NBER WORKING PAPER SERIES

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Working Paper 29434
http://www.nber.org/papers/w29434

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
October 2021, Revised March 2022

We thank Nicola Borri (the discussant) and participants in the Franco-German Fiscal seminar that was held in Paris, 9-10 November, 2021. The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research.

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JEL No. E13,E43,E44,E50,E62,E63,H30,H62,H63,H68

**ABSTRACT**

In a deterministic overlapping-generations economy with production and physical capital, the market value of debt is not necessarily equal to the present discounted value of future budget surpluses: it can be positive without any budget surpluses being in the offing, because debt incorporates a rational bubble. Yet the dynamics of debt remain a function of the dynamics of the primary budget deficit. The true fiscal cost of excessive government debt issuance cannot be assessed just from the current rate of interest or any current macroeconomic variable. Rather, it should be assessed in a dynamic context reflecting anticipated deficits and population growth going forward. As a way to study their joint behavior, we specify the variation of a structural deficit in the form of an underfunded social-security scheme. We define debt capacity as the level of debt that can be just sustained without a change of policy all the way to an unstable steady state. When it starts below the capacity, the debt converges to a stable steady state, in which the bubble is sustained. Above capacity the bubble unravels and the deficit cannot be financed. In several realistic scenarios occurring in economies, we calculate the needed policy response, which is the true "fiscal cost" of exceeding debt capacity.

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Over the last seventy-five years, the debt of high-income nations has mostly increased, while prospects for government surpluses are dimming.[1] Specifically, the 2019 Long-Term Budget Outlook by the U.S. Congressional Budget Office predicts steady deficits, if not rising ones, for thirty years. In the same vein, the slowdown or even decline in population growth (Jones (2020)) as well the reduction in technological progress mirrored in reduced GDP growth rates would seem to make it unlikely that successful debt reduction is possible in the foreseeable future.[2] In this paper, we show that, if a government's debt starts below a well-defined ceiling, deficits can persist and interest rates can remain low.

Working in a similar framework, Blanchard (2019) suggests that, in the current situation of low interest rates, an increase of government debt may be feasible and beneficial.[3] He writes: "If the interest rate paid by the government is less than the growth rate, then the intertemporal budget constraint facing the government no longer binds. What the government can and should do in this case is definitely worth exploring."

When the discount rate on a safe security is lower than the perpetual rate of growth of its payoffs, there are three possibilities for its price: it is equal to plus or to minus infinity, or, if it is finite, the price contains a positive bubble that offsets a negative infinite present value of future payoffs. In the "high-income" countries of today, the riskless rate is below the growth rate while it is likely simultaneously that the governments will not default and that their future primary budgets will remain in deficit.[4] Yet the prices of government bonds in the financial market are finite and positive. Our first goal is to provide an explanation of this paradoxical situation. In this regard, we establish the basic idea that government bond prices necessarily include a bubble.

One important aspect that Blanchard did not indicate is how deeply the government can go into debt. Our second goal, therefore, is to show that, even if a bubble is present in the price of government debt, which means that a Ponzi scheme is being run, the amount of debt that can be placed in the market is not infinite; there is still an upper bound on it. We call "debt capacity" the ceiling on the outstanding amount of debt. When staying within debt capacity, the bubble makes it possible for the total market value of government debt to be strictly positive even when the government budget is forever in deficit, which seems to reflect today's situation.

[1] Historically, sovereigns have borrowed to finance wars. Only in the 19th century governments started to systemically borrow to build ports, railways, roads, schools and universities. Despite this history, there are only three successful debt reduction episodes: Great Britain after the Napoleonic Wars, the United States in the last third of the 19th century, and France in the decades leading up to 1913; see Eichengreen, El-Ganainy, Esteves, and Mitchener (2019).

[2] Only a few countries seem to defy this trend. With its "debt brake" constitutional amendment, Germany is actively doing that: "The federal and state budgets shall in general be balanced without proceeds from borrowing. The federal and state governments can provide for rules to take into account the effects of deviations from normal cyclical developments, as well as a derogation for natural disasters or exceptional emergency situations that are beyond the control of the state and significantly affect the state's financial situation. A corresponding repayment plan must be provided for any derogation." *Constitution of the Federal Republic of Germany*, Article 109, Section 3.

[3] See also Furman and Summers (2020).

[4] While there is some uncertainty about whether interest rates are below growth in the data, Blanchard (2019) contains strong and robust evidence in favor of the view that interest rates are below growth. For instance, in the United States since 1950 even the nominal 10-year rate with an average of 5.6 percent is below the nominal GDP growth with an average of 6.3 percent.

In a recent talk, Sims (2020),[5] refers to debt issuance when the rate of interest is below the rate of growth as "zero fiscal cost debt." Yet, he states: "When the real interest rate on debt is below the growth rate of the economy, the government can issue and roll over debt forever without backing it by new taxes and still see the debt-to-GDP ratio shrink over time. But this only applies when the government makes a one-time increase in debt unaccompanied by increased taxation. This is not the situation we're in, even when it concerns pandemic relief spending. Given past, steady spending increases without tax increases, we unfortunately cannot view pandemic relief spending as a single, wartime increase in debt. Ultimately, these steady increases will affect the interest rate on debt and require dynamic solutions." In other words, the zero fiscal cost of today may foreshadow a much larger one later. The true fiscal cost of excessive government debt issuance cannot be assessed just from the current rate of interest or any current macroeconomic variable, or on the basis of an exogenous future path of the rate of interest. Rather, it should be assessed in a dynamic context reflecting anticipated deficits and population growth, going forward. For that, one must determine the future path of the rate of interest and one needs a dynamic model of growth and capital accumulation, with interest rate being affected by the government primary deficit/surplus directly or indirectly. Even when the market value of the debt is not equal to the present discounted value of future surpluses, the dynamics of the debt remain a function of the dynamics of the budget deficit. Our third goal is to develop such a model. We study debt sustainability with a primary government deficit that varies with the growth of the economy. Extant models assume a fixed path for the deficit. We fix the relation that ties the budget to the economy by assuming that it arises from a social-security scheme; it is then driven by the fundamentals of the economy, namely, the terms of the scheme, population growth, preferences and production technology.

Several economic settings allow the presence of a bubble. In this paper, we consider the overlapping-generations (OLG) setting. Its initiator, Samuelson (1958), had already indicated that the "social contrivance of money" was welfare improving because it bridged the non coincidence of needs of different generations. Wallace (1980) mused at the time of his writing that "neither [Samuelson] nor most economists seem to take it seriously as a model of fiat money."[6] The reason was that money is needed obviously as an every-day device, not one used across generations as a form of social security.[7] But an application of the very same argument to government debt (alongside money or without it) seems vividly relevant and realistic given that all of us will die while the government will exist forever. Along that line, Diamond (1965) sets the debt per capita to be constant and introduces the taxes needed to pay for the cost of financing. The stable steady state he obtains is not "efficient" (i.e., welfare maximizing) for the well-known reason that each generation, in order to finance their retirement, saves in excess of what they would if the welfare of all generations were optimized. In Tirole (1985), there is no deficit and, therefore, also no government debt; however, Tirole

[5]See Christopher Sims: "How to worry about government debt" at https://bcf.princeton.edu/events/christopher-sims-how-to-worry-about-government-debt/.

[6]Contributions pertaining to OLG as a model of fiat money include: Shell (1971), Wallace (1980), and Kocherlakota (1998) among many others.

[7]A variation on the same reasoning appears in Kiyotaki and Wright (1989), which is the foundation of the so-called "New Monetarism" school. Instead of generations overlapping partially, traders with different needs meet randomly and, for that reason, benefit from a universal medium of exchange.

shows that a rational, intergenerational bubble (a finite one in his case) can exist and lead to an unstable steady state, at which welfare is optimal (i.e., is at the Golden Rule). We show that the bubble in his model can be interpreted as a one-time issuance of government debt to finance a one-time government expenditure. In addition, we allow further issuance dynamically to finance additional government deficit each period. Chalk (2000) was perhaps the first to point out that government debt is determined by the real side of the economy and that there is an upper bound on its size.[8] Yet, he also assumes that the exogenously determined deficit per capita (occasioned by a wasteful expenditure) is constant. Further, in his variant of Diamond's model, agents work all periods and, therefore, there is no role for social security.[9]

Brunnermeier, Merkel, and Sannikov (2021) adopt another setting. Instead of overlapping generations, they consider an incomplete market model featuring infinitely-lived log-utility agents. The capital owned by each agent is exposed to counter-cyclical idiosyncratic risk because a "skin-in-the game" constraint requires the agent to hold more than a minimum amount of his capital, while the rest is tradable. Each agent is equipped with a strictly positive personal stochastic discount factor and applies a transversality condition written with it. When doing that, all agents value any traded security at one and the same market price and, relative to each agent's own discount factor, there is no bubble. However, if one compares the market price with the present value of future surpluses calculated with an unweighted average stochastic discount factor, which does not put a price on idiosyncratic risk, there is a deviation, which can be viewed as a bubble, although that bubble differs conceptually from ours. In particular, the bubble causes riskless government debt to trade at a higher price than it would under the average discount factor. According to the authors, it is possible, when the present value of future surpluses is negative, for the bubble on government debt to give that debt a strictly positive market value.[10]

In the absence of a bubble, it is also conceivable to explain the current valuation of government bonds by recognizing that the government budget is a risky cash flow. A recent study by Jiang, Lustig, Nieuwerburgh, and Xiaolan (2021) aims to explain, as we do, the valuation gap of government debt, which is the gap between the value of the debt observed in the financial market and the present value of future primary surpluses. It features a thorough empirical investigation of the stochastic process of government surplus, – carefully estimating separate processes for government income and government expenditures –, postulates an exponential affine stochastic discount factor for infinitely lived investors and derives the risk premium of government debt. The authors dismiss a rational bubble as a possible explanation of the gap on the grounds that the value of the debt would become infinite if the transversality condition on debt were violated. We show below that, in the presence of a perpetual deficit and a rate of interest lower than the rate of growth, the debt per capita in an OLG model can be positive and finite at all times while containing a bubble that is positive infinite. The reason is that the present value of future surpluses is at minus infinity, the sum of the two remaining finite. They also state that "In rational bubble models, the debt/GDP ratio declines over time," which is inconsistent with a valuation gap that grows

[8]For a treatise on dynamics and policies in overlapping-generations models see De la Croix and Michel (2002).

[9]In Appendix B, we compare our model to previously published ones.

[10]See also Brunnermeier, Merkel, and Sannikov (2020).

empirically faster than GDP. In our model, the debt/GDP ratio can rise either temporarily or permanently.

Specifically, Jiang, Lustig, Nieuwerburgh, and Xiaolan (2021) propose a convenience yield on government debt as the explanation of the valuation gap but acknowledge that it falls short of the mark quantitatively. They also acknowledge that the convenience yield does not explain a positive value for the debt when the primary surplus is permanently negative (see their Figure 10, Panel (a)). In our paper, we propose an explanation that is complementary to theirs and show that the more traditional OLG rational bubble is capable of explaining that last fact.

Blanchard (2019), Brunnermeier, Merkel, and Sannikov (2021) and Jiang, Lustig, Nieuwerburgh, and Xiaolan (2021) all analyze a stochastic economy. We choose to first study the debt capacity of a government in a deterministic economy because our investigations and policy experiments hinge more on the dynamics of debt than on the nature of the shocks. They are equally meaningful and more transparent in the simpler, deterministic setting. We now describe these investigations.

Under conditions in which the rate of interest is below the growth rate, the steady states of our model, when they exist, come in a pair. That pair is the foundation of debt capacity. One steady state is unstable; for a given initial capital stock, there is one unique saddle path emanating from a unique level of government debt that leads to it; we call "debt capacity" the threshold level at the initial point in time. One steady state is stable (and requires a lower deficit); for a given initial capital stock, many paths, from a range of values for an initial value of government debt, lead to it.[11] In our variant of the Diamond model, the value of the interest rate is not sufficient to determine whether debt is sustainable or desirable. The amount of outstanding debt (captured by the Debt-to-GDP ratio) also matters. It is for this reason that the notion of debt capacity and the knowledge of its value are essential.

We extend the model and its concept of debt capacity to two policy-relevant settings. First, one might be concerned that a high, potentially explosive level of nominal debt produces high inflation. To investigate this issue, we allow the government, in addition to collecting taxes and paying benefits, to intervene in the money market following a Taylor rule, as a way to fix the nominal rate of interest. Explosive real debt also leads to explosive inflation and seignorage revenues increase debt capacity markedly.

Second, as debt capacity depends on growth, one might hope that endogenous productivity increases would raise the limit. We, therefore, ask whether a government can increase its debt capacity by subsidizing innovation which ultimately raises productivity and growth. To answer this question, we adjust our framework along the lines of a revised Romerian approach but let the government finance R&D in addition to paying for social security. Overall, our numerical illustrations under these varied scenarios show that public R&D spending does not lift debt capacity miraculously.[12]

[11]For some combination of parameter values, the two steady states coincide, in which case stability prevails when approaching the steady state from one side only.

[12]Compare to the statement by Prime Minister Draghi and President Macron in the *Financial Times*, December 23, 2021: "We need to have more room for manoeuvre and enough key spending for the future and to ensure our sovereignty. Debt raised to finance such investments, which undeniably benefit the welfare of future generations and long-term growth, should be favoured by the fiscal rules, given that public spending of this sort actually contributes to debt sustainability over the long run."

In policy experiments, we use the notion of debt capacity to explore the government responses that are needed in case debt, for whatever reason, exceeds it. These capture the complete fiscal cost of exceeding debt capacity today. We also explain how debt could come to exceed debt capacity. For that, we develop a scenario of unexpected population-growth declines. We again calculate the needed policy response. The more delayed the response, the larger it has to be. Our model can arguably be interpreted as saying that it is better to implement a policy that reduces the debt automatically during normal times, as a way to aim towards the stable steady state.

That part of our paper is closest to the vast literature that developed tax and social-security policy scenarios in an OLG context.[13] The applied models of that literature are much more detailed and incorporate more features than we have here, going as far as to imbed tax calculators corresponding to the tax law of a specific year (see Moore and Pecoraro (2018)), or to include a population of 55 generations (Diamond and Zodrow (2006)). Their purpose is advisory; it is to forecast the effects of specific aspects of tax reforms: general lowering of taxes, or replacement of one form of tax by another. Many of these papers take the rate of interest as exogenous. They also take the amount of debt of a base year as given, without any emphasis on the largest viable amount of debt. Some address the issue of the financing of a tax reform (Diamond (2005), Diamond and Viard (2008)) but all of them consider scenarios in which, at the tail end of the tax reform, a change in tax will restore budget balance, so that the so-called "intertemporal budget constraint" of the government binds with a value of debt equal to present discounted value of future surpluses. In other words, a no-Ponzi condition and a return to a stable steady state is assumed implicitly or explicitly. The lesson from our paper regarding advisory work is that the set of scenarios to be considered can and should be greatly expanded to allow for the permanent presence of a bubble, and for a definition of debt sustainability and debt capacity that is grounded on the unstable steady state.

