NBER WORKING PAPER SERIES

PUBLIC DEBT BUBBLES IN HETEROGENEOUS AGENT MODELS WITH TAIL RISK

Narayana R. Kocherlakota

Working Paper 29138
http://www.nber.org/papers/w29138

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
August 2021

I thank Xiaonan Ma for valuable research assistance and participants at the NBER Macro-Public Finance SI 2021 meeting for their comments. The views expressed herein are those of the author and do not necessarily reflect the views of the National Bureau of Economic Research.

Public Debt Bubbles in Heterogeneous Agent Models with Tail Risk
Narayana R. Kocherlakota
NBER Working Paper No. 29138
August 2021
JEL No. E62,H62,H63

ABSTRACT

This paper studies the public debt implications of a class of Aiyagari (1994)-Bewley (1977)-Huggett (1993) (ABH) models of incomplete insurance in which agents face a near-zero probability of a highly adverse outcome. In generic models of this kind, there exists a public debt bubble, so that the government is able to borrow at a real interest rate that is perpetually below the economic growth rate. Given an equilibrium with a public debt bubble, the primary deficit and the level of debt are both strictly increasing in the real interest rate and in the fraction of government expenditures used for lumpsum transfers. There is no upper bound on the deficit level or long-run debt level that is sustainable in equilibrium. In a public debt bubble, regardless of its size, agents are better off in the long run if the government chooses policies that give rise to a larger debt and primary deficit.

Narayana R. Kocherlakota
Department of Economics
University of Rochester
202 Harkness Hall
P.O. Box 270156
Rochester, NY 14627
and NBER
nkocherl@ur.rochester.edu

1 Introduction

In the current century, many advanced economies (including the US) have experienced:

- Real interest rates that have been, and are expected to remain, below growth rates for many years.
- Debt levels that are large and positive, and seem likely to remain high by historical standards.
- Primary deficit levels that have been, and are expected to remain, positive for many years

Like Blanchard (2019)[1], I see these data as (at least potentially) representing a public debt bubble (Brunnermeier, et al. (2020)), in the sense that the large positive value of government debt seems to exceed the present value of government primary surpluses (which is large and negative). Apparently, people are willing to hold public debt because they anticipate that governments will be able to repay it primarily through the issue of new debt.

Motivated by this observation, I study a class of Aiyagari (1994)-Bewley (1977)-Huggett (1993) (ABH) models. I begin with a baseline model in which agents experience uninsurable idiosyncratic taste and income shocks that evolve according to stochastically independent Markov chains. I alter the baseline model by adding tail risk so that, at any date, agents in the model face an auxiliary state that can occur with probability p and in which the (constant) marginal utility of consumption is ν/p. This auxiliary state can be interpreted as being any kind of eventuality in which the agent is desperate for resources, such as a health shock. I focus on the properties of perturbations in which p is near zero.

I show in this augmented model that (for any sufficiently high value of ν) there is a public debt bubble: agents are willing to buy and hold one-period government bonds even though the bonds pay a constant real interest rate r that is perpetually below the growth rate g

[1] See also Jiang, et. al. (2020).

(which is set equal to zero). In this situation, the government is able to run a perpetual positive primary deficit. It uses a fraction α of that deficit to finance lump-sum transfers and the remainder to finance government purchases. I treat r and α as the government's policy variables, and the time path of government debt as the endogenous response to those variables.

I obtain the following main results when p (the transition probability to the auxiliary state) is near zero:

- The value of outstanding government debt in any period is an increasing function of the real interest rate r and of the redistribution parameter α. The value of government debt in any period grows without bound as α nears 1, and the long run debt level grows without bound as r rises to zero.

- The size of the primary deficit in any period grows as r rises to zero. The size of the primary deficit in any period grows without bound as α nears 1.

- For any period t, the expected utility from private consumption is strictly increasing in the size of α. For any period t that is sufficiently large, the expected utility from private consumption[2] is strictly increasing in the real interest rate r.

- For any period t, government purchases are strictly increasing in the real interest rate r and are independent of the size of α.

I first prove the results in a simple setup without any private sector borrowing or any physical capital. I then go on to show that they remain valid when agents can borrow or they can accumulate capital.

The basic intuition underlying these results is that when the marginal utility ν is large, agents have a high precautionary demand for savings. Given that large demand, they are

[2]This conclusion is based on the assumption that all agents begin life with identical wealths. In contemporaneous research, Aguiar, Amador, and Arellano (2021) study the possibility of Pareto improvements in ABH economies when $r < g$ under the assumption that agents' initial wealths are distributed as in a stationary equilibrium. Brumm, et al. (2021) provide examples of economies in which $r < g$, but it is suboptimal to expand debt.

willing to give up current consumption to buy risk-free bonds that pay a real return $r < 0$. As r nears zero (from below), the debt and primary deficit both increase because agents are willing to give up more consumption to buy bonds and (more importantly) their cumulated bondholdings depreciate more slowly. These higher debts/deficits allow for more risk-sharing and allow the government to undertake more purchases. In a similar vein, as the distribution parameter α nears 1 from below, the agents have more resources available that they use to buy government bonds. Those increases in private saving push upward on deficits and debts.

The model's main policy implication is straightforward. If there is a public debt bubble, so that $r < 0$, the level of public debt is necessarily too small (regardless of how big it is). The government can raise societal welfare by increasing its debt and its deficit through devoting a larger fraction of public expenditures to redistribution (as opposed to purchases) or (in the long run) by raising the real interest rate (that is, lowering its bond price).

