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PRIORITY-BASED ASSIGNMENT WITH RESERVES AND QUOTAS

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ABSTRACT

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Abstract

We study priority-based assignment problems with distributional and diversity objectives. Our work provides an axiomatic characterization of a general class of choice rules which are based on type-specific reserves and quotas. The choice rules in the class differ by the order in which applicants are considered for units reserved for different types. We show that a particular reserves- and quotas-based choice rule, where all applicants are first considered for units reserved for their own types, uniquely minimizes priority violations in this class.

1 Introduction

Allocation of scarce resources commonly involves priority-based rationing. Priorities are decided on the basis of merit, needs, property rights or some other legal or ethical consideration. For example, schools prioritize pupils based on neighborhood or sibling enrollment status, colleges prioritize students based on academic performance and health official prioritize patients based on urgency or medical condition. An assignment violates an individual's priority, if she prefers another object to her assignment and the object is assigned to a lower priority individual. Avoiding priority violations is a major policy goal in all applications above as such violations are perceived unfair, illegal or unethical.

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A purely priority-based assignment completely eliminates priority violations. However, such a solution may result in undesirable distributional outcomes. For example, when a school district grants a high priority to neighborhood children, priority-based assignment may result in high-income families attending best-performing schools, which are potentially located in affluent neighborhoods, while low-income families may end up in worst-performing ones (Ryan and Heise, 2002). Additionally, school districts are typically restricted by court-ordered desegregation guidelines and seek to achieve a diverse student body (Armor and Rossell, 2002). Therefore, in a school district with racially segregated neighborhoods, a neighborhood priority-based rationing would produce racially segregated schools, violating districts' policies and court orders. Likewise, schools may require to maintain student balance across ability range, with sufficient representation of students with low, middle or high reading scores (Abdulkadiroğlu et al., 2005). Diversity is also a major consideration for college admissions. A purely merit-based admission is opposed by advocates of affirmative action policies, who argue that racial minorities and other marginalized groups face discrimination and are historically shut out of education opportunities (Maxwell and Garcia, 2019). Finally, in pandemic rationing a priority-based assignment may fail to guarantee vaccines to individuals representing multiple and conflicting ethical considerations, such individuals with disabilities, highest survival probabilities, health workers or representatives of disproportionately affected groups (Pathak et al., 2020a; Persad et al., 2020).

To address distributional concerns, priority-based assignment problems are commonly controlled by reserves and quotas for different groups. Reserves guarantee a certain number of objects for the members of a group. Quotas bound the number of objects allocated to the members of a group by a certain number. Reserves and quotas are commonly applied for race-neutral or race-based affirmative action policies in public school assignment (Abdulkadiroğlu and Sönmez, 2003; Hafalir et al., 2013; Dur et al., 2018) or college admissions (Abdulkadiroğlu, 2005; Aygun and Bó, 2020). For example, exam schools in Chicago reserve a certain proportion of seats for representatives of different income groups (Dur et al., 2020; Ellison and Pathak, 2021). COVID-19 vaccine allocation is another application of a reserves system. Vaccine allocation guideline of the National Academy of Sciences, Engineering and Medicine (NASEM, 2020) recommends reserving 10 percent of vaccines for people from hard-hit areas, defined by a socioeconomic vulnerability index. Many US states that have adopted reserves for vaccine allocation.[1]

Reserves and quotas do not uniquely pin down the assignment: there are many assignments

[1]Details on reserves systems for COVID-19 vaccine allocation can be found here: https://www.covid19reservesystem.org/policy-impact.

that respect given reserves and quotas. Consider the following example. There is a single school with two seats and three applicants. The school's objective is to guarantee at least one seat for an applicant from a low-income group. To achieve this goal, the school reserves a seat for low-income applicants and the remaining seat goes to the general pool. We will refer to them as reserved seat and open seat, respectively. Otherwise, the school assigns seats in a priority order which may reflect the applicants' exam scores or neighborhood/sibling status. Suppose that the first and third highest priority applicants are from a low-income group, whereas the second highest priority applicant is not. If applicants are considered for the open seat first, then the open seat goes to the highest priority applicant, who is a low-income applicant. The remaining seat is reserved, so it goes to the remaining low-income applicant. The resulting assignment violates the priority of the second highest priority applicant. Now suppose that applicants are first considered for the reserved seat. In that case the highest priority applicant will be assigned to the reserved seat, and the second highest priority applicant will be assigned to the open seat. No applicant's priority is violated in that case.

As the example demonstrates the amount of priority violations depends on the order in which the applicants are considered for different seats. Our work aims at providing axiomatic foundation for pinning down the 'right' order and the corresponding reserves- and quotas-based rule. We first study a single school's decision of choosing the set of applicants to be admitted to the school. This problem is particularly relevant when allocation of different units are administered in a decentralized manner. Examples include public school admissions systems where each school admits from its applicant independently of other schools (e.g. New York City High School Admissions prior to 2003, Abdulkadiroğlu et al. (2009)), decentralized college admissions (Che and Koh, 2016) and administration of COVID-19 vaccine during the pandemic where US jurisdictions/states receive there vaccines in proportion to their populations and allocate vaccines based on their own guidelines (Galston and Kamarck, 2021).

Formally, a choice rule of a school selects a subset of from each set of applicants. We characterize a class of choice rules which we call generalized reserves-and-quotas rules. These rules differ by the order in which applicants are considered for seats reserved for different types. Most of the rules that have been studied in the literature (e.g., Abdulkadiroğlu (2005), Hafalir et al. (2013), Sönmez and Switzer (2013), Echenique and Yenmez (2015), Dur et al. (2018), Kominers and Sönmez (2016) and Pathak et al. (2020a)) are in this class. As the class is large, many of the generalized reserves-and-quotas rules potentially result in unnecessary priority violations. As a solution, we show that the choice rule where applicants are first considered for seats reserved for their types, which we call regular reserves-and-quotas rule, uniquely minimizes priority violations among all generalized reserves-and-quotas rules. Our characterization result counterpoints the advocacy for implementing alternative rules which may result in distributional outcomes that favor certain types, such as benefiting low-income public school applicants (Dur et al., 2020), assigning more students to neighborhood schools (Dur et al., 2018) or allocating more vaccines or scarce medical supplies to reserves beneficiaries (Pathak et al., 2020a). We show that the improved distributional outcomes only happen at the expense of creating more priority violations. Such violations are not justified or explained in any sense. Our result potentially suggests that improved distributional outcomes should be achieved not through alternative implementations, but through the regular reserves-and-quotas rule with potentially larger reserves or smaller quotas. In addition to its axiomatic foundation, the benefit of the latter approach is transparency in meeting policy goals. The regular reserves-and-quotas rule directly addresses the policy maker's fairness and distributional objectives: namely, it minimizes priority violations among non-wasteful assignments that respect the reserves and quotas. Therefore, unlike the alternatives, any remaining priority violation is fully explained by reserves and quotas policy.

Many priority-based assignment problems such as public school assignment or college admissions feature multiple alternatives and the final assignment is determined in centralized clearinghouses. In the second part of our paper we analyze assignment problems in this centralized environment. Assignments are a commonly achieved through a sequential algorithm, such as Deferred Acceptance (DA) (Gale and Shapley, 1962), where at each step applicants apply to their most preferred schools among those that have not rejected them, each school accepts a subset of applicants using some choice rule and rejects the rest. Our characterization results for the decentralized setting do not necessarily extend to the centralized one. Due to the dynamic nature of the application process, creating more priority violations in some earlier step may prevent creating more priority violations in later steps. Thus, the final assignment may not necessarily minimize priority violations when all schools apply the regular reserves-and-quotas rule at each step. We extend our axiomatic framework to the setting with multiple alternatives and study the possibility of minimizing priority violations in certain classes of assignments. We show that these problems are NP-hard, and therefore computationally intractable.

We supplement our theoretical results with simulations analysis to quantify the amount of priority violations under different choice rules. We consider a school choice setup with an affirmative action policy where each school reserves a proportion of its seats for applicants from low- and high-income groups. We obtain considerably less priority violations under the regular reserves-and-quotas rule compared to the alternative choice rule studied in the literature (Dur et al., 2018, 2020). The results therefore suggest that the theoretical predictions for the single school problem are robust to the environment with multiple schools.

The remainder of this work is organized as follows. Section 2 reviews the related literature. Section 3 introduces the model. Section 4 provides an axiomatic characterization of generalized reserves-and-quotas rules. Section 5 characterizes the regular reserves-and-quotas rule as priority violations minimizer and priority rank maximizer. Section 6 discusses the implications for centralized assignment with multiple alternatives. Section 7 reports simulation results. Proofs are in Appendix A.

2 Related Literature

Affirmative action in school choice has been studied by Abdulkadiroğlu and Sönmez (2003). Their model uses type-specific quotas for promoting diversity. Kojima (2012) gives an example where imposing quotas on majority students may have a perverse effect of hurting minority students. This happens when a rejected majority student at one school applies and displaces a minority student at another school. Hafalir et al. (2013) show by computer simulations that the adverse effects of the quotas rule is a likely phenomenon, rather than an exception. As an alternative solution, the authors propose a reserves-based rule. Kominers and Sönmez (2016) study a class of reserves-based rules that generalize the one in Hafalir et al. (2013). Under those rules, certain number of units are reserved for each type and applicants receive higher priority at their corresponding units. Applicants are considered for different units and are accepted based on unit-specific priority order. Axiomatic characterizations of these generalized reserves rules have been given by Pathak et al. (2020a) and Delacrétaz (2020). The authors use an axiom which requires the assignment to respect unit-specific priorities. However, in many priority-based assignment problems respecting unit-specific priority rankings is not a direct policy objective but a means to achieve desirable distributional outcomes. Instead, these problems typically feature a baseline priority ranking which is common across all units (Abdulkadiroğlu, 2005; Hafalir et al., 2013; Ehlers et al., 2014; Echenique and Yenmez, 2015). Our characterization result uses axioms which directly incorporate policy objectives such as bounding the number of baseline priority violations and meeting the distributional constraints.

