $\beta(\underline{C}) = 1$ so that $a(\underline{C}) = \frac{1}{\kappa}$. Second, the drift of C in (K.3), denoted $\mu_C(C)$, must be zero, i.e., $\mu_C(\underline{C}) = 0$. Third, $P(\underline{C}) = 0$. Furthermore, these requirements can be met jointly only if $\alpha(\underline{C}) = P\left(\overline{C}\right) - \overline{C} - P(\underline{C}) = P\left(\overline{C}\right) - \overline{C}$. Inserting this expression for $\alpha(\underline{C})$, $\beta(\underline{C}) = 1$, $Y(\underline{C}) = -\underline{C}$ as well as $a(\underline{C}) = 1/\kappa$ into (K.3), we obtain

$$\mu_C(\underline{C}) = \mu + \frac{1}{2\kappa} - \frac{\rho r\sigma^2}{2} + r\underline{C} + \pi\left(\frac{1 - e^{-\rho r[P(\overline{C}) - \overline{C}]}}{\rho r}\right).$$

Notice that $P(\overline{C}) - \overline{C}$ is characterized above in (K.6), so that

$$\mu_C(\underline{C}) = \mu + \frac{1}{2\kappa} - \frac{\rho r\sigma^2}{2} + r\underline{C} + \pi\left(\frac{1 - e^{-\rho[\mu - \lambda\overline{C} + a^B - 0.5\kappa(a^B)^2 - 0.5\rho r\sigma^2(\beta^B)^2]}}{\rho r}\right).$$

Using Lemma 4, we can solve $\mu_C(\underline{C}) = 0$ (after some algebra) for

$$C^S = \underline{C} = \frac{w\left(\frac{\pi}{r}\exp\left\{\rho r\left[\frac{\lambda}{r}\overline{C} + \frac{\chi}{r} + \frac{\pi}{\rho r^2} - \frac{\rho}{2}\sigma^2\right]\right\}\right) - \frac{\pi}{r}}{\rho r} - Y^A, \tag{K.7}$$

where $w(\cdot)$ is the primary branch of the Lambert-w function and $\chi = \frac{1}{2\kappa} - \left(a^B - \frac{\kappa(a^B)^2}{2} - \frac{\rho r \sigma^2 (\beta^B)^2}{2}\right)$. Above expression (K.7) simplifies to C^S from the baseline, when $\kappa \to \infty$.

It follows that $\underline{C} = \min\left\{C^S, -L\right\}$. When $\underline{C} = -L$, the firm is liquidated at $C = \underline{C}$ and $\tau < \infty$ almost surely. When $\underline{C} < -L$, the firm is never liquidated and $\tau = \infty$.

Finally, it is intuitive that cash flow-based financing capacity $-C^S$ decreases with κ, as higher monitoring cost implies that the intermediary can add less value to the firm and reduces the intermediary's valuation of the firm.

L Micro-foundation with CARA Preferences

We now present a micro-foundation of the intermediary's payoff in (3) as well as the cost function (7). The intermediary is risk-averse with CARA preferences over consumption c with risk-aversion of $\rho > 0$, $u(c) = -\frac{1}{\rho}\exp(-\rho c)$. The intermediary can maintain savings on its own account, denoted by S_t. Savings accrue interest at rate r and are subject to changes induced by transfers to ($dI_t < 0$) and from ($dI_t > 0$) the firm and consumption c_t, so that

$$dS_t = rS_t dt + dI_t - c_t dt, \tag{L.1}$$

where the payout process dI_t is stipulated by the contract $\mathcal{C}$.[34] As such, the intermediary has essentially deep pockets, but capital provision by the intermediary is costly in a sense that the intermediary is risk-averse. We normalize the balance of savings at $t = 0^-$ to zero, i.e., $S_{0^-} = 0$, where time $t = 0^-$ denotes the time before the contract is written and any transfers are made. Savings must satisfy the standard transversality condition $\lim_{t\to\infty} \mathbb{E}\left[e^{-rt}S_t\right] = 0$, ruling out Ponzi schemes. Let U_0 be the intermediary's utility for a given contract $\mathcal{C}$, that is,

$$U_0 = \max_{c_t} \mathbb{E}\left[\int_0^\infty e^{-rt} u\left(c_t\right) dt\right] \quad \text{s.t.} \quad \text{(L.1)} \quad \text{and} \quad \lim_{t\to\infty} \mathbb{E}\left[e^{-rt}S_t\right] = 0. \tag{L.2}$$

Given $\mathcal{C}$, the intermediary chooses consumption c_t to maximize its lifetime utility, with optimal consumption denoted by c_t^*.