The theoretical possibility of public debt bubbles has been known since the classic work of Samuelson (1958) and Diamond (1965). However, such bubbles are often discounted or even ignored as being the product of unduly stylized models.[3] This paper focuses on the ABH class of models of incomplete insurance that are now foundational for much of modern macroeconomics. It shows that arbitrarily large public debt bubbles emerge as equilibria in a wide class of ABH models. In this sense, the possibility of large public debt bubbles can be seen as a theoretically robust consequence of the need for incomplete insurance against idiosyncratic tail risks.

2 ABH Models with Tail Risks

In this section, I describe a generic ABH model of incomplete insurance, augmented so that agents face a low probability of an extreme risk.

[3] There is, of course, a large literature in macroeconomics on rational bubbles. See Martin and Ventura (2018) for a recent useful survey and Miao (2014) for an elegant introduction to the underlying economic theory.

2.1 Base Model

There is a unit measure of agents. Agents are characterized by states that evolve over time according to identical and stochastically independent finite-state Markov chains. The state space for the Markov chain is given by $\{1, 2, 3, ..., J\}$. The transition matrix $\hat{\Gamma}$ is an aperiodic and irreducible $(J \times J)$ Markov matrix, with unique stationary density $\hat{\mu}$.

An agent in state i is endowed with y_i units of consumption (which is nondurable and nonstorable). At any date, an agent in state i has a momentary utility function u_i defined over current consumption, which satisfies the usual restrictions:

$$u_i', -u_i'' > 0$$

$$lim_{c \to 0} u_i'(c) = \infty$$

$$lim_{c \to \infty} u_i'(c) = 0$$

At each date, agents seek to maximize the expected value of the sum of current and discounted future momentary utilities, with a common discount factor given by $\beta \in (0, 1)$.

2.2 An Auxiliary State

I now change the above model economy by adding a state labelled 0. In state 0, agents are endowed with $y_0 > 0$ units of consumption, and have linear utility functions over consumption:

$$u_0(c) = \nu c/p > 0$$

where $\nu > 0$ and $0 < p < 1$.

As in the original economy, agents' states evolve over time according to stochastically independent Markov chains. An agent in state i has probability p of transiting to state 0 and probability $(1-p)\hat{\Gamma}_{ij}$ of transiting to any state $j > 0$. An agent in state 0 has probability ρ of staying in state 0 and probability $(1-\rho)\hat{\mu}_j$ of transiting to state j. The stationary density

$(\mu_j(p))_{j=0}^J$ of this augmented transition matrix is given by:

$$\mu_0(p) = \frac{p}{p + (1-\rho)}$$
$$\mu_j(p) = (1 - \mu_0(p))\hat{\mu}_j, j = 1, .., J$$

where, as noted earlier, $\hat{\mu}$ is defined to be the stationary density of the original (unperturbed) transition matrix $\hat{\Gamma}$. Suppose that a fraction $\mu_j(p)$ of agents have initial state j, for any $j = 0, 1, 2, ..., J$. In this perturbed model, the *relative* fractions of agents with state $j > 0$ are constant and are the same as in the original model.

In what follows, I consider what happens when the parameter p is near zero. Under this restriction, when the fractions of the agents in the various initial states are governed by $(\mu_j(p))_{j=0}^J$, the stochastic evolution of the individual states in the perturbed model is close to that in the base model. However, the need for consumption in state 0 becomes increasingly urgent as p is increasingly near zero (in the sense that the marginal utility ν/p of consumption in that state converges to infinity).

2.3 Bubble Conditions

I assume that ν is sufficiently large that:

$$u_j'(y_j) < \frac{\beta\nu}{1-\beta}, j = 1, ..., J \tag{1}$$

This condition makes possible the existence of bubbles, in the sense that it ensures that any agent in state $j > 0$ is willing to hold one-period bonds with a negative real interest rate. Note that for any specification of the baseline model, there exists an unbounded open interval of ν that satisfy (1).

3 Individual Bond Demands

In this section, I characterize individual bond demands. Specifically, I suppose that the government sets a constant price $q > 1$ for one-period bonds that pay off one unit of consumption. All agents begin life with $\bar{B}_1 \geq 0$ one-period bonds. Thereafter, they are able to buy as many bonds as they wish at this price, but (for now) are assumed to be unable to borrow. The government makes a non-negative lump-sum transfer τ_t to all agents in period t. (Later, I tie this transfer back to aggregate bond issuance, but that connection is irrelevant for the determination of individual bond demands.)

Consider an agent who is initially in state i. That agent chooses consumption and bond-holdings at each date so as solve the problem P*:

$$max_{(c,b)}\ E\{\sum_{t=1}^{\infty} \beta^{t-1} u_{s_t}(c_t)|s_1 = i\}$$

$$c_t(s^t) + qb_{t+1}(s^t) \leq y_{s_t} + b_t(s^{t-1}) + \tau_t \text{ for all } t \geq 1, \text{ w.p. } 1$$

$$b_t(s^{t-1}) \geq 0 \text{ for all } t \geq 1, \text{ w.p. } 1$$

$$b_1 = \bar{B}_1$$

where s^t represents the history at date t of current and past states. The expectation operator is calculated via the Markov chain probabilities defined earlier.

