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SUSTAINABILITY IN A RISKY WORLD

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Sustainability in a Risky World  
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**ABSTRACT**

How much consumption is “sustainable”? We view sustainability as a requirement that welfare should not be expected to decline over time. We impose this requirement as a constraint on the consumption-savings-investment problem, and study its implications for saving, risky investment, and the social rate of time preference. The constraint does not distort portfolio choice, but it imposes an upper bound on the sustainable rate of time preference and the sustainable consumption-wealth ratio, which we show must lie between the riskless interest rate and the expected return on optimally invested wealth (and if risky wealth evolves according to a geometric Brownian motion, it must lie exactly halfway between the two). For plausible parameter values, the sustainable consumption-wealth ratio is considerably higher than both the riskless interest rate and the consumption-wealth ratio permitted by the Ramsey rule of zero social time preference.

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Do ethical considerations restrict the rate at which society consumes, or its preference for the present over the future? Economists have answered this question in different ways.

One view is that preferences, social or individual, must be taken as given. If society discounts the future at a high rate, strongly preferring present consumption over future consumption, that preference must be respected; and if it leads to high consumption today, declining over time, that outcome must be accepted.

An alternative view, famously expressed by Ramsey (1928), is that at least for long-term discounting over the lifetimes of multiple generations, society should not discount the future at all because to do so is unethically to privilege the generation alive today over those yet unborn. Recently, this view has found powerful expression in the *Stern Review* (Stern, 2006), which argues for aggressive action to combat climate change in large part on the basis of a social rate of time preference close to zero.

A third view is that social choices over consumption and saving should be subjected to an external “sustainability” constraint. Sustainability was defined by the World Commission on Environment and Development (1987) as a consumption plan that “meets the needs of the present without compromising the ability of future generations to meet their own needs.” Economists including Pezzey (1992), Solow (1993), Howarth (1995), Arrow et al. (2004), Asheim (2007), and Llavador et al. (2015) have formalized this as a requirement that social value—the expected discounted value of utility from the present to the infinite future—should not decline over time. In the words of Solow (1993),

A sustainable national economy is one that allows every future generation the option of being as well off as its predecessors. The duty imposed by sustainability is to bequeath to posterity not any particular thing . . . but rather to endow them with whatever it takes to achieve a standard of living at least as good as our own and to look after their next generation similarly.

The concept of sustainability as a constraint, rather than an objective, is consistent with the moral philosophy of Rawls (1999).<sup>1</sup> It can be understood as a prior principle that an ethical society should impose on itself because it would be agreed to by an individual who does not know into

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<sup>1</sup>Rawls (1999, §6) writes, “In justice as fairness, on the other hand, persons accept in advance a principle of equal

which of a sequence of generations they will be born. Since the time of birth is “morally arbitrary”, it should not influence expected utility.

As Arrow et al. (2004) discuss, in a deterministic economy with a single form of capital that has a constant riskless rate of return the sustainability constraint requires that the social rate of time preference does not exceed the exogenous riskless interest rate. When the constraint binds, the constrained rate of time preference equals the riskless interest rate, implying that society consumes the riskless return generated by its wealth and leaves the capital stock unchanged. Wealth, consumption, the utility and marginal utility of current consumption, and social value are then all constant over time. Sustainable consumption is only feasible when the riskless interest rate is positive, and then the constrained social rate of time preference is also positive. The sustainability constraint responds to the availability of an investment opportunity with a positive rate of return by allowing a greater rate of time preference and higher current consumption than would be required by Ramsey.<sup>2</sup>

In this paper we extend the concept of sustainability to allow for risk. In a risky economy, with an uncertain return on capital, it is not possible to guarantee that social value remains constant over time. Instead, we impose a weaker sustainability constraint that social value—expected utility, which is itself a random variable because it is a function of current wealth—should not be *expected* to decline over time. This constraint, which has also been suggested though not formally analyzed by Howarth (1995), acknowledges the reality that social welfare is subject to random shocks, some of which cannot be controlled. In the deterministic case, our constraint reduces to the one considered by Arrow et al. (2004).

We study a continuous-time model with two forms of capital, one safe and one risky, so that

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liberty and they do this without knowledge of their more particular ends. They implicitly agree, therefore, to conform their conceptions of the good to what the principles of justice require, or at least not to press claims which directly violate them. . . . The principles of right, and so of justice, put limits on which satisfactions have value; they impose restrictions on what are reasonable conceptions of one’s good. In drawing up plans and deciding on aspirations men are to take these constraints into account.”

<sup>2</sup>By adjusting the rate of time preference to the available rate of return, the sustainability constraint responds to a famous critique of Ramsey made by Koopmans (1960, 1967). Koopmans (1967) summarized his argument by writing: “The moral is, in my opinion, that one cannot adopt ethical principles without regard to . . . the anticipated technological possibilities. Any proposed optimality criterion needs to be subjected to a mathematical screening, to determine whether it does indeed bear on the problem at hand, under the circumstances assumed. More specifically, too much weight given to generations far in the future turns out to be self-defeating. It does nobody any good. How much weight is too much has to be determined in each case.”

society faces an asset allocation problem as well as a consumption-savings decision. While we use the terminology of financial economics—referring, for example, to assets, wealth, consumption, and saving—we emphasize that these financial concepts should be thought of in broad terms. The risky asset, for example, could represent the Earth itself, while consumption should be understood as a catch-all for, among other things, the rate at which society consumes, rather than sustains, the biosphere. In these terms, the consumption-savings decision we consider is intended as a modelling metaphor that encompasses questions of resource depletion, environmental degradation, and so on. Having said that, we should be clear that our framework does not address certain important aspects of sustainability, notably issues related to the economics of exhaustible resources (as studied in a deterministic setting by Dasgupta and Heal (1974), Solow (1974), and Hartwick (1977)).

We assume that the two assets have i.i.d. returns, with a flexible specification for the risky return that is driven both by a Brownian motion and by a Poisson jump process. The assumption of i.i.d. returns is consistent with an endogenous growth or  $Ak$  model of capital accumulation with constant returns. It implies that there is a unique consumption-wealth ratio at which the sustainability constraint is binding.<sup>3</sup> We assume that society has a standard time-separable power utility function defined over aggregate consumption, and we impose the sustainability constraint on this. We also consider an important special case where the safe asset is in zero net supply, so risk is inescapable for society and the risk premium on the risky asset must adjust to ensure a risky portfolio share equal to one.

Our main results are as follows.

First, when the sustainability constraint binds, the sustainable consumption-wealth ratio does not depend on the rate of time preference of an unconstrained representative individual in our economy.<sup>4</sup>

Second, the constraint binds whenever individuals' time preference rate exceeds the sustainable consumption-wealth ratio. In this case, we show that sustainability can justify defining a “social

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<sup>3</sup>In a model with diminishing returns to capital, by contrast, any constant savings rate can be sustainable although different savings rates imply different levels of steady-state consumption.

<sup>4</sup>This is consistent with the view of Rawls (1999, §45), who writes: “Of course, a present or near future advantage may be counted more heavily on account of its greater certainty or probability . . . But none of these things justifies our preferring a lesser present to a greater future good simply because of its nearer temporal position . . . The just savings principle for society must not, then, be affected by pure time preference, since as before the different temporal position of persons and generations does not in itself justify treating them differently.”

rate of time preference” that is less than individuals’ time preference rates. A social planner who optimizes using this social rate of time preference will choose to set the consumption-wealth ratio at the sustainable level.

Third, the sustainable consumption-wealth ratio exceeds the riskless interest rate but is less than the expected return on optimally invested wealth. In the absence of jumps, the sustainable consumption-wealth ratio lies exactly at the midpoint between these two rates of return; equivalently, it is the riskless interest rate plus one-half the squared Sharpe ratio on the risky asset divided by risk aversion. In the presence of jumps, the solution is defined implicitly but is straightforward to calculate numerically.

Fourth, the sustainable consumption-wealth ratio is higher than the consumption-wealth ratio required by the Ramsey zero-time-preference rule by a factor of risk aversion divided by risk aversion minus one. The difference is therefore small for very high levels of risk aversion, but substantial for moderate levels of risk aversion normally considered plausible.

Fifth, the sustainability constraint does not distort portfolio choice, which is always the same whether or not the constraint binds. In the absence of jumps, the portfolio rule is the classic one derived by Merton (1969, 1971).

The risky model we consider is different in two fundamental ways from the riskless model considered by the previous literature. First, in the presence of risk a zero drift for social value does not imply zero drifts for consumption, wealth, or the marginal utility of consumption. We show that consumption, wealth, and marginal utility all have positive drifts in the constrained equilibrium. Intuitively, this is because risky investment causes the distribution of consumption and wealth to spread out over time, imposing more risk on later generations. To prevent risk from reducing the welfare of later generations relative to earlier ones, later generations must be compensated by higher average levels of consumption and wealth. This implies that our sustainability constraint is different from the zero drifts in consumption and wealth, or log consumption and log wealth, imposed by Campbell and Sigalov (2022). The Campbell and Sigalov constraints generally distort portfolio choice, whereas our sustainability constraint does not.

Second, the sustainable rate of time preference is not the same as the sustainable discount rate that society should apply to riskless investment projects. That discount rate is given by the riskless interest rate in the sustainable equilibrium, which is lower than the sustainable rate of time

preference when the economy is exposed to risk. As a salient example, investments to mitigate climate change should be discounted at low rates if the investments are riskless and the sustainable equilibrium has a low riskless interest rate. They should be discounted at even lower rates if climate investments pay off in bad states of the world—that is, if they are analogous to insurance policies—an important point emphasized by Weitzman (2009) and Gollier (2021).