Intermediary Consumption Problem. Given $\mathcal{C}$, the intermediary's continuation utility defined as $U_t \equiv \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)} u(c_s^*) ds\right]$. Further, U_t can be expressed as W_t in certainty equivalent monetary terms, with $W_t \equiv \frac{-\ln(-\rho r U_t)}{\rho r}$, and we work with W_t instead of U_t. Note that W_t is the intermediary's total continuation payoff in monetary terms with law of motion

[34] Endowing the intermediary with the possibility to accumulate savings ensures that it can smooth consumption and consume at a rate c_t even if payouts dI_t are not smooth. Consumption c_t and savings balance S_t can both take positive and negative values.

derived below in Online Appendix L.2

$$dW_t = \left[\frac{\rho r}{2}\left(\beta_t\sigma\right)^2 + \pi\left(\alpha_t - \frac{1-e^{-\rho r\alpha_t}}{\rho r}\right)\right]dt + \beta_t\sigma dZ_t + \alpha_t(d\Pi_t - \pi dt), \tag{L.3}$$

where α_t and $\sigma\beta_t$ are the loadings of dW_t on the martingales $(d\Pi_t - \pi dt)$ and dZ_t respectively. The first term in the drift of (L.3) captures the intermediary's required compensation for being exposed to Brownian cash flow risk, while the second term captures the intermediary's required compensation for being exposed to shocks $d\Pi_t$. Both terms are unambiguously positive, so that W_t increases in expectation, $\mathbb{E}[dW_t] \geq 0$. Summarizing, we have:

Proposition 7. *The intermediary's optimal consumption satisfies $c_t^* = rW_t$. The intermediary's certainty equivalent payoff W_t, defined abovem, follows the dynamics (L.3).*

Section L.1 and Section L.2 provide the proof of Proposition 7 in two parts: Part I analyzes the intermediary's optimal consumption and Part II derives the law of motion (L.3). The intermediary's certainty equivalent payoff W_t consists of two sources. First, the intermediary has savings S_t it has accumulated up to time t. Second, it expects to receive payouts from the firm after t, which it values at $Y_t \equiv W_t - S_t$. If the intermediary leaves the firm at t, dollar continuation payoff is S_t. If it stays with the firm and follows the contract, its expected continuation payoff is $W_t = Y_t + S_t$. The intermediary is better off leaving the firm if and only if $Y_t < 0$, so the optimal contract must respect $Y_t \geq 0$.

Combining (L.1) and (L.3) and using optimal consumption $c_t = c_t^* = rW_t$, we obtain

$$dY_t = dW_t - dS_t = \left[rY_t + \frac{\rho r}{2}\left(\beta_t\sigma\right)^2 - \pi\left(\frac{1-e^{-\rho r\alpha_t}}{\rho r}\right)\right]dt + \beta_t\sigma dZ_t + \alpha_t d\Pi_t - dI_t. \tag{L.4}$$

Integrating (L.4) against time and taking expectations, we obtain

$$Y_t = \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)}\left\{dI_s - \left[\frac{\rho r}{2}\left(\beta_s\sigma\right)^2 + \pi\left(\alpha_s - \frac{1-e^{-\rho r\alpha_s}}{\rho r}\right)\right]ds\right\}\right]. \tag{L.5}$$

As desired, the integral representation for Y_t in (L.5) coincides with the integral representation for Y_t in (3) upon choosing k_s according to (7). Likewise, when k_t is determined according to (7), then the law of motion of Y_t in (6) coincides with (L.4).

L.1 Proof of Proposition 7 Part I — Optimal Consumption

We first state an auxiliary Lemma:

