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Zhengyang Jiang
Hanno Lustig
Stijn Van Nieuwerburgh
Mindy Z. Xiaolan

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ABSTRACT

Governments face a trade-off between insuring households who pay taxes and receive transfers and bondholders against aggregate output risk. Insulating bondholders by keeping the debt risk-free imposes tight restrictions on the expected primary surplus process and its covariance with aggregate shocks. Through counter-cyclical debt issuance, the government can protect taxpayers against adverse economic shocks over short horizons, but not over longer horizons. The restrictions imposed by risk-free debt on expected surpluses are rejected in US data. If the portfolio of U.S. Treasurys were risk-free, then Treasury investors would have been expecting large primary surpluses since the GFC. Such surpluses never materialized even though r < g.

Zhengyang Jiang
Kellogg School of Management
Northwestern University
2211 Campus Drive
Evanston, IL 60208
and NBER
zhengyang.jiang@kellogg.northwestern.edu

Hanno Lustig
Stanford Graduate School of Business
655 Knight Way
Stanford, CA 94305
and NBER
hlustig@stanford.edu

Stijn Van Nieuwerburgh
Columbia University
Graduate School of Business
Kravis Hall, office 796
665 West 130th Street
New York, NY 10027
and NBER
svnieuwe@gsb.columbia.edu

Mindy Z. Xiaolan
University of Texas at Austin
2110 Speedway B6600
Austin, TX 78703
mindy.xiaolan@mccombs.utexas.edu

1 Introduction

For a model that relies heavily on dark matter, cross-equation restrictions appear highly informative about the parameters of the fundamental dynamics relative to fundamental data alone. Chen, Dou, and Kogan (2024a).

Is the U.S. federal government on a sustainable fiscal path? That is a central and increasingly important question for the US. The growing literature on fiscal sustainability typically assumes that the government can fund deficits at the risk-free rate. In other words, the portfolio of outstanding Treasurys is assumed to have a portfolio beta of zero (Blanchard, 2019; Mehrotra and Sergeyev, 2021; Aguiar, Amador, and Arellano, 2024; Mian, Straub, and Sufi, 2021; Ball and Mankiw, 2023). This paper shows that this assumption imposes tight restrictions on the data that are rejected and will likely lead one to overestimate the government's debt capacity, even when the risk-free rate (r) is lower than the growth rate g.

We use a tractable dynamic asset pricing model with priced shocks to output growth to analyze the implications of the risk-free debt restriction. First, we infer the expectations of rational bond investors about future primary surpluses that are implied by risk-free U.S. government debt. The expected surplus levels and dynamics implied by the risk-free debt restriction and the mean-reversion in the debt/output ratio cannot be reconciled with either realized or projected surpluses. Second, to keep the debt risk-free, the government has to shift aggregate risk from bondholders to taxpayers. While the government can provide some insurance to taxpayers over short horizons by issuing more debt in response to negative output shocks, the government is highly limited in its ability to insure taxpayers over intermediate and longer horizons. It can only provide long-run insurance to taxpayers by issuing risky debt, which will then earn a higher rate of return.

U.S. Treasurys occupy a central position in the global financial system as the global safe asset, but the valuation of Treasurys is hard to rationalize based on the observed dynamics of the U.S. federal government's primary surpluses (Jiang, Lustig, Van Nieuwerburgh, and Xiaolan, 2024b). Rather than starting from historical surplus data, this paper imposes that the debt is risk-free and backs out the expected surplus process that is consistent with this assumption. We interpret the rejection of the risk-free debt model as a restatement of the U.S. government debt valuation puzzle.

The starting point of our analysis is the standard valuation model, in which the value of government debt is backed by future primary surpluses provided that a no-bubble condition holds in the market for government debt. We impose that government debt is risk-free: debt is a zero-beta asset. We calibrate a mean-reverting process for the debt/output ratio that matches the data. The risk-free debt model implies a process for the expected primary surplus that only depends on the debt/output dynamics, the risk-free rate r, and the expected output growth rate g. Other asset pricing parameters that govern the pricing kernel dynamics are irrelevant.

For each year in the postwar era, we back out the sequence of implied expected surpluses by feeding in the observed debt/output history. The risk-free debt model implies that U.S. Treasury investors have been pricing in large primary surpluses since the start of the Great Financial Crisis, even though $r < g$ in our calibration. The substantial expansion of debt in the Covid-19 era further increases the future expected surpluses necessary to keep the debt risk-free. As of the end of 2022, keeping the debt risk-free requires bondholders to expect cumulative primary surpluses of 37% of GDP over the decade from 2023 until 2032. In sharp contrast, the CBO projects a cumulative surplus of -24% over this same time period, amounting to a 61% gap with the model.

In addition to the expected future surplus, we also characterize the restrictions that the risk-free debt assumption imposes on the impulse-response function of the surplus process to an output shock. In response to a negative output shock, the government can only run deficits for a few years. This prediction is also at odds with the protracted deficits we have seen in the U.S. (and many other developed nations) in response to the GFC or the Covid crises. Instead of seeing surpluses, U.S. bondholders experienced a real return of -25% between March 2020 and October 2023, illustrating the riskiness of the debt.

The risk-free debt model has a lot of dark matter in the sense of Chen et al. (2024a). The cross-equation restrictions, risk-free debt, correct pricing of government debt, the debt dynamics, and the transversality condition, are highly informative about the (expected) surplus dynamics, relative to a model that only relies on the historical surplus dynamics. That should be a source of concern about "in-sample overfitting and out-of-sample validity" when the cross-equation restrictions are potentially misspecified. These restrictions imply that, if the U.S. federal debt is in fact priced accurately and the debt is risk-free, then the surplus will just have to adjust massively in the future. Sine the GFC the model-implied expected surpluses have been far above ex-post realized surpluses, casting doubt on the plausibility of such a large fiscal correction in the future. This model is rejected if we impose that agents have rational expectations.

Next, we focus on the risk characteristics of the surplus. In contrast to the expected surpluses, the asset pricing parameters do matter for the risk characteristics of surpluses. The Treasury's bond portfolio is backed by a long position in a claim to tax revenues which exceeds its short position in a claim to government spending. Both claims are exposed to output risk. We measure the risk exposures by their betas. To render the entire Treasury portfolio risk-free (zero beta), the claim to tax revenues needs to have a lower beta than the spending claim. Recast in the language of Modigliani-Miller, the tax revenue claim can be regarded as the government's unlevered asset; government debt and the claim to government spending its liabilities. Therefore, just as the equity of a firm has to be riskier than its asset in order to generate risk-free corporate debt, the government's spending claim has to be riskier than the tax claim to generate risk-free government debt. The spending beta has to be higher than the tax beta to manufacture a zero-debt beta. Manufacturing risk-free debt in the presence of permanent output shocks is a feat of financial engineering.

The tax claim has a low beta if the present discounted value of future tax revenues increases in bad times. Since the taxpayers pay the taxes, they have a short position in the tax revenue claim. From their perspective, a low-beta tax claim is a risky tax liability. As a result, when the government insures bondholders from aggregate risk by keeping the debt risk-free, it shifts that risk onto taxpayers in the form of a risky tax liability. It cannot simultaneously insure bondholders and taxpayers. The larger the amount of outstanding government debt, the larger the gap between the tax beta and the spending beta needs to be to keep the debt risk-free. The trade-off between insuring taxpayers and bondholders steepens.

We develop a sufficient statistic for how much insurance the government can provide to taxpayers over intermediate horizons by issuing more debt in response to bad shocks. Specifically, the cash-flow beta of the cumulative surpluses generated over the next h periods, $\beta_t^{S,CF}(h)$, measures the possibilities of insuring taxpayers over horizon h. If the debt is risk-free ($\beta^D = 0 = \beta_t^{S,CF}(\infty)$), then the surplus claim cannot be risky over long horizons. There is no possibility to insure taxpayers over long horizons. Put differently, any long-run protection the government offers taxpayers against adverse aggregate shocks must be funded with risky debt.

Over shorter horizons, the government can insure taxpayers by issuing debt in bad times and backloading their aggregate risk exposure. When debt is risk-free, our sufficient statistic $\beta_t^{S,CF}(h)$ is pinned down by the risk properties of a debt strip, an asset with stochastic payoffs equal to the outstanding debt at time $t + h$. The risk premium of this debt strip reflects the cyclical nature of the debt issuance decisions between t and $t+h$. The risk premium on the debt strip over short horizons is negative if the debt/output ratio is sufficiently counter-cyclical. By issuing more debt in response to a negative output shock, the government can run deficits for a limited time and provide some insurance to taxpayers. The cash-flow beta of the cumulative surpluses $\beta_t^{S,CF}(h)$ is positive for small h. Put differently, the tax claim can be riskier than the spending claim over short horizons ($\beta_t^{T,CF}(h) > \beta_t^{G,CF}(h)$). Over intermediate and longer horizons, the surplus claim has to become sufficiently safe for investors (risky for households) to exactly offset the long-run output risk priced into the debt issuance process. Hence, taxpayer insurance provision is limited in magnitude and duration. By increasing the persistence of the debt/output ratio and its counter-cyclicality, the government can increase the horizon over which surpluses are risky and households are insured. The only way for the government to escape this trade-off is by saving rather than borrowing.

Moreover, governments, like the U.S., have an incentive to manufacture safe debt, because safe debt earns sizeable convenience yields. Krishnamurthy and Vissing-Jorgensen (2012) estimate convenience yields on U.S. Treasuries of around 75 bps per year; Jiang, Krishnamurthy, and Lustig (2021); Koijen and Yogo (2020) estimate even higher convenience yields that foreign investors derive from their holdings of dollar safe assets. We show that seigniorage revenues from convenience yields can alleviate the tension between insuring taxpayers and bondholders in the short-run, but only at the expense of aggravating it in the long-run.

Our main results are robust. They carry over to richer asset pricing models with disaster risk and to settings where output risk is transitory rather than permanent. In models with only transitory output shocks, the debt issuance process is subject to significant long-run interest rate risk. The government must make the surplus process safer for investors (riskier for taxpayers) in order to offset the long-run interest rate risk and keep the debt risk-free.

Related Literature. The large and protracted decline in long-term real rates spurred a debate around debt sustainability (Blanchard, 2019; Mehrotra and Sergeyev, 2021; Aguiar et al., 2024; Mian et al., 2021; Ball and Mankiw, 2023; Reis, 2022). We explicitly introduce aggregate risk and risk premia into the analysis. We show that when the government commits to a stationary debt/output policy, the transversality condition (TVC) for debt is naturally satisfied in a world with priced permanent output shocks. The condition

$r > g$, analyzed in models without aggregate risk, is neither necessary nor sufficient for the TVC to be satisfied in economies with priced permanent output risk. Instead, we need to add an output risk premium: $r + rp > g$. Contemporaneous work by van Wijnbergen, Olijslagers, and de Vette (2020) and Barro (2023) highlights this point in models with production and disaster risk, respectively. Jiang, Sargent, Wang, and Yang (2024a) also introduce aggregate risk and they allow for default in a model with tax distortions.

Even when the TVC is satisfied, the government can potentially sustain steady-state primary deficits when $r < g$. This is not a free lunch. Rather, taxpayers have to insure bondholders by paying more in taxes in bad states. Even when $r < g$, rational investors will expect the government to run large primary surpluses whenever the debt exceeds its long-run mean.