The bubble condition (1) implies that there exists $\bar{p} > 0$ and $\bar{q} > 1$ such that:

$$u_j'(y_j) < \frac{\beta\nu}{\bar{q} - \beta(1-\bar{p})}, j = 1, ..., J$$

Consider the open interval:

$$\Lambda \equiv (0, \bar{p}) \times (1, \bar{q}) \tag{2}$$

The following proposition shows that, for any (p, q) in the interval Λ, the solution to the problem P* is quite simple. An agent in state 0 in period t consumes all available wealth (from bond payoffs and income). An agent in any state $i > 0$ in period t consumes $c_i^*(p, q)$, where:

$$u_i'(c_i^*(p,q)) = \frac{\beta\nu}{q - \beta(1-p)}.$$

Proposition 1. *If (p, q) is in Λ, as defined in (2), there is a unique solution to the problem P^*. In that solution (c^*, b^*):*

$$s_t \in \{1, ..., J\} \Rightarrow \begin{cases} c_t^*(s^t) = u_{s_t}'^{-1}(\frac{\beta\nu}{q-\beta(1-p)}) \\ b_{t+1}^*(s^t) = (y_{s_t} - c_{s_t}^* + \tau_t + b_t^*(s^{t-1}))/q \end{cases}$$

and:

$$s_t = 0 \Rightarrow \begin{cases} c_t^*(s^t) = b_t^*(s^{t-1}) + y_0 + \tau_t \\ b_{t+1}^*(s^t) = 0 \end{cases} .$$

Proof. In Appendix. □

4 Results About Aggregates

In this section, I examine the behavior of aggregate debt and deficits when all agents solve problem P*. In doing so, I assume that the fraction of agents with initial state $i \in \{0, 1, 2, ..., J\}$ is equal to $\mu_i(p)$ (the stationary density). I assume too that a law of large numbers holds, so that the fraction of the agents who are in state i at date t who transit to state j is governed by the transition matrix. It follows immediately that, at any date, the fraction of agents with state i is given by $\mu_i(p)$. I assume that all agents begin life with $\bar{B}_1 \geq 0$ units of maturing bonds.

4.1 Debt

Let $\bar{B}_t(p,q,\alpha)$ represent the cross-sectional average of agents' bondholdings at the end of date $(t-1)$, given that all agents are solving problem P* when the transfer is defined via:

$$\tau_t(p,q,\alpha) = \alpha(q\bar{B}_{t+1}(p,q,\alpha) - \bar{B}_t(p,q,\alpha)), 0 \le \alpha < 1.$$

The government uses the remainder of the revenue to fund purchases. For now, I simply assert that the transfer is positive. We shall verify the validity of this assertion, for p near zero, in the next section.

The following proposition provides approximations to $\bar{B}_t$ for p near zero.

Proposition 2. *Suppose (p,q) is in Λ defined in (2). Define:*

$$\Delta_i^*(p,q) = y_i - u_i'^{-1}\left(\frac{\beta\nu}{q-\beta(1-p)}\right), i = 1, ..., J$$

which (from the definition of Λ) is positive for all i. Then for p near zero:

$$\bar{B}_{t+1}(p,q,\alpha) \approx \frac{\left(\sum_{i=1}^J \hat{\mu}_i \Delta_i^*(0,q)\right)}{(1-\alpha)} \frac{(1-1/q^t)}{q-1} + \bar{B}_1/q^t$$

Proof. In Appendix. □

The following corollary establishes key properties of the function $\bar{B}_t(p,q,\alpha)$ for p near zero.

Corollary 1. *Suppose (p,q) is in Λ (as defined in (2)). The period t debt level is strictly decreasing in q and is strictly increasing in α. For any $t \ge 2$:*

$$lim_{\alpha\to 1} lim_{p\to 0} \bar{B}_t(p,q,\alpha) = \infty$$

For any $\alpha \in (0,1)$, and for p near zero, the long-run debt level is unbounded as a function of

$q:$

$$lim_{q\to 1} lim_{t\to\infty} lim_{p\to 0} \bar{B}_t(p,q,\alpha) = \infty.$$

Proof. The sign of the partial derivative with respect to α is straightforward and:

$$\Delta_i^*(p,q) = y_i - u_i'^{-1}(\frac{\beta\nu}{q-\beta(1-p)})$$

is clearly a strictly decreasing function of q. Hence, the sign of the partial derivative with respect to q is governed by the sign of the derivative of $f(q) = \frac{(1-1/q^t)}{(q-1)}$. This can be assessed by checking the sign of:

$$\begin{aligned} f'(q) &= \frac{tq^{-t-1}(q-1)-(1-q^{-t})}{(q-1)^2} \\ &= \frac{(((t+1)q-t)-q^{t+1})}{q^{t+1}(q-1)^2}. \end{aligned}$$

The numerator of this expression equals 0 if $q = 1$; its derivative is $(t+1)-(t+1)q^t < 0$ if $q > 1$. It follows that $f'(q) < 0$ for $q > 1$ and so the period t debt level is strictly decreasing in q.

The limits are readily verified. □

The unboundedness of debt is, in part, a consequence of the linearity of utility in the auxiliary state 0. It is this property of the model that leads agents in that state to consume all of their bondholdings, regardless of how large they are.

4.2 Deficits

In this subsection, I describe the temporal behavior of the primary deficit for different values of the bond price q and the redistribution parameter α.

