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Optimal Foreign Reserves and Central Bank Policy Under Financial Stress
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ABSTRACT

We study the interaction between optimal foreign reserves accumulation and central bank international liquidity provision in a small open economy under financial stress. Firms and households finance investment and consumption by borrowing from domestic financial intermediaries (banks), which in turn borrow from abroad. Binding financial constraints can cause the domestic rate of interest to rise above the world rate and the real exchange rate to depreciate, leading to inefficiently low investment and consumption. A role then emerges for a central bank that accumulates reserves in order to provide liquidity if financial frictions bind. The optimal level of international reserves in this context depends, among other variables, on the term premium, the depth of financial markets, ex ante financial uncertainty and the precise way the central bank intervenes. The model is consistent with both the increase in international reserves observed during the period 2004-2008 and with policy intervention after the Lehman bankruptcy.

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1 Introduction

The period surrounding the recent global financial crisis, which peaked around the Lehman bankruptcy in September 2008, has motivated lively and important debates on macroeconomic policy. Two observations have attracted particular attention, especially in emerging economies. First, prior to the crisis, several central banks had accumulated significantly high amounts of international reserves. A justification for such accumulation, often mentioned by central bankers, was the need to build a "war chest" of international liquidity, to be available in case of sudden outflows of capital, similar to those experienced in the late 1990s. Second, when the global crisis did erupt, those same central banks were in a good position to mitigate its impact, in particular by providing foreign currency liquidity that compensated the foreign credit crunch, prevented the collapse of the financial sector, and calmed the markets.[1]

The first observation has motivated a recent literature that focuses on accounting for the observed levels of foreign reserves accumulation. That literature has served to identify variables, such as the probability of a "sudden stop" or the degree of financial development, that are key determinants of optimal levels of reserves and were ignored by older treatments which tied optimal reserve levels to international trade considerations. At the same time, the second observation has spurred a host of studies on central bank responses to financial crunches and sudden stops, especially on the so called "unconventional policies" designed to increase the availability of credit in a financial crisis.

While these two literatures have delivered important and useful lessons, they have evolved mostly in parallel. It is not hard, however, to argue that the two issues, optimal foreign reserves accumulation and policy responses to crises, are intimately connected. For one thing, as mentioned, central bankers often say that they hoard reserves *because* they plan to use them to finance liquidity assistance in case of a crisis. For another, reserves accumulation may affect private sector behavior, inducing domestic agents to borrow more, and potentially placing the financial system in a more fragile situation. Finally, it may be the case that the probability

[1] For a discussion of Latin American experiences, see Céspedes, Chang, and Velasco (2014).

of crisis itself may be affected by the level of international reserves and by the existence and nature of liquidity policies that the central bank implements during crises.

Accordingly, this paper analyzes the interaction between optimal reserves accumulation and central bank liquidity policies under financial duress, in a small open economy in which the probability of financial crisis is endogenously determined. This exercise yields several interesting lessons about the benefits and costs of international reserves, their link to the precise details of *ex post* central bank responses to crises and of financial intermediation.

We develop our arguments in a small open economy model that extends that of Céspedes, Chang, and Velasco (2017, hence CCV). In the model, domestic agents cannot borrow directly from international lenders, but instead they can obtain credit from financial intermediaries or banks, which in turn can borrow from abroad. Because of incentive or enforcement problems, however, banks' foreign debts are limited to a multiple of their net worth. The collateral constraint may or may not bind in equilibrium. If it does, there is a credit crunch that causes the domestic interest rate to rise above the world rate, the real exchange rate to depreciate, and investment to fall.

Our model departs from that in CCV in two key respects. First, the severity of the collateral constraint can be affected by exogenous shocks, which can be thought of as exogenous "sudden stops" of international capital flows. Second, whether the collateral constraint binds or not also depends on inherited debt, which is determined by households' initial consumption and borrowing decisions, and is therefore endogenous. As a consequence, under *laissez faire*, crises can occur with a probability that depends on the basic parameters of the economy.

In this setting, we allow the central bank to accumulate international reserves that can be used to finance liquidity assistance to domestic agents if there is a crisis. Specifically, the central bank has the option of borrowing long term in order to acquire short term international assets. Such option can come at a cost, but enables the central bank to provide liquidity to domestic agents in case of need. Because liquidity provision in a crisis reduces inefficiencies in investment, it can be welfare improving for the central bank to take advantage of the option and hoard some reserves, even if they involve some financial cost.

Significantly, the central bank has a role because, as we show, domestic banks would not find it individually optimal to issue long term debt to buy short term assets. This occurs because individual banks would not internalize that the increase in short term assets would result in a lower interest spread and more efficient investment when financial constraints bind. In this sense, a central bank policy of reserves accumulation is socially necessary.

It is also the case in our model that, in equilibrium, reserves accumulation and the ensuing central bank credit provision in crises can provide incentives for households to take more debt initially, which in turn reduces the effectiveness of liquidity provision. But such effects do not generally eliminate the benefits of reserves and ex post credit policy.

We derive several additional implications. As stressed, a policy of building reserves to provide liquidity when financial constraints bind can be welfare improving. This is in spite of the fact that reserves are costly and, as stressed in CCV, liquidity provision is ineffective when financial constraints do not bind. Such a policy implies, in particular, a reduction in the probability of crisis.

On the other hand, while it is possible to eliminate crises completely by building a large enough chest of reserves, it is not optimal to do so if reserves have a cost. This is because, in this model, when reserves are substantial, the probability of a crisis becomes too small, and the benefits of an additional unit of reserves are outweighed by its financial cost.

Given these findings, we identify the welfare maximizing amount of reserves and its determinants. The optimal level of reserves turns out to depend on the economy's fundamentals and also, and more novel, on the details about the specific liquidity policy that the central bank implements in a crisis. We show, for example, that the optimal level of reserves increases when their cost falls. This is as expected, of course, but underscores that the model is consistent with the view that one of the factors behind the observed reserves buildup in the last decade was the ample availability of international liquidity prior to the global crisis. Also, we find that an increase in uncertainty easily implies that optimal reserves are higher. This is consistent with arguments often advanced by central bankers to justify reserves accumulation.

As for the impact of policy details, we show that optimal reserves depend on whether the central bank policy in a crisis takes the form of lending to banks (liquidity facilities, in the terminology of Gertler and Kiyotaki 2010) or to firms and households (direct lending). This result follows because the benefits of reserves are given by their effectiveness in alleviating financial constraints when they become binding and, as demonstrated by CCV, direct lending is less effective than liquidity facilities because of leverage. On the other hand, we also find that, with direct lending, optimal reserves can be smaller or larger than under liquidity facilities. This reflects the fact that the optimal level of reserves is determined not by their net value (which is unambiguously smaller under direct lending) but by their marginal value (which can be smaller or larger, depending on fundamental elasticities).

Our paper is closely related to a large body of literature that focuses on the optimal amount of international reserves. An classic contribution is Heller (1966),in which shocks to the trade balance, such as a fall in foreign demand, are the main motive to hold external reserves. The optimal level of international reserves is given by the amount that minimizes the total cost of adjustment taking into consideration the cost of holding liquid international reserves and the probability that there will be a need for that level of reserves. Our paper is in the same spirit, but does not assign as big a role to trade factors. Instead, financial frictions take center stage.

The role of reserves in mitigating the impact of financial shocks has been a focus of recent studies. A prominent and influential one is Jeanne and Ranciere (2011) which builds a model in which the accumulation of reserves is viewed as an insurance device against a sudden stop. They find that the optimal level of reserves then depends on the probability of the sudden stop, the consumer´s risk aversion, and the opportunity cost of holding reserves. Their model, however, takes several variables, such as consumption and savings, and the probability of a sudden stop, as either exogenous or ad hoc functions of reserves policy. In our model, consumption, savings, and investment are derived as equilibrium outcomes, which must then reflect expectations about central bank policy and reserves accumulation. In addition, the probability of crisis is endogenous, and interacts with private decisions, accounting in particular for the possibility of self insurance.[2]

Our paper is also related to recent studies on policy responses to financial crises, and especially to the literature on "unconventional" central bank policy. An early survey is Gertler and Kiyotaki (2010), which compares the different kinds of liquidity provision policies implemented by advanced country central banks in the midst of the global crisis. For emerging economies, CCV develops a similar comparison, emphasizing the importance of occasionally binding financial constraints. CCV also show the equivalence between sterilized foreign exchange intervention and liquidity provision, thus connecting the analysis of unconventional policy with an ongoing reconsideration of the implications of foreign exchange intervention.[3] Relative to this literature, our paper emphasizes that it is often the case, in emerging economies, that the ability of central banks to alleviate crises depends on their access to *international* liquidity, which they cannot themselves produce and, therefore, they must arrange for in advance, by hoarding reserves.