The question of which generalized reserves-and-quotas rule should be used has received wide attention in academia and practice (Dur et al., 2018; Pathak et al., 2020b,a). The regular reserves-and-quotas corresponds to the generalized reserves-and-quotas rule where applicants are first considered for units reserved for their types. Papers like Dur et al. (2018) and Pathak et al. (2020b) argue that the beneficiaries may be better-off under an alternative implementation where reserves beneficiaries are first considered for non-reserved units instead. Our characterization result counterpoint the objections for the regular reserves rule and the advocacy for other rules in the literature. In particular, it implies that the improved distributional outcomes of the alternative rules in papers like Dur et al. (2018) and Pathak et al. (2020b) can only happen at the expense of creating more priority violations.

Our work is related papers that provide axiomatic foundation for the regular reserves-and-quotas rule. In a setting without quotas, such results have been provided by Echenique and Yenmez (2015), Sönmez and Yenmez (2019a), Sönmez and Yenmez (2019b) and Grigoryan (2020). The first paper uses an axiom called saturated priority compatibility, which says that an individual can cause a priority violation only if the number of individuals of her type who receive a unit is weakly smaller than the reserves.[2] Imamura (2020) generalizes the characterization by Echenique and Yenmez (2015) to an environment with quotas. Sönmez and Yenmez (2019a) and Sönmez and Yenmez (2019b) study a setting with complex types and without quotas, and characterize a version of the regular reserves-and-quotas rule with an axiom of priority rank maximality, i.e., choosing the set of individuals with highest priorities. In a setting with quotas, we give two characterizations of the regular reserves-and-quotas rule with axioms of priority violations minimality and a stronger version of priority rank maximality than the ones in Sönmez and Yenmez (2019a) and Sönmez and Yenmez (2019b). Our axioms directly incorporate the policy objectives of reducing priority violations and choosing highest priority individual whenever possible. Like our paper, Grigoryan (2020) characterizes the regular reserves-and-quotas rule with an axiom of priority violations minimality. However, the author studies a setting with weak priorities and without quotas. None of the papers above discusses the implication of their characterization results for centralized assignment problems with multiple schools.

Finally, our work is related to papers that study computational complexity of assignment problems with distributional constraints, such as Biró et al. (2010) and Hamada et al. (2016). Both papers show that minimizing priority violations in the class of reserves-respecting assignments is NP-hard. However, their models and NP-hardness results differ from ours. First and most importantly, we model reserves as *soft lower bounds*, i.e., we allow that the number of applicants of a certain types is smaller than the corresponding type-specific

[2]Doğan (2017) corrects the characterization result of Echenique and Yenmez (2015).

reserves, as long as no applicant of that type prefers that school to her assignment. In contrast, Biró et al. (2010) and Hamada et al. (2016) model reserves as *hard lower bounds*.[3] Second, we also prove NP-hardness of minimizing priority violations for a proper subclass of reserves-respecting assignments, namely, those eliminate priority violations among same type applicants.

3 The Model

There is a finite set of applicants $\mathcal{A}$ and a single school $\mathcal{S}$ with capacity q. For a given subset of applicants $A \subseteq \mathcal{A}$, the school is facing a decision of choosing some of the applicants in A, without exceeding its capacity. Formally, a choice rule of the school is a mapping $\mathcal{C} : 2^{\mathcal{A}} \to 2^{\mathcal{A}}$ such that for any $A \subseteq \mathcal{A}$, $\mathcal{C}(A) \subseteq A$ and $|\mathcal{C}(A)| \leq q$.

The school orders applicants according to a priority ranking $\succ$ which is a complete, transitive and anti-symmetric binary relation on $\mathcal{A}$. Applicants are categorized into different types according to mapping $\tau : \mathcal{A} \to T$, where T is a finite set. For a subset $A \subseteq \mathcal{A}$, let

$$A_t = \big\{a \in A : \tau(a) = t\big\}$$

be the set of type-t applicants in A.

For a subset $A \subseteq \mathcal{A}$, we say that a's priority is violated at $\mathcal{C}(A)$, if $a \in A \setminus \mathcal{C}(A)$ and there is $a' \in \mathcal{C}(A)$ with $a \succ a'$. In that case, we say that a' causes a priority violation for a.

4 Generalized Reserves-and-Quotas Rules

In this section, we provide an axiomatic foundation for a general class of reserves- and quotas-based choice rules. The axioms reflect the policy objectives of promoting diversity and restricting priority violations.

Let vectors $(r_t)_{t\in T}$ and $(q_t)_{t\in T}$ denote the reserves and quotas, respectively. Assume $\sum_{t\in T} r_t \leq q \leq \sum_{t\in T} q_t$.

To define the class of generalized reserves-and-quotas rules, we first divide school S into $T+1$

[3] Biró et al. (2010) allow some schools not to meet the lower bounds if and only if the school is closed and accepts no applicant.

different slots $\{s_t\}_{t\in T\cup\{0\}}$, where each s_t has a capacity r_t and s_0 has a capacity $q - \sum_{t\in T} r_t$. We abuse notation to define $S = \{s_t\}_{t\in T\cup\{0\}}$. Priority ranking $\succ_t$ of slot s_t is such that all type-t applicants have higher priorities than non-type-t applicants, and otherwise it agrees with $\succ$.

For each $A \subseteq \mathcal{A}$, an application order $\rhd_{a,A}$ of $a \in A$ is a complete, transitive and anti-symmetric binary relation on S. For any $s, s' \in S$, $s \rhd_{a,A} s'$ denotes that s has a higher position in application order of a than s'.[4]

For given applicant orders $(\rhd_{a,A})_{a\in A, A\subseteq \mathcal{A}}$, we describe the corresponding **generalized reserves-and-quotas rule** $\mathcal{C}^g$. For any $A \subseteq \mathcal{A}$, $\mathcal{C}^g(A)$ is determined as follows.

Step $k = 0$. Meeting Quotas.
For each $t \in T$, all slots reject any type$-t$ applicant that does not have one of the q_t highest priorities among type$-t$ applicants.

Step $k \geq 1$. Application Steps.
Each applicant $a \in A$ applies to the highest rank slot according to $\rhd_{a,A}$ among those that have not rejected her yet. For each $t \in T$, slot s_t considers all its applicants, tentatively accepts the highest $\succ_t$ priority ones up to its capacity r_t and rejects the rest. If there is a rejection, proceed to Step $k + 1$. Otherwise, the procedure terminates and its output $\mathcal{C}^g(A)$ equals to the union over $t \in T$ of applicants tentatively assigned to s_t at the end of Step k.

Before providing axiomatic foundation for the generalized reserves-and-quotas rules, we provide an example demonstrating that different application orders may result in different sets of chosen applicants with different degrees of priority violations.

Example 1. *Suppose* $\mathcal{A} = \{a_1, a_2, a_3\}$, $\tau(a_1) = \tau(a_3) = t \neq t' = \tau(a_2)$, $a_1 \succ a_2 \succ a_3$, $q = q_t = q_{t'} = 2$, $r_t = 1$ *and* $r_{t'} = 0$.

In other words, there are three applicants considered for admission at a school with two seats. Applicants a_1 *and* a_3 *are of the same type* t*, which may be thought of as the low-income status. Applicants* a_2 *is of high-income type* t'*. Lower indexed applicants have higher priorities. The school has an affirmative action policy in the form a unit reserves for low-income type* t*.*

By the description of generalized reserves-and-quotas rules, the school is divided into three slots $s_t, s_{t'}$ *and* s_0*. Since* $s_{t'}$ *has a capacity of* $r_{t'} = 0$*, we may ignore that slot. Let us call* s_t

[4] Kominers and Sönmez (2016) only consider cases when $\rhd_{a,A}$ is constant for all $a \in A$. This restriction eliminates certain desirable reserves-and-quotas rules, such as the regular one which we introduce in the next section.

and s_0 the reserved slot and the open slot, respectively.

Consider two application orders:

1. *applicants first apply to the open seat, then to the reserved seat,*
2. *applicants first apply to the reserved seat, then to the open seat.*

Under the first application orders, all applicants are first considered for the open slot. The open accepts a_1 and rejects the remaining two applicants. The rejected applicants apply to the reserved slot and a_3 is accepted since she is the only remaining low-income applicant. Thus, the chosen applicants under the first application orders are a_1 and a_3.

Now consider the second application orders. All applicants are first considered for the reserved slot, who accepts a_1 and rejected the remaining applicants. The rejected applicants apply to the open slot and a_2 is accepted as the highest priority remaining applicant. The chosen applicants under the second application orders are a_1 and a_2.

We now characterize the class of generalized reserves-and-quotas rule with four axioms. The first axiom guarantees that at least r_t and no more than q_t seats for representatives of each type $t \in T$.