Jiang et al. (2024b) establish that the market value of the outstanding Treasurys equals the present discounted value (PDV) of future surpluses in the absence of arbitrage opportunities and after imposing a transversality condition (TVC). They estimate a process for the historical primary surpluses and price a claim to these surpluses using a state-of-the-art dynamic asset pricing model. If all currently outstanding and future government bonds are accurately priced and the TVC holds, then the value of debt has to be equal to the PDV of surpluses. Jiang et al. (2024b) discover a large positive gap between the market value of debt and the PDV of surpluses. This implies one of two things. Either, the gap indicates a violation of the TVC, a solution of the government debt valuation puzzle advocated by Brunnermeier, Merkel, and Sannikov (2024). Or the asset pricing model is misspecified; the market overprices government bonds.

The latter possibility is in line with a long literature that has argued that Treasuries are expensive relative to Agency bonds (Longstaff, 2004), corporate bonds (Bai and Collin-Dufresne, 2019), TIPS (Fleckenstein, Longstaff, and Lustig, 2014), and foreign bonds (Du, Im, and Schreger, 2018; Jiang et al., 2021). Jiang et al. (2024b) argue that the portfolio of Treasurys collectively may be overpriced relative to the underlying collateral, i.e. the future primary surpluses of the federal government. In similar vein, Van Binsbergen (2024) highlights the tension between bond prices and duration-matched equity prices.

In related work, Collin-Dufresne, Hugonnier, and Perazzi (2023) adopt the same mean-reversion dynamics for the debt/GDP ratio as this paper and a similar asset pricing model to Jiang et al. (2024b) to solve for the properties of the surplus process consistent with the government bond valuation equation. Our paper considers a simpler asset pricing model which allows us to analytically characterize the trade-off between insuring taxpayers and bondholders at different horizons. Our main point is that the characteristics of the surplus process implied by the mean-reversion in debt/output are empirically implausible.

If the debt/output ratio is mean-reverting and the debt is close to risk-free, then most of the variation in the debt/output ratio should be attributable to future surpluses. The debt/output ratio should the best predictor of future government surpluses, but it is not. High surpluses drive the debt/output ratio back to its mean. However, the US debt/output ratio does not predict future surpluses at any horizon (Jiang, Lustig, Van Nieuwerburgh, and Xiaolan, 2024c).

The rest of this paper is organized as follows. Section 2 derives the general trade-off between insuring taxpayers and bondholders. Section 3 introduces a specific asset pricing model in which we quantify the trade-off. Section 4 derives restrictions on expected surpluses when debt is risk-free. Section 5 characterizes the trade-off over shorter horizons by characterizing the covariance of expected surpluses with aggregate shocks when debt is risk-free. Section 6 investigates whether convenience yields can relax the trade-off. Section 7 considers two extensions. Section 8 concludes. The appendix contains the proofs and the details of the model extensions.

2 General Fiscal Trade-Off

We start with a general characterization of the trade-off that the government faces between insuring taxpayers and bondholders, expressing it in terms of the riskiness of the claims to government tax revenue and spending.

Following Jiang et al. (2024b), we model government debt as a portfolio of nominal debt with different maturities. Let $D_t = \sum_{h=1}^{H} P_t^{\$}(h) Q_t^{\$}(h)$ define the market value of the aggregate government debt portfolio, where $P_t^{\$}(h)$ is the price at time t of a nominal zero-coupon bond that pays \$1 at time $t+h$, and $Q_t^{\$}(h)$ is the outstanding face value of this bond at time t. We assume the existence of a pricing kernel $M_{t,t+j}$. Define the ex-dividend present values of tax and spending claims as P_t^T and P_t^G:

$$P_t^T = \mathbb{E}_t \left[\sum_{j=1}^{\infty} M_{t,t+j} T_{t+j} \right], \qquad P_t^G = \mathbb{E}_t \left[\sum_{j=1}^{\infty} M_{t,t+j} G_{t+j} \right].$$

Let R_{t+1}^D, R_{t+1}^T and R_{t+1}^G denote the holding period returns on the aggregate government debt portfolio, the tax claim, and the spending claim, respectively:

$$R_{t+1}^D = \frac{D_{t+1} + S_{t+1}}{D_t}, \qquad R_{t+1}^T = \frac{P_{t+1}^T + T_{t+1}}{P_t^T}, \qquad R_{t+1}^G = \frac{P_{t+1}^G + G_{t+1}}{P_t^G}. \tag{1}$$

When the transversality condition (TVC) holds, the government debt portfolio return is the return on a portfolio that goes long in the tax claim and short in the spending claim. Expressed in terms of the expected excess return over a short-term risk-free Treasury rate:

$$\mathbb{E}_t \left[R_{t+1}^D - R_t^f \right] = \frac{P_t^T}{D_t} \mathbb{E}_t \left[R_{t+1}^T - R_t^f \right] - \frac{P_t^G}{D_t} \mathbb{E}_t \left[R_{t+1}^G - R_t^f \right]. \tag{2}$$

By rearranging equation (2), we obtain the following expression for the risk premium on the tax claim:

$$\mathbb{E}_t \left[R_{t+1}^T - R_t^f \right] = \frac{P_t^G}{D_t + P_t^G} \mathbb{E}_t \left[R_{t+1}^G - R_t^f \right] + \frac{D_t}{D_t + P_t^G} \mathbb{E}_t \left[R_{t+1}^D - R_t^f \right], \tag{3}$$

where we used the intertemporal government budget condition $D_t = P_t^T - P_t^G$. On the one hand, governments typically want to engineer a counter-cyclical spending claim, i.e. they want to spend more in recessions to stabilize the economy. That counter-cyclicality makes the spending claim's risk premium $\mathbb{E}_t \left[R_{t+1}^G - R_t^f \right]$ low. On the other hand, they also want to engineer a pro-cyclical tax claim, because they want to reduce the tax burden in recessions. As a result, the tax claim is risky; it's risk premium $\mathbb{E}_t \left[R_{t+1}^T - R_t^f \right]$ is high. When the debt value D_t is positive, the fraction $\frac{P_t^G}{D_t + P_t^G}$ is between 0 and 1. Then, for equation (3) to hold, it requires a high risk premium on the government debt portfolio, $\mathbb{E}_t \left[R_{t+1}^D - R_t^f \right] > \mathbb{E}_t \left[R_{t+1}^T - R_t^f \right]$. As the debt risk premium measures the aggregate risk borne by bondholders, the government's debt portfolio needs to be risky. In other words, when fiscal policy provides insurance to taxpayers, in the form of counter-cyclical spending and pro-cyclical taxation, it shifts risk onto the bondholders.

Given a pricing kernel M, we define the return beta of an asset i as:

$$\beta_t^i = \frac{-cov_t \left(M_{t+1}, R_{t+1}^i \right)}{var_t(M_{t+1})}. \tag{4}$$

By the investor's Euler equation, this beta determines the conditional risk premium of this asset $\mathbb{E}_t \left[R_{t+1}^i - R_t^f \right] = \beta_t^i var_t[M_{t+1}] / \mathbb{E}_t[M_{t+1}]$, where $var_t[M_{t+1}] / \mathbb{E}_t[M_{t+1}]$ is the market price of risk. Let β_t^D, β_t^T, β_t^G, and β_t^Y denote the beta of the bond portfolio, the tax claim, the spending claim, and the aggregate output claim respectively. We assume $\beta_t^Y > 0$, so that the output claim has a positive risk premium. The following proposition characterizes the relationship of these risk exposures.

Proposition 1. *The beta on the tax claim is a weighted average of the beta of the spending claim and the beta of the debt:*

$$\beta_t^T = \frac{P_t^G}{D_t + P_t^G}\beta_t^G + \frac{D_t}{D_t + P_t^G}\beta_t^D. \tag{5}$$

The proof is in Appendix A.1. If the government wants to provide insurance to the households by choosing counter-cyclical spending and pro-cyclical taxation, $\beta_t^G < \beta_t^Y < \beta_t^T$, then the debt cannot be made risk-free.

Corollary 1. *In order for the government debt to be risk-free (i.e., $\beta_t^D = 0$), the beta of the tax claim needs to equal:*

$$\beta_t^T = \frac{P_t^G}{D_t + P_t^G}\beta_t^G.$$

With $D_t > 0$, $\beta_t^T < \beta_t^G$.

If the government has a positive amount of risk-free debt $D_t > 0$, there is no scope to insure taxpayers. Instead, the taxpayers provide insurance to the bondholders. First, consider the case in which the spending claim has a positive beta ($\beta_t^G > 0$). Then, a government that wishes to engineer risk-free debt must do so by lowering the beta of the tax claim relative to that of the spending claim: $\beta_t^T < \beta_t^G$. A low beta for the tax claim means that tax revenue must fall by less than GDP in a recession. Tax rates, the ratio of tax revenue to GDP, must rise in recessions.

Second, consider the case in which the spending claim has a negative beta ($\beta_t^G < 0$); government spending rises in bad times. To ensure risk-free debt, the tax claim must also have a negative beta when $D_t > 0$ ($\beta_t^T < 0$). Tax payments must increase during recessions.

In both cases, taxpayers are insuring the bondholders by enduring "wrong-way around" taxation. The more debt there is outstanding (higher D_t), the lower the beta of the tax claim needs to be relative to that of the spending claim. Put differently, the more debt there is outstanding, the steeper the trade-off between insuring taxpayers and bondholders becomes. The restriction on the tax betas is generic; it holds true regardless of the specific dynamics of the tax and spending process. Below, we will derive restrictions on the tax revenue/output process by committing to a specific, realistic process for debt/output and spending/output ratios. As we will show, the only way the government can escape the trade-off and provide insurance to bondholders while keeping the debt risk-free is by saving—choosing $D_t < 0$.

There is an analogy to the well-known Modigliani-Miller relationship between the expected return on the firm's assets, on the one hand, and the firm's debt and equity claims, on the other hand. Here, the tax revenue claim can be regarded as the government's asset. The government's liability is split into the government debt and the claim to government spending, which can be interpreted as the government's equity. To manufacture risk-free debt, the spending claim has to be a levered version of the tax claim. Therefore, just like the equity of a firm has to be riskier than its assets in order to generate risk-free debt, the government's spending claim has to be riskier than its tax revenue claim to generate risk-free debt: $\beta^T < \beta^G$.

3 Trade-off with Asset Pricing Model

We now illustrate the implications of keeping government debt safe for the properties of the tax process. We do so in a simple economy that faces aggregate output risk. Throughout, we assume that there are no arbitrage opportunities in the debt markets and that the TVC for government debt is satisfied. We show that the cross-equation restrictions that result from insisting on risk-free debt are highly informative about the surplus dynamics, or equivalently about the dynamics for taxes given a process for government spending.

3.1 Setup

To derive closed form-solutions, we use a simple model. We adopt an exogenous stochastic discount factor (SDF) with plausible asset pricing implications. This SDF prices payoffs from the perspective of the investors buying government debt. Output is subject to permanent shocks.