The present paper's perspective on the benefits of reserves accumulation contrasts with the one suggested by CCV and recently developed by Bocola and Lorenzoni (2018). In the models of CCV and Bocola and Lorenzoni, there may be multiple equilibria ex post. The central bank can then eliminate bad equilibria by implementing an appropriate lending of last resort policy. But the latter is only credible if the central bank has accumulated a large enough stock of international reserves

As discussed, in our paper foreign exchange reserves are seen as a "war chest" that central banks can use in case of a financial crunch. This perspective complements recent studies on *ex ante* taxes or subsidies on international borrowing, leverage constraints, foreign borrowing limits, and other *macroprudential* policies. Notable contributions include Bianchi (2011), Bianchi and Mendoza (2018), and Korinek and Simsek (2016). For a useful discussion between ex ante

[2] Other noteworthy recent contributions include Caballero and Panageas (2004) and Durdu, Mendoza, and Terrones (2009).

[3] For further development, see Chang (2018).

and ex post policies, see Benigno, Chen, Otrok, Rebucci, and Young (2013) and Jeanne and Korinek (2017).

Section 2 describes the model that serves as the setting for the analysis. Section 3 derives equilibrium under laissez faire. Reserves accumulation that finances liquidity provision is introduced in section 4. Section 5 discusses the determinants of the optimal level of reserves. Some final remarks are collected in section 6.

2 The Model

To convey our main ideas, we study a simple small open economy whose main features are described in this section. There are three periods and two goods, one tradable and another nontradable. The real exchange rate is defined as the relative price of nontradables in terms of tradables. The economy is inhabited by a representative household that owns domestic financial intermediaries (banks) and firms. Initially, households borrow from the home banks in order to finance consumption; in turn, banks finance their loans by borrowing from international capital markets. In the second period, households roll over their debts and firms buy productive capital, financing investment borrowing from domestic banks. Banks then borrow further from world capital markets to repay their previous debt and finance their new loans to households and firms. Crucially, in the second period, banks' international debt is limited by a collateral constraint that depends on their own net worth. The constraint may or may not bind. If it does, the domestic interest rate rises above the world rate, and the exchange rate depreciates. In that case, investment turns to be inefficiently low. In addition, initial consumption and debt depend on the likelihood of binding financial constraints, as they determine the expected cost of credit to households.

2.1 Households

Households consume only tradables, and only at $t = 0$ and $t = 2$. Preferences are given by expected utility, given by:

$$U(C_0) + \beta E(C_2)$$

where C_t denotes consumption in period t, β is a discount factor, $U(.)$ is (for simplicity) a CRRA function, and $E(.)$ is the expectation operator.

To express budget constraints, we take tradables as the numeraire. We assume that the only sources of income for households are profits from firms and banks, which are paid only at $t = 2$. Hence, to finance consumption at $t = 0$, households borrow from domestic banks an amount $L_0^h = C_0$ at an interest rate R_0. In period $t = 1$, households refinance their debts at interest rate R_1, borrowing $L_1^h = R_0 L_0^h$. Finally, in period $t = 2$ households receive profits from banks and firms (denoted by Π^b and Π^f), repay their debts, and consume, so

$$C_2 = \Pi^b + \Pi^f - R_1 L_1^h$$

Finally, because there are no frictions in intermediation in the initial period, in any equilibrium the domestic interest rate R_0 must equal the world rate, denoted by R_0^*. Combining these considerations, the household's budget constraint can be written as:

$$R_0^* R_1 C_0 + C_2 = \Pi^b + \Pi^f \tag{1}$$

The household takes as given the interest rates R_0^* and R_1 as well as profits Π^b and Π^f. In equilibrium, the interest rate at which the household will have to roll over its debt in period 1, R_1, will be random. Denoting its expectation by $E(R_1)$, the first order condition for initial consumption is:

$$U'(C_0) = \beta R_0^* E(R_1) \tag{2}$$

The interpretation is the usual one: the household chooses initial consumption and debt by equating the marginal utility of initial consumption to its discounted expected price. Here, the relevant price is given by the two period interest rate, $R_0^* R_1$. Note that an increase in $E(R_1)$ leads to smaller initial consumption and borrowing.

2.2 Firms

At $t = 2$, competitive domestic firms produce Y_2 tradable goods with capital K_2 via a Cobb Douglas function:

$$Y_2 = AK_2^\alpha$$

where $0 < \alpha \leq 1$.

For production at $t = 2$, a typical firm purchases capital at $t = 1$, financing the purchase with a loan from domestic banks at rate R_1. Letting Q_1 denote the price of capital, the firm must then borrow from domestic banks an amount L_1^f given by:

$$L_1^f = Q_1 K_2$$

At $t = 2$, firms repay their debts to domestic banks, and send profits to the household, so:

$$\begin{aligned} \Pi^f &= Y_2 - R_1 L_1^f \\ &= AK_2^\alpha - R_1 Q_1 K_2 \end{aligned}$$

Hence the profit maximizing demand for capital is given by:

$$\alpha A K_2^{\alpha-1} = R_1 Q_1 \tag{3}$$

The interpretation is again standard: investment equates the marginal product of capital to its cost. But note that the cost of capital includes not only its price Q_1 but also the associated financial cost, given by the domestic interest rate R_1.

2.3 Capital production

Capital is produced in period 1 via a CES aggregator function:

$$K_2 = \left[\gamma^{1/\eta} I_H^{(\eta-1)/\eta} + (1-\gamma)^{1/\eta} I_W^{(\eta-1)/\eta}\right]^{\eta/(\eta-1)}$$

where I_H and I_W denote inputs of nontradables and tradables, η is the elasticity of substitution between them, and γ is a constant in $[0, 1]$. As usual, we assume that $\eta > 0$, and that $\eta = 1$ is the Cobb Douglas case:

$$K_2 = \kappa I_H^{\gamma} I_W^{1-\gamma}$$

with $\kappa = 1/\gamma^{\gamma}(1-\gamma)^{1-\gamma}$..

Recalling that tradables is the numeraire, if $\eta \neq 1$, the price of capital is then:

$$Q = \left[\gamma X^{1-\eta} + (1-\gamma)\right]^{1/(1-\eta)}$$

where X is the price of nontradables in terms of the tradable, which we will refer as the *real exchange rate*. For $\eta = 1$,

$$Q = X^{\gamma} \tag{4}$$

The optimal input of nontradables is then given by:

$$I_H = \gamma \left(\frac{Q}{X}\right)^{\eta} K_2$$

while the demand for tradables is given by:

$$I_W = (1-\gamma) Q^{\eta} K_2$$

2.4 Banks

As indicated before, in each period $t = 0, 1$, domestic banks borrow from world capital markets at rates R_t^*, and lend to domestic households and firms at rates R_t. We assume that banks are competitive. Financial intermediation is frictionless at $t = 0$, so equilibrium requires that $R_0 = R_0^*$, as mentioned. In contrast, banks are subject to a collateral constraint at $t = 1$. This problem was discussed at length in CCV: we summarize its implications here.

At $t = 0$, banks borrow some amount D_0 from world capital markets and lend to households. At $t = 1$, banks roll over loans to households, lend to firms, repay their own debts, and borrow further from the world market to finance domestic loans. Also, banks receive endowments T and N of tradables and nontradables respectively.

Hence, the amount of loans that the bank can extend at $t = 1$ is given by:

$$\begin{aligned} L_1 &= T + X_1 N + D_1 + R_0^* L_0^h - R_0 D_0 \\ &= T + X_1 N + D_1 \end{aligned}$$

the last equality being warranted because, in any equilibrium, $L_0^h = D_0$ and $R_0 = R_0^*$.

Finally, at $t = 2$, banks collect debt repayments from households and firms, repay their own debts, and send profits to households:

$$\Pi^b = R_1 L_1 - R_1^* D_1$$

As mentioned, at $t = 1$, also, banks face the financial constraint

$$R_1 L_1 - R_1^* D_1 \geq \theta R_1 L_1 \tag{5}$$

where θ is a random variable realized at period $t = 1$. This constraint can be justified in various ways. For example, one can assume that in period $t = 1$ domestic bankers can default on their foreign debt and divert a fraction θ of the payments made to the bank by firms. International lenders will then only accept contracts that satisfy the above constraint. From this perspective, a high realization of θ may reflect an exogenous tightening of international financial conditions. This can be seen as a *sudden stop.*

The assumption that θ is random is a key departure from CCV. For concreteness, we assume that θ can take n values, denoted by $\theta^{(s)}$, $s = 1, ...n$, each with probability $\pi_s > 0$, and that this is the only source of uncertainty in the model. We also impose $\theta^{(1)} = \underline{\theta} > 0$ and $\theta^{(n)} = \bar{\theta} < 1$. (Note that, to alleviate notation, the superscript s will be omitted when it is not needed.)