Axiom 1 (Reserves- and quotas-respecting)**.** *A choice rule is* ***reserves- and quotas-respecting*** *if for any $A \subseteq \mathcal{A}$ and $t \in T$,*

$$\min\{|A_t|, r_t\} \leq \left|\mathcal{C}(A) \cap A_t\right| \leq q_t.$$

The second axiom says that the school accepts applicants unless there are no empty seats or type-specific quotas are binding. In the literature, this property is also called acceptant (Echenique and Yenmez, 2015) or capacity-filling (Doğan et al., 2020).

Axiom 2 (Non-wasteful)**.** *A choice rule $\mathcal{C}$ is* ***non-wasteful*** *if for any $A \subseteq \mathcal{A}$,*

$$a \in A \setminus \mathcal{C}(A) \text{ and } |\mathcal{C}(A)| < q \text{ imply } \left|\mathcal{C}(A)_{\tau(a)}\right| = q_{\tau(a)}.$$

The third axiom says that priority violations occur only between students of different types.

Axiom 3 (Within-type priority compatibility)**.** *A choice rule $\mathcal{C}$ is* ***within-type priority compatible*** *if for any $A \subseteq \mathcal{A}$, whenever a causes priority violation for a' at $\mathcal{C}(A)$, then $\tau(a) \neq \tau(a')$.*

Our next axiom limits the number of priority violations that each type can cause.

Axiom 4 (Beyond-reserves priority compatibility)**.** *A choice rule $\mathcal{C}$ is* ***beyond-reserves priority compatible*** *if for any $A \subseteq \mathcal{A}$ and $t \in T$,*

$$\left|\left\{a \in A_t : a \text{ causes priority violation at } \mathcal{C}(A)\ \right\}\right| \leq r_t.$$

In other words, Axiom 4 guarantees that the number of type-t applicants that cause priority violations is bounded by the corresponding reserves r_t for type-t. Our first result characterizes the entire class of generalized reserves-and-quotas rules with Axioms 1-4.

Theorem 1. *A choice rule satisfies Axioms 1-4 if and only if it is a generalized reserves-and-quotas rule.*

Axioms 1-4 may not be strong enough to preclude some potentially undesirable choice rules. For example, the generalized reserves-and-quotas rule corresponding to the first application orders in Example 1 chooses a_1 and a_3, violating the priority of a_2. In contrast, the generalized reserves-and-quotas rule corresponding to the second application orders chooses applicants a_1 and a_2 and no applicant's priority is violated. This motivates our next section's analysis of choosing the 'right' application orders and the corresponding generalized reserves-and-quotas rule. We show that particular application orders induce a choice rule that minimizes priority violations among all reserves- and quotas-respecting and non-wasteful choice rules.

5 Regular Reserves-and-Quotas Rules

A choice rule $\mathcal{C}$ is the **regular reserves-and-quotas rule** for reserves and quotas vectors r and q, if for any $A \subseteq \mathcal{A}$, $\mathcal{C}(A)$ is determined as follows:

1. Select up to r_t of highest priority applicants of each type t. Let $A' \subseteq A$ denote the set of all selected applicants.
2. From the remaining applicants $A \setminus A'$, accept highest priority applicants up to the capacity without violating type-specific quotas.

We say an application order $\triangleright_{a,A}$ is **regular** if

$$s_{\tau(a)} \triangleright_{a,A} s, \forall s \in S \setminus \{s_{\tau(a)}\}, \forall a \in A \subseteq \mathcal{A}.$$

Proposition 1. *A choice rules is the regular reserves-and-quotas rule if and only if it is a generalized reserves-and-quotas rule for some regular application orders.*

The only restriction regularity puts on application orders is that each type-t applicant first applies to the slot reserved for her type. The order in which applicants apply to the open slot and the slots reserved for other types does not affect the outcome. As a corollary of Theorem 1 and Proposition 1, we conclude that regular reserves-and-quotas rule satisfies Axioms 1-4. We provide two characterizations of regular reserves-and-quotas rules that single them out among all choice rules satisfying the axioms above.

A choice rule $\mathcal{C}$ **creates less priority violations** than choice rule $\mathcal{C}'$ if at any $A \subseteq \mathcal{A}$, the number of applicants whose priority is violated at $\mathcal{C}(A)$ is weakly smaller than the number of applicants whose priority is violated at $\mathcal{C}'(A)$. A choice rule $\mathcal{C}$ is **priority violations minimal** in the class Γ of choice rules, if $\mathcal{C} \in \Gamma$ and it creates less priority violations than any choice rule in Γ.

Theorem 2. *Regular reserves-and-quotas rule is priority violations minimal in the class of choice rules satisfying Axioms 1 and 2.*

We say a choice rule $\mathcal{C}$ **priority dominates** choice rule $\mathcal{C}'$ if for any $a, a'A \subseteq \mathcal{A}$,

$$a \in \mathcal{C}(A) \setminus \mathcal{C}'(A) \text{ and } a' \in \mathcal{C}'(A) \setminus \mathcal{C}(A) \text{ implies } a \succ a'.$$

In other words, priority domination says that if from each of the chosen sets we remove applicants that appear in both of them, any applicant in the first set has a higher priority than any applicant in the second one. A choice rule $\mathcal{C}$ is **priority rank maximal** in the class Γ of choice rules, if $\mathcal{C} \in \Gamma$ and it priority dominates any choice rule in Γ.

Theorem 3. *Regular reserves-and-quotas rule is the unique priority rank maximal choice rule in the class of choice rules satisfying Axioms 1-3.*

In the class of non-wasteful choice rules, priority rank maximality implies priority violations minimality. Therefore, the regular reserves-and-quotas rule is the unique priority violations minimal rule in the class of choice rules satisfying Axioms 1-3, and consequently, in the class of generalized reserves-and-quotas rules.

The results above provide an axiomatic foundation for using the regular reserves-and-quotas rule. A potential argument for applying an alternative generalized reserves-and-quotas rule instead of the regular one is that the former may potentially achieve better distributional outcomes for certain groups. Consider Example 1. The second application order in the example corresponds to the regular reserves-and-quotas rule. Let us refer to the generalized reserves-and-quotas rule corresponding to the first application order as alternative rule. The alternative rule assigns more low-income applicants compared to the regular reserves-and-quotas rule. However, this potentially 'improved' distribution outcome comes at the expense of violating a_2's priority under the alternative rule. This violation cannot be explained and can easily be avoided by changing the application order. Instead, the same distributional outcome can be achieved by increasing low-income reserves from one to two and applying the regular reserves-and-quotas rule. In contrast to the alternative rule, the violation of a_2's priority is fully explained. a_3 is chosen over a_2, because the reserves would not be respected otherwise.

That the outcome of some generalized reserves-and-quotas rule can also be achieved by the regular reserves-and-quotas rule is not peculiar to the example above, but rather a general phenomenon.

Proposition 2. *For any set of reserves and quotas, consider an arbitrary reserves- and quotas-respecting, non-wasteful and within-type priority compatible choice rule $\mathcal{C}$ and a subset $A \subseteq \mathcal{A}$. Let $\mathcal{C}^r$ be the regular reserves-and-quotas rule corresponding to the reserves vector $r_t = \left|\mathcal{C}(A) \cap A_t\right|$. Then, $\mathcal{C}^r(A) = \mathcal{C}(A)$.*

Since generalized reserves-and-quotas rules are reserves- and quotas-respecting, non-wasteful and within-type priority, this result holds for generalized reserves-and-quotas rules as well. Proposition 2 follows from that within-type priority compatibility of $\mathcal{C}$ implies that the choice rule selects r_t highest priority type-t applicants for each $t \in T$. Also, non-wastefulness of $\mathcal{C}$ implies that it selects applicants up to the school's capacity, unless type specific quotas are binding. When reserves are $(r_t)_{t \in T}$, $\mathcal{C}^r$ also selects r_t highest priority type-t applicants for each $t \in T$. Thus, the set of selected applicants coincide under both choice rules.

The example and Proposition 2 potentially suggest that the regular reserves-and-quotas rule resolves the conflict between distributional objectives and respecting priorities in the most transparent way. Alternative generalized reserves-and-quotas rules may create 'unexplained' priority violations, while the regular reserves-and-quotas rule can achieve the same distributional outcomes, and any remaining priority violation is explained by the choice of reserves and quotas.

6 Multiple Schools

In this section we assume $|S| > 1$. A choice rule $C_s : 2^{\mathcal{A}} \to 2^{\mathcal{A}}$ for each $s \in S$ is defined as before, i.e., it satisfies $\mathcal{C}_s(A) \subseteq A$ and $\mathcal{C}_s(A) \leq q_s$ for all $A \subseteq \mathcal{A}$, where q_s is the capacity of school s. A preference ranking P_a of applicant a is a complete, transitive and anti-symmetric binary relation on S. We assume that $\sum_{s\in S} q_s \geq |\mathcal{A}|$. This is without loss of generality as we can add a school that no applicant prefers and that has unlimited capacity.

An assignment is a mapping $\mu : \mathcal{A} \to S$, such that $\left|\mu^{-1}(s)\right| \leq q_s$ for all $s \in S$. Each school has type-specific reserves $r_s = (r_{st})_{t\in T}$ satisfying $\sum_{t\in T} r_{st} \leq q_s$. Let $r = (r_s)_{s\in S}$. For simplicity, assume there are no type-specific quotas, i.e., $q_{st} = q_s$ for all $t \in T$. We extend the following axioms to assignments.

