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ABSTRACT

We apply the method of constrained asset share estimation (CASE) to test the mean-variance efficiency (MVE) of the stock market. This method allows conditional expected returns to vary in unrestricted ways, given investor preferences. We also allow conditional variances to follow an ARCH process. The data estimate reasonably the coefficient of relative risk aversion, though are unable to reject investor risk neutrality. We reject the restrictions implied by MVE, although changing conditional variances improve statistically upon measured market efficiency. We find that unrestricted asset-share and ARCH models help forecast excess returns. Once MVE is imposed, however, this forecasting ability disappears.

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This paper uses a technique that we call Constrained Asset Share Estimation (CASE) to test the conditional mean-variance efficiency (MVE) of the U.S. stock market. The technique is useful in time-series tests of simple asset pricing models because it allows estimated expected returns to vary in an unrestricted way. It was first applied in a macroeconomic context in which the "market" portfolio included not only equities, but also money, bonds and real estate.[1] It has since been applied more widely to other portfolios and has been extended to allow for variation in conditional second as well as first moments, as in an autoregressive-conditional-heteroskedasticity (ARCH) model.[2]

There is still a need for a clear statement of the advantages of the CASE method over earlier tests of the MVE hypothesis for the stock market. Briefly, these advantages are of three sorts. First, the technique does not impose the condition that expected returns are constant over time, or even that they change in a slowly moving way. Rather it allows expected returns to vary freely, as they must, for example, whenever new information which may not be observed by the econometrician becomes available to the investor.[3] In addition, in some of the tests below we allow

[1] See Frankel (1982, 1985a), Frankel and Dickens (1984), Frankel and Engel (1984), and Wills (1982).

[2] Ferson, Kandel and Stambaugh (1987) and Rayner (1986) test the constant-variance version on stock portfolios. Bodurtha and Mark (1988), Bollerslev, Engle and Wooldridge (1987) and Engel and Rodrigues (1989), test a version which allows for changing conditional second moments on portfolios, respectively, of stocks, domestic bonds, and short-term bills denominated in different currencies.

[3] In the tests below, expected excess returns are allowed to vary in a completely general way as functions of the asset shares, requiring only that a set of preference parameters consistent with the Hara class of utility functions remain constant.

second moments to vary according to an ARCH process.[4] Allowing for such variation in conditional moments is essential for a properly specified test of MVE. In fact, there is considerable evidence that both the conditional expectation and conditional variance of excess returns contain important predictable components.[5]

The second advantage of this method is that, by allowing the CAPM betas to evolve along with the characteristics of the underlying assets, longer time series can be used to test MVE. In the past, tests of unconditional MVE coped with changing conditional moments by using short test periods, usually 5 years or less. There are two problems with this procedure. First, there appears to be a substantial amount of conditional variation in both first and second moments over forecast horizons of much less than 5 years.[6] Second, while limiting time-series samples to 5 years makes the assumption of constant conditional moments more believable, it also reduces the power of tests of MVE. Low power can potentially explain the lack of any measured relationship between risk and return in tests of MVE.[7] The use of longer time series also reduces the need to develop small-sample test statistics, such as that suggested by Shanken (1987). With large time-series samples, the distributions of conventional test statistics are likely to be closer to their asymptotic approximations.

The third advantage implicit in the CASE method is that it nests MVE in a more general, but economically meaningful, theory of portfolio determination. In contrast, most tests of the null hypothesis of MVE have no clear alternative hypothesis. This feature is particularly important because many tests do in fact reject MVE; when one rejects the null hypothesis it is crucial to have some idea of what the alternative is. In some of the tests below, the alternative to MVE is that investors' portfolio shares are linearly related to expected returns, and possibly to conditional variances as well, but that investors do not compute covariances with the market portfolio in the

[4] The ARCH process does not allow second moments to vary *freely* however. It is analogous to estimating the first moments by an ARIMA process, in which this period's expectation is related to recent realizations, rather than by the CASE technique, in which expectations can vary freely.

[5] See, for example, Fama and French (1988) and Poterba and Summers (1987) for evidence on the predictability of stock market returns, and Bollerslev (1985) and Bollerslev, Engle and Wooldridge (1988) for evidence on the predictability of conditional variances of excess returns. These findings coupled with the results of Hansen and Richard (1987), who show that the conditionally and unconditionally mean-variance efficient frontiers are generally different, suggest that such variation in conditional moments is important for tests of MVE.

[6] Fama and French (1988) document substantial mean reversion at forecast horizons of 3-5 years. Pindyck (1984) and Poterba and Summers (1986) find evidence of high-frequency variation in conditional stock-market variances.

[7] See, for example, Schwert (1983), Gibbons, Ross and Shanken (1986), MacKinlay (1987), and Gibbons and Shanken (1987).

precise way that MVE would imply they should.[8]

Our tests below emphasize the nested nature of the hypotheses we consider. We pay special attention to the importance of ARCH vs. MVE vs. the asset shares themselves in explaining risk premia. The broad findings can be summarized as follows. First, we find that stock-market shares by themselves have statistically significant explanatory power in predicting monthly excess stock returns. This is what we would expect if the stock market is mean-variance efficient and if required returns change over time. However, we reject the restrictions implied by constant-variance MVE. Moreover, the ability of asset shares to forecast future excess returns *disappears* once the MVE restrictions are imposed. Something very different than MVE appears to be responsible for asset shares' ability to predict stock returns. Indeed, for a majority of the portfolios we construct, higher conditionally expected returns are associated with *lower* value shares.