The collateral constraint may or may not bind in equilibrium. If it does, we will sometimes say that there is a *financial crisis.* As we will see, a financial crisis will be more likely when θ turns out to be high. Importantly, however, the financial constraint may not bind even if θ is high; in other words, a *sudden stop* does not necessarily lead to a *financial crisis.*

If θ is such that the collateral constraint does not bind, the cost of borrowing R_1 must be equal to R_1^*. In that case, the incentive constraint reduces to $L_1 - D_1 \geq \theta L_1$, the bank makes zero profits, and lends any quantity less than or equal to the multiple $1/\theta$ of its net worth:

$$L_1 \in [0, \frac{1}{\theta}(T + X_1 N)]$$

If the collateral constraint binds, $R_1 > R_1^*$. Combining the budget constraint of the bank with the binding collateral constraint, the bank´s supply of loans is:

$$L_1 = \frac{1}{1 - (1 - \theta)\phi}(T + X_1 N)$$

where $\phi = R_1/R_1^*$ is the interest rate spread. Loan supply is a multiple of the bank´s net worth: the leverage ratio,$1/\left(1 - (1 - \theta)\phi\right)$, is greater than one and finite in equilibrium (assuming that $\phi < 1/(1 - \theta)$). The previous expression indicates that banks leverage their capital to finance loans.

As in CCV, a real exchange rate depreciation (a fall in X_1) reduces bank´s net worth and, in a more novel aspect, an increase in the spread will increase the leverage ratio. As we will discuss in the next section, a real exchange rate depreciation will increase the spread and therefore, the leverage ratio.

3 Laissez Faire Equilibrium

In this section we describe equilibrium under laissez faire. As the only source of uncertainty is θ, it is convenient to start with the analysis of the continuation equilibrium, from $t = 1$, which depends not only on the realized value of θ but also on the economy's inherited debt D_0. It turns out that, given θ and D_0, the collateral constraint may or may not bind. Hence the distribution of continuation outcomes, and in particular of R_1, depends on D_0. In turn, D_0 is determined by initial consumption and savings choices which, as we have seen, reflect expectations about R_1. Equilibrium is then determined by a fixed point problem. The solution has no closed form but is not too hard to illustrate numerically, as we do at the end of this section.

3.1 Continuation Equilibrium

Consider the economy from $t = 1$ on, after $\theta = \theta^{(s)}$ is realized. At this point, the economy has an initial level of debt $D_0 = C_0$. From then on, the continuation economy is essentially the same as in CCV, so their results apply here. For convenience, here we describe CCV's analysis for the case $\eta = 1$. Extending the analysis to $\eta \neq 1$ is straightforward.

With $\eta = 1$,the demand for nontradables is given by $I_H = \gamma(Q/X)K_2 = \gamma X^{\gamma-1}K_2$, while the supply of nontradables is equal to the bank's endowment N. Equilibrium in the nontradables market in period $t = 1$ is then given by $I_H = N$, which implies a key link between investment and the real exchange rate:

$$K_2 = \left(\frac{N}{\gamma}\right) X_1^{1-\gamma} \tag{6}$$

But recall that the demand for capital depends on its price, Q, and the domestic interest rate R_1, as given by (3). Combining (3) with the preceding equation, and using (4), we obtain:

$$R_1 = \alpha A \left(\frac{\gamma}{N}\right)^{1-\alpha} \left(\frac{1}{X_1}\right)^{1-\alpha(1-\gamma)} \tag{7}$$

This is a key expression that connects domestic interest rates and the real exchange rate. An increase in R_1 reduces the demand for capital, and therefore, the demand for nontradables. The fall in the demand for nontradables generates a real exchange rate depreciation (a fall in X_1).

To complete the solution, we turn to the market for domestic loans. The demand for loans in period $t = 1$ is given by the value of investment plus the amount of debt $D_0 = C_0$ that households must roll over:

$$L_1 = L_1^f + L_1^h = Q_1 K_2 + R_0^* C_0$$

From (6) and (4), however, we know that the value of investment depends on the real exchange rate:

$$Q_1 K_2 = \left(\frac{N}{\gamma}\right) X_1$$

Combining the last two expressions we see that loan demand is given by:

$$L_1 = R_0^* C_0 + \left(\frac{N}{\gamma}\right) X_1$$

Intuitively, a depreciation of the real exchange rate reduces loan demand in equilibrium. In addition, higher initial consumption in $t = 0$ increases loan demand in period $t = 1$.

The supply of loans depends on whether the credit constraint binds or not. To proceed, note that if the credit constraint does not bind, the domestic interest rate must equal the world rate, i.e. $R_1 = R_1^*$. Then (7) gives the equilibrium exchange rate, denoted by X_{1f}:

$$R_1^* = \alpha A \left(\frac{\gamma}{N}\right)^{1-\alpha} \left(\frac{1}{X_{1f}}\right)^{1-\alpha(1-\gamma)}$$

It is easy to see that X_{1f} is also the *frictionless* equilibrium exchange rate, i.e., the one that would obtain in the absence of the collateral constraint (5). Likewise, the price of capital and investment will be at their frictionless equilibrium values, which we denote with an f subscript.

These observations and those of the previous section now imply that, when the financial constraint does not bind, the supply of loans must be given by any L_1 such that:

$$L_1 \in [0, \frac{1}{\theta}(T + X_{1f}N)]$$

If the collateral constraint binds instead, $R_1 > R_1^*$, and the bank´s supply of loans will be:

$$L_1 = \frac{1}{1-(1-\theta)\phi}(T + X_1 N)$$

where $\phi = R_1/R_1^* \geq 1$ is the spread between the domestic loan rate and the world rate. Noting that, by (7), ϕ is a function of the real exchange rate, the preceding equation gives loan supply as a function of X_1.

It follows that, when financial constraints bind, the real exchange rate must be given by:

$$R_0^* C_0 + \left(\frac{N}{\gamma}\right) X_1 = \frac{1}{1-(1-\theta)\phi}(T + X_1 N) \tag{8}$$

with

$$\phi = R_1/R_1^* = \left(\frac{X_{1f}}{X_1}\right)^{\gamma+(1-\alpha)(1-\gamma)}$$

Finally, we see that the collateral constraint will not bind in the continuation equilibrium if, at frictionless values, the demand for loans does not exceed the bank's credit limit:

$$R_0^* C_0 + Q_{1f} K_{2f} \leq \frac{1}{\theta}(T + X_{1f}N)$$

i.e. if $\theta \leq \hat{\theta}$, where the threshold $\hat{\theta}$ is given by:

$$\hat{\theta} = \frac{T + X_{1f}N}{R_0^* C_0 + Q_{1f}K_{2f}} \tag{9}$$

Summarizing, if $\theta \leq \hat{\theta}$, the continuation equilibrium is the frictionless outcome. But if $\theta > \hat{\theta}$, investment, the domestic interest rate, and the exchange rate all adjust for the economy to satisfy the collateral constraint. Intuitively, when the collateral constraint binds, investment falls below its frictionless value, the interest rate spread widens, and the exchange rate depreciates. This looks, in other words, like a crisis.

As mentioned, the analysis here replicates that in CCV. But in this model, and crucially, $\hat{\theta}$ is determined endogenously. In particular, we see that $\hat{\theta}$ falls, and the probability of a crisis increases, with C_0. This reflects, of course, that an increase in C_0 increases the amount that households must roll over and, hence, the economy's demand for external credit in $t = 1$.

More generally, the continuation equilibrium depends on the realization of θ and on initial consumption C_0. In particular, in any continuation equilibrium,

$$\begin{aligned} R_1 &= R_1^* \text{ if } \theta \leq \hat{\theta} \\ &= \rho(C_0, \theta) \;\text{ if } \theta > \hat{\theta} \end{aligned}$$

where $\rho(C_0, \theta)$ is the value of R_1 that solves (8) and $\hat{\theta}$ is given in (9).