Lemma 6. *Take a process $\hat{I}$ and $s_1, s_2 \in \mathbb{R}$. Consider the problem*

$$U_t := U_t(c) = \max_{\{c_s\}_{s\geq t}} \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)}u(c_s)ds\right] \tag{L.6}$$

subject to $dS_s(c) = rS_s(c)ds + d\hat{I}_s - c_s ds,\ S_t(c) = s_1,$ *and* $\lim_{s\to\infty}\mathbb{E}e^{-r(s-t)}S_s(c) = 0,$

where we explicitly denote the dependence of savings S on the consumption path c. Next, consider the problem

$$\tilde{U}_t := \tilde{U}_t(\tilde{c}) = \max_{\{\tilde{c}_s\}_{s\geq t}} \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)}u(\tilde{c}_s)ds\right] \tag{L.7}$$

$$\text{subject to } d\tilde{S}_s(\tilde{c}) = r\tilde{S}_s(\tilde{c})ds + d\hat{I}_s - \tilde{c}_s ds,\ \tilde{S}_t(\tilde{c}) = s_2,\ \text{and } \lim_{s\to\infty}\mathbb{E}e^{-r(s-t)}\tilde{S}_s(\tilde{c}) = 0.$$

Then, for $\Delta^S := s_2 - s_1$, the optimal consumption processes c and $\tilde{c}$, solving (L.6) and (L.7) respectively, satisfy $\tilde{c}_t = c_t + r\Delta^S$ so that $\tilde{U}_t = e^{-\rho r\Delta^S}U_t$.

Proof. To start with, note that with $\tilde{c}_s = c_s + r\Delta^S$,

$$\tilde{U}_t(\tilde{c}) = \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)}u(c_s + r\Delta^S)ds\right] = e^{-\rho r\Delta^S}\mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)}u(c_s)ds\right] = e^{-\rho r\Delta^S}U_t(c), \tag{L.8}$$

where the first equality uses $\tilde{c}_s = c_s + r\Delta^S$ and the second equality uses

$$u(c_s + r\Delta^S) = -\frac{e^{-\rho(c_s + r\Delta^S)}}{\rho} = e^{-\rho r\Delta^S}\left(-\frac{e^{-\rho c_s}}{\rho}\right) = e^{-\rho r\Delta^S}u(c_s). \tag{L.9}$$

Next, suppose there exists a different consumption process $c' \neq \tilde{c}$, solving (L.7), with

$$\tilde{U}_t(c') > \tilde{U}_t(\tilde{c}) = e^{-\rho r\Delta^S}U_t(c), \tag{L.10}$$

and the transversality condition $\lim_{s\to\infty}\mathbb{E}e^{-r(s-t)}\tilde{S}_s(c') = 0$ holds under the consumption process c'. Define the consumption process c'' via $c''_t = c'_t - r\Delta^S$. As c' is different from $\tilde{c}$, it follows that c'' is different from c. As under the consumption path c' the transversality condition $\lim_{s\to\infty}\mathbb{E}e^{-r(s-t)}\tilde{S}_s(c') = 0$ holds, it follows that under the consumption path c'' the transversality condition $\lim_{s\to\infty}\mathbb{E}e^{-r(s-t)}S_s(c'') = 0$ holds too. In addition, note that the payoff under the consumption path c'' equals

$$U_t(c'') := \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)}u(c''_s)ds\right] = e^{\rho r\Delta^S}\tilde{U}_t(c') > e^{\rho r\Delta^S}e^{-\rho r\Delta^S}U_t(c) = U_t(c), \tag{L.11}$$

where the second equality applies (L.9), which yields $u(c'_s) = e^{-\rho r\Delta^S}u(c''_s)$ and $u(c''_s) = e^{\rho r\Delta^S}u(c'_s)$, and the inequality uses (L.10). However, $U_t(c'') > U_t(c)$ contradicts the fact that c solves problem (L.6). The assertion follows. □

Using Lemma 6, we can now complete the argument by showing that optimal consumption satisfies $u(c_t) = rU_t$ and $c_t = rW_t$. According to Lemma 6, the marginal value of an additional unit of savings S_t at time t for the intermediary is given by $\left[\frac{\partial}{\partial\Delta^S}e^{-\rho r\Delta^S}U_t\right]|_{\Delta^S=0} = -\rho rU_t$. Optimal consumption smoothing implies that along the optimal path the first order $u'(c_t) = \left[\frac{\partial}{\partial\Delta^S}e^{-\rho r\Delta^S}U_t\right]|_{\Delta^S=0}$. has to hold at all times $t \geq 0$. That is, in optimum, the intermediary's marginal utility $u'(c_t)$ has to be equal to the marginal value of an additional unit of savings, $\left[\frac{\partial}{\partial\Delta^S}e^{-\rho r\Delta^S}U_t\right]|_{\Delta^S=0}$. Next, observe that $u'(c_t) = -\rho u(c_t)$ and use the above relations to obtain $u(c_t) = rU_t$. Inverting $u(c_t) = rU_t$ yields $c_t = rW_t$.