One might conjecture that MVE holds and that these results are an artifact of the maintained assumption that conditional variances are constant. Indeed, we find that the data reject the hypothesis that the market is mean-variance efficient with a constant variance against the alternative that the market is mean-variance efficient with a conditional covariance matrix that evolves according to an ARCH process. Time-varying second moments therefore move the mean-variance efficient frontier closer to the market portfolio. This is good news for ARCH, but not for MVE: we cannot reject the hypothesis that the ARCH-MVE model can explain *no portion* of excess returns. Nevertheless, the data produce a sensible estimate of the coefficient of relative risk aversion of 2, with a standard error of about 1.5, so that, while we cannot reject the hypothesis that investors are risk neutral, we can reject hypotheses that they are strongly risk loving or risk averse.

Finally, we test a generalized ARCH specification, which does not impose MVE, against the null hypothesis that the market is conditionally mean-variance efficient and that conditional variances evolve according to an ARCH process. Once again we reject the restrictions imposed by by conditional MVE.

The paper is structured as follows. Sections 1 and 2 briefly describe the model and the data, respectively. Section 3 tests for constant-variance MVE. We introduce our ARCH specification in section 4, and test an unrestricted model as well as an ARCH-MVE system. Section 5 summarizes

[8] One possibility is that the managers of pension funds and the other funds that hold most equities are concerned only with minimising the variance of their own performance, rather than computing covariances with the aggregate portfolios held by individuals as they in theory should.

our general nesting procedure for the hypotheses of interest and offers our conclusions.

1. The model

Mean-variance efficiency implies that the vector of conditional risk premia is a linear combination of the asset shares in the portfolio, with the weights proportional to the conditional variance of asset returns:

$$E_t(r_{t+1}) = \rho_t \Omega_t \lambda_t, \tag{1}$$

where $E_t(r_{t+1})$ is the expected excess return above the riskless rate on an $N \times 1$ vector of assets conditional on all information available at time t, Ω_t is the conditional variance of returns between t and $t+1$, λ_t is the $N \times 1$ vector of portfolio weights, with $\sum_{i=1}^{N} \lambda_{t,i} = 1$, and ρ_t is a preference parameter – the coefficient of relative risk aversion. If the aggregate stock portfolio is the "market" portfolio, MVE is equivalent to the CAPM. To see this, note that the right-hand side of (1) is equivalent to the risk-adjusted conditional expected return on the aggregate (or market) portfolio,

$$E_t(r_{t+1}) = \beta_t E_t(m_{t+1}) = \beta_t \lambda_t' E_t(r_{t+1}),$$

where

$$\beta_t = \frac{\Omega_t \lambda_t}{\lambda_t' \Omega_t \lambda_t} = \frac{\text{cov}(m_{t+1}, r_{t+1})}{\text{var}(m_{t+1})}.$$

This expression makes it clear that the vector of sub-portfolio β_ts varies both with the shares of assets in the portfolio, λ_t, and the conditional covariance matrix, Ω_t, and thus may move substantially over short time intervals. Also, note that given preferences and Ω_t, (1) is a complete model of expected excess returns: MVE implies that asset shares are sufficient statistics for optimal forecasts of excess returns.

Under rational expectations, we can replace the vector of expected excess returns with the actual returns by including a prediction error that is orthogonal to all information at time t:

$$r_{t+1} = \rho_t \Omega_t \lambda_t + \epsilon_{t+1}, \tag{2}$$

where $\epsilon_{t+1} = r_{t+1} - E_t(r_{t+1})$. The insight in Frankel (1982) was that information about the conditional covariance matrix of returns can be obtained from the error terms, since under MVE

$$\Omega_t = E_t(\epsilon_{t+1}\epsilon_{t+1}'). \tag{3}$$

MVE therefore imposes a set of restrictions that are highly nonlinear in that they constitute proportionality between the coefficient matrix and the variance-covariance matrix of the error term in (2).

To evaluate (3), we must take a position on whether Ω_t is constant over time. In sections 3 and 4 below, we assume that Ω_t is constant and that it follows an ARCH process, respectively. We test the hypotheses that MVE holds against more general alternatives in which investors forecast excess returns as a function of asset shares and past prediction errors. The exact specifications for the alternative hypotheses are discussed in sections 3 and 4. We also test the MVE hypotheses above, as well as the more general alternatives, against an even more restrictive null hypothesis: that investors expect conditional excess returns to be zero. The results of these tests are also discussed in sections 3 and 4. Section 5 presents a diagram which makes it easy to see the results of our nested hypothesis tests.

2. The data

Our tests use monthly stock returns from the New York and American Stock Exchanges from January 1955 to December 1984. Because of the computational difficulties in estimating (2) we were forced to reduce the size of the cross section.[9] In the tests below we aggregate stocks into $N = 11$ (and sometimes 7) industry portfolios.

Table 1 describes the aggregation of stocks into industry portfolios. The returns for each portfolio are value-weighted average returns. The $N \times 1$ vector of portfolio shares, λ_t, is the value of the stocks in the portfolios as a fraction of the total value of all stocks. Because it is desirable to group together equities that have highly correlated returns, we tried to put similar industries into the same portfolio.[10] Stambaugh (1982) aggregates into 20 industries, roughly by type of final output. We further aggregate into 11 industries, combining some of Stambaugh's catagories. Table 1 shows Stambaugh's 20 industries, as well as the 11-industry aggregation that we use to perform our maximum likelihood tests of MVE. Table 1 also reports a 7-industry aggregation that we use for the ARCH estimation in section 4.

