NBER WORKING PAPER SERIES

CHAOS AND UNPREDICTABILITY IN DYNAMIC SOCIAL PROBLEMS

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Working Paper 28347
http://www.nber.org/papers/w28347

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
January 2021

I thank seminar participants at Cornell, the University of Washington in St. Louis, and the Office of the Chief Economist at Microsoft. Aviv Caspi and Neelanjan Datta provided outstanding research assistance. The views expressed herein are those of the author and do not necessarily reflect the views of the National Bureau of Economic Research.

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JEL No. D71,D72,D78,H23,Q50

ABSTRACT

We study a dynamic model of environmental protection in which the level of pollution is a state variable that strategically links policy making periods. Policymakers are forward looking but politically motivated: they have heterogeneous preferences and do not fully internalize the cost of pollution. This type of political economy model is often reduced to a "modified" planner's problem, and yields predictions that are qualitatively similar to a planner's constrained optimum, albeit with a bias: too much pollution in the steady state (or, in other applications, too little investment in public goods, too much public debt, etc.). We highlight conditions under which this reduction is not possible, and the dynamic time inconsistency generated by the political process is responsible for a new type of distortion. Under these conditions, there are equilibria in which, for a generic economy and generic initial conditions, the state evolves in complex cycles, or unpredictable chaotic dynamics. Depending on the fundamentals of the economy, these equilibria may generate ergodic distributions that consistently overshoot the planner's steady state of pollution, or that fluctuate around it.

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1 Introduction

Most economically relevant decisions are taken in dynamic environments. This includes decisions on public debt, investments in durable public goods such as infrastructure and research, and investments aimed at reducing public bads such as environmental policy. There is however still a limited understanding of these problems. Research has traditionally focused attention either on very simple environments such as two period models, or identified environment in which well-behaved equilibria exist, focusing exclusively on their simple dynamics. Dynamic political equilibria are often reduced to "modified" planners' solutions and thus described as constrained optima: what distinguishes the political equilibria from the planner's solution is typically just a simple "one dimensional" bias: too much debt, too little investment, or too little environmental protection.

Describing political equilibria as simple stationary processes is a convenient and sometimes necessary first step. To a casual observer, however, it may seem odd to describe dynamic political decisions as in a predictable steady state, or embedded in an equally predictable process of convergence to a steady state. Even in peacetime, public decisions are hardly predictable beyond the very short term and prone to large swings even in the absence of significant shocks. In this paper we ask if there could be fundamental reasons to expect political equilibria to be qualitatively different than planner's solutions. Are there reasons for expecting chaotic and unpredictable behavior from dynamic political processes that we are not seeing in existing models? Could it be that we ignore these phenomena because we focus on equilibria that are easier to analyze or overly simple environments?

To address these questions, we study a very simple political economy model of environmental protection.[1] An incumbent policy maker selects the level of environmental protection and thus faces a trade-off: public policies that improve the state of the environment increase the utility of all citizens in future periods (pollution is a pure public bad); these policies however reduce the resources that can be used for public expenditures favoring the incumbent's constituency. We study the political equilibrium of this economy assuming that two parties alternate in power as incumbents.[2]

[1] The results can be extended to economic environments different than the specfic example studied in detail here. Alternative interpretations and versions of our model to which our results apply are discussed in Section 5.1.

[2] The specific way of describing the political process is not important for the results of this paper. As discussed

When the incumbent policy-maker is a benevolent planner, the economy described above has a unique equilibrium with a simple dynamics in which the state of the environment monotonically converges to a unique steady state. When policies are selected in a political equilibrium, however, the set of equilibria is very different. Under some conditions, we can still have a "planner-like" solution with monotonic convergence to a unique steady state, except that in the political equilibrium we have a bias leading to too little environmental protections. But in the political equilibrium we generally also have equilibria with much more complex dynamics. Indeed we show that if conflict of interest between the parties is sufficiently large (as measured by the size of the externalities that are not internalized by the incumbent), we always have equilibria that do not converge to a steady state at all, but that display instead persistent cycles with complex or aperiodic dynamics.

The presence of persistent cycles with complex dynamics is sometimes equated with chaotic behavior, but it does not necessarily imply that behavior is unpredictable and thus truly chaotic.[3] Generally these "complex" cycles are "invisible" for two reasons: first, the set of initial conditions that generate them have measure zero; and, second, even if the exact right initial condition is chosen, the cycles are unstable. It is therefore significant that in our main results we show that "true" chaotic behavior can arise in equilibrium.[4] In these equilibria, starting from almost any initial condition, the economy is in an aperiodic cycle that "wanders around" in a given set of possible states with positive measure; and the evolution of the economy is sensitive to initial conditions. The existence of these equilibria, moreover, does not require knife hedge parametrizations, but holds instead for generic economies. The problem in these economies is not that there are multiple equilibria; the problem is that even knowing the equilibrium, the dynamics is effectively unpredictable, despite the complete absence of random shocks.

A key variable for the existence of chaotic behavior is the degree of time inconsistency in the policy making decision process. We show that as the degree of time inconsistency converges to zero, the set over which cycles and/or chaotic behavior can occur converges, ceteris paribus,

in Section 4.2 and 5.1, the general feature of the political environment that is key to the results is the time inconsistency of the dynamic decision that is induced by the political process.

[3] By the so called Sharkovsky theorem, if there is a cycle of period 3, then there are cycles of any order. This is sometime equated to the presence of chaotic behavior. As discussed below, however, the complex cycles implied by the Sharkovsky theorem are typically "invisible" for all practical purposes (see Collet and Eckmann [1980] for a discussion).

[4] Many definitions of chaotic behavior are available in the theory of nonlinear dynamics. We associate "true" chaotic behavior with the classical topological definition in Devaney [1989] and/or the concept of "ergodic chaos." We will discuss in greater detail these definitions in Sections 2 and 4.

to zero: so chaos persists, but it is confined to narrower sets of possible states (thus limiting the unpredictability of the system). Given any degree of time inconsistency, however, the set of chaotic behavior can be arbitrarily large, so significant problems of unpredictability may be present even in economies with mild time inconsistency problems in policy making.

A limitation of the results described above is that the chaotic behavior we characterize is not typical of all equilibria of our dynamic economy, but instead of the specific class of equilibria that we characterize. Our results can be collectively interpreted as an *impossibility result*: for the simple yet natural economy we consider, it is impossible to predict equilibrium behavior in the sense that there are always chaotic equilibria that make it impossible. Equilibria with complex dynamics, moreover, highlight a new source of inefficiency generated in political equilibria that has no correspondent in standard planner's problems: the instability of policies even in the absence of external shocks. We will show that both in equilibria with persistent cycles and in equilibria with chaotic behavior, we do not necessarily have a simple "one dimensional" bias (in our case "too little" environmental protection). Under conditions that will be characterized, the state of the economy (pollution in our application) fluctuates around the planner's optimum in these equilibria. In these equilibria, pollution is excessive on average, but it may recurrently dip below the planner's first best. Naturally, this may be even worse than reaching a constant steady state with excessive pollution.

1.1 Related Literature

This paper connects three lines of research. The first is the political economy literature on dynamic social choice problems. Since the celebrated impossibility result by Arrow, an important literature has studied how intransitivity in social choices can lead to "chaotic" behavior in simple dynamic processes.[5] The term "chaos" here is not used in a technical sense, but instead to mean that given any initial and terminal points in the policy space (even if Pareto dominated), it is possible to find an exogenous sequence of policy alternatives such that an assembly deliberating by majority rule would move from the first to the second. A key feature of the works in this literature is that they do not consider strategic games, since they assume that voters are fully myopic, thus voting over the alternatives that are offered to them ignoring dynamic effects in the following periods. They, moreover, assume an exogenously given order of the alternatives in the Condorcet cycle,

[5] See McKelvey [1975, Schoffield [1978], among others.

thus leaving unexplained where it comes from and what determines the policies.

The more recent literature in political economy has opted for modelling dynamic political problems as full fledged stochastic games with rational forward looking agents, as we do in our work. This literature, however, has either focused on simple two periods environments in which cycles or complex dynamics is impossible, or focused on "well behaved" equilibria admitting steady state.[6] Among the exceptions, there are Boyland, Ledyard and McKelvey [1996] and, more recently, Bai and Lagunoff [2011] and Battaglini, Palfrey and Nunnari [2012]. The first paper considers a model in which simple cycles with finite orbit may emerge when the policy-maker selects policies that can be defeated by the smallest possible majority, and s/he can commit for at least 3 periods into the future: the length of the commitment period determines the length of the cycle in this model.[7] Bai and Lagunoff [2011] study a dynamic political game in which policies at t affect political turnover at $t+1$. They show conditions under which the equilibrium may converge to a stable steady state following a dampened cycle. Battaglini et al. [2012] considers a model of free riding in which n agents independently contribute to a public good: they show the existence of Markov equilibria with dampened cycles, or with cycles of period 2 in which the agents perpetually increase and then decrease the level of the public good.[8] None of these papers, nor to our knowledge any other in the political economy literature, has linked political distortions and, more generally, time inconsistency in dynamic decision making to the existence of complex limit cycles and/or chaotic behavior.[9]

The second literature to which our paper contributes is the literature studying complex dynamical systems and chaos in market economies (see Baumol and Benhabib [1989] for a survey). The turnpike theorem shows that cycles and chaos is impossible in growth models with one sector

[6] Focusing only on works concerning environmental policies and dynamic public goods for brevity, positive theories are presented by Levhari and Mirman [1980], Fershtman and Nitzan [1991], Marx and Matthews [2000], Dutta and Radner [2004], Harstad [2012], Battaglini, Nunnari and Palfrey [2014], Harrison and Lagunoff [2019] who, among others, develop pure models of free riding; and Battaglini and Coate [2014], Battaglini and Harstad [2016, 2020] who, among others, consider other political economy problems.

[7] The authors also show that no cycles is possible if the policy makers can not commit to a policy.

[8] The NBER working paper Battaglini et al. [2012] was published as Battaglini et al. [2014], but some of the results on cyclical equilibria were omitted in the 2014 version.

[9] A different (and less related to our work) body of research is the literature on the so called "political business cycles." This literature looks at models in which fluctuations in economic activity are generated by recurrent stimuli right before an election by an incumbent attempting to signal his/her competence to influence the electoral outcome; or right after an election as the uncertainty on the type of the winning party is resolved. These are typically stationary models with no underlying state variable, in which fluctuations coincide with the electoral cycle, not with a long term evolution of a state variable. See Alesina [1988], among others, for a survey.

with a sufficiently high discount factor (see Scheinkman [1976], McKenzie [1976]). Starting from the eighties, however, an important literature in economics has shown that limit cycles and complex dynamics can emerge in a variety of economic environments: in two sector growth models, in overlapping generation models, in models with market imperfections, and even competitive equilibria if the discount factor is sufficiently small.[10] These models derive sufficient conditions for the existence of limit cycles and of a specific form of chaos: that is, equilibria that display chaotic behavior, but only starting from a specific countable set of initial conditions. As mentioned, this definition of chaos has important limits since it is not sufficient for "observable chaos:" it does not guarantee chaotic behavior except for given initial conditions; moreover, it is typically compatible with the existence of a unique stable cycle, to which the state converges from any initial point, except at most for a set of measure zero. Proving the existence of natural economic models with robust chaotic dynamics have proven to be elusive: while examples exist in the literature, they rely on very specific and non-generic choice of parameters,[11] technologies,[12] or on the assumption of very low discount factors.[13] This has led researchers to doubt that observable chaos in economic models is other than a pathological phenomenon (Grandmont [1985], Melese and Transue [1986, 1987]). The theoretical contribution of this paper to this literature is twofold. First, we establish a link between the presence of time inconsistency and the presence of complex cycles and chaos that has not been identified in the existing literature. But, secondly, and most importantly, we present natural economic environments in which robust chaotic behavior emerges *generically*: that is, for any generic selection of parameters, the economy has an equilibrium that is topologically conjugate to a dynamical system displaying robust chaotic behavior (technically speaking with a positive Lyapunov exponent and admitting, so called, ergotic chaos).

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[10] Cycles and chaos in two sector growth models have been first shown in a seminal paper by Benhabib and Nishimura [1985]. Overlapping generation models are studied, among others, by Gale [1973], Benhabib and Day [1982], Grandmont [1985], Reichlin [1986]. Models with market imperfections are studied by Bewley [1986] and Woodford [1988]. Boldrin and Montrucchio [1986] have presented an anti Turnpike theorem for economies with sufficiently small intertemporal discount factors. One of the very first papers to show cycles and chaotic behavior is Day [1982], who however assumes an exogenous policy function that is inconsistent with intertemporal utility maximization.

[11] Most of the model cited in footnote 8 may generate equilibria with observale chaotic behavior as defined above. These models however, require very specific nongenertic choices of parameters for observable chaos, so they can not be considered as generic examples of economies displaying chaotic behavior.

[12] Day and Shafer [1983], for example, show an example with ergodic chaos assuming an economy in which demand is assumed to be piecewise linear.

[13] Boldrin and Montrucchio [1986], Denneckere and Pelikan [1986], Nishimura, Sorger and Yano [2000], among others, consider infinite horizon economies with low discount factors.

Finally, our work contributes to the literature on dynamic decision with time inconsistency. Morris and Postlewaite [1997] is one of the first papers to point out that equilibria with time inconsistent decision makers look different than planners' solutions. They present an example in which, contrary to what happens with a time consistent decision maker, the unique equilibrium is discontinuous. For the most part, the literature has focused on the solution of the decision problem of an hyperbolic consumer with per period utility that is strictly supermodular in the state variable (the level of assets) at t and $t+1$. This implies a monotonic investment function and indeed savings or dissavings at all wealth levels, thus ruling out cycles from the start.[14] In the problem we study the objective function is not strictly supermodular and the equilibrium strategy is not monotonic in the state, consistent with savings and dissavings depending on the state. To our knowledge no paper in this literature has shown examples in which time inconsistency may generate cycles and chaotic behavior.

2 Model

Consider an economy in which two parties alternate in power, that we call $Left$ and $Right$. Each party is associated to a constituency of citizens. We assume that there is a continuum of citizens and we normalize the size of the constituency of each party to one. The party in power at time t selects a policy p_t from a set of feasible policies P. We assume that the policy generates immediate costs or benefits for the citizens' and it also contributes to a long term state variable x that also affects the citizens utility. For example, p_t may be a polluting activity (say fracking) that generates economic benefits to all or a subset of the citizens, but that contributes to global warming. Specifically, we assume that, for any t, p_t generates an immediate utility Kp_t for the constituency of the party in power and a utility $\alpha K p_t$ for the party out of power, where $K > 0$ and $\alpha \in (-\infty, 1]$. When $\alpha \in [0, 1]$ the policy benefits both the constituency of the party in power and the constituency of the party out of power, albeit less for the latter if $\alpha < 1$. This scenario may occur when the government can select a policy that is biased in favor of its constituency; or

[14] The first characterization for linear equilibria with savings at all wealth levels is presented by Phelps and Pollak [1968]. A general characterization of the Euler equation for these problems is presented in Harris and Laibson [2001]. Krussell and Smith [2003] have shown conditions under which a multiplicity of equilibria with different steady state may exist. Chatterjee and Eyigungor [2014] have shown that, with constant marginal productivity r, in all equilibria the policy function is monotonic in the state, not differentiable, and it implies dissaving at all levels of wealth in the standard problem in which the discount factor $\delta = 1/(1+r) - 1$. Cao and Werning [2018] have extended this result assuming linear technology but allowing the previous inequality to be violated, showing that depending on r there can be either saving or dissavings at all wealth levels.

when it can redistribute the benefit of the policy to favor its constituency. When $\alpha \in (-\infty, 0)$, instead, the policy benefits the constituency of the party in government, but generates negative externalities for the rest of the citizens.