Assumption 1. *(a) Let Y_t and $y_t = \log Y_t$ denote output and its log. All output shocks are i.i.d. and permanent:*

$$y_{t+1} = \mu + y_t + \sigma\varepsilon_{t+1},$$

where ε_{t+1} denotes the innovation to output growth that is i.i.d. normally distributed with mean zero and standard deviation one.
(b) The log SDF is given by:

$$m_{t,t+1} = -\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1}.$$

(c) The government only issues one-period real risk-free debt.

The one-period risk-free rate in this simple model is constant and equal to ρ. The market price of risk, or equivalently the maximum Sharpe ratio, is also constant and equal to $Std_t(m_{t+1}) = \gamma$. We choose this SDF because it delivers tractable expressions yet is rich enough to deliver quantitatively meaningful results. It can be micro-founded by bond investors with CRRA preferences, in which case γ reflects the product of the output growth volatility and the coefficient of relative risk aversion. All of our results go through in a richer model with output disasters, which has the virtue of implying a lower risk aversion coefficient for the same maximum Sharpe ratio (see Section 7.1 and Appendix C).

3.2 Constant Debt-Output

To build intuition for the general trade-off between insurance of bondholders and taxpayers, we start by considering the simplest case of constant spending/output and debt/output ratio policies.

Assumption 2. *(a) The government commits to a constant spending/output ratio $x = G_t/Y_t$.*
(b) The government commits to a constant debt/output ratio $d = D_t/Y_t$.

Under Assumption 2, the government budget constraint implies a counter-cyclical process for tax revenue-to-GDP (the tax rate):

$$\frac{T_t}{Y_t} = \frac{G_t}{Y_t} - \frac{D_t}{Y_t} + R^f_{t-1}\frac{D_{t-1}}{Y_t} = x - d\left(1 - \exp\left\{-(\mu - \rho + \sigma\varepsilon_t)\right\}\right).$$

To perfectly insure the bondholders by keeping the debt risk-free, the government must make the tax revenue claim counter-cyclical: $\partial(T/Y)/\partial\varepsilon < 0$. When the growth rate of output is low ($\varepsilon < 0$), tax revenue needs to increase as a fraction of GDP. Tax rates must rise in recessions. The magnitude of the counter-cyclical exposure is increasing in the debt-to-GDP ratio d.

Similarly, the primary surplus/output ratio is counter-cyclical:

$$s_t = \frac{S_t}{Y_t} = \frac{T_t - G_t}{Y_t} = -d\left(1 - \exp\left\{-(\mu - \rho + \sigma\varepsilon_t)\right\}\right). \tag{6}$$

We have that $\partial s_t/\partial \varepsilon_t < 0$. When the unconditional growth rate of output exceeds the risk-free rate modulo a Jensen term ($\mu > \rho - \frac{1}{2}\sigma^2$), the government runs a primary deficit on average. But when shocks are negative enough ($\mu - \rho < -\sigma\varepsilon_t$), the government must run a primary surplus.[1]

This simple model also places tight restrictions on the persistence of surpluses. The conditional auto-covariance of the surplus/output ratio is zero: $cov_t(s_t, s_{t-1}) = 0$.

The restrictions on the surplus and tax processes described above are independent of the SDF model. Next, we use the SDF to value the debt as the expected present-discounted value of future surpluses.

Proposition 2. *Under Assumptions 1 and 2, if the transversality condition holds and the primary surplus satisfies (6), the government debt value is the sum of the values of the surplus strips:*

$$D_t = \mathbb{E}_t\left[\sum_{j=1}^{\infty} M_{t,t+j} S_{t+j}\right] = \sum_{j=1}^{\infty} \mathbb{E}_t\left[M_{t,t+j} S_{t+j}\right] = dY_t.$$

The proof is in Appendix A.2. This proposition confirms that the (ex-dividend) value of outstanding debt in period t is indeed a constant fraction of output. The proof first solves for the price of a claim to a single future surplus realization (a surplus strip), and then adds up the surplus strip prices across all horizons.

In this equation, the government surpluses are not discounted at the risk-free rate even though the debt is risk-free. To see why, consider the valuation equation for debt as a function of surplus/output ratios over some finite horizon h plus the residual value of the debt at time $t+h$:

$$D_t = \mathbb{E}_t\left[\sum_{j=1}^{h} M_{t,t+j} Y_{t+j} s_{t+j}\right] + \mathbb{E}_t\left[M_{t,t+h} Y_{t+h} \frac{D_{t+h}}{Y_{t+h}}\right].$$

The debt/output ratio $\frac{D_{t+h}}{Y_{t+h}} = d$ in the second term is constant. The TVC for government

[1]When $\sigma \to 0$, the government always runs deficits. But $\mu > \rho$ now implies a violation of the TVC, as we show below.

debt in this model is given by:

$$\lim_{h\to\infty} \mathbb{E}_t \left[M_{t,t+h} D_{t+h}\right] = \lim_{h\to\infty} \exp\left\{h(\mu - \rho + \frac{1}{2}\sigma^2 - \gamma\sigma)\right\} dY_t. \tag{7}$$

This TVC is satisfied if and only if $\mu - \rho + \frac{1}{2}\sigma^2 - \gamma\sigma < 0$. This requires that the output risk premium ($\gamma\sigma$) is high enough either because the amount of permanent output risk (σ) is high enough and/or the price of that output risk (γ) is high enough. This condition ensures that the term $\mathbb{E}_t \left[M_{t,t+h} Y_{t+h}\right] \to 0$ as $h \to \infty$. Since the output risk premium is akin to the unlevered equity risk premium, and since the asset pricing literature suggests that the equity risk premium is large, the TVC is likely to be satisfied. The textbook condition for a TVC violation, $\rho < \mu$, is neither necessary nor sufficient in an economy with permanent output risk. So, it is not the case that the government can always run deficits when $\rho < \mu$, at least not without violating the TVC.

The TVC highlights the importance of modeling the dynamics of future outstanding debt. While the debt at t, D_t, is risk-free under Assumption 1, meaning that its value does not change in response to news revealed between t and $t+1$, the value of debt outstanding at some future date $t+h$, the debt strip D_{t+h}, is a stochastic variable, even when the debt-to-output ratio is a constant, since Y_{t+h} is random. More generally, as long as the debt quantity and the output are co-integrated, future debt inherits the permanent risk in output. As evident in (7), the risk premium associated with the debt strip at time $t+h$ is crucial in determining whether the TVC is satisfied.

Next, we turn to the main result characterizing the expected return and beta of the tax claim.

Proposition 3. *(a) The ex-dividend values of the spending and revenue claims are given by:*

$$P_t^G = x\frac{\xi_1}{1-\xi_1} Y_t,$$

$$P_t^T = \left(d + x\frac{\xi_1}{1-\xi_1}\right) Y_t,$$

with $\xi_1 = \exp\left\{\mu - \rho + 0.5\sigma^2 - \gamma\sigma\right\}$.

(b) The risk premia and betas on the tax claim and the spending claim satisfy:

$$\mathbb{E}_t \left[R_{t+1}^T - R_t^f\right] = \frac{x\frac{\xi_1}{1-\xi_1}}{d + x\frac{\xi_1}{1-\xi_1}} \mathbb{E}_t \left[R_{t+1}^G - R_t^f\right], \tag{8}$$

$$\beta^T = \frac{x\frac{\xi_1}{1-\xi_1}}{d + x\frac{\xi_1}{1-\xi_1}}\beta^G < \beta^G. \quad (9)$$

The proof is in Appendix A.3. The constant ξ_1 is the price/dividend ratio of a one-period output strip, a claim to GDP next year. The expected return on this output strip is given by $\mathbb{E}_t\left[R^Y_{t+1}\right] = \frac{\exp(\mu+\sigma^2/2)}{\exp(-\rho-\gamma\sigma+\mu+\sigma^2/2)} = \exp(\rho+\gamma\sigma)$. Hence, the (log of the multiplicative) output risk premium is constant and equal to $\gamma\sigma$. Since spending is a constant fraction of output, the risk premium on the spending claim equals that of the output claim: $\mathbb{E}[R^G - R^f] = \mathbb{E}[R^Y - R^f]$. The beta of the spending claim equals the beta of the output claim: $\beta^G = \beta^Y > 0$.

The investor in government debt is long in a tax revenue claim and short in a spending claim. To make the debt risk-free, as long as the debt/output ratio d is positive, we need to render the government tax revenue process safer than the spending process. A positive d implies the fraction $\frac{x\frac{\xi_1}{1-\xi_1}}{d+x\frac{\xi_1}{1-\xi_1}}$ is between 0 and 1, which requires the return on the tax claim to be less risky than the return on the output claim: $0 < \beta^T < \beta^Y$. When output falls, tax revenues must fall by less. The tax rate increases. In other words, there is no scope to insure taxpayers. As the debt/output ratio d increases, the government needs to make the tax revenue increasingly safe. The tax claim is really a portfolio of a claim to government spending and risk-free debt. The larger the debt/output ratio d, the safer the tax claim needs to be. As the debt/output ratio approaches infinity, the beta of the tax claim tends to 0.

3.3 Counter-cyclical Debt-Output

The previous section showed that there is no scope for insuring taxpayers at any horizon in the presence of permanent output shocks when the debt/output ratio is constant. Next, we assume that the government commits to a state-contingent policy for the debt/output ratio which features persistence and counter-cyclicality. Can the government systematically issue more risk-free debt, instead of raising taxes, when the economy is hit by an adverse shock, in order to provide more insurance to taxpayers? We show below that insisting on risk-free debt continues to impose tight constraints on surpluses.

Assumption 3. *The government commits to a policy for the debt/output ratio $d_t = D_t/Y_t$ given* *by:*

$$\log d_t = \phi_0 + \sum_{p=1}^{P} \phi_p \log d_{t-p} - \lambda \varepsilon_t - \frac{1}{2}\lambda^2,$$

where $\lambda > 0$ so that the debt-output ratio increases in response to a negative output shock ε_t.

All results from Section 2 continue to hold and are a straightforward generalization of the results from the constant debt/output ratio case of Section 3.2. The value of the spending claim is unchanged. The value of the tax claim now depends on the time-varying debt/output ratio d_t:

$$P_t^G = x\frac{\xi_1}{1-\xi_1}Y_t, \qquad P_t^T = \left(d_t + x\frac{\xi_1}{1-\xi_1}\right)Y_t.$$

The tax claim's conditional beta satisfies: $\beta_t^T = \frac{x\frac{\xi_1}{1-\xi_1}}{d_t + x\frac{\xi_1}{1-\xi_1}}\beta_t^G$. With positive debt outstanding ($d_t > 0$), $\beta_t^T < \beta_t^G$. Taxpayer insurance possibilities remain limited.

An empirically realistic description of the debt/output ratio in the U.S. data is an AR(2) process for its logarithm:

$$\log d_t = \phi_0 + \phi_1 \log d_{t-1} + \phi_2 \log d_{t-2} - \lambda \varepsilon_t - \frac{1}{2}\lambda^2. \tag{10}$$

We derive the results for this case of $P = 2$, but note that all results go through for the more general version of Assumption 3.