[9] If there are N assets, the computation involves a parameter matrix of dimension $N(N-1)/2 \times N(N-1)/2$ that must be repeatedly inverted. Engel and Rodrigues (1988) offer a Wald test version of the CASE test that gets around this problem, and allows one to consider larger vectors of assets. We apply it in Section 3 below.

[10] On the other hand, we would not want to include together the suppliers of intermediate products and the producers of final output in the same industry. When steel prices rise, the cost of producing autos increases so that it is possible that steel producers' profits rise when auto manufacturers profits decline.

The value shares, λ_t, are used to predict excess returns between time t and $t+1$. The share are measured monthly from the last day of January 1955 to the last day of November 1984 (35 observations), while the returns are calculated as the dividend plus appreciation over the previou month beginning the last day of February 1955 and ending the last day of December 1984. A returns are nominal excess returns above the return on one-month Treasury bills recorded b Ibbotson Associates (1986).

3. Tests of MVE with constant conditional variances.

If relative risk aversion and the return covariance matrix are constant, $\rho_t\Omega_t = \rho\Omega$, we ca write demands for assets as a function of their own rate of return and returns on all other equitie We would have,

$$\lambda_t = \mathbf{B}E_t(r_{t+1}), \quad (\text{}$$

where $\mathbf{B}$ is an $N \times N$ matrix of coefficients. By inverting the system of equations in (4), we obta a measure of expected excess returns,

$$E_t(r_{t+1}) = \mathbf{A}\lambda_t, \quad (\text{}$$

where $\mathbf{A} = \mathbf{B}^{-1}$. This system of equations is a generalization of a static model of MVE. MV imposes the restriction that the matrix of coefficients $\mathbf{A}$ be proportional to the variance of t forecast error, ϵ_{t+1}. Using *ex post* returns, (5) can be written:

$$r_{t+1} = \mathbf{A}\lambda_t + \epsilon_{t+1}. \quad (\text{}$$

Although the values of the equities are endogenous variables in an economic sense, they are st uncorrelated with the prediction errors, which under rational expectations are uncorrelated wi all information available at time t. (Under the null hypothesis that MVE holds precisely, predicti errors are the only source of errors that enter the equation.[11]) Thus the system in (6) can estimated consistently using ordinary least squares, equation by equation.[12]

Table 2 reports the results from estimating the unconstrained system of equations (6). Few the coefficients individually are significantly different from zero. Unsurprisingly, the R^2s are

[11] For example, Engel and Rodrigues (1989) show how iid measurement error in the rates of return could be included in residual, ϵ_{t+1}.

[12] Note that the N asset shares, $\lambda_{t,1} \ldots \lambda_{t,N}$, are perfectly collinear because they sum to 1. This does not pose a problen the estimation of (7), however, because the equations do not include a constant term.

very high, and none exceeds .10. We can reject at the 95 percent level the hypothesis that the asset shares have no explanatory power for excess stock returns. The log-likelihood value for the 11 equation system is 8709.35. The log-likelihood when all 121 coefficient are constrained to be zero is 8592.57. Twice the difference is distributed as χ^2_{121}. The value of the statistic is 233.56 compared with a critical value of 147.39.[13][14]

There is mixed support for one of our assumptions – that forecasts are rational. This assumption implies that there is no serial correlation in forecast errors. We performed Breusch-Godfrey tests for serial correlation from orders 1 to 20. We report the chi-square statistics only for the tests of the existence of 20th order autoregressive or moving average errors. In only four of the regressions can we reject the null hypothesis of no serial correlation up to 20th order at the 95 percent level.

Under the MVE hypothesis, this unconstrained system of inverted asset demand equations is not estimated efficiently. If we impose more structure on the system we can hope to improve the precision of our parameter estimates. So we will estimate the system of equations in (6) imposing the MVE constraints:

$$r_{t+1} = \rho\Omega\lambda_t + \epsilon_{t+1}, \tag{7}$$

so that $\mathbf{A} = \rho\Omega$. The N equation system (7) must be estimated by maximum likelihood techniques, imposing an unusual cross-equation restriction – between the matrix of coefficients in the regressions and the variance matrix of the regression errors. Note that the assumption that Ω is constant is not the same as the usual assumption in MVE tests of constant betas and expected returns. As we saw in the previous section, even with a constant covariance matrix, the betas, and hence the expected returns on all securities including the aggregate or "market" portfolio, will vary over time in a general, unrestricted way.[15]

Table 3 reports the maximum likelihood results of (7). The log-likelihood value is necessarily lower than the log-likelihood for (6) because (7) is a restricted form of (7): 8593.68 (as compared to the unrestricted log-likelihood of 8709.35). We also report a chi-square statistic for the restrictions

[13] The 99 percent critical value is 159.32.

[14] The only prior beliefs we have about the coefficients is that the return on asset j should be positively related to the share of asset j in the total portfolio. If we think of the market portfolio as comprised only of stocks, then in equilibrium investors will demand a higher return from a given stock portfolio the more of it they are required to hold. Table 2 shows that in 8 out of the 11 regressions this own-coefficient is negative (and significantly negative for industries 2 and 7). It is not significantly positive in any of the regressions.

[15] Frankel (1985a)

implied by (8). We impose 120 restrictions on the unconstrained system (121 coefficients are constrained to be proportional to their corresponding elements in the variance matrix). The test statistic is distributed χ^2_{120}, and its value is 231.34. We can easily reject the hypothesis of MVE at the 99 percent level. Comparing the results from table 3 to table 2, it is easy to see the source of the rejection. When the coefficients are constrained, they are much smaller than when they are unconstrained. Under the MVE constraints, an increase in the share of an asset has a much smaller impact on risk premia.