The balance of the paper is organized as follows. Section 1 presents a deterministic model with perpetual refinancing of a social-security driven government deficit. Section 2 contains our definition of debt capacity and a study of the comparative statics of it. Section 3 contains policy experiments and describes the consequences of a decline in the population growth rate. In Section 4, we turn to two extensions of the model: first, we examine the implications for inflation of a high level of nominal government debt and consider the role of the central bank; second, we envision the possibility that the government may sustain its debt by subsidizing R&D, in addition to supporting a social-security scheme. The final section contains the conclusion.

# 1 A deterministic model of perpetual refinancing

## 1.1 The components of the system

We build an overlapping-generations model with population growth and physical capital accumulation. The economy comprises a production sector, a household sector and a gov-

[13] That literature issued from Summers (1981) and Auerbach and Kotlikoff (1987). For a thorough survey, see Zodrow and Diamond (2013).

ernment sector.

**The production function** is

$$Y_t = F(K_t, \Lambda_t)$$

where $K_t$ and $\Lambda_t$ are the inputs of capital and labor. The function $F$ is continuous, twice differentiable, homogeneous of degree 1 in its two arguments, strictly increasing and strictly concave in each. To exploit the homogeneity, we write

$$k_t \triangleq \frac{K_t}{L_t}; f(k_t) \triangleq F(k_t, 1)$$

In applications, the production function will be of the constant-elasticity of substitution (CES) type

$$Y_t = A \times \left[\alpha K_t^{\rho} + (1-\alpha)\Lambda_t^{\rho}\right]^{\frac{1}{\rho}}; \rho < 1 \tag{1}$$

where $\rho = (\eta - 1)/\eta$ and $\eta$ is the elasticity of technical substitution (ETS) between capital and labor, and $A > 0$, $0 < \alpha < 1$, $K > 0$, $\Lambda > 0$. At time 0, the economy starts with a capital stock equal to $K_0$. At $t \neq 0$, the capital stock $K_t$ is set aside at time $t-1$ and chosen by the generation born at time $t-1$. It depreciates at the rate $\delta < 1$ per period.

**The households/investors:** Like in Diamond (1965), we introduce the following notation and assumptions: $c_t^t$ is the consumption at date $t$ of one household of the generation born at date $t$ while $c_{t+1}^t$ is the consumption at date $t+1$ of the generation born at date $t$. $L_t$ is the exogenous number of individuals in the generation born at time $t$. Their lives are summarized with two periods and they work at the first point in time only; their supply of labor is inelastic. $L_t$ grows at the constant rate $n$ per period. The number $n$ stands for population growth but possibly also for labor-augmenting technical progress. It is a catch-all for all forms of exogenous perpetual growth. We examine technical progress explicitly in Section 4.2.

Generations are born with an endowment of only one kind: their labor force. They collect a wage bill $w_t L_t$. At time 0, there are only "old" people born notionally at time $-1$ with arbitrary consumption $c_0^{-1}$.

The two-points in time utility functions of all generations are the same:

$$U\left(c_t^t, c_{t+1}^t\right) = u\left(c_t^t\right) + \beta u\left(c_{t+1}^t\right); t \geq 0$$

where $u$ is a continuous, twice differentiable, strictly increasing and strictly concave function. In applications, the function $u$ will be a power function $u(c) = c^{1-\zeta}/(1-\zeta); \zeta > 0$ with, therefore, an elasticity of intertemporal substitution (EIS) equal to $1/\zeta$.

**The financial market:** Two assets are traded in the financial market, the maturity of which is immaterial. One is a bond, which is a claim on the government and the other is the direct ownership of the capital that serves as input into the production system, and which can be rented to production facilities. In this deterministic world, the young households are indifferent between physical capital and government debt, so that we let them choose not each one separately but their sum which is called "savings" $s_t$. In total, they save an amount $s_t L_t$ at time $t$.

In other words,

$$s_t \times L_t \triangleq K_{t+1} + G_{t+1}$$

in which $G_{t+1}$ is the time-$t$ exiting amount of government held, and which can be restated on a per capita basis as:

$$s_t \triangleq (1+n)(k_{t+1} + g_{t+1}) \tag{2}$$

where $g_t \triangleq G_t/L_t$.

The two assets being perfect substitutes in demand, they bring the same rate of return. The one-period rate of return or rate of interest quoted at time $t$ is called $r_{t+1}$.

**Taxation and spending:** Taxation is in the form of a contribution to the social-security system. The time-$t$ young make a total social-security contribution of $L_t \tau w_t$, where $\tau$ is the social-security tax rate.

Government spending is in the form of social-security defined benefits paid to the old households on the basis of the wages they were earning when young. Specifically, at time $t$ the old receive a total social-security benefit of $L_{t-1}\theta w_{t-1}$, where $\theta$ is the social-security benefit ratio. In Appendix A, we show that social security is a welfare-improving form of spending.[14] Throughout we consider the case in which the primary budget deficit is structural: $\tau < \theta \times (1+n)$.

With this notation, the simultaneous flow budget constraints at time $t$ are as follows: for the young household,

$$c_t^t + s_t = (1-\tau) w_t$$

for the old household,

$$c_t^{t-1} = s_{t-1} \times (1+r_t) + \theta \times w_{t-1}$$

for the government,

$$-G_{t+1} + \theta \times w_{t-1} \times L_{t-1} = \tau \times w_t \times L_t - (1+r_t) G_t$$

where $G_t$ is the total debt with which the government enters time $t$ and $G_{t+1}$ is the debt with which it exits time $t$.

**Market clearing:** The labor market clears

$$\Lambda_t = L_t$$

and the market for goods clears

$$L_t \times c_t^t + L_{t-1} \times c_t^{t-1} + K_{t+1} = F(K_t, L_t) + (1-\delta) K_t$$

## 1.2 Difference equations and steady states

By writing the first-order conditions of a household, who owns the production facility and rents it out, and imposing market clearing, the difference-equations system governing the

[14] See the policy recommendation of L. Summers, *Washington Post*, Jan 7, 2020.

evolution of the economy, stated on a per capita basis, is[15]

$$
\begin{aligned}
\frac{\frac{\partial}{\partial c_{t+1}^t} U\left(c_t^t, c_{t+1}^t\right)}{\frac{\partial}{\partial c_t^t} U\left(c_t^t, c_{t+1}^t\right)} &= \frac{1}{1+r_{t+1}} && (3)\\
f'\left(k_t\right) - \delta &= r_t && (4)\\
f\left(k_t\right) - k_t f'\left(k_t\right) &= w_t && (5)\\
c_t^t + s_t &= (1-\tau)\, w_t && (6)\\
c_t^{t-1} &= s_{t-1} \times \left(1+r_t\right) + \theta \times w_{t-1} && (7)\\
-(1+n)\, g_{t+1} + \theta \times w_{t-1} \frac{1}{1+n} &= \tau w_t - \left(1+r_t\right) g_t && (8)\\
c_t^t + \frac{1}{1+n} c_t^{t-1} + (1+n)\, k_{t+1} &= f\left(k_t\right) + (1-\delta)\, k_t && (9)
\end{aligned}
$$

For each generation, it would be suboptimal to finish its life with total savings of capital and government-debt holdings greater than zero. And default is not allowed, so that their total savings at the end of their life are set exactly to zero. These are the only terminal conditions of optimality that are present in the model; they have already been coded into the system of equations above. In this overlapping-generations setting, no terminal conditions should be imposed at "the end of time." No so-called transversality conditions apply at infinity.[16,17] For that reason, the system is a "forward" system of equations, with initial conditions to be specified only. It is almost entirely backward looking; this is a case of rational myopia, in which future events beyond one period need not be considered by economic agents. We introduce an important exception to the myopia principle in Remark 1 below.

Equations (4) and (5) allow us to define $r_t$ and $w_t$ as functions $r\left(k_t\right)$ and $w\left(k_t\right)$:

$$r\left(k_t\right) = f'\left(k_t\right) - \delta; \;\; w\left(k_t\right) = f\left(k_t\right) - k_t f'\left(k_t\right)$$

The function $r\left(k_t\right)$ is strictly decreasing and bounded below by $-\delta$. Substituting Equation (7) shifted forward and Equation (6) into Equation (3) gives

$$\beta \frac{u'\left(s_t \times \left(1+r_{t+1}\right) + \theta \times w_t\right)}{u'\left((1-\tau)\, w_t - s_t\right)} = \frac{1}{1+r_{t+1}}$$

and, given the monotonicity of the function $u'$ and customary Inada assumptions allows us to find a solution for $s_t$. Define a supply-of-savings or demand-for-assets function

$$s_t = s\left(w_t, r_{t+1}\right)$$

[15] In accordance with Walras law, the system made of equations ((2),( 3)-(9)) contains a redundant equation: (3)-(9) implies (2) and ((2),( 3)-(8)) implies (9).

[16] They apply as necessary conditions of optimality when an agent *with an infinite lifetime* maximizes his lifetime utility. There is no such agent in this economy.

[17] As a technical point, one could note that, if the "end of time" were finite, some terminal conditions could be imposed, and the solution could be calculated backward, resulting in a unique initial situation. Increasing the end of time forever in the backward solution with a range of terminal values gives the same result as does the forward solution.

At the end of its two-period life each household consumes its entire wealth, including the value of the capital stock, which is part of its savings, which it sells to the young, leaving nothing behind.

Equations (6), (7) and (9) form a linear system

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & \frac{1}{1+n} & 0 \end{bmatrix} \times \begin{bmatrix} c_t^t \\ c_t^{t-1} \\ s_t \end{bmatrix} = \begin{bmatrix} (1-\tau)\, w_t \\ s_{t-1} \times (1+r_t) + \theta \times w_{t-1} \\ f(k_t) + (1-\delta)\, k_t - (1+n)k_{t+1} \end{bmatrix}$$

which can be solved easily to provide the demand for savings (supply of assets)

$$s_t = \begin{bmatrix} 1 & \frac{1}{1+n} & -1 \end{bmatrix} \times \begin{bmatrix} (1-\tau)\, w_t \\ s_{t-1} \times (1+r_t) + \theta \times w_{t-1} \\ f(k_t) + (1-\delta)\, k_t - (1+n)k_{t+1} \end{bmatrix}$$

We are ready to reduce the eight-equation system to two difference equations or even one of second-order. That can be done in two, equivalent ways. We can, first, equate demand and supply and get an equation relating $k_{t+1}$ to $k_t$ and $k_{t-1}$, which is, therefore, a stand-alone second-order difference equation:[18]

$$\begin{aligned} s\left[w(k_t), r(k_{t+1})\right] &= \begin{bmatrix} 1 & \frac{1}{1+n} & -1 \end{bmatrix} \\ &\times \begin{bmatrix} (1-\tau)\, w(k_t) \\ s\left[w(k_{t-1}), r(k_t)\right] \times (1+r(k_t)) + \theta \times w(k_{t-1}) \\ f(k_t) + (1-\delta)\, k_t - (1+n)k_{t+1} \end{bmatrix} \end{aligned} \tag{10}$$

Initial conditions are set by the initial capital stock $k_0 = K_0/L_0$. Suppose for a moment that $k_1 = K_1/L_1$ were also given. Equation (10) would then provide the path of the capital stock autonomously and the evolution of government debt would just follow from difference Equation (8).

Since, however, $k_0$ and $k_1$ are not given jointly, debt must be part of initial conditions and the government budget equation (8) must be brought in. For easier interpretation, define the social-security deficit as[19]

$$d_t \triangleq d(k_{t-1}, k_t) = \frac{\theta}{1+n} w(k_{t-1}) - \tau \times w(k_t) \tag{11}$$

The two-equation systems is

$$\begin{aligned} s\left[w(k_t), r(k_{t+1})\right] &= (1+n)(k_{t+1} + g_{t+1}) \\ (1+n)g_{t+1} &= (1+r(k_t))\, g_t + d(k_{t-1}, k_t) \end{aligned} \tag{12}$$

Formulation (10) is sufficient for the study of steady states while formulation (12) high-

[18] As in the linear multiplier-accelerator model of Samuelson (1939).

[19] There are two ways to specify social-security deficit $d_t$ in our model: benefits indexed on wage earned while working (Equation (11)) or benefits indexed on current wage $d_t = \theta w(k_t)/(1+n) - \tau w(k_t)$. The first way is more consistent with reality but causes the model to be second order in capital (involving $k_{t-1}, k_t, k_{t+1}$).

lights the interaction along a path between government debt and the capital stock, and is more traditional in the literature.[20] In Appendix B, we show the novelty of the our formulation as compared to extant models.

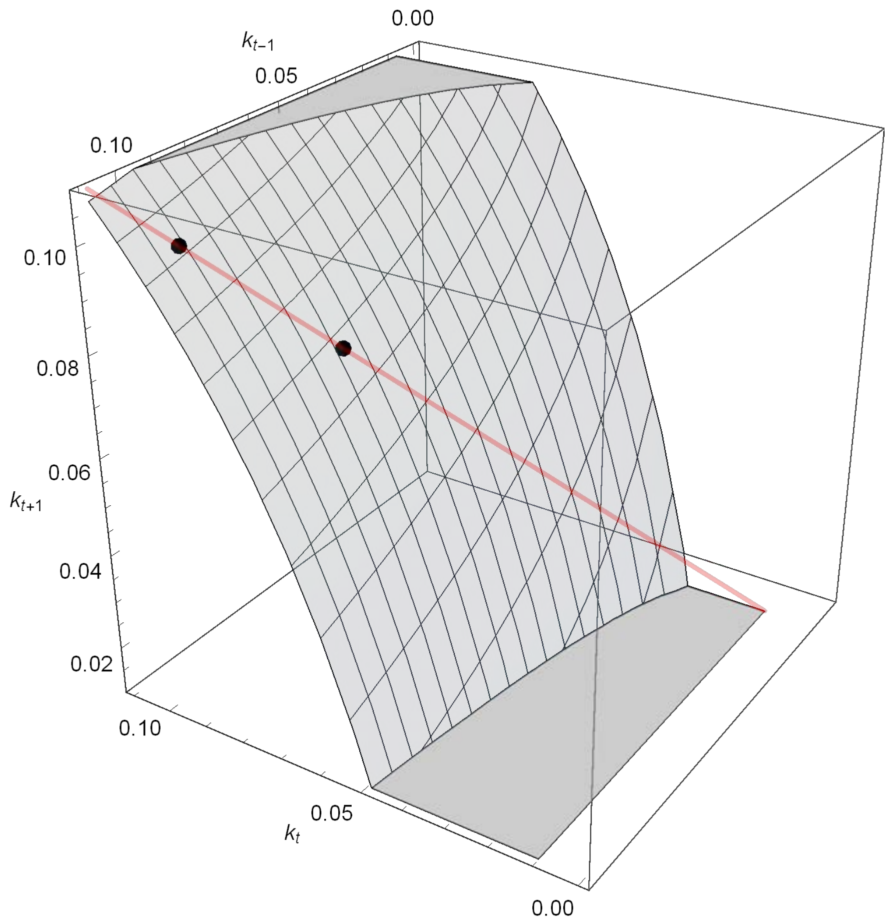


Figure 1: **The dynamics of the capital stock. Illustration with log utility and Cobb-Douglas production function**. The surface represents the relationship $k_{t+1}(k_t, k_{t-1})$. Parameter values are: $n = (1 + 0.02)^{25} - 1$, $\alpha = 0.2$, $\beta = 0.99^{25}$, $\delta = 1 - (1 - 0.1)^{25}$, $\theta = 0.165$, $\tau = 0.1$.