We can restate (10) as follows:

$$\log d_t = \bar{d} + \phi_1(\log d_{t-1} - \bar{d}) + \phi_2(\log d_{t-2} - \bar{d}) - \lambda \varepsilon_t.$$

The expected value of the debt/output ratio at some future date $t + j$ equals:

$$\mathbb{E}[d_{t+j}] = \exp\left(\bar{d} + \psi_{1,j}(\log d_t - \bar{d}) + \psi_{2,j}(\log d_{t-1} - \bar{d}) + \frac{1}{2}\lambda^2 \sum_{k=0}^{j-1} \psi_{1,k}^2\right),$$

where the autocorrelation coefficients are defined recursively:

$$\begin{aligned} \psi_{1,j} &= \phi_1 \psi_{1,j-1} + \phi_2 \psi_{1,j-2}, \\ \psi_{2,j} &= \phi_1 \psi_{2,j-1} + \phi_2 \psi_{2,j-2}, \end{aligned} \tag{11}$$

initialized at $\psi_{1,0} = 1$, $\psi_{1,1} = \phi_1$, $\psi_{2,0} = 0$, and $\psi_{2,1} = \phi_2$.

We consider two cases for the debt/output process. First, if the roots of the characteristic equation $1 - \phi_1 z - \phi_2 z^2 = 0$ lie outside the unit circle, then the debt/output process is stationary. Second, if one or both roots are smaller than one, then the debt/output process is a random walk (non-stationary). In both cases, a positive λ means that the debt/output ratio increases when the shock ε_t is negative, implying a counter-cyclical debt policy.

Under what conditions is the transversality (TVC) satisfied? How persistent and counter-cyclical can debt be without violating TVC?

Proposition 4. *Under Assumptions 1 and 3 with $P = 2$,*
(a) If the roots of the characteristic equation $1 - \phi_1 z - \phi_2 z^2 = 0$ lie outside the unit circle, the TVC condition is satisfied if and only if $\log(\xi_1) = \mu - \rho + \sigma^2/2 - \gamma\sigma < 0$.
(b) If one or both roots are smaller than one, then the TVC condition is satisfied if and only if: $\log(\xi_1) + \lambda(\gamma - \sigma) = \mu - \rho + \sigma^2/2 - \gamma\sigma + \lambda(\gamma - \sigma) < 0$.

The proof is in Appendix A.4.

For the stationary case, the TVC is satisfied whenever the price-dividend ratio of a claim to next period's output is less than one. That is, when investors are willing to pay less than Y_t today for a claim to Y_{t+1}. This requires the risk-adjusted discount rate to exceed the growth rate of GDP (modulo a Jensen adjustment). This condition can be satisfied even when $\rho < \mu$, as long as the risk premium $\gamma\sigma$ is large enough. In this case, the government can run steady-state deficits, even though the TVC is satisfied. By raising taxes in bad time, it keeps the debt risk-free. The average deficit is not a free lunch: the larger the steady-state deficits, the larger the increase in taxes required in bad times. Taxpayers shoulder the burden for the insurance provision to bondholders. As such, the average deficit can be understood as the compensation to the taxpayers for their insurance provision and risk taking.

For the random walk case, the same condition ensures that the TVC is satisfied when the government does not pursue counter-cyclical stabilization ($\lambda = 0$). When the government does pursue counter-cyclical stabilization ($\lambda > 0$), then the TVC is only satisfied if:

$$\gamma\sigma - \lambda\gamma + \lambda\sigma > \mu - \rho + \frac{1}{2}\sigma^2 \Leftrightarrow \lambda < \frac{\rho + \gamma\sigma - \mu - \frac{1}{2}\sigma^2}{\gamma - \sigma}.$$

The left-hand side of the first inequality is now lower than before (when $\lambda = 0$) when the Sharpe ratio of the economy exceeds the volatility of output ($\gamma > \sigma$). When debt issuance is sufficiently counter-cyclical, $\lambda > \sigma$, the left-hand side is decreasing in the maximum Sharpe ratio γ. For high enough γ, the TVC is violated. Intuitively, when investors are risk averse enough, the insurance provided by the counter-cyclical debt issuance policy makes government debt a terrific hedge. The price of a claim to the debt outstanding in the distant future $d_{t+T}Y_{t+T}$ fails to converge to zero. This is the first important insight contributed by asset pricing theory. If output is subject to permanent, priced risk and we want to rule out arbitrage opportunities, then there have to be limits to the government's ability to pursue counter-cyclical debt issuance. This bound on λ is shown in the second inequality. When the government exceeds this bound, it has granted itself an arbitrage opportunity.

3.4 Model Calibration

Table 1 proposes a calibration of the model that matches basic features of post-war U.S. data. We set γ to 1.29, which measures the maximum annual Sharpe ratio in the economy. The asset pricing literature suggests that this is a reasonable value given high average excess returns on a broad set of risky assets. The mean and the standard deviation of annual output growth are set to their empirical counterparts for real GDP growth between 1947 and 2023: $\mu = 3.00\%$, and $\sigma = 2.35\%$. The real risk-free rate is set to its sample average: $r = 1.62\%$, obtained from the annual yield on the five-year Treasury from 1961–2023. Spending accounts for a fraction $x = 17.56\%$ of GDP in post-war data. We note that this calibration features a risk-free rate below the growth rate of output. However, the TVC is satisfied because the growth rate is still below the risk-adjusted discount rate: $\mu - r + \frac{1}{2}\sigma^2 - \gamma\sigma = \log(\xi_1) = -1.62\% < 0$, with an output risk premium $\gamma\sigma$ of 3.00%.

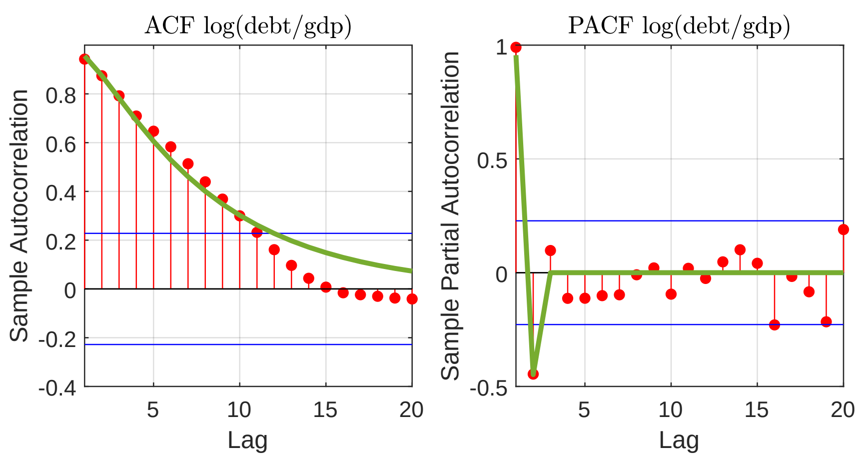
To calibrate the debt/output process, Figure 1 reports the sample autocorrelation function (ACF) and partial autocorrelation function (PACF) of the log government debt/output ratio (red dots). They are estimated on the post-war U.S. sample (1947–2023). The PACF function indicates that an AR(2) process fits the data well. Lags beyond two years in the PACF are not statistically different from zero. We estimate the coefficients of the AR(2) process by matching four moments: the autocorrelation of order 1, the autocorrelation of order 10, the partial autocorrelation of order 1, and the partial autocorrelation of order 2. We minimize the sum of the squared differences between the empirical and model-implied moments. The point estimates for ϕ_1 and ϕ_2 are 1.38 and -0.45, respectively, as listed in panel B of Table 1. Both roots lie outside the unit circle, so that the debt/output process is stationary. The green line in Figure 1 shows that the estimated AR(2) process

Table 1: Benchmark Calibration for U.S.

Panel A: Preferences and Output Dynamics		
γ	1.29	maximum annual Sharpe ratio
r	1.62%	real risk-free rate
μ	3.00%	mean of growth rate of output
σ	2.35%	std. of growth rate of output
$rp = \gamma\sigma - \frac{1}{2}\sigma^2$	3.00%	GDP risk premium in log
Panel B: Debt/Output Ratio Dynamics		
λ	$2.53 \times \sigma$	sensitivity of debt/output to output innovations
$d = \exp\{\phi_0/(1-\phi_1-\phi_2)\}$	37.92%	mean of debt/output
ϕ_1	1.38	first-order coeff of debt/output
ϕ_2	-0.45	second-order coeff of debt/output
Panel C: Government Spending/Output Ratio Dynamics		
b^g	$1.42 \times \sigma$	sensitivity of spending/output to output innovations
φ_1^g	0.86	AR(1) coeff of spending/output
$x = \exp\left\{\varphi_0^g/(1-\varphi_1^g)\right\}$	17.56%	mean of govt. spending/output

provides a good fit to the U.S. post-war data.

Figure 1: Autocorrelation in Debt/Output



The figure plots the sample autocorrelation and partial autocorrelation of the U.S. log government debt/output ratio. The sample is annual from 1947 until 2023. The estimates in the data are denoted by red dots and the model-implied moments are denoted by the solid green line. The parameters are listed in panel B of Table 1.

We set ϕ_0 to match the post-war mean of the debt/output ratio of 37.92%. Finally, we set $\lambda = 2.53 \times \sigma$ to match the slope coefficient in a regression of the debt/output ratio innovations on the GDP growth rate in the post-war U.S. sample. A one percentage point increase in GDP growth lowers the debt/output ratio by 2.53 percentage points.

We use this calibration to demonstrate the quantitative implications in the rest of the paper.

4 The Dynamics of Implied Surpluses

We now characterize the constraints that risk-free debt imposes on the surplus process. We begin by characterizing expected future surpluses in Section 4.1. Then we characterize the response of realized surpluses to an adverse output shock in Section 4.2. These moments are particularly powerful because they do not depend on the properties of the SDF (other than through the risk-free rate), but only on the deviation of the current debt/output ratio from its long-run mean. We then go further and characterize the constraint that risk-free debt places on the coveriance of expected future cumulative (discounted) surpluses over an intermediate horizon h with output shocks in Section 5.

4.1 Implied Expected Surpluses

The surplus/output ratio in period $t+j$ for $j \geq 1$ is given by:

$$s_{t+j} = \frac{S_{t+j}}{Y_{t+j}} = d_{t+j-1} \exp(\rho - \mu - \sigma \varepsilon_{t+j}) - d_{t+j}.$$

The following proposition characterizes the expected future surpluses and their sensitivity to the debt/output ratio under the AR(2) specification for the debt/output ratio.

Proposition 5. *If Assumptions 1 and 3 hold with $P = 2$,*

$$\mathbb{E}_t[s_{t+j}] = \mathbb{E}_t[d_{t+j-1}] \left[\exp(\rho - \mu + \sigma^2/2) - \exp(\log \mathbb{E}_t[d_{t+j}] - \log \mathbb{E}_t[d_{t+j-1}]) \right], \quad (12)$$

and

$$\frac{\partial \mathbb{E}_t[s_{t+j}]}{\partial(\log d_t - \bar{d})} = \psi_{1,j-1} \mathbb{E}[s_{t+j}] + (\psi_{1,j-1} - \psi_{1,j}) \mathbb{E}[d_{t+j-1}],$$

where

$$\mathbb{E}_t[d_{t+j-1}] = \exp\left(\bar{d} + \psi_{1,j-1}(\log d_t - \bar{d}) + \psi_{2,j-1}(\log d_{t-1} - \bar{d}) + \frac{1}{2}\lambda^2 \sum_{k=0}^{j-2} \psi_{1,k}^2 \right).$$

The proof is presented in Appendix A.5 and the autocorrelation coefficients are defined in (11). Surprisingly, the expression for the expected surplus does not depend on the market price of risk $Var_t[M_{t+1}]/\mathbb{E}_t[M_{t+1}]$, governed in our asset pricing model by the parameter γ. Rather, it only depends on the risk-free rate, the moments of the growth rate of output, and the deviations of the debt/output ratio from its long-run mean. Our proof would still go through for a more complicated pricing kernel. The pricing kernel only matters for expected future surpluses through its effect on the risk-free rate in the model where debt is risk-free.