The primary deficit at any date is:

$$D_t^{primary}(p,q,\alpha) = q\bar{B}_{t+1}(p,q,\alpha) - \bar{B}_t(p,q,\alpha)$$

The following proposition characterizes the primary deficit when p is near zero.

Proposition 3. *Suppose (p,q) in Λ (as defined in (2)). If p is near zero, the primary deficit $D_t^{primary}(p,q,\alpha)$ at any date is approximately:*

$$\frac{(\sum_{i=1}^{J} \hat{\mu}_i \Delta_i^*(0,q))}{1-\alpha}.$$

where:

$$\Delta_i^*(0,q) = y_i - u_i'^{-1}(\frac{\beta\nu}{q-\beta}).$$

The approximate deficit is a strictly decreasing function of q. For p near zero, the primary deficit converges to infinity as α nears 1:

$$lim_{\alpha\to 1} lim_{p\to 0} D_t^{primary}(p,q,\alpha) = \infty.$$

Proof. In Appendix. □

When p is near zero, the primary deficit is unbounded from above as a function of α. Intuitively, the private sector's consumption is independent of α because it uses any transfer from the government to buy more bonds. So, the primary deficit has to satisfy:

$$(1-\alpha)D_t^{primary}(p,q,\alpha) = (Y - C)$$

where Y (the aggregate endowment) and C (aggregate private consumption) are independent of α. The unboundedness result follows immediately.

4.3 Welfare

In this subsection, I examine how, when p is near zero, social welfare depends on the bond price q and the distribution parameter α. The social welfare is derived from two sources: agents' expected utility from private consumption and the societal benefits of government purchases.

4.3.1 Expected Utility

I first consider the momentary expected utility from private consumption of an agent who is "behind the veil of ignorance" and is assigned initial state i with probability $\mu_i(p)$. A subtlety is that we need to weigh two effects when p is near zero: the consumption being given to agents in state 0 is nearing zero with p, and the marginal utility from consumption in state 0 converges to infinity as p nears zero. I show that the agent's momentary expected utility in period t is strictly decreasing in the bond price q, if t is sufficiently large, and strictly increasing in the redistribution parameter α in any period t.

Any agent who is in state $i > 0$ in period t consumes $c_i^*(p, q)$. An agent who is in state 0 consumes:

$$y_0 + \frac{p(1-\mu_0(p))}{\mu_0(p)} \frac{\bar{B}_t(p,q,\alpha)}{(1-\mu_0(p))} + \alpha(q\bar{B}_{t+1}(p,q,\alpha) - \bar{B}_t(p,q,\alpha))]$$

Hence, an agent's (momentary) expected period t utility is:

$$\begin{aligned} W_t(p,q,\alpha) = &\sum_{i=1}^{J} \mu_i(p) u_i(c_i^*(p,q)) \\ &+ \nu \frac{\mu_0(p) y_0}{p} \\ &+ \nu \bar{B}_t(p,q,\alpha) + \frac{\alpha\nu\mu_0(p)(q\bar{B}_{t+1}(p,q,\alpha) - \bar{B}_t(p,q,\alpha))}{p} \end{aligned}$$

The following proposition establishes that, when p is near zero, the expected utility from private consumption is strictly increasing in α, and, for sufficiently large t, is strictly decreasing in q.

Proposition 4. *Suppose* (p, q) *is in* Λ *(as defined in (2)). Define* $W_t^0(q, \alpha) = lim_{p \to 0} W_t(p, q, \alpha)$*.* W_t^0 *is strictly increasing in* α *for any* (q, t)*. For* $t \geq \frac{(2-\beta)}{(1-\beta)}$*, and for any* α*,* W_t^0 *is strictly decreasing in* q*.*

Proof. In Appendix. □

4.3.2 Government Purchases

Government purchases at any date are given by:

$$G_t(p, q, \alpha) = (1 - \alpha)(q\bar{B}_{t+1}(p, q, \alpha) - \bar{B}_t(p, q, \alpha)).$$

For p near zero, purchases are approximately:

$$\begin{aligned} G_t(p, q, \alpha) \approx (1 - \alpha) lim_{p \to 0} D_t^{primary}(p, q, \alpha) &= (1 - \alpha) \frac{(\sum_{i=1}^{J} \hat{\mu}_i \Delta_i^*(0, q))}{1 - \alpha} \\ &= \sum_{i=1}^{J} \hat{\mu}_i \Delta_i^*(0, q) \end{aligned}$$

The approximate level of government purchases is strictly decreasing in q, because agents in state $i > 0$ give up more consumption for bonds when their price q is lower. The approximate level of government purchases is independent of α, because agents in state $i > 0$ simply use any increase in their transfers to buy more bonds.

5 Extensions

In this section, I discuss the robustness of the findings to the inclusion of private credit and capital accumulation. I show in a numerical example that per capita debtholdings can be large when the real interest rate is negative, even if utility is strictly concave in state 0.

5.1 Private Credit

In this subsection, I extend the above results to a model in which agents are able to hold negative amounts of bonds, up to a limit $-B_{max}$, $B_{max} > 0$. The government fixes a bond price q in $(1, \bar{q})$ and p lies in $(0, \bar{p})$ (so that $(p, q) \in \Lambda$, as defined in (2)).