L.2 Proof of Proposition 7 Part II — Martingale Representation

Take the intermediary's continuation value $U_t = \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)} u(c_s)ds\right]$ under any consumption process c_t (possibly, $c_t = c_t^*$). Define

$$A_t = \mathbb{E}_t\left[\int_0^\infty e^{-rs}u(c_s)ds\right] = \int_0^t e^{-rs}u(c_s)ds + e^{-rt}U_t. \tag{L.12}$$

By construction, $A = \{A_t\}$ is a martingale. By the martingale representation theorem, there exist stochastic processes $\hat{\alpha} = \{\hat{\alpha}_t\}$ and $\beta = \{\beta_t\}$ such that

$$e^{rt}dA_t = (-\rho r U_t)\beta_t\big(dX_t - \mu dt\big) + (-\rho r U_t)\hat{\alpha}_t(d\Pi_t - \pi dt), \tag{L.13}$$

where $dZ_t = \frac{dX_t - \mu dt}{\sigma}$ is the increment of a standard Brownian Motion and $(d\Pi_t - \pi dt)$ is the increment of a compensated Poisson process. We differentiate (L.12) w.r.t. time t to obtain an expression for dA_t, then plug this expression into (L.13) and solve (L.13) to get $dU_t = rU_t dt - u(c_t)dt + (-\rho r U_t)\beta_t\big(dX_t - \mu dt\big) + (-\rho r U_t)\hat{\alpha}_t(d\Pi_t - \pi dt)$. With the optimal consumption policy $c_t = c_t^*$, satisfying $u(c_t) = rU_t$, this simplifies to

$$dU_t = (-\rho r U_t)\beta_t\big(dX_t - \mu dt\big) + (-\rho r U_t)\hat{\alpha}_t(d\Pi_t - \pi dt), \tag{L.14}$$

which is a martingale in that $\mathbb{E}[dU_t] = 0$. Next, to derive the law of motion of $W_t = W(U_t) := \frac{-\ln(-\rho r U_t)}{\rho r}$, note that $W'(U) = \frac{1}{-\rho r U}$, $W''(U) = \frac{1}{\rho r U^2}$, and $W(U - \rho r U\hat{\alpha}) - W(U) = -\frac{\ln(1-\rho r\hat{\alpha})}{\rho r}$. We now use Itô's Lemma in its version for jump processes and calculate via (L.14)

$$\begin{aligned} dW(U_t) &= W'(U_t)\rho r U_t \pi \hat{\alpha}_t dt + W'(U_t)(-\rho r U_t)\beta_t \sigma dZ_t + W''(U_t)\left(\frac{(\rho r U_t)^2(\beta_t\sigma)^2}{2}\right)dt \\ &+ \big[W(U_t - \rho r U_t\hat{\alpha}_t) - W(U_t)\big]d\Pi_t = -\pi\hat{\alpha}_t dt + \beta_t\sigma dZ_t + \frac{\rho r}{2}(\beta_t\sigma)^2 dt - \frac{\ln(1-\rho r\hat{\alpha}_t)}{\rho r}d\Pi_t. \end{aligned}$$

Setting $\alpha_t := -\frac{\ln(1-\rho r\hat{\alpha}_t)}{\rho r} \iff \hat{\alpha}_t = \frac{1-e^{-\rho r\alpha_t}}{\rho r}$ and rewriting, (L.3) follows.

M Refinancing Costs

Similar to Décamps et al. (2011) or Bolton et al. (2011), we can incorporate fixed costs $\phi \in (0,\infty)$ of equity issuance in addition to infrequent capital market access. The jump in existing shareholders value upon refinancing is now $J(C) = P(\overline{C}) - P(C) - (\overline{C} - C) - \phi - \alpha(C)$. We require, as in the baseline, $J(C) \geq -P(C)$, due to shareholders' limited liability. With fixed equity financing costs, the firm raises equity in state C upon refinancing if and only if the total gains from refinancing in the HJB equation (16) are positive, i.e.,

$$\max_{C^*,\alpha\in\mathcal{S}(C^*,C)} \left\{P'(C)\left[\alpha - k_\Pi(\alpha)\right] + \left[P(C^*) - P(C) - (C^* - C + \alpha)\right]\right\} \geq 0 \tag{M.1}$$

It follows that $C^* = \overline{C}$. In this case, $\alpha(C)$ must satisfy $\alpha(C) \leq P(\overline{C}) - (\overline{C} - C) - \phi$. If (M.1) does not hold, the firm does not refinance upon capital market access and the choice of $\alpha(C)$ becomes irrelevant. Overall, we can write $\alpha(C) = \min\{P(\overline{C}) - (\overline{C} - C) - \phi, \alpha_U(C)\}$. This results in no refinancing upon $d\Pi = 1$ on some interval $\left[\tilde{C}, \overline{C}\right]$ for some $\tilde{C} \in \left[\underline{C}, \overline{C}\right]$.