Our main analysis defines utility over aggregate consumption, in effect treating each generation equally regardless of population. This is only equivalent to treating each individual equally if population is constant over time. Population growth creates notoriously difficult issues for intertemporal ethics (Parfit 1984, Dasgupta 2001), particularly when population is itself a choice variable. However we show that if population growth is exogenous and constant, then we can modify the sustainability constraint to prevent the expected utility of an individual from declining over time. This is equivalent to subtracting the rate of population growth from all rates of return, and therefore from the sustainable consumption-wealth ratio and the sustainable rate of time preference.

The literature on discounting and sustainability is enormous, and we do not attempt a complete review here. Dasgupta (2008, 2021) and Zeckhauser and Viscusi (2008) provide recent surveys. Within the literature on climate change, there has been debate between those such as Cline (1992) and Stern (2006, 2016) who argue for a very low social rate of time preference, and Nordhaus (1994) who uses a higher rate of time preference. Our analysis implies that a substantial rate of time preference can be consistent with the ethical criterion of sustainability in a risky world.

The organization of the paper is as follows. Section 2 sets up our unconstrained continuous-time model with portfolio choice over a safe and a risky asset. Section 3 introduces the sustainability constraint and solves the constrained model. This section also briefly considers the effect of population growth. Section 4 imposes the equilibrium condition that the risky asset share equals one. Section 5 compares the sustainable consumption-wealth ratio with the consumption-wealth ratio implied by the Ramsey rule of a zero social rate of time preference. Section 6 concludes. An appendix presents details of key derivations.

# 1 Unconstrained Consumption and Portfolio Choice

We consider a representative investor faced with two assets, one riskless and one risky. The investor chooses society's aggregate consumption,  $C_t$ , and risky portfolio share,  $\alpha$ , to maximize the expected discounted value of a power utility function,

$$U_0 = \mathbf{E}_0 \int_0^\infty e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma} dt. \quad (1)$$

We take as given this representation of utility derived from aggregate consumption. It can be derived from individual utility of individual consumption under assumptions that permit aggregation across consumers. For example, we could assume that individuals have a constant probability of death following Blanchard (1985), that they have power utility defined over their own consumption, that they are unable to annuitize their wealth, and that the wealth of those who die is allocated to an equal number of newly born individuals. In this case utility at each point of time is both the welfare of the generation born at that moment and the welfare of all agents alive at that time. This microfoundation for equation (1) assumes a constant population; we consider a case with population growth in Section 2.4.

We assume that the rate of time preference  $\rho > 0$ . If individuals have a constant probability of death and do not care about their descendants, then as Blanchard (1985) shows,  $\rho$  is the sum of the pure individual rate of time preference and the probability of death. In a more general model with intergenerational altruism,  $\rho$  will also be affected by (and declining in) the degree of altruism.

We assume that the coefficient of relative risk aversion  $\gamma > 1$ . In Appendix B we show that all the results extend in the expected way to the log utility case,  $\gamma = 1$ . It would also be easy to handle the case  $0 < \gamma < 1$ , but as this case requires occasional sign flips in our logic below, we rule it out to streamline the exposition.

The riskless asset has gross return  $R_f$ . It will generally be convenient to think in terms of the log riskless rate,  $r_f = \log R_f$ .

We require assumptions about the return on wealth that is invested rather than consumed. As sustainability is inherently a long-run issue, we abstract away from high frequency variation in mean, volatility, and so on by modelling returns as i.i.d. It will also be convenient to work in continuous



time for tractability. We therefore model log asset values as Lévy processes. We specialize slightly, within this class of processes, by ruling out the possibility that infinitely many jumps occur in a finite time interval. This allows us to specify the risky asset value using a combination of a Brownian motion and a Poisson process.

In our most general formulation, we assume that the risky asset has constant expected excess return  $\mu = \log(ER/R_f) > 0$ , Brownian volatility  $\sigma$ , and is exposed to jumps arriving according to a Poisson counting process  $N_t$  with constant arrival rate  $\omega$ ; we assume that  $\mu$ ,  $\sigma$ , and  $\omega$  are each constant. We write  $W_t$  for wealth at time  $t$  and  $\theta = C_t/W_t$  for the consumption-wealth ratio. Under our assumptions,  $\theta$  is also constant. Thus

$$\frac{dC}{C} = \frac{dW}{W} = \left[ r_f + \alpha \underbrace{(\mu + \omega EL)}_{\hat{\mu}} - \theta \right] dt + \alpha \sigma dZ - \alpha L dN, \quad (2)$$

where we suppress time subscripts on random variables for simplicity.

Jumps are captured by the third term on the right-hand side of equation (2). When a jump occurs, an agent who is fully invested in the risky asset loses a fraction  $L$  of her capital. We assume that  $L$  is a random variable that is drawn in i.i.d. fashion each time a jump occurs. We also assume (with one eye on the equilibrium we study below, in which  $\alpha = 1$ ) that  $L$  is strictly less than 1, so that someone who invests fully in the risky asset is not bankrupted. We can allow  $L$  to take negative values; these represent good news for the risky asset. We write  $\hat{\mu} = \mu + \omega EL$  to denote the expected excess return in the absence of jumps.

It turns out that almost all of our qualitative conclusions apply whether or not there are jumps, so in the body of the paper we will simplify the exposition by ruling out jumps. In that case, equation (2) specializes to

$$\frac{dC}{C} = \frac{dW}{W} = (r_f + \alpha \mu - \theta) dt + \alpha \sigma dZ. \quad (3)$$

We handle jumps in the Appendix: Appendix A.1 generalizes the results in Section 1, and Appendix A.2 generalizes those in Section 2. We will highlight points at which jumps make a difference as they occur (most notably in Section 3, where we explicitly allow for jumps in the body of the paper).

As the coefficients in equation (3) are constants, it follows in the usual way that

$$C_t = C_0 \exp \left\{ \left( r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2 - \theta \right) t + \alpha\sigma Z_t \right\}, \quad (4)$$

and hence that

$$E_0 C_t^{1-\gamma} = W_0^{1-\gamma} \theta^{1-\gamma} \exp \left\{ (1-\gamma) \left( r_f + \alpha\mu - \frac{1}{2}\gamma\alpha^2\sigma^2 - \theta \right) t \right\}. \quad (5)$$

where we have used the fact that  $C_0 = \theta W_0$ .

The objective function (1) can therefore be evaluated explicitly, as

$$U_0 = \frac{W_0^{1-\gamma}}{1-\gamma} \frac{\theta^{1-\gamma}}{\rho - (1-\gamma) \left( r_f + \alpha\mu - \frac{1}{2}\gamma\alpha^2\sigma^2 - \theta \right)}. \quad (6)$$

The unconstrained optimal investment and consumption choices are identified by maximizing utility with respect to  $\alpha$  and  $\theta$ . Maximizing with respect to  $\theta$ , we find that the optimal consumption-wealth ratio is

$$\theta_{\text{unc}} = \frac{\rho + (\gamma - 1) \left( r_f + \alpha\mu - \frac{1}{2}\gamma\alpha^2\sigma^2 \right)}{\gamma}. \quad (7)$$

We assume that  $\theta_{\text{unc}}$  is positive when  $\alpha$  and  $\theta$  are chosen optimally. This condition implies that the denominator of (6) is positive, and hence that the integral in the definition of expected utility converges.

Maximizing with respect to  $\alpha$ , the optimal risky portfolio share is given by the classic Merton formula

$$\alpha = \frac{\mu}{\gamma\sigma^2}. \quad (8)$$

Using (8) to eliminate  $\mu$  in (7), we can also write

$$\theta_{\text{unc}} = \frac{\rho + (\gamma - 1) \left( r_f + \frac{1}{2}\gamma\alpha^2\sigma^2 \right)}{\gamma}. \quad (9)$$

## 2 A Sustainability Constraint

Equation (6) shows that expected utility at time  $t$ ,  $U_t$ , is proportional to  $W_t^{1-\gamma}/(1-\gamma)$ . Expected utility is itself a random variable, because it is a function of current wealth. To understand how expected utility evolves over time, we multiply by  $1-\gamma$  (which is negative under our maintained assumption that  $\gamma > 1$ ) and work with a rescaled variable  $X_t = W_t^{1-\gamma}$ . This follows the process

$$\frac{dX}{X} = (1-\gamma) \left( r_f + \alpha\mu - \theta - \frac{1}{2}\gamma\alpha^2\sigma^2 \right) dt + (1-\gamma)\alpha\sigma dZ. \quad (10)$$

If the consumption-wealth ratio,  $\theta$ , is sufficiently large then the drift of  $X$  (by which we mean, heuristically,  $\frac{1}{dt} \mathbb{E} \frac{dX}{X}$ ) is positive, and hence expected utility has negative drift: the optimal consumption-investment decision induces declining expected utility over time, on average.

We formalize the notion of sustainability, following Rawls (1999), by imagining representatives of each generation agreeing on a time-invariant consumption-investment policy that gives each generation the same expected utility. This implies that at time  $t$  the representative agent will solve the consumption-investment problem studied above, subject to the extra constraint that the drift of expected utility—or, equivalently, the drift term in equation (10)—should be zero. If the representative agent is thought of as the currently living generation in an infinite dynasty, then the constraint is appropriate if she does not want her descendants to expect a lower quality of life than she does.<sup>5</sup>

The sustainability constraint therefore implies—writing  $\theta_{\text{con}}$  for the case in which the constraint binds—that

$$\theta_{\text{con}} = r_f + \alpha\mu - \frac{1}{2}\gamma\alpha^2\sigma^2. \quad (11)$$

Similarly, we write  $\theta_{\text{unc}}$  for the case in which the constraint does not bind, so that the optimal choice is given by equation (9). The optimal consumption-wealth ratio is independent of  $\rho$  if the constraint binds; this is not true of the unconstrained case.