The relaxation of the credit constraints affects the behavior of the agents in state $i = 0$, who now choose consumption and bonds as:

$$c_0^* = y_0 + b + qB_{max}$$
$$b' = -B_{max}$$

In contrast, the agents who are in state $i > 0$ make the same consumption choices $c_i^*(p, q) = u_i'^{-1}(\frac{\beta\nu}{q-\beta(1-p)})$ as in the absence of borrowing. This consumption choice remains budget-feasible because:

$$\begin{aligned} b' &= (y_i - c_i^* + b)/q \\ &\geq (y_i - c_i^* - B_{max}/q) \\ &\geq -B_{max}/q \\ &> -B_{max}. \end{aligned}$$

As before, I am interested in the behavior of aggregates when p is near zero. But we have just seen that the relaxation of credit constraints only changes the behavior of agents in state 0, and the fraction of those agents becomes vanishingly small as p converges to zero. It follows that even when agents can borrow, for p near zero, the evolution of aggregate debt is approximately:

$$\bar{B}_t(p, q, \alpha) \approx \frac{\sum_{i=1}^{J} \hat{\mu}_i \Delta_i^*(0, q)}{(q-1)(1-\alpha)}(1 - 1/q^t) + \bar{B}_1/q^t$$

where (as before):

$$\Delta_i^*(0,q) = y_i - u_i'^{-1}(\frac{\beta\nu}{q-\beta}).$$

This (approximate) aggregate debt level is a strictly decreasing function of q and, in the long run, is an unbounded function of q. The approximate primary deficit is given by:

$$\frac{\sum_{i=1}^J \hat{\mu}_i \Delta_i^*(0,q)}{(1-\alpha)}$$

which is again strictly decreasing in q and strictly increasing in α. It remains true that, for any given q, the primary deficit when p is near 0 is unbounded from above for α close to 1.

Relaxing the credit constraint does affect how the bond price q impacts the expected momentary utility from private consumption for p near zero. When $B_{max} > 0$, then the per capita consumption of agents in state 0 is given by:

$$\begin{aligned} & y_0 + (q-\rho)B_{max} \\ & + \frac{p}{\mu_0(p)}(\bar{B}_t(p,q,\alpha) + \mu_0(p)B_{max}) \\ & + \alpha(q\bar{B}_{t+1}(p,q,\alpha) - \bar{B}_t(p,q,\alpha))] \end{aligned}$$

The second term captures the ability of agents in state 0 to expand their consumptions by issuing bonds at price q. It represents a new channel whereby agents can be made better off through higher bond prices (lower interest rates). However, this effect is temporary if the real interest rate is sufficiently close to zero: as in the proof of Proposition 4, it can be shown that W_t^0 is strictly decreasing in q for q sufficiently near 1 and t sufficiently large.

5.2 Adding Capital

Policymakers are often concerned that increases in government debt may serve to crowd out other more productive forms of wealth accumulation. In this subsection, I add capital to the model in Section 3 (without credit constraints), as in Aiyagari (1994). I show that debt increases (and commensurate interest rate increases) do in fact reduce steady-state capital. However, if there is a debt bubble, that reduction is in fact welfare-increasing, because agents spend fewer resources on making up for losses due to depreciation of the capital stock. Throughout the subsection, I assume that $\alpha = 0$ (so that all government revenue is spent on public consumption).

Suppose that consumption goods at each date are produced by firms using capital and labor, according to the constant returns to scale production function:

$$F(K, N)$$

where $F_K > 0$ and $F_{KK} < 0$. As in Aiyagari (1994), the various agents supply labor inelastically, with their maximal labor in state i equal to $\bar{N}_i$ (states evolve according to the independent Markov chains described in Section 2). The initial cross-sectional distribution of initial states is governed by the stationary density $\{\mu_i(p)\}_{i=0}^{J}$, so that per-capita labor remains constant at $\bar{N}(p) = \sum_{i=0}^{J} \mu_i(p)\bar{N}_i$.

Suppose the government chooses a bond price $q > 1$. Define $k(q)$ to be the capital-labor ratio when capital has the same return as these bonds, so that:

$$\frac{1}{q} - 1 = F_K(k(q), 1) - \delta.$$

I suppose that q is sufficiently close to one that:

$$F(k(q), 1) - \delta k(q) > 0.$$

This guarantees that there is sufficient output to replace the capital lost to depreciation.

To ensure that there is a debt bubble given q (and given p near zero), we need ν to be sufficiently large to satisfy the following $(J + 1)$ conditions:

$$u_i'(F_N(k(q),1)\bar{N}_i)) < \frac{\beta\nu}{q-\beta}, i = 1, ..., J$$
$$\sum_{i=1}^{J} \hat{\mu}_i u_i'^{-1}(\frac{\beta\nu}{q-\beta}) < F(k(q),1)\bar{N} - \delta k(q)\bar{N}$$

where $\bar{N} = \sum_{i=1}^{J} \hat{\mu}_i \bar{N}_i$ is per capita maximal labor for the agents in positive states. The first J conditions ensure that agents in positive states want to save, either through capital or bonds, when the real interest rate is negative (but near zero). The final condition ensures that their per capita consumption is less than output net of investment.