We now derive the lower boundary C^S under survival. Notice that, as in the baseline with $\phi = 0$, we have $P(C^S) = 0$, $\mu_C(C^S) = 0$, and $\beta(C^S) = 1$. In addition, when the firm finds it optimal to raise equity upon capital market access $d\Pi = 1$, i.e., when (M.1) holds at $C = C^S$, then $J(C^S) = 0$ which pins down $\alpha(C^S) = P(\overline{C}) - \overline{C} + C^S$. We can then solve for the lower boundary boundary conditional on survival:[35]

$$C^S = \min\left\{\frac{w\left(\frac{\pi}{r}\exp\left\{\rho r\left[\frac{\lambda}{r}\overline{C} + \phi + \frac{\pi}{\rho r^2} - \frac{\rho}{2}\sigma^2\right]\right\}\right) - \frac{\pi}{r}}{\rho r}, 0\right\} - Y^A.$$

The lower boundary is then $\underline{C} = \min\{C^S, -L\}$. When $\underline{C} = C^S$, the firm is never liquidated and raises new equity financing in state $\underline{C}$ upon market access. When $\underline{C} = -L$, the firm liquidates once C reaches $-L$. In the limit $\pi \to \infty$ (continuous access to equity financing at fixed cost ϕ), we get $\underline{C} = -\max\{[P(\overline{C}) - \overline{C}] - \phi, L\}$, but $\overline{C} > 0$ as both intermediary financing and refinancing are costly. If ϕ and L are sufficiently small, the firm refinances at a lower boundary $\underline{C} < -L$. If $\underline{C} = C^S$, the firm raises refinances once C reaches $\underline{C}$. The firm finances cash flow shortfalls against future promises Y for $C < 0$. Y may exceed the liquidation value of assets L, leading to cash flow-based financing. Our qualitative results go through as long as refinancing is costly (e.g., $\phi > 0$) or not frequently available ($\pi < \infty$).[36]

N Linear Cost and Stochastic Discount Factor

Consider $k_Z(\beta) = \rho\beta$ with $\beta \geq 0$ and $k_\Pi(\alpha) = 0$, so $k_t = \sigma^2 k_Z(\beta_t)$. This can be micro-founded by assuming that the intermediary applies a stochastic discount factor S_t with price of risk $\rho \geq 0$, i.e., $dS_t = S_t(-rdt - \rho\sigma dZ_t)$, so that $Y_t = \mathbb{E}_t\left[\int_t^\tau \frac{S_s}{S_t} dI_s\right]$. We obtain

$$Y_t = \mathbb{E}_t\left[\int_t^\tau \frac{S_s}{S_t} dI_s\right] = \mathbb{E}_t^Q\left[\int_t^\tau e^{-r(s-t)} dI_s\right] = \mathbb{E}_t\left[\int_t^\tau e^{-r(s-t)}\left(dI_s - \sigma^2\rho\beta_s ds\right)\right].$$

The expectation $\mathbb{E}_t^Q$ is taken under the risk-neutral measure with $d\tilde{Z}_t = dZ_t - \rho\sigma dt$ as the increment of a standard Brownian motion; $\mathbb{E}_t$ is taken under the physical measure with dZ_t as increment of a Brownian motion. The remainder of the analysis remains similar as in the baseline. For instance, the HJB equation (16) applies for any $k_Z(\beta)$ and $k_\Pi(\alpha)$, including the specification of this section.

[35] When $\{\cdot\} = 0$, then $C^S = -Y^A$ and the firm does not raise equity financing at C^S upon $d\Pi = 1$.

[36] For completeness, one could also add — next to the fixed cost of refinancing ϕ — a variable "flotation" cost of refinancing $\hat{\phi}$, as in Bolton et al. (2011). Under these circumstances, the firm would choose a refinancing target $C^* < \overline{C}$, but the remainder of the findings would likely remain similar.