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<sup>5</sup>One might imagine imposing other types of constraint. We could, for example, allow for a type of risk aversion over future expected utility by requiring that some concave function of future expected utility should have non-decreasing expectation. This is analytically intractable in the constant relative risk aversion (power) case, however; and indeed it is infeasible in the limit as risk aversion over future expected utility approaches infinity, as it would require expected utility—and hence wealth itself—to be non-decreasing, which is not possible unless society can entirely eliminate risk.

If the constraint binds, we can use it to eliminate  $\theta$  from the objective function (6), giving

$$U_{\text{con},0} = \frac{W_0^{1-\gamma} (r_f + \alpha\mu - \frac{1}{2}\gamma\alpha^2\sigma^2)^{1-\gamma}}{1-\gamma} \frac{1}{\rho}. \quad (12)$$

Conveniently, the optimal investment choice is unaffected by the presence of the sustainability constraint. Maximizing equation (12) with respect to  $\alpha$ , we find the same first-order condition as before, equation (8). (As a corollary, the sustainable strategy is Pareto-efficient because it is identical to the unconstrained-optimal strategy for some choice of  $\rho$ .)

Intuitively, it is not optimal to distort portfolio choice in the presence of a sustainability constraint because doing so affects expected utility in the same way in all periods. Distorting the portfolio choice decision away from the unconstrained optimum therefore does not relax the constraint; nor (by definition) does it directly benefit the objective function. By contrast, in papers such as Dybvig (1995) or Campbell and Sigalov (2022) that feature constraints on consumption as opposed to welfare, it may be optimal to distort portfolio choice relative to the unconstrained optimum in order to relax the constraint.

We can use the Merton formula (8) to eliminate  $\mu$  from equation (11), giving

$$\theta_{\text{con}} = r_f + \frac{1}{2}\gamma\alpha^2\sigma^2. \quad (13)$$

Hence the constrained consumption-wealth ratio  $\theta_{\text{con}}$  exceeds the riskfree interest rate  $r_f$  in a risky economy. Equation (13) can be rewritten, again using condition (8), as

$$\theta_{\text{con}} = \frac{1}{2} \times r_f + \frac{1}{2} \times \underbrace{[(1-\alpha)r_f + \alpha(r_f + \mu)]}_{\text{expected return on optimally invested wealth}}. \quad (14)$$

This shows that the constrained consumption-wealth ratio lies exactly halfway between the riskless return and the expected return on optimally invested wealth. (This finding does depend, to some extent, on the specialness of geometric Brownian motion: we show in the Appendix that in the presence of jumps the constrained consumption-wealth ratio must lie somewhere between the riskless return and the expected return on wealth, with the precise location determined by the arrival rate and distribution of jump sizes.)

Alternatively, we can use (8) to substitute  $\alpha$  out of equation (11), giving

$$\theta_{\text{con}} = r_f + \frac{1}{2} \frac{\mu^2}{\gamma \sigma^2}. \quad (15)$$

This exhibits the maximum sustainable consumption-wealth ratio as the riskless interest rate plus one half the squared Sharpe ratio of the risky asset, divided by risk aversion. This can be much larger than the riskless interest rate: for example, if the riskless rate is 2%, the Sharpe ratio of the risky asset is 0.4, and risk aversion is 2, then the constrained consumption-wealth ratio is 6%. The constrained consumption-wealth ratio only approaches the riskless rate, its value in a deterministic model, as risk aversion becomes extremely high.

## 2.1 Comparing the constrained and unconstrained solutions

The optimal sustainable consumption-wealth ratio,  $\theta_{\text{sus}}$ , is given by whichever of  $\theta_{\text{con}}$  and  $\theta_{\text{unc}}$  is smaller. If the unconstrained case features a lower consumption-wealth ratio, then it certainly satisfies the constraint and delivers higher utility. If not, the unconstrained case does not satisfy the constraint, so that  $\theta_{\text{con}}$  is the best we can do. Thus

$$\theta_{\text{sus}} = \min \{ \theta_{\text{unc}}, \theta_{\text{con}} \}. \quad (16)$$

Equivalently,  $\theta_{\text{con}}$  is the highest possible sustainable consumption-wealth ratio.

It follows from equations (9) and (13) that

$$\theta_{\text{unc}} = \frac{1}{\gamma} \rho + \left( 1 - \frac{1}{\gamma} \right) \theta_{\text{con}}. \quad (17)$$

This equation also applies in the presence of jumps.

Equations (16) and (17) have several interesting implications. First, the sustainability constraint binds if and only if  $\rho > \theta_{\text{con}}$  (or, equivalently, if and only if  $\rho > \theta_{\text{unc}}$ ). Related to this, we can show that in the absence of a sustainability constraint,

$$E_0 X_t = X_0 e^{(\rho - \theta_{\text{unc}})t}. \quad (18)$$

If impatience is sufficiently high that  $\rho > \theta_{\text{unc}}$ , then  $X_t = W_t^{1-\gamma}$  is expected to grow without limit in an unconstrained equilibrium and expected utility is expected to decline without limit.<sup>6</sup> The sustainability constraint, which rules out declining expected utility, binds in this circumstance.

Second, equation (17) shows that the moderating influence of  $\rho$  makes  $\theta_{\text{unc}}$  less sensitive than  $\theta_{\text{con}}$  to changes in other parameters of the model, holding  $\rho$  fixed.

Third, equation (17) implies that the behavior of an extremely risk-averse individual is little affected by the presence or absence of a sustainability constraint, as  $\theta_{\text{unc}} \approx \theta_{\text{con}}$  if  $\gamma$  is large. This reflects the fact that highly concave utility leads an agent to choose a flat consumption path that is close to sustainable, regardless of the level of  $\rho$ .

Fourth, equations (16) and (17) show that  $\theta_{\text{sus}}$  and  $\theta_{\text{unc}}$  can easily be calculated from knowledge of  $\theta_{\text{con}}$ , so we can focus our analysis on the determinants of  $\theta_{\text{con}}$ .

Finally, we can use this analysis to analyse how and why the social discount rate used in, say, the *Stern Review* (2016), might differ from the discount rate  $\rho$  that enters an individual’s utility function. The *Stern Review* emphasizes the importance of the discount rate in making welfare comparisons across generations separated by long tracts of time, eventually settling on a social discount rate of zero. Weitzman’s (2007) review of the *Review* describes this as “a decidedly minority paternalistic view” and worries that “For most economists, a major problem . . . is that people are not observed to behave as if they are operating with [the social discount rate]  $\delta \approx 0$ ”.

In our setting, individuals unconstrained by sustainability will use the discount rate  $\rho > 0$  in calculations. Might sustainability justify a lower social discount rate suitable for use in a *Stern Review*-like exercise?<sup>7</sup>

To answer this question, define the social discount rate  $\hat{\rho}$  via the equation

$$\theta_{\text{unc}}(\hat{\rho}) = \theta_{\text{sus}}. \tag{19}$$

With this definition,  $\hat{\rho}$  is the hypothetical discount rate that should be used by a social planner

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<sup>6</sup>Expected wealth will decline toward zero if  $\rho$  is sufficiently large; but if  $\rho$  is sufficiently close to  $\theta_{\text{con}}$  then wealth has positive drift despite the negative drift in expected utility.

<sup>7</sup>Weitzman (2007) points out that model uncertainty provides another justification.

who wants to impose sustainability. From equation (16), this is equivalent to defining

$$\theta_{\text{unc}}(\hat{\rho}) = \min \{ \theta_{\text{unc}}(\rho), \theta_{\text{con}} \}. \quad (20)$$

If the sustainability constraint is not binding, so that  $\theta_{\text{unc}}(\rho) \leq \theta_{\text{con}}$ , then we can set  $\hat{\rho} = \rho$ . If the constraint does bind, then we must ensure that  $\theta_{\text{unc}}(\hat{\rho}) = \theta_{\text{con}}$ . By equation (17), this requires that  $\hat{\rho} = \theta_{\text{con}}$ . As the constraint binds precisely when  $\rho > \theta_{\text{con}}$ , we can summarize all this by saying that the hypothetical social planner should set

$$\hat{\rho} = \min \{ \rho, \theta_{\text{con}} \}. \quad (21)$$

If the sustainable consumption-wealth ratio is lower than the unconstrained time discount rate  $\rho$ , this represents an alternative justification for using a social discount rate lower than an individual's discount rate,  $\rho$ ; nonetheless, it suggests a higher social discount rate than does the Ramsey rule, which sets  $\hat{\rho}$  equal to zero. We return to this issue in Section 4.

## 2.2 Sustainable drifts in wealth and marginal utility

A binding sustainability constraint implies zero drift in expected utility, but this does not imply a zero drift in wealth. Instead wealth drifts upwards over time under our assumption that  $\gamma > 1$ , because

$$\frac{dW}{W} = \frac{1}{2}\gamma\alpha^2\sigma^2 dt + \alpha\sigma dZ, \quad (22)$$

and  $\frac{1}{2}\gamma\alpha^2\sigma^2$  is positive. Indeed, as we are assuming that  $\gamma > 1$ , we have the stronger result that *log* wealth drifts upwards over time under a binding sustainability constraint:

$$d \log W = \frac{1}{2}(\gamma - 1)\alpha^2\sigma^2 dt + \alpha\sigma dZ. \quad (23)$$

These facts illustrate the distinction between our sustainability constraint and the arithmetic and geometric constraints considered by Campbell and Sigalov (2022), which impose zero drift in wealth or in log wealth, respectively. Their constraints distort portfolio choice, unlike the sustainability constraint we consider.