3.2 Equilibrium and Implications

Initial consumption and debt are determined by the Euler condition of the household, (2), which becomes:

$$U'(C_0) = \beta R_0^* \left[R_1^* F(\hat{\theta}) + \sum_{\theta^{(s)} > \hat{\theta}} \rho(C_0, \theta)\pi_s \right]$$

where, recalling our assumptions about the distribution of θ,

$$F(\hat{\theta}) = \sum_{\theta^{(s)} \leq \hat{\theta}} \pi_s$$

is the probability of no crisis.

Noting that $\hat{\theta}$ is a function of C_0 (by (9)), the expression can be seen as an equation in the single unknown C_0. Given a solution C_0, the continuation equilibrium is determined as in the preceding subsection. In particular, a solution C_0 determines $\hat{\theta}$ and the probability of crisis, which is therefore endogenous.

Whether the probability of crisis is zero, one, or something in between, depends on the parameters of the model, and especially on the distribution of θ. If the distribution of θ is very favorable (for instance, if $\underline{\theta}, \overline{\theta}$ are close to zero), the collateral constraint never binds, and the continuation outcome is the frictionless equilibrium, regardless of the realization of θ. In other cases, the probability of crisis is positive, and it can be one if the distribution of θ is sufficiently adverse.

In order to give a flavor of the behavior of the model, we compute outcomes for particular parametrizations. We parametrize the model so that the frictionless C_0 is always equal to one. The details of the parametrization are presented in section 5. In a baseline parametrization, C_0 in laissez faire is 0.9635.

Aspects of the continuation equilibrium are depicted in Figure 1.[4] The figure shows that, in the continuation equilibrium, financial constraints do not bind if θ is low, that is, if $\theta \leq \hat{\theta} = 0.3527$. In that case, the real exchange rate, the domestic interest rate, and investment are all at their frictionless values. For values of θ larger than $\hat{\theta}$, financial constraints bind. In that region, the exchange rate depreciates, investment falls, and the interest spread increases. As expected, these effects are stronger the larger is θ.

The laissez faire equilibrium depends on the parameters of the model in an intuitive way.

[4]Figures are placed at the end of the paper.

To illustrate, Figure 2 describes the equilibrium crisis probability, the expected interest rate ($E(R_1)$) and initial debt and consumption, as functions of the expected value of θ, keeping the dispersion of θ constant (i.e. taking $\underline{\theta} = E(\theta) - h, \bar{\theta} = E(\theta) + h$, the figure describes equilibrium outcomes as we vary $E(\theta)$, keeping h constant).

As expected, if the mean value of θ is small enough (less than 0.3 in the figure), financial constraints never bind. In this case, R_1 is always equal to $R_1^* = 1$, so that $E(R_1) = 1$. Initial debt is then equal to its frictionless value (one). As the mean value of θ rises, the probability of binding financial constraints goes up. Consequently, the equilibrium distribution of R_1 shifts to the right, and $E(R_1)$ goes up. Finally, initial consumption and debt go down, as expected future domestic interest rates increase.

Interestingly, an increase in ex ante uncertainty can result in an increase in the probability of crisis. This is depicted in Figure 3, which displays equilibrium outcomes as functions of h, keeping $E(\theta)$ constant. For this parameterization, if uncertainty is sufficiently small (i.e. h is less than 0.01), financial constraints never bind, and crisis do not occur. If, on the contrary, uncertainty is large enough, crises happen with positive probability. $E(R_1)$ goes up, reflecting this fact.

Notably, the figure shows that initial debt and consumption fall in response to an increase in uncertainty. This is in response, of course, to higher expected interest rates. A crucial fact is that the endogenous fall in initial debt is not sufficient, by itself, to eliminate the country's exposure to crises.

4 Optimal Reserves and Ex Post Policy

In our model economy, binding financial constraints can cause the domestic rate of interest to rise above the world rate. This implies that investment can be too low ex post, and also that initial consumption and debt can be inefficiently small. As noted in CCV, the resulting inefficiencies can be mitigated if the central bank provides liquidity when there is a financial crisis. However, in order to do so, the central bank must have ready access to the necessary international liquidity.

Arguably, considerations of this kind have been a main motivation for the accumulation of international reserves in emerging economies. Hence it is of interest to examine the implications of reserves accumulation in our model, and to ask about the determinants of the optimal quantity of reserves. We turn to this question in the rest of the paper.

4.1 Equilibrium With Reserves Accumulation

To allow for the accumulation of reserves, we assume from now on that the government or central bank has access to long term loans in the world market at $t = 0$: if it borrows F dollars at $t = 0$, it must repay $(1+\tau)R_0^* R_1^* F$ dollars at $t = 2$, where $\tau \geq 0$ is a "term premium". (It will become clear that the more interesting case is $\tau > 0$).

The central bank can invest its F reserves in the world market and earn R_t^* in periods $t = 0$ and $t = 1$. Therefore, in our setup, the central bank invests its international reserves in "liquid instruments", as central banks do in reality.

In this context, in period $t = 1$, it has the option to use the reserves to enact a policy aimed at alleviating financial frictions. To simplify the discussion, we assume that, if the financial constraint binds at $t = 1$, the central bank can lend but not borrow at that point.

In this setting, several questions emerge: What are the implications of reserves accumulation for equilibrium? What is the optimal level of reserves, and what are its determinants? Interestingly, as we will see, the answers depend on the policies that the central bank implements at $t = 1$. We initially focus on the case in which, at $t = 1$, the central bank provides a loan, of size $M \leq R_0^* F$, to domestic banks (in terms of Gertler and Kiyotaki (2010), the central bank provides *liquidity facilities*). In CCV we showed that liquidity facilities are more effective than providing loans directly to households or firms, and that sterilized foreign exchange intervention is equivalent to liquidity provision of one or the other kind (depending on the nature of the sterilizing credit operation). The implications of different types of ex post policy for optimal reserves accumulation are discussed in the next section.

As in CCV, we assume that central bank loans to domestic banks carry the world interest rate R_1^* and that the repayment of these loans can be enforced perfectly. This means that the banks' collateral constraint changes to

$$R_1 L_1 - R_1^*(D_1 + M) \geq \theta R_1 L_1 - R_1^* M$$

which implies that loans' supply is now constrained by

$$L_1 \leq \frac{1}{1-(1-\theta)\phi}(T + X_1 N + M)$$

Intuitively, there may be two cases. One can conjecture that the central bank might choose F large enough to be able to eliminate completely the possibility of financial frictions. This is indeed possible if F is large enough.

To see how large F needs to be to eliminate crises, note that if crises never occur, $C_0 = C_{0f}$, that is, the initial consumption and debt must be at their frictionless level. Since investment and capital will also be at their frictionless levels, it follows that loan demand in period 1 will be $R_0^* C_{0f} + Q_{1f} K_{2f}$, and the exchange rate will be X_{1f}. Now, the minimum F that eliminates crises, which we will denote by $\bar{F}$, must be such that the collateral constraint just binds when θ is at its highest possible value, which we have denoted by $\bar{\theta}$. This requires that,

$$R_0^* C_{0f} + Q_{1f} K_{2f} = \frac{1}{\bar{\theta}}(T + X_{1f} N + R_0^* \bar{F})$$

or

$$R_0^* \bar{F} = \bar{\theta}(R_0^* C_{0f} + Q_{1f} K_{2f}) - (T + X_{1f} N)$$

Clearly, then, if the central bank accumulates $F = \bar{F}$ reserves, then crises will not occur. But this may be too costly if $\tau > 0$. In fact, it is easy to check that such a policy results in expected utility given by $EU_f - \tau R_0^* R_1^* \bar{F}$, .where EU_f is expected utility in the *frictionless* equilibrium. This utility level falls with τ and $\bar{F}$, and suggests that other choices of $F < \bar{F}$ result in higher welfare.

More generally, we will ask what is the *optimal* level of reserves for a central bank that chooses F (and M) to maximize the household's expected utility, taking into account the cost of reserves accumulation (in the spirit of Heller (1966)).

First, we formally argue that the optimal level of reserves is less than $\bar{F}$:

Proposition 1 *If the term premium $\tau = 0$, it is optimal to choose $F = \bar{F}$, driving the probability of crisis to zero. If $\tau > 0$, it is not optimal to eliminate crises completely.*

The proof of the proposition is obvious if $\tau = 0$. In the case $\tau > 0$, start by noting that, in any continuation equilibrium, at $t = 2$ the central bank receives $R_1^* M$ from banks, and $R_1^*(R_0^* F - M)$ from investing the remainder in the world market. It therefore makes a loss of $\tau R_1^* R_0^* F$, which we assume is covered with a lump sum tax on households. Since the tax does not depend on M, we can assume without loss of generality that $M = R_0^* F$.