We define a steady state as a situation in which $K_t/L_t$ is a constant $k$ over time. A steady state must solve Equation (10) with $k_{t+1} = k_t = k_{t-1}$. Figure 1 illustrates the way in which steady states are determined.

[20]Formulation (12) can be shifted forward to produce a first-order system in $(k_{t+1}, g_{t+2})$:

$$\begin{aligned} s\left(w\left(k_t\right), r\left(k_{t+1}\right)\right) &= (1+n)\left(k_{t+1}+g_{t+1}\right) \\ (1+n) g_{t+2} &= \left(1+r\left(k_{t+1}\right)\right) g_{t+1}+d\left(k_{t+1}, k_{t+1}\right) \end{aligned}$$

**Example 1.** *Particularizing the problem, there exists an analytical solution for the steady states in the special case of logarithmic utility (isoelastic with* $\zeta = 1$*) and Cobb-Douglas production function: Equation (1) with* $\rho \to 0$ *gives*

$$F(k,1) = Ak^{\alpha};\ w = A(1-\alpha)k^{\alpha};\ r = A\alpha k^{\alpha-1} - \delta \tag{13}$$

*It is a bit simpler to rewrite the equation system with* $x \triangleq r + \delta$ *as an unknown. The equation is cubic in* $x$*. There exist explicit but fairly cumbersome formulae for the roots of a cubic equation. Here, we just give a numerical example. Suppose that a period of the model is equal to twenty-five years and that:* $n = (1+0.02)^{25} - 1$, $\alpha = 0.2$, $\beta = 0.99^{25}$, $\delta = 1 - (1-0.1)^{25}$, $\theta = 0.165$, $\tau = 0.1$.
*The cubic equation of* $x$ *has two positive real roots:*
*Root 1:* $x = 1.27512$ *(*$3.343\%$*/year),* $k = 0.0987074$, $r = 1.198\%$*/year, locally stable.*
*Root 2:* $x = 1.55597$ *(*$3.825\%$*/year),* $k = 0.0769635$, $r = 1.968\%$*/year, locally unstable.*
*Roots 1 and 2 are displayed in Figure 1.*
*In the absence of social security and debt (*$\theta = 0, \tau = 0$*), there would have been only one root at* $k = 0.144993$.

**Example 2.** *With logarithmic utility, CES production (1) with* $\rho = -1$ *(ETS equal to* $1/2$*), and other parameters as in the previous example there exists only one real root of the steady-state equation, which is:*
$x = 1.56976$ *(*$3.847\%$*/year),* $k = 0.196178$, $r = 2.002\%$*/year. With these parameters, there is no solution with* $r < n$*. But see Section 2.1 for a broader study of existence with CES production.*

In general cases, non linear Equation (10) with $k_{t+1} = k_t = k_{t-1} = k$ is not in polynomial form. One can still solve it numerically with the limitation that one does not know how many roots it has in total. For that reason, we assume that the relationship between $k_{t+1}$ and the two arguments $(k_t, k_{t-1})$, which is implicit in (10), is concave against $k_t$ and that this concavity dominates, in case they are contrary, over the curvature against $k_{t-1}$. Then there exist at most two steady states, the steady state with the lower value of $k$ being locally unstable.

So far, we have derived the law of motion of $k$ starting from given $k_0$ and $k_1$. This solution procedure has allowed us to determine the steady states without any consideration of the behavior of government debt. It remains to endogenize $k_1$ if $k_0$ is given or $k_0$ if $k_1$ is given. Assume that the initial amount of debt, set by history, is contractually denominated as a real amount.[21] Then $k_1$ actually follows from $k_0$ and the definition of savings (2) written at time 0:

$$s\left[f(k_0) - k_0 f'(k_0),\, f'(k_1) - \delta\right] = (1+n)(k_1 + g_1)$$

where it is most convenient to take $g_1$ (as opposed to $g_0$) as initial condition for the government debt.[22] In total, initial conditions are set by the initial capital stock $k_0 = K_0/L_0$ and the amount of debt $g_1 = G_1/L_1$ with which the government leaves time 0 or, equivalently, by

[21] The alternative of nominal denomination is explored in Section 4.1 below.

[22] Otherwise, one would have to specify in an ad hoc fashion the amount of benefits paid to the old generation at time 0.

$k_1$ and $g_1$. The law of motion of government debt then follows from the government budget constraint, Equation (8).

Rolling over Equation (8) one gets

**Proposition 1.** *The value of debt is*

$$\begin{aligned} g_1 &= \frac{1}{1+n}\sum_{t=1}^{T-1}\frac{-d_t}{\prod_{u=1}^{t}\frac{1+r_u}{1+n}} + \frac{g_T}{\prod_{u=1}^{T-1}\frac{1+r_u}{1+n}}; \forall T > 1 \\ &= \lim_{T\to+\infty} \quad \underbrace{\frac{1}{1+n}\sum_{t=1}^{T-1}\frac{-d_t}{\prod_{u=1}^{t}\frac{1+r_u}{1+n}}}_{\text{PV of current and future surpluses}} + \underbrace{\frac{g_T}{\prod_{u=1}^{T-1}\frac{1+r_u}{1+n}}}_{\text{Bubble}} \end{aligned} \tag{14}$$

The proof is in Appendix C. Proposition 1 gives a decomposition of the time-1 value of debt: with $r(k) < n$ and $d_t > 0$, the first component becomes negative infinite, and the second positive infinite as $T \to +\infty$.

Their sum can very well be finite. Indeed, call $g$ a steady-state per-capita value of government debt; as per Equation (8):

$$g = \frac{-d(k)}{r(k) - n}; d(k) = \frac{\theta}{1+n} w(k) - \tau \times w(k) \tag{15}$$

where the formula is valid if $n \neq r(k)$. In the Cobb-Douglas special case,

$$g = \frac{y}{r(k) - n}\left(\tau - \frac{\theta}{1+n}\right)(1-\alpha) \tag{16}$$

When there exists a steady state $k$ of the capital stock per capita, then there exists a steady state value $g$ of government debt per capita.

In a steady state, it may very well happen that the government's social-security budget per capita is permanently in deficit ($d > 0$), while, if $r(k) < n$, government debt still has positive market value. That, in a nutshell, is the point made by Blanchard (2019). Obviously, although the formula looks like the Gordon formula or the summation of a geometric series, the value of government debt, in the case $r(k) < n$, cannot be equal to the present value of future primary surpluses. The government debt contains a positive rational bubble, along the lines of Tirole (1985).

With a zero deficit, Tirole (1985) shows that there exists a starting value of the bubble for which a saddle path leads to a steady-state, non-zero, per-capita finite bubble, which lasts forever and brings about the efficient (Golden rule) outcome $r(k) = n$. Below that starting value, the bubble eventually reaches zero (at the stable steady state). In contrast, in our model, in every period with a jointly determined social-security deficit new debt is issued and old debt is rolled over. In a steady state, the per-capita value of debt is given by the Gordon growth formula (15). Whenever there is a deficit, $d > 0$ that is, and the interest rate is smaller than growth, the government finances each and every deficit with more debt. Deficits occur forever, total debt per capita remains positive in market value and finite and, hence, contains a rational bubble. Deficits in the steady states imply that

$r(k) < n$ and vice versa, a configuration that best reproduces current economic conditions. Whether in the stable or in the unstable steady state, if there is a permanent deficit, the bubble component of debt is positive and larger, in absolute value, than the present value of surpluses. To the opposite, when the steady-state social-security scheme produces surpluses, we have $r(k) > n$.[23]

# 2 The definition of debt capacity

In this section, we continue to assume that the initial amount of debt, set by history or reset by fiat, is contractually denominated as a real amount and we turn to the important matter of global convergence or divergence.

Consider a parametrization in which a stable steady state produces a deficit period by period financed by a never ending issuance of debt while the real rate of interest is lower than the real rate of growth of the economy. The economy might be (already) at the stable steady state or on its way to it. Figure 2 illustrates the fact that many paths with many levels of deficit-to-GDP (or debt to GDP) ratios all lead to the same stable steady state.

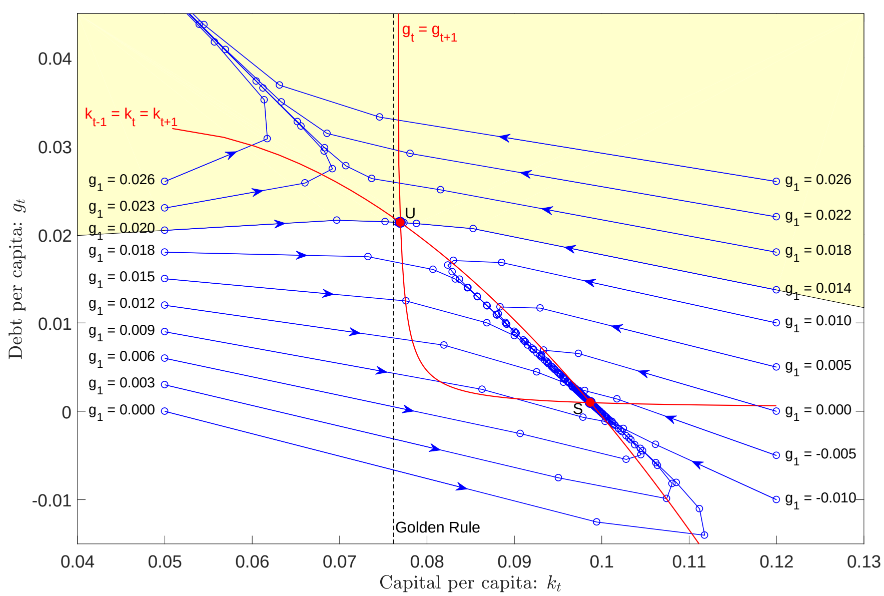


Figure 2: **The paths of the debt per capita and the capital stock per capita for two initial values $k_1$ and several initial values $g_1$.** Illustration with log utility and Cobb-Douglas production function. Parameter values are: $n = (1 + 0.02)^{25} - 1$, $\alpha = 0.2$, $\beta = 0.99^{25}$, $\delta = 1 - (1 - 0.1)^{25}$, $\theta = 0.165$, $\tau = 0.1$. The stable steady state is marked "S" and the unstable one "U".

**Example 3.** *Example 1 continued: the joint dynamics of the capital stock and the debt are illustrated in the diagram of paths Figure 2. The diagram also shows the two loci* $g_t = g_{t+1}$

[23] Farmer and Zabczyk (2020) state that, if the economy is dynamic efficient ($r > n$), the bubble term vanishes as long as per-capita debt $g_s$ does not explode as $s \to +\infty$.

*and $k_{t-1} = k_t = k_{t+1}$.*[24] *Their two intersections are the two steady states, where the stable steady state is marked "S" and the unstable one "U".*

**Definition 1.** *Debt capacity for a given level of $k_0$ is the highest level of $g_1$ such that convergence occurs* without any change of policy parameters $(\theta, \tau)$.

Debt capacity is also the level of debt today that would lead to the unstable steady state along a saddle path. Provided debt starts from any level *strictly* within capacity, it converges to the stable steady state. The limitation on the level of debt exists despite the myopia of each individual generation. It is not due to concern about the welfare of future generations. Old generations and young generations trade with each other and hence one might think that through this trading current generations care, at least indirectly, for future generations. But they simply do not do that in an OLG setting. Please, note that our definition is not based on the existence or inexistence of a locally stable steady state. There exists one anyway but the non linearity in the model is such that the economy does not converge to it when debt starts above capacity.

To make more concrete the threshold between convergence and divergence, we display in Figure 3, for initial conditions $(k_1, g_1)$, the contour that separates the area of divergence (upper, shaded area) from the area of convergence.

**Example 4.** *Example 1 continued: we include in the top plot in Figure 3 a diagram of paths in the plane of the ratio of debt $g$ to annual output $(y/25)$ versus the real rate of interest $r$. The diagram shows that, for the higher of the two initial capital stocks displayed, a $g_1$ of* 0.0137 *(which is about* 52.45% *of annual output), and for the lower one a $g_1$ of* 0.020 *(which is* 89% *of annual output), lead to the unstable steady state. At the unstable steady state itself, the debt is* 89.17% *of annual output. At the stable steady state, the debt is about* 4% *of annual output. We also display in the bottom plot in Figure 3 the paths in the plane of the ratio of budget deficit to output $d/y$ versus the real rate of interest $r$. The ratio $d/y$ can be viewed as a Maastricht criterion. We see that the deficit ratio is* 0.046% *(*1.15% *of* annual *output) in both steady states.*[25]

An economy can start above debt capacity, on a seemingly explosive path, if it can be anticipated that the government will, at some point, increase the tax rate, $\tau$, decrease the social-security benefit ratio, $\theta$, or both, in order to help return to a sustainable steady state. In Section 3 below, we examine such policy responses.

When the debt does not converge, one can call it "divergent" or "explosive". But it is more properly called "unsustainable" or, at most, "potentially explosive." As it exploded, the debt, inclusive of its bubble, would crowd out physical capital, since total saving cannot explode. Hence it could also be forecast that physical capital would become negative, which is impossible. Since this would be known to the last generation that lived just before this happened, the debt and the capital could not be sold to them; it would have zero market value. It then could not be sold to the previous generation and so on; hence, any equilibrium

[24] These are the loci as per the second formulation. As per the first formulation, the locus $k_{t-1} = k_t = k_{t+1}$ is made of two vertical straight lines and the locus $g_t = g_{t+1}$ of two horizontal lines.

[25] For the Cobb-Douglas production function, $w(k)/f(k)$ is independent of $k$ and the steady-state deficit-output ratio is $(\theta/(1+n) - \tau)(1-\alpha)$.