A binding sustainability constraint also does not imply zero drift in marginal utility. The process for marginal utility,  $M = W^{-\gamma}$ , is

$$\frac{dM}{M} = \frac{1}{2}\gamma\alpha^2\sigma^2 dt - \gamma\alpha\sigma dZ, \quad (24)$$

so the drift is positive. (This is also true in the presence of jumps, so long as  $\gamma > 1$ . Note further that the drift in marginal utility equals the drift in wealth in the Brownian case, but differs from it in the general case with jumps.)

The positive drift in marginal utility in the constrained economy is another way to understand the result that the constrained consumption-wealth ratio exceeds the riskfree interest rate. The first-order condition for optimal investment in a riskless asset implies that discounted marginal utility drifts downward at the riskfree interest rate, and hence that the drift in undiscounted marginal utility is the constrained social rate of time preference (equivalently, the constrained consumption-wealth ratio) less the riskfree interest rate. That is, we have

$$\mathbb{E}\frac{dM}{M} = \theta_{\text{con}} - r_f. \quad (25)$$

Our solutions have this property, as can be verified by comparing the right hand sides of equations (13) and (24).

With power utility, the driftless variable  $X = W^{1-\gamma}$  is the product of marginal utility and wealth:  $X = MW$ . At first sight it might seem surprising that  $X$  has no drift while both  $M$  and  $W$  have positive drift. But we must also take into account comovement in  $M$  and  $W$ , whose effect is visible in the product rule:

$$\frac{dX}{X} = \frac{dM}{M} + \frac{dW}{W} + \frac{dM}{M} \frac{dW}{W}. \quad (26)$$

The product is

$$\frac{dM}{M} \frac{dW}{W} = -\gamma\alpha^2\sigma^2 dt, \quad (27)$$

and the fact that this quantity is negative makes it possible for  $X = MW$  to be driftless even though  $M$  and  $W$  each have positive drift:

$$\mathbb{E}\frac{dX}{X} = \mathbb{E}\frac{dM}{M} + \mathbb{E}\frac{dW}{W} + \mathbb{E}\left(\frac{dM}{M} \frac{dW}{W}\right) = 0. \quad (28)$$



Table 1: Numerical examples in the Brownian case

Baseline calibration sets  $r_f = 1\%$ ,  $\mu = 8\%$ ,  $\sigma = 20\%$ .

$\gamma$	$\theta_{\text{con}}$	$\alpha$	E $\frac{dW}{W}$	E $d \log W$	E $\frac{dW^{-\gamma}}{W^{-\gamma}}$
1	0.09	2	0.08	0	0.08
2	0.05	1	0.04	0.02	0.04
5	0.026	0.4	0.016	0.0128	0.016
10	0.018	0.2	0.008	0.0072	0.008

We conclude by noting one other implication of the upwards drift in log wealth visible in equation (23). Although the sustainability constraint requires that the average future value of expected utility  $U_t$  is the same as expected utility today, as one looks into the far-distant future expected utility is overwhelmingly likely to be higher than its current value. This counterintuitive fact, which echoes the result of Martin (2012), follows from the fact that  $\log W_t$  has positive drift, so that expected utility at time  $t$  approaches its upper bound of zero almost surely as  $t$  approaches infinity. The invariance of expected utility to the horizon is achieved due to the presence of a vanishingly small number of paths in which expected utility in the future is arbitrarily low.

### 2.3 Numerical examples

In this section we present some numerical examples to illustrate the properties we have discussed. Table 1 reports numerical results for a Brownian model without jumps, in which the riskless interest rate  $r_f$  equals 1%, the risk premium  $\mu$  equals 8%, and the standard deviation of the risky asset  $\sigma$  equals 20%, implying a Sharpe ratio of 0.4. The four rows of the table consider risk aversion coefficients  $\gamma$  of 1, 2, 5, and 10. The columns report the constrained consumption-wealth ratio  $\theta_{\text{con}}$ , the risky portfolio share  $\alpha$ , and the corresponding drifts in wealth, log wealth, and marginal utility.

In a Brownian model, the risky portfolio share is inversely proportional to risk aversion. Given our assumed parameters an investor with log utility ( $\gamma = 1$ ) sets  $\alpha = 2$ , leveraging the risky asset two for one. The constrained consumption-wealth ratio is 9% and the corresponding expected growth rate of wealth is 8%. However, the expected growth rate of log wealth is zero since this is the growth rate of utility for a log investor so the sustainability constraint sets it to zero. The expected growth

Table 2: Numerical examples with jumps

Baseline calibration sets  $r_f = 2\%$ ,  $\mu = 4\%$ ,  $\sigma = 10\%$ ,  $\omega = 4\%$ ,  $L = 0.4$ ,  $\gamma = 2$ .

	$\gamma$	$r_f$	$\mu$	$\sigma$	$\omega$	$L$	$\theta_{\text{con}}$	$\alpha$	$E \frac{dW}{W}$	$E d \log W$	$E \frac{dW^{-\gamma}}{W^{-\gamma}}$
Baseline	2	0.02	0.04	0.10	0.02	0.40	0.045	1.11	0.019	0.010	0.025
High $\gamma$	5						0.031	0.49	0.009	0.007	0.011
Low $\gamma$	1						0.065	1.83	0.028	0	0.045
High $r_f$		0.04					0.065	1.11	0.019	0.010	0.025
Low $r_f$		0					0.025	1.11	0.019	0.010	0.025
High $\mu$			0.08				0.10	1.56	0.045	0.026	0.080
Low $\mu$			0.02				0.027	0.66	0.006	0.003	0.007
High $\sigma$				0.15			0.035	0.72	0.014	0.007	0.015
Low $\sigma$				0.06			0.055	1.36	0.019	0.011	0.035
High $\omega$					0.04		0.040	0.88	0.016	0.008	0.020
Low $\omega$					0		0.060	2.00	0.040	0.020	0.040
High $L$						0.60	0.037	0.72	0.012	0.007	0.017
Low $L$						0.20	0.056	1.73	0.034	0.017	0.036
Negative $L$						-0.40	0.054	1.74	0.036	0.017	0.034

rate of marginal utility is the difference between the constrained consumption-wealth ratio and the riskfree rate, or 8%, and in the Brownian model it also equals the expected growth rate of wealth, as we noted earlier.

As risk aversion increases, the constrained consumption-wealth ratio and the risky portfolio share both decline. For example, when  $\gamma = 2$  the constrained consumption-wealth ratio is 5% and the risky portfolio share  $\alpha = 1$ . However, the constrained consumption-wealth ratio declines slowly and is still 1.8%, almost twice the riskless interest rate, when  $\gamma = 10$ . The expected growth rates of wealth and marginal utility also decline with risk aversion, but the expected growth rate of log wealth is hump-shaped in risk aversion, first increasing and then ultimately declining towards zero.

Table 2 reports numerical results for the more general model studied in the Appendix, which allows for jumps. Here, in the benchmark case the riskless interest rate  $r_f$  equals 2%, the risk premium  $\mu$  equals 4%, and the Brownian standard deviation of the risky asset  $\sigma$  equals 10%. In addition a jump of size  $L = 40\%$  occurs with a probability  $\omega$  of 4% per period. The benchmark case sets  $\gamma = 2$ , in which case the optimal constrained consumption-wealth ratio is 4.5% and the

risky portfolio share  $\alpha = 1.11$ . These values are fairly close to those in the Brownian case reported in Table 1, indicating that the calibration with jumps is broadly comparable in its overall level of risk.

The remaining rows of the table consider variations of the benchmark model with higher and lower risk aversion, then higher and lower riskfree rates, risk premium, Brownian volatility, jump probability, and jump size. The results are intuitive and in all cases imply a sustainable consumption-wealth ratio well above the riskfree interest rate, or equivalently a substantial positive expected growth rate of marginal utility.

## 2.4 Sustainability with population growth

The analysis so far has imposed sustainability on a social welfare function defined over aggregate consumption. This is equivalent to sustainability of individual utility only if the population is constant. In this section we modify our analysis to make individual utility sustainable given exogenous, constant population growth at rate  $g$ .

If there is population growth, then wealth at time  $t$  is shared between more people. Normalizing the initial population size to 1, the wealth of an average individual at time  $t$  is  $W_t e^{-gt}$ , where  $g > 0$  is the population growth rate. To ensure that such an average individual's expected utility is nondecreasing, we require that  $\tilde{X}_t$  has nonpositive drift, where  $\tilde{X}_t = e^{g(\gamma-1)t} X_t$ .

This condition also ensures nondecreasing expected utility for any class of individuals who have a constant share of the wealth of society. For example, a Blanchard (1985) model with population growth implies that a newborn person has lower wealth than the average currently living person, because more people are born than die at each instant; however, with a constant population growth rate the wealth share of newborn individuals is constant over time. Thus, the constraint that  $\tilde{X}_t$  has nonpositive drift ensures that the expected utility of newborn individuals does not decline over time.