Now, taking into consideration the aforementioned tax, it is straightforward to check that that in any continuation equilibrium,

$$C_2 = AK_2^\alpha - R_1^* I_w + R_1^* T - R_1^* R_0^* C_0 - \tau R_0^* R_1^* F$$

In the above expression, K_2, I_w, and C_0 are seen as functions of F, determined by the resulting equilibrium. Keeping this in mind, the expected value of choosing F is

$$\begin{aligned} V(F) &= U(C_0) + \beta E C_2 \\ &= U(C_0) - \beta R_1^* R_0^* C_0 + \beta E\left[AK_2^\alpha - R_1^*(I_w - T)\right] - \beta\tau R_0^* R_1^* F \end{aligned}$$

If K_2, I_w, and C_0 are differentiable with respect to F, $V(F)$ will have a derivative given by:

$$\begin{aligned} V'(F) &= [U'(C_0) - \beta R_1^* R_0^*] \frac{dC_0}{dF} + \beta E \left\{ \frac{d}{dI_w} [AK_2^\alpha - R_1^* I_w] \cdot \frac{dI_w}{dF} \right\} - \beta \tau R_0^* R_1^* \\ &= [U'(C_0) - \beta R_1^* R_0^*] \frac{dC_0}{dF} + \beta E \left\{ (R_1 - R_1^*) \frac{dI_w}{dF} \right\} - \beta \tau R_0^* R_1^* \end{aligned}$$

the second equality being justified by the fact that, in any continuation equilibrium, $d(AK_2^\alpha)/dI_w = R_1$

This expression gives the marginal value of international reserves, and it is quite instructive. A marginal increase in F will, in this model, help reducing R_1 at times of crisis and, hence, induce an increase in C_0. The impact on the household's utility is then given by dC_0/dF times the "wedge" $U'(C_0) - \beta R_1^* R_0^*$, which is a measure of the cost of the distortion in initial consumption. In addition, the increase in F and fall in R_1 leads to an increase in borrowing for investment. The marginal benefit is given by dI_w/dF multiplied by the wedge between the marginal product of tradables input, given by R_1, and the world cost of borrowing tradables, R_1^*.

Now consider the behavior of $V'(F)$ as F is close to $\bar{F}$. It is easy to see that there must be an $\varepsilon > 0$ such that, if F is in the interval $(\bar{F} - \varepsilon, \bar{F})$, financial constraints bind only if $\theta = \theta^{(n)} = \bar{\theta}$. By the Implicit Function Theorem, one can then show that $V(F)$ is continuous and differentiable on the semi open interval $(\bar{F} - \varepsilon, \bar{F}]$, the derivative being given by

$$V'(F) = [U'(C_0) - \beta R_1^* R_0^*] \frac{dC_0}{dF} + \beta (R_1^n - R_1^*) \frac{dI_w^n}{dF} \pi_n - \beta \tau R_0^* R_1^*$$

where $R^n = \rho(C, \theta^{(n)})$ and I_w^n the value of I_w when $\theta = \theta^{(n)}$.

Recall that, if $F = \bar{F}$, then $U'(C_0) = U'(C_{0f}) = \beta R_0^* R_1^*$ and $R_1 = R_1^*$. The Appendix also shows dI_w^n/dF exists and is finite at $F = \bar{F}$. Hence $V'(\bar{F}) = -\ \beta \tau R_0^* R_1^* < 0$, and it cannot be optimal to set $F = \bar{F}$.■

The above argument is similar to others in international trade and public finance: at $F = \bar{F}$, the benefit of reducing F by a marginal amount is positive, while the cost, in terms of allowing for a crisis to occur with small probability, is zero to first order.

The situation is depicted in Figure 4. The upper panel graphs the probability of crisis as a function of F. As F increases, the probability falls, and it becomes zero if F is large enough. The lower panel is a graph of $V(F)$, that is, of expected utility as a function of reserves. It is apparent that the optimal level of reserves is less than $\bar{F}$, which allows for a positive probability of crisis (about one half, in the figure). This is because the marginal benefit of reserves becomes smaller than their cost, given by the term premium.

4.2 Solving for Equilibrium

For $\tau > 0$, and $0 < F < \bar{F}$, the analysis of equilibrium involves a straightforward extension of the laissez faire case. Fix F in that range $0 < F < \bar{F}$. In equilibrium, there is $\hat{\theta}$ in $(\underline{\theta},\bar{\theta})$ such that financial constraints bind if $\theta > \hat{\theta}$. This $\hat{\theta}$ must then satisfy:

$$R_0^* C_0 + Q_{1f} K_{2f} = \frac{1}{\hat{\theta}}(T + X_{1f} N + R_0^* F)$$

recalling that Q_{1f}, K_{2f}, and X_{1f} refer to frictionless values. Note that $\hat{\theta}$ depends on F and also on C_0 . We now show how to find C_0, given F.

In the continuation equilibrium, given $\theta \leq \hat{\theta}$, financial frictions do not bind, and final consumption is

$$C_2 = AK_{2f}^{\alpha} - R_1^*(I_{wf} - T) - R_1^* R_0^* C_{0f} - \tau R_0^* R_1^* F$$

If $\theta > \hat{\theta}$, the loan market equilibrium exchange rate is determined by

$$R_0^* C_0 + Q_1 K_2 = \frac{1}{1 - (1 - \theta)\phi}(T + X_1 N + R_0^* F)$$

with Q_1, K_2, I_w, ϕ, and R_1 are determined once X_1 is given, as discussed in the previous section.

So, given C_0 and F, one can compute $\hat{\theta}$ and the continuation equilibrium, including R_1. To complete the derivation, C_0 is pinned down by the Euler equation:

$$U'(C_0) = \beta R_0^* E R_1$$

This analysis allows us to find equilibrium and expected utility associated with any $0 < F < \bar{F}$.

It is instructive to study the condition for the optimal amount of reserves: rewrite $V'(F) = 0$ as

$$\left\{ [U'(C_0) - \beta R_1^* R_0^*] - \beta R_0^* E[(R_1 - R_1^*)\eta El_D^X \frac{I_w}{R_0^* C_0}] \right\} \frac{dC_0}{dF} = \beta \tau R_0^* R_1^*,$$

where $El_D^X = -\frac{C_0 dX_1}{X_1 dC_0}$ is the elasticity of the exchange rate X with respect to the initial debt $D_0 = C_0$, and η is the elasticity of substitution between home and foreign goods in the production of capital.

The previous expression it is clear that an optimal amount of reserves must balance, on the one hand, the impact of an additional unit of international reserves on initial consumption and, on the other hand, the effect of the necessary increase in external debt on relative prices and investment.

In this model, the central bank accumulates reserves in order to provide international liquidity if financial frictions bind. The availability of reserves and central bank policy affects private decisions, and in particular it can lead households to increase their initial consumption and debt. This can be optimal, in spite of the fact that, if θ turns out to be large, the increased inherited debt exacerbates financial distortions and reduces the amount of investment. In this sense, optimal reserves accumulation must balance underborrowing ex ante against the possibility of ex post overborrowing.

4.3 The Role of the Government

Before leaving this section, it is worth emphasizing that, in this model, government intervention is necessary, in the sense that the private sector would be unable to accomplish the same outcomes.

Suppose that domestic banks have access to the same international options as the government. That is, any bank can borrow, say F', for two periods, at interest cost $(1+\tau)R_0^*R_1^*$. This option would enable the bank to increase its loans in $t = 1$ by R_0^*F'. On the other hand, its profits at $t = 2$ are reduced by the service of the long term debt, so:

$$\begin{aligned}\Pi^b &= R_1L_1 - R_1^*D_1 - (1+\tau)R_0^*R_1^*F' \\ &= R_1L_1 - R_1^*(L_1 - (T + X_1N + R_0^*F')) - (1+\tau)R_0^*R_1^*F' \\ &= (R_1 - R_1^*)L_1 + R_1^*(T + X_1N) - \tau R_0^*R_1^*F'\end{aligned}$$

This expression makes it clear that borrowing F' cannot increase bank profits, and must reduce them if $\tau > 0$. This is clear in states of nature in which $R_1 = R_1^*$, as then profits reduce to $\Pi^b = R_1^*(T + X_1N) - \tau R_0^*R_1^*F'$. If $R_1 > R_1^*$, there is an additional, negative impact, in that increases F' reduces the binding credit limit on L_1.[5]

We conclude that no individual bank would set $F' > 0$. The private banking sector would not accumulate liquidity in this model, even if doing so may turn out to be collectively beneficial ex post.