Axiom 5. *An assignment μ is **reserves-respecting** if for any $a \in \mathcal{A}$ and $s \in S$, $sP_a\mu(a)$ implies that $\left|\mu^{-1}(s) \cap \mathcal{A}_{\tau(a)}\right| \geq r_{s\tau(a)}$.*

Axiom 6. *An assignment μ is **non-wasteful** if for any $a \in \mathcal{A}$ and $s \in S$, $sP_a\mu(a)$ implies $\left|\mu^{-1}(s)\right| = q_s$.*

Axiom 7. *An assignment μ is **within-type priority compatible** if for any $a \in \mathcal{A}$ and $s \in S$, $sP_a\mu(a)$ implies that $a' \succ_s a$ for all $a' \in \mu^{-1}(s) \cap I_{\tau(a)}$.*

The classes of assignments satisfying Axioms 5-7 is large. In particular, it includes the set of stable assignments in a setting with distributional constraints (e.g., Abdulkadiroğlu (2005), Hafalir et al. (2013)). Within those assignments, we are interested in ones that minimize or reduce priority violations. We say that the priority of applicant a is violated at μ, if there is an $s \in S$ such that $sP_a\mu(a)$ and $a \succ_s a'$ for some $a' \in \mu^{-1}(s)$. We say an assignment μ is **priority violations minimal** in the class of assignments $\mathcal{M}$ if $\mu \in \mathcal{M}$ and for every $\mu' \in \mathcal{M}$ the number of applicants whose priority is violated at μ is weakly smaller than the number of applicants whose priority is violated under μ'. Multiple schools complicate the problem of finding priority violations minimal assignments.

Theorem 4. *Finding a priority violations minimal assignment in the class of assignments satisfying Axioms 5 and 6 is an NP-hard problem, even when there are only two types.*

Theorem 5. *Finding a priority violations minimal assignment in the class of assignments satisfying Axioms 5-7 is an NP-hard problem.*

In practice, an assignment is typically found by applying the following Deferred Acceptance (DA) algorithm.

Deferred Acceptance (DA)

Step $k \geq 1$. Each applicant applies to her most preferred school that has not rejected her yet. Let A^s denote the applicants who apply to school s. Each $s \in S$ tentatively accepts applicants in $\mathcal{C}_s(A^s)$ and rejects the rest. If there is a rejection, proceed to Step $k+1$. Otherwise, the procedure terminates and tentative acceptances are finalized.

Notice that the outcome of the DA is reserves-respecting, non-wasteful and within-type priority compatible when all schools use a reserves-respecting, non-wasteful and within-type priority compatible choice rules, respectively. In particular, all these properties are satisfied when all schools apply the regular reserves-and-quotas rule.

As our NP-hardness results suggest, the outcome of DA where all schools apply the regular reserves-and-quotas rule is not priority violations minimal in the class of assignments satisfying Axioms 5 and 6 (and 7). This is true even when all applicants share common preferences and all schools share common priorities.[5] We discuss these complications in Appendix C. We provide simulations results for evaluating the performance of the regular reserves-and-quotas rule in terms of reducing priority violations compared to an alternative alternative generalized reserves-and-quotas rule. Simulations show that the superior performance of the regular reserves-and-quotas rule established in Section 5 generalizes to the setting with multiple schools.

7 Simulations

The previous section highlights the complications related to analytical comparisons of the DA outcome under different choice rules. Therefore, we implement these comparisons in a simulated public school assignment environment. Our simulations feature 17,000 applicants and 200 schools (programs). Preferences and priorities are generated to resemble a real public school assignment data from a US school district. Each school grants highest priority to applicants who have a sibling attending the school, the second highest priority to applicants residing in the school's neighborhood, and the lowest priority to remaining applicants. Schools use a common random tie-breaker to obtain strict ranking over applicants. Recall that our notion of priority violation considers only original priorities, not the random tie-breakers. Each school reserves proportion α of seats for low-income applicants and equal

[5] We show in the Appendix that a positive result can be obtained in this rather restrictive environment only of we change the way we count priority violations.

proportion α of seats for high-income ones. The remaining seats are not reserved for anyone and we refer to those as open seats. We report results for $\alpha \in \{0.2, 0.3, 0.4\}$.

Applicants' income groups are determined as follows. First, we run DA without reserves and quotas to determine overdemanded schools, i.e., those that reject an applicant during the algorithms' implementation.[6] For each applicant we draw a number η uniform randomly from the unit interval. The applicants' income level is $\eta + \beta$ if the applicant lives in a neighborhood of an overdemanded school and η if she does not. Thus, the average income of applicants in neighborhoods of overdemanded schools is β and the average income of applicants not in neighborhoods of overdemanded schools is normalized to zero. We report results for $\beta \in \{0.1, 0.2, 0.5\}$.

School assignment is determined by the DA algorithm. We consider two choice rules: the regular reserves-and-quotas rule and an alternative rule. As described in Section 5, under the regular reserves-and-quotas applicants are considered for seats reserved for their types first. Under the alternative rule, applicants are considered for seats reserved for their types last. We run 100 simulations for each parameter value. Table 1 reports average number of priority violation instances. Standard errors are in the parentheses.

Most importantly, simulations reveal that the regular reserves-and-quotas rule creates substantially less priority violations compared to the alternative one.

For example, when only 20% of seats are reserved for low- and high-income applicants each, i.e. $\alpha = 0.2$, and neighborhoods with over-demanded schools have an average income that is 0.1 higher than the average of the remaining neighborhoods, i.e. $\beta = 0.1$, the average number of applicants whose priority is violated under the regular reserves-and-quotas is 0.05. In comparison, this number under the alternative rule is 30.36. The number of priority violations instances are increasing in the reserves' size α. This direction is intuitive as under larger reserves, or equivalently, stricter distributional constraints, schools accept more applicants based on their types, as opposed to priorities.

Second, these numbers are higher for larger income parameter β. This is because with larger β, more of the low-income applicants are residing outside of neighborhoods with over-demanded schools. Therefore, reserves play a larger role in determining the assignment to over-demanded schools, and consequently, there more applicants' priorities are violated.

[6] In our simulations, the number of applicants living at a neighborhood of an overdemanded school is 7,240.

α	β	Regular Reserves-and-Quotas Rule	Alternative Rule
0.2	0.1	0.05 (0.30)	30.36 (11.00)
	0.2	0.31 (1.31)	57.00 (15.62)
	0.5	26.00 (16.76)	225.31 (39.20)
0.3	0.1	0.50 (1.12)	51.43 (14.78)
	0.2	3.25 (4.23)	100.35 (18.94)
	0.5	174.11 (40.47)	452.08 (56.91)
0.4	0.1	10.37 (8.00)	76.50 (20.12)
	0.2	40.18 (17.07)	165.15 (25.67)
	0.5	546.32 (55.58)	703.87 (50.31)

Table 1: Average Number of Applicants Whose Priority is Violated

However, the conclusion that the regular reserves-and-quotas rule creates substantially less priority violations persists across the parameter space.

8 Conclusion

We study priority-based assignment with reserves and quotas. There are three main contributions. First, we characterize generalized reserves-and-quotas rules with four axioms that reflect the policy maker's objectives of respecting priorities and achieving diversity. Second, we show that there is a unique choice rule that selects highest priority applicants and minimizes priority violations among all generalized reserves-and-quotas rules. Our findings provide axiomatic foundation for the regular reserves-and-quotas rule. Third, we study priority violations minimization in two-sided markets with multiple schools and quantify the performance of the regular reserves-and-quotas rule compared to an alternative rule through simulations.

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A Proofs of Main Results

A.1 Proof of Theorem 1

The proof has two parts. Part 1 shows that any choice rule satisfying Axioms 1-4 is a generalized reserves-and-quotas rule for some application orders. Part 2 shows that any generalized reserves-and-quotas rule satisfies Axioms 1-4.

Part 1. Suppose $\mathcal{C}$ satisfies Axioms 1-4. Consider an arbitrary $A \subseteq \mathcal{A}$. Since the choice rule is (reserves- and) quotas- respecting and within-type priority compatible, for each $t \in T$, it never selects any type$-t$ who does not have one of the q_t highest priorities among type$-t$ applicants. Thus, without loss of generality, we may restrict attention to the case where $|A_t| \leq q_t$ for all $t \in T$. Also, since $\mathcal{C}$ is non-wasteful, we may assume $|A| \geq q$.

For each $t \in T$, let $A'_t \subseteq A_t$ be the set of type-t applicants who cause priority violations at $\mathcal{C}(A)$. Since $\mathcal{C}$ is beyond-reserves priority compatible, $|A'_t| \leq r_t$ for any $t \in T$. If $|A_t| \leq r_t$, let $\bar{A}_t = \mathcal{C}(A) \cap A_t$. Otherwise, let $\bar{A}_t \subseteq \mathcal{C}(A) \cap A_t$ be such that $|\bar{A}_t| = r_t$ and $A'_t \subseteq \bar{A}_t$. For an applicant $a \in A$, we construct the application order $\triangleright_{a,A}$ as follows:

- If $a \in \bar{A}_{\tau(a)}$ or $a \in A \setminus \mathcal{C}(A)$, then $s_{\tau(a)} \triangleright_{a,A} s$ for any $s \in S$.
- If $a \in \mathcal{C}(A) \setminus \bar{A}_{\tau(a)}$, then $s \triangleright_{a,A} s_{\tau(a)}$ for any $s \in S$.
- The order between any two slots in $S \setminus \{s_{\tau(a)}\}$ is arbitrary.