The policy also contributes to a state variable x_t that we interpret as the stock of pollution or the state of the environment. The stock of pollution evolves according to:

$$x_{t+1} = (1-\gamma)x_t + p_t, \tag{1}$$

where γ is the rate of depreciation of x_t. We assume the policy takes values in a set $P = [-l, \infty)$, where $l \geq 0$ is a non negative bound representing the limits of how "reversible" is the polluting process generating x.

In this economy, an allocation is an infinite nonnegative sequence $z = (p_\infty, x_\infty)$ where $p_\infty = (p_1, ..., p_t, ...)$, $x_\infty = (x_1, ..., x_t, ...)$ and x_1 is exogenously given. We refer to $z_t = (p_t, x_t)$ for $t \geq 1$ as the allocation in period t. Citizens are identical except for the party whose constituency they belong to. Let $u_i(p_t, o(t))$ be the utility of a citizen in the constituency of party i when the party in office is $o(t)$, so $u_i(p_t, o(t)) = Kp_t$ if $i = o(t)$ and $u_i(p_t, o(t)) = \alpha K p_t$ otherwise. We assume that the intertemporal utility at $t = 1$ of an agent in party i's constituency is:

$$U^i(z_\infty) = \sum_{t=1}^{\infty} \delta^{t-1} \left[u_i(p_t, o(t)) - e(x_t) \right],$$

where δ is the discount factor, $e(\cdot)$ is the impact of the state x_t on the citizens' utility. We assume $e(\cdot)$ is continuous, twice differentiable and strictly convex on $\Re$ with minimum at a finite point $\widehat{x} \in \Re$. A positive $\widehat{x}$ may arise when p generates positive technological spillovers. For most of the analysis we indeed assume $e(x) = (\beta/2)(x - \widehat{x})^2$. In this case, the economy is fully characterized by the vector of parameters $\omega = (\delta, K, \alpha, l, \gamma, \beta, \widehat{x})$. The set of possible values that ω can take is denoted Ω. We will relax the assumption that $e(x)$ take a specific functional form in Section 5.1.

We assume that, in every period t, party $l \in \{Right, Left\}$ has a probability 1/2 to be in power. This assumption reflects the idea that the two parties have the same constituency, so the identity of the majority party at t is determined by chance. The result below do not depend on this assumption, that is made here for simplicity. We will describe in Section 5.2 alternative ways of modeling policy making. As we will discuss in greater detail below, what really matters is that the preferences for the policy-maker at time t are time inconsistent. There are many environments in which this feature may occur.

We study the symmetric Markov perfect equilibria of this economy, in which the parties use the same strategy, and in each period t these strategies are time-independent functions of the state x_t.[15] A strategy is a function $p(\cdot)$, where $p(x)$ is the policy of the party in power when the state is x. Naturally, once $p(x)$ is defined, then the state variable at $t+1$ is automatically defined as: $y(x) = (1-\gamma)x + p(x)$. In the following it will be more convenient to define equilibria in terms of $y(x)$. For the remainder of the paper we refer to $y(x)$ as the *investment function*. Associated with any Markov perfect equilibrium of the game is a value function, $v(x)$, which specifies the expected discounted future payoff to an agent when the state is x.

We are interested in studying the long term properties of the allocation and whether this behavior is predictable given the initial conditions or not. An equilibrium $y(x)$ and a support X for the state variable x define a dynamical system. We define $y^0(x) = x$, $y^1(x) = y(x)$ and the kth iterate $[y]^k(x)$ as $[y]^k(x) = y([y]^{k-1}(x))$. For any starting condition x_0, iteration of $y(x)$ naturally define an orbit $\{x_0, x_1, ...x_t, x_{t+1}, ...\}$ in which $x_t = [y]^t(x_0)$. A cycle is a set $\{x_0, x_1, ..., x_k\}$ such that $x_i = y(x_{i-1})$ for all $i \geq 1$ and $x_0 = y(x_k)$. Any element of a cycle with k element satisfies the condition $x_i = [y]^k(x_i)$ and is called a periodic point of period k. The simplest, and most widely studied, case of cycle is a *steady state*, which is a cycle of period 1. In this case, behavior remains constant at the steady state. When there is a cycle of period $k > 1$, the system rotates around the points of the cycle. We define a cycle $\{x_0, x_1, ..., x_k\}$ to be stable if for all periodic points x_i there exists an open neighborhood U of x_i such that for all $x \in U$, we have $y^{mk}(x) \in U$ for any integer $m \geq 1$ and $\lim_{m\to\infty} y^{mk}(x) = x_0$. When a cycle is stable a small perturbation to the state variable does not alter the long run behavior of the system.

A standard definition of chaotic behavior is provided by Devaney [1989]. We say that an investment function y is *transitive* in X if for any open U, $V \subset X$, there exists a k such that $y^k(U) \cap V \neq \emptyset$. Intuitively, a topologically transitive map "wanders" around the support, moving under iteration from one arbitrarily small neighborhood to any other. We moreover say that $y(x)$ has a *dense orbit* in X if the set of periodic points of $y(x)$ is dense in X. When the orbit is dense, the investment function recurrently returns arbitrarily close to any point in X. Given this, we have:

[15] The main result of our analysis is in proving the existence of equilibria with cycles, and/or unpredictableand chaotic behavior. The focus on Markov equilibria, therefore, is without loss of generality and makes the results stronger as it relies on simpler strategies.

Definition 1. *An investment function y exhibits topological chaos in a set X if it is transitive and it has a set of periodic points that is dense in X.*

To understand this definition note that if the conditions of Definition 1 are satisfied, then y is extremely sensitive to initial conditions. We say that y has *sensitive dependence on initial conditions* if there exists a δ such that, for any $x \in X$ and any neighborhood N of x there exists a $z \in N$ and a $m \geq 0$ such that $|f^m(x) - f^m(z)| > \delta$. It can be shown that the conditions in Definition 3 imply sensitive dependence.[16] A function y that exhibits topological chaos, therefore, not only has a dense set of periodic points and wanders around the support, it gives different predictions for arbitrarily close initial conditions. This form of chaotic behavior is incompatible with the existence of stable cycles.[17]

An alternative approach to define chaotic behavior consists in directly looking at the long term behavior of the state variable. A dynamical system is said to display *ergodic chaos* on X if the ergodic distribution of the states generated by iterating y converges to an absolutely continuous invariant distribution on X. As we will see, our equilibria will satisfy both the topological and the ergodic definition of chaos. We will formally define and discuss ergodic chaos in Section 4.1.

All these definition pertain to a given investment function $y(x)$. A number of specific examples of dynamical systems displaying stable cycles of chaotic behavior have been provided in the literature on complex dynamics. We are interested in characterizing environments in which chaotic behavior can emerge in entire classes of economies. *Let* ω be the vector of parameters characterizing the economy described above with domain Ω.

Definition 2. *We say that the economy is chaotic in Ω if and only if for all $\omega \in \Omega$ except at most for a subset of measure zero, there is an equilibrium y of the economy that is chaotic as in Definition 1.*

The interest in Definition 2 lies in the fact that when an economy is chaotic according to Definition 1 we should not *expect* to be able to predict the dynamic behavior of the parties in the economy: there exists always an equilibrium in which this behavior is unpredictable even if there

[16] Devaney [1989] originally included sensitive dependence on initial conditions in the definition of topological chaos; Banks et al. [1992] subsequently proved that it is implied by transitivity and a dense orbit.

[17] An alternative definition of topological chaos is provided by Li and Yorke [1975]. If a y is chaotic according to Devaney's definition, then it is chaotic according to Li and Yorke's definition. While Li and Yorke's definition is easier to verify, it does not imply sensitive dependence and it is compatible with situations in which the chaotic behavior is "invisible" as discussed above. We will return to this definition in Section 4.1, where we will show an example of this phenomenon.

are no shocks. The existence of a chaotic equilibrium, moreover, does not depend on the exact configuration of the parameters describing the economy. In a chaotic economy, the unpredictability is purely generated by the strategic interactions of the parties in the economy.

As a benchmark, it is useful to characterize the optimal policy when it is selected by a utilitarian planner who maximizes the sum of utilities for all citizens (henceforth, the planner). The planner solves the following problem:

$$V(x) = \max_{y \geq (1-\gamma)x - l} \{\Gamma(x, y; \alpha, \gamma) + \delta V(y)\} \tag{2}$$

where $\Gamma(x, y; \alpha, \gamma) = (1+\alpha) K \cdot (y - (1-\gamma)x) - 2e(x)$ and $V(y)$ is the planner's continuation function. Note that $\Gamma(x, y; \alpha, \gamma)$ is continuous, differentiable in $y \in \Re$ for a given $x \in \Re$, and concave for $x, y \in \Re$, strictly with respect to x alone. By a standard argument, we can show that there exists a unique $V^*(x)$ satisfying (2) that is strictly concave and differentiable. From the envelope theorem, we have that at a point with an interior solution: $[V^*]'(x) = -(1+\alpha)(1-\gamma)K - 2e'(x)$. Under the assumption that $e(x) = (\beta/2)(x - \widehat{x})^2$, we have that the policy that maximizes (2) ignoring the constraints is:

$$Y^* = \widehat{x} + \frac{1+\alpha}{2\beta}\left(\frac{1}{\delta} - (1-\gamma)\right) K.$$

Let us define $\widehat{x}(l, \gamma) = (Y^* + l)/(1-\gamma)$. The optimal policy function is therefore: $Y^*(x) = Y^*$ for $x \leq \widehat{x}(l, \gamma)$ and $Y^*(x) = (1-\gamma)x - l$ for $x > \widehat{x}(l, \gamma)$. It is easy to see that $Y^*(x)$ admits a unique steady state, equal to Y^*. The dynamics is therefore very simple: the planner immediately jump to the optimal value Y^* if possible, i.e. if $x \leq \widehat{x}(l, \gamma)$. When the state is too high, that is when $x > \widehat{x}(l, \gamma)$, the planner can not achieve the first best, so we have a corner solution $Y^*(x) = (1-\gamma)x - l$.

3 Will Society reach a steady state?

We now turn to the study of the equilibria of the game in which policies are chosen by the incumbent party (henceforth, the incumbent). The incumbent's problem can be written as follows:

$$\max_{y \geq (1-\gamma)x - l} \{K[y - (1-\gamma)x] - e(x) + \delta v(y)\}. \tag{3}$$

The incumbent maximizes the expected utility of her constituency taking $v(y)$ as given, thus ignoring the cost/benefit for the constituency of the other party, $\alpha K[y - (1-\gamma)x] - e(x)$. In equilibrium, the expected continuation in state $x \in \Re$, $v(x)$, must satisfy:

$$\begin{aligned} v(x) &= \frac{1}{2}\left[K(y(x)-(1-\gamma)x)+\delta v(y(x))\right] \\ &+\frac{1}{2}\left[\alpha K(y(x)-(1-\gamma)x)+\delta v(y(x))\right]-e(x). \end{aligned} \quad (4)$$

If x is the state, each party suffers a disutility $e(x)$ for sure; with probability 1/2 the incumbent remains in office and selects $y(x)$, obtaining $K(y(x)-(1-\gamma)x)+\delta v(y(x))$; with probability 1/2 the party is no longer in office and receives only $\alpha K\left(y(x)-(1-\gamma)x\right)+\delta v(y(x))$, since the policy $y(x)$ is selected by the other party. An equilibrium is characterized by a pair of functions $y(\cdot)$ and $v(\cdot)$ such that for all states x, $y(x)$ solves (3) given $v(x)$ and $v(x)$ solves (4) given $y(x)$.

The incumbent's trade-off consists in increasing current utility by increasing y (say, by polluting more); and by reducing future's utility through the effect of y on the expected continuation function $v(y)$. There are two key differences between (3)-(4) and the planner's problem (2). The first is that, as we mentioned above, in any given period the incumbent selects a policy that maximizes the expected utility of his/her constituency alone, ignoring the spillover effects on the cosntituency of the other party. The second (and most important) difference is that the value of the incumbent's problem (3) does not coincide with the incumbent's continuation value function (4) except in the special case in which $\alpha = 1$. The value of (3) is the expected value for the incumbent; in the continuation of the game, however, the incumbent at t will remain incumbent only with probability 1/2. This feature make the incumbent's problem time inconsistent since her objective function when selecting the policy does not coincide with the expected continuation value. In the planner's solution the marginal effect of the state on the expected continuation value is independent of expected future policy $y(x)$:

$$\begin{aligned} \left[V^*\right]'(x) &= -\left(1+\alpha\right)\left(1-\gamma\right)K-2e'(x)+\left[\left(1+\alpha\right)K+\delta\left[V^*\right]'\left(Y^*(x)\right)\right]\left[Y^*\right]'(x) \\ &= -\left(1+\alpha\right)\left(1-\gamma\right)K-2e'(x) \end{aligned}$$

where $\left[Y^*\right]'(x)$ is the derivative of the planner's policy function and the second equality follows from the envelope theorem. In the political equilibrium, however, the standard envelope theorem is not directly applicable, making the optimal decision for the incumbent critically dependent on her expectation of future behavior of the other party. The incumbent's value function (4) can be written as:

$$v(x) = [K(y(x) - (1-\gamma)x) + \delta v(y(x))] - e(x) \tag{5}$$
$$-\frac{1}{2}(1-\alpha)K \cdot [y(x) - (1-\gamma)x]$$

where the first line on the right hand side is the objective function that is maximized by the incumbent at $t+1$, and the second line collects the wedge between the incumbent's objective function and the expected continuation value. Applying the envelope theorem to the first term in (5),[18] we have:

$$v'(x) = -e'(x) - (1+\alpha)K(1-\gamma)/2 - (1-\alpha)Ky'(x)/2. \tag{6}$$

The key feature of this expression is that the marginal change in the value function depends on the expected policies selected by future incumbents, i.e. $y(x)$. If the incumbent at t expects the incumbent at $t+1$ (herself or the opponent) to rapidly increase y as a function of x (i.e. a high positive $y'(x)$), then she will have higher incentives to keep the state x low at t; similarly, if s/he expects the future incumbent to clean up x or increases it slowly (i.e. a low or negative $y'(x)$), then she will have higher incentives to pollute. The important question for predicting behavior in a political equilibrium is what kind of expectations on $y(x)$ are consistent with equilibrium behavior.