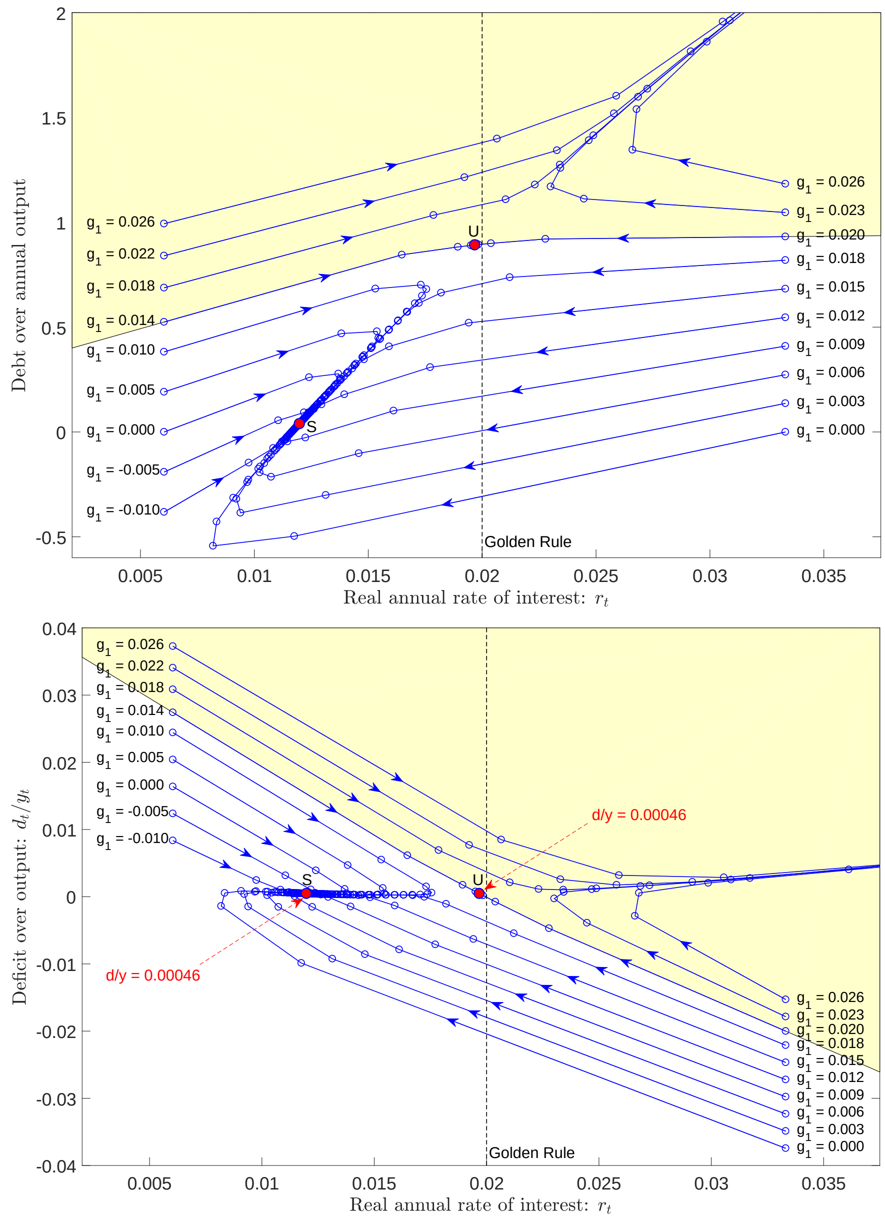


Figure 3: **The paths of the debt ratio and the real interest rate and of the deficit ratio and the real interest rate for two initial values $k_1$ and several initial values** $g_1$. Illustration with log utility and Cobb-Douglas production function. Parameter values are: $n = (1 + 0.02)^{25} - 1$, $\alpha = 0.2$, $\beta = 0.99^{25}$, $\delta = 1 - (1 - 0.1)^{25}$, $\theta = 0.165$, $\tau = 0.1$. The stable steady state is marked "S" and the unstable one "U". $g$ is debt per capita; $y$ is output per capita over 25 years and $d$ is deficit over 25 years.

with explosive debt would unravel. This means that rational explosive paths cannot even begin: the debt cannot be sold, or has zero market value, even at the initial date.

**Remark 1.** *Although, when convergent, the debt is calculated forward (see the paragraph after Equations (3) to (9)), the bounds to be placed on it are forward looking, and, therefore, calculated backward. That is, indeed, true for the saddle path that leads to the unstable steady state. By the same token, when no steady state exists in our model, there can be no debt at all, as all paths would be divergent.*

## 2.1 Existence

We now explore the existence of an unstable steady state and the comparative statics of it, under the assumption that the utility function of the private sector is isoelastic ($u(c) = c^{1-\zeta}/(1-\zeta); \zeta > 0$). The savings function is then explicit:

$$s(w_t, r_{t+1}) = w_t \frac{\beta^{\frac{1}{\zeta}}(1-\tau) - \theta(1+r_{t+1})^{-\frac{1}{\zeta}}}{\beta^{\frac{1}{\zeta}} + (1+r_{t+1})^{1-\frac{1}{\zeta}}} \tag{17}$$

More importantly, the wage $w_t$ appears in it as a factor. That allows us to recast the steady-state equation (10) with $k_{t+1} = k_t = k_{t-1} = k$, in a way that places the preference parameters on one side of the equation and the production parameters on the other side, provided one puts $r$ in the role of the unknown variable:

$$s(1, r) \times \frac{n-r}{1+n} = \kappa(r)\frac{n-r}{w(\kappa(r))} - \tau + \frac{\theta}{1+n} \tag{18}$$

where $\kappa(r)$ is defined as the function inverse of $f'(k) - \delta$ (a decreasing function). Equivalently, since $r = n$ is evidently not a solution as long as $-\tau + \theta/(1+n) \neq 0$, we can write the above as:

$$s(1, r) \times \frac{1}{1+n} = \kappa(r)\frac{1}{w(\kappa(r))} + \frac{-\tau + \frac{\theta}{1+n}}{n-r} \tag{19}$$

Savings divided by wage is equal to capital over wage plus government debt over wage. The function $s(1, r)$ depends on preference parameters $\beta$ and $\zeta$ (and fiscal parameters $\tau$, $\theta$) while $\kappa(r)$ and $w(\kappa(r))$ depend only on production parameters, and $-\tau + \theta/(1+n)$ is a composite budget deficit parameter.

We restrict our search to the class of steady states in which the rate of interest $r$ satisfies $r < n$. The reason we invoke for this restriction is relevance to the current government-debt situation of developed countries. Referring to steady-state equation (15) above, these countries run a deficit ($d > 0$) and their debt has positive market value ($g > 0$). It must be, therefore, that $r < n$, with the result that the three terms of Equation (19) are all positive.[26]

When is there a steady state in that class? The answer to the question depends very much on the EIS $1/\zeta$ and on the ETS $\eta = 1/(1-\rho)$ between capital and labor.

[26] Of course, we cannot assume that these countries are in a steady state. The assumptions we make is that the path of the rate of interest followed by an economy cannot get above $n$ when initial conditions are $r_0 < n$ and $g_0$ or $g_1$ is below debt capacity. We verify these assumptions when we find a solution $r$ below $n$.

The savings function (17) on the left-hand side of formulation (19) is well-known to be monotonic in the rate of interest. It is monotonically increasing when the EIS is greater than 1, so that the substitution effect dominates the income effect of the rate of interest, and monotonically decreasing otherwise. As the EIS changes, the graph of the function in the $(r, s)$ plane pivots around the point $(1/\beta - 1, (1 - \tau - \theta) / (1 + \beta^{-1}))$.

Assume further that the production function is CES, as in Equation (1) above. For that production function, the ratio $w/k$ is equal to

$$(1-\alpha)\,\frac{1}{k}\,(1+\alpha\times(k^{\rho}-1))^{\frac{1-\rho}{\rho}}$$

which is a decreasing function of $k$. Hence, on the right-hand side of formulation (19), the first term is a decreasing function of $r$. The second term, however, rises sharply as $r$ approaches $n$ from below. As a result, the graph of the right-hand side goes through a minimum point. If and when the savings function of the left-hand side passes below that minimum, there does not exist a steady state. If and when the savings function of the left-hand side crosses the graph of the right-hand side above that minimum, there exist two steady states, in both of which $r < n$.

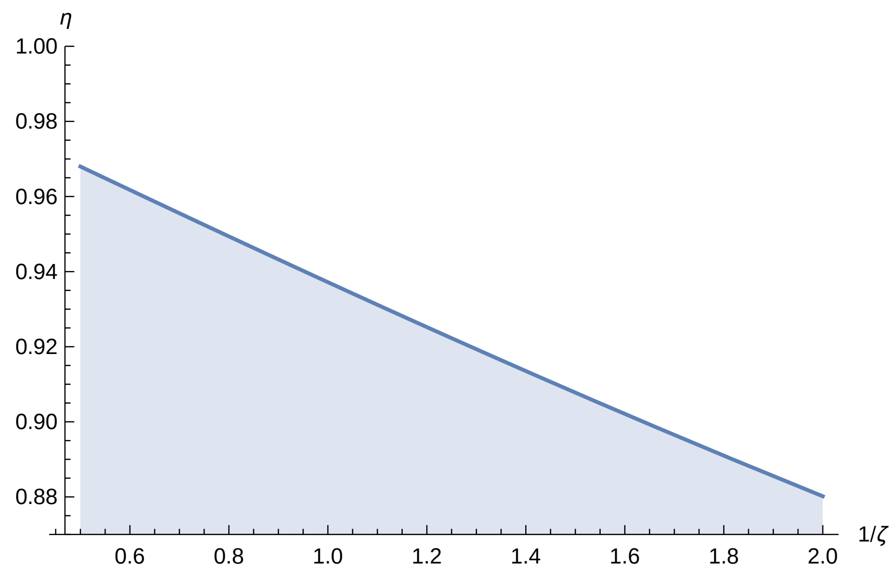


Figure 4: **Condition of existence of steady states with** $r < n$. For a given value of the elasticity of intertemporal substitution $1/\zeta$, the elasticity of technical substitution $\eta$ must not fall in the shaded area below the frontier. The utility function is the isoelastic and the production function is CES. Parameter values other than EIS and ETS are: $n = (1 + 0.02)^{25} - 1$, $\alpha = 0.2$, $\beta = 0.99^{25}$, $\delta = 1 - (1 - 0.1)^{25}$, $\theta = 0.165$, $\tau = 0.1$.

In Figure 4 we display the domain of existence of a steady state such that $r < n$, in the space of $1/\zeta$ (the EIS) on the $x$-axis and $\eta$ (the ETS) on the $y$-axis. We have chosen a range for the EIS that is empirically relevant (see Hall (1988), Attanasio and Weber (1995) and Kaltenbrunner and Lochstoer (2010)). Given the other parameter values indicated in the caption of the figure, the resulting minimal values of the ETS are not far below 1, which is the traditional Cobb-Douglas reference case. Below that frontier there does not exist any debt capacity; government debt is explosive or unsustainable for any initial debt or asset level.

### 2.2 Comparative statics

Within the domain of existence, we plot on the $x$-axis of the top plot in Figure 5 the values of the rate of interest $r$ at the locally stable (solid graph segments) and locally unstable (dotted graph segments) steady states for several values of the EIS and ETS. The value of the rate of interest at the stable steady state is a decreasing function of the ETS, the more so as the EIS is higher. As the ETS rises the value of the rate of interest at the unstable steady state rapidly approaches $r = n$, the more so as the EIS is higher.

In the bottom plot of Figure 5, we display (as the dotted graph segments) the corresponding values of the long-run debt capacity, which is the debt per capita at the unstable steady state. For low values of the ETS $\eta$, which are close to the frontier of non existence of a steady state, the debt capacity is not far from zero but it rises rapidly as the ETS rises toward its higher levels, the more so as the EIS is higher.

## 3 Policy experiments and demographic scenarios

We now make use of the concept of debt capacity to run policy experiments. If future policies that are stabilizing are anticipated, wrongly or rightly, debt may start above capacity on a seemingly explosive path. The stabilizing responses that are needed represent the true fiscal cost of exceeding debt capacity. We illustrate such scenarios in Figure 6. As before, the debt capacity per capita at the initial point in time is $g_1 = 0.0137$ (debt/annual output $= 52.45\%$), which is on the saddle path leading to the steady state marked U. If the debt starts above that level, it embarks on a seemingly explosive path that is rectified after one period of 25 years, by means of an increase in the wage tax rate $\tau$, in order to put the economy on another saddle path. Notice a very important effect of this rectification. The steady state that follows rectification features a rate of interest that is above the Golden rule rate equal to the rate of population growth $n = 2\%$/year. Correspondingly, the new steady-state primary budget is in surplus. The reason is that the initial steady state was close to $n$, leaving little room for extra debt. All the same, the delay in the response generally goes in the direction of forcing a higher tax increase.

In Table 1, we calculate the needed quantitative responses. When the response is immediate ($M = 0$), an initial debt equal to 114.61% of annual output requires a tax of 11.83%, which is above the 10% considered so far and leads to a steady-state rate of interest equal to 2.6%/year. When the response is delayed to the next generation ($M = 1$), the tax rate for the same initial debt must be raised to 12.81%, which leads to a much larger steady-state surplus and a larger steady-state interest equal to 2.83%/year. We see again that, as the government delays the policy response, the new tax rate required to prevent the government debt from exploding may be sufficiently high to cause a switch to $r > n$ and to a huge primary surplus.

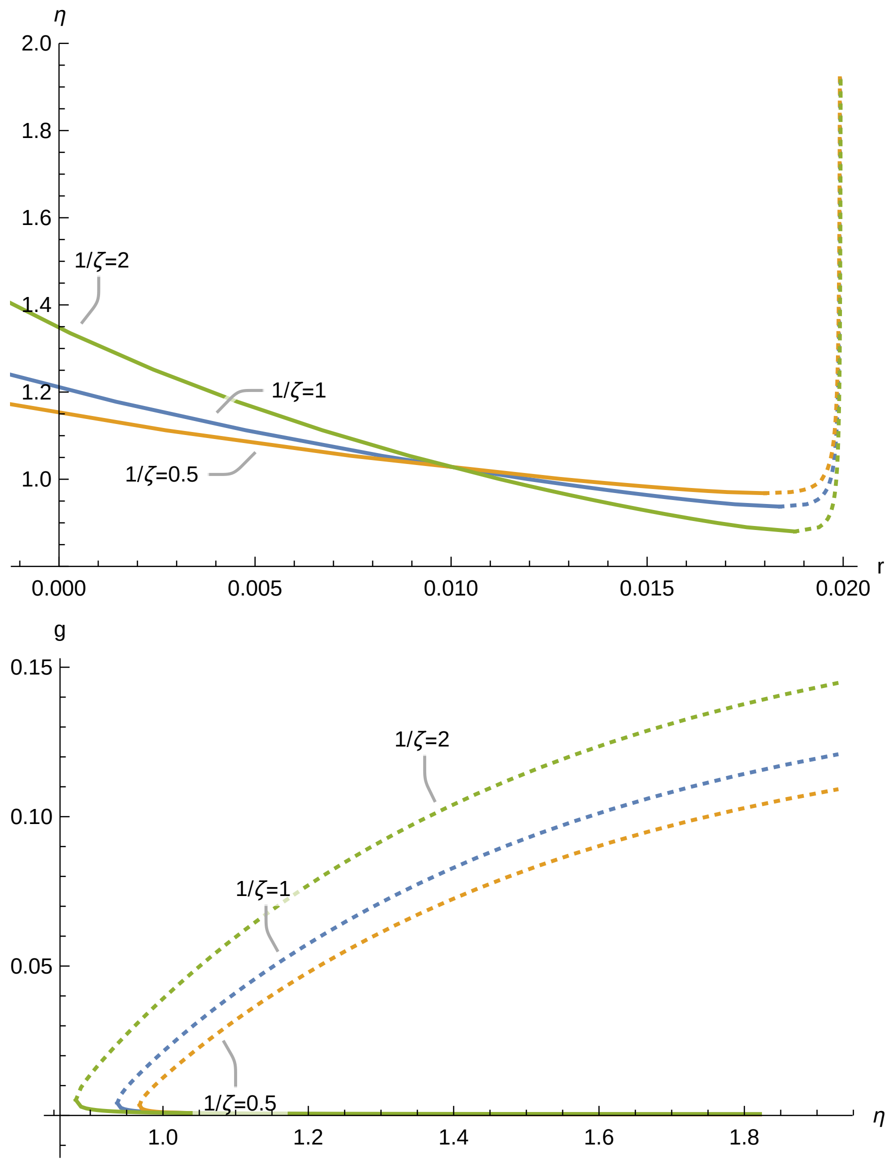


Figure 5: **Steady-state values of the yearly rate of interest $r$ (on the $x$-axis) of the top plot and debt per capita (bottom plot) for different values of the elasticity of intertemporal substition $1/\zeta$ and of the elasticity of technical substitution** $\eta = 1/(1-\rho)$. The solid graph segments show the values at the stable steady state and the dotted segments the values at the unstable steady state. The utility function is the isoelastic and the production function is CES. Parameter values other than EIS and ETS are: $n = (1+0.02)^{25} - 1$, $\alpha = 0.2$, $\beta = 0.99^{25}$, $\delta = 1 - (1-0.1)^{25}$, $\theta = 0.165$, $\tau = 0.1$.