Let $\mathcal{C}^g$ denote the generalized reserves-and-quotas rule corresponding to the constructed application orders. In what follows we show that $\mathcal{C}^g = \mathcal{C}$.

Since the DA outcome is independent of the order of applications, suppose applicants in $\mathcal{C}(A)$ apply before applicants in $A \setminus \mathcal{C}(A)$. Note that since $|\mathcal{C}(A)| \leq q$, all applicants in $\mathcal{C}(A)$ will be accepted by some slot in S. Consider an arbitrary $a \in A \setminus \mathcal{C}(A)$. It is sufficient to show that a is rejected by all slots. By construction of application orders, a first applies to $s_{\tau(a)}$. Since $\mathcal{C}$ is reserves-respecting and $a \in A \setminus \mathcal{C}(A)$, there are at at least $r_{\tau(a)}$ type $\tau(a)$ applicants in $\mathcal{C}(A)$. Moreover, since $\mathcal{C}$ is within-type priority compatible, all these applicants have a higher priority than a. In particular, applicants in $\bar{A}_{\tau(a)}$ have higher priority than a. By construction of application orders, these applicants apply to $s_{\tau(a)}$ before any other slot. Therefore, at least $r_{\tau(a)}$ applicants with higher priorities than a are tentatively assigned to $s_{\tau(a)}$ at the time when she applies there. Hence, a is rejected by $s_{\tau(a)}$.

We now show that a is rejected from all slots $s \in S \setminus \{s_{\tau(a)}\}$. By definition, no applicant in $\mathcal{C}(A) \setminus \cup_{t \in T} \bar{A}_t$ causes a priority violation. Thus, all those applicants have higher priorities than a. Since $a \in A \setminus \mathcal{C}(A)$ and $\mathcal{C}$ is non-wasteful, $|\mathcal{C}(A)| = q$. Therefore, all slots $s \in S \setminus \{s_{\tau(a)}\}$ are tentatively filled with higher priority applicants than a, and a is rejected by all of them.

Part 2. Consider an arbitrary $A \subseteq \mathcal{A}$ and application orders $(\triangleright_{a,A})_{a \in A}$. Let $\mathcal{C}^g$ be the generalized reserves-and-quotas rule corresponding to these application orders. We verify that $\mathcal{C}^g$ satisfies Axioms 1-4.

Respecting reserves: Consider an arbitrary $A \subseteq \mathcal{A}$. That $\left|\mathcal{C}^g(A) \cap A_t\right| \leq q_t$ for all $t \in T$ is immediate from the description of $\mathcal{C}^g$. Moreover, since type-t applicants have the highest $\succ_t$ priority at s_t, at least r_t of them are guaranteed to be accepted by s_t. Thus, $|\mathcal{C}^g(A) \cap A_t| \geq \min\left\{|A_t|, r_t\right\}$.

Non-wastefulness: Non-wastefulness is immediate from the description of the generalized reserves-and-quotas rule.

Within-type priority compatibility: Suppose a' causes priority violation for a at A. By contradiction, suppose $\tau(a) = \tau(a')$. Then, $a \succ_t a'$ for all $t \in T$. This contradicts that $\mathcal{C}^g(A)$ chooses a', but not a.

Beyond-reserves priority compatibility: By contradiction, suppose there is a set of type-t applicants A'_t causing priority violations at A such that $|A'_t| > r_t$. Let a be the highest priority applicant in $A \setminus \mathcal{C}(A)$. Then, each $a' \in A'_t$ causes priority violation for a. Let $A''_t \subseteq A'_t$ be the set of applicants who are assigned to s_t. Since $|A''_t| \leq r_t$, there exists an $a'' \in A'_t \setminus A''_t$. This contradicts that a'' is selected over a at $\mathcal{C}^g(A)$.

A.2 Proof of Proposition 1

Let $\mathcal{C}^g$ be a generalized reserves-and-quotas rule corresponding to some regular application order. We show that $\mathcal{C}^g$ equals the regular reserves-and-quotas rule $\mathcal{C}^r$.

Consider an arbitrary $A \subseteq \mathcal{A}$ with $|A_t| \leq q_t$ for all $t \in T$ and $|A| \geq q$. It is immediate that both $\mathcal{C}^g$ and $\mathcal{C}^r$ are non-wasteful. Therefore,

$$|\mathcal{C}^g(A)| = q = |\mathcal{C}^r(A)|. \tag{1}$$

Let A' be the set of applicants selected at the first stage of the implementation of $\mathcal{C}^r$. First, we show that $A' \subseteq \mathcal{C}^g(A)$. Consider an arbitrary $a \in A'$. By description of $\mathcal{C}^r$, a is one of the $r_{\tau(a)}$ highest priority type applicants in $A_{\tau(a)}$, and therefore, she is one of the $r_{\tau(a)}$ highest $\succ_{\tau(a)}$ priority applicants at $s_{\tau(a)}$. Thus, in Step 1 of $\mathcal{C}^g$, a applies to $s_{\tau(a)}$ and is never rejected by the school. This establishes that $a \in \mathcal{C}^g(A)$.

Now, by contradiction, suppose $\mathcal{C}^g(A) \neq \mathcal{C}^r(A)$. By equation 1, there is an applicant $a \in \mathcal{C}^r(A) \setminus \mathcal{C}^g(A)$. Since $A' \subseteq \mathcal{C}^g(A)$, it should be that $a \notin A'$. Therefore, a is selected at the second stage of $\mathcal{C}^r$'s implementation. By description of $\mathcal{C}^r$, a is one of the $\min\big\{|A|, q\big\} - \sum_{t\in T} \min\big\{|A_t|, r_t\big\}$ highest priority applicants in $A \setminus A'$. This contradicts that a is not selected at $\mathcal{C}^g(A)$.

A.3 Proof of Theorem 2

Let $\mathcal{C}^r$ be the regular reserves-and-quotas rule. We show that $\mathcal{C}^r$ is priority violations minimal in the class of reserves- and quotas-respecting and non-wasteful choice rules.

That $\mathcal{C}^r$ is reserves- and quotas-respecting and non-wasteful is immediate from its description. Let $\mathcal{C}$ be an arbitrary reserves- and quotas-respecting and non-wasteful choice rule. Consider an arbitrary $A \subseteq \mathcal{A}$ with $|A| \geq q$. Since both $\mathcal{C}^r$ and $\mathcal{C}$ are non-wasteful,

$$\big|\mathcal{C}^r(A)\big| = \min\Big\{q, \sum_{t\in T} \min\big\{|A_t|, q_t\big\}\Big\} = \big|\mathcal{C}(A)\big|. \tag{2}$$

Let $a \succeq a'$ denote that $a \succ a'$ or $a = a'$. Also, let $\bar{a}$ and $\bar{a}'$ be the lowest priority applicants in $\mathcal{C}^r(A)$ and $\mathcal{C}(A)$, respectively. Then, by equation 2, the number of applicants whose priority is violated at $\mathcal{C}^r(A)$ and $\mathcal{C}(A)$ is $\big|\big\{a \in A : a \succ \bar{a}\big\}\big| - \big|\mathcal{C}^r(A)\big| + 1$ and $\big|\big\{a \in A : a \succ \bar{a}'\big\}\big| - \big|\mathcal{C}(A)\big| + 1$, respectively. Thus, by equation 2, it is sufficient to show that $\bar{a} \succeq \bar{a}'$.

By contradiction, suppose $a \succ \bar{a}$ for all $a \in \mathcal{C}(A)$. Since $\mathcal{C}$ is reserves-respecting, there are at least $\min\big\{|A_t|, r_t\big\}$ type-t applicants in A for each $t \in T$. Thus, $\bar{a}$ cannot be selected at the first stage of $\mathcal{C}^r$'s implementation, since then she would have a weakly higher priority than at least one individual in $\mathcal{C}(A)$. Thus, $\bar{a}$ is selected at the second stage of $\mathcal{C}^r$'s implementation. For any $a \in \mathcal{C}(A) \setminus \mathcal{C}^r(A)$, it should be that $\big|\mathcal{C}^r(A)_{\tau(a)}\big| = q_{\tau(a)}$, since otherwise $\bar{a}$ would not have been selected over a at the second stage of $\mathcal{C}^r$'s implementation. Let $T' := \big\{\tau(a) : a \in \mathcal{C}(A) \setminus \mathcal{C}^r(A)\big\}$. Then,

$$\min\Big\{q, \sum_{t\in T} \min\big\{|A_t|, q_t\big\}\Big\} = \big|\mathcal{C}(A)\big| \leq \sum_{t\in T'} q_t + \Big|\big\{\mathcal{C}(A) : \tau(a) \in T \setminus T'\big\}\Big|$$

$$= \sum_{t\in T'} q_t + \Big|\big\{a \in \mathcal{C}^r(A) \cap \mathcal{C}(A) : \tau(a) \in T \setminus T'\big\}\Big|$$

$$= \sum_{t \in T'} \big|\mathcal{C}^r(A)_t\big| + \Big|\big\{a \in \mathcal{C}^r(A) \cap \mathcal{C}(A) : \tau(a) \in T \setminus T'\big\}\Big|$$

$$\leq \big|\mathcal{C}^r(A)\big| = \min\Big\{q, \sum_{t \in T} \min\big\{|A_t|, q_t\big\}\Big\},$$

where the second equality follows from that $\big\{a \in \mathcal{C}(A) : \tau(a) \in T \setminus T'\big\} = \big\{a \in \mathcal{C}^r(A) \cap \mathcal{C}(A) : \tau(a) \in T \setminus T'\big\}$ by definition of T'. Thus,

$$\big|\mathcal{C}(A)\big| = \sum_{t \in T} q_t + \big|\big\{a \in \mathcal{C}(A) : \tau(a) \in T \setminus T'\big\}\big|,$$

and consequently,

$$\big|\mathcal{C}(A) \cap A_t\big| = q_t = \big|\mathcal{C}^r(A) \cap A_t\big| \text{ for all } t \in T'. \tag{3}$$

Moreover, by definition of T',

$$\big|\mathcal{C}(A) \cap A_t\big| \leq \big|\mathcal{C}^r(A) \cap A_t\big| \text{ for all } t \in T \setminus T'. \tag{4}$$

Equations 2-4 imply that

$$\big|\mathcal{C}^r(A) \cap A_t\big| = \big|\mathcal{C}(A) \cap A_t\big| \text{ for all } t \in T.$$

Since $\mathcal{C}^r$ selects the highest priority applicants of each type, this contradicts that $a \succ \bar{a}$ for all $a \in \mathcal{C}(A)$.