Under these conditions, when p is near zero, there is a steady-state equilibrium in which agents hold capital (approximately equal to) $k(q)\bar{N}$ and a positive amount of public debt $\bar{B}(q)$. In that equilibrium, the government uses its primary deficit to finance a positive level of government purchases:

$$\begin{aligned}(q-1)\bar{B}(q) &= G(q) \\ &= F(k(q)\bar{N},\bar{N}) - \delta k(q)\bar{N} - \sum_{i=1}^{J} \hat{\mu}_i u_i'^{-1}(\frac{\beta\nu}{q-\beta}).\end{aligned}$$

When p is near zero, lowering the bond price q toward 1 (and raising the real interest rate toward zero) serves to:

- Lower the capital stock $k(q)\bar{N}$
- Raise $G(q) = F(k(q)\bar{N},\bar{N}) - \delta k(q)\bar{N} - \sum_{i=1}^{J} \hat{\mu}_i u_i'^{-1}(\frac{\beta\nu}{q-\beta})$ (because $F_K - \delta < 0$).
- Raise the debt level $\bar{B}(q) = G(q)/(q-1)$ without bound.
- Raise the expected utility from private consumption (as in Proposition 4) by transferring resources from agents in positive states to those in state 0.

It is true in this model that larger amounts of government debt crowd out private capital.

But, with a debt bubble, crowding out private capital *raises* the amount of resources available for private and public consumption, because the marginal product of capital is lower than the depreciation rate.

In the above model, the marginal product of capital net of depreciation ($MPK - \delta$) is equal to r. In contrast, Reis (2020) studies a class of models in which:

$$(MPK - \delta) > g > r \tag{3}$$

where g represents the growth rate of the economy (which was always assumed to be zero in this paper). In Reis' models, production is undertaken by individual entrepreneurs who face financial frictions. As in earlier work (including Kocherlakota (2009)), Reis shows that these frictions can give rise to equilibria which satisfy (3). I abstract from these frictions in the current paper in order to stay close to the original ABH framework.

Unlike what I find in this paper, Reis shows that there is an upper bound on public debt. This difference in results is, at least in part, attributable to differing assumptions about the "spenders" in the models. In my paper, the spenders (agents in the auxiliary state 0) have linear utility. In Reis' paper, spenders (entrepreneurs with desirable investment opportunities) have diminishing marginal utility (specifically log utility).

5.3 Related Model with Curvature in Utility

The bulk of the paper focuses on a class of ABH models in which utility is linear in a low-probability state. In this subsection, I illustrate in a numerical example how large debt bubbles can also emerge in an ABH model in which agents' utility functions are always strictly concave.

In the example, there is a unit measure of agents. Their individual states evolve according to a Markov chain with two possible outcomes labelled $\{0, 1\}$. In state 1, agents have endowments equal to one unit of consumption and momentary utility functions given by $ln(c)$. In state 0, agents have endowments equal to zero units of consumption and momentary utility functions given by $10ln(c)$. The transition matrix is given by:

$$\Gamma_{00} = 0$$

$$\Gamma_{01} = 1$$

$$\Gamma_{11} = 0.975$$

$$\Gamma_{10} = 0.025$$

so that the stationary probability density is $\mu_1 = 0.976$ and $\mu_0 = 0.024$. I suppose that the agents have a common discount factor 0.97. Thus, agents in state 1 face a low probability risk of having a large income loss and a large need to consume.

The government sells one-period riskless debt in every period at a constant price 1.02 (meaning that, at each date, bond purchasers give up 1.02 units of consumption for 1 unit of consumption in the following period). As in the original model in Section 2, agents are not able to borrow. Unlike in the body of the paper, I focus on stationary equilibria, in which the joint distribution of bondholdings and individual states is time-invariant. The government uses the constant positive revenue from its bond sales at each date for purchases (that enter agents' utility separably, if at all).

In the stationary equilibrium, agents in state 0 buy no government bonds (because they know that they will be receiving a high endowment in the next period). Agents in state 1 increase their bondholdings over time until they re-enter state 0 (so that the distribution of bondholdings of state 1 agents is geometric). Despite the curvature in utility in state 0, the desire to guard against a temporary but highly adverse risk gives rise to to large per capita bondholdings - specifically, approximately 2.65 times per capita income. The maximal bondholdings are approximately 3.14 times per capita income. Both of these numbers would be even larger if the government used some of its revenue for transfers, as opposed to purchases.

6 Conclusion

The paper makes two contributions. From a theoretical perspective, it shows that given a generic ABH model economy, public debt bubbles arise as equilibria in perturbations of that model. Government policy choices can engender arbitrarily large debt and deficit levels in these equilibria. From a policy perspective, the paper shows that as long as there is a public debt bubble (in this class of models), agents are better off in the long run if the government changes its policy choices so as to increase the debt and deficit.

References

[1] Aiyagari, S. Rao, 1994, "Uninsured Idiosyncratic Risk and Aggregate Saving," *Quarterly Journal of Economics* 109, 659-684.

[2] Aguiar, M., Amador, M., and Arellano, C., 2021, "Micro Risks and Pareto Improving Policies with Low Interest Rates," NBER working paper 28996.

[3] Bewley, Truman, 1977, "The Permanent Income Hypothesis: A Theoretical Formulation," *Journal of Economic Theory* 16, 252-292.

[4] Blanchard, Olivier, 2019, "Public Debt and Low Interest Rates," *American Economic Review* 109, 1197-1229.

[5] Brumm, J., Feng, X., Kotlikoff, L. and Kubler, F., 2021, "When Interest Rates Go Low, Should Debt Go High?" NBER working paper 28951.

[6] Brunnermeier, M., Merkel, S., and Sannikov, Y., 2020, "The Fiscal Theory of the Price Level with a Bubble," Princeton University working paper.

[7] Diamond, P., 1965, "National Debt in a Neoclassical Growth Model," *American Economic Review* 55, 1126-1150.