Consider the case where the risk-free equals the expected growth rate in levels ($\rho = \mu - \sigma^2/2$). Then, whenever the debt/output ratio exceeds it long-run mean, investors expect positive surpluses at all horizons. As the horizon grows large $j \to \infty$, $\psi_{1,j}$, $\psi_{2,j} \to 0$, the expected surplus shrinks to its steady-state value $\exp\left(\overline{d}\right)(\exp(\rho - \mu + \sigma^2/2) - 1)$. This expression is zero when the risk-free equals the expected growth rate in levels. The higher the persistence of the debt process, the slower the decay in expected future surpluses as the horizon j increases.

When the debt exceeds its long-run mean and the debt is risk-free, the only way to bring the debt/output ratio back to its mean is to run large surpluses. Only the gap between the long-run real risk-free rate and the expected growth rate of output matters. Consider again the case in which the steady-state surplus is zero ($\rho = \mu - \sigma^2/2$). Then

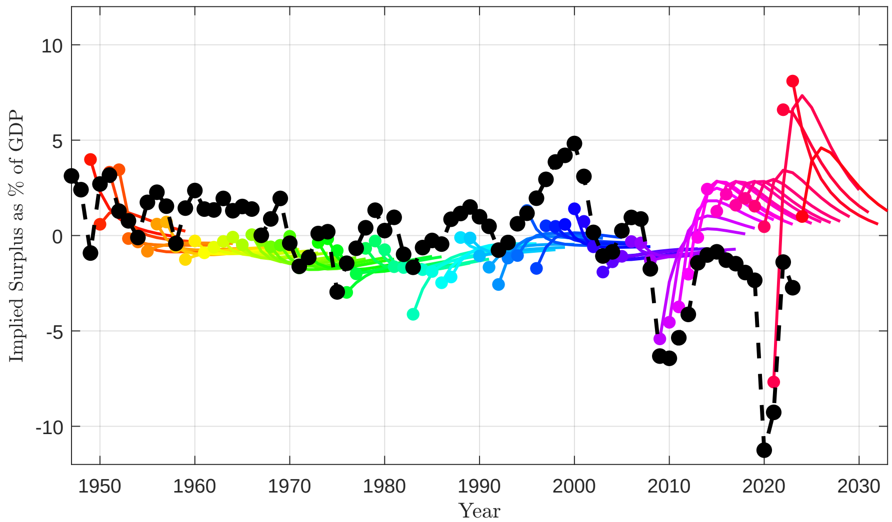
$$\mathbb{E}_t[s_{t+j}] = \mathbb{E}_t[d_{t+j-1}] \left[1 - \exp(\log \mathbb{E}_t[d_{t+j}] - \log \mathbb{E}_t[d_{t+j-1}])\right].$$

The faster the rate at which the debt reverts back to its mean, the larger the surpluses required.

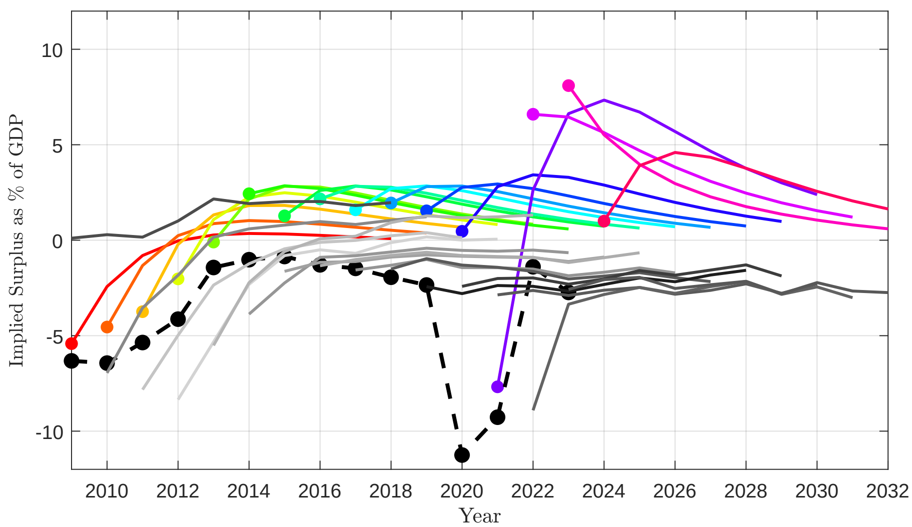
When debt is risk-free, the debt/ouptut ratio in the current and the past year are the only predictors of future surpluses. This implication is counterfactual. There is no evidence in the US data that a high debt/output ratio predicts high primary surpluses (Jiang et al., 2024c).

To illustrate the implications of this proposition in the data, we feed in the realized debt/output ratio time-series for the U.S. and, in each year, compute the expected surpluses for the following ten years $\mathbb{E}[s_{t+j}]$ for $j = 1, \ldots, 10$ from the expression in Proposition 5 and the parameters in Table 1. Figure 2 plots these implied future surpluses as a fraction of output $\mathbb{E}[s_{t+j}]$ as the colored line segments. It also plots the realized surplus/output ratio as the dashed black line. The debt/output ratio in the U.S. has exceeded its unconditional mean since 1991, $\log d_t - \bar{d} > 0$. Since then, the risk-free debt model im-

Figure 2: Expected Surpluses



(a) Full Time-Series



(b) Since the GFC

Panel (a) plots the model-implied expected future primary surpluses as a fraction of GDP ($\mathbb{E}[s_{t+j}]$) in Prop. 5 evaluated for the next 10 years given the actual debt/GDP ratio (solid line), and the realized surpluses (s_{t+j}) (dashed line). The parameters are given in Table 1. Panel (b) zooms in the post-GFC period, and includes the CBO forecasts in greyscale.

plies expected surpluses that must exceed the steady-state surplus. The expected surplus is larger the farther the economy is from the steady-state debt/output ratio $\bar{d}$. The higher the debt level $\mathbb{E}[d_{t+j-1}]$, the higher the additional surplus required for a given increase in debt since the autocorrelation coefficients $\psi_{1,j-1} - \psi_{1,j} > 0$ decline for high enough j, as shown in the second part of proposition 5. These surpluses are necessary to push the debt/output ratio back towards its long-run mean.

Since the GFC, a wide gap has opened up between the expected surpluses implied by risk-free debt and the realized surpluses, as shown in the bottom panel of Figure 2. Take the COVID-19 pandemic as an example. The primary deficit was 11% of GDP in 2020, 9.3% in 2021, and 1.4% in 2022. At the end of 2022, the marketable debt/output ratio stood at 83%, down from a peak of 103% in 2020, but well above its long-run mean. To keep the debt risk-free, the model prediction at the end of 2022 is that the U.S. government should be running a primary surpluses of 8.1% of GDP in 2023. Thereafter, the predicted surplus gradually falls to 0.6% by 2032.

In sharp contrast with the model predictions, the primary surplus was −2.7% of GDP in 2023. That's a gap of 11% between the realized and the predicted surplus. The CBO projects a primary surplus of -1.9% of GDP in 2024. Thereafter, primary surpluses are expected to fall further as Social Security, Medicare, and Medicaid expenditures rise. The grey line segments in the bottom panel display the 10-year CBO projections at each point in time. The key observation is that both realized and CBO-projected primary surpluses are far from the surpluses required to keep the debt safe. Keeping the debt risk-free after 2022 requires running cumulative primary surpluses of 28% over the decade from 2023 until 2032. Over this same period, the CBO projects a cumulative *deficit* of 24%. The gap in the surplus/output ratio between the model and the CBO projections is 5.3% per annum.

Appendix B shows that the differences between the surpluses implied by the risk-free debt model, on the one hand, and both the realized surpluses and the CBO-projected surpluses, on the other hand, are statistically significantly different from zero. The surplus data strongly reject the risk-free debt model.

As a robustness check, Appendix B also shows that the gap between model-implied and realized surpluses remains largely the same when we allow for the gap between the risk-free rate and the growth rate of the economy to vary over time. Specifically, using the CBO's projections for future interest rates and GDP growth rates rather than a constant $\mu - \rho$ in (12), results in similar model-implied surpluses.

4.2 Impulse Responses of Realized Surpluses

How much latitude does a government have to stabilize the economy by lowering taxes in response to an adverse output shock if it wants to keep the debt safe? The answer is given by the impulse-response functions (IRF) of the surpluses with respect to an output shock. The IRF can be computed in closed form and does not depend on the SDF.

Proposition 6. *If Assumptions 1 and 3 hold with $P = 2$, the TVC is satisfied, and $\rho = \mu$, the IRF of the surplus output ratio, evaluated at $\varepsilon_\tau = 0$ for $\tau \leq j$, is given by:*

$$\begin{aligned} \frac{\partial s_{t+j}}{\partial \varepsilon_{t+1}} &= (\lambda - \sigma)\exp(\bar{d}), && \textit{for j=1,} \\ &= \lambda(\phi_1 - 1)\exp(\bar{d}), && \textit{for j=2,} \\ &= \lambda(\psi_{1,j-1} - \psi_{1,j-2})\exp(\bar{d}), && \textit{for } j \geq 3. \end{aligned}$$

The proof is in Appendix A.6. The definition of the autocorrelation coefficients $\psi_{1,j}$ is given in (11). This result can easily be generalized to any AR(P) process, with appropriately redefined autocorrelation coefficients $\psi_{1,j}$.

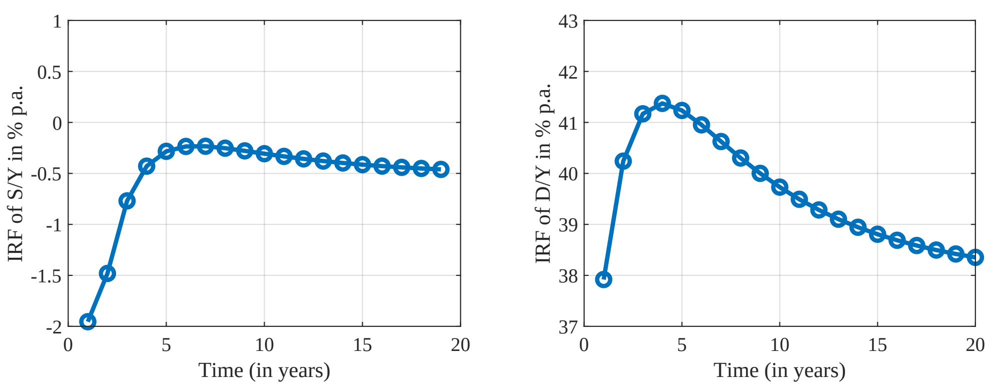
In the year of the shock, the derivative is positive as long as the debt is counter-cyclical enough: $\lambda > \sigma$. That is, a negative shock to output is countered by debt issuance that is sufficiently large to deliver a primary deficit in the initial year without jeopardizing the risk-free nature of the debt. In the second year, the government can run another primary deficit following an adverse shock when $\phi_1 > 1$. In the case of a stationary AR(1), this is not feasible, but it is the case for the stationary AR(2) process for debt/output we estimated in the U.S. data.