5 Determinants of the Optimal Level of Reserves

In this section we explore some determinants of the optimal level of reserves, emphasizing those that have not been identified in previous work (such as Jeanne and Ranciere 2011). For the analysis, we resort to numerical exercises.

We parametrize the model in order to illustrate its qualitative implications, rather than providing quantitative lessons, for which this model is not yet ready. Most of the parameters are set at conventional values. For example, we choose a baseline value of one for the world real

[5] The collateral constraint becomes $R_1L_1 - R_1^*D_1 - (1+\tau)R_0^*R_1^*F' \geq \theta R_1L_1$, which leads to $L_1 \leq (1/(1 - \phi(1-\theta))[T + X_1N - \tau R_0^*F']$.

interest rate R_t^*, $t = 0, 1$. The elasticity of substitution between tradables and nontradables in capital production, η, is set to 1.4, while the share of the nontradable good in the production of capital, γ , is set to 0.5. The capital share α is assumed to be 0.8. In the baseline parametrization we assume that θ is distributed uniformly between 0.36 and 0.44 and τ, the term premium, is equal to 0.02. We assume that $U(C) = C^{1-\sigma}/(1-\sigma)$, with $\sigma = 2$. The nontradable endowment is equal to one while the tradables endowment is assumed to be zero. Choices for other parameters are discussed below.

5.1 The Term Premium

In this model, an increase in the term premium τ will generally lead to a fall in the optimal amount of reserves, and hence an increase in the probability of crises. This is illustrated by Figure 5.

In the figure, the solid line graphs expected utility as a function of reserves F in the absence of a term premium or, equivalently, assuming that $\tau = 0$. If $\tau > 0$, total expected utility is just the vertical difference between the solid graph and $\beta\tau R_0^* R_1^* F$. The optimal level of reserves is then given by the point on the solid graph that has slope $\beta\tau R_0^* R_1^*$. The dashed (green) ray in the graph has that slope for $\beta = R_t^* = 1$ and $\tau = 0.02$, while the dotted (red) ray assumes $\tau = 0.04$. In each case, the optimal value of F is that which maximizes the distance between the solid line and the corresponding ray. It is then obvious that an increase in τ leads to a smaller value for the optimal F. This accords with intuition and the previous results of e.g. Jeanne and Ranciere (2011).

Interestingly, as noted in the introduction, during the period 2004-2008 there was a significant decrease in the term premium in global financial markets. Our model then suggests that that decrease may have been a main factor underlying the observed increase in foreign reserves before the global crisis.

5.2 Mean Value of Shocks

An increase in the expected value of θ, keeping the dispersion of θ constant, typically increases the probability of binding financial constraints. One would then expect the benefits of liquidity provision in crises become stronger, and the optimal amount of reserves to increase.

Figure 6 provides an illustration. The solid line (blue with circles) graphs $V(F)$ for $E(\theta) = 0.38$, while the dashed dotted (green) line is $V(F)$ for $E(\theta) = 0.40$. As anticipated, for any F, expected utility is higher when $E(\theta)$ is lower. Also, for F sufficiently high, the two lines converge. This is because a sufficiently large F completely eliminates crises in both cases, and secures the same outcome (the frictionless one).

For this parametrization, the figure also tell us that optimal reserves are higher for higher $E(\theta)$. It should be noted, however, that this result may not be general since, as apparent from the figure, it depends on details about the curvature of $V(F)$ or, more precisely, on how $V'(F)$ changes with $E(\theta)$. This underscores the crucial point that the optimal level of reserves depends not on their total benefits, but on their *marginal* benefits, relative their marginal cost.

For a suggestive interpretation, one can take smaller $E(\theta)$ as a characteristic of more developed financial markets. In this sense, the model is consistent with the view that financial development justifies smaller international reserves accumulation. Evidence for such a view is presented by Dominguez (2010).

5.3 Impact of Ex Ante Uncertainty

Now consider the case of a mean preserving spread of θ. We have assumed that θ is uniformly distributed in $[\underline{\theta}, \overline{\theta}]$, with $\underline{\theta} = E(\theta) - h$, and $\overline{\theta} = E(\theta) + h$. Intuitively, for given $E(\theta)$, an increase in h must increase the probability of crisis, and justify higher reserves. An illustration is given by Figure 7, which fixes $E(\theta)$ (at 0.4) and plots $V(F)$ for $h = 0.04$ (solid blue line) and $h = 0.06$ (broken green line). The analysis of the figure is the same as that of changes in $E(\theta)$. In this case, higher uncertainty justifies increased accumulation of reserves.

In our model, increased ex ante uncertainty raises the probability of crises. In response, it is optimal for the central bank to accumulate more reserves, so as to be ready to provide liquidity to domestic markets in case the crisis actually occurs. This story is quite consistent with observed increases in international reserves in the period prior to the Lehman bankrupcy. In fact, the increase in uncertainty regarding future financial conditions was stated by central banks as a crucial element to accelerate international reserves accumulation.

5.4 Reserves accumulation and ex post tools

In our model, the reason for the central bank to accumulate reserves, even if reserves are costly, is to alleviate financial constraints, if they bind, using reserves to provide credit to domestic agents. It follows that the optimal quantity of reserves should depend on the effectiveness of government intervention and the precise way the central bank intervenes.

In order to illustrate this point, consider the implications of changing the way the central bank provides liquidity in a crisis. So far we have assumed that, when the collateral constraint binds, the central bank lends its reserves to domestic banks (liquidity facilities). Instead, assume a policy of *direct lending*: in a crisis, the central bank lends directly to households and firms.

In any continuation equilibrium, the analysis of CCV applies, and implies that direct lending is less effective than liquidity facilities. The argument is as follows: we assume that at $t = 1$, the central bank increases the domestic supply of credit by its reserves R_0^*F; in period $t = 2$, the government collects R_1F in debt repayments and pays its foreign debt. Extending the arguments of previous sections, it can be shown that if $R_1 = R_1^*$, total loan supply is given by

$$L_1 \in [0, \frac{1}{\theta}(T + X_{1f}N) + R_0^*F]$$

while, if $R_1 > R_1^*$, loan supply is the sum of domestic bank loans plus government loans:

$$L_1 = \frac{1}{1-(1-\theta)\phi}(T + X_1N) + R_0^*F$$

Under direct lending, each dollar of reserves is used at $t = 1$ to increase loan supply by one unit. In contrast, with liquidity facilities, loan supply in a crisis is $(T+X_1N+R_0^*F)/[1-(1-\theta)\phi]$ and, therefore, larger for any given F. This reflects leverage: under liquidity facilities, each dollar in reserves is lent to banks, which then leverage it to borrow $1/[1-(1-\theta)\phi] > 1$ dollars in the world market. Hence direct lending is less effective, and results in lower welfare than liquidity facilities, for any given amount of reserves.

The implications for the optimal amount of reserves, however, are ambiguous. This is (again) because, while welfare is lower under direct lending than under liquidity facilities for any F, the difference in the optimal level of reserves depends on the comparison of the *marginal* benefit of reserves under each policy, relative their cost.

This is illustrated by Figure 8. The upper panel plots $V(F)$ under liquidity facilities (solid line) and direct lending (broken line), under the assumption $\tau = 0.02$. As implied by our analysis, the two policies deliver the same expected welfare for $F = 0$ (i.e. no intervention) and F large enough (since crises are eliminated in both cases). For intermediate values of F, liquidity facilities deliver higher welfare than direct lending. Also, in that case, the optimal level of reserves is higher with a direct lending policy.

The lower panel of Figure 8 assumes a higher $\tau = 0.04$. One can check that, for any F, the height of each curve in that panel is lower than the corresponding one in the upper panel. The key observation, however, is that the optimal level of reserves is now smaller under direct lending. The intuition is that, if τ is higher, reserves under both policies must fall because of the higher cost. However, the fall is more pronounced under direct lending, which is the less effective policy.

6 Final Remarks

Our analysis can obviously be extended in several interesting and potentially fruitful directions. One is to allow for endogenous currency mismatches. In our model we could, for example, assume that in the initial period the household can borrow in either tradables or nontradables.

In such a situation, real exchange movements would redistribute wealth between domestic banks and households, altering equilibrium outcomes when financial frictions bind. While this extension is then straightforward, the solution is involved, so we leave it for future work.

A second direction would be to develop a multiperiod version of our model and examine implications for dynamics. Such an extension would also allow the model to be calibrated or estimated. This would also be a substantial project.