[8] Huggett, M., 1993, "The Risk-free Rate in Heterogeneous-Agent Incomplete-Insurance Economies," *Journal of Economic Dynamics and Control* 17, 953-969.

[9] Jiang, Z., Lustig, H., van Nieuwerburgh, S., and Xiaolan, M., 2020, "The U. S. Public Debt Valuation Puzzle," National Bureau of Economic Research working paper 26853.

[10] Kocherlakota, N., 2009, "Bursting Bubbles: Consequences and Cures," Federal Reserve Bank of Minneapolis working paper.

[11] Martin, A. and Ventura, J., 2018, "The Macroeconomics of Rational Bubbles: A User Guide," *Annual Review of Economics,* 505-539.

[12] Miao, J., 2014, "Introduction to Economic Theory of Bubbles," *Journal of Mathematical Economics,* 130-136.

[13] Reis, R., 2021, "The Constraint on Public Debt When r < g but g < m," LSE working paper

[14] .Samuelson, P., 1958, "An Exact Consumption-Loan Model of Interest With or Without the Social Contrivance of Money," *Journal of Political Economy* 66, 467–82.

Appendix

In this appendix, I collect the proofs of the propositions.

Proof of Proposition 1

We first check the Euler equation if the agent is in state $i > 0$. The agent's Euler equation is satisfied because:

$$\begin{aligned} q u_i'(c_i^*) - \beta(1-p) \sum_{j=1}^{J} \hat{\Gamma}_{ij} u_j'(c_j^*) - \beta\nu \\ = \frac{q\beta\nu}{q-\beta(1-p)} - \frac{\beta^2(1-p)\nu}{q-\beta(1-p)} - \beta\nu \\ = \frac{q\beta\nu - \beta^2(1-p)\nu - q\beta\nu + \beta^2(1-p)\nu}{q-\beta(1-p)} \\ = 0. \end{aligned}$$

In state 0, the agent's Euler inequality is satisfied because:

$$\begin{aligned} q\nu/p - \beta(1-\rho)\frac{\beta\nu}{q-\beta(1-p)} - \beta\rho\nu/p \\ > \beta\nu/p - \beta(1-\rho)\frac{\beta\nu}{\beta-\beta(1-p)} - \beta\rho\nu/p \text{ (b/c } q > \beta) \\ = 0 \end{aligned}$$

We also need to verify the transversality condition that:

$$liminf_{t\to\infty} \beta^{t-1} u'(c_t^*) B_{t+1} = 0 \text{ w.p. } 1$$

But this is readily confirmed by noting that, for $p > 0$, state 0 occurs infinitely along almost every sample path.

Proof of Proposition 2

Let $B_{i,t+1}(p,q,\alpha)$ be the average bondholdings at the end of period t of agents in state $i > 0$. At any date t, these satisfy the recursion:

$$\begin{aligned} qB_{i,t+1}(p,q,\alpha) &= \frac{\sum_{j=1}^{J} \mu_j(p)\Gamma_{ji}(p)B_{jt}(p,q,\alpha)}{\mu_i(p)} + \Delta_i^*(p,q) + \tau_t(p,q,\alpha), i = 1, ..., J \\ \tau_t(p,q,\alpha) &= \alpha(q\bar{B}_{t+1}(p,q) - \bar{B}_t(p,q)) \end{aligned}$$

where $B_{j1} = \bar{B}_1, j = 1, ..., J$. If we multiply the first expression by $\mu_i(p)$ and sum over i, we get:

$$\begin{aligned} q\bar{B}_{t+1}(p,q,\alpha) &= (1-p)\bar{B}_t(p,q,\alpha) + \sum_{i=1}^{J} \mu_i(p)\Delta_i^*(p,q) + (1-\mu_0(p))\tau_t(p,q,\alpha) \\ \tau_t(p,q,\alpha) &= \alpha(q\bar{B}_{t+1}(p,q,\alpha) - \bar{B}_t(p,q,\alpha)) \end{aligned}$$

We can rewrite as:

$$\begin{aligned} & q\bar{B}_{t+1}(p,q,\alpha)(1-(1-\mu_0(p))\alpha) \\ = \bar{B}_t(p,q,\alpha)((1-p) - \alpha(1-\mu_0(p))) + & \sum_{i=1}^{J} \mu_i(p)\Delta_i^*(p,q) \end{aligned}$$

This implies in turn that:

$$\begin{aligned} \bar{B}_{t+1}(p,q,\alpha) &= \frac{\sum_{i=1}^{J} \mu_i(p)\Delta_i^*(p,q)}{q(1-(1-\mu_0(p))\alpha)} \frac{(1-\lambda(p,q,\alpha)^t)}{(1-\lambda(p,q,\alpha))} + \bar{B}_1\lambda(p,q,\alpha)^t \\ &= \frac{\sum_{i=1}^{J} \mu_i(p)\Delta_i^*(p,q)}{(q-1)(1-(1-\mu_0(p))\alpha) + p}(1-\lambda(p,q,\alpha)^t) + \bar{B}_1\lambda(p,q,\alpha)^t \end{aligned}$$

where:

$$\lambda(p,q,\alpha) = \frac{(1-p-(1-\mu_0(p))\alpha)}{(1-(1-\mu_0(p))\alpha)q}$$

Note that $0 < \lambda(p, q, \alpha) < 1$ for (p, q) in Λ.