If one were willing to accept the MVE estimates on the basis of prior beliefs, they yield in some ways much more plausible asset pricing equations. We noted that in the unconstrained regressions we frequently found that an increase in an asset share would actually decrease that asset's expected return. That is not possible with the constrained MVE estimates.

Also, the point estimate of the coefficient of relative risk aversion, ρ, is very plausible – 2.03. It is very close to the "Samuelson presumption" of a likely value for average risk aversion. The coefficient is not estimated precisely, however, as it is not statistically different from zero at the 95 percent level. But its 95 percent confidence interval ranges only up to about 5.3 – still a believable estimate for average risk aversion.

On the other hand, the constrained model does a very poor job of predicting excess returns. The failure to reject the hypothesis that $\rho = 0$ implies that asset shares provide no statistically significant explanatory power for risk premia under the MVE restrictions, because the coefficients on the shares are all multiples of ρ. Above we mentioned that the log-likelihood when the coefficients are all constrained to be zero is 8592.57. The likelihood under the MVE restrictions is only 8593.68 – a meager increase of 1.11. MVE vitiates the predictive power of the asset shares alone.

The estimates reported in Tables 2 and 3 calculate the shares as a fraction of total equity investment. If, however, there are positive net holdings of the riskless asset, then the shares should properly be calculated as a fraction of total equity investment plus the total net value of the riskless asset. The riskless asset could have a positive net value if the government issues riskless short-term bonds, and investors consider government bonds to be additions to net wealth (so that they do not fully discount future tax liabilities) or if the government issues money. We estimated the model under the assumption that the relevant measure of the net supply is the value of all government bonds (which are calculated by Cox, 1985), and again under the assumption that the value of outstanding Treasury bills measure the net supply of the riskless asset. In both cases, there was almost no change in the estimates.

We considered two other formulations for the coefficient of relative risk aversion, besides assuming that it is constant. In the first, we assumed constant absolute risk aversion. In that case, $\rho_t = bW_t$ where b is the coefficient of absolute risk aversion and W_t is the value of all equities at time t. In the second, we considered a more general formulation consistent with the Hara class of utility functions, $\rho_t = a + bW_t$. If $b = 0$ we have the constant relative risk aversion case, and if $a = 0$ we have constant absolute risk aversion. Again, however, these versions of the model failed to improve the constrained model's performance.[16]

3.1. A wald test of MVE with constant conditional variances.

Maximum likelihood estimation of MVE is a difficult task. The constraints between the coefficients and the variance cause grave problems in finding the maximum of the likelihood function. The estimation is expensive and time consuming. The entire system must be estimated simultaneously, which in the case of the 11-asset system means simultaneously estimating 122 coefficients. The complexity of the problem increases with the square of the number of equations and assets. If we were to estimate the model even for all 20 of Stambaugh's original portfolios, it would mean maximizing a very messy function over 401 parameters.

If we are interested in testing MVE, but not in actually obtaining the constrained coefficient estimates, we do not need to estimate the constrained set of equations. A Wald test can be performed using only the unrestricted model. In this case, the unconstrained model (6) is particularly easy to estimate, because it requires only equation-by-equation ordinary least squares. Engel and Rodrigues (1988) provide an expression for the Wald statistic for the MVE restrictions.

The Wald statistic is not difficult to compute even for large collections of assets. We can test the MVE restrictions for the entire set of 20 industry portfolios composed by Stambaugh. We again reject the MVE restrictions easily. The test statistic is distributed χ^2_{19}, and has a value of 58.99, well above the 99 percent level.[17]

[16] In order to save space, we do not report these results.

[17] The comparable Wald test for the 11-asset aggregation yields a statistic distributed as χ^2_{10} equal to 22.76. This also rejects the MVE restrictions at the 99 percent level. These particular tests restrict only the diagonal elements of the return covariance matrix, and yet they reject easily.

The estimates of this section provide little support for MVE of the stock market. In all of the tests performed, the restrictions that MVE places on a more general asset demand model are strongly rejected.

4. Tests of MVE with ARCH conditional variances.

In the estimates reported in section 3, we assumed that return covariance matrix, Ω_t, was constant over time. Because it has become clear in recent years that conditional variances show a considerable amount of variation, we turn to a model of time-varying conditional variances.

In simple regression models, the presence of heteroskedasticity often does not affect the consistency of the coefficient estimates, although it does cause standard calculations of test statistics to be inconsistent. When the MVE restrictions are imposed, however, changes in variances imply changes in coefficient estimates, which in turn imply changes in expected excess returns. The coefficient on the asset shares in the constrained model must move over time if Ω_t does, so holding Ω_t constant leads to inconsistent coefficient estimates.

Inspection of (2) makes it easy to see why it is important to allow for variation in Ω_t. There are two possible sources of variation in expected returns if the measure of relative risk aversion is constant: changes in asset shares, λ_t, and changes in Ω_t. Suppose, for example, that favorable news about a stock is announced. One could easily think of cases in which the price is pushed up, increasing the stock's share in the aggregate portfolio, even though its expected return is now lower with the news. If the market is mean-variance efficient, this can happen when the riskiness of the asset declines – its own variance falls, or its variance with other assets declines. But, for the jth asset, this is exactly a change in the jth row of Ω_t.

The burgeoning econometric literature that proposes general corrections for heteroskedasticity is not applicable to this model. That literature relies generally on procedures in which consistent estimates of the residuals are obtained before any heteroskedasticity correction is made, and those estimated residuals are used to construct heteroskedasticity-consistent statistics. In our MVE tests, we must correct for time-varying variances when we estimate the regression coefficients because the coefficients move with the variance. In order to do this, we need an explicit model of the variance process.