Noting that

$$\frac{d\tilde{X}}{\tilde{X}} = g(\gamma - 1) dt + \frac{dX}{X} = (1 - \gamma) \left( r_f + \alpha\mu - \theta - \frac{1}{2}\gamma\alpha^2\sigma^2 - g \right) dt + (1 - \gamma)\alpha\sigma dZ, \quad (29)$$

the condition that  $\tilde{X}_t$  has nonpositive drift modifies the sustainability constraint (11) to

$$\theta \leq r_f + \alpha\mu - \frac{1}{2}\gamma\alpha^2\sigma^2 - g. \quad (30)$$

Equation (30) subtracts the population growth rate  $g$  from the previous formula for the sustainable consumption-wealth ratio. Sustainability of individual utility is a more demanding requirement in the presence of population growth. However, for realistic population growth rates the central message of the paper remains unchanged: in the presence of risk, the sustainable consumption-wealth ratio exceeds the riskfree interest rate.

### 3 Sustainability Without a Riskless Asset

We might alternatively deal with the risky share,  $\alpha$ , by requiring that it should equal one in equilibrium. In this case we are implicitly normalizing the risky asset return to equal the return on society's invested wealth, and are considering a situation where society cannot eliminate risk no matter how much it might wish to do so. In such an equilibrium, the riskless interest rate adjusts to make society content to bear the inescapable risk of its invested wealth.

Setting  $\alpha = 1$  in equation (8), the risk premium satisfies the familiar equation

$$\mu = \gamma\sigma^2. \quad (31)$$

Setting  $\alpha = 1$  in equation (13),

$$\theta_{\text{con}} = r_f + \frac{1}{2}\gamma\sigma^2. \quad (32)$$

These two equations imply, as in equation (14), that the sustainable consumption-wealth ratio has a simple relationship to observable rates of return: it is the average of the interest rate,  $r_f$ , and the expected return on wealth,  $r_f + \mu$ :

$$\theta_{\text{con}} = \frac{1}{2}r_f + \frac{1}{2}(r_f + \mu). \quad (33)$$

As noted in the discussion following equation (14), the neatness of this result is special to the

Brownian motion setting, but in the presence of jumps we can derive bounds on the sustainable consumption-wealth ratio. Depending on the nature of the jumps, either the upper or lower bound can be arbitrarily close to being tight.

To analyze this more general case, we adapt the methods of Martin (2013). It will be convenient to write  $L = 1 - e^{-J}$  where  $J$  is a random variable that can take positive or negative values, and which has moment-generating function (MGF)  $\mathbf{m}(x) = \text{E}e^{Jx}$ . (Large positive  $J$  is very bad news, large negative  $J$  is very good news.) Setting  $\alpha = 1$  in equation (44) of the Appendix, we can generalize the risk premium (31) to allow for jumps:

$$\mu = \gamma\sigma^2 + \omega \{ \mathbf{m}(\gamma) - \mathbf{m}(\gamma - 1) - [\mathbf{m}(0) - \mathbf{m}(-1)] \}. \quad (34)$$

Correspondingly, we can generalize the sustainable consumption-wealth ratio to allow for jumps by setting  $\alpha = 1$  in equation (50) of the Appendix:

$$\theta_{\text{con}} = r_f + \frac{1}{2}\gamma\sigma^2 + \omega \left[ \mathbf{m}(\gamma) - \mathbf{m}(\gamma - 1) - \frac{\mathbf{m}(\gamma - 1) - \mathbf{m}(0)}{\gamma - 1} \right]. \quad (35)$$

The terms that appear inside the square brackets on the right hand side of equation (35) are collectively positive because  $\mathbf{m}(x)$  is convex. This fact supplies a lower bound on the maximum sustainable consumption-wealth ratio,  $\theta_{\text{con}}$ , that is greater than the riskless rate.

To derive a bound in the opposite direction, we can use equation (34) to rewrite equation (35) as

$$\theta_{\text{con}} = \mu + r_f - \frac{1}{2}\gamma\sigma^2 - \omega \left[ \frac{\mathbf{m}(\gamma - 1) - \mathbf{m}(0)}{\gamma - 1} - \frac{\mathbf{m}(0) - \mathbf{m}(-1)}{1} \right]. \quad (36)$$

The terms in square brackets on the right hand side of equation (36) are also positive (by the convexity of  $\mathbf{m}(x)$  once again), so equation (36) supplies an upper bound on the maximum sustainable consumption-wealth ratio that is lower than the return on the risky asset. To sum up, we have

$$r_f + \frac{1}{2}\gamma\sigma^2 \leq \theta_{\text{con}} \leq \mu + r_f - \frac{1}{2}\gamma\sigma^2. \quad (37)$$

In the pure Brownian case,  $\mu = \gamma\sigma^2$  so the bounds (37) collapse to give equation (32) once again. More generally, the sustainable consumption-wealth ratio must lie between the riskless return and the expected return on wealth no matter what we assume about the nature of the jump

size distribution or arrival rate. Moreover, the bounds (37) are best possible, in the sense that different assumptions on the jump arrival rate and size distribution can drive  $\theta_{\text{con}}$  arbitrarily close either to the upper or to the lower bound.

For example, we can choose the jump arrival rate and size distribution so that the terms in square brackets in (36) are close to zero while the terms in square brackets in (35) are not; then the upper bound is tight. This happens if the average slope of the MGF of  $J$  is much smaller in magnitude between  $-1$  and  $\gamma - 1$  than it is between  $\gamma - 1$  and  $\gamma$ , as can happen in the frightening case with occasional bad news jumps. Conversely, it is possible to arrange for the terms in square brackets in (35) to be close to zero, while those in (36) are not; then the lower bound is tight. This happens if the average slope of the MGF of  $J$  is (negative, and) much larger in magnitude between  $-1$  and  $0$  than it is between  $0$  and  $\gamma$ , as can happen if there are occasional good news jumps.

### 3.1 Numerical examples without a riskless asset

We illustrate these properties numerically in two different ways. Figure 1 shows the constrained consumption-wealth ratio and the upper and lower bounds given in equation (37) for a model with risk aversion  $\gamma = 2$ , an expected return on the risky asset  $r_f + \mu = 6\%$ , Brownian volatility  $\sigma = 10\%$ , a jump probability  $\omega = 2\%$ , and jumps of deterministic size  $L$ . The horizontal axis shows different values for  $L$ , where positive values correspond to negative jumps (losses) in wealth, and negative values correspond to positive jumps in wealth. In the left panel, the constrained consumption-wealth ratio  $\theta_{\text{con}}$  is plotted along with the expected risky asset return  $r_f + \mu$  (constant at 6%) and the riskfree interest rate  $r_f$ . The constrained consumption-wealth ratio is halfway between the two returns in the Brownian case; it is closer to the risky asset return in the bad-jump region where  $L > 0$ , and closer to the riskfree interest rate in the good-jump region where  $L < 0$ . In the right panel, the constrained consumption-wealth ratio is plotted along with the upper and lower bounds from equation (37). The bounds are tight in the Brownian case, and widen out as the absolute jump size increases.

Table 3 reports numerical results for variations of this model. Here, in the benchmark case the jump size  $L = 0.4$ . The constrained consumption-wealth ratio equals 4.5% and the riskfree interest rate is only 2.6%. The remaining rows of the table consider variations of the benchmark model with higher and lower risk aversion, then higher and lower risky asset returns, Brownian volatility,

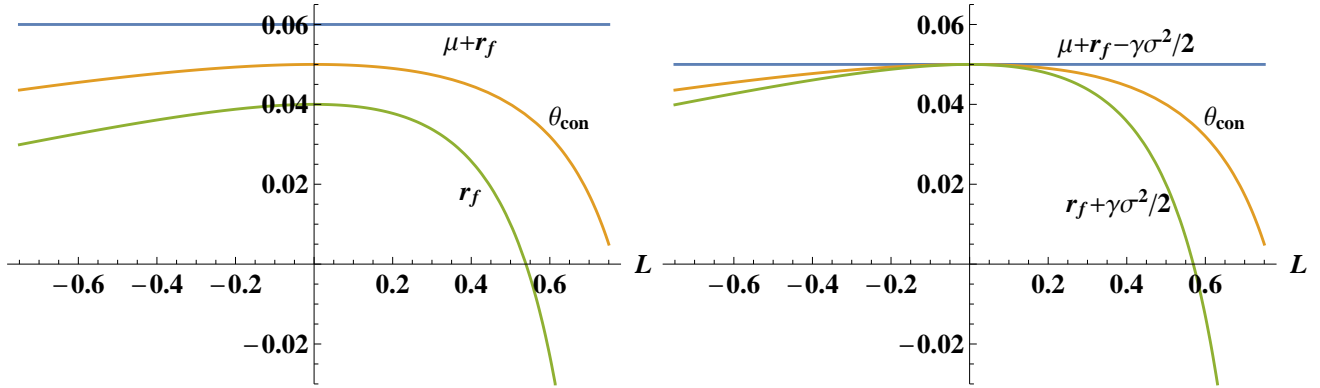


Figure 1:  $\theta_{\text{con}}$  and the upper and lower bounds for various deterministic jump sizes  $L$ , with  $\gamma = 2$ ,  $\sigma = 0.1$ ,  $\omega = 0.02$ ,  $\mu + r_f = 0.06$ . Jumps are bad news if  $L$  is positive and good news if  $L$  is negative.

jump probability, and jump size. The results are intuitive and in all cases imply a sustainable consumption-wealth ratio well above the riskfree interest rate, or equivalently a substantial positive expected growth rate of marginal utility.

## 4 Sustainability and the Ramsey Rule

We have interpreted sustainability as requiring that expected utility should not be allowed to decline over time. Following Rawls (1999), one can imagine representatives of each generation attempting to agree (at time 0, behind the veil of ignorance) on a savings policy that gives each generation the same expected utility. Put another way, these representatives maximize the expected utility of the worst-off generation, following a “max-min” criterion. Equation (6) shows that this is equivalent to ensuring that  $E_0 W_t^{1-\gamma}/(1-\gamma)$  is independent of  $t$ , or (as the consumption-wealth ratio is constant) to ensuring that  $E_0 C_t^{1-\gamma}/(1-\gamma)$  is constant over time.