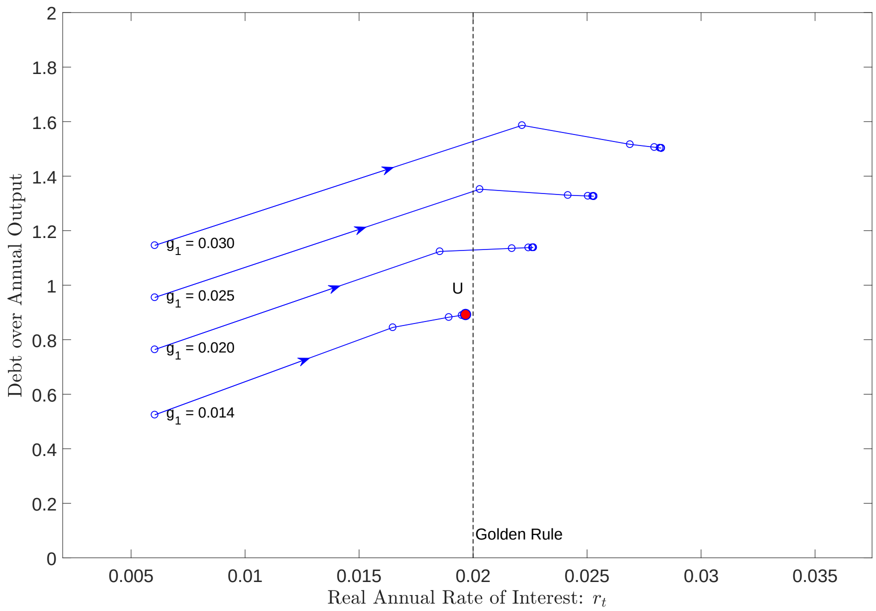


Figure 6: **The paths of the debt ratio and the rate of interest when debt starts above capacity and a policy response takes place at time** 2**.** Parameter values are: $n = (1+0.02)^{25} - 1$, $\alpha = 0.2$, $\beta = 0.99^{25}$, $\delta = 1-(1-0.1)^{25}$, $\theta = 0.165$, $\tau = 0.1$. Initial capital per capita is $k = 0.12$ with $r = 0.006$, which is a high amount of capital and a low rate of interest in the context of our model illustration. Debt starts on or above the debt-capacity saddle path ($g_1 = 0.0137$, which is a debt-output ratio equal to 0.5245). The policy response is indicated in the column labelled "$\tau$" in Table 1.

| | Initial debt per capita | Initial debt /annual output | $\tau$ | Steady-state $r$/year | Steady-state debt /annual output | Steady-state deficit /output |
|---|---|---|---|---|---|---|
| $M = 0$ | 0.0300 | 1.1461 | 0.1183 | 0.0260 | 1.3714 | −0.3552 |
| | 0.0250 | 0.9551 | 0.1111 | 0.0240 | 1.2394 | −0.2098 |
| | 0.0200 | 0.7641 | 0.1052 | 0.0221 | 1.0940 | −0.0934 |
| | 0.0137 | 0.5245 | 0.1000 | 0.0197 | 0.8917 | 0.0115 |
| $M = 1$ | 0.0300 | 1.1461 | 0.1281 | 0.0283 | 1.5030 | −0.5508 |
| | 0.0250 | 0.9551 | 0.1156 | 0.0253 | 1.3264 | −0.3007 |
| | 0.0200 | 0.7641 | 0.1068 | 0.0226 | 1.1385 | −0.1248 |
| | 0.0137 | 0.5245 | 0.1000 | 0.0197 | 0.8917 | 0.0115 |
| $M = 2$ | 0.0300 | 1.1461 | 0.1569 | 0.0336 | 1.7442 | −1.1263 |
| | 0.0250 | 0.9551 | 0.1264 | 0.0279 | 1.4821 | −0.5157 |
| | 0.0200 | 0.7641 | 0.1099 | 0.0236 | 1.2130 | −0.1856 |
| | 0.0137 | 0.5245 | 0.1000 | 0.0197 | 0.8917 | 0.0115 |

Table 1: **Tax rate, interest rate, and deficit responses to over-capacity initial debt.** Parameter values are: $n = (1+0.02)^{25} - 1$, $\alpha = 0.2$, $\beta = 0.99^{25}$, $\delta = 1-(1-0.1)^{25}$, $\theta = 0.165$, $\tau = 0.1$. Initial capital per labor is $k = 0.12$ with $r = 0.006$, which is a high amount of capital and a low rate of interest in the context of our model illustration. Debt starts on or above the debt-capapcity saddle path. The policy response can take place at the intitial point, $M = 0$, or with a delay, $M = \{1, 2\}$.

| | $\tau$ | Steady-state $r$/year | Steady-state debt /annual output | Steady-state deficit/output |
|---|---|---|---|---|
| | Initial drop to 1%/year | | | |
| $M = 0$ | 0.1288 | 0.0101 | 0.6990 | −0.0021 |
| $M = 1$ | 0.1339 | 0.0130 | 1.0458 | −0.104 |
| $M = 2$ | 0.1500 | 0.0180 | 1.5215 | −0.4275 |
| | Initial drop to 1.5%/year then to 1%/year | | | |
| $M = 0$ | 0.1301 | 0.0111 | 0.8206 | −0.0282 |
| $M = 1$ | 0.1399 | 0.0152 | 1.2721 | −0.2245 |
| $M = 2$ | 0.1673 | 0.0218 | 1.7919 | −0.7729 |

Table 2: **Tax rate, interest rate, debt and deficit responses to declining population growth.** Parameter values are: $n = (1+0.02)^{25} - 1$, $\alpha = 0.2$, $\beta = 0.99^{25}$, $\delta = 1-(1-0.1)^{25}$, $\theta = 0.165$, $\tau = 0.1$. Initial capital per labor is $k = 0.12$ with $r = 0.006$, which is a high amount of capital and a low rate of interest in the context of the model. Debt starts on the saddle path, which means at capacity (debt per capita $g_1 = 0.0137$, which is debt/annual output = 0.5245 ). At the initial point in time, or at the initial point and then again at the second point in time, the annual population growth rate drops to the levels indicated. The policy response can take place immediately, $M = 0$, or with a delay, $M = \{1, 2, 3\}$.

We also explain how debt could come to exceed debt capacity. Debt may become unsustainable, for instance, because of a drop in the growth rate of the population. For that reason, we turn to a major concern that one might have regarding the debt capacity of a government. In real life, population growth has been in decline in every single industrialized economy. In our model, when the population growth rate falls, the debt capacity shrinks and the economy may move to an exploding path, which, as we saw, would actually unravel. Here again, in case of explosion, the government could no longer sell its debt unless it increased taxes, or promised to do so, and thereafter embarked on a new saddle path.

To generalize the principle that the later the policy response, the larger it has to be, we develop scenarios of population-growth declines, starting debt exactly on the saddle path and varying the timing of the response. Specifically, we start with a high $k$ (0.12) and with a debt at capacity (debt over annual output equal to 52.45%). In the first scenario, the rate of population growth is 1%/year instead of 2%/year. The policy response can take place immediately ($M = 0$) or with a delay of several quarter centuries ($M = \{1, 2\}$). The top panel of Table 2 indicates the value to which the government must raise the contribution or tax rate $\tau$ in order to stay on a saddle path and avoid an unsustainable situation. It is clear from this panel that, if the government increases the tax rate with delay ($M > 0$), it must raise it more: changing the tax rate in the same period as the decline in $n$ requires an increase from 10% to 12.9% while two periods after the decline in the population growth rate an 15% tax is required.

In the second scenario (bottom panel of Table 2), the rate of population growth is 1.5%/year initially and drops 1%/year at the next generation. If this demographic evolution is anticipated by the government so that it acts right at the initial point in time, the tax needed for sustainability is only 13%, whereas, if it waits till the second drop in population growth, the tax must be as high as 16.7%. In both scenarios, the need arises in case of delay to generate a huge steady-state budget surplus.

The true fiscal cost of excessive government debt issuance should be assessed in a dynamic context reflecting anticipated deficits and population growth going forward. A switch to surpluses may be needed in the future, either through increased tax rates, reduced social-security benefits, or both, all being politically painful. Our model can arguably be interpreted as saying that it is better to implement a policy that reduces the debt automatically during normal times, as a way to aim towards the stable steady state, so that a safety margin remains in case growth drops.

# 4 Extensions

We have shown so far how debt capacity can be defined on the basis of a very simple OLG model with growth. In this section, we want to enrich the model, give it more policy substance and open the way towards future implementation. Specifically, we consider two additional forms of intervention by the government. First, we aim to increase our model's degree of realism by introducing nominal considerations. For that, we allow the government (not distinguished from the central bank) to issue nominal debt and to buy and sell it as a way to implement a form of monetary policy. Second, we ask whether a government can increase its debt capacity by subsidizing innovation which ultimately raises productivity and growth.

## 4.1 Money and the role of the central bank

One might be concerned that a high, potentially explosive level of nominal debt would produce high inflation. The prospect of the money vs. bond trades of the central bank, and of monetization of the debt, might modify the debt capacity of the government, because households must hold money whereas they choose freely to hold bonds.

To investigate these issues, we now assume realistically that the initial amount of debt, set by history, is contractually denominated as a nominal amount, this debt being now the consolidated debt of the government and the central bank. We allow the government to intervene in the money market, buying and selling bonds, as a way to fix the nominal rate of interest in accordance with a Taylor rule, in addition to collecting taxes and paying benefits. As before, government debt is a one-period debt. Let the nominal rate of interest be $i_t$.

Let $M_{1,t}$ be the young households' total money demand in *real* units (total nominal money balances deflated by the price level $P_t$) and $m_{1,t} = M_{1,t}/L_t$ is the *per capita* money demand in real units. They need it because they must turn into cash their wage, which is paid to a bank account. Cash can be withdrawn by taking trips to the bank. Each trip costs a fixed real amount $\nu$. Old households do not demand money: the social-security benefits are paid directly in cash. The per capita cost of trips to the bank is refunded to the young households, as a real *lump-sum amount* coming from outside resources and equal to:

$$\zeta_{1,t} = (1 - \tau) \times w_t \times \frac{\nu}{2 \times m_{1,t}}$$

The simultaneous budget constraints at time $t$ are as follows:
for the young household,

$$c_t^t + s_t + m_{1,t} = (1 - \tau) \times w_t \times \left(1 - \frac{\nu}{2 \times m_{1,t}}\right) + \zeta_{1,t} \tag{20}$$

for the old household,[27]

$$c_t^{t-1} = s_{t-1} \times (1 + i_t) \frac{P_{t-1}}{P_t} + m_{1,t-1} \frac{P_{t-1}}{P_t} + \theta \times w_{t-1} \tag{21}$$

for the government

$$-G_{t+1} - M_{2,t} + \theta w_{t-1} L_{t-1} = \tau w_t L_t - (1 + i_t) \frac{P_{t-1}}{P_t} G_t - \frac{P_{t-1}}{P_t} M_{2,t-1}$$

or,

$$-(1 + n) g_{t+1} - m_{2,t} + \theta \times w_{t-1} \frac{1}{1 + n} = \tau \times w_t - (1 + i_t) \frac{P_{t-1}}{P_t} g_t - \frac{P_{t-1}}{P_t} \frac{1}{1 + n} m_{2,t-1} \tag{22}$$

where $M_{2,t}$ is the total money supply in *real* units and $m_{2,t} = M_{2,t}/L_t$, $G_t$ is the total debt

[27] Please, observe that seignorage is not refunded.

in *real* units (total nominal debt deflated by $P_t$) with which the government enters time $t$ and $G_{t+1}$ is the debt in real units with which it exits time $t$, and $g_t \triangleq G_t/L_t$.

The behavior of the government is dictated by the following Taylor rule:

$$1 + i_{t+1} = (1 + \bar{\imath}) \times \left( \frac{\frac{P_{t+1}}{P_t}}{1 + \bar{\pi}} \right)^{\phi} ; \phi \geq 0; \phi \neq 1 \tag{23}$$

The first-order conditions of the firms (Equations (4) and (5)) are as they were in Section 1. As for the young households, they follow

$$\frac{\frac{\partial}{\partial c_{t+1}^t} U\left(c_t^t, c_{t+1}^t\right)}{\frac{\partial}{\partial c_t^t} U\left(c_t^t, c_{t+1}^t\right)} = \frac{1}{1 + i_{t+1}} \frac{P_{t+1}}{P_t} \tag{24}$$

$$m_{1,t} = \sqrt{\frac{(1 - \tau) \times w_t \times \left(\frac{\nu}{2}\right)}{1 - \frac{1}{1+i_{t+1}}}} \tag{25}$$

Given money demand (25), the savings decision that follows from condition (24) is again given by formula (17) but with constant parameters $\tau$ and $\theta$ replaced by variables $\tau_t^*$ and $\theta_t^*$ that absorb the effect of cash holdings including seignorage, and are defined as:

$$\tau_t^* \triangleq \tau + \frac{m_t}{w_t}$$

$$\theta_t^* \triangleq \theta + \frac{P_t}{P_{t+1}} \frac{m_t}{w_t}$$

Market clearing is still as it was in Section 1, but $m_{1,t} = m_{2,t}$ is an additional market-clearing equation. The initial conditions are set by the initial capital stock $k_0 = K_0/L_0$ and the amounts of *nominal* debt and money $G_0 \times P_0$ and $M_{2,0} \times P_0$ (these products are given numbers) with which the government leaves time 1.

**Example 5.** *(Example 1 continued with the addition of money) The resulting difference equations system, made of Equations (12) and (25) with $\tau$ and $\theta$ replaced by variables $\tau_t^*$ and $\theta_t^*$, can be simulated. The resulting paths are shown in Figure 7 displaying the ratio of debt $g$ to annual output $(y/25)$ vs. the nominal rate of interest.*

If ever debt exceeded debt capacity and the real rate were on a putative explosive path,[28] the capital stock would approach zero and the real rate of interest would rise forever, as in the model without money. When $\phi > 1$, the nominal rate of interest and, with it, the rate of inflation would increase forever, in a form of hyperinflation. When $\phi < 1$, the opposite would be true.[29] This shows that the degree of reaction of monetary policy makes a massive inflationary difference in the behavior of the economy. Here, we depart in an important way from models that rely on linearization, which conclude (this is the "Taylor principle") that $\phi > 1$ is necessary for local and, therefore, global stability. As noted above, our analysis of sustainability is not based on the existence of a local steady state.