Of course, our model is partial equilibrium in the sense that it does not indicate the nature of the exogenous variables that determine asset prices. It takes the stochastic processes of returns as given, and computes the mean-variance efficient portfolio from these. In particular, it gives us no indication of how variances should change over time.

We choose to model variances empirically following Engle's (1982) ARCH process. The ARCH takes the conditional variance of this period's forecast error to be a function of past forecast errors. It is not based on any theoretical notion of how the general equilibrium of the economy works. It is an *ad hoc* model that seems to work well in practice.

The univariate representation of a first-order ARCH would be $\sigma^2_{t,i} = \alpha + \gamma\epsilon^2_{t,i}$. The variance of the forecast error of the ith stock between time t and $t+1$ is given by $\sigma^2_{t,i}$, and $\epsilon^2_{t,i}$ is the square of the forecast error made between time $t-1$ and t. This equation states that if we make a large forecast error in one period, the variance of our forecast for the next period will be greater (assuming $\gamma > 0$).

In this section, we apply a multi-equation version of ARCH to the MVE problem. Because of the difficulty in estimating large ARCH systems, we have further aggregated the assets into the 7 portfolios described in table 1. Even with only 7 equations to estimate, the dimension of the ARCH problem can be quite large. For example, even if we restrict ourselves to first-order ARCH in which the variances and covariances this period are related only to the squares and cross-products of forecast errors from the previous period, the problem is unmanageably large. There are 28 independent elements in the covariance matrix. If each element were linearly related to the 28 lagged squares and cross products of the forecast errors, there would be 812 variables to estimate. More general forms of ARCH would relate the variance to more than one lag of the cross-products of forecast errors, or to lagged variances (as in Bollerslev's (1986) GARCH).

Given the complexity of estimating the MVE-ARCH system, and given the limited amount of data, it is helpful to lower the number of ARCH coefficients. Our test of MVE uses a parsimonious version of ARCH, in which the model,

$$E_t(r_{t+1}) = \rho\Omega_t\lambda_t, \tag{8}$$

has return variance given by:

$$\Omega_t = P'P + G\epsilon_t\epsilon_t'G.$$

We treat as parameters the upper triangular matrix P, and the diagonal matrix G. Under thi formulation, each element of Ω_t is linearly related to its corresponding component in the matri of cross-products of lagged forecast errors. There are only 35 coefficients to estimate. A furthe advantage of the ARCH in (9) is that it enforces positive semi-definiteness on the covariance matri Ω_t. This turns out to be helpful in estimating the constrained model by maximum likelihood.

The unrestricted form of the inverted system of asset demand equations is given by:

$$E_t(r_{t+1}) = \mathbf{A}_t\lambda_t. \qquad (9$$

MVE imposes the restriction that $\mathbf{A}_t = \rho\Omega_t$, where Ω_t is the conditional variance of r_{t+1}. I practice, if MVE is to be nested in the general system of asset demands, then the elements of $\mathbf{A}_t$ i the general system might be related to the same variables that Ω_t is assumed to be related to, bu in an arbitrary way. More specifically, we assume that in the unrestricted model, the coefficien matrix $\mathbf{A}_t$ evolves according to:

$$\mathbf{A}_t = Q'Q + F\epsilon_t'\epsilon_t F, \qquad (10$$

where Q is upper triangular and F is diagonal, and the conditional covariance matrix of return Ω_t, is given by (8). The MVE constraint, that $\mathbf{A}_t = \rho\Omega_t$, imposes 34 additional constraints on t unconstrained asset demand equations in (9).

For our restricted ARCH-MVE model in (8), the log-likelihood for observation t is given by

$$\mathcal{L} = -(7/2)\ln(2\pi) - (1/2)|\Omega_t| - (1/2)(r_{t+1} - \rho\Omega_t\lambda_t)'\Omega_t^{-1}(r_{t+1} - \rho\Omega_t\lambda_t), \qquad (1$$

where Ω_t is defined in (8), and $\epsilon_t = r_t - \rho\Omega_{t-1}\lambda_{t-1}$. Maximization of (11) is difficult for sever reasons. First is the constraint between coefficients and variances. Second is the recursive natu of the problem (so that the likelihood at t, defined above, depends on all observations from 1 to Third is the large number of parameters to estimate simultaneously. We estimated the system a modified version of a maximum likelihood program available in the Gauss programming packa It uses a technique based on the Berndt, Hall, Hall and Hausman (1974) algorithm.

Before turning to the results of the ARCH estimation, it is useful first to examine the co strained MVE estimates on the 7 equation system when Ω_t is constrained to be constant, as in t previous section. Table 4 shows that the 7-equation system performs much like its 11-equation cou terpart. The estimate of the relative risk aversion parameter is close to 2.0. However, it is still statistically different from zero, which indicates that the asset share data with the MVE constrai imposed do a poor job of explaining expected returns. The log-likelihood with MVE imposed 5558.56. This compares to a log-likelihood of 5603.56 for the corresponding constant-coefficie unconstrained system of asset demand equations. In this case, MVE imposes 27 constraints on general system. The test statistic is distributed χ^2_{27}, with a size of 70.00. The MVE constrai can be rejected strongly at the 99 percent level.