We can also consider the possibility that the representatives aim to maximize average utility across generations. Given the discussion in the previous paragraph, this is equivalent to maximizing

$$\int_{t=0}^{\infty} E_0 \frac{C_t^{1-\gamma}}{1-\gamma} dt = E_0 \int_{t=0}^{\infty} \frac{C_t^{1-\gamma}}{1-\gamma} dt.$$

Table 3: Numerical examples without a riskless asset

Baseline calibration sets  $\mu + r_f = 6\%$ ,  $\sigma = 10\%$ ,  $\omega = 2\%$ ,  $L = 0.4$ ,  $\gamma = 2$  in equilibrium with  $\alpha = 1$ .

	$\gamma$	$\mu + r_f$	$\sigma$	$\omega$	$L$	$\theta_{\text{con}}$	$r_f$	$\mu$	$E \frac{dW}{W}$	$E d \log W$	$E \frac{dW^{-\gamma}}{W^{-\gamma}}$
Baseline	2	0.06	0.10	0.02	0.40	0.045	0.026	0.034	0.015	0.008	0.019
High $\gamma$	5					0.009	-0.085	0.145	0.051	0.043	0.094
Low $\gamma$	1					0.053	0.045	0.015	0.007	0	0.008
High $\mu + r_f$		0.10				0.085	0.066	0.034	0.015	0.008	0.019
Low $\mu + r_f$		0.02				0.005	-0.014	0.034	0.015	0.008	0.019
High $\sigma$			0.15			0.032	0.001	0.059	0.028	0.014	0.031
Low $\sigma$			0.06			0.051	0.039	0.021	0.009	0.005	0.012
High $\omega$				0.04		0.039	0.012	0.048	0.021	0.011	0.028
Low $\omega$				0		0.050	0.040	0.020	0.010	0.005	0.010
High $L$					0.50	0.040	0.010	0.050	0.020	0.011	0.030
Low $L$					0.30	0.047	0.034	0.026	0.013	0.006	0.014
Negative $L$					-0.40	0.048	0.036	0.024	0.012	0.006	0.012

This is the problem faced by an unconstrained agent with pure time preference rate  $\rho = 0$ . It leads to the savings rule proposed by Ramsey (1928), who argued on ethical grounds that the rate of pure time preference should be zero. Setting  $\rho = 0$  in equation (9), we arrive at the Ramsey consumption-wealth ratio

$$\theta_{\text{Ramsey}} = \frac{\gamma - 1}{\gamma} \left( r_f + \frac{1}{2} \gamma \alpha^2 \sigma^2 \right).$$

There is therefore a simple relationship between the sustainable consumption-wealth ratio (13) and the Ramsey consumption-wealth ratio:

$$\theta_{\text{Ramsey}} = \frac{\gamma - 1}{\gamma} \theta_{\text{con}}. \quad (38)$$

The same relationship continues to hold in the presence of jumps, as we show in the Appendix.

The two rules are similar at very high levels of risk aversion, but the Ramsey rule is substantially more conservative at plausible values of  $\gamma$ , as illustrated in Table 4. Holding wealth fixed, the Ramsey rule dictates 10% less consumption than our sustainable rule if  $\gamma = 10$ ; 20% less consumption if  $\gamma = 5$ , and 50% less consumption if  $\gamma = 2$ . In the log utility case  $\gamma = 1$ , the Ramsey rule cannot be implemented at all, as it sets the consumption-wealth ratio equal to the rate of time



Table 4: Sustainable consumption-wealth ratio versus Ramsey consumption-wealth ratio

Baseline Brownian calibration with  $r_f = 1\%$ ,  $\mu = 8\%$ ,  $\sigma = 20\%$ .

<i>Sustainable rule</i>					<i>Ramsey rule</i>				
$\gamma$	$\theta_{\text{con}}$	$E \frac{dW}{W}$	$E d \log W$	$E \frac{dW^{-\gamma}}{W^{-\gamma}}$	$\gamma$	$\theta_{\text{Ramsey}}$	$E \frac{dW}{W}$	$E d \log W$	$E \frac{dW^{-\gamma}}{W^{-\gamma}}$
1	0.09	0.08	0	0.08	1	0	0.17	0.09	-0.01
2	0.05	0.04	0.02	0.04	2	0.025	0.065	0.045	-0.01
5	0.026	0.016	0.0128	0.016	5	0.021	0.021	0.018	-0.01
10	0.018	0.008	0.0072	0.008	10	0.016	0.010	0.009	-0.01

preference—that is, to zero.

Unsurprisingly, Table 4 also shows that the Ramsey rule implies faster average growth of wealth and log wealth, and lower growth of marginal utility, than the sustainable consumption rule. For all levels of risk aversion, the Ramsey rule has marginal utility declining at the riskless interest rate, assumed to be 1% in Table 4. Again the differences between the Ramsey and sustainable policies are substantial for plausible levels of risk aversion.

## 5 Conclusion

We have argued, in the spirit of Koopmans (1960, 1967), that the implication of an ethical criterion—sustainability—for social discounting and consumption decisions depends on the production technology available to society. Specifically, in a risky world with a binding sustainability constraint, the sustainable social rate of time preference and consumption-wealth ratio, which equal one another, are not equal to either the riskless interest rate or the risky return on invested wealth, but lie in between these two. In the special case where invested wealth has only Brownian risk and no jump risk, the sustainable social rate of time preference is the equal-weighted average of the riskless interest rate and the risky return.

We have made this point in the context of a model in which the parameters governing the distribution of returns are known. We have therefore ignored parameter uncertainty, a phenomenon emphasized by Weitzman (2001). We have also ignored the possibility that returns may not be

i.i.d., because expected returns or risks change over time. Models with non-i.i.d. returns in general imply time-varying consumption growth and a term structure of discount rates. When consumption growth is persistent, this term structure is generally downward-sloping for safe investments and upward-sloping for risky ones as in the long-run risk model of Bansal and Yaron (2004). Gollier (2002) emphasizes the potential importance of a downward-sloping term structure of discount rates for social discounting. Our i.i.d. model has discount rates that are invariant to the horizon of an investment.

Although we have emphasized the sustainable social rate of time preference, we conclude by noting that this is not the same as the appropriate social discount rate that should be applied to an investment project. That discount rate depends on the project's risk. For a riskless project, the appropriate discount rate is the riskless interest rate, which is lower than the sustainable social rate of time preference in a risky world; and for a project that has the same risk as society's invested wealth, the appropriate discount rate is the expected risky return, which is higher than the sustainable social rate of time preference. Some previous discussions of social discounting have obscured these distinctions by ignoring the risk that society faces. Our analysis is deliberately simple in order to achieve clarity about these issues.

## References

- Arrow, Kenneth, Partha Dasgupta, Lawrence Goulder, Gretchen Daily, Paul Ehrlich, Geoffrey Heal, Simon Levin, Karl-Göran Mäler, Stephen Schneider, David Starrett, and Brian Walker, 2004, “Are We Consuming Too Much?”, *Journal of Economic Perspectives* 18, 147–172.
- Asheim, Geir B., 2007, *Justifying, Characterizing and Indicating Sustainability*, Springer, Dordrecht, The Netherlands.
- Bansal, Ravi, and Amir Yaron, 2004, “Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles”, *Journal of Finance* 59, 1481–1509.
- Blanchard, Olivier J., 1985, “Debt, Deficits, and Finite Horizons”, *Journal of Political Economy* 93, 223–247.
- Campbell, John Y. and Roman Sigalov, 2022, “Portfolio Choice with Sustainable Spending: A Model of Reaching for Yield”, *Journal of Financial Economics* 143, 188–206.
- Cline, William R., 1992, *The Economics of Global Warming*, Institute for International Economics, Washington, DC.
- Cont, Rama, and Peter Tankov, 2004, *Financial Modelling with Jump Processes*, Chapman & Hall/CRC, Boca Raton, FL.
- Dasgupta, Partha, 2001, *Human Well-Being and the Natural Environment*, Oxford University Press, Oxford, UK.
- Dasgupta, Partha, 2008, “Discounting Climate Change”, *Journal of Risk and Uncertainty* 37, 141–169.
- Dasgupta, Partha, 2021, *The Economics of Biodiversity: The Dasgupta Review*, HM Treasury, London, UK.
- Dasgupta, Partha, and Geoffrey Heal, 1974, “The Optimal Depletion of Exhaustible Resources”, *Review of Economic Studies* 41, 3–28.
- Dybvig, Philip H., 1995, “Duesenberry’s Ratcheting of Consumption: Optimal Dynamic Consumption and Investment Given Intolerance for any Decline in Standard of Living”, *Review of Economic Studies* 62, 287–313.
- Gollier, Christian, 2002, “Discounting an Uncertain Future”, *Journal of Public Economics* 85, 149–166.