In equilibrium the policy must solve (3). From the first order necessary condition we have $K = -\delta v'(x)$. Ignoring for the moment the policy constraint, in an interior optimum both this condition and (6) must be satisfied. Combining them we obtain:

$$y'(x) = [2/\delta - (1+\alpha)(1-\gamma) - 2e'(x)/K]/(1-\alpha). \tag{7}$$

Let us now define the function $y(x,c)$ by integrating (7) under the assumption that $e(x) = (\beta/2)(x-\widehat{x})^2$:

$$y(x,c) = [2/\delta - (1+\alpha)(1-\gamma) + 2\beta\widehat{x}/K] \cdot x/(1-\alpha) - \beta x^2/((1-\alpha)K) + c \tag{8}$$

where c is a free constant. If the agents expect the other players to invest according to (8), then $y(x,c)$ satisfies the first order necessary condition for optimality for the incumbent in state x. It

[18] The assumption of differentiability here is without loss of generality since, as we will show in Proposition 1, the equilibrium is almost everywhere differentiable (and always in the relevant region). We assume here differentiability only for ease of notation, since the same argument regarding the dependence of differentials of v on y can be made without assuming differentiability.

follows that $y(x,c)$ is an equilibrium if the necessary condition is sufficient and if the expectation that $y(x,c)$ is used is correct. It is easy to see that this can not always be the case. The policy function $y(x,c)$ defined in (8) surely violates the constraint $y \geq (1-\gamma)x - l$ in states that are sufficiently large or sufficiently small.[19] The following Proposition 1, however, shows that if time inconsistency in the planner's problem is sufficiently high, then there is an equilibrium that coincides to $y(x,c)$ in an interval $\mathcal{X}^*$ for some appropriately chosen constant c; and that the state remains in $\mathcal{X}^*$ in all periods except at most for an initial transition period. In these equilibria, therefore, the dynamics of the economy is ultimately characterized by (8).

An important feature of (8) is that it is a nonlinear, non-monotonic function of the state variable. The function is increasing in x for low values, but then becomes decreasing for $x > \frac{K}{2\beta}\left[\frac{2}{\delta} - (1+\alpha)(1-\gamma) + \frac{2\beta}{K}\widehat{x}\right]$, no matter what the value of c is. Besides proving the existence of equilibria in which (8) describes the dynamics of the economy, Proposition 1 shows that there are indeed equilibria in which this nonlinearity is translated in a complex dynamics for the state variable.

Let R be the ratio $R{=}\beta/K$. This ratio captures the temptation for an incumbent to abuse its position in selecting the policy. The numerator measures the cost of the externality generated by x on society; the denominator is the private benefit of the policy for the incumbent. Define the threshold:

$$R^*(\alpha) = \frac{4\delta(1-\alpha)(2-\gamma) + \delta(1+\alpha)(1-\gamma)\gamma - 2\gamma}{2\delta(\widehat{x}\gamma + l)} \tag{9}$$

and the sets:

$$\begin{aligned} \mathcal{C}^* &= \left[\frac{(3-\varphi_1)(1+\varphi_1)}{4\varphi_2}, \frac{(4-\varphi_1)(2+\varphi_1)}{4\varphi_2}\right] \\ \text{and } \mathcal{X}^*(c) &= \left[\frac{\varphi_1 - 1 - \sqrt{(\varphi_1-1)^2 + 4\varphi_2 c}}{2\varphi_2}, \frac{\varphi_1^2 + 4\varphi_2 c}{4\varphi_2}\right] \end{aligned} \tag{10}$$

where we denote $\varphi_1 = \frac{1}{1-\alpha}\left[\frac{2}{\delta} - (1+\alpha)(1-\gamma) + \frac{2\beta}{K}\widehat{x}\right]$ and $\varphi_2 = \beta/[(1-\alpha)K]$, the coefficients of, respectively, the linear and qudratic term in (8). We say that an orbit *generically converges* to a cycle if it converges to a cycle for all initial states $x^0 \in \Re$, except at most for a subset of measure zero. We have:

Proposition 1. *Consider an economy with $R \geq R^*(\alpha)$:*

[19] Indeed, for any c, we have $\lim_{x\to\pm\infty}[y(x,c) - (1-\gamma)x + l] < 0$.

- *For any $c^* \in \mathcal{C}^*$, there is an equilibrium $y^*(x, c^*)$ in correspondence of which the state variable is in $\mathcal{X}^*(c^*)$ for all periods except at most for a finite transition period, and such that $y^*(x, c^*) = y(x, c^*)$ as defined in (8) for all $x \in \mathcal{X}^*(c^*)$.*

- *For any $c^* \in \mathcal{C}^*$, moreover, each equilibrium $y^*(x, c^*)$ defines an orbit that either generically converges to a stable cycle of finite period $m \geq 2$ in $\mathcal{X}^*(c^*)$, or that is aperiodic in $\mathcal{X}^*(c^*)$.*

The first part of the proposition shows that (8) essentially describes an equilibrium of our economy (where the behavior may not be described by (8) only for a finite number of periods); and the second part shows that all these equilibria do not converge to a deterministic steady state. This result should be contrasted with the planner's solution described in the previous section, in which we have a unique equilibrum converging to a deterministic steady state.

Figure (1) illustrates two example of equilibria for the same economy (i.e. the same identical parametrization): in the first, the state variable converges to stable cycles of period 2; in the second, of period 3.[20] Each of the two equilibria corresponds to specific selections of the value of the constant c in (8).[21] As we will show in the next section, not all equilibria converge to stable cycles, but equilibria with stable cycles of this type exist for *any* economy satisfying the condition of Proposition 1, i.e. $R \geq R^*(\alpha)$.

As a term of comparison, the figure also shows the steady state that would be reached in a planner's economy. In the equilibrium with a cycle of period 2, the state variable cycles at levels always higher than the planner's steady state. In the equilibrium with a cycle of period 3, the state cycles around the steady state reached in the planner's solution: undershooting the planner's steady state for one period by about 5.5%, then overshooting it for two periods, by 6.5% and 23% respectively. We discuss in detail what type of dynamics (i.e. whether we have stable cycles or not) can be achieved in equilibrium in the next section. Here we complete the discussion of Proposition 1 commenting on the characteristics of the economy that guarantee the existence of equilibria with non-converging orbits.

A number of parameters in the model contribute to making it easier or more difficult to have equilibria with a non converging orbit. For example, it is clear that if both $\gamma = 0$ and $l = 0$, then it

[20] In this example,the parameters describing the economy are assumed to be $\delta = 0.95$, $\alpha = .8$, $\gamma = .1$, $l = 0.5$, $K/\beta = 2$ and $\widehat{x} = 1$.

[21] The equilibria coincide with (8) in $\mathcal{X}^*(c^*) = [x^*_-, x^*_+]$, but the bounds x^*_-, x^*_+ defining this region are endogenous and thus different in the two equilibria.

is impossible to construct cycles or non converging orbits. The reason is that in this case the policy constraint $x^{t+1} \geq (1-\gamma)x^t - l$ forces the policy to be monotonically increasing over time, since $x^{t+1} = y(x^t) \geq x^t$ for $\gamma = l = 0$. And indeed, consistently with this observation, we have that $R^*(\alpha) \to \infty$ as both $\gamma \to 0$ and $l \to 0$. Remarkably, however, cycles and non-converging orbits exist even for arbitrarily small γ and l if we choose the other parameters $\delta, \widehat{x}$ and α appropriately.

Two other important variables are the discount factor δ and the ideal point for society $\widehat{x}$. The threshold $R^*(\alpha)$ is increasing in δ, so the smaller is the discount factor the easier is to satisfy the sufficient condition in Proposition 1. It is indeed interesting to note that a small enough discount factor is sufficient for the existence of non converging equilibria. A small discount factor, however is not necessary, and the sufficient condition can be satisfied for any δ. On the contrary, the threshold $R^*(\alpha)$ is decreasing in $\widehat{x}$, so a larger ideal point makes nonconverging equilibria easier to achieve. Non-converging orbits are however possible even if $\widehat{x} \leq 0$.

The most interesting variable in $R^*(\alpha)$ is α, which measures the degree of time inconsistency in the economy. We postpone the discussion of the relationship between time consistency and nonconverging equilibria to Section 4.2.

4 Will Society be predictable?

4.1 Characterization of the dynamics

When the equilibrium orbit converges to a cycle as in Figure 1, the equilibrium is inefficient, but it is predictable since the orbit follows a well defined deterministic path. Unpredictability becomes a problem only when the orbit is aperiodic and chaotic (as defined in Section 2). In this case, the equilibrium is unpredictable because even an arbitrarily small error in measuring the state at time t implies a significant error in predicting the state at $t+T$, to the extent that observing the state at t may be irrelevant. Proposition 1 does not specify whether the equilibrium dynamics is cyclical; and, if it is cyclical, the period of the orbit. We study these questions in this section: what kind of dynamics can we generate as we vary c in the set $\mathcal{C}^*$?

To address this question, it is useful to "re-scale" (8) by an homeomorphism h.[22] Let us denote the composition of two functions by $f \circ g(x) = f(g(x))$.

[22] A function between two topological spaces I and J, $g : I \to J$, is said to be a *homeomorphism* if it is one-to-one, onto, continuous, and its inverse h^{-1} is also continuous.

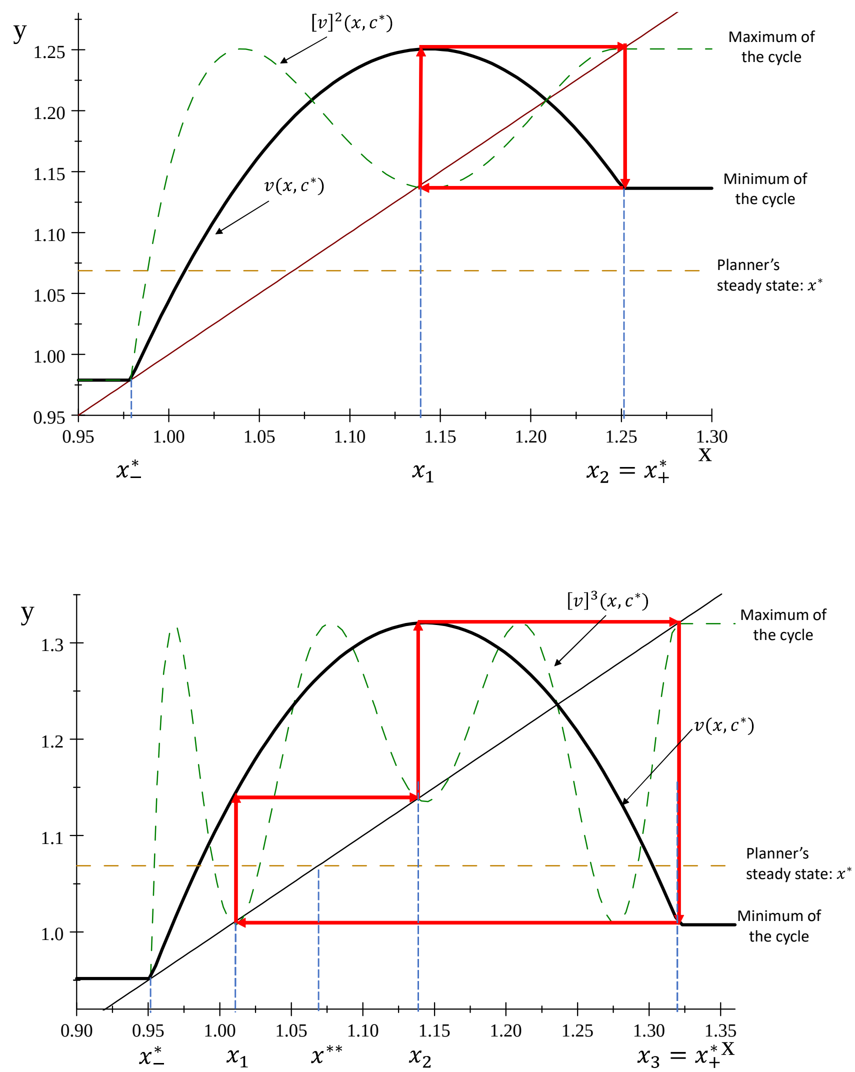

Figure 1: One economy, two equilibria with stable cycles of periods $m = 2$ and $m = 3$. The solid (black) line are the investment function $y(x)$, the dashed (green) line are the iterated maps $[y]^2(x)$ and $[y]^3(x)$. Note that the bounds x_-^*, x_+^* defining $\mathcal{X}^*(c^*) = [x_-^*, x_+^*]$ are different in the two equilibria.

Definition 2. *Let $f : A \to A$ and $g : B \to B$ be two maps, we say that f and g are topologically conjugate if there exist a homeomorphism $h : A \to B$ such that $h \circ f = g \circ h$.*

It is important to establish whether two functions f and g are topologically conjugate, because topologically conjugate functions have the same dynamical properties. We have that $[f]^n = \left[h^{-1} \circ g \circ h\right]^n = h^{-1} \circ g^n \circ h$, so if x is a fixed point of $[f]^n$, then $h(x)$ must be a fixed point of $[g]^n$, since we have $[g]^n \circ h(x) = h \circ [f]^n (x) = h(x)$. Indeed, the function h gives a one-to-one correspondence between the periodic points of f and g. Periodic and aperiodic for f go over via h to qualitatively similar orbits for g; and f is topologically chaotic a' la Devaney and admits an absolutely continuous ergodic distribution if and only if the same is true for g.[23] We can therefore study the properties of f by studying g.

An adequate re-scaling of (8) by an homeomorphism simplifies the analysis of the equilibria of Proposition 1 because it allows us to link equilibrium dynamics to the dynamics of the logistic function $L_\eta(x) = \eta x(1-x)$, one of the few nonlinear functions for which the dynamics has been extensively studied (see for instance Ulam and von Neumann [1947], Ruelle [1977], Jakobson [1981], among others). Naturally, an equilibrium will never be conjugate to the logistic $L_\eta(x)$ on the entire real line, since $L_\eta(x)$ is an unbounded function while the equilibrium must satisfy the feasibility constraint $y \geq (1-\gamma)x - l$. To characterize the equilibrium dynamics, however, it is sufficient that we have conjugacy on a superset of the support of the states reached in equilibrium. We say that an equilibrium $y(x;c)$ is *topologically conjugate on its support* to L_η if there is a set X_S such that $[y]^m (x;c) \in X_S$ for all $m \geq \underline{m}$ for some finite $\underline{m}$ and $x \in X_S$, and $y(x;c)$ is topologically conjugate to L_η on X_S. We have:

Lemma 1. *Assume $R \geq R^*(\alpha)$ as defined in Proposition 1. For any $\eta \in [3,4]$, there is a constant:*

$$c(\eta; \varphi_1, \varphi_2) = (\varphi_1/2)(1 - \varphi_1/2)/\varphi_2 - \eta/2(1-\eta/2)/\varphi_2 \tag{11}$$

such that $c(\eta; \varphi_1, \varphi_2) \in \mathcal{C}^*$ as defined in (10), and $y^*(x; c(\eta; \varphi_1, \varphi_2))$ is an equilibrium that is topologically conjugate to L_η on $\mathcal{X}^*(c(\eta; \varphi_1, \varphi_2))$.