A.4 Proof of Theorem 3

We shows that $\mathcal{C}^r$ is the unique priority rank maximal rule in the class of reserves- and quotas-respecting, non-wasteful and within-type priority compatible choice rules.

That $\mathcal{C}^r$ is within-type priority compatible is immediate from its description. Let $\mathcal{C}$ be an arbitrary reserves-respecting, quotas respecting, non-wasteful and within-type priority compatible choice rule. To establish Part 2, it is sufficient to show that $\mathcal{C}^r$ priority dominates $\mathcal{C}$. Uniqueness would follow from the definition of priority rank maximality.

Consider an arbitrary $A \subseteq \mathcal{A}$ with $|A_t| \leq q_t$ for all $t \in T$ and $|A| \geq q$. Without loss of generality, suppose $\mathcal{C}^r(A) \neq \mathcal{C}(A)$. Since both $\mathcal{C}^r$ and $\mathcal{C}$ are non-wasteful,

$$\big|\mathcal{C}^r(A)\big| = q = \big|\mathcal{C}(A)\big|.$$

Thus, $\mathcal{C}^r(A) \neq \mathcal{C}(A)$ implies that there are $a, a' \in A$ such that

$$a \in \mathcal{C}^r(A) \setminus \mathcal{C}(A) \text{ and } a' \in \mathcal{C}(A) \setminus \mathcal{C}^r(A).$$

Since $\mathcal{C}$ is reserves-respecting and $a \notin \mathcal{C}(A)$, there are at least $r_{\tau(a)}$ applicants in $\mathcal{C}(A)_{\tau(a)}$. Moreover, since $\mathcal{C}$ is within-type priority compatible, all $r_{\tau(a)}$ highest priority applicants in $A_{\tau(a)}$ are in $\mathcal{C}(A)$. Therefore, a is not one of the $r_{\tau(a)}$ highest priority type applicants in $A_{\tau(a)}$. Similarly, since $\mathcal{C}^r$ is reserves-respecting, a' is not one of the $r_{\tau(a')}$ highest priority applicants in $A_{\tau(a')}$. Hence, neither a nor a' is selected at the first stage of $\mathcal{C}^r$' implementation. Since a is selected over a' at the second stage of $\mathcal{C}^r$'s implementation and $\left|\mathcal{C}^r(A)_{\tau(a')}\right| < q_{\tau(a')}$, we conclude that $a \succ a'$.

A.5 Proof of Theorem 4

Complexity theory categorizes problems into classes based on the amount of computational resources needed to solve them. NP-hard problems are commonly considered as the class of computationally intractable problems. This is because there are no known polynomial time algorithms for solving NP-hard problems, and it is widely conjectured and believed that none exists. A typical way of proving NP-hardness of a problem is constructing a polynomial time reduction from a well-known NP-hard problem. A reduction transforms one problem to another in a way that solving the second problem solves the first problem.

We prove the NP-hardness of finding a priority violations minimal reserves-respecting and non-wasteful assignment by constructing a polynomial time reduction from the Complete Bipartite Subgraph (*CBS*) problem. For a given bipartite graph (V_1, V_2, E) and an integer $k \leq \min\{|V_1|, |V_2|\}$, the *CBS* problem asks whether there are subsets $U_1 \subseteq V_1$ and $U_2 \in V_2$ such that $|U_1| = k$, $|U_2| \geq k$ and $(u_1, u_2) \in E$ for all $u_1 \in U_1$ and $u_2 \in U_2$. This problem is NP-hard (Garey and Johnson, 1979).

Consider an arbitrary *CBS* instance (V_1, V_2, E) and k. We construct the corresponding tuple $\left(\mathcal{A}, S, T, \tau, r, (\succ_s)_{s\in S}, (P_a)_{a\in\mathcal{A}}\right)$ as follows:

- $\mathcal{A} = A \cup \underline{A} \cup \bar{A}$, where $A = \{a_{vn}\}_{v\in V_2, n=1,2,\ldots,N}$, $N > |V_1|$ is some large number,[7] $|\underline{A}| = k$, $|\bar{A}| = |V_1| - k$, and A, $\underline{A}$ and $\bar{A}$ are pairwise disjoint.

  For each $v \in V_2$, we may think of applicants in $\{a_{vn}\}_{n=1,2,\ldots,N}$ as N copies of each other. For any $m, n = 1, 2, \ldots, N$, we say applicants $a_{vn} \in A$ and $a_{um} \in A$ are *distinct* if $v \neq u$.

- $S = \{s_v\}_{v\in V_1} \cup \{\bar{s}\}$,

[7] E.g., $N = |V_1| + 1$ would work.

- Capacities are

$$q_s = \begin{cases} |\mathcal{A}| & \text{if } s = \bar{s}, \\ 1 & \text{otherwise.} \end{cases}$$

- $T = \{t, t'\}$ and for all $a \in A$,

$$\tau(a) = \begin{cases} t & \text{if } a \in A, \\ t' & \text{otherwise.} \end{cases}$$

- $r_{st} = 0$ and $r_{st'} = 1$ for all $s \in S \setminus \{\bar{s}\}$.

- Schools have common priorities $\succ$ satisfying

$$\bar{a} \succ a \succ \underline{a}$$

  for all $\bar{a} \in \bar{A}$, $a \in A$ and $\underline{a} \in \underline{A}$.

- Preferences satisfy

  - for each $s_v \in S \setminus \{\bar{s}\}$ and $a_{un} \in A$,

$$\bar{s} P_{a_u} s_v \text{ if and only if } (v, u) \in E,$$

  - for each $s_v \in S \setminus \{\bar{s}\}$ and $a \in \bar{A} \cup \underline{A}$,

$$s_v P_a \bar{s}.$$

All unspecified aspects of the tuple $\big(\mathcal{A}, S, T, \tau, r, (\succ_s)_{s \in S}, (P_a)_{a \in \mathcal{A}}\big)$ are arbitrary.

We first prove the following claim.

Claim 1. *An assignment is priority violations minimal (in some class of assignments) only if it minimizes the number of distinct applicants in A whose priority is violated.*

The proof of the Claim 1 is straightforward. Notice that for each $v \in V_2$, applicants in $\{a_{vn}\}_{n=1,2,\ldots,N}$ are identical, i.e., the priority of one of them is violated if and only if all their priorities are violated. Thus, priority violations for applicants in $\underline{A} \cup \bar{A}$ are of secondary importance since their number is bounded by N. Therefore, any priority violations minimizing assignment should minimize the number of distinct applicants in A whose priority is violated.

Thus, to establish NP-hardness of finding priority violations minimal reserves-respecting and non-wasteful assignment it is sufficient to prove the following claim.

Claim 2. *There are subsets* $U_1 \subseteq V_1, U_2 \subseteq V_2$ *such that* $|U_1| = k$, $|U_2| \geq k$ *and* $(u_1, u_2) \in E$ *for all* $u_1 \in U_1, u_2 \in U_2$ *if and only if there is a reserves-respecting and non-wasteful assignment such that the number of distinct applicants in* A *whose priority is violated is weakly smaller than* $|V_2| - k$.

We now prove Claim 2. Fix and arbitrary reserves-respecting and non-wasteful assignment μ. Note that all schools in $S \setminus \{\bar{s}\}$ have a single seat reserved for type-t' applicants, and there are exactly $|V_2| = \left|S \setminus \{\bar{s}\}\right|$ type-t' applicants (namely those in $\underline{A} \cup \bar{A}$). Moreover, all these applicants prefer schools in $S \setminus \{\bar{s}\}$ to $\bar{s}$. Therefore, all schools in $S \setminus \{\bar{s}\}$ are filled with applicants in $\underline{A} \cup \bar{A}$. Consequently, all applicants in A are assigned to $\bar{s}$.