[28]Here again, explosive paths will unravel.

[29]In both cases, there would still exist a stable steady state. But that is not our focus at this point.

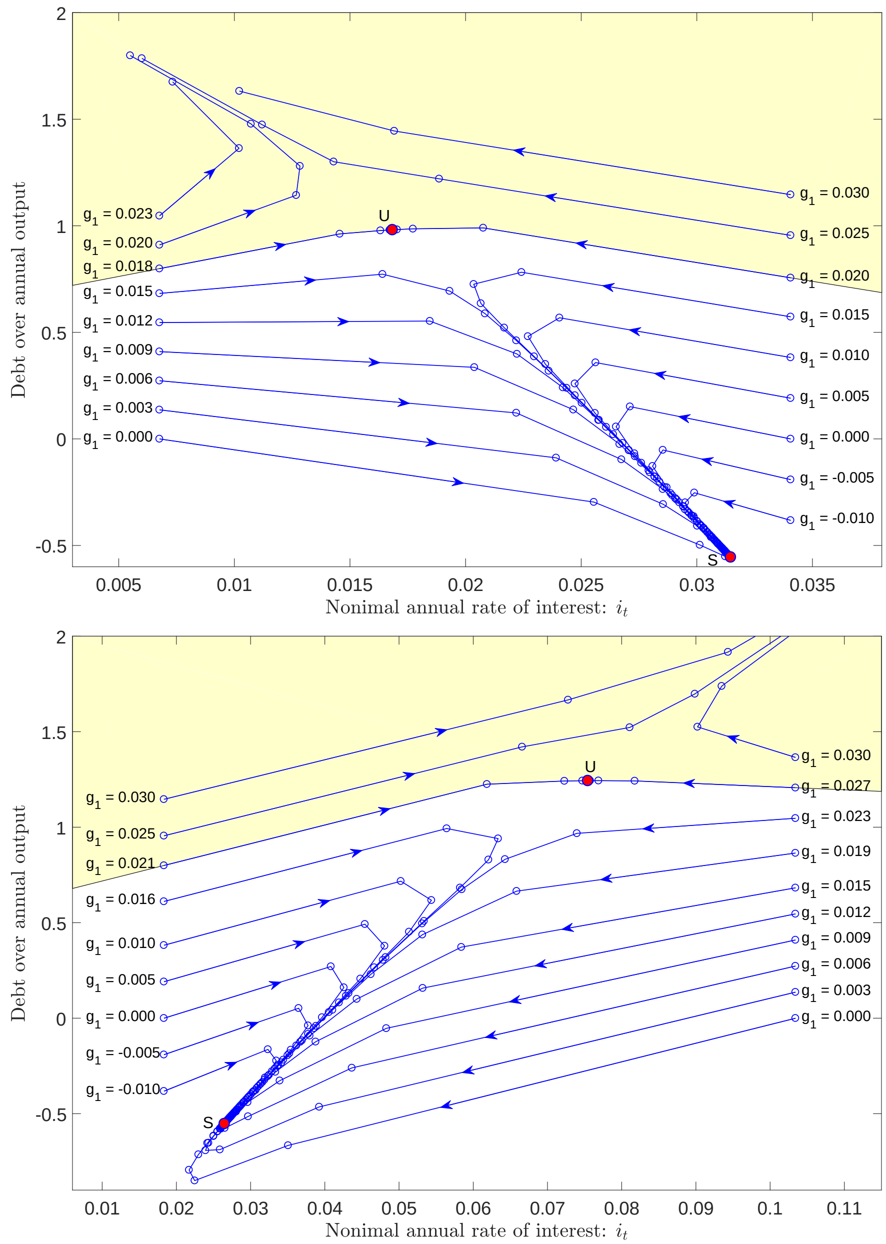


Figure 7: **The paths of the debt ratio and the nominal interest rate.** Illustration with log utility and Cobb-Douglas production function. Parameter values are: $n = (1+0.02)^{25}-1$, $\alpha = 0.2$, $\beta = 0.99^{25}$, $\delta = 1-(1-0.1)^{25}$, $\theta = 0.165$, $\tau = 0.1$, $\phi = 0.5$ (top plot) and $\phi = 1.5$ (bottom plot), $\bar{\imath} = 0.03$, $\bar{\pi} = 0.02$, $\bar{\imath} = 0.03$, $\nu = 0.0002$. The starting points $(k_1, g_1)$ of the paths (and the starting price level $P_0$) are derived from the true initial conditions $(k_0, G_0 \times P_0, M_{2,0} \times P_0)$. The stable steady-state is marked "S" and the unstable one "U".

Monetary policy also affects debt capacity, as displayed in Tables 3 and 4, in which we calculate capacity at the unstable steady state and at the initial point, respectively. At a steady state, all per-capita quantities are constants (let $1+\pi \triangleq P_{t+1}/P_t$); therefore, we can drop the time script in Equations (12) (written with $\tau^*$ and $\theta^*$):

$$\begin{aligned} \frac{g}{w} &= \frac{1}{n-r}\left(\frac{\theta^*}{1+n} - \tau^*\right) \\ \frac{1}{1+n} s\left(1, r, \tau^*, \theta^*\right) &= \frac{k}{w} + \frac{g}{w} \end{aligned}$$

where:

$$\begin{aligned} \tau^* &= \tau + \frac{m}{w} \\ \theta^* &= \theta + \frac{1}{1+\pi}\frac{m}{w} \\ m &= \left\{\frac{1}{2}\nu\left(1-\tau\right) w \left[1 - \frac{1}{1+\bar{\imath}}\left(\frac{1+\pi}{1+\bar{\pi}}\right)^{-\phi}\right]^{-1}\right\}^{\frac{1}{2}} \end{aligned}$$

Combining these equations, we obtain the steady-state condition for the real rate of interest $r$:

$$\frac{1}{1+n} s\left(1, r, \tau^*, \theta^*\right) = \frac{k}{w} + \frac{1}{n-r}\left(\frac{\theta^*}{1+n} - \tau^*\right)$$

For the Cobb-Douglas production function, the steady-state condition reads:

$$\frac{1}{1+n} s\left(1, r, \tau^*, \theta^*\right) = \frac{1}{r+\delta}\frac{\alpha}{1-\alpha} + \frac{1}{n-r}\left(\frac{\theta^*}{1+n} - \tau^*\right)$$

so that

$$\frac{g}{y} = \underbrace{\frac{1}{n-r}}_{g/d} \underbrace{\underbrace{\left(\frac{\theta^*}{1+n} - \tau^*\right)}_{d/w} \underbrace{\left(1-\alpha\right)}_{w/y}}_{d/y}$$

Money balances affect debt capacity through two channels. Implicitly, cash holding $m$ affects capital accumulation. Explicitly, it affects debt capacity $g/y$ through the term $d/w$, which contains seignorage revenues:

$$\frac{d}{w} = \frac{\theta^*}{1+n} - \tau^* = \underbrace{\left(\frac{\theta}{1+n} - \tau\right)}_{d/w \text{ without cash}} + \underbrace{\left(\frac{1}{\left(1+n\right)\left(1+\pi\right)} - 1\right)\frac{m}{w}}_{\text{cash effect}}$$

The top panel of Table 3 shows for $\phi > 1$ how money affects the unstable steady state: it increases markedly the steady-state debt capacity for all the values displayed of the cost parameter $\nu$ and of the policy parameter $\bar{\imath}$. This is true whether the government debt ratio is viewed as being $g$/annual $y$ or $(m + g)$ /annual $y$. For instance, for $\phi = 1.5$, $\nu = 0.0002$ and $\bar{\imath} = 0.01$, the debt to annual output ratio goes from 89% in the economy without money ($\nu = 0$) to 128% in the economy with money, while debt plus money goes from 89% to 157%. This is because the government budget turns from a deficit of 1% to a surplus of 25% arising from seignorage revenues. The nominal rate of interest is increased moderately while the real rate of interest is increased very moderately under the effect of money hoarding, which is crowding out physical capital a little.

Money also affects debt capacity at time 1 differently depending on the value of the initial capital stock ($k_1 = 0.05$ vs. $k_1 = 0.12$) with corresponding real interest rate. The effect is in line with the steady-state effects that we just outlined. See Table 4. For instance, for $\phi = 1.5$, $\nu = 0.0002$ and $\bar{\imath} = 0.01$, debt over annual output goes from 52% to 97% for the higher value of the initial capital stock, and from 93% to 123% for the lower one.

When $\phi < 1$, debt capacity at the unstable steady state is again increased by the presence of money but less strongly than when $\phi > 1$. For instance, for $\phi = 0.5$, $\nu = 0.0002$ and $\bar{\imath} = 0.03$,[30] the debt to output ratio goes from 89% in the economy without money ($\nu = 0$) to 98% in the economy with money, while debt plus money goes from 89% to 146%. Again the government budget turns from a deficit to a surplus arising from seignorage revenues. The nominal rate of interest is reduced while the real rate of interest is increased very moderately under the effect of money hoarding.

The effect at time 1 is once again in line with the steady-state effects. See Table 4. For instance, for $\phi = 0.5$, $\nu = 0.0002$ and $\bar{\imath} = 0.03$, debt over annual output goes from 52% to 76% for the higher value of the initial capital stock, and from 93% to 80% for the lower one.

## 4.2 Innovation

In the model developed so far, the only source of perpetual growth is population growth (or exogenous technical progress). However, in today's economy, productivity, thanks to innovation, keeps rising. As a second extension, we would like to know whether the government can increase its debt capacity by subsidizing innovation and fostering growth. To determine to what extent innovation can modify a government's debt capacity, we borrow a model from the literature on endogenous growth but let the government finance R&D, in addition to paying for social security. Strictly speaking, growth by innovation is endogenous when R&D is decided by the private sector. Our focus, however, is on government expenditure. We choose a model of innovation that allows policy to have a direct effect on growth.[31]

The modifications to the specification of the economy are as follows.

**The households/investors:** intermediate goods are produced in varieties $i$, the cardinal number $B_t$ of which grows like population ($B_t = L_t$). This assumption captures the fact

[30] When $\phi = 0.5$, existence of the equilibrium demands higher values of the parameter $\bar{\imath}$ than in the case $\phi = 1.5$.

[31] See Aghion and Howitt (1998), Dinopoulos and Thompson (1998), Peretto (1998), and Young (1998). We use a reduced form formulated by Jones (1999) that is also used in the empirical paper of Laincz and Peretto (2006). Ferraro and Peretto (2020) develop a model of quality-improving innovation and firm entry and exit that allows them to study the evolution of government debt, like we do. However, they consider the case of no physical capital and of infinitely-lived households that satisfy a no-Ponzi scheme requirement. They find one steady state.

| $\phi = 1.5$ | | | | | | | |
|---|---|---|---|---|---|---|---|
| | $\bar{\imath} = 0.01$ | | | $\bar{\imath} = 0.02$ | | $\bar{\imath} = 0.03$ | |
| | $\nu = 0$ | $\nu = 0.0002$ | | $\nu = 0$ | $\nu = 0.0002$ | $\nu = 0$ | $\nu = 0.0002$ |
| $m$ / annual $y$ | 0 | 0.2873 | | 0 | 0.2933 | 0 | 0.3042 |
| cost / annual $y$ | NaN | 0.2702 | | NaN | 0.2645 | NaN | 0.2548 |
| $g$ / annual $y$ | 0.8917 | 1.2778 | | 0.8917 | 1.2653 | 0.8917 | 1.2434 |
| $d$ / annual $y$ | 0.0115 | −0.2566 | | 0.0115 | −0.2496 | 0.0115 | −0.2376 |
| $(m+g)$ / annual $y$ | 0.8917 | 1.5651 | | 0.8917 | 1.5586 | 0.8917 | 1.5476 |
| annual $r$ | 0.0197 | 0.0247 | | 0.0197 | 0.0246 | 0.0197 | 0.0245 |
| annual $i$ | 0.1029 | 0.1194 | | 0.0814 | 0.0973 | 0.0603 | 0.0754 |
| $\phi = 0.5$ | | | | | | | |
| | $\bar{\imath} = 0.03$ | | | $\bar{\imath} = 0.04$ | | $\bar{\imath} = 0.05$ | |
| | $\nu = 0$ | $\nu = 0.0002$ | | $\nu = 0$ | $\nu = 0.0002$ | $\nu = 0$ | $\nu = 0.0002$ |
| $m$ / annual $y$ | 0 | 0.4744 | | 0 | 0.3637 | 0 | 0.3235 |
| cost / annual $y$ | NaN | 0.1619 | | NaN | 0.2122 | NaN | 0.2392 |
| $g$ / annual $y$ | 0.8917 | 0.9810 | | 0.8917 | 1.1374 | 0.8917 | 1.2066 |
| $d$ / annual $y$ | 0.0115 | −0.1260 | | 0.0115 | −0.1857 | 0.0115 | −0.2185 |
| $(m+g)$ / annual $y$ | 0.8917 | 1.4554 | | 0.8917 | 1.5011 | 0.8917 | 1.5301 |
| annual $r$ | 0.0197 | 0.0231 | | 0.0197 | 0.0239 | 0.0197 | 0.0243 |
| annual $i$ | 0.0202 | 0.0168 | | 0.0399 | 0.0357 | 0.0600 | 0.0553 |

Table 3: **Debt capacity with money at the unstable steady state.** Illustration with log utility and Cobb-Douglas production function. Parameter values are: $n = (1+0.02)^{25}-1$, $\alpha = 0.2$, $\beta = 0.99^{25}$, $\delta = 1-(1-0.1)^{25}$, $\theta = 0.165$, $\tau = 0.1$ and $\bar{\imath} = 0.03$ and $\bar{\pi} = 0.02$. The starting points $(k_1, g_1)$ of the paths (and the starting price level $P_0$) are derived from the true initial conditions $(k_0, G_0 \times P_0, M_{2,0} \times P_0)$.

| $\phi = 1.5$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $\bar\imath = 0.01$ | | | $\bar\imath = 0.02$ | | | $\bar\imath = 0.03$ | |
| $\nu = 0$ | $k_1 = 0.05$ | $k_1 = 0.12$ | | $k_1 = 0.05$ | $k_1 = 0.12$ | | $k_1 = 0.05$ | $k_1 = 0.12$ |
| saddle $g_1$ | 0.020471 | 0.013729 | | 0.020471 | 0.013729 | | 0.020471 | 0.013729 |
| saddle $g_1$ / annual $y_1$ | 0.931720 | 0.524497 | | 0.931720 | 0.524497 | | 0.931720 | 0.524497 |
| saddle $m_1$ / annual $y_1$ | 0 | 0 | | 0 | 0 | | 0 | 0 |
| $\nu = 0.0002$ | $k_1 = 0.05$ | $k_1 = 0.12$ | | $k_1 = 0.05$ | $k_1 = 0.12$ | | $k_1 = 0.05$ | $k_1 = 0.12$ |
| saddle $g_1$ | 0.026981 | 0.025377 | | 0.026808 | 0.024051 | | 0.026511 | 0.020929 |
| saddle $g_1$ / annual $y_1$ | 1.228034 | 0.969492 | | 1.220124 | 0.918837 | | 1.206619 | 0.799574 |
| saddle $m_1$ / annual $y_1$ | 0.293928 | 0.273720 | | 0.299228 | 0.281976 | | 0.308679 | 0.297504 |
| $\phi = 0.5$ | | | | | | | | |
| | $\bar\imath = 0.03$ | | | $\bar\imath = 0.04$ | | | $\bar\imath = 0.05$ | |
| $\nu = 0.0002$ | $k_1 = 0.05$ | $k_1 = 0.12$ | | $k_1 = 0.05$ | $k_1 = 0.12$ | | $k_1 = 0.05$ | $k_1 = 0.12$ |
| saddle $g_1$ | 0.017554 | 0.019787 | | 0.023343 | 0.023070 | | 0.025129 | 0.024728 |
| saddle $g_1$ / annual $y_1$ | 0.798957 | 0.755931 | | 1.062419 | 0.881348 | | 1.143719 | 0.944707 |
| saddle $m_1$ / annual $y_1$ | 0.519752 | 0.413702 | | 0.382012 | 0.332045 | | 0.335905 | 0.299802 |