The government may be able to run a third year of deficits if $\psi_{1,2} - \psi_{1,1} > 0$ or equivalently if $\phi_1^2 + \phi_2 > \phi_1$. For our parameter choices, this condition is satisfied, but the sign generally depends on parameter values. The surplus response two years after the shock is always smaller than the response one year after the shock. In other words, the government's ability to run a third year of deficits in response to the negative output shock is either limited or gone. The IRF flips sign either 2 or 3 years after the shock ($j = 3$ or $j = 4$). The derivative remains negative thereafter and shrinks in absolute value as j increases. The government must revert to running persistently higher primary surpluses.

The left panel of Figure 3 illustrates the response of the surplus/output ratio to a negative shock to output for our benchmark calibration. We recall this calibration features a primary deficit in the steady-state. The deficit/output ratio in the year of the shock (year 1) is followed by another sizeable deficit in year 2. However, the deficit must shrink dramatically in year 3 and turn into a lower-than-average deficit starting in year 4 and beyond, before eventually returning to the steady-state value. The right panel shows the corresponding debt dynamics, which display a hump-shaped response. A state-contingent and persistent debt issuance policy enables the government to delay the fiscal adjustment. Returning the debt to its long-run mean requires generating higher-than-average surpluses three and more years after the shock.

In summary, keeping the debt risk-free still imposes severe restrictions on the primary surplus dynamics. Running sizeable primary deficits for more than two years is incompatible with risk-free debt. The observed fiscal responses to large adverse shocks, such as the GFC and the Covid-19 pandemic, of large and sustained deficits are inconsistent with the surplus responses predicted by a model of risk-free government debt.

Figure 3: IRF of Surplus/Output and Debt/Output in Model



The figure plots the Impulse Response Function of the surplus/output ratio S/Y (left panel) and the debt to output ratio D/Y (right panel). The parameters are given in Table 1.

5 Covariance of Implied Surpluses with Aggregate Shocks

The previous section characterized the response of realized surpluses to an adverse output shock. These results do not depend on the properties of the pricing kernel, other than the risk-free rate. In this section, we analyze the covariance of expected future surpluses over a finite horizon with output shocks when the debt is to remain risk-free. We refer to this covariance as the cash flow beta of surpluses. In the presence of permanent shocks, the government can only insure taxpayers over a limited period of time; the cash-flow beta of surpluses can only be positive over short horizons. The results in this section depend on the properties of the pricing kernel.

When the TVC is satisfied, the debt return innovation reflects news about the present discounted value of future government surpluses:

$$D_t(\mathbb{E}_{t+1} - \mathbb{E}_t)R^D_{t+1} = (\mathbb{E}_{t+1} - \mathbb{E}_t)[\sum_{j=1}^{\infty} M_{t+1,t+j}S_{t+j}].$$

When the debt is risk-free, the left-hand side of the above expression is zero, and there is no news about future surpluses:

$$(\mathbb{E}_{t+1} - \mathbb{E}_t)\sum_{j=1}^{\infty} M_{t+1,t+j}S_{t+j} = 0.$$

This puts tight restrictions on risk-adjusted surpluses. We now explore these restrictions over finite horizons.

5.1 Cash-Flow Betas with Risk-free Debt

We define the conditional beta of a generic h-period stream of discounted cash flows Z as:

$$\beta_t^{Z,CF}(h) \equiv -\frac{cov_t\left(M_{t+1}, \mathbb{E}_{t+1}\sum_{j=1}^{h} M_{t+1,t+j}Z_{t+j}\right)}{D_t var_t(M_{t+1})}.$$

We refer to this object as the cash-flow beta for short. The cash-flow beta converges to the return beta (from Section 2) as the horizon h goes to infinity.

The cash-flow beta of the discounted sum of surpluses over the next h periods, $\beta_t^{S,CF}(h)$, is a sufficient statistic for how much insurance the government can provide to taxpayers over the next h periods. The following proposition states that, when the current debt D_t is risk-free, the risk properties of the surpluses over a given horizon h are completely determined by riskiness of the debt h periods hence.

Proposition 7. *Under Assumptions 1 and 3 with $P = 2$, when debt is risk-free, $\beta_t^{S,CF}(h)$ is equal to the beta of the debt h periods from now:*

$$\beta_t^{S,CF}(h) = \frac{cov_t\left(M_{t+1}, \mathbb{E}_{t+1}M_{t+1,t+h}D_{t+h}\right)}{D_t var_t(M_{t+1})}$$

$$= \frac{\mathbb{E}_t[M_{t+1}]}{D_t var_t(M_{t+1})} \mathbb{E}_t[M_{t+1,t+h} d_{t+h} Y_{t+h}](\exp\{\gamma(\psi_{1,h-1}\lambda - \sigma)\} - 1).$$

and $sign\left(\beta_t^{S,CF}(h)\right) = sign\left(\gamma(\psi_{1,h-1}\lambda - \sigma)\right).$

The proof is in Appendix A.7. This result can easily be generalized to any higher-order autoregressive process for debt, with $\psi_{1,j}$ denoting the appropriately-redefined autocorrelation function. It immediately follows form this proposition that, when the debt has a negative (positive) risk premium, $\beta^D < 0$ ($\beta^D > 0$), the cash flow beta of the surplus is smaller (greater) than the beta of the h-period debt strip.

Analogously, we define the cash-flow beta of discounted government spending and of tax revenues over a horizon h. We do so for a richer process for the spending to output ratio than we have considered hitherto:

$$\log x_t = \varphi_0^g + \varphi_1^g \log x_{t-1} - b_g \varepsilon_t - \frac{1}{2} b_g^2. \tag{13}$$

The parameter estimates for the U.S. postwar data are in Panel C of Table 1. The positive estimate for b_g indicates that the spending/output ratio is counter-cyclical in the data.

Corollary 2. *Under the assumptions of proposition 7 and the government spending process* (13)*, the cash-flow betas* $\beta_t^{G,CF}(h)$ *and* $\beta_t^{T,CF}(h)$ *satisfy:*

$$\beta_t^{G,CF}(h) = -\sum_{j=1}^{h} \frac{\mathbb{E}_t[M_{t+1}]}{D_t var_t[M_{t+1}]} \mathbb{E}_t[M_{t+1,t+j} x_{t+j} Y_{t+j}](\exp\left\{\gamma((\varphi_1^g)^{j-1} b_g - \sigma)\right\} - 1).$$
$$\beta_t^{T,CF}(h) = \beta_t^{S,CF}(h) + \beta_t^{G,CF}(h).$$

The proof is in Appendix A.8. The properties of the $\beta_t^{G,CF}(h)$ depend on the persistence and cyclicality of the exogenous spending/GDP process. The properties of $\beta_t^{T,CF}(h)$ depend on the risk properties of both the debt claim and the spending claim.

Constant Debt-Output To build intuition for the result in proposition 7, we return to the simple case of a constant debt/output ratio: $\lambda = 0$. Proposition 7 then implies:

$$\beta_t^{S,CF}(h) = \frac{\mathbb{E}_t[M_{t+1}]}{D_t var_t[M_{t+1}]} \mathbb{E}_t[M_{t+1,t+h} d Y_{t+h}](\exp\{-\gamma\sigma\} - 1). \tag{14}$$

The cash-flow beta of the surplus is negative for all horizons h since $\gamma\sigma > 0$. This means that in bad times, the future (discounted) surplus/output ratios go up. When spending/output is constant (or also goes up in bad times), future tax revenues/output must go up. The government cannot insulate taxpayers from adverse output shocks. Rather, the taxpayers insure the bondholders.

Panel A of Figure 4 plots the risk premium on a claim to cumulative surpluses over the next h periods in the left panel (red circles). This risk premium equals the surplus beta multiplied by the market price of risk, $\beta_t^{S,CF}(h) \times \frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]}$. The cumulative risk premium at horizon h is the sum of the individual strip risk premia up until horizon h. The negative risk premium over short horizons indicates that short-run surpluses are a hedge. Since taxpayers are short the surplus claim (they pay the taxes and receive the transfer spending), their tax-minus-transfer liability is risky. When the debt/output ratio is constant and there is no possibility to raise the debt in response to an adverse shock, the surplus/output ratio must rise on impact. This makes the one-period surplus claim a hedge. The year-2 surplus claim in contrast earns a small positive risk premium, reflecting the underlying output risk, so that the cumulative 2-period surplus risk premium is higher than the 1-period surplus risk premium. As $h \to \infty$, the sum of discounted surpluses converges to the current value of debt D_t. Insisting on risk-free debt ($\beta_t^D = 0$) implies that $\beta_t^{S,CF}(h) \to 0$ as $h \to \infty$. The red line in the left panel converges to zero from below for large h.

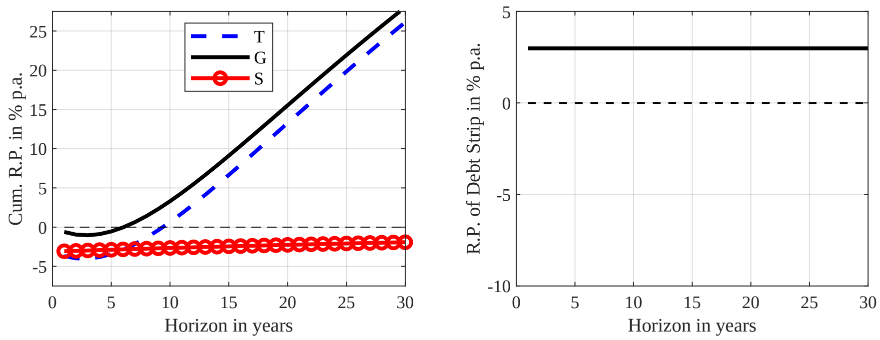
The solid black line in the left panel plots the risk premium on the claim to cumulative government spending over the next h periods. It equals the cash-flow beta of the h-period spending claim multiplied by the market price of risk. Since the spending/output dynamics are exogenously given, the spending beta does not depend on the debt policy. The counter-cyclical nature of spending/output ($b^g < 0$) makes the risk premium negative at short horizons. At longer horizons, the spending risk premium turns positive reflecting the long-run output risk in the spending claim, since the spending/output ratio is stationary.