We can now use Lemma 1 to derive the two examples presented in Figure 1. It is well known that for values $\eta \in \left[3, 1+\sqrt{6}\right]$, the logistic has a unique stable cycle of period 2.[24] The

[23] See Devaney [1989], Ch. 1.7.

[24] See Devaney (1989), among others.

equilibrium in the top panel of Figure 1 is constructed by setting c in (8) equal to $c\,(7/2;\varphi_1,\varphi_2)$. As η increases beyond $1+\sqrt{6}$, cycles of order $2m$ for any $m \geq 1$ emerge; and for $\eta > \eta_\infty = 3.5699$ there are isolated values of μ for which cycles with odd periods appear (as the value of the example in Figure 1 where $m=3$).[25] Specifically, the equilibrium with a stable cycle of period 3 in Figure 1 is indeed constructed setting $c = c\,(3.835;\varphi_1,\varphi_2)$.[26] It is important to note that the existence of these equilibria does not depend on knife-edge assumptions about the parameters. By Lemma 1, for any economy with $R \geq R^*(\alpha)$, equilibria with these properties can be easily constructed by simply solving the equations $c = c\,(7/2;\varphi_1,\varphi_2)$ and $c = c\,(3;\varphi_1,\varphi_2)$.

In addition to stable cycles, the literature has also identified specific values of η in $[\eta_\infty, 4]$ for which $L_\eta(x)$ displays chaotic behavior, for example $\eta = 4$ (see Ulam and von Neumann [1947]); or the Ruelle's constant η^*, which is approximately 3.6785735.[27] In Proposition 2 below we show that both $c\,(4;\varphi_1,\varphi_2)$ or $c\,(\eta^*;\varphi_1,\varphi_2)$ are in $\mathcal{C}^*$: we can therefore generate equilibria with the same qualitative properties setting $c = c\,(4;\varphi_1,\varphi_2)$ or $c\,(\eta^*;\varphi_1,\varphi_2)$. The important observation is that while the values for which the logistic displays cycles of period 3 or chaotic behavior are very specific, the set of economies that admit an equilibrium with, say, cycle of period 3 or chaotic behavior are completely generic. Proposition 1 and Lemma 1 prove that for *any* generic economy satisfying $R \geq R^*(\alpha)$, there exists an equilibrium with these properties: the value $c\,(\eta;\varphi_1,\varphi_2)$ that achieves these properties is not assumed in the environment, is endogenously determined in equilibrium.

Summarizing these considerations, we have the following result.

Proposition 2. *Assume an economy with $R \geq R^*(\alpha)$ as defined in (9):*

- *For every value $m \geq 2$, there is an equilibrium $y^*(x, c^*)$ associated to a point $c^* \in \mathcal{C}^*$ with a unique stable cycle of period at least m. The orbit of this equilibrium is in $\mathcal{X}^*(c^*)$ and characterized by the fixed points of $[y]^m\,(x;c^*)$, as defined in (8).*

[25] The existence of stable cycles of order 3 is particularly important because by the Sarkovski Theorem they imply the existence of cycles of any order. This has sometimes been equated to the presence of chaos. This is however not a completely legitimate interpretation. The logistic has a unique stable cycle and the dynamics converges to it starting from all points in its support except from a subset of measure zero. The additional cycles are unstable cycles that exist only for initial values in a set of measure zero. These cycles are often referred to as "invisible" since for all practical purposes they are unobserved.

[26] For the fact that the logistic with constant equal to 3.835 generates a stable cycle of period 3 (and for specific intervals for which the logistic has stable cycles of higher periods) see, for example, May and Oster [1976].

[27] Ruelle's constant η^* is the the only real solution η^* of $(\eta^*-2)^2(\eta^*+2) = 16$. See Ruelle [1977]).

- *There is also a non empty subset of $\mathcal{C}^D \subset \mathcal{C}^*$ such that the associated equilibria $y^*(x, c^*)$ with $c^* \in \mathcal{C}^D$ display topological chaos a' la Devaney on $\mathcal{X}^*(c^*)$.*

Using the same economy as in Figure 1, the top panel of Figure 2 presents the orbit of two chaotic equilibria by setting c equal to, respectively, $c\,(4; \varphi_1, \varphi_2)$ and to $c\,(\eta^*; \varphi_1, \varphi_2)$, where η^* is Ruelle's constant discussed above. From Lemma 1, the first equilibrium is topologically conjugate to $L_4(x)$, the famous chaotic case first studied by Ulam and von Neumann [1947]; while the second is topologically conjugate to $L_{\eta^*}(x)$. In both cases, the orbit of the state variable described by the equilibrium fluctuates above and below the steady state reached by the planner (which is the same in both examples since the fundamentals are unchanged). The difference between the equilibria of Figure 2 and those of Figure 1 is that now the oscillations are aperiodic and thus unpredictable. It is easy to compute examples showing that even a marginal perturbation in the initial state implies significantly different behavior in the long run.

Given Proposition 2, it is natural to ask what the properties of the long term distribution of states induced by iterations of y are. A particularly important property is whether the distribution is absolutely continuous, invariant and ergodic.[28]

Definition 3. *We say that a dynamical system displays ergodic chaos if there is an absolutely continuous probability measure that is ergodic and invariant.*

When we have ergodic chaos, the behavior of the dynamical system in the long term is completely independent of the starting point, and it can be described by a distribution function.

Ulam and von Neumann [1947] have famously shown that $L_4(x)$ admits an ergodic distribution that can be characterized in closed form and is equal to the arcsine distribution with density:

$$\lambda(x) = \frac{1}{\pi\sqrt{x(1-x)}}$$

While this is the only case for which the ergodic distribution of the logistic has been characterized (and one of the very few dynamical system for which it can be characterized), subsequent work has shown that there is a set of positive measure of values of η such that $L_\eta(x)$ admits an ergodic distribution (one of which is Ruelle's number η^*).[29] By Lemma 1, for each of these values, there

[28] A distribution μ is said to be invariant if $y_*\mu = \mu$, where $y_*\mu$ is the push forward measure $y_*\mu(A) = \mu(y^{-1}(A))$. A distribution is ergodic if $\lim_{T\to\infty} \sum_{k=1}^{T} \varphi([y]^k\,(x)) = \int \varphi d\mu$ for almost all x.

[29] See Jacobson [1981] and Collet and Eckmmann [1981] for a detailed discussion.

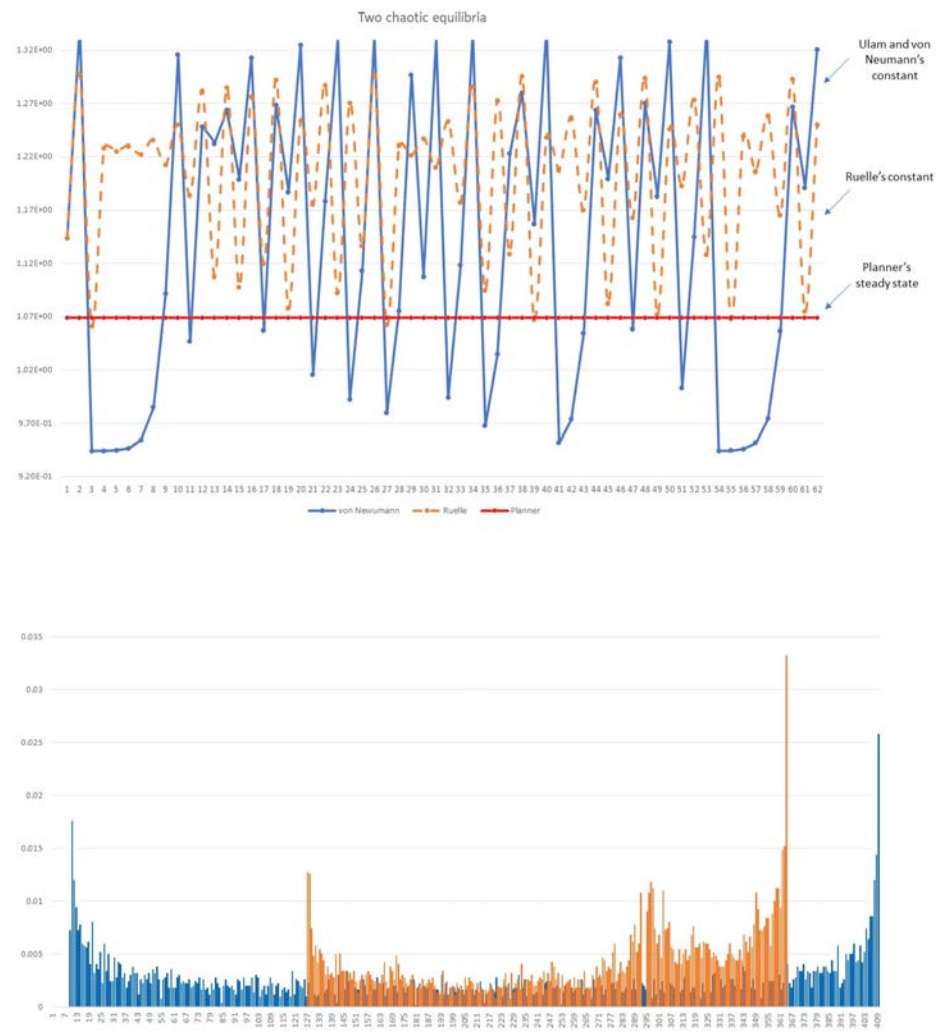

Figure 2: Two equilibria with ergodic chaos existing for the same economy. The first equilibrium is topologically conjugate to von Neumann and Ulam's example, the second to Ruelle's example. The top panel displays the state trajectories; the second the ergodic distributions.

is an equilibrium of the policy game that qualitatively inherits the same properties. The case with $c\,(4;\varphi_1,\varphi_2)$ inherits the qualitative properties of Ulam and von Neumann's case, though in a translated space by a specific homeomorphism. This implies that the distribution generated by the equilibrium correspondent to $c\,(4;\varphi_1,\varphi_2)$ can be characterized in closed form, though now its "shape" depends on the fundamentals of the economy. Define the following density $\mu^*(x;R,\alpha)$ on $\mathcal{X}^*$:

$$\mu(x;R,\alpha)=\frac{2R}{\pi\sqrt{16\,(1-\alpha)^2-\left(2Rx-\left[\frac{2}{\delta}-(1+\alpha)\,(1-\gamma)+2R\cdot\widehat{x}\right]\right)^2}}. \tag{12}$$

We have:

Proposition 3. *Assume an economy with $R\geq R^*(\alpha)$. There is also a subset of $\mathcal{C}^E\subset\mathcal{C}^*$ with positive measure such that the equilibria $y(x,c^*)$ with $c^*\in\mathcal{C}^E$ display ergodic chaos on the compact on support $\mathcal{X}^*(c^*)$. Among these equilibria, the equilibrium $y^*(x,c\,(4;\varphi_1,\varphi_2))$ admits the invariant distribution $\mu(x;R,\alpha)$ on $\mathcal{X}^*(c\,(4;\varphi_1,\varphi_2))$ defined in (12).*

It is important to highlight the key difference between Proposition 2-3 and the previous examples in the economic literature deriving equilibria displaying chaotic behavior. In this literature, examples of chaotic behavior have been obtained under very specific assumption of the parameters of the environment.[30] The existence result presented above holds for *any* economy satisfying the condition of Proposition 1. While the fact that we have topological chaos in the logistic with specific values $\eta=4$ or η^* may appear as a mathematical curiosity, in Proposition 2 we show that topological and ergodic chaos exists for very large, non-generic economies.[31] The reason for this is because as the parameters of the economy change, the characterization of Proposition 1 and Lemma 1 shows we have enough latitude to select an equilibrium with the desired properties.

4.2 Time inconsistency and the "size" of the chaotic region

The degree of time inconsistency impressed on the economy by the political system plays a particularly important role in determining "how much" chaos we can observe in equilibrium. In our

[30] Examples of economies with ergodic chaos are presented, among others, by Day [1982], Day and Pianigiani [1989], Woodford [1988], Boldrin and Montrucchio [1986], Nishimura, Sorger and Yano [2000]. The first three papers present examples that require nongeneric parametrizations of the economy or the technology. Boldrin and Montrucchio [1986], Nishimura, Sorger and Yano [2000] consider infinite horizon economies with low discount factors.

[31] In Section 5.1 we show that the results in Proposition 2-3 do not depend on the assumption of a specific functional form for $e(\cdot)$.

environment, time inconsistency consists in the discrepancy between the value of the objective function that is maximized by the incumbent when selecting the policy (i.e. (3)); and the expected continuation value at t before the incumbent at $t+1$ is determined (i.e. (4)). In the planner's problem there is no difference between these two functions. By comparing (3) and (5), we can see that in the political game, instead, the functions differ by:

$$-\frac{(1-\alpha)K}{2}\left[y(x)-(1-\gamma)x\right].$$

This discrepancy depends on α, but it converges to zero as $\alpha \to 1$. The parameter $\alpha \in (-\infty, 1]$, therefore, captures time inconsistency: as $\alpha \to 1$, time inconsistence converges to zero; as we reduce α, time inconsistency is increased.[32] In this limit case as $\alpha \to 1$, $v'(x)$ is independent of $y'(x)$ and the equilibrium qualitatively looks like the planner's problem. In this case the optimal policy for the incumbent ignoring the feasibility constrain is $y^*_{\alpha=1} = \widehat{x} + \frac{K}{2\beta}\cdot(1/\delta - (1-\gamma))$, and so the equilibrium is:

$$y^*_{\alpha=1}(x) = \begin{cases} \widehat{x} + \frac{K}{2\beta}\cdot(1/\delta-(1-\gamma)) & x \leq \widehat{x} \\ (1-\gamma)x - l & x > \widehat{x} \end{cases},$$

where $\widehat{x} = \left(y^*_{\alpha=1} + l\right)/(1-\gamma)$. This simple dynamical system has a unique steady state that differs from the planner's steady state only because $y^*_{\alpha=1}(x) < Y^*$.

The effect of α on the "degree of chaos" in the equilibria constructed in the previous sections can be seen from the size of $\mathcal{X}^*(c)$: $\|\mathcal{X}^*(c)\| = \overline{x}(c) - \underline{x}(c)$, where $\underline{x}(c)$ and $\overline{x}(c)$ are the bounds defined in (10). Using the bounds on c in $\mathcal{C}^*$, it is easy to see that $\|\mathcal{X}^*(c)\| \leq 4(1-\alpha)K/\beta \to 0$ as $\alpha \to 1$. It follows that as $\alpha \to 1$ we can still have chaos, but the size of the set in which the state can "wander around" collapses to zero.