Table 4: **Debt capacity with money at the initial time 1.** Illustration with log utility and Cobb-Douglas production function. Parameter values are: $n = (1+0.02)^{25} - 1$, $\alpha = 0.2$, $\beta = 0.99^{25}$, $\delta = 1 - (1-0.1)^{25}$, $\theta = 0.165$, $\tau = 0.1,$, $\bar\imath = 0.03$ and $\bar\pi = 0.02$. The starting points $(k_1, g_1)$ of the paths (and the starting price level $P_0$) are derived from the true initial conditions $(k_0, G_0 \times P_0, M_{2,0} \times P_0)$.

that more people generate more varieties; see Jones (1999). For continuity with the previous specifications, in which the population growth rate $n$ stood for all exogenous forms of growth, we now reduce that number to make room for endogenous growth.[32]

Before they can be consumed or reinvested, varieties of intermediate goods are turned into a final good, the amount produced being $Y_t$:

$$Y_t = \left( \int_0^{B_t} X_{i,t}^{\frac{1}{\theta_Y}} \, di \right)^{\theta_Y}$$

where $\theta_Y > 1$, $X_{it}$ is the input of each variety of intermediate good and $\int_0^{B_t} di = B_t$. We look for an equilibrium that is symmetric across varieties: $\int_0^{B_t} X_{i,t} di = B_t X_{i,t} = X_t$. In other words, total production of intermediate goods is $X_t = B_t X_{i,t}$.

**The production function** for each variety $i$ is

$$X_{i,t} = A_t F\left(K_{i,t}, \Lambda_{i,Y,t}\right)$$

where $K_{i,t}$ and $\Lambda_{i,Y,t}$ are the inputs of physical capital and labor into the production process of variety $i$, $A_t > 0$ is knowledge capital applicable in a non rival way to the production of

[32]That is, we solve for a new number $n$ such that, when research activity is at a zero level, the overall growth rate, including population growth and varieties growth, remains what it was before. See Footnote 33.

all varieties. With symmetric use of capital and labor for the production of each variety,

$$X_{i,t} = A_t \frac{F(K_t, \Lambda_{Y,t})}{B_t}$$

so that

$$\begin{aligned} X_t &= A_t F(K_t, \Lambda_{Y,t}) \\ Y_t &= \left( B_t X_{i,t}^{\frac{1}{\theta_Y}} \right)^{\theta_Y} = B_t^{\theta_Y} X_{i,t} = B_t^{\theta_Y - 1} A_t F(K_t, \Lambda_{Y,t}) \end{aligned}$$

**The production and accumulation of knowledge capital:** is controlled by government expenditure. It evolves as

$$A_{t+1} - A_t = \frac{\theta_A L_{A,t} A_t}{B_t}$$

where $\theta_A > 0$ is the productivity of labor in knowledge production and $L_{A,t}$ is the amount of labor devoted by the government to knowledge production. As the number of varieties rises, more research labor is required to increase knowledge.

**Taxation and spending**: the government budget constraint becomes

$$-G_{t+1} + \theta w_{t-1} L_{t-1} + w_t L_{A,t} = \tau w_t L_t - (1 + r_t) G_t$$

**Market clearing:** the labor market clears

$$\Lambda_{Y,t} + L_{A,t} = L_t$$

and the market for final goods clears

$$L_t c_t^t + L_{t-1} c_t^{t-1} + K_{t+1} = B_t^{\theta_Y - 1} A_t F(K_t, \Lambda_{Y,t}) + (1 - \delta) \times K_t$$

**Difference equations and steady states:** Suppose that the government pays for a constant proportion $s_A$ of labor to be involved in R&D: $L_{A,t} = s_A \times L_t$. With that, the difference-equations system governing the evolution of the economy, stated on a per capita basis, still contains Equations (3,6,7) together with:

$$B_t^{\theta_Y-1}A_t f'\left(\frac{k_t}{1-s_A}\right)-\delta=r_t \tag{26}$$

$$B_t^{\theta_Y-1}A_t\left[f\left(\frac{k_t}{1-s_A}\right)-\frac{k_t}{1-s_A}f'\left(\frac{k_t}{1-s_A}\right)\right]=w_t \tag{27}$$

$$-(1+n)\,g_{t+1}+\theta w_{t-1}\frac{1}{1+n}+w_t s_A=\tau w_t-(1+r_t)\,g_t \tag{28}$$

$$c_t^t+\frac{1}{1+n}c_t^{t-1}+(1+n)\,k_{t+1}=(1-s_A)\,B_t^{\theta_Y-1}A_t f\left(\frac{k_t}{1-s_A}\right)+(1-\delta)\times k_t \tag{29}$$

$$A_{t+1}-A_t=\frac{\theta_A L_{A,t}A_t}{B_t} \tag{30}$$

Equations (26) and (27) allow us to define $r_t$ and $w_t$ as functions $r\,(k_t,A_t,B_t)$ and $w\,(k_t,A_t,B_t)$. The savings function $s_t=s\,(w_t,r_{t+1})$ is unchanged. Proceeding to equate demand and supply, as we did in Section 1, we get an equation relating $k_{t+1}$ to $k_t$ and $k_{t-1}$

$$\begin{aligned} s\left[w\left(k_t,A_t,B_t\right),r\left(k_{t+1},A_t,B_t\right)\right]=\left[\begin{array}{ccc}1 & \frac{1}{1+n} & -1\end{array}\right] \\ \times\left[\begin{array}{c}(1-\tau)\,w\left(k_t,A_t,B_t\right)\\ s\left[w\left(k_{t-1},A_{t-1},B_{t-1}\right),r\left(k_t,A_t,B_t\right)\right]\times\left(1+r\left(k_t,A_t,B_t\right)\right)+\theta\times w\left(k_{t-1},A_{t-1},B_{t-1}\right)\\ (1-s_A)\,B_t^{\theta_Y-1}A_t f\left(k_t\frac{1}{1-s_A}\right)+(1-\delta)\,k_t-(1+n)k_{t+1}\end{array}\right]\end{aligned} \tag{31}$$

The evolution of the debt follows from Equation (28).

Given the similarity of equation system (31) with the previous one (10), one can safely state that, under similar conditions, there will be again two steady states, in each of which, however, growth per capita is no longer zero. That is, there are two "expansion paths" with the same growth rate (see below the calculation of the growth rates) but differing interest rates. One of them is stable as all paths that start in the debt-capacity region (to be determined) approach it; the other is unstable as all paths that do not start within the debt-capacity region diverge from it.

We turn to the calculation of growth rates on a steady-state path. The stock of knowledge capital $A_t$ evolves autonomously as does the population. From Equation (30), for a constant policy $s_A$, its growth rate, denoted $\varpi_A$, is equal to $\theta_A s_A$ (independently of a steady-state assumption). For a Cobb-Douglas production function, Equation (26) says that $B_t^{\theta_Y-1}A_t\left[k_t/\left(1-s_A\right)\right]^{\alpha-1}$ is constant:

$$(1+\varpi_A)\,(1+n)^{\theta_Y-1}\,(1+\varpi)^{\alpha-1}=1$$

which gives the steady-state rate of growth $\varpi$ of capital per capita $k_t$ (a rate which, in

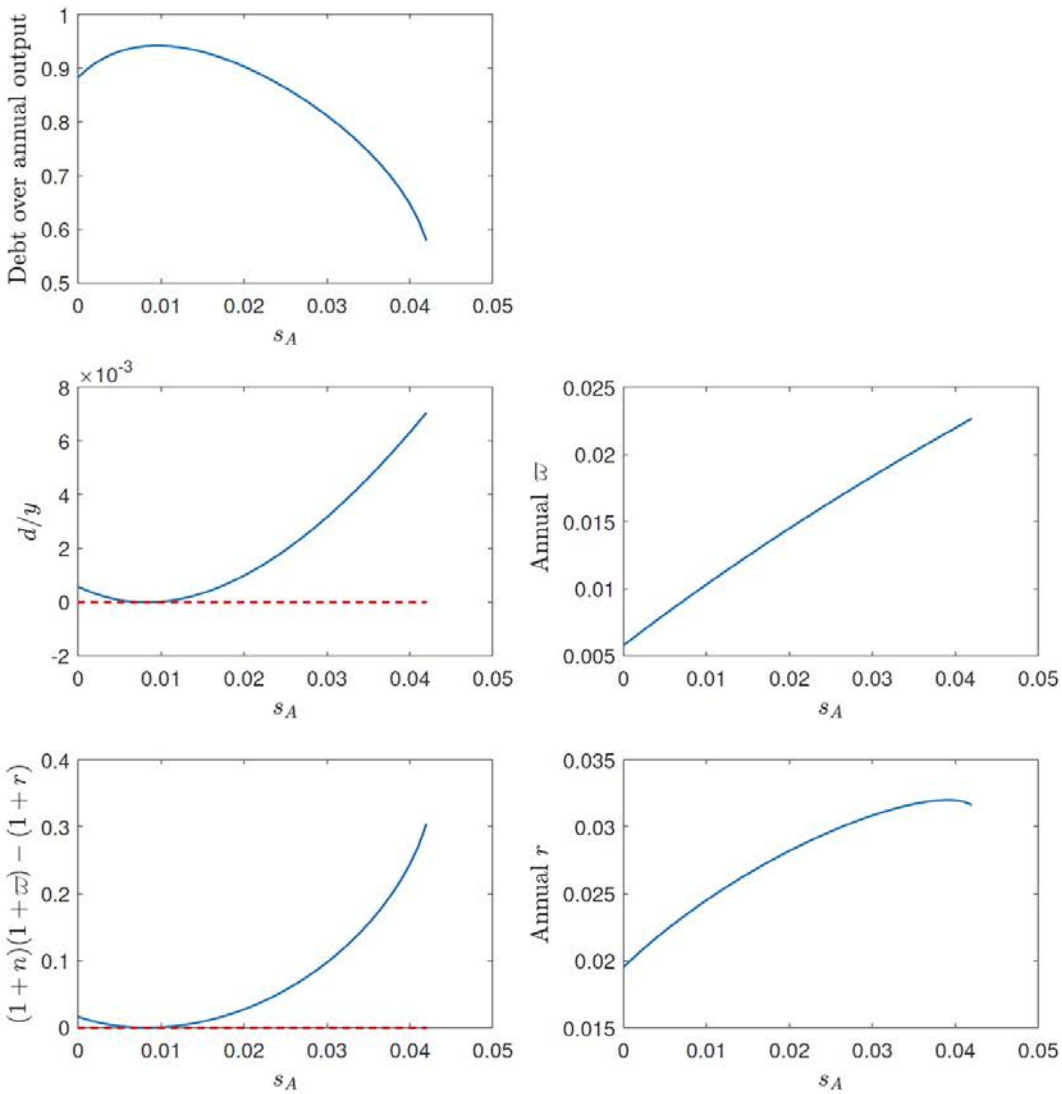


Figure 8: **Debt capacity over annual output, deficit over output, growth condition, rate of growth per year and rate of interest per year as a function of $s_A$ at the unstable steady state.** Illustration with log utility and Cobb-Douglas production function. The lines stop at value of $s_A$ for which the unstable steady-state does not exist. In these examples we set the population growth rate such that the compounded growth rate is at 0.02year for $s_A = 0$; see footnote 33. We use $\alpha = 0.2$ and $\theta_Y = 4/3$ implying $n_{new} = 0.01408/year$ (0.4183 over 25 years), and $\theta_A = 9.4$. The other parameter values are identical to what they are in the other figures.

previous sections, was equal to 0)[33]

$$1+\varpi=\left[\left(1+\varpi_A\right)(1+n)^{\theta_Y-1}\right]^{\frac{1}{1-\alpha}}=\left[\left(1+\theta_A s_A\right)(1+n)^{\theta_Y-1}\right]^{\frac{1}{1-\alpha}}$$

Output per capita $y$, debt per capita $g$, deficit per capita $d$ and the wage rate $w$ all grow at that same rate at any steady state.

As we did before in the case of a Cobb-Douglas production function, where:

$$\begin{aligned} r\left(k_t, A_t, B_t\right) &= B_t^{\theta_Y-1} A_t \alpha\left(\frac{k_t}{1-s_A}\right)^{\alpha-1}-\delta \\ w\left(k_t, A_t, B_t\right) &= B_t^{\theta_Y-1} A_t \times\left(\frac{k_t}{1-s_A}\right)^{\alpha}-\frac{k_t}{1-s_A}\left[r\left(k_t, A_t, B_t\right)+\delta\right] \end{aligned}$$

we can write the equation for the steady-state interest rate $r$ (analogous to Equation (19))

$$\begin{aligned} \frac{1}{(1+n)(1+\varpi)} s(1, r) &= \frac{1-s_A}{r+\delta} \frac{\alpha}{1-\alpha} \\ &+\frac{1}{(1+n)(1+\varpi)-(1+r)}\left(\frac{\theta}{(1+n)(1+\varpi)}-\left(\tau-s_A\right)\right) \end{aligned} \tag{32}$$

and calculate the steady-state debt-capacity ratio, which is debt per output at the unstable steady-state, as we did before in Equation (16):

$$\frac{g}{y}=\underbrace{\frac{1}{(1+n)(1+\varpi)-(1+r)}}_{\text{total growth rate minus interest rate}} \underbrace{\left(\frac{\theta}{(1+n)(1+\varpi)}-\left(\tau-s_A\right)\right)}_{d/w} \underbrace{\frac{1-\alpha}{1-s_A}}_{w/y} \tag{33}$$

Equation (33) relates debt over output to deficit over output, where the deficit in the form $d/w$ is adjusted by $1-\alpha$, as before but also by $1/\left(1-s_A\right)$ for the fact that some labor is diverted from the production of goods to the production of knowledge. That deficit is discounted, with the inclusion of a bubble, in a manner that is analogous to formula (15) of the basic model. The denominator is positive when the debt contains a bubble. We view the sign of the denominator as a "growth condition". One would expect two opposing effects: R&D enhances growth but deepens the deficit of the government. Specifically, one would hope for a hump shaped relation where initially increasing $s_A$ the growth effect of an increase in R&D dominates and thus the debt capacity increases. For some large enough value of $s_A$ the deficit would then increase faster than growth implying that the debt capacity declines.