The extent of taxpayer insurance is captured by $\beta_t^{T,CF}(h)$. The blue dashed line in the left panel plots $\beta_t^{T,CF}(h)$ multiplied by the market price of risk, the risk premium on a claim to the next h periods of tax revenue. When this risk premium is negative, taxpayers are providing insurance rather than receiving insurance. The risk premium is negative until about year 10 for our parameters, after which it turns positive. The positive risk premium on longer-dated tax strips reflects cointegration between tax revenues and output and a positive risk premium for output risk. The tax beta $\beta_t^{T,CF}(h)$ in the left panel is below the spending beta $\beta_t^{G,CF}(h)$ at all horizons. As $h \to \infty$, these cash-flow betas

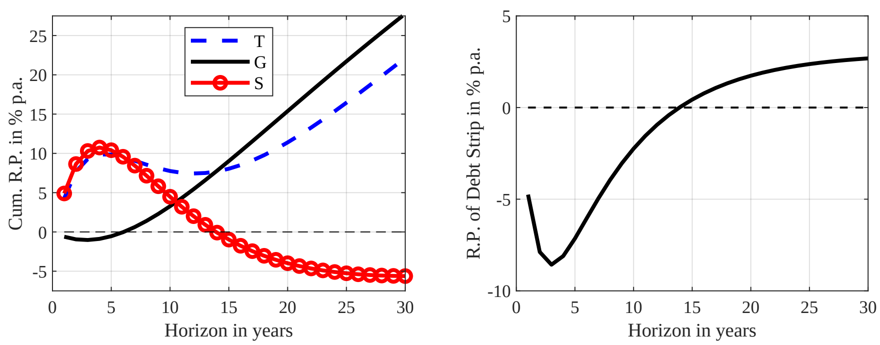
Figure 4: Risk Premia Across Horizons

The figure plots the risk premium of cumulative discounted cash flows, $\beta_t^{i,CF}(h) \times \frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]}$, in the left panel against the horizon h, for $i \in \{S, G, T\}$. The right panel plots the risk premium on the debt strips in (15). The parameters are given in Table 1, and the risk premia are evaluated at the long-run mean log spending/output and log debt/output ratios. Panel A considers the special case where the debt/output ratio is constant ($\lambda = 0$).

Panel A: Constant Debt/Output ($\lambda = 0$)



Panel B: $AR(2)$ Debt/Output ($\phi_1 = 1.38, \phi_2 = -0.45$)



converge to the return betas β_t^T and β_t^G. As we discussed in Corollary 1, $\beta_t^T < \beta_t^G$ was the condition to keep the debt risk-free.

On the right-hand side of Panel A, we report the risk premium on the debt strips. Specifically, it is the valuation of the debt strip scaled by its expectation at time t and multiplied by the market price of risk $\frac{var_t(M_{t+1})}{\mathbb{E}_t[M_{t+1}]}$:

$$RP_t^{Dstrip}(h) = -\frac{cov_t\left(M_{t+1}, \frac{\mathbb{E}_{t+1}[M_{t+1,t+h}D_{t+h}]}{\mathbb{E}_t[M_{t+1,t+h}D_{t+h}]}\right)}{var_t(M_{t+1})} \frac{var_t(M_{t+1})}{\mathbb{E}_t[M_{t+1}]}. \tag{15}$$

When the debt/output ratio is a constant, this risk premium on the debt strip is also a constant, given by $1 - \exp\{-\gamma\sigma\}$, where $\gamma\sigma > 0$ is the output risk premium. By Proposition 7, the risk premium on the h-period debt strip is inversely related to the risk premium on the cumulative surpluses over the next h periods. To offset the output risk in the debt strips, the risk premium on the surplus has to be negative. The surplus claim must be safe for the government, risky for the taxpayers.

AR(2) for Debt/Output In our preferred case of an AR(2) for debt/output, the sign of the cash flow beta of the surplus is determined by $\gamma(\psi_{j-1}\lambda - \sigma)$. If $\lambda > \sigma$, the initial surplus beta is positive (since $\psi_0 = 1$). The second surplus beta is larger still since $\psi_1 = \phi_1 > 1$. The third surplus beta remains positive and is larger than the second beta if $\psi_2 > \psi_1$ or $\phi_1(\phi_1 - 1) + \phi_2 > 0$. This condition is satisfied for our point estimates $\phi_1 = 1.38$ and $\phi_2 = -0.45$. For these parameter values, the fourth surplus beta is lower than the third, the fifth lower than the fourth, etc. Eventually the surplus cash-flow beta crosses over into negative territory. Panel B of Figure 4 shows this occurs in year 13.

The cash-flow betas for government spending are unaffected by the debt dynamics and the same as in the case of constant debt/output. The cash-flow betas for tax revenues follow a similar pattern as those for the surplus at short horizons. At longer horizons, the stationarity of the tax revenue/output ratio and the long-run output risk dictate the positive risk premium on the tax claim.

In sum, the positive surplus betas for horizons up to 13 years indicate that taxpayers can be temporarily insulated from adverse output shocks when debt/output follows an AR(2) process with counter-cyclical debt issuance. Risk premia on debt strips, shown in the right panel, are negative for 13 years. The slow expansion and repayment of the debt in response to an adverse shock allows the government to postpone fiscal rectitude. The cumulative surplus can be risky over a horizon h, providing insurance to the taxpayer, only if this risk is offset by the safety of debt issuance at time $t + h$. But as h increases, the expression $\gamma(\sigma - \psi_{h-1}\lambda)$, which controls the debt strip risk premium, turns positive and converges to $\gamma\sigma$, the risk premium on the output strip. Insurance provision to the tax-payer is necessarily short-lived because of the long-run risk in the debt strips. Far-out surpluses inherit the permanent output risk.

5.2 Escaping the Trade-off with Government Savings

So far, we have analyzed the case where the government borrows ($D > 0$). When, instead, the government saves ($D < 0$), it can insure taxpayers at all horizons and escape the trade-off.

We consider a government which saves at the risk-free rate. We use the following stochastic AR-process in logs:

$$\log d_t = \phi_0 + \phi_1 \log d_{t-1} + \phi_2 \log d_{t-2} + \lambda \varepsilon_t - \frac{1}{2}\lambda^2, \tag{16}$$

where λ now enters with a positive sign. Savings in levels is given by: $-D_t = -\exp(\log d_t)$. The results in Proposition 7 go through. Because $D < 0$, the government now has a short position in permanent output risk because the value of taxes is smaller than the value of spending. As a result, to manufacture risk-free savings, the surpluses have to contribute enough long-run output risk.

In the simplest case in which the savings/output ratio is constant ($\lambda = 0$), Proposition 7 implies:

$$\beta_t^{S,CF}(h) = \frac{\mathbb{E}_t[M_{t+1}]}{-D_t var_t[M_{t+1}]} \mathbb{E}_t[M_{t+1,t+h} dY_{t+h}](1 - \exp\{-\gamma\sigma\}). \tag{17}$$

The sufficient statistic for taxpayer insurance possibilities, $\beta_t^{S,CF}(h)$, is positive at all horizons since $\gamma\sigma > 0$. This means that, the surplus/output ratio declines in bad times. When spending/output is constant (or goes up) in bad times, tax revenues/output must also decline. By lowering tax collections in bad times, the government can insure taxpayers against adverse output shocks at all horizons. In fact, it has to do so, because its savings are risk-free.

In the benchmark case where the savings/output ratio follows an AR(2), the short-run surplus risk premium is around 10% as shown in the left panel of Figure 5. The cumulative surplus risk premium remains positive at every horizon, indicating that surpluses are risky over all horizons. Since taxpayers are short the surplus claim, they receive insurance at every horizon. As $h \to \infty$, the sum of discounted surpluses converges to the current value of savings D_t. Insisting on risk-free savings ($\beta_t^D = 0$) implies that $\beta_t^{S,CF}(h) \to 0$. The red line in the left panel converges to zero from above.

6 Revisiting the Trade-off with Convenience Yields

Some governments are endowed with the ability to issue safe government debt at prices that exceed their fair market value. Typically, the debt of such government serves the role of a special, safe asset for domestic or foreign investors. U.S. Treasuries currently fill the role of the world's safe asset; the U.K. and the Dutch Republic enjoyed that status in earlier eras (Chen, Jiang, Lustig, Van Nieuwerburgh, and Xiaolan, 2024b). We investigate whether the resulting "convenience yields" relax the trade-off between insuring bondholders and taxpayers. In order to collect convenience yields, the government needs to manufacture safe debt. This justifies our emphasis on the $\beta^D = 0$ case throughout this paper (or $\beta^D < 0$, which makes all results stronger).

The convenience yield κ_t is defined as a wedge in the investors' Euler equation for government bonds: $\mathbb{E}_t\left[M_{t,t+1}R^D_t\right] = \exp(-\kappa_t)$.

6.1 The Trade-off With Return Betas over Long Horizons

Let $K_{t+j} = (1 - e^{-\kappa_{t+j}})D_{t+j}$ be the amount of interest the government does not need to pay in period $t + j$ thanks to the convenience yield. The current value of government debt reflects the present value of all convenience yields earned on future debt. The value of the Treasury's seigniorage revenue claim is:

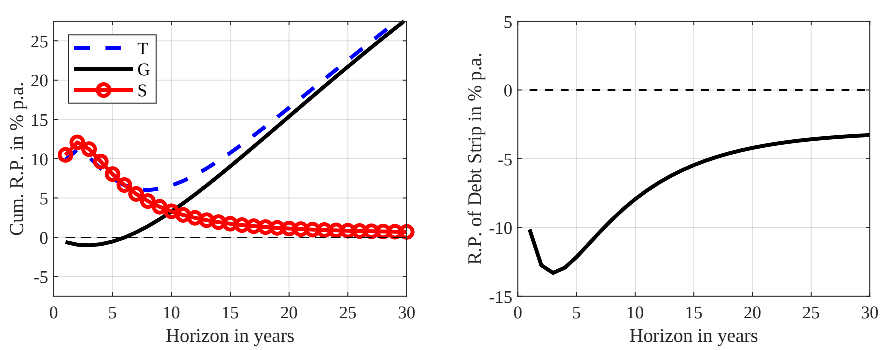
$$P^K_t = \mathbb{E}_t\left[\sum_{j=0}^{\infty} M_{t,t+j}(1 - e^{-\kappa_{t+j}})D_{t+j}\right].$$

Jiang et al. (2024b) show that the value of the government debt equals the sum of the present value of future tax revenues plus future seigniorage revenues minus future gov-

Figure 5: Risk Premia Across Horizons with Saving

The figure plots the risk premium of cumulative discounted cash flows, $\beta^{i,CF}_t(h) \times \frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]}$, in the left panel against the horizon h. The right panel plots minus the risk premium on the debt/savings strips: $1 - \exp\left\{\gamma(\phi^{h-1}\lambda - \sigma)\right\}$. The parameters are given in Table 1, and the risk premia are evaluated at the long-run mean log spending/output and log debt/output ratios.

$AR(2)$ Savings/Output



ernment spending:

$$D_t = \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j} (T_{t+j} + (1 - e^{-\kappa_{t+j}}) D_{t+j} - G_{t+j}) \right] = P_t^T + P_t^K - P_t^G,$$

provided that the TVC for debt holds.