Table 5 reports the results of the MVE restrictions imposed on the ARCH system. There two hypotheses to test here. The first asks whether we can reject the constant-variance MVE mo in favor of the ARCH-MVE. A rejection would imply that time-varying variances statistically red the distance between the stock-market portfolio and the mean-variance efficient frontier. Suc rejection would lead us to the other interesting hypothesis: can we reject the restrictions imp by MVE on the unrestricted ARCH system in (9) and (10)? This would involve a test of hypothesis that $Q = P$ and $F = G$.

The log-likelihood for the ARCH-MVE model in (9) is 5573.97. The constant variance vers of MVE is a special case of this ARCH model, in which the G matrix from (9) is constrained tc zero. This imposes 7 constraints on the ARCH system. Our test statistic is 30.82 and is distribu χ^2_7: we reject the constant-variance restrictions at the 99 percent level. ARCH therefore impr significantly on the constant-variance form of MVE.

Four of the 7 ARCH coefficients (elements of the G matrix) are significantly different f zero at the 95 percent level. These coefficients are all quite small in magnitude. The squar each element gives the coefficient relating the variance in each equation to its own lagged squ forecast error. Only one of the squared components of G is greater than .10.

The point estimate of ρ is 1.91 – again close to the Samuelson value of 2.0. Once again, estimate is not statistically different from zero at the 95 percent level (although it is now signifi at the 80 percent level). The most important question is whether the ARCH-MVE model is restrictive relative to the general ARCH system given in (9) and (10). This system will pro a log-likelihood value at least as large as the value we reported above – 5603.56 – for the ver of the unconstrained model in which $\mathbf{A}_t$ is constant. But even if its likelihood was no larger this, the size of the test statistic (distributed χ^2_{34}) for testing the MVE constraints on the A model would be 59.18. MVE would therefore be rejected at the 99 percent level. So we do not even need to estimate the unconstrained asset-pricing equations with $\mathbf{A}_t$ varying over time to know that MVE is rejected.

We conclude that while letting the variance change over time is important in improving the explanatory power of MVE, it does not improve it enough relative to an unconstrained system of asset-demand equations.

5. Summary and Conclusions

Figure 1 provides a graphical summary of our nested hypothesis tests. At the top of the figure is the most unrestricted model we consider, the unrestricted ARCH model in equations (9) and (10). At the bottom of the figure is the most restrictive model, that asset shares are of no help in explaining required returns, or equivalently, that risk aversion is zero. For each pair of models, the line connecting them reports the results of a test of whether the lower model (the null hypothesis) can be rejected in favor of the upper model (the alternative hypothesis). It is easy to see that both of the MVE formulations – the constant-variance case in equation (7) and the ARCH case in (8) – are rejected when compared with any more general alternative hypothesis. Worse, there is no evidence in favor of these MVE models even when they are pitted as alternative hypotheses against the straw-man model in which asset shares don't matter at all ($\mathbf{A}_t = 0$ in equation (9)).

There are several ways to rationalize these results. One would be that the true asset pricing model is not the CAPM, but rather the APT, a version of the intertemporal CAPM, or even the one-period CAPM plus some other omitted variable. A second explanation for the results would rely on the Roll (1977) critique. If the stock market is very unlike the true "market" portfolio, we would not expect to find MVE, even if the CAPM holds.[18]

Indeed, under this explanation, the asset shares and ARCH processes cannot be accurately observed. A third explanation of the results would be that the residuals in (2) lead to poor measures of the conditional variances. If "peso problems" affect stock market returns, the estimated residuals will be biased. Imposing the MVE restrictions only compounds the problems. For example, it is well known that in the five years following the stock-market boom of August 1982, the market rose at an average annual rate of 22 percent. Few would argue in retrospect that it is possible to obtain from this period *ex post*, valid measures of *ex ante* expected risk and return. Thus if the model in (8) were true we would expect that unconstrained asset shares and ARCH would predict excess returns, but this could be erased by imposing the MVE restrictions which are not exactly satisfied in our sample.

[18] Similar results were found, however, when money, bonds, and real estate were allowed into the portfolio (Frankel, 1985a and Frankel and Dickens, 1984) and when foreign assets were allowed (Frankel, 1982, and Frankel and Engel, 1984).

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Table 1

Industry Portfolios and S.E.C. Codes

	Industry	S.E.C. Codes
1.	Mining	10, 11, 12, 13, 14
2.	Food and Beverages	20
3.	Textile and Apparel	22, 23
4.	Paper Products	26
5.	Chemical	28
6.	Petroleum	29
7.	Stone, Clay and Glass	32
8.	Primary Metals	33
9.	Fabricated Metals	34
10.	Machinery	35
11.	Appliances, Elec. Equip.	36
12.	Transportation Equipment	37
13.	Misc. Manufacturing	38, 39
14.	Railroads	40
15.	Other Transportation	41, 42, 44, 45, 47
16.	Utilities	49
17.	Department Stores	53
18.	Other Retail Trade	50-52, 54-59
19.	Bank., Fin., Real Estate	60-67
20.	Miscellaneous	1,4,15-17,21,24,25,27,30,31,46,48,70, 73,75,78,79,80,82,89,99

11 Portfolios (Combinations of the 20 portfolios)

Portfolio	Industry Portfolios
1	1, 20
2	2, 3, 4
3	5
4	6
5	7, 8, 9
6	10
7	11
8	12-15
9	16
10	17, 18
11	19

7 Portfolios (Combinations of the 20 portfolios)

Portfolio	Industry Portfolios
1	1, 2, 3, 4, 20
2	5, 7, 8, 9
3	6
4	10, 11
5	12-15
6	16
7	17-19

Table 2

Estimated Coefficients from Unconstrained OLS Regressions

Dependent Variable: Excess rate of return on asset j
Independent Variables: Shares of asset j in total portfolio

	λ^1	λ^2	λ^3	λ^4	λ^5	λ^6	λ^7	λ^8	λ^9	λ^{10}	λ^{11}
Equation 1											
	-0.14	0.19	0.26	-0.06	-0.11	0.14	-0.70	0.08	0.21	-0.35	0.26
	(0.12)	(0.82)	(0.30)	(0.26)	(0.32)	(0.25)	(0.44)	(0.22)	(0.32)	(0.25)	(0.44)
R^2 = .023	Breusch-Godfrey statistic (20 lags) = 42.79*										
Equation 2											
	-0.11	-2.29*	0.64*	-0.29	-0.28	0.44	-1.12*	0.16	0.59*	0.83	2.06
	(0.13)	(0.86)	(0.32)	(0.27)	(0.34)	(0.26)	(0.46)	(0.23)	(0.22)	(0.57)	(1.24)
R^2 = .050	Breusch-Godfrey statistic (20 lags) = 23.38										
Equation 3											
	-0.20	-1.05	0.12	-0.04	-0.32	0.14	-1.20*	-0.02	0.46*	1.16	2.05
	(0.13)	(0.89)	(0.33)	(0.28)	(0.35)	(0.27)	(0.47)	(0.24)	(0.23)	(0.59)	(1.29)
R^2 = .047	Breusch-Godfrey statistic (20 lags) = 16.99										
Equation 4											
	0.15	-0.55	0.74	-0.82*	-0.81*	0.14	-1.01	0.44	-0.01	-0.60	2.79
	(0.16)	(1.09)	(0.40)	(0.34)	(0.43)	(0.33)	(0.58)	(0.29)	(0.28)	(0.72)	(1.57)
R^2 = .027	Breusch-Godfrey statistic (20 lags) = 21.74										
Equation 5											
	-0.25	-1.00	0.83*	-0.25	-0.81	0.18	-1.68*	0.50	0.41	-0.02	2.20
	(0.16)	(1.07)	(0.39)	(0.34)	(0.42)	(0.33)	(0.57)	(0.29)	(0.28)	(0.71)	(1.55)
R^2 = .044	Breusch-Godfrey statistic (20 lags) = 30.71										
Equation 6											
	-0.10	-0.19	0.46	-0.40	-0.68	-0.45	-0.28	0.37	0.18	-0.06	1.99
	(0.15)	(1.04)	(0.38)	(0.33)	(0.41)	(0.32)	(0.56)	(0.28)	(0.27)	(0.69)	(1.51)
R^2 = .046	Breusch-Godfrey statistic (20 lags) = 20.41										

Table 2 (continued)

λ^1	λ^2	λ^3	λ^4	λ^5	λ^6	λ^7	λ^8	λ^9	λ^{10}	λ^{11}
Equation 7										
-0.17	-2.72*	0.83*	-0.26	0.71	0.44	-2.15*	0.37	0.75*	1.21	3.15
(0.17)	(1.13)	(0.41)	(0.36)	(0.44)	(0.35)	(0.60)	(0.30)	(0.29)	(0.75)	(1.63)
R^2 = .066				Breusch-Godfrey statistic (20 lags) = 17.38						
Equation 8										
-0.14	-0.85	0.25	-0.10	-0.43	0.08	-1.41*	-0.04	0.62	0.94	1.80
(0.14)	(0.93)	(0.34)	(0.29)	(0.36)	(0.29)	(0.49)	(0.25)	(0.24)	(0.62)	(1.34)
R^2 = .067				Breusch-Godfrey statistic (20 lags) = 21.10						
Equation 9										
-0.09	-0.77	0.50	-0.10	-0.12	0.18	-0.64	-0.04	0.30	0.07	0.82
(0.12)	(0.80)	(0.30)	(0.25)	(0.31)	(0.25)	(0.43)	(0.21)	(0.21)	(0.53)	(1.16)
R^2 = .032				Breusch-Godfrey statistic (20 lags) = 35.07*						
Equation 10										
-0.11	-0.38	0.20	-0.10	-0.27	0.01	-0.56	0.06	0.41	-0.02	1.05
(0.16)	(1.06)	(0.39)	(0.33)	(0.42)	(0.33)	(0.56)	(0.28)	(0.28)	(0.70)	(1.53)
R^2 = .027				Breusch-Godfrey statistic (20 lags) = 44.68*						
Equation 11										
-0.04	-0.25	0.13	0.19	0.09	0.24	0.13	0.19	0.54	-0.20	-1.31
(0.14)	(0.95)	(0.35)	(0.30)	(0.37)	(0.29)	(0.50)	(0.25)	(0.25)	(0.63)	(1.37)
R^2 = .027				Breusch-Godfrey statistic (20 lags) = 42.42*						

* = significant at 95% level

(standard errors in parentheses)

<u>Table 3</u>

CAPM Estimation, constant Ω, 11 assets

$$r_{t+1} = \rho(P'P)\lambda_t + \varepsilon_{t+1}$$

$$Var_t(\varepsilon_{t+1}) = P'P$$

*** Log Likelihood = -8593.684711 ***

The estimate of the coefficient ρ:

2.0319
(1.6130)

The estimate of the upper triangular matrix P:

.0398	.0322	.0334	.0385	.0411	.0346	.0404	.0331	.0257	.0316	.0374
(.0018)	(.0021)	(.0023)	(.0028)	(.0026)	(.0026)	(.0030)	(.0025)	(.0022)	(.0029)	(.0023)
	.0274	.0197	-.0033	.0166	.0198	.0223	.0189	.0089	.0252	.0047
	(.0011)	(.0015)	(.0023)	(.0019)	(.0025)	(.0022)	(.0019)	(.0017)	(.0025)	(.0015)
		.0204	.0042	.0044	.0097	.0097	.0078	-.0046	.0015	.0000
		(.0008)	(.0024)	(.0015)	(.0019)	(.0019)	(.0014)	(.0018)	(.0018)	(.0018)
			.0360	-.0029	-.0019	-.0032	-.0017	.0018	-.0073	.0125
			(.0014)	(.0016)	(.0021)	(.0019)	(.0015)	(.0018)	(.0019)	(.0016)
				.0276	.0058	.0102	.0090	-.0046	.0003	.0051
				(.0011)	(.0019)	(.0019)	(.0013)	(.0017)	(.0017)	(.0016)
					.0304	.0068	.0050	-.0025	.0021	-.0009
					(.0011)	(.0016)	(.0015)	(.0017)	(.0018)	(.0014)
						.0272	.0063	.0000	.0063	.0020
						(.0011)	(.0014)	(.0017)	(.0019)	(.0017)
							.0214	.0020	.0094	.0027
							(.0010)	(.0018)	(.0018)	(.0017)
								.0272	.0032	.0050
								(.0011)	(.0017)	(.0014)
									.0287	.0006
									(.0013)	(.0013
										.0219
										(.0007

(standard errors in parentheses)

Table 4

CAPM Estimation, Constant Ω, 7 assets

$$r_{t+1} = \rho(P'P)\lambda_t + \varepsilon_{t+1}$$

$$Var_t(\varepsilon_{t+1}) = P'P$$

*** Log Likelihood = -5558.561247 ***

The estimate of the coefficient ρ:

2.02778
(1.46639)

The estimate of the upper triangular matrix P:

.03842	.03935	.03711	.04015	.03640	.02695	.03782
(.00150)	(.00185)	(.00246)	(.00213)	(.00206)	(.00190)	(.00197)
	.02075	-.00471	.01708	.01571	-.00389	.00485
	(.00075)	(.00225)	(.00140)	(.00149)	(.00153)	(.00123)
		.03757	-.00296	-.00242	-.00033	.00140
		(.00115)	(.00143)	(.00120)	(.00166)	(.00117)
			.02435	.00762	-.00136	.00342
			(.00092)	(.00137)	(.00156)	(.00117)
				.02206	.00269	.00735
				(.00087)	(.00149)	(.00108)
					.02801	.00408
					(.00103)	(.00109)
						.02034
						(.00075)

(standard errors in parentheses)

Table 5

CAPM Estimation, ARCH, 7 assets

$$r_{t+1} = \rho \Omega_t \lambda_t + \epsilon_{t+1}$$

$$Var_t(\epsilon_{t+1}) = \Omega_t = P'P + G\epsilon_t \epsilon_t' G$$

*** Log Likelihood = -5573.969787 ***

The estimate of the coefficient ρ:

1.91212
(1.47685)

The estimate of the upper triangular matrix P:

.03714	.03883	.03364	.04036	.03738	.02700	.03700
(.00152)	(.00189)	(.00274)	(.00213)	(.00204)	(.00191)	(.00200)
	.02050	-.00278	.01648	.01486	-.00395	.00494
	(.00077)	(.00233)	(.00150)	(.00158)	(.00160)	(.00130)
		.03541	-.00285	-.00127	.00084	.00082
		(.00116)	(.00157)	(.00124)	(.00182)	(.00128)
			.02405	.00687	-.00140	.00308
			(.00095)	(.00138)	(.00160)	(.00122)
				.02118	.00253	.00747
				(.00096)	(.00158)	(.00109)
					.02779	.00391
					(.00109)	(.00112)
						.01971
						(.00082)

The estimates of the diagonal elements of G:

.19819	.13305	.31874	.06267	-.03718	.15481	.17706
(.03958)	(.04684)	(.06162)	(.04517)	(.04355)	(.09843)	(.04668)

(standard errors in parentheses)

Tests of the model

$r_{t+1} = A_t \lambda_t + \varepsilon_{t+1}$; $E_t [\varepsilon_{t+1}\varepsilon_{t+1}'] = \Omega_{t+1}$

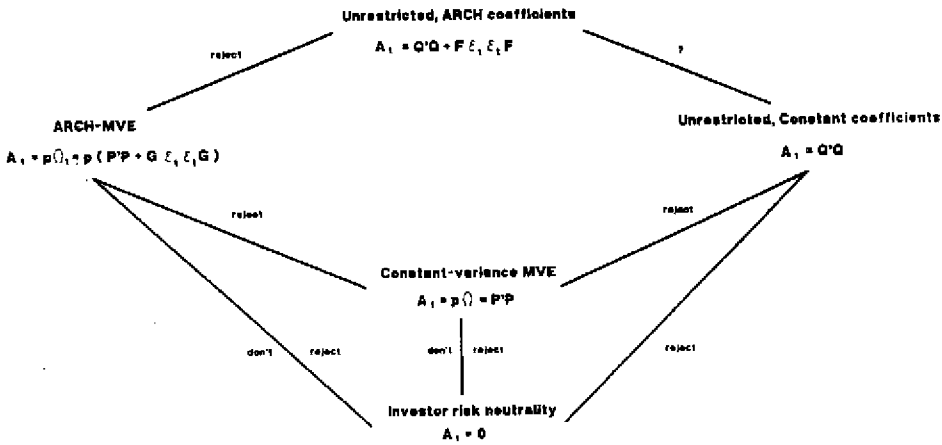

Figure 1