Applicants in A have higher priorities than applicants in $\underline{A}$, and lower priorities than applicants $\bar{A}$. Therefore, only applicants in $\underline{A}$ can cause priority violations for applicants in A. More precisely, the priority of $a \in A$ is violated at μ if and only if $\mu(\underline{a}) P_a \bar{s}$ for some $\underline{a} \in \underline{A}$. Equivalently, the priority of $a \in A$ is not violated at μ if and only if $\bar{s} P_a \mu(\underline{a})$ for all $\underline{a} \in \underline{A}$. By construction of preferences, the last condition holds if and only if $(v, u) \in E$ for all $u \in U_1 \subseteq V_1$, where v is the index corresponding to applicant a and U_1 is the set of indices corresponding to schools $\left\{\mu(\underline{a}) : \underline{a} \in \underline{A}\right\}$. Thus, $|U_1| = |\underline{A}| = k$. Denoting by $U_2 \subseteq V_2$ the set of indices of applicants in A whose priority is not violated, we have that $(u_1, u_2) \in E$ for all $u_1 \in U_1$ and $u_2 \in U_2$. Hence, $|U_2| \geq k$ if and only if the number of distinct applicants in A whose priority is violated is weakly smaller than $|V_2| - k$. This completes the proof of the claim, and therefore, of Theorem 4.

A.6 Proof of Theorem 5

Like with Theorem 4, we prove Theorem 5 by constructing a polynomial time reduction from the *CBS* problem.

Consider an arbitrary *CBS* instance (V_1, V_2, E) and k. We construct the corresponding tuple $\left(\mathcal{A}, S, T, \tau, r, (\succ_s)_{s \in S}, (P_a)_{a \in \mathcal{A}}\right)$ as follows:

- $\mathcal{A} = A \cup \underline{A}$, where $A = \{a_{vn}\}_{v \in V_2, n=1,2,\ldots,N}$, $N > |V_1|$ is some large number, $\underline{A} = \{\underline{a}_v\}_{v \in V_1}$, and A and $\underline{A}$ are disjoint.

  For any $m, n = 1, 2, \ldots, N$, we say applicants $a_{vn} \in A$ and $a_{um} \in A$ are *distinct* if $v \neq u$.

- $S = \{s_v\}_{v \in V_1} \cup \{\underline{s}\} \cup \{\bar{s}\}$.
- Capacities are

$$q_s = \begin{cases} |\underline{A}| - k = |V_1| - k & \textit{if } s = \underline{s}, \\ |\mathcal{A}| & \text{if } s = \bar{s}, \\ 1 & \text{otherwise.} \end{cases}$$

- $T = \{t\} \cup \{t_v\}_{v \in V_1}$ and for all $a \in A$,

$$\tau(a) = \begin{cases} t & \text{if } a \in A, \\ t_v & \text{if } a = \underline{a}_v, v \in V_1. \end{cases}$$

- $r_{s_v t} = r_{s_v t_u} = 0$ for all $v, u \in V_1, v \neq u$ and $r_{s_v t_v} = 1$.
- Schools have common priorities $\succ$ satisfying

$$a \succ \underline{a}$$

  for all $a \in A$ and $\underline{a} \in \underline{A}$.
- Preferences satisfy
  - for each $s_v \in S \setminus \{\underline{s}, \bar{s}\}$, and $a_{un} \in A$, $\bar{s} P_{a_u} \underline{s}$, and

$$\bar{s} P_{a_u} s_v \text{ if and only if } (v, u) \in E,$$

  - for $s_v, s_u \in S \setminus \{\underline{s}, \bar{s}\}, v \neq u$, and $\underline{a}_v \in \underline{A}$,

$$\underline{s} P_{a_v} s_v P_{a_v} \bar{s} P_{a_v} s_u..$$

All unspecified aspects of the tuple $\big(\mathcal{A}, S, T, \tau, r, (\succ_s)_{s \in S}, (P_a)_{a \in \mathcal{A}}\big)$ are arbitrary.

The proof of the following claim is analogous to that of Claim 1.

Claim 3. *An assignment is priority violations minimal (in some class of assignments) only if it minimizes the number of distinct applicants in A whose priority is violated.*

Thus, to establish NP-hardness of finding priority violations minimal reserves-respecting, non-wasteful and within-type priority compatible assignment it is sufficient to prove the following claim.

Claim 4. *There are subsets $U_1 \subseteq V_1, U_2 \subseteq V_2$ such that $|U_1| = k$, $|U_2| \geq k$ and $(u_1, u_2) \in E$ for all $u_1 \in U_1, u_2 \in U_2$ if and only if there is a reserves-respecting, non-wasteful and within-type priority compatible assignment μ such that the number of distinct applicants in A whose priority is violated is weakly smaller than $|V_2| - k$.*

We now prove Claim 4. Fix and arbitrary reserves-respecting, non-wasteful and within-type priority compatible assignment μ. Since all applicants in A prefer $\bar{s}$ to $\underline{s}$ and $\bar{s}$ has enough seats to accommodate all applicants, no applicant in A will be assigned to $\underline{s}$ at the non-wasteful assignment μ. Moreover, applicants in $\underline{A}$ rank school $\underline{s}$ as a first choice. Therefore, $\underline{s}$ is entirely filled with applicants in $\underline{A}$ at the non-wasteful assignment μ. However, the school has only $|\underline{A}| - k$ seats. Thus, exactly k of the applicants in $\underline{A}$ are not assigned to $\underline{s}$ at any non-wasteful assignment. Moreover, each applicant $\underline{a}_v \in \underline{A}$ ranks $s_v \in S \setminus \{\underline{a}, \bar{a}\}$ as most preferred choice after $\underline{s}$, and the school has a single seat reserved for that applicant. Thus, any applicant in $\underline{A}$ that is not assigned to $\underline{s}$ is assigned to the single reserved seat of the corresponding school at any reserves-respecting allocation. The remaining schools in $S \setminus \{\underline{s}, \bar{s}\}$ are filled by applicants in A.

Consider an applicant $a \in A$ whose priority is violated at μ. Since μ is within-type priority compatible, it should be that the priority violation is respect to in $S \setminus \{\underline{s}, \bar{s}\}$ where some applicant $\bar{A}$ is assigned. That is, the priority of $a \in A$ is violated if and only if $\mu(\underline{a}) P_a \bar{s}$ for some $\underline{a} \in \underline{A}$. Equivalently, the priority of $a \in A$ is not violated at μ if and only if $\bar{s} P_a \mu(\underline{a})$ for all $\underline{a} \in \underline{A}$. By construction of preferences, the last condition holds if and only if $(v, u) \in E$ for all $u \in U_1 \subseteq V_1$, where v is the index corresponding to applicant a and U_1 is the set of indices corresponding to schools $\left\{\mu(\underline{a}) : \underline{a} \in \underline{A}\right\}$. Thus, $|U_1| = |\underline{A}| = k$. Denoting by $U_2 \subseteq V_2$ the set of indices of applicants in A whose priority is not violated, we have that $(u_1, u_2) \in E$ for all $u_1 \in U_1$ and $u_2 \in U_2$. Hence, $|U_2| \geq k$ if and only if the number of distinct applicants in A whose priority is violated is weakly smaller than $|V_2| - k$. This completes the proof of the claim, and therefore, of Theorem 5.

B Independence of Axioms in Theorem 1

When any of the Axioms 1-4 is relaxed, we provide examples of choice rules that are not generalized reserves-and-quotas rules and satisfy the remaining three axioms.

- Consider the choice rule $\mathcal{C}$ that only accepts up to r_t highest priority applicants of each type $t \in T$. Then, $\mathcal{C}$ satisfies Axioms 1,3 and 4, but not 2.

- Consider the choice rule $\mathcal{C}$ that accepts up to q of highest priority applicants. Then, $\mathcal{C}$ satisfies Axioms 2,3 and 4, but not 1.

- Suppose $A = \{a_1, a_2\}$, $a_1 \succ a_2$, $\tau(a_1) = \tau(a_2) = t$ and $r_t = q_t = q = 1$. Consider the choice rule $\mathcal{C}$ that chooses a_2 whenever she is available and chooses a_1 whenever a_2 is not available, but a_1 is. Then, the rule satisfies Axioms 1,2 and 4, but not 3.

- Suppose $\mathcal{A} = \{a_1, a_2\}$, $a_1 \succ a_2$, $\tau(a_1) = t \neq t' = \tau(a_2)$ and $r_t = r_{t'} = 0$ and $q_t = q_{t'} = q = 1$. Consider the choice rule $\mathcal{C}$ that chooses a_2 whenever she is available and chooses a_1 whenever a_2 is not available, but a_1 is Then, the rule satisfies Axioms 1,2 and 3, but not 4.

C Additional Results for Multiple Schools

Suppose all applicants have a common preference ranking P and all schools have a common priority ranking $\succ$. We show that the outcome of DA where all schools apply the regular reserves-and-quotas rule may not be priority violations minimal in the class of reserves-respecting and non-wasteful (and within-type priority compatible) assignments even for this simple case with common preferences and priorities.

Example 2. *Suppose $\mathcal{A} = \{a_1, a_2, a_3, a_4\}$, $S = \{s_1, s_2\}$, $q_{s_1} = 1, q_{s_2} = 2$, $\tau(a_1) = t_1 \neq \tau(a_2) = \tau(a_4) = t_2 \neq \tau(a_3) = t_3$, $s_1 P s_2$, $a_1 \succ a_2 \succ a_3$, $r_{s_1 t_2} = r_{s_2 t_2} = 1$ and $r_{s_2 t_1} = r_{s_2 t_3} = r_{s_1 t_3} = 0$. The last two conditions say that the single seat at school s_1 and one of the seats in s_2 are reserved for type-t_2 applicants. If all schools apply the regular reserves-and-quotas rule, DA will proceed as follows:*

Step 1. *All applicants apply to s_1. The school accepts a_2 as she is the highest priority type-t_2 applicant.*

Step 2. *All rejected applicants apply to s_2. The school accepts a_1 and a_4 as they are the highest priority and the remaining highest priority type-t_2 applicants, respectively.*

Now consider an alternative reserves-respecting and non-wasteful (and within-type priority compatible) assignment where a_2 is assigned to s_1, and a_3 and a_4 are assigned to s_2. Then, the only applicant whose priority is violated is a_1. Thus, the outcome of DA where all schools apply the regular reserves-and-quotas rule is not priority violations minimal in the class of reserves-respecting and non-wasteful (and within-type priority compatible) assignments.