By substituting $p = 0$, we can conclude that when p is near zero:

$$\bar{B}_{t+1}(p, q, \alpha) \approx \frac{\sum_{j=1}^{J} \hat{\mu}_i \Delta_i^*(0, q)(1 - 1/q^t)}{(q-1)(1-\alpha)} + \bar{B}_1/q^t$$

Proof of Proposition 3

The proof of Proposition 2 established that:

$$\bar{B}_{t+1}(p, q, \alpha) = \frac{\sum_{i=1}^{J} \mu_i(p) \Delta_i^*(p, q)}{(q-1)(1-(1-\mu_0(p))\alpha) + p}(1 - \lambda(p, q, \alpha)^t) + \bar{B}_1 \lambda(p, q, \alpha)^t.$$

where:

$$\lambda(p, q, \alpha) = \frac{(1 - p - (1 - \mu_0(p))\alpha)}{(1 - (1 - \mu_0(p))\alpha)q}$$

When p is near zero, $\lambda(p, q, \alpha) \approx 1/q$. It follows that:

$$\begin{aligned} D_t^{primary}(p, q, \alpha) &= q\bar{B}_{t+1}(p, q, \alpha) - \bar{B}_t(p, q, \alpha) \\ &\approx \frac{(\sum_{i=1}^{J} \hat{\mu}_i \Delta_i^*(0, q))}{1 - \alpha} \text{for } p \text{ near zero.} \end{aligned}$$

The infinite limit (with respect to α) follows immediately.

Proof of Proposition 4

Consider $W_t^0(q, \alpha)$ net of the the utility derived from the endowment y_0. It is equal to:

$$lim_{p \to 0} \sum_{i=1}^{J} \mu_i(p) u_i(y_i - \Delta_i^*(p, q)) + \nu \bar{B}_t(p, q, \alpha) + \alpha \frac{\nu \mu_0(p)}{p}(q\bar{B}_{t+1}(p, q, \alpha) - \bar{B}_t(p, q, \alpha))]$$

$$= \sum_{i=1}^{J} \hat{\mu}_i u(y_i - \Delta_i^*(0, q)) + \frac{\nu \{\sum_{i=1}^{J} \hat{\mu}_i \Delta_i^*(0, q)\}(1 - 1/q^{t-1})}{(q-1)(1-\alpha)} + \nu \bar{B}_1/q^{t-1} + \nu \frac{\alpha \{\sum_{i=1}^{J} \hat{\mu}_i \Delta_i^*(0, q)\}}{(1-\alpha)(1-\rho)}$$

$$= \sum_{i=1}^{J} \hat{\mu}_i u(y_i - \Delta_i^*(0, q)) + \psi_{t-1}(\alpha, q) \sum_{i=1}^{J} \hat{\mu}_i \Delta_i^*(0, q) + \nu \bar{B}_1/q^t$$

where $\psi_t(\alpha, q) = \nu \frac{\frac{\alpha}{1-\rho} + \frac{(1-q^{-t})}{q-1}}{(1-\alpha)}$. Here, I have exploited the observation that $lim_{p\to 0}\mu_0(p)/p = 1/(1-\rho)$.

The derivative of $W_t^0(\alpha, q)$ with respect to q is:

$$-\nu\bar{B}_1(t-1)/q^t + (\sum_{i=1}^{J}\hat{\mu}_i\Delta_i^*(0,q))\frac{\partial\psi_{t-1}}{\partial q}(\alpha,q) + \sum_{i=1}^{J}\hat{\mu}_i[-u_i'(y_i - \Delta_i^*(0,q)) + \psi_{t-1}(\alpha,q)]\frac{\partial\Delta_i^*}{\partial q}(0,q)$$

The first term of this expression is clearly negative for $t > 1$. In the second term, the sign of the derivative of ψ_t with respect to q is the same as the sign of:

$$\begin{aligned} & tq^{-t-1}(q-1) - (1-q^{-t}) \\ &= (t+1)q^{-t} - tq^{-t-1} - 1 \\ &= q^{-t}((t+1) - t/q - q^t) \end{aligned}$$

which is negative for $t \geq 1$ (because the derivative of the expression in parentheses w.r.t. q is $tq^{-2}(1-q^{t+1})$, which is negative for $q > 1$).

In the third term, it can be shown that:

$$\begin{aligned} & \psi_{t-1}(\alpha,q) - u_i'(c_i^*(0,q)) \\ &= \nu\frac{\frac{\alpha}{1-\rho} + \frac{(1-q^{-t+1})}{q-1}}{(1-\alpha)} - \frac{\beta\nu}{q-\beta} \\ & > 0 \end{aligned}$$

if t is sufficiently large ($t > \frac{(2-\beta)}{(1-\beta)}$).[4] Recall that:

$$\Delta_i^*(0,q)) = y_i - u_i'^{-1}(\frac{\beta\nu}{q-\beta}), i = 1, ..., J$$

Hence, Δ_i^* is strictly decreasing in q. It follows that, for t sufficiently large, the third term of

[4]This lower bound on t ensures that $\frac{(1-q^{-t+1})}{(q-1)} > \frac{\beta}{(q-\beta)}$ for all $q > 1$.

the derivative of W_t^0 with respect to q is negative, and so the overall derivative is negative.

The sign of the derivative of $W_t^0(\alpha, q)$ with respect to α is the same as the sign of the derivative of ψ_{t-1} with respect to α, which is positive.