- Gollier, Christian, 2021, “The Welfare Cost of Ignoring the Beta”, unpublished paper, Toulouse School of Economics.
- Hartwick, John M., 1977, “Intergenerational Equity and the Investing of Rents from Exhaustible Resources”, *American Economic Review* 67, 972–974.
- Howarth, Richard B., 1995, “Sustainability Under Uncertainty: A Deontological Approach,” *Land Economics* 71, 417–427.
- Koopmans, Tjalling, 1960, “Stationary Ordinal Utility and Impatience”, *Econometrica* 28, 287–309.
- Koopmans, Tjalling, 1967, “Objectives, Constraints, and Outcomes in Optimal Growth Models”, *Econometrica* 35, 1–15.
- Llavador, Humberto, John E. Roemer, and Joaquim Silvestre, 2015, *Sustainability for a Warming Planet*, Harvard University Press, Cambridge, MA.
- Martin, Ian W. R., 2012, “On the Valuation of Long-Dated Assets”, *Journal of Political Economy* 120, 346–358.
- Martin, Ian W. R., 2013, “Consumption-Based Asset Pricing with Higher Cumulants”, *Review of Economic Studies* 80, 745–773.
- Merton, Robert C., 1969, “Lifetime Portfolio Selection Under Uncertainty: The Continuous-Time Case”, *Review of Economics and Statistics* 51, 247–257.
- Merton, Robert C., 1971, “Optimum Consumption and Portfolio Rules in a Continuous-Time Model”, *Journal of Economic Theory* 3, 373–413.
- Nordhaus, William D., 1994, *Managing the Global Commons: The Economics of Climate Change*, MIT Press, Cambridge, MA.
- Parfit, Derek, 1984, *Reasons and Persons*, Oxford University Press, Oxford, UK.
- Pezzey, John, 1992, “Sustainability: An Interdisciplinary Guide,” *Environmental Values* 1, 321–362.
- Ramsey, Frank P., 1928, “A Mathematical Theory of Saving”, *Economic Journal* 38, 543–559.
- Rawls, John, 1999, *A Theory of Justice*, Revised edition, Oxford University Press, Oxford, UK.
- Solow, Robert M., 1974, “Intergenerational Equity and Exhaustible Resources”, *Review of Economic Studies* 41, 29–46.

- Solow, Robert M., 1993, “An Almost Practical Step Toward Sustainability,” *Resources Policy*, 19, 162–172
- Stern, Nicholas H., 2006, *Stern Review: The Economics of Climate Change*, Cambridge University Press, Cambridge, UK.
- Stern, Nicholas H., 2016, *Why Are We Waiting? The Logic, Urgency, and Promise of Tackling Climate Change*, MIT Press, Cambridge, MA.
- Weitzman, Martin L., 2007, “A Review of *The Stern Review on the Economics of Climate Change*,” *Journal of Economic Literature* 45, 703–724.
- Weitzman, Martin L., 2009, “On Modeling and Interpreting the Economics of Catastrophic Climate Change”, *Review of Economics and Statistics* 91, 1–19.
- World Commission on Environment and Development, 1987, *Our Common Future*, Oxford University Press, New York, NY.
- Zeckhauser, Richard J., and W. Kip Viscusi, 2008, “Discounting Dilemmas: Editors’ Introduction”, *Journal of Risk and Uncertainty* 37, 95–106.

## A Appendix

### A.1 Unconstrained Consumption and Portfolio Choice

It follows by applying Itô’s formula for semimartingales to (2) that

$$d \log C = \left( r_f + \alpha \hat{\mu} - \frac{1}{2} \alpha^2 \sigma^2 - \theta \right) dt + \alpha \sigma dZ + \log(1 - \alpha L) dN. \quad (39)$$

See, for example, Proposition 8.19 of Cont and Tankov (2004). Heuristically, we can derive it by writing

$$d \log C = \frac{1}{C} dC - \frac{1}{2!} \frac{1}{C^2} (dC)^2 + \frac{1}{3!} \frac{2}{C^3} (dC)^3 - \frac{1}{4!} \frac{6}{C^4} (dC)^4 + \dots$$

and using the relationships  $dt dN = dZ dN = 0$  and  $dN^k = dN$  for all  $k > 0$ , in addition to the standard properties of  $dZ$  and the fact that  $\log(1 + x) = x - x^2/2 + x^3/3 - x^4/4 + \dots$  if  $|x| < 1$ , which holds when  $x = -\alpha L$  because the agent will never risk bankruptcy.

Integrating forwards, exponentiating, using  $C_0 = \theta W_0$ , and raising to the power  $1 - \gamma$ , we have

$$C_t^{1-\gamma} = W_0^{1-\gamma} \theta^{1-\gamma} \exp \left\{ (1-\gamma) \left( r_f + \alpha \widehat{\mu} - \frac{1}{2} \alpha^2 \sigma^2 - \theta \right) t + \alpha (1-\gamma) \sigma Z_t \right\} \prod_{i=1}^{N_t} (1 - \alpha L_i)^{1-\gamma}. \quad (40)$$

Writing  $L$  for a representative of the i.i.d.  $L_i$ , we have

$$\mathbb{E} C_t^{1-\gamma} = W_0^{1-\gamma} \theta^{1-\gamma} \exp \left\{ (1-\gamma) \left( r_f + \alpha \widehat{\mu} - \frac{1}{2} \gamma \alpha^2 \sigma^2 - \theta \right) t + \omega \mathbb{E} \left[ (1 - \alpha L)^{1-\gamma} - 1 \right] t \right\}. \quad (41)$$

This holds because  $N_t$ ,  $Z_t$ , and  $L_i$  are independent. By the law of iterated expectations, the fact that  $N_t$  is a Poisson random variable with parameter  $\omega t$ , the i.i.d. nature of the  $L_i$ , and the series definition of the exponential function,

$$\begin{aligned} \mathbb{E} \prod_{i=1}^{N_t} (1 - \alpha L_i)^{1-\gamma} &= \mathbb{E} \left[ \mathbb{E} \left( \prod_{i=1}^{N_t} (1 - \alpha L_i)^{1-\gamma} \mid N_t \right) \right] \\ &= \sum_{n=0}^{\infty} e^{-\omega t} \frac{(\omega t)^n}{n!} \mathbb{E} \prod_{i=1}^n (1 - \alpha L_i)^{1-\gamma} \\ &= \sum_{n=0}^{\infty} e^{-\omega t} \frac{(\omega t)^n}{n!} (\mathbb{E} [(1 - \alpha L)^{1-\gamma}])^n \\ &= \exp \left\{ \omega \mathbb{E} [(1 - \alpha L)^{1-\gamma} - 1] t \right\}. \end{aligned}$$

Hence the objective function can be evaluated explicitly, as

$$U_0 = \frac{W_0^{1-\gamma} \theta^{1-\gamma}}{1 - \gamma \rho - (1 - \gamma) \left( r_f + \alpha \widehat{\mu} - \frac{1}{2} \gamma \alpha^2 \sigma^2 - \theta \right) - \omega \mathbb{E} [(1 - \alpha L)^{1-\gamma} - 1]}. \quad (42)$$

The optimal investment and consumption choices are identified by maximizing (42) with respect to  $\alpha$  and  $\theta$ . The optimal consumption-wealth ratio is

$$\theta_{\text{unc}} = \frac{\rho + (\gamma - 1) \left( r_f + \alpha \widehat{\mu} - \frac{1}{2} \gamma \alpha^2 \sigma^2 \right) - \omega \mathbb{E} [(1 - \alpha L)^{1-\gamma} - 1]}{\gamma}. \quad (43)$$

We assume that  $\theta_{\text{unc}}$  is positive when  $\alpha$  and  $\theta$  are chosen optimally. It is easy to check that this implies that the denominator of (42) is positive, and hence that the integral in the definition of expected utility converges.

The optimal risky portfolio share is defined implicitly by

$$\widehat{\mu} - \alpha\gamma\sigma^2 = \omega\mathbb{E} [L(1 - \alpha L)^{-\gamma}]. \quad (44)$$

Using (44) to eliminate  $\widehat{\mu}$  in (43), we can also write

$$\theta_{\text{unc}} = \frac{\rho + (\gamma - 1)(r_f + \frac{1}{2}\gamma\alpha^2\sigma^2) - \omega\mathbb{E} [(1 - \alpha\gamma L)(1 - \alpha L)^{-\gamma} - 1]}{\gamma}. \quad (45)$$

## A.2 A Sustainability Constraint

Equation (42) shows that expected utility at time  $t$ ,  $U_t$ , is proportional to  $W_t^{1-\gamma}/(1-\gamma)$ . As in the main text, we multiply by  $1-\gamma$  (which is negative under our maintained assumption that  $\gamma > 1$ ) and work with a rescaled variable  $X_t = W_t^{1-\gamma}$ . This follows the process

$$\frac{dX}{X} = (1-\gamma) \left( r_f + \alpha\widehat{\mu} - \theta - \frac{1}{2}\gamma\alpha^2\sigma^2 \right) dt + (1-\gamma)\alpha\sigma dZ + [(1-\alpha L)^{1-\gamma} - 1] dN. \quad (46)$$

To see this, note that

$$\begin{aligned} \frac{dX}{X} &= (1-\gamma) \frac{dW}{W} + \frac{\gamma(\gamma-1)}{2} \left( \frac{dW}{W} \right)^2 - \frac{\gamma(\gamma-1)(\gamma+1)}{6} \left( \frac{dW}{W} \right)^3 + \dots \\ &= (1-\gamma) \left( r_f + \alpha\widehat{\mu} - \theta - \frac{1}{2}\gamma\alpha^2\sigma^2 \right) dt + (1-\gamma)\alpha\sigma dZ + \\ &\quad + \left[ (\gamma-1)\alpha L + \frac{\gamma(\gamma-1)}{2}\alpha^2 L^2 + \frac{\gamma(\gamma-1)(\gamma+1)}{6}\alpha^3 L^3 + \dots \right] dN \\ &= (1-\gamma) \left( r_f + \alpha\widehat{\mu} - \theta - \frac{1}{2}\gamma\alpha^2\sigma^2 \right) dt + (1-\gamma)\alpha\sigma dZ + [(1-\alpha L)^{1-\gamma} - 1] dN. \end{aligned}$$

The drift of  $dX/X$  (by which we mean, heuristically,  $\frac{1}{dt} \mathbb{E} \frac{dX}{X}$ ) is therefore

$$(1-\gamma) \left( r_f + \alpha\widehat{\mu} - \theta - \frac{1}{2}\gamma\alpha^2\sigma^2 \right) + \omega \mathbb{E} [(1-\alpha L)^{1-\gamma} - 1], \quad (47)$$

where we have used the fact that  $\mathbb{E} dN = \omega dt$ . If the consumption-wealth ratio,  $\theta$ , is sufficiently large then  $X$  has positive drift, and hence expected utility has negative drift: the optimal consumption-investment decision induces declining expected utility over time, on average.