The example above is peculiar: it relies on the fact that we define priority violations minimality by the number of individuals whose priority is violated. However, at the alternative assignment in Example 2 the priority of a_1 is violated with respect to both schools s_1 and s_2. This motivates our analysis of alternative priority violations minimality notions. We say that an applicant-school pair (a, s) is a **priority violations instance** is $s \in S$ such that $sP_a\mu(a)$ and $a \succ_s a'$ for some $s \in S$ and $a' \in \mu^{-1}(s)$. We say an assignment μ is **priority violation instances minimal** in the class of assignments $\mathcal{M}$ if $\mu \in \mathcal{M}$ and for every $\mu' \in \mathcal{M}$ the number of priority violation instances is weakly under μ than under μ'. In that case we can state the following positive result.

Proposition 3. *In the setting with common preferences and priorities, the outcome of the DA where all schools apply the regular reserves-and-quotas rule is priority violation instances minimal in the class of assignments satisfying Axioms 5 and 6 (and 7).*

Proof. Suppose schools are indexed by preferability, i.e. $s_1Ps_2Ps_3...$, etc. Let K be such that

$$\sum_{k=1}^{K-1} q_{s_k} < |\mathcal{A}| \leq \sum_{k=1}^{K} q_{s_k}.$$

The condition implies that s_K is the least preferred school that fills a seat at a reserves-respecting and non-wasteful (and within-type priority compatible) assignment.

Let μ be the outcome of the DA algorithm where all schools apply the regular reserves-and-quotas rule. The feasibility of μ is immediate from the description of regular reserves-and-quotas rule and the DA algorithm.For an integer an k, let a_k^μ denote the lowest priority applicant that is assigned to s_k or a more preferred school at μ. Then, the number of priority violation instances at μ is

$$\sum_{k=1}^{K} \Big(\big|\{a \in \mathcal{A} : a \succ a_{k-1}^\mu\}\big| - \sum_{j=1}^{k} q_{s_j} + 1.$$

Also let μ' be an arbitrary feasible allocation and let $a_k^{\mu'}$ the lowest priority applicant that is assigned to s_k or a more preferred school at μ'. Then, the number of priority violation instances at μ'

$$\sum_{k=1}^{K} \Big(\big|\{a \in \mathcal{A} : a \succ a_{k-1}^{\mu'}\}\big| - \sum_{j=1}^{k} q_{s_j} + 1.$$

Thus, to show that there are weakly less priority violations instances at feasible assignment μ than at feasible assignment μ', it is sufficient to show that $a_k^\mu \succeq a_k^{\mu'}$ for all $k \in \{1, 2, ..., K-1\}$.

By contradiction, suppose $a_k^{\mu'} \succ a_k^{\mu}$. Denote $s_l := \mu(a_k^{\mu})$. Also, let $\tilde{A} := \{a \in A : \mu(a) P s_{l+1}\}$ and $\tilde{A}' := \{a \in A : \mu'(a) P s_{l+1}\}$. By feasibility of μ and μ',

$$|\tilde{A}| = \sum_{j=1}^{l} q_{s_l} = |\tilde{A}'|. \tag{5}$$

By equation 5 and by that $a_k^{\mu} \notin \tilde{A}'$, we have $\tilde{A}' \setminus \tilde{A} \neq \emptyset$. Then, $a \succeq a_k^{\mu'} \succ a_k^{\mu}$. Note that a_k^{μ} is not one of the $\sum_{j=1}^{l} r_{\tau(a_k^{\mu})}$ highest priority applicants in $A_{\tau(a_k^{\mu})}$. This is true, since otherwise, by feasibility of μ', a_k^{μ} would be assigned to a school weakly more preferred to s_l at μ'. Thus, a_k^{μ} is selected by school s_l over a at the second stage of the regular reserves-and-quotas rule's implementation. This contradicts $a \succ a_k^{\mu}$. $\square$

Although restrictive, the assumption of common preferences and priorities is a standard simple case. The assumption is heavily used for tractability in various contexts, such as college admissions, school choice and labor market (e.g., Becker (1973), Bulow and Levin (2006), Abdulkadiroğlu et al. (2011), Bodoh-Creed and Hickman (2018)). The following examples demonstrate that Proposition 3 no longer holds once we relax the assumptions of common preferences or common priorities.

Example 3 (Relaxing Common Preferences)**.** *Suppose* $\mathcal{A} = \{a_1, a_2, a_3, a_4\}, S = \{s_1, s_2, s_3\}$, $q_{s_1} = q_{s_2} = 1, q_{s_3} = 4$, $T = \{t_1, t_2, t_3\}$, $\tau(a_1) = t_1 \neq \tau(a_2) = t_2 \neq \tau(a_3) = \tau(a_4) = t_3$, $r_{t_2 s_2} = 1$ *and reserves are zero otherwise. Applicant* a_1 *prefers* s_1 *to* s_3 *to* s_2*,* a_2 *prefers* s_1 *to* s_2 *to* a_3*, and* a_3 *and* a_4 *prefer* s_2 *to* s_3 *to* s_1*. Schools have a common priority ranking* $a_3 \succ a_4 \succ a_1 \succ a_2$*.*

If all schools apply the regular reserves-and-quotas rule, DA will proceed as follows:
Step 1. *Applicants* a_1 *and* a_2 *apply to* s_1*. The school accepts* a_1 *and rejects* a_2*. Applicants* a_3 *and* a_4 *apply to* s_2*. The school admits* a_3 *and rejects* a_4*.*
Step 2. *Applicant* a_2 *applies to* s_2*. The school's single seat is reserved for* a_2*, therefore the school accepts* a_2 *and rejects* a_3*. Applicant* a_4 *applies to* s_3 *and is accepted by the school.*

Step 3. *Applicant* a_3 *applies to* s_3 *and is accepted by the school.*

The assignment is finalized at $\mu(a_1) = s_1, \mu(a_2) = s_2$ *and* $\mu(a_3) = \mu(a_4) = s_3$*. There are four priority violations instances at* μ*:* $(a_3, s_1), (a_4, s_1)$*,* (a_3, s_2) *and* (a_4, s_2)*.*

Now suppose s_1 *accepts* a_2 *instead of* a_1 *in Step 1 of DA. Then, in Step 2,* a_1 *would apply to* s_3 *and* a_3 *and* a_4 *would apply to* s_2*. The latter would reject* a_4*, who would apply and get assigned to* s_3 *at Step 3. The resulting assignment would be* $\mu'(a_1) = \mu'(a_4) = s_3, \mu'(a_2) = s_1$

and $\mu'(a_3) = s_2$, which is reserves-respecting and non-wasteful (and within-type priority compatible). There are only three priority violation instances at μ': $(a_1, s_1), (a_3, s_1)$ and (a_4, s_1).

Example 4 (Relaxing Common Priorities)**.** *Suppose $\mathcal{A} = \{a_1, a_2, a_3, a_4\}, S = \{s_1, s_2, s_3\}$, $q_{s_1} = q_{s_2} = 1, q_{s_3} = 4$,$T = \{t_1, t_2, t_3\}$, $\tau(a_1) = t_1 \neq \tau(a_2) = t_2 \neq \tau(a_3) = \tau(a_4) = t_3$, $r_{t_2 s_2} = 1$ and reserves are zero otherwise. All applicants prefer s_1 to s_2 to s_3. School s_1's priority ranking is $a_1 \succ_{s_1} a_2 \succ_{s_1} a_3 \succ_{s_1} a_4$, s_2's priority ranking is $a_1 \succ_{s_2} a_3 \succ_{s_2} a_4 \succ_{s_2} a_2$, and s_3's priority ranking is arbitrary.*

If all schools apply the regular reserves-and-quotas rule, DA will proceed as follows:
Step 1. *All applicants apply to s_1. The school accepts a_1 and rejects the rest.*
Step 2. *All rejected applicants apply to s_2. The schools' single seat is reserved for a_2. Therefore, the school admits a_2 and rejected the rest.*
Step 3. *a_3 and a_4 apply to s_3, and both are accepted.*
The assignment is finalized at $\mu(a_1) = s_1, \mu(a_2) = s_2$ and $\mu(a_3) = \mu(a_4) = s_3$. There are two priority violations instances at μ: (a_3, s_2) and (a_4, s_2).

Now suppose s_1 accepts a_2 instead of a_1 in Step 1 of DA. Then, in Step 2, a_1 would apply to s_2 and get accepted by the school. In Step 3 a_3 and a_4 would apply to s_3. The resulting assignment would be $\mu'(a_1) = s_2, \mu'(a_2) = s_1$ and $\mu'(a_3) = \mu'(a_4) = s_2$, which is reserves-respecting and non-wasteful (and within-type priority compatible). There is a unique priority violations instance at μ': (a_1, s_1).