The plots in Figure 8 confirm this intuition. We display the way debt capacity changes as one varies the policy parameter $s_A$. The graphs are drawn for $\theta_A=9.4$ (the degree to

[33]In these equations, as explained above, $n$ is reduced to a number $n_{new}$ such that

$$\left(1+\left.\varpi\right|_{s_A=0}\right)\left(1+n_{new}\right)=1+n_{old}$$

$$\left(1+n_{new}\right)^{\frac{\theta_Y-1}{1-\alpha}+1}=1+n_{old}$$

For $1+n_{old}=(1.02)^{25}$, $n_{new}=0.4183$ ( over 25 years), which is $0.01408/year$.

which the growth of knowledge responds to R&D labor input) and for $\theta_Y = 1.33$, which corresponds to an elasticity of substitution between varieties equal to 4, a number accepted often by macroeconomists (see, for instance, Galí (2015)). The behavior of debt capacity against $s_A$ is the result of several competing effects. First, in the denominator, an increase in R&D spending opens a race between the total growth rate $(1+n)(1+\varpi)-1$ and the rate of interest $r$, both of which increase with $s_A$, as the right-hand panels indicate. Second, in the deficit, which is the numerator, the race is between the rate of growth, which lightens the burden of benefits paid to the old and the rate of spending on R&D. We see that the steady-state debt capacity rises from 89.17% (see Example 4) to a highest value of about 94% for about the value of $s_A$ that minimizes the deficit (second panel on the left-hand side) and also, approximately, for the value that minimizes the discount factor in the denominator (third panel on the left-hand side).[34]

Overall, we do see a hump-shaped relation between $g$/annual $y$ and $s_A$ but the hump occurs for small values of $s_A$ of about 1%. Most high-income countries already spend more than 2% on R&D. Therefore, for most parameter configurations, steady-state debt capacity is not increased, or is even reduced, by an increase in public R&D spending beyond what it is already. Overall, this exercise does not show that public R&D spending miraculously lifts debt capacity. It remains conceivable that spending on education would have a larger impact.[35]

# 5 Conclusion

In an overlapping-generations economy with capital accumulation and a realistic social-security scheme, where debt covers deficits from the scheme, debt is welfare improving and can have positive market value even if the government budget is forever in deficit. This is because government debt contains a rational bubble. Of two steady states we found, the unstable one has higher debt to GDP ratio and is closer to the Golden-rule economy. In that sense, it appears to be a good idea to let the debt rise if it is not already at its capacity level.

We have shown that, even when the cost of financing is very low, one cannot push the level of debt beyond some amount. We have defined debt capacity as the level of debt that leads to an unstable steady state. Whenever the market value of debt is below debt capacity, the debt converges to a stable steady state. If it is above, it is unsustainable. We have followed the economy along an explosive path and shown that government debt crowds out physical capital to extinction, so that by anticipation such paths actually unravel, which means that debt is unsustainable.

Steady states, however, may not exist. When none exists, there is no capacity for debt. We have explored the issue of existence among equilibria for which the real rate of interest is below the real rate of growth, in accordance with the situation in today's world. And,

[34]For $9.4 < \theta_A \leq 1$, there exists a range of values of $s_A$ over which the deficit turns into a surplus and the denominator $(1+n)(1+\omega)-(1+r)$ switches sign, indicating an absence of bubble. Over that range, that is, the unstable steady-state rate of interest is larger than the total growth rate. We have chosen $\theta_A < 9.4$ for the reason, invoked earlier, that we expect the governments of high-income countries to remain in deficit forever.

[35]See Akcigit, Pearce, and Prato (2020).

when steady states exist, we have shown how debt capacity varies with the parameters of the model, the key ones – beyond population growth and structural deficit – being the elasticity of intertemporal substitution in the lifetime utility and the elasticity of substitution between capital and labor.

We have used this basic idea to run policy experiments. If future policies that are stabilizing are anticipated, wrongly or rightly, debt may start above capacity on a seemingly explosive path, which may be slow and last several generations. The stabilizing responses that are needed sooner or later represent the true "fiscal cost" of exceeding debt capacity. By way of illustration, we have examined demographic scenarios, which can lead to debt becoming unsustainable.[36]

We have extended the model and its concept of debt capacity to two policy-relevant settings. First, we have shown in a monetary version of the economy that debt explosion means inflation explosion as well, although the seignorage revenue does increase debt capacity. Second, adding growth by innovation to our model, we have shown that, in all cases considered, a government R&D subsidy raises debt capacity for small subsidy amounts but, for larger subsidy rates, worsens it.

The first policy implication of our model is that it is not enough to compare interest rates to growth rates to draw any conclusion about the sustainability of debt; amounts of debt outstanding also matter. They must remain within debt capacity.

Secondly, the debt capacity is related to parameters of the economy. When near a cliff edge, it is useful to find out where the edge is located. Our model is a first attempt, in a realistic policy setting, at locating the edge.

Are the high-income country government debt levels close to debt capacity right now? Careful econometric estimation of our model will have to be carried out before precise, quantitative answers can be given. We have provided some illustrative numerical examples; these suggest that the current debt levels of high-income countries are close to debt capacity right now. We see steady increases in deficits and government debt levels relative to GDP. We do not, so far, see that social-security benefits, or other governmental services, or governmental spending more generally, are being reduced. To make things worse, population growth rates are declining and even turning negative or are, at least, predicted to turn negative. All of these developments but, especially, the reduced population and economic growth rates, leave little debt capacity to spare.

[36] There is evidence suggesting that productivity has declined over the last few decades. The effect of such a decline is similar.

# Appendixes

## A Social security

In this appendix, we provide the rationale for having chosen to incorporate social security in our model as the form of government spending. We verify that, when the rate of interest is below the rate of growth, a social-security scheme, balanced or unbalanced, can be welfare improving, which is the reason for which we chose that form of government spending as an illustration. We focus on the steady-state lifetime welfare, which we define as in Diamond (1965). In either the stable or unstable steady state, the lifetime utility of one person is constant, generation after generation.

Five configurations are considered here, the first two being viewed as benchmarks: the Diamond (1965) equilibrium with no social security and no debt, the bubbly equilibrium of Tirole (1985) with zero deficit and no debt, the equilibrium with balanced security and zero government debt as in Blanchard and Fischer (1989), the equilibrium with balanced social security with pure roll-over government debt, and finally equilibrium with social security in deficit, financed by government debt.

The Diamond equilibrium is inefficient for the well-known reason that each generation, in order to finance their retirement, saves in excess of what they would if the welfare of all generations were optimized. As there is too much capital,[37] the steady-state utility is strictly smaller than in the Golden-rule equilibrium, which can be reached in the Tirole bubbly, zero-deficit equilibrium. These two facts are reflected in our Figure 9 (Example 1 continued) by the solid green line, which is below the solid blue line.

Because the government is infinitely lived, it alone can issue debt that can be perpetually refinanced and, for that reason, can contain a bubble component. When the stock of capital is too high,[38] our Figure 9, – plotted against the level of benefits and for two levels of contributions (taxes) of 5% and 10% –, illustrates the fact that a budget deficit generated by social security and financed by debt can be a welfare-improving form of spending, relative to the competitive Diamond equilibrium.

The figure shows the special case of the equilibrium with balanced social security and no debt. That configuration can only approach the welfare optimum and never be equal to it.

More importantly, the figure shows that, with deficit social security, the unstable steady state, when it exists and assuming one can stay there, produces a larger utility per labor than the competitive equilibrium of Diamond (1965). For the special cases with zero deficit, such as $\theta/(1+n) = \tau = 0.05$ and $\theta/(1+n) = \tau = 0.10$, the unstable steady state with bubbly debt can actually reach the Golden Rule equilibrium, as does Tirole's bubble.

The stable steady states of equilibria with deficit social security also produce a welfare improvement, but only for sufficiently high values of the benefits. For lower values of the

[37]This comes with the caveat that in case of endogenous productivity growth, there are two kinds of capital: physical capital and knowledge capital. Knowledge capital may be too low, while physical capital is too high. See Section 4.2.

[38]If, to the opposite, the stock of capital were below the welfare optimum (the rate of interest is above the one in the welfare optimum), the proceeds of government debt issue could, of course, be used for investment. But that is not the case considered here.

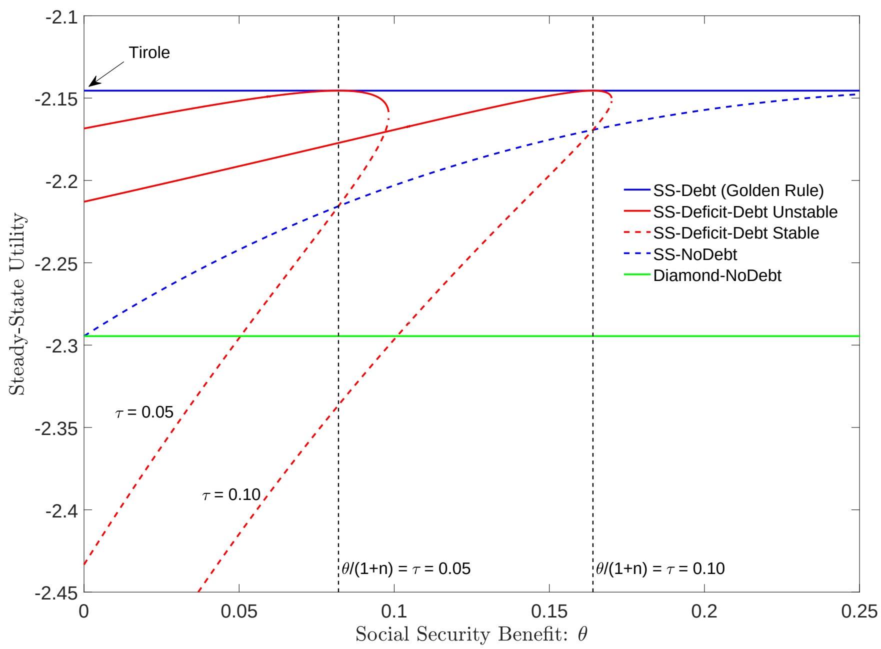


Figure 9: **Steady-state utilities.** Illustration with log utility and Cobb-Douglas production function. The parameter values are identical to what they are in the other figures. In the plot, we vary the social-security benefit, $\theta$, and show the resulting steady-state utilities in the Diamond and Tirole models and in our stable and unstable steady-states, and for a special case without debt. Five configurations are considered: the Diamond (1965) equilibrium with no social security and no debt, the bubbly equilibrium of Tirole (1985) with zero deficit and no debt, the equilibrium with fully funded social security and zero government debt, the equilibrium with fully funded social security with pure roll-over government debt, and finally equilibrium with social security in deficit, financed by government debt.

benefits, it is also possible for the stable steady state to exhibit smaller utility per capita than the competitive equilibrium of Diamond (1965).

# B Comparison of the second formulation (12) with previous models

In an attempt to enhance policy realism, our model generalizes a number of preexisting models described below:

- **Diamond (1965):** $d(k_{t-1}, k_t) = -\varphi$. The deficit is negative; a constant tax on the young yields a government surplus.

$$\begin{aligned} s(w(k_t), r(k_{t+1})) &= (1+n)(k_{t+1}+g_{t+1}) \\ (1+n)g_{t+1} &= (1+r(k_t))\,g_t - \varphi \end{aligned}$$

- **Tirole (1985):** $d(k_{t-1}, k_t) \equiv 0$**.** A bubble $g$ is present while the government pays and collect nothing.

- **Chalk (2000)**: $d(k_{t-1}, k_t) = d > 0$. The deficit arises from a constant, wasteful expenditure.

$$\begin{aligned} s(w(k_t), r(k_{t+1})) &= (1+n)(k_{t+1}+g_{t+1}) \\ (1+n)g_{t+1} &= (1+r(k_t))\,g_t + d \end{aligned}$$

- Tirole (1985) focuses on a special case of Diamond with $d \equiv 0$. Chalk (2000) focuses on a special case opposite to Diamond where $d > 0$ and is constant (i.e., negative tax $\varphi < 0$).

- **De la Croix and Michel (2002)** contains a synthesis of the models that existed at the time of their writing.

- **Our Model**: $d(k_{t-1}, k_t) = \frac{\theta}{1+n} w(k_{t-1}) - \tau w(k_t)$. A tax $\tau w(k_t)$ is levied on the young and a benefit $\frac{\theta}{1+n} w(k_{t-1})$ is paid to the old.

$$\begin{aligned} s(w(k_t), r(k_{t+1})) &= (1+n)(k_{t+1}+g_{t+1}) \\ (1+n)g_{t+1} &= (1+r(k_t))\,g_t + d(k_{t-1}, k_t) \end{aligned}$$

# C Proof of Proposition 1

The government budget or debt evolution is

$$-G_{t+1} + \theta_t w_{t-1} L_{t-1} = \tau_t w_t L_t - (1+r_t)\,G_t$$

which we first rewrite as

$$- (1+n)\, g_{t+1} + \theta_t w_{t-1} \frac{1}{1+n} = \tau_t w_t - (1+r_t)\, g_t$$

then as

$$(1+n)\, g_{t+1} = (1+r_t)\, g_t + d_t \qquad \text{where} \qquad d_t = \theta_t w_{t-1} \frac{1}{1+n} - \tau_t w_t$$

Rearranging leads to

$$g_t = \frac{1+n}{1+r_t} \left( g_{t+1} - \frac{d_t}{1+n} \right)$$

Rolling over from $t = 1$ to $T > 1$ leads to

$$\begin{aligned}
g_1 &= \frac{1+n}{1+r_1} \left( g_2 - \frac{d_1}{1+n} \right) \\
&= \frac{1+n}{1+r_1} \frac{1+n}{1+r_2} g_3 - \frac{1+n}{1+r_1} \frac{1+n}{1+r_2} \frac{d_2}{1+n} - \frac{1+n}{1+r_1} \frac{d_1}{1+n} \\
&= \frac{1+n}{1+r_1} \frac{1+n}{1+r_2} \frac{1+n}{1+r_3} g_4 - \frac{1+n}{1+r_1} \frac{1+n}{1+r_2} \frac{1+n}{1+r_3} \frac{d_3}{1+n} - \frac{1+n}{1+r_1} \frac{1+n}{1+r_2} \frac{d_2}{1+n} - \frac{1+n}{1+r_1} \frac{d_1}{1+n} \\
&\vdots \\
g_1 &= \frac{1}{1+n} \sum_{t=1}^{T-1} \frac{-d_t}{\prod_{u=1}^{t} \frac{1+r_u}{1+n}} + \frac{g_T}{\prod_{u=1}^{T-1} \frac{1+r_u}{1+n}} \qquad \forall T > 1 \\
g_1 \prod_{u=1}^{T-1} \frac{1+r_u}{1+n} &- \frac{1}{1+n} \sum_{t=1}^{T-1} \prod_{u=1}^{T-1} \frac{1+r_u}{1+n} \frac{-d_t}{\prod_{u=1}^{t} \frac{1+r_u}{1+n}} = g_T \\
g_T &= g_1 \prod_{u=1}^{T-1} \frac{1+r_u}{1+n} + \frac{1}{1+n} \sum_{t=1}^{T-1} \left( \prod_{u=t+1}^{T-1} \frac{1+r_u}{1+n} \right) d_t
\end{aligned}$$

Finally, letting $T \to +\infty$ completes the proof.

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