The sustainability constraint therefore implies that the expression (47) equals zero, and hence—writing  $\theta_{\text{con}}$  for the case in which the constraint binds—that

$$\theta_{\text{con}} = r_f + \alpha\hat{\mu} - \frac{1}{2}\gamma\alpha^2\sigma^2 + \omega \frac{\text{E}[(1 - \alpha L)^{1-\gamma} - 1]}{1 - \gamma}. \quad (48)$$

Similarly, we write  $\theta_{\text{unc}}$  for the case in which the constraint does not bind, so that the optimal choice is given by equation (45). The optimal consumption-wealth ratio is independent of  $\rho$  if the constraint binds; this is not true of the unconstrained case.

If the constraint binds, we can use it to eliminate  $\theta$  from the objective function (42), giving

$$U_{\text{con},0} = \frac{W_0^{1-\gamma} \left( r_f + \alpha\hat{\mu} - \frac{1}{2}\gamma\alpha^2\sigma^2 + \omega \frac{\text{E}[(1-\alpha L)^{1-\gamma}-1]}{1-\gamma} \right)^{1-\gamma}}{1-\gamma \rho}. \quad (49)$$

Conveniently, the optimal investment choice is unaffected by the presence of the sustainability constraint. Maximizing equation (49) with respect to  $\alpha$ , we find the same first-order condition as before, equation (44).

We can use the condition (44) to eliminate  $\hat{\mu}$  from equation (48), giving

$$\theta_{\text{con}} = r_f + \frac{1}{2}\gamma\alpha^2\sigma^2 + \omega \frac{\text{E}[(1 - \alpha\gamma L)(1 - \alpha L)^{-\gamma} - 1]}{1 - \gamma}. \quad (50)$$

When risk and the risky portfolio share are positive, the second and third terms on the right hand side of equation (50) are positive. Hence (50) shows that the constrained consumption-wealth ratio  $\theta_{\text{con}}$  exceeds the riskfree interest rate  $r_f$  in a risky economy. Conversely, it follows from conditions (44) and (50) that  $\theta_{\text{con}}$  is less than the expected return on optimally invested wealth,  $r_f + \alpha\mu$ .

Comparing equations (45) and (50), we see that (17) holds, as in the main text.

A binding sustainability constraint implies zero drift in expected utility, but this does not imply a zero drift in wealth. Instead wealth drifts upwards over time under our assumption that  $\gamma > 1$ , because

$$\frac{dW}{W} = \left\{ \frac{1}{2}\gamma\alpha^2\sigma^2 - \frac{\omega}{1-\gamma} \text{E}[(1 - \alpha L)^{1-\gamma} - 1] \right\} dt + \alpha\sigma dZ - \alpha L dN, \quad (51)$$



so the drift is

$$\mathbb{E} \frac{dW}{W} = \left( \frac{1}{2} \gamma \alpha^2 \sigma^2 + \frac{\omega}{\gamma - 1} \mathbb{E} [(1 - \alpha L)^{1-\gamma} - 1 + \alpha L(1 - \gamma)] \right) dt, \quad (52)$$

and both terms in the brackets are positive.

We can also show the stronger result that log wealth drifts upwards over time under a binding sustainability constraint. We have

$$\mathbb{E} d \log W = \left( \frac{1}{2} (\gamma - 1) \alpha^2 \sigma^2 + \frac{\omega}{\gamma - 1} \mathbb{E} [(1 - \alpha L)^{1-\gamma} - 1 - (1 - \gamma) \log(1 - \alpha L)] \right) dt, \quad (53)$$

and again both terms in the brackets are positive.

The process for marginal utility,  $M = W^{-\gamma}$ , is

$$\frac{dM}{M} = \left\{ \frac{1}{2} \gamma \alpha^2 \sigma^2 + \frac{\omega \gamma}{1 - \gamma} \mathbb{E} [(1 - \alpha L)^{1-\gamma} - 1] \right\} dt - \gamma \alpha \sigma dZ + [(1 - \alpha L)^{-\gamma} - 1] dN, \quad (54)$$

so the drift is

$$\mathbb{E} \frac{dM}{M} = \left( \frac{1}{2} \gamma \alpha^2 \sigma^2 + \frac{\omega}{1 - \gamma} \left\{ \gamma \mathbb{E} [(1 - \alpha L)^{1-\gamma} - 1] + (1 - \gamma) \mathbb{E} [(1 - \alpha L)^{-\gamma} - 1] \right\} \right) dt, \quad (55)$$

which is positive when  $\gamma > 1$ . The drift in marginal utility equals the drift in wealth in the Brownian case where  $L = 0$ , but differs from it in the general case with jumps.

As in the main text, we can check that

$$\mathbb{E} \frac{dM}{M} = \theta_{\text{con}} - r_f \quad (56)$$

by comparing the right hand sides of equations (50) and (55).

In the presence of population growth, noting that

$$\frac{d\tilde{X}}{\tilde{X}} = g(\gamma - 1) dt + \frac{dX}{X} = (1 - \gamma) \left( r_f + \alpha \hat{\mu} - \theta - \frac{1}{2} \gamma \alpha^2 \sigma^2 - g \right) dt + (1 - \gamma) \alpha \sigma dZ + [(1 - \alpha L)^{1-\gamma} - 1] dN, \quad (57)$$

the condition that  $\tilde{X}_t$  has nonpositive drift modifies the sustainability constraint (48) to

$$\theta \leq r_f + \alpha \hat{\mu} - \frac{1}{2} \gamma \alpha^2 \sigma^2 + \omega \frac{\mathbb{E} [(1 - \alpha L)^{1-\gamma} - 1]}{1 - \gamma} - g. \quad (58)$$

Equation (58) subtracts the population growth rate  $g$  from the previous formula for the sustainable consumption-wealth ratio.

Setting  $\rho = 0$  in equation (45), we arrive at the Ramsey consumption-wealth ratio

$$\theta_{\text{Ramsey}} = \frac{\gamma - 1}{\gamma} \left\{ r_f + \frac{1}{2} \gamma \alpha^2 \sigma^2 + \omega \frac{\text{E} [(1 - \alpha \gamma L) (1 - \alpha L)^{-\gamma} - 1]}{1 - \gamma} \right\}.$$

Comparing this expression with the sustainable consumption-wealth ratio (50), we recover the relationship (38) derived in the main text.

## B The log utility case

With log utility, the investor's objective function is

$$U = \text{E} \int_0^{\infty} e^{-\rho t} \log C_t dt, \quad \text{where } \rho > 0.$$

It follows from equation (39) that

$$\log C_t = \log C_0 + \left( r_f + \alpha \hat{\mu} - \frac{1}{2} \alpha^2 \sigma^2 - \theta \right) dt + \alpha \sigma Z_t + \sum_{i=1}^{N_t} \log (1 - \alpha L_i),$$

and hence

$$\text{E} \log C_t = \log C_0 + \left( r_f + \alpha \hat{\mu} - \frac{1}{2} \alpha^2 \sigma^2 - \theta \right) dt + \omega \text{E} [\log (1 - \alpha L)] t.$$

Thus the objective function can be evaluated explicitly as

$$U = \frac{\log W_0 + \log \theta}{\rho} + \frac{r_f + \alpha \hat{\mu} - \frac{1}{2} \alpha^2 \sigma^2 - \theta + \omega \text{E} [\log (1 - \alpha L)]}{\rho^2}.$$

Maximizing with respect to  $\theta$  and  $\alpha$  we find the first-order conditions for an unconstrained optimum,

$$\theta = \rho \quad \text{and} \quad \hat{\mu} - \alpha \sigma^2 = \omega \text{E} [L (1 - \alpha L)^{-1}].$$

The objective function at time  $t$  is affine in  $\log W_t$ , so the sustainability condition requires that

$d \log W_t$ , or equivalently  $d \log C_t$ , is driftless, i.e. that

$$\theta \leq r_f + \alpha \hat{\mu} - \frac{1}{2} \alpha^2 \sigma^2 + \omega \mathbb{E} [\log (1 - \alpha L)].$$

We define the constrained solution as before, giving

$$\theta_{\text{con}} = r_f + \alpha \hat{\mu} - \frac{1}{2} \alpha^2 \sigma^2 + \omega \mathbb{E} [\log (1 - \alpha L)].$$

When the constraint binds, we have

$$U = \frac{\log W_0 + \log \theta}{\rho},$$

so  $\alpha$  is chosen to maximize the constrained consumption-wealth ratio. We end up with the same first-order condition as in the unconstrained case. Thus the optimal investment choice is the same in the constrained and unconstrained cases, as before. Equations (16) and (17) also hold as before. Thus, all the results stated previously for risk aversion  $\gamma > 1$  carry over to the log case where  $\gamma = 1$ .