As time inconsistency increases, the potential degree of unpredictability impressed by the two incumbents may become arbitrarily large. To make this point, let us define $\underline{X}(\alpha, y_\alpha)$ and $\overline{X}(\alpha, y_\alpha)$ to be, respectively, the infimum and the supremum of the set of points reached by iteration of an equilibrium y_α:

$$\begin{aligned} \underline{X}(\alpha, y_\alpha) &= \inf\left\{x \,\middle|\, x = \left[y_\alpha\right]^k(x_0) \text{ for } x_0 \in \Re \;\; k \geq 1\right\}, \\ \overline{X}(\alpha, y_\alpha) &= \sup\left\{x \,\middle|\, x = \left[y_\alpha\right]^k(x_0) \text{ for } x_0 \in \Re \;\; k \geq 1\right\}. \end{aligned}$$

[32] When $\alpha = 1$, the effect of private expenditure is the same on the constituencies of the incumbent and the other party. The equilibrium policy is still not the utilitarian policy because the incumbent does not internalize the negative externality of x; but it is time consistent.

Define $\Delta^*(\alpha, y_\alpha) = \overline{X}(\alpha, y_\alpha) - \underline{X}(\alpha, y_\alpha)$ to be the size of the chaotic region in an equilibrium y_α. When $\Delta^*(\alpha, y_\alpha) > \Delta > 0$, then the state will unpredictably wander around in a set of at least length Δ. We have:

Proposition 4. *For any $\Delta > 0$ there is an equilibrium with a chaotic region of size Δ if $R \in [R^*(\alpha), 4(1-\alpha)/\Delta]$.*

Proposition 4 tells us that equilibria with a chaotic regions at least size Δ are possible if two conditions are met: R must be sufficiently large to allow for non linear dynamics, $R \in R^*(\alpha)$ (i.e. the condition of Proposition 1); but then R can not be too large, else the chaotic region is going to be small. Recall that $R = \beta/K$. As discussed in Section 3, for an equilibrium with nonlinear dynamics the social cost of x (as measured by β) relative to its private value (as measured by K) needs to be sufficiently large. When β is large relative to K, however, the incumbent is unwilling to reach high levels of x: thus the range of states x that can be reached in equilibrium is limited. The "size" of this region depends on α and other features of the economy. A decrease in α increases both the upperbound, and the lowerbound of $[R^*(\alpha), 4(1-\alpha)/\Delta]$. For a given α, however, the lower bound $R^*(\alpha)$ decreases in $\widehat{x}$. For any α such that $R \leq 4(1-\alpha)/\Delta$, therefore, we can always find a $\widehat{x}^*$ such that the chaotic region has size at least Δ if $\widehat{x} \geq \widehat{x}^*$.

5 Discussion and extensions

In Section 5.1 we extend the analysis to allow for a generic functional form for the cost function $e(x)$. In Section 5.2 we show that the type of equilibria with cycles and chaotic behavior characterized above are relevant in a variety of alternative economic environments with time inconsistency: including single agent decision problems with hyperbolic discounting; dynamic bargaining problems; and dynamic free riding problems.

5.1 General functional forms

In the previous analysis we assumed a quadratic cost function $e(x)$. The quadratic case allowed us to fully characterize the range of dynamic paths that can be observed in equilibrium. We could even derive the ergodic distribution of a political equilibrium in closed form. The qualitative results presented in the previous analysis, however, continue to hold if we allow for more general functional forms for $e(x)$.

From (5) it is immediately clear that independently from the functional form $e(x)$, cycles and chaotic behavior exist only with time inconsistency, as measured by α. As $\alpha \to 1$, the second term in (8) disappears and the value function becomes independent of the future expected policy, thus making it impossible to support a nonlinear investment function $y(x)$.

When $\alpha < 1$, the analysis is complicated by the fact that there is still limited understanding of the evolution of complex, nonlinear dynamical systems. We can however generalize the conditions presented above providing easily verifiable conditions for cycles and chaotic behavior. From the necessary condition in (7), we obtain:

$$y(x,c) = \left[2/\delta - (1+\alpha)(1-\gamma)x\right]/(1-\alpha) - (2/K)\cdot e(x) + c, \tag{13}$$

which corresponds, for a general $e(\cdot)$, to (8). Let us maintain the assumption that $e(x)$ is a differentiable, with continuous derivative, and strictly concave in x. If we define x^* and $x^*_-(c)$ to be, respectively, the maximizer and the smaller fixed point of $y(x,c)$, then $y(x,c)$ is differentiable and C^1-unimodal in $[x_-(c), y(x^*,c)]$ for any c such that $f(x^*,c) \geq x^*$.[33]

Li and Yorke [1978] have shown that, if continuous in x, a dynamical system $y(x)$ has cycles of any order in X if there is a $x' \in X$ such that: $[y]^3(x') < x' < y(x') < [y]^2(x')$. To show that an equilibrium with complex cycles exists, we therefore only need to show that there exists a c^* such that this condition is satisfied for $y(x;c)$, as defined in (13). Once we show that (13) satisfies Li and Yorke's condition, it is then immediate to derive sufficient conditions that guarantee that the incumbent's reaction function is consistent with the constraints on the state space. As in Proposition 1, we can indeed modify (13) to allow for constraints in the state space by constructing a function $y^*(x,c^*)$ as follows:

$$y^*(x,c^*) = \begin{cases} \max\left\{y(x^*_-(c),c), y(x,c)\right\} & x \leq y(x^*,c^*) \\ \max\left\{[y]^2(x^*,c^*), (1-\gamma)x - l\right\} & x > y(x^*,c^*) \end{cases}. \tag{14}$$

An equilibrium exists if $\left[x^*_-(c), y^*(x^*,c^*)\right]$ is sufficiently large so that $y(x,c)$ maps $\left[x^*_-, x^*_+\right]$ to itself, but sufficiently small that $y(x,c) \geq (1-\gamma)x - l$ for any $x \in \left[x^*_-(c), y^*(x^*,c^*)\right]$. This, for instance, is always satisfied if l is sufficiently large. We omit to record this result as a formal

[33] A strictly concave, C^1 function is C^1 unimodal in a set $[a,b]$ if $f(x^*) \geq x^*$ at its critical point x^*. Note that the function $y(x;c)$ in (13) has a unique maximizer at x^* independent of c, and two fixed-points $x^*_-(c)$, $x^*_-(c)$ with $x^*_-(c) < x^*_- < x^*_-(c)$.

proposition because we present a stronger result in Proposition 5 below. Here is a simple example in which Li and Yorke's condition can be applied when $e(\cdot)$ is exponential:

Example 6.1. Assume the same economy as in the examples of Figure 1 (i.e. $\alpha = .8$, $\delta = .95$, $\gamma = .5$, $l = .5$), but now assume the cost function is $e(x) = -\eta e^x$ with $\eta = 2$. We prove here that this economy has an equilibrium with cycles of any order when we set $c^* = 3.5$. In this case, (13) becomes $6.6111x - e^x - 3.5$, which has a unique maximizer at $x^* = \log(6.6111)$, maximum at $y(x^*, c^*) = 2.3756$, and lower fixed-point $x_-(c^*) = 1.239$. If we define $y^*(x, c^*)$ as in (14), it is easy to see that $y^*(x)$ maps $[x_-, y(x^*, c)]$ to itself and that the constraint is satisfied by construction. With respect to Li and Yorke's conditions, for $x' = [y^*]^{-1}(x',, c^*) = 1.4787$, we have: $1.4479 = [y^*]^3 (x', c^*) < x' = 1.4787 < y^*(x',, c^*) = 1.8888 < [y^*]^2 (x',, c^*) = 2.3756$. It follows that $y^*(x,, c^*)$ admits cycles of any order $k \geq 2$.

Li and Yorke's condition also guarantees a form of topological chaos; that is, there is a subset W of X (called the "scrambling set") for which we observe chaotic behavior: starting from any $x \in W$ points, the system does not converge to any periodic point.[34] As discussed above, a problem with this definition of chaos is that it does not exclude the possibility in which the "scrambling set" W has measure zero, and indeed that, starting from any point in X except at most a subset of measure zero, the equilibrium trajectory converges to a unique stable cycle.[35] While Li and Yorke's condition provides a sufficient condition for the existence of complex cycles, it does not tell us whether a dynamical system has a stable cycle and if it does, it does not reveal its order. If there is a stable cycle, then we can't really see the system as "chaotic" or "unpredictable".

Fully characterizing necessary and sufficient conditions for ergodic chaos for a general functional form $e(x)$ is more challenging, since only limited results exist (see Collet and Eckmann [1980] for a survey). A new, easily verifiable sufficient condition for the existence of ergodic chaos can however be derived in our model by exploiting the equilibrium construction leading to (13) and the degrees of freedom we have by selecting the constant c.

A dynamical system y is said to be *S-unimodal* if it is unimodal and it has nonpositive

[34] Formally, Li and Yorke [1978] define topological chaos as a situation in which: a. for every integer $k > 2$, there is a periodic point x_k with period k; b. There is an uncountable set $W \subset N(X)$ such that: i) if $x, y \in W$ with $x_1 \neq x_2$, then: $\liminf_{m\to\infty} d([y]^m (x_1), [y]^m (x_2)) = 0$ and and $\limsup_{m\to\infty} d([y]^m (x_1), [y]^m (x_2)) > 0$; ii) and if $x \in W$ and $y \in P(X)$, then $\limsup_{m\to\infty} d([y]^m (x_1), [y]^m (x_2)) > 0$. See Collet and Eckmann [1980].

[35] An example of such a situation is the equilibrium with period 3 in Figure 1: this equilibrium is chaotic according to Li and Yorke's definition, but it admits a unique stable cycle. The remaining cycles still exist, but are invisible because they can be observed only by starting from a subset of measure zero.

Schwarzian derivative at all noncritical points (i.e. except for x^* such that $f'(x^*) = 0$).[36] Many common functions satisfy this condition including, for instance, any polynomial of degree larger than or equal to 2 with real valued critical points (and thus the the quadratic used in Section 4), and the exponential function. The following result provides a sufficient condition for ergodic chaos when $y(x, c)$ is S-unimodal. Define $\Delta^{2,3}(c) = [y]^3 (x^*, c) - [y]^2 (x^*, c)$, this is the gap between the third and the second iteration starting from the critical point x^*, it is an easily computed function of c given $y(x, c)$. We have:

Proposition 5. *Assume that* $y(x, c)$ in (13) *has negative Schwarzian derivative with respect to* x*, and that there is a* c' *and* c'' *with* $c'' > c'$ *such that*

$$\Delta^{2,3}(c') \cdot \Delta^{2,3}(c'') \leq 0 \text{ and } [y^*]' \left([y^*]^2 (x^*, c''), c'' \right) > 1,$$

then there exists a c^* *such that* $y(x^*, c^*)$ *displays ergodic chaos on* $[[y^*]^2 (x^*, c^*), y(x^*, c^*)]$.

Proposition 6 extends a theorem by Misiurewicz [1981] using the equilibrium construction leading to (13). Misiurewicz [1981] proved that if a dynamical system y is S-unimodal and such that the iterates $[y]^n (x^*)$ of the critical point x^* converge to an unstable cycle, then y has exactly one absolutely continuous invariant measure.[37] Misiurewicz's original condition can be used to construct simple examples with ergodic chaos, but the examples typically imply non-generic parameter configurations, since they require the knife-edge case in which that iterations from x^* enter an unstable cycle.[38] In our case we can construct equilibria with ergodic chaos that exist for generic economies because c is endogenous and it can generically be selected to make sure that there is an n and a c^* such that $[y]^n (x^*, c^*)$ enters an unstable cycle. The inequalities in Proposition 5 are sufficient conditions for this to be possible. Once we have shown that the condition of Proposition 5 is satisfied, we can indeed modify (13) to allow for constraints in the state space by constructing a function $y^*(x, c^*)$ as in (14). Again, an equilibrium exists if

[36] The Schwarzian derivative of a thrice continuously differentiable function is defined as $DS(f) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2$. See Collet and Eckmann [1980] for details.

[37] Misiurewicz [1981] actually proved that if y has no weakly stable orbit, and there exists an open neighborhood of U of x^* such that $[v]^n (x) \notin U$, then y has exactly one absolutely continuous invariant measure. The statement above is a corollary of this result (see Corollary 6 in Grandmont [1992] and Theorem II.8.3 and Corollary II.8.4 in Collet and Eckmann [1980]).

[38] An example of this is von Neumann and Ulam's example of logistic L_η with $\eta = 4$. In this this case $[y]^2 (x^*) = 0$, which is also the smaller fixed point of L_4: this point is an unstable steady state, so a unstable cycle of period 1. Any arbitrarily small perturbation of η, however, compromises the property of the existence of an absolute continuous ergodic distribution.

$\left[x_{-}^{*}(c), y(x^{*}, c^{*})\right]$ is sufficiently large so that $y(x, c^{*})$ maps $\left[x_{-}^{*}, x_{+}^{*}\right]$ to itself, but sufficiently small that $y(x, c^{*}) \leq (1-\gamma)\, x - l$ for any $x \in \left[x_{-}^{*}(c), y(x^{*}, c^{*})\right]$.

The following examples show that the condition of Proposition 5 is easy to apply in specific examples. Proposition 5 can be used to provide an alternative proof of the existence of an equilibrium with ergodic chaos in the same cases of the economy of Section 4.[39] Our first example derives the existence of an equilibrium with ergodic chaos for the economy of the examples in Figure 1 and 2.

Example 6.2 (Quadratic). Assume $e(x) = (\beta/2)\,(x - x)^2$ and, as in the example of Figure 1 and 2, $\alpha = .8$, $\delta = .95$, $\gamma = .5$, $l = .5$. It is easy to verify that for $c'' = 11.78$, $\Delta^{2,3}(c'')$ is $0.164\,78 > 0$; and for $c' = 11.6$ it is equal to $\Delta^{2,3}(c') = -5.4138 < 0$. Moreover, $[y^{*}]'\left([y^{*}]^{2}(x^{*}, c''), c''\right) = 1.8785$, so the conditions of Proposition 6 are verified. Indeed in correspondence to $c^{*} = 11.739$, we have the equilibrium of Figure 1 and 2 that is topologically conjugate to the Ulam and von Neumann [1947]'s example. This equilibrium corresponds to the point $c^{*} = 11.739$ in correspondence of which we solve $\Delta^{2,3}(c^{*}) = 0$.

The next example allows for a completely different functional form, the exponential.

Example 6.3 (Exponential). Assume, as in Example 6.1, that $\alpha = .8$, $\delta = .95$, $\gamma = .5$, $l = .5$ and $e(x) = -2e^{x}$. Again, it is easy to verify that for $c' = 5.58$, $\Delta^{2,3}(c') = 0.50003 > 0$; and for $c'' = 5.8$, $\Delta^{2,3}(c'') = -.282\,3 < 0$. Moreover, $[y^{*}]'\left([y^{*}]^{2}(x^{*}, c''), c''\right) = 4.305\,6 > 1$. The equilibrium is found in closed form setting c^{*} such that $\left[[y^{*}]^{3}(x^{*}, c^{*}) - [y^{*}]^{2}(x^{*}, c^{*})\right] = 0$, so $c^{*} = 5.6924515$, implying $y(x, c) = 6.026\,3x - 5.0e^{x} + 5.6924515$.

It is obviously not always the case that an economy displays cycles and unstable behavior. As we said, it is impossible as $\alpha \to 1$ and clearly as $\gamma \to 0$, $l \to 1$ for the reasons discussed in Section 5. For $\alpha < 1$ and l sufficiently large, however, Proposition 5 makes clear that it is a phenomenon that does not depend on the selection of specific functional forms. We should note that the condition in Proposition 5 is only sufficient and indeed it can be easily extended using the same logic if we are willing to check conditions on higher iterates of $y(x; c)$. The condition in Proposition 5 is indeed sufficient to have the second iterate enter an unstable cycle. It is easy to

[39] Proposition 1-5 is less general than Proposition 6 as it requires the more demanding assumption that $e(x) = (\beta/2)\,(x - x)^2$. These proof, however, allows us to reach a much more precise characterization of the possible dynamics, as discussed in Section 4.

verify that the second example in Figure 1 (the one constructed with Ruelle's constant) has the property that it is the third iterate to enters an unstable cycle.

5.2 Alternative economic models

In the previous analysis we have assumed a simple dynamic political economy model in which two parties alternate in power. It is easy to see that the logic behind the type of equilibria with cycles and chaotic behavior studied above applies to a variety of other important economic problems. In the following we briefly illustrate how to the analysis can be applied to a single agent decision making problem with hyperbolic discounting, a dynamic model of legislative bargaining, and a dynamic multi-agent problems with free riding.

5.2.1 Hyperbolic discounting

In the case of a single decision maker with β-δ preferences as in Phelps and Pollak [1968], the policy solves:[40]

$$\max_{y\geq(1-\gamma)x-l}\left\{K\left[y-(1-\gamma)x\right]-e(x)+\beta\delta v(y)\right\}, \tag{15}$$

where the only difference with (3) is that there is an additional term, the hyperbolic discount factor $\beta<1$. The decision maker plays a game against his future selves. The expected continuation $v(x)$, must satisfy: $v(x)=K(y(x)-(1-\gamma)x)-e(x)+\delta v(y(x))$, where $y(x)$ is the expected future policy and β does not appear. This expression can be written as:

$$v(x)=\max_{y\in[(1-\gamma)x,\overline{x}]}\left\{K\left[y-(1-\gamma)x\right]-e(x)+\beta\delta v(y)\right\}+(1-\beta)\,\delta v(y(x)). \tag{16}$$

Condition (15) and (16) corresponds to conditions (3) and (5) presented above. The second term in (16), $(1-\beta)\,\delta v(y(x))$, is the time inconsistency gap: i.e. the difference between the decision-maker's objective function and the expected value function. Because of this additional term, the shape of the expected value function directly depends on the expected future investment function $y(x)$ as in (5). Differentiating (16) and using the first order necessary condition $K/(\beta\delta)=-v'(x)$ from (15), we obtain a condition analogous to (7) in the analysis of Section 4: $y(x)=\left[\frac{1-(1-\gamma)\delta\beta}{\delta(1-\beta)}\right]\cdot x-\frac{\beta}{(1-\beta)k}\cdot e(x)+c$.[41]

[40] The parameter β was used in Sections 3 and 4 in the definition of $e(x)$. We use it it for the time preferences since this is the traditional notation for hyperbolic discounting and there is no risk of confusion since $e(\cdot)$ is can be assumed here to be a general convex function as in Section 5.1.

[41] The existence of a differentiable equilibrium in this setting can be proven as in the analysis of the previous sections.

5.2.2 Legislative bargaining

Consider version of the model presented in the previous sections in which a governmental policy in state x consists in y (i.e. the new state of pollution) as above, and a vector of transfers $\mathbf{s} = (s_1, ..., s_n)$. For simplicity here we assume that there are no other taxes/transfers and that the total direct benefit from pollution, i.e. $K[y - (1 - \gamma)x]$, is directly appropriated by the government. The government redistributes $K[y - (1 - \gamma)x]$ as part of the policy: so $(y, \mathbf{s})$ must satisfy the budget constraint $\sum_i s_i = K[y - (1 - \gamma)x]$ in state x.[42] The bargaining protocol in a period t is as in Battaglini and Coate [2007, 2008] and Battaglini [2014]: at each stage τ of the protocol, a party (the proposer) is selected to propose a policy $(y, \mathbf{s})$ with probability $1/n$; if the proposal is accepted, then the policy is implemented and the legislature adjourns, meeting again in period $t + 1$ with new state y; if the proposal fails, a new party is randomly selected at stage $\tau + 1$ of period t with probability $1/n$ to make a new proposal and the process repeats in the same state x.[43] We assume that the time lost between proposals is negligible.

In a symmetric equilibrium the party selected to make the proposal makes a transfer s to $q - 1$ randomly selected other parties to win a minimal winning coalition, and zero to the parties outside the selected minimal winning coalition. The variables y and $\mathbf{s}$ are selected to maximize the proposer's expected utility: $K[y - (1 - \gamma)x] - (q - 1)s + \delta v(y)$. The "bribe" s must satisfy the incentive constraint that guarantees that the parties in the minimal winning coalition vote accept the proposal $y, \mathbf{s}$, so $s \geq v(y(x)) - \delta v(y)$. Given this, it can be shown that the equilibrium policy y solves:

$$\max_{y \geq (1-\gamma)x - l} \left\{ \frac{K[y - (1 - \gamma)x]}{q} + \delta v(y) \right\}, \tag{17}$$

On the other hand, in a symmetric equilibrium, the value function is $v(x) = \frac{K[y(x) - (1-\gamma)x]}{n} + \delta v(y(x))$, that can be written as:

$$\max_{y} \left\{ \frac{K[y - (1 - \gamma)x]}{q} + \delta v(y) \right\} - K \left(\frac{n - q}{q} \right) \left[\frac{y(x) - (1 - \gamma)x}{n} \right].$$

Once again, condition (15) and (16) corresponds to conditions (3) and (5) presented above.

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[42] For a more sophisticated version of this model in which the direct benefit of pollution are appropriated by the citizens and trasfers are financed by tax revenues, see battaglini and Coate [2007].

[43] This version of the bargaining protocol is presented in Battaglini [2014]. The timing in Battaglini and Coate [2007,2008] is slightly different, but it generates the same equilibrium conditions. The discussion below, therefore, applies to these cases as well.

5.2.3 Dynamic free riding

Battaglini et al. [2014] propose a dynamic model of free riding with n agent in which a public good is accumulated by independent voluntary contributions by the agents. Each agent is endowed with a per period budget of W/n to allocate in each period t, either to private consumption or to increase the stock of the public good y. Battaglini et al. [2014] shows that, in state x, each player selects the individual contributions as if they could directly choose the new state y to maximize $u(y) + \left[\frac{W+(1-\gamma)x}{n} - y\right] + \delta v(y)$ under a feasibility constraints $y \in F(x)$. As in (16) the value function diverges from each player's objective function and can be written as:

$$v(x) = \max_{y \in F(x)} \left[u(y) + \left[\frac{W + (1-\gamma)x}{n} - y\right] + \delta v(y)\right] + \left(1 - \frac{1}{n}\right) y(x),$$

where $y(x)$ is the expected equilibrium investment function. While Battaglini et al. [2014] focus on monotonic equilibria with no cycles, the fact that the second term in the value function $v(x)$ directly depends on $y(x)$ allows to construct in this environment too cycles of any order and chaotic equilibria as in the previous sections.

6 Conclusions

In this paper we have presented a simple dynamic model of environmental protection in which the level of pollution is a state variable that strategically links policy making periods, and policies are chosen by forward looking, but politically motivated decision makers. We asked the question: under what conditions can such a simple model generate cycles and complex, unpredictable dynamics? Complex dynamics are impossible when policies are selected by a benevolent, time consistent policy maker. In the presence of time inconsistency generated by the political process, however, simple sufficient conditions guarantee the existence of equilibria with cycles of any order and even chaos for generic economies.

A limitation of our results is that the chaotic behavior we characterize is not typical of all equilibria of our dynamic economy, but instead of the specific class of equilibria that we characterize. Still, they show a simple yet realistic environment in which predicting dynamic public policies is impossible in the sense that there are always chaotic equilibria that make it impossible. The problem is not that there are multiple equilibria, but that even knowing the equilibrium, the dynamics are effectively unpredictable, despite the complete absence of random shocks. Equilibria with complex dynamics, moreover, highlight a new source of inefficiency generated in political equilibria that has no correspondent in standard planner's problems: the instability of policies even in the absence of external shock. This implies that we do not necessarily have a simple "one dimensional" bias (in our case "too little" environmental protection): the state of the economy (pollution in our application) tends to be excessive on average, but it may recurrently dip below the planner's first best.

The key feature behind complex dynamics in our environment is the degree of time inconsistency induced by the political system. Even when the negative externalities are not fully internalized, if the decision making problem is time consistent, complex dynamics are impossible as in a planner solution. With time inconsistency, the size of the chaotic region converges to zero in the equilibria that we have characterized if we keep the other features of the environment constant (the size of the negative externality, the discount factor etc.); for any level of time inconsistency, however, the size of the chaotic region may be very large, depending on the other features of the economy.

The results presented above can be extended and generalized in many directions. Dynamic problems with time inconsistency are common in many economic environments different than the environmental protection example we have studied in this paper. The results presented above can be applied to these environments as well. These include alternative positive models of dynamic policy making (such as models in which policies are the outcome of non-cooperative bargaining, or dynamic models with free riding; or models with individual decision makers with hyperbolic discounting. A full analysis of these environments is left for future research, but specific solved examples are available from the author.

In the previous analysis we restricted attention to Markov equilibria and quasi-linear preferences. With respect to the equilibrium concept, there is no reason to restrict attention to Markov equilibria if the goal is to show the existence of equilibria with chaotic dynamics. The advantage of focusing on Markov equilibria is that it shows chaotic behavior may exist even with simple strategies that do not require memory. With respect to the utilities, assuming quasilinear preferences provide a simple environment for which equilibria can be fully characterized and analyzed. Stochastic games are notoriously difficult to analyze, and even proving existence of Markov equilibria is challenging in general environments. Using a less restrictive equilibrium concept, may help extend the analysis to environments beyond the quasi-linear case. Again, we leave the analysis of these extension for future research.

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7 Appendix

7.1 Proof of Proposition 1

Let us rewrite (8) as $y(x,\kappa) = \varphi_1 x - \varphi_2 x^2 + \kappa$, where $\varphi_1 = \frac{1}{1-\alpha}\left[\frac{2}{\delta} - (1+\alpha)(1-\gamma) + \frac{2\beta}{K}\widehat{x}\right]$ and $\varphi_2 = \beta/[(1-\alpha)K]$. For any real number κ satisfying $\kappa \geq \left[4 - (\varphi_1 - 1)^2\right]/4\varphi_2$, define $\widehat{x}_-$, $\widehat{x}_+$ as, respectively, the lowest and the largest fixed points of $y(x,\kappa)$. We have:

$$\widehat{x}_- = \frac{\varphi_1 - 1 - \sqrt{(\varphi_1-1)^2 + 4\varphi_2\kappa}}{2\varphi_2},\ \widehat{x}_+ = \frac{\varphi_1 - 1 + \sqrt{(\varphi_1-1)^2 + 4\varphi_2\kappa}}{2\varphi_2}.$$

Define the function:

$$Y(x;\kappa,\underline{x},\overline{x}) = \begin{cases} \max\{y(\underline{x},\kappa), y(x,\kappa)\} & x \leq \overline{x} \\ \max\{y(\overline{x},\kappa), (1-\gamma)x - l\} & x > \overline{x} \end{cases}, \tag{18}$$

where $\underline{x} = \widehat{x}_-$ and $\overline{x} = y(\varphi_1/(2\varphi_2),\kappa)$. We now show that there exists a non empty set of values of κ such that $Y(x;\kappa,\underline{x},\overline{x})$ is an equilibrium investment function with the properties stated in Proposition 1. We start with a technical lemmata. Lemma 1.1 shows that there is a non empty set:

$$\mathcal{C}^* = \left[\frac{4-(\varphi_1-1)^2}{4\varphi_2}, \frac{9-(\varphi_1-1)^2}{4\varphi_2}\right],$$

such that for any $\kappa \in \mathcal{C}^*$, $Y(x;\kappa,\underline{x},\overline{x})$ maps $\mathcal{X}^* = [\underline{x},\overline{x}]$ into itself:

Lemma 1.1. *Let $\kappa \in \mathcal{C}^*$, then for any $x \in [\underline{x},\overline{x}]$, $Y(x;\kappa,\underline{x},\overline{x}) \in [\underline{x},\overline{x}]$.*

Proof. See online appendix. ■

It can be verified that when $Y(x;\kappa,\underline{x},\overline{x})$ is the investment function with $\kappa \in \mathcal{C}^*$, then the state will eventually enter in $[\underline{x},\overline{x}]$ in a finite number of steps starting from any point $x \notin [\underline{x},\overline{x}]$. This observation together with Lemma 1.1. implies that when the players strategy is (18), then the state will be in $\mathcal{X}^* = [\underline{x},\overline{x}]$ in all periods except at most for a finite transition period. The next lemma shows that the investment function (18) is actually feasible for all x when $R > R^*(\alpha)$ as defined in (9).

Lemma 1.2. *If (9) is satisfied, then $Y(x;\kappa,\underline{x},\overline{x})$ is feasible for all x.*

Proof. See online appendix. ■

We now show that $Y(x;\kappa,\underline{x},\overline{x})$ is an optimal policy for an incumbent. The objective function of (3) can be written as $P(x_{t+1}) = Kx_{t+1} + \delta v(x_{t+1})$, since $y = x_{t+1}$ and $x_t = x$ is given for the incumbent at t. In the following, we write the objective function as a generic state x as $P(x) = Kx + \delta v(x)$. We now show that given $Y(x;\kappa,\underline{x},\overline{x})$, the objective function $P(x)$ of (3) is concave; almost everywhere differentiable; and differentiable in $\mathcal{X}^*$, implying that the first order necessary condition with respect to $y = x_{t+1}$ is sufficient for optimality in problem (3).[44] To this goal, first note that in $\mathcal{X}^*$, we have $Y(x;\kappa,\underline{x},\overline{x}) = y(x,\kappa)$, which is differentiable. This implies that $P(x)$ is differentiable in this interval. The objective function $P(x)$ is also obviously differentiable in $x < \underline{x}$ and $x > \overline{x}$. In $[\underline{x},\overline{x}]$ the derivative objective function is such that:

$$P'(x) = K + \delta v'(x) = K - \delta\left[e'(x) + \frac{(1+\alpha)K\,(1-\gamma)}{2} + \frac{(1-\alpha)K}{2}y'(x,\kappa)\right] = 0.$$

Consider now for $x > \overline{x}$. Note that:

$$\overline{x} = y(\frac{\varphi_1}{2\varphi_2},\kappa) \geq \frac{1}{4\varphi_2}\left[\varphi_1^2 + 1 - (\varphi_1 - 1)^2\right] = \frac{\varphi_1}{2\varphi_2} = x^*$$

where $x^* = \arg\max_z y(z,\kappa)$. Since $\overline{x} \geq \varphi_1/(2\varphi_2)$, $y(x,\kappa)$ is concave, and $y'(\varphi_1/(2\varphi_2),\kappa) = 0$, we conclude that $y'(\overline{x},\kappa) \leq 0$. We therefore have:

$$\begin{aligned} P'(x) &\leq K - \delta\left[e'(x) + \frac{(1+\alpha)K\,(1-\gamma)}{2} + \frac{(1-\alpha)K}{2}(1-\gamma)\right] \\ &\leq K - \delta\left[e'(x) + \frac{(1+\alpha)K\,(1-\gamma)}{2} + \frac{(1-\alpha)K}{2}y'(x,\kappa)\right] = 0 \end{aligned} \tag{19}$$

for any $x > \overline{x}$. Naturally, $\underline{x} < \varphi_1/(2\varphi_2)$, so for $x < \overline{x}$:

$$P'(x) = K - \delta\left[e'(x) + \frac{(1+\alpha)K\,(1-\gamma)}{2}\right] \geq K - \delta\left[\begin{array}{c} e'(x) + \frac{(1+\alpha)K(1-\gamma)}{2} \\ + \frac{(1-\alpha)K}{2}y'(x,\kappa) \end{array}\right] = 0. \tag{20}$$

Conditions (19) and (20) imply that $P(x)$ achieves a maximum at any point in $\mathcal{X}^*$. To see the concavity of $P(x)$, note that it is continuous, concave with positive derivative in $x < \underline{x}$, flat in $x \in [\underline{x},\overline{x}]$, and concave with negative derivative in $x > \overline{x}$. To see that $Y(x;\kappa,\underline{x},\overline{x})$ is an optimal policy for the incumbent note that (19) and (20), $P(x)$ achieves a maximum in $\mathcal{X}^* = [\underline{x},\overline{x}]$ and that, when $\kappa \in \mathcal{C}^*$, $Y(x;\kappa,\underline{x},\overline{x}) \in \mathcal{X}^*$ for any x for which it is feasible; and given the state x a point y in $\mathcal{X}^*$ is not feasible, the policy is at a constrained optimum.

[44] Note that for problem (3), the choice variable is $y = x_{t+1}$. We write the objective function of this problem as $P(x)$.

We finally show that for any $\kappa \in \mathcal{C}^*$, $Y(x;\kappa,\underline{x},\overline{x})$ does not admit a stable steady state. Note that $Y(x;\kappa,\underline{x},\overline{x}) = x$ at the point:

$$\widehat{x} = \left[(\varphi_1 - 1) + \sqrt{(\varphi_1 - 1)^2 + 4\varphi_2\kappa}\right]/2\varphi_2.$$

We have $Y'(\widehat{x};\kappa,\underline{x},\overline{x}) = 1 - \sqrt{(\varphi_1 - 1)^2 + 4\varphi_2\kappa} \leq 0$, so $\widehat{x}$ is an unstable steady state. ■

7.2 Proof of Lemma 1

We proceed in three steps.

Step 1. We first observe that $y_c = y(x,c)$, as defined in (8), is conjugate to $Q_k = x^2 + k$ for $k = (\varphi_1/2)(1-\varphi_1/2) - \varphi_2 c$ by the homeomorphism $\xi(x) = \varphi_1/2 - \varphi_2 x$. To see this note that $\xi \circ y_c(x) = \varphi_1/2 + \varphi_2^2 x^2 - \varphi_1\varphi_2 x - c\varphi_2$, moreover:

$$\begin{aligned} Q_k \circ \xi(x) &= [\varphi_1/2 - \varphi_2 x]^2 + (\varphi_1/2)(1-\varphi_1/2) - \varphi_2 c \\ &= \varphi_1/2 + \varphi_2^2 x^2 - \varphi_2(\varphi_1 + c) = \xi \circ y_c(x). \end{aligned}$$

So we have $Q_k \circ \xi = \xi \circ y_c$. Similarly, we can show that L_η is conjugate to Q_k with $k = \eta/2(1-\eta/2)$ by the homeomorphism $h_\eta = -\eta x + \eta/2$.

Step 2. Let us now define $c(\eta;\varphi_1,\varphi_2)$ such that $\eta/2(1-\eta/2) = (\varphi_1/2)(1-\varphi_1/2) - \varphi_2 \cdot c(\eta;\varphi_1,\varphi_2)$, that is:

$$c(\eta;\varphi_1,\varphi_2) = \frac{1}{\varphi_2}\left[(\varphi_1/2)(1-\varphi_1/2) - \eta/2(1-\eta/2)\right].$$

We can then write:

$$\begin{aligned} L_\eta &= h_\eta^{-1} \circ Q_{\eta/2(1-\eta/2)} \circ h_\eta = h_\eta^{-1} \circ \left[\xi \circ y_{c(\eta;\varphi_1,\varphi_2)} \circ \xi^{-1}\right] \circ h_\eta \\ &= \left[h_\eta^{-1} \circ \xi\right] \circ y_{c(\eta;\varphi_1,\varphi_2)} \circ \left[\xi^{-1} \circ h_\eta\right] = \left[\xi^{-1} \circ h_\eta\right]^{-1} \circ y_{c(\eta;\varphi_1,\varphi_2)} \circ \left[\xi^{-1} \circ h_\eta\right] \\ &= z_\eta^{-1} \circ y_{c(\eta;\varphi_1,\varphi_2)} \circ z_\eta \Leftrightarrow z_\eta \circ L_\eta = y_{c(\eta;\varphi_1,\varphi_2)} \circ z_\eta \end{aligned}$$

where $z_\eta = \xi^{-1} \circ h_\eta$. This implies that L_η is topologically conjugate to $y(x, c(\eta;\varphi_1,\varphi_2))$ through the homeomorphism z_η.

Step 3. From Proposition 1, $y(x,c)$ with $c = c(\eta;\varphi_1,\varphi_2)$ is an equilibrium if $c(\eta;\varphi_1,\varphi_2) \in \mathcal{C}^*$ as defined in (10). We have:

$$\frac{1}{\varphi_2}\left[(\varphi_1/2)(1-\varphi_1/2) - \eta/2(1-\eta/2)\right] \geq \frac{4 - (\varphi_1 - 1)^2}{4\varphi_2} \Leftrightarrow \eta/2(1-\eta/2) \leq -\frac{3}{4}.$$

Moreover, we need:

$$\frac{1}{\varphi_2}\left[(\varphi_1/2)(1-\varphi_1/2)-\eta/2(1-\eta/2)\right] \leq \frac{9-(\varphi_1-1)^2}{4\varphi_2} \Leftrightarrow \eta/2(1-\eta/2) \geq -2.$$

We can therefore construct an equilibrium that is conjugate to L_η if: $-2 \leq \eta/2(1-\eta/2) \leq -\frac{3}{4}$. We conclude that we can construct an equilibrium that is conjugate to L_η if $3 \leq \eta \leq 4$ as desired.

■

7.3 Proof of Proposition 2

The result follows from the argument in the text. For the existence of values η_k in $[3,4]$ such that L_{η_k} has a stable cycle of period k or displays topological chaos, see for example Rasband [1990], ch. 2.3. For specific values that generate stable cycles up to the order 11 see Metropolis et al. [1973]. ■

7.4 Proof of Proposition 3

The fact that we have a set of positive measures of values in $\mathcal{C}^*$ such that an equilibrium with ergodic distribution exists follows from Lemma 1 and the discussion in Section 3. We proceed to the characterization of the ergodic distribution for $c = c(4;\varphi_1,\varphi_2)$. Let μ be the measure that is invariant under L_4, so that $\mu = L_{4*}\mu$. It is well known that μ is the arcsine law $x = 1/\left(\pi\sqrt{x(1-x)}\right)$. Let us define the so called "push forward" measure $z_{4*}\mu$ by: $z_{4*}\mu(A) := \mu\left(z_4^{-1}(A)\right)$, where z_4 is the homeomorphism such that $y \circ z_4 = z_4 \circ L_4$, defined in the proof of Lemma 1. We have:

$$z_{4*}\mu = z_{4*}\left[L_{4*}\mu\right] = \left(z_4 \circ L_4\right)_* \mu = \left(y \circ z_4\right)_* \mu = y_*\left(z_{4*}\mu\right)$$

where in the second and fourth equalities we use the definition of the push forward measure, and in the third the fact that $y = z_4 \circ L_4$. So we have: $y_*\left(z_{4*}\mu\right) = z_{4*}\mu$. To find $z_{4*}\mu$ note that $z_4 = \left(\varphi_1 - 4\right)/\left(2\varphi_2\right) + \left(4/\varphi_2\right)x$, so $x = \varphi_2 z_4/4 - (\varphi_1 - 4)/8$. It follows that:

$$\begin{aligned}
\mu^*(x,\alpha,R) &= \frac{1}{\pi\sqrt{x(1-x)}\,|z_4'(x)|} = \frac{b(\alpha,R)}{4\pi\cdot\sqrt{\left(\frac{\varphi_2}{4}x-\frac{\varphi_1-4}{8}\right)\left(1-\frac{\varphi_2}{4}x+\frac{\varphi_1-4}{8}\right)}} \\
&= \frac{2R}{\pi\,(1-\alpha)\cdot\sqrt{16-\left(\frac{2R}{(1-\alpha)}x-\frac{1}{1-\alpha}\left[\frac{2}{\delta}-(1+\alpha)(1-\gamma)+2R\widehat{x}\right]\right)^2}}.
\end{aligned}$$

Which gives us (12) in the statement of Proposition 3. ■

7.5 Proof of Proposition 4

It is sufficient to show that if $R^*(\alpha) \leq R \leq 4(1-\alpha)/\Delta$, then there is an equilibrium y_α in correspondence of which $\Delta^*(\alpha, y_\alpha) \geq \Delta$. Consider $y^*(x, c^*)$ as constructed in Proposition 1 with $c^* = \frac{(4-\varphi_1)(2+\varphi_1)}{4\varphi_2}$. Note that $y(x, c^*)$ has maximum at $\frac{\varphi_1}{2\varphi_2} \in \mathcal{X}^*(c^*)$, and, by construction, $y(\frac{\varphi_1}{2\varphi_2}, c^*) = \left(\varphi_1^2 + 4\varphi_2 c^*\right)/(4\varphi_2)$, so $\overline{X}(\alpha, y_\alpha) \geq \left(\varphi_1^2 + 4\varphi_2 c^*\right)/(4\varphi_2)$. Moreover, it can be verified that,

$$[y]^2\left(\frac{\varphi_1}{2\varphi_2}, c^*\right) = \left(\varphi_1 - 1 - \sqrt{(\varphi_1-1)^2 + 4\varphi_2 c^*}\right)/(2\varphi_2),$$

so $\underline{X}(\alpha, y_\alpha) \leq \left(\varphi_1 - 1 - \sqrt{(\varphi_1-1)^2 + 4\varphi_2 c^*}\right)/(2\varphi_2)$. We conclude that:

$$\begin{aligned} \Delta^*(\alpha, y^*(x, c^*)) &= \overline{X}(\alpha, y^*(x, c^*)) - \underline{X}(\alpha, y^*(x, c^*)) \\ &\geq \frac{1}{4\varphi_2}\left[\begin{array}{c} \varphi_1^2 + 4\varphi_2 \cdot c^* \\ -\left(\varphi_1 - 1 - \sqrt{(\varphi_1-1)^2 + 4\varphi_2 \cdot c^*}\right) \end{array}\right] = 4(1-\alpha)(K/\beta). \end{aligned} \tag{21}$$

From Proposition 1, a sufficient condition for $y^*(x, c^*)$ to exist is that $R \geq R^*(\alpha)$, where $R^*(\alpha)$ is defined in (9). From (21), a sufficient condition for $\Delta^*(\alpha, y^*(x, c^*)) \geq \Delta$ is $R \leq 4(1-\alpha)/\Delta$. We conclude that if $R^*(\alpha) \leq R \leq 4(1-\alpha)/\Delta$, then we have an equilibrium with chaotic area larger or equal than Δ. ■

7.6 Proof of Proposition 5

The function $y(x; c)$ defined in (13) is strictly concave and C^1-unimodal in $[x_-(c), y(x^*, c)]$ for any c. Moreover by the assumptions of the proposition, it is trice continuously differentiable with negative Schwartzian derivative in $[x_-(c), y(x^*, c)]$ for any c. This implies that, for any c, it satisfies assumptions $S1$-$S3$ and $S5$ for Corollary 6 in Grandmont [1990]. By continuity and the fact that $\Delta^{2,3}(c') \cdot \Delta^{2,3}(c'') \leq 0$, there must be a $x^* \in [c', c'']$ such that $[y^*]^3(x^*, c) = [y^*]^2(x^*, c)$, thus implying that the second iteration is a steady state of $y(x; c)$. From strict concavity and $[y^*]'([y^*]^2(x^*, c''), c'') > 1$, we have:

$$[y^*]'\left([y^*]^2(x^*, c^*), c^*\right) > [y^*]'\left([y^*]^2(x^*, c''), c''\right) > 1.$$

So $y(x; c^*)$ enters an unstable cycle of period 1 at the second iteration. Note moreover that $y(x; c^*) > x$ for all $x \in [x_-(c), x^*]$, thus it satisfies assumption $S4''$ of Corollary 6 in Grandmont [1990]. By this result, we therefore conclude that $y(x; c^*)$ has a unique absolutely continuous invariant ergodic measure in $[x_-(c), y(x^*, c)]$. ■

8 Online appendix

8.1 Proof of Lemma 1.1

Assume $\kappa \in \mathcal{C}^*$, we show here that for any $x \in [\underline{x}, \overline{x}]$, $Y(x; \kappa, \underline{x}, \overline{x}) \in [\underline{x}, \overline{x}]$. We proceed in two steps.

Step 1. We first prove that for $\kappa \in \mathcal{C}^*$, then $[y]^2\left(\frac{\varphi_1}{2\varphi_2}, \kappa\right) \geq \underline{x}$, where $[y]^k(x, \kappa)$ is the kth iteration of y, $[y]^k(x, \kappa) = y(y^{k-1}(x, \kappa), \kappa)$. A sufficient condition for $[y]^2\left(\frac{\varphi_1}{2\varphi_2}, \kappa\right) \geq \underline{x}$ is:

$$\frac{\varphi_1 - \sqrt{\varphi_1^2 + 4\varphi_2(\kappa - \underline{x})}}{2\varphi_2} \leq \frac{\varphi_1^2}{4\varphi_2} + \kappa \leq \frac{\varphi_1 + \sqrt{\varphi_1^2 + 4\varphi_2(\kappa - \underline{x})}}{2\varphi_2}. \tag{22}$$

Note that $\kappa \geq \frac{1}{4\varphi_2}\left[4 - (\varphi_1 - 1)^2\right]$ implies $\frac{\varphi_1^2}{4\varphi_2} + k \geq \frac{\varphi_1}{2\varphi_2}$. It follows that the first inequality in (22) is always satisfied. So we need:

$$\frac{\varphi_1^2}{4\varphi_2} + \kappa - \frac{\varphi_1 + \sqrt{\varphi_1^2 + 4\varphi_2(\kappa - \underline{x})}}{2\varphi_2} \leq 0. \tag{23}$$

We now show that this condition is satisfied for any:

$$\kappa \in \left[\frac{4 - (\varphi_1 - 1)^2}{4\varphi_2}, \frac{9 - (\varphi_1 - 1)^2}{4\varphi_2}\right]. \tag{24}$$

To this goal we proceed by induction:

Step 1.1. Given $\kappa \geq \frac{1}{4\varphi_2}\left[4 - (\varphi_1 - 1)^2\right] = \kappa_0$, we have:

$$\underline{x} = \frac{\varphi_1 - 1 - \sqrt{(\varphi_1 - 1)^2 + 4\varphi_2\kappa}}{2\varphi_2} \leq \frac{\varphi_1 - 3}{2\varphi_2}. \tag{25}$$

So $[y]^2\left(\frac{\varphi_1}{2\varphi_2}, \kappa\right) \geq \underline{x}$, if

$$\frac{\varphi_1^2}{4\varphi_2} + \kappa \leq \frac{\varphi_1 + \sqrt{\varphi_1^2 + 4\varphi_2(\kappa - \frac{\varphi_1 - 3}{2\varphi_2})}}{2\varphi_2}. \tag{26}$$

After a change in variable, (26) can be written as $\xi^2 - 2\xi - 6 \leq 0$, where $\xi = \sqrt{\varphi_1^2 + 4\varphi_2(\kappa - \frac{\varphi_1 - 3}{2\varphi_2})}$. It follows that we need: $\sqrt{\varphi_1^2 + 4\varphi_2(\kappa - \frac{\varphi_1 - 3}{2\varphi_2})} \leq 1 + \sqrt{7}$, or $\kappa \leq \left[3 + 2\sqrt{7} - (\varphi_1 - 1)^2\right]/4\varphi_2$. We therefore conclude that $[y]^2\left(\frac{\varphi_1}{2\varphi_2}, \kappa\right) \geq \underline{x}$ is satisfied for any:

$$\kappa \in \left[\frac{4 - (\varphi_1 - 1)^2}{4\varphi_2}, \frac{3 + 2\sqrt{7} - (\varphi_1 - 1)^2}{4\varphi_2}\right], \tag{27}$$

which gives us a nonempty set.

Step 1.2. We now prove that if $[y]^2(\frac{\varphi_1}{2\varphi_2},\kappa) \geq \underline{x}$ in for any $\kappa \in \left[\frac{4-(\varphi_1-1)^2}{4\varphi_2},\kappa_n\right]$ for some $\kappa_n < \frac{9-(\varphi_1-1)^2}{4\varphi_2}$, then we can find a $\kappa_{n+1} > \kappa_n$ such that the property $[y]^2(\frac{\varphi_1}{2\varphi_2},\kappa) \geq \underline{x}$ is satisfied in $\kappa \in \left[\frac{4-(\varphi_1-1)^2}{4\varphi_2},\kappa_{n+1}\right]$. From the previous step, we know that this property is true for $\kappa_1 = \frac{3+2\sqrt{7}-(\varphi_1-1)^2}{4\varphi_2}$. Let us assume we have proven it up to some $\kappa_n \in \left[\frac{3+2\sqrt{7}-(\varphi_1-1)^2}{4\varphi_2},\frac{9-(\varphi_1-1)^2}{4\varphi_2}\right)$. Note that if $\kappa \geq \kappa_n$, then we have:

$$\underline{x} \leq \frac{\varphi_1 - 1 - \sqrt{(\varphi_1-1)^2+4\varphi_2\kappa_n}}{2\varphi_2} = \frac{\varphi_1 - 2 - S_n}{2\varphi_2}, \tag{28}$$

where $S_n = 1 - \sqrt{(\varphi_1-1)^2+4\varphi_2\kappa_n}$ is the slope of $y(x,\kappa_n)$ at the fixed-point $\widehat{x}_+^n$.

Note that (23) is implied by:

$$\varphi_1^2 + 4\varphi_2\left(\kappa - \frac{\varphi_1-2+S_n}{2\varphi_2}\right) - 2\,(2-S_n) - 2\sqrt{\varphi_1^2+4\varphi_2(\kappa - \frac{\varphi_1-2+S_n}{2\varphi_2})} \leq 0.$$

After a change in variable, this condition can be written as

$$\begin{aligned} \xi^2 - 2\xi - 2\,(2-S_n) &\leq 0, \\ &\Leftrightarrow \xi \leq 1 + \sqrt{1+2\,(2-S_n)} \end{aligned}$$

where $\xi = \sqrt{\varphi_1^2+4\varphi_2(\kappa - \frac{\varphi_1-2+S_n}{2\varphi_2})}$. So we need:

$$\begin{aligned} \sqrt{\varphi_1^2+4\varphi_2(\kappa - \frac{\varphi_1-2+S_n}{2\varphi_2})} &\leq 1 + \sqrt{1+2\,(2-S_n)} \\ &\Leftrightarrow \kappa \leq \frac{3+2\sqrt{3+2\sqrt{(\varphi_1-1)^2+4\varphi_2\kappa_n}}-(\varphi_1-1)^2}{4\varphi_2} = \kappa_{n+1}. \end{aligned}$$

We have the result if $\kappa_{n+1} > \kappa_n$. For this we need:

$$3+2\sqrt{3+2\sqrt{(\varphi_1-1)^2+4\varphi_2\kappa_n}}-(\varphi_1-1)^2 > 4\varphi_2\kappa_n.$$

It is easy to see that this inequality is satisfied for $4\varphi_2\kappa_n + (\varphi_1-1)^2 \leq 9$, or: $\kappa_n < \left[9-(\varphi_1-1)^2\right]/(4\varphi_2)$, which is always satisfied since we are assuming it in the induction step.

Step 1.2. The sequence κ_n is bounded above by $\left(9-(\varphi_1-1)^2\right)/(4\varphi_2)$, thus it converges to $\kappa_\infty = \left(9-(\varphi_1-1)^2\right)/(4\varphi_2)$. Since $[y]^2(\frac{\varphi_1}{2\varphi_2},\kappa)$ is continuous in κ, we have that $[y]^2(\frac{\varphi_1}{2\varphi_2},\kappa) \geq \underline{x}$ for any $\kappa \in \left[\left(4-(\varphi_1-1)^2\right)/(4\varphi_2), \left(9-(\varphi_1-1)^2\right)/(4\varphi_2)\right]$, thus proving the result.

Step 2. We now prove that $Y(x;\kappa,\underline{x},\overline{x}) \in \mathcal{X}^*$ for any x in $\mathcal{X}^* = [\underline{x},\overline{x}]$ and κ satisfying $\kappa \in \mathcal{C}^*$. To see this, first note that for any $x \in \mathcal{X}^*$, we have $Y(x;\kappa,\underline{x},\overline{x}) \leq \max_z y(z,\kappa) = y(\frac{\varphi_1}{2\varphi_2},\kappa) = \overline{x}$, where the equality follows from the fact that $y(z,\kappa)$ achieves a maximum at $\varphi_1/(2\varphi_2)$, and the second equality from the definition of $\overline{x}$. Then note that for any $x \in \mathcal{X}^*$:

$$Y(x;\kappa,\underline{x},\overline{x}) \geq \min_{z\in\{\underline{x},\overline{x}\}} y(z,\kappa) \geq \min\{y(\overline{x},\kappa),\underline{x}\} \geq \underline{x}, \tag{29}$$

where the first inequality follows from the concavity of $Y(x;\kappa,\underline{x},\overline{x})$ in $\mathcal{X}^*$, and the second from $y(\underline{x},\kappa)=\underline{x}$, and the last inequality from $y(\overline{x},\kappa)=[y]^2\left(\frac{\varphi_1}{2\varphi_2},\kappa\right)\geq \underline{x}$ when κ satisfies $\kappa\in\mathcal{C}^*$. We conclude that $Y(x;\kappa,\underline{x},\overline{x})\in\mathcal{X}^*$ for any x in $\mathcal{X}^*=[\underline{x},\overline{x}]$. ■

8.2 Proof of Lemma 1.2

Assume (9) is satisfied, we show here that then $Y(x;\kappa,\underline{x},\overline{x})$ is feasible for all x. Define x_l^-, x_l^+ the points at which y intersects $(1-\gamma)x-l$, that is:

$$\begin{aligned} x_l^- &= \frac{\varphi_1-(1-\gamma)-\sqrt{(\varphi_1-(1-\gamma))^2+4\varphi_2(\kappa+l)}}{2\varphi_2}, \\ x_l^+ &= \frac{\varphi_1-(1-\gamma)+\sqrt{(\varphi_1-(1-\gamma))^2+4\varphi_2(\kappa+l)}}{2\varphi_2}. \end{aligned}$$

We have $y(x,\kappa)\geq(1-\gamma)x-l$ for $x\in\left[x_l^-,x_l^+\right]$. Consider x_l^- first. We have:

$$x_l^- - \underline{x} = \frac{1}{2\varphi_2}\left[\gamma+\sqrt{(\varphi_1-1)^2+4\varphi_2\kappa}-\sqrt{(\varphi_1-(1-\gamma))^2+4\varphi_2(\kappa+l)}\right]. \tag{30}$$

We need to have $x_l^- - \underline{x}\leq 0$. Note first that:

$$\begin{aligned} (\varphi_1-1)^2+4\varphi_2\kappa &\leq (\varphi_1-(1-\gamma))^2+4\varphi_2(\kappa+l) \\ &\Leftrightarrow \gamma^2+2\gamma(\varphi_1-1)+4\varphi_2 l\geq 0. \end{aligned}$$

Assume first that (31) is satisfied. In this case, the square parenthesis in (30) is increasing in κ and $x_l^- - \underline{x}$ can be bounded above inserting the upperbound of $\mathcal{C}^*$:

$$\begin{aligned} x_l^- - \underline{x} &\leq \frac{1}{2\varphi_2}\left[\gamma+\sqrt{(\varphi_1-1)^2+9-(\varphi_1-1)^2}-\sqrt{(\varphi_1-(1-\gamma))^2+4\varphi_2 l+9-(\varphi_1-1)^2}\right] \\ &= \frac{1}{2\varphi_2}\left[\gamma+3-\sqrt{9+\gamma^2+2\gamma(\varphi_1-1)+4\varphi_2 l}\right]. \end{aligned}$$

So we have $x_l^- - \underline{x}\leq 0$ if $\varphi_1\geq 4-2\varphi_2 l/\gamma$. Consider now the case: $\gamma^2+2\gamma(\varphi_1-1)+4\varphi_2 l<0$. Now $x_l^- - \underline{x}$ can be bounded above inserting the lowerbound of $\mathcal{C}^*$:

$$\begin{aligned} x_l^- - \underline{x} &\leq \frac{1}{2\varphi_2}\left[\gamma+\sqrt{(\varphi_1-1)^2+4-(\varphi_1-1)^2}-\sqrt{(\varphi_1-(1-\gamma))^2+4\varphi_2 l+4-(\varphi_1-1)^2}\right] \\ &= \frac{1}{2\varphi_2}\left[\gamma+2-\sqrt{4+\gamma^2+2\gamma(\varphi_1-1)+4\varphi_2 l}\right]. \end{aligned}$$

So we have $x_l^- - \underline{x} \leq 0$ if $\varphi_1 \geq 3 - 2\varphi_2 l/\gamma$. It follows that a sufficient condition is that $\varphi_1 \geq 4 - 2\varphi_2 l/\gamma = \varphi_{11}^*$.

Consider now x_l^+. We have:

$$x_l^+ - \overline{x} = \frac{1}{4\varphi_2}\left[2\varphi_1 - 2(1-\gamma) + 2\sqrt{(\varphi_1 - (1-\gamma))^2 + 4\varphi_2(\kappa + l)} - \left(\varphi_1^2 + 4\varphi_2\kappa\right)\right].$$

We need $x_l^+(\varphi_1) - \overline{x} \geq 0$. The right hand side is concave in κ, so it is minimized at one of the extremes. If the minimum is at the lowerbound, we have:

$$\begin{aligned} x_l^+ - \overline{x} &= \frac{1}{4\varphi_2}\left[2\varphi_1 - 2(1-\gamma) + 2\sqrt{(\varphi_1 - (1-\gamma))^2 + 4\varphi_2(\kappa + l)} - \left(\varphi_1^2 + 4\varphi_2\kappa\right)\right] \\ &\geq \frac{1}{4\varphi_2}\left[-5 + 2\gamma + 2\sqrt{\gamma^2 + 2(\varphi_1 - 1)\gamma + 4 + 4\varphi_2 l}\right]. \end{aligned}$$

It follows that $x_l^+ - \overline{x} \geq 0$ if $\varphi_1 \geq \frac{3}{8\gamma}[3 - 4\gamma] - 2\frac{\varphi_2 l}{\gamma}$. If the minimum is at the upperbound, we have:

$$x_l^+ - \overline{x} \geq \frac{1}{4\varphi_2}\left[-10 + 2\gamma + 2\sqrt{\gamma^2 + 2(\varphi_1 - 1)\gamma + 9 + 4\varphi_2 l}\right].$$

Which can be written as $\varphi_1 \geq \frac{1}{\gamma}[8 - 4\gamma] - 2\frac{\varphi_2 l}{\gamma}$. It follows that a sufficient condition for $x_l^+ - \overline{x} \geq 0$ is that $\varphi_1 \geq \frac{1}{\gamma}[8 - 4\gamma] - 2\frac{\varphi_2 l}{\gamma} = \varphi_{12}^*$. Note that $\varphi_{12}^* - \varphi_{11}^* = 8(1/\gamma - 1) > 0$. We conclude that $Y(x;\kappa,\underline{x},\overline{x})$ is feasible for all $x \in [\underline{x},\overline{x}]$ if $\varphi_1 \geq \varphi_{12}^*$. Using the definitions of φ_1 and φ_2, the condition becomes:

$$\begin{aligned} \frac{1}{1-\alpha}\left[\frac{2}{\delta} - (1+\alpha)(1-\gamma) + \frac{2\beta}{K}\widehat{x}\right] &\geq \frac{1}{\gamma}[8 - 4\gamma] - 2\frac{l}{\gamma}\cdot\frac{\beta}{(1-\alpha)K} \\ \Leftrightarrow \quad \frac{\beta}{K} &\geq \frac{4\delta(1-\alpha)(2-\gamma) + \delta(1+\alpha)(1-\gamma)\gamma - 2\gamma}{2\delta(\widehat{x}\gamma + l)} = R^*(\alpha). \end{aligned}$$

It follows that $y(x,\kappa) \geq (1-\gamma)x - l$ for $x \in [\underline{x},\overline{x}]$ if $R \geq R^*(\alpha)$. Moreover $Y(x;\kappa,\underline{x},\overline{x})$ obviously satisfies the constraint $y \geq (1-\gamma)x - l$ for $x > \overline{x}$. Finally we have that $(1-\gamma)x - l \leq (1-\gamma)\underline{x} - l \leq y(\underline{x},\kappa) = Y(x;\kappa,\underline{x},\overline{x})$ in $x < \underline{x}$. ■