Extending the Modigliani-Miller approach of Section 2 to the world with convenience yields, government debt is equivalent to a portfolio that goes long in the tax revenue claim and the seigniorage revenue claim and short in the spending claim. The government debt risk premium becomes:

$$\mathbb{E}_t \left[R_{t+1}^D - R_t^f \right] = \frac{P_t^T}{D_t} \mathbb{E}_t \left[R_{t+1}^T - R_t^f \right] + \frac{P_t^K}{D_t} \mathbb{E}_t \left[R_{t+1}^K - R_t^f \right] - \frac{P_t^G}{D_t} \mathbb{E}_t \left[R_{t+1}^G - R_t^f \right],$$

where R_{t+1}^D, R_{t+1}^T,R_{t+1}^K and R_{t+1}^G are the returns on the bond portfolio, the tax claim, the seigniorage claim, and the spending claim, respectively.

Taking the government spending and debt return process as given, we explore the implications for the properties of the tax claim. We impose that debt is risk-free ($\beta^D = 0$) because only safe debt earns convenience yields.

Proposition 8. *In the absence of arbitrage opportunities, if the TVC holds and the debt is risk-free ($\beta^D = 0$), then the expected excess return on the tax claim is the unlevered expected excess return on the spending claim and the seigniorage claim:*

$$\mathbb{E}_t \left[R_{t+1}^T - R_t^f \right] = \frac{P_t^G}{D_t + P_t^G - P_t^K} \mathbb{E}_t \left[R_{t+1}^G - R_t^f \right] - \frac{P_t^K}{D_t + P_t^G - P_t^K} \mathbb{E}_t \left[R_{t+1}^K - R_t^f \right].$$

The beta of the tax claim is given by: $\beta_t^T = \frac{P_t^G}{D_t + P_t^G - P_t^K} \beta_t^G - \frac{P_t^K}{D_t + P_t^G - P_t^K} \beta_t^K$.

The proof is a straightforward extension of the proof of Prop. 1 in Appendix A.1. Consider the special case where the convenience yield seigniorage process has a zero return beta ($\beta^K = 0$); the stream of seigniorage revenues is uncorrelated with the SDF. Then the implied beta of the tax revenue process exceeds the beta without seigniorage revenue because $P_t^K > 0$:

$$\beta_t^T = \frac{P_t^G}{D_t + P_t^G - P_t^K} \beta_t^G > \frac{P_t^G}{D_t + P_t^G} \beta_t^G,$$

The higher tax beta means that the presence of convenience yields expands insurance provision to taxpayers.

If the seigniorage revenue beta is negative ($\beta^K < 0$), then the proposition shows that β_t^T is higher still so that even more taxpayer insurance is possible. Conversely, $\beta^K > 0$ shrinks taxpayer insurance possibilities. In sum, the extent to which convenience yields relax the trade-off depends on the properties of the seigniorage revenue stream they generate.

6.2 The Trade-off With Cash-Flow Betas over Short Horizons

We now explore how the trade-off over finite horizons is affected by the presence of convenience yields. We do so under the assumption that seigniorage revenue from convenience is proportional to the debt outstanding.

Assumption 4. *The convenience yield κ is constant.*

We define the cash flow beta of future discounted seigniorage revenue as:

$$\beta_t^{K,CF}(h) \equiv -\frac{(1-e^{-\kappa}) \cdot cov_t \left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t) \sum_{j=1}^{h} M_{t+1,t+j} D_{t+j} \right)}{D_t var_t(M_{t+1})}.$$

Proposition 9. *When debt is risk-free ($\beta_t^D = 0$), then the cash-flow beta of surpluses is determined by the cash-flow beta of seigniorage revenues and beta of the debt outstanding h periods from now:*

$$\beta_t^{S,CF}(h) = \frac{cov_t \left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t) M_{t+1,t+h} D_{t+h} \right)}{D_t var_t(M_{t+1})} - \beta_t^{K,CF}(h).$$

The proof is a straightforward extension of the proof of Prop. 7 in Appendix A.7. To keep the debt risk-free ($\beta_t^D = 0$) while delivering a risky surplus over short horizons ($\beta_t^{S,CF}(h) > 0$), the government must resort to issuing more debt when marginal utility growth is high ($cov_t \left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t) M_{t+1,t+h} D_{t+h} \right) > 0$). When it earns seignorage revenue, this debt issuance produces a safe seigniorage revenue stream over short horizons ($\beta_t^{K,CF}(h) < 0$), increasing $\beta_t^{S,CF}(h)$ and expanding taxpayer insurance possibilities.

However, over long horizons, debt is co-integrated with output. Since seigniorage revenue is proportional to debt outstanding under Assumption 4, the seigniorage claim inherits the long-run risk from output; $\beta_t^{K,CF}(h)$ turns positive as $h \to \infty$. The return beta of the seigniorage claim equals its cash-flow beta as the horizon becomes large: $\beta_t^K = \beta_t^{K,CF}(\infty)$. The cointegration argument suggests that $\beta_t^K > 0$ is the relevant case in a model with permanent output risk.

6.3 Quantifying the Impact of Seigniorage Revenue from Convenience

We now quantify the impact of convenience yields on the trade-off for the asset pricing model from the previous section with permanent risk and AR(2) dynamics for debt/output. Under the proportional convenience yields Assumption 4, the seigniorage revenue beta becomes:

$$\beta_t^{K,CF}(h) \equiv -(1-e^{-\kappa}) \sum_{j=1}^{h} \frac{\mathbb{E}_t[M_{t+1}]}{D_t var_t(M_{t+1})} \mathbb{E}_t[M_{t+1,t+j} d_{t+j} Y_{t+j}] (\exp\{\gamma(\psi_{1,j-1}\lambda - \sigma)\} - 1).$$

Figure 6 plots the risk premium on the cumulative seigniorage claim in the left panel. It is the product of $\beta_t^{K,CF}(h)$ and the market price of risk. The three lines refer to different values for the convenience yield $(1-e^{-\kappa})$, ranging from 1% to 3%.[2] In the short run, the seignorage revenue claim is safe and hence earns a negative risk premium. The larger κ, the more negative the seigniorage risk premium at short horizons. As a result, the seigniorage revenue relaxes the trade-off between insuring bondholders and taxpayers over short horizons. This is shown in the right panel, which plots the risk premium on the cumulative surplus claim $\beta_t^{S,CF}(h) \times \frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]}$, with $\beta_t^{S,CF}(h)$ computed from Proposition 9. In the short run, the cumulative surplus claim risk premium is more positive the higher the convenience yield.

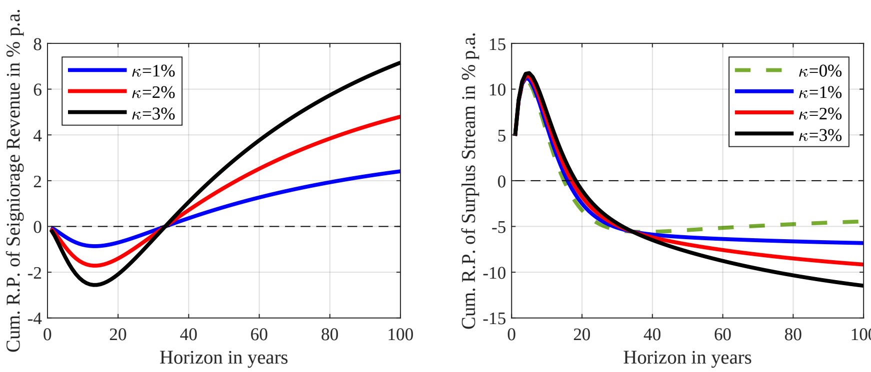
In the long run, the seigniorage revenue is risky. Since seignorage revenue is proportional to debt outstanding, and debt is cointegrated with output, the long-run risk premium on the seigniorage revenue claim is dominated by long-run output risk. Seigniorage revenue inevitably adds long-run output risk to the debt. This effect is stronger the higher the convenience yield. As the right panel of Figure 6 shows, insurance provision to taxpayers worsens over horizons beyond 20 years.

The return beta of the seigniorage revenue stream equals its cash-flow beta at horizon $h = \infty$. Since $\beta_t^{K,CF}(\infty) > 0$, so is $\beta_t^K > 0$. As shown in Proposition 8, convenience yields then lower the return beta of the tax claim β^T. In sum, convenience yields, even large ones, are no panacea. They allow for more taxpayer insurance over short horizons but come at the expense of less insurance in the longer-run.

[2] Krishnamurthy and Vissing-Jorgensen (2012) estimate convenience yields on U.S. Treasuries of around 0.75% per year. Using the deviations from CIP in Treasury markets, Jiang, Krishnamurthy, and Lustig (2018); Jiang et al. (2021); Koijen and Yogo (2020) estimate convenience yields that foreign investors derive from their holdings of dollar safe assets of around 2% per year.

Figure 6: Risk Premia on Seigniorage and Surplus Claims in Convenience Yields Model

The left panel plots the risk premium of cumulative discounted seigniorage revenue, $\beta_t^{K,CF}(h) \times \frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]}$, against the horizon h. The right panel plots the risk premium of cumulative discounted surpluses, $\beta_t^{S,CF}(h) \times \frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]}$, against the horizon h. The parameters are given in Table 1, and the risk premia are evaluated at the long-run mean log spending/output and log debt/output ratios. Convenience yields $(1 - e^{-\kappa})$ range from 0% to 3%.



7 Extensions

7.1 Richer Asset Pricing Model with Disasters

The asset pricing model of Section 3 recognized the importance of permanent shocks to output and a high enough market price of risk in order to deliver realistic asset pricing implications. As is well-known in the asset pricing literature, if one commits to constant relative risk aversion preferences for the representative investor, a realistic value for the market price of risk (maximum Sharpe ratio) implies an implausibly high coefficient of relative risk aversion. To generate realistic discount rates on risky claims with empirically plausible risk aversion, we consider an economy that is subject to rare disaster risk in output in the tradition of Rietz (1988); Barro (2006). Appendix C shows that all theoretical results in Section 3 and 4 carry through. It also shows that a calibration of the disaster model delivers quantitatively similar results to those of the benchmark model.

7.2 Insurance Trade-off in Models with Transitory Risk

Modern asset pricing has consistently found that permanent shocks to output account for most of the variance of the pricing kernel, and receive a high price of risk in securities market (e.g., Alvarez and Jermann, 2005; Hansen and Scheinkman, 2009; Bansal and Yaron, 2004; Borovička, Hansen, and Scheinkman, 2016; Backus, Boyarchenko, and

Chernov, 2018). Models without large permanent shocks produce bond risk premia that exceed equity risk premia. We apply this basic insight from the asset pricing literature to the fiscal policy literature to study the allocation of aggregate risk between taxpayers and bondholders. Insulating bondholders from permanent output shocks imposes severe restrictions on the feasible tax processes.

Even in environments with only transitory shocks (which feature counter-factually high interest rate risk), the government is highly limited in its ability to prove taxpayer insurance over intermediate horizons. It must shift the significant long-run interest rate risk onto taxpayers to keep the debt risk-free. In business cycle models, shocks to output are typically transitory, as output fluctuates around potential output. The models in the optimal fiscal policy literature similarly imply that equilibrium output does not have a unit root component.[3] Surprisingly, we find that the trade-off between insuring taxpayers and bondholders is just as strong in models with transitory output risk as in our benchmark model with permanent output risk. The standard intuition that the government is able to smooth transitory shocks on behalf of households to provide substantially more insurance fails. The reason is that environments with transitory shocks feature substantial long-