Finally, it may be of interest to introduce nominal rigidities so one can examine the interaction of reserves accumulation not only with unconventional central policy but also with conventional policy. It may be the case, for example, that reserves are instrumental in giving the central bank additional policy tools in case the conventional tool, the policy interest rate, falls to its lower bound of zero. These ideas seem worth pursuing in future research.

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Appendix

Here we prove the claims at the end of the proof of Proposition 1 in the text. It will be useful to define some terms with more precision. Under our assumptions, for given $F \geq 0$, a *competitive equilibrium* is given by:

- an initial debt/consumption value C_0
- *continuation* equilibrium values, summarized by the real exchange rate X_{1s} and interest rate R_{1s}, $s = 1, ..n$
- a cutoff value $\hat{\theta}$ such that such that financial constraints do not bind if $\theta \leq \hat{\theta}$, but they bind if $\theta > \hat{\theta}$

These values must solve:

1. The cutoff property:

$$R_0^* C_0 + Q_{1f} K_{2f} = \frac{1}{\hat{\theta}} (T + X_{1f} N + R_0^* F)$$

2. Optimal initial consumption:

$$u'(C_0) = \beta R_0^* \left[\sum_{s=1}^{n} R_{1s} \pi_s \right]$$

3. Continuation equilibrium equations: if $\theta^s \leq \hat{\theta}$,

$$R_{1s} = R_1^*$$

and, if not, R_{1s} and X_{1s} solve:

$$R_0^* C_0 + \left(\frac{N}{\gamma} \right) X_{1s} = \frac{1}{1 - (1 - \theta) \phi_s} (T + X_{1s} N + R_0^* F)$$

with

$$\phi_s = R_{1s}/R_1^* = \left(\frac{X_{1f}}{X_{1s}}\right)^{\gamma+(1-\alpha)(1-\gamma)}$$

As in the text, $\bar{F}$ be the value of reserves at which all constraints cease to be binding. It is obvious from the above equations that there is some $\varepsilon > 0$ such that, for any F in $(\bar{F} - \varepsilon, \bar{F})$, constraints only bind when $\theta = \theta^n$.This means that, for any F in $(\bar{F} - \varepsilon, \bar{F})$, C_0 and the continuation equilibrium in state $s = n$ satisfy the following three equations:

$$C_0^{-\sigma} = \beta R_0^* R_1^* \left[(1 - \pi_n) + \pi_n \phi_n\right]$$

$$\left[R_0^* C_0 + \left(\frac{N}{\gamma}\right) X_{1n}\right] \left[1 - (1 - \theta)\phi_n\right] - T - X_{1n} N = R_0^* F$$

$$\phi_n = \left(\frac{X_{1f}}{X_{1n}}\right)^{\gamma+(1-\alpha)(1-\gamma)}$$

Importantly, these equations are also satisfied at $F = \bar{F}$. The Implicit Function Theorem then implies that we can define (C_0, X_{1n}, ϕ_{1n}) as functions of F in $(\bar{F} - \varepsilon, \bar{F}]$, and also examine their derivatives at each such F, by examining the above equations.

Now, regarding C_0 and ϕ_n as functions of X_{1n} (using the first and third equations of the preceding system), we can see the left hand side of the second equation as a function of X_{1n}, say $\Gamma(X_{1n})$. The derivative of X_{1n} with respect of F then exists if $\Gamma'(X_{1n})$ is not zero at $X_{1n} = X_{1f}$ (and $F = \bar{F}$). It is straightforward to show that this is the case.

(In case, here is a proof of the last assertion: the derivative is

$$\begin{aligned}
&\frac{N}{\gamma}\theta - N \\
&+\frac{d\phi_n}{dX_{1n}} \left\{ \left[R_0^* C_{0f} + \left(\frac{N}{\gamma}\right) X_{1f}\right] (1 - \theta) + \theta R_0^* \frac{dC_0}{d\phi_n} \right\}
\end{aligned}$$

where we have used that, at $F = \bar{F}$, all variables are at their frictionless values, and $\frac{d\phi_n}{dX_{1n}}$ and $\frac{dC_0}{d\phi_n}$ are evaluated at frictionless values as well.

Now, from

$$\frac{X_{1n}}{\phi_n}\frac{d\phi_n}{X_{1n}} = -(\gamma + (1-\alpha)(1-\gamma))$$

it follows that, at frictionless values,

$$\frac{d\phi_n}{dX_{1n}} = -(\gamma + (1-\alpha)(1-\gamma))\frac{1}{X_{1f}}$$

In turn, since

$$\frac{\phi_n}{C_0}\frac{dC_0}{d\phi_n} = -\frac{1}{\sigma}\pi_n$$

it follows that, at frictionless values,

$$\frac{dC_0}{d\phi_n} = -\frac{1}{\sigma}\pi_n C_{0f}$$

Combining all of this, the derivative can be written as

$$\begin{aligned}\Gamma'(X_{1n}) &= N[\frac{1}{\gamma}(\theta + (1-\theta)(\gamma + (1-\alpha)(1-\gamma) - 1] \\ &\quad +(\gamma + (1-\alpha)(1-\gamma))\frac{R_0^* C_{0f}}{X_{1f}}\left[(1-\theta) + \theta\pi_n\frac{1}{\sigma}\right]\end{aligned}$$

It is straightforward to check that both terms are strictly positive. Hence $\Gamma'(X_{1n}) > 0$, which means that $dX_{1n}/dF > 0$ and finite, as claimed.)

The existence of dX_{1n}/dF obviously implies the existence of dI_w^n/dF. This completes the proof.

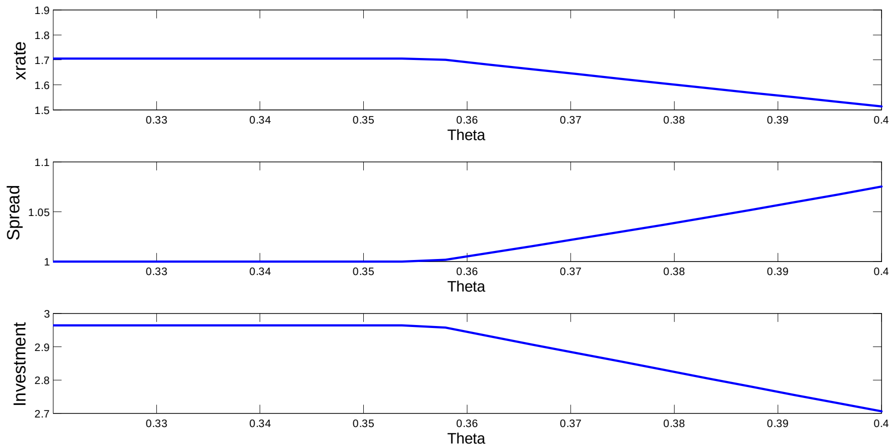

Figure 1: Continuation Equilibrium

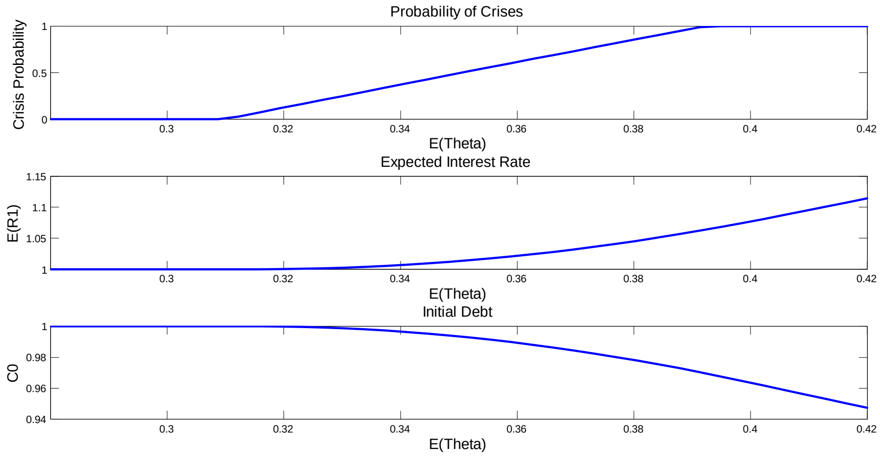

Figure 2: Equilibrium and $E(\theta)$

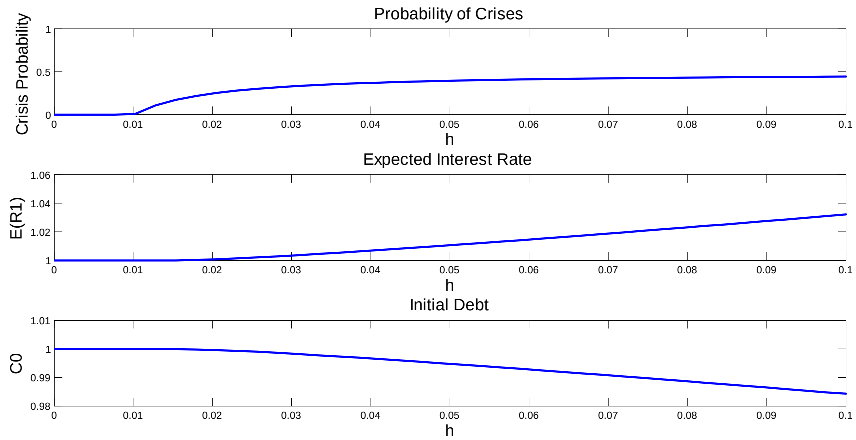

Figure 3: Equilibrium and Uncertainty

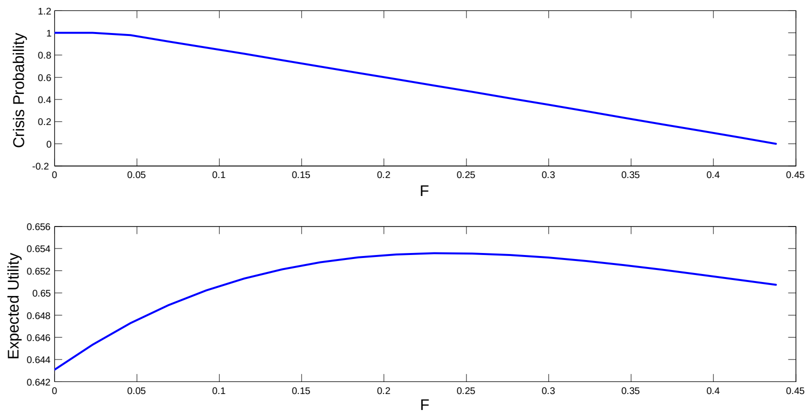

Figure 4: Reserves, Crisis Probability, and Expected Utility

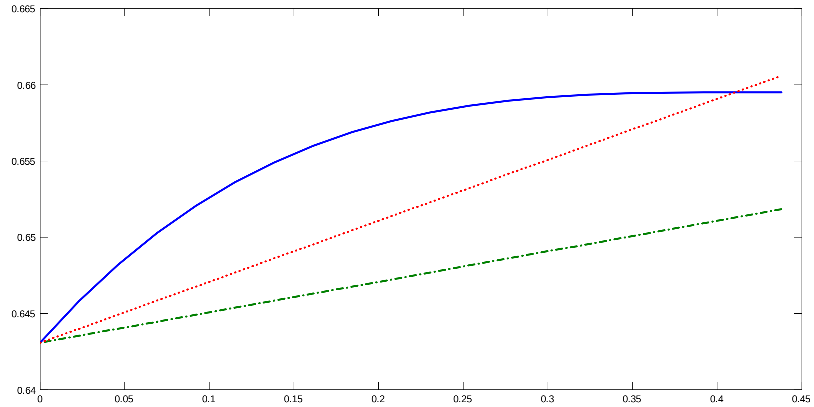

Figure 5: The Term Premium

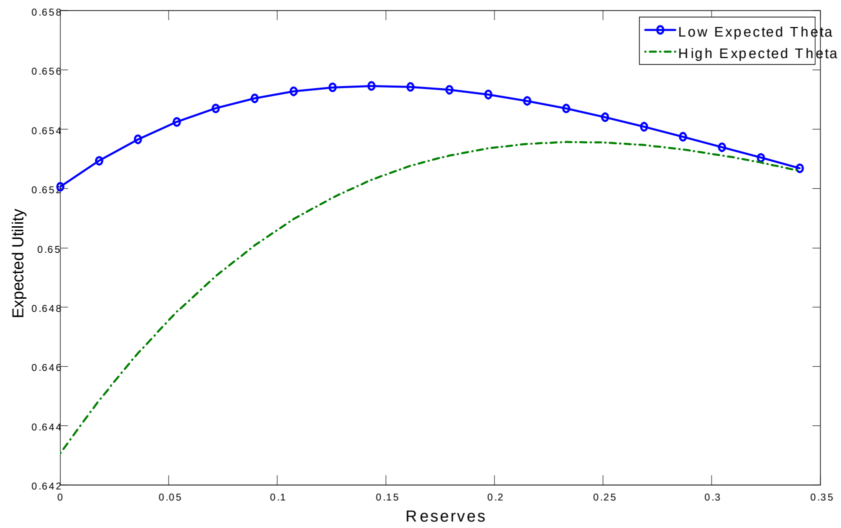

Figure 6: Reserves and $E(\theta)$

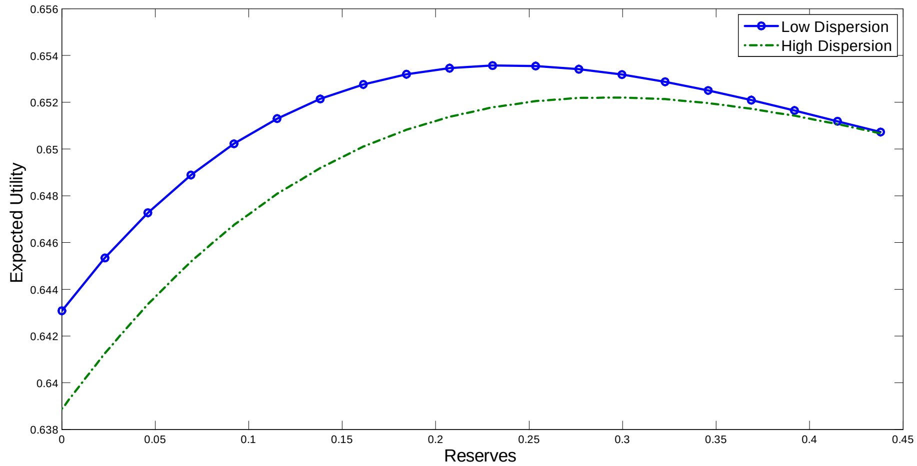

Figure 7: The Impact of Uncertainty

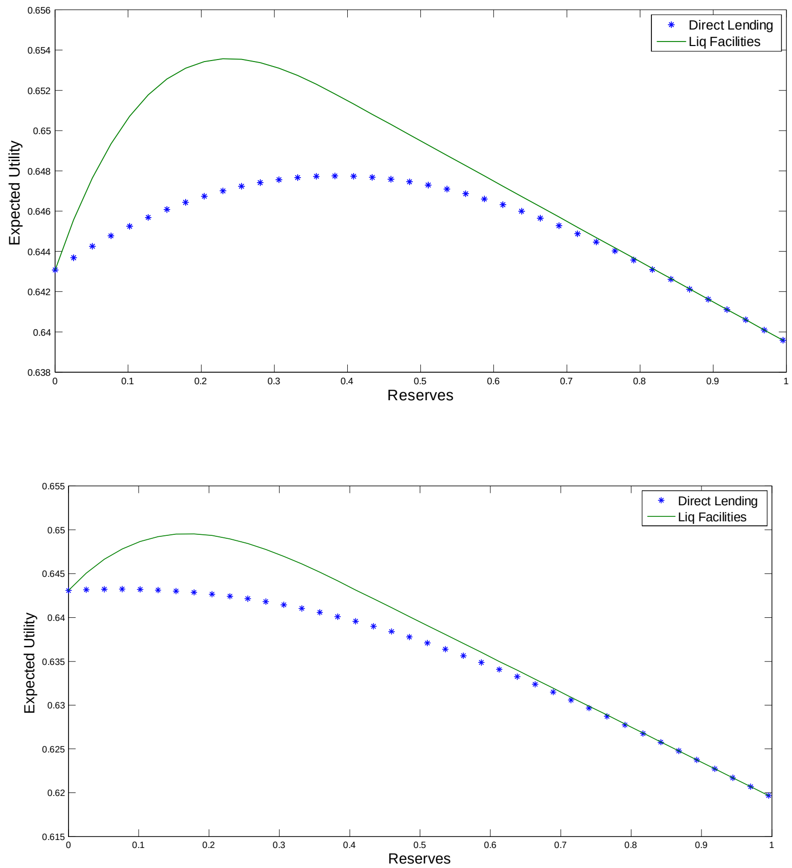

Figure 8: Optimal Reserves and Ex Post Policies, $\tau = 0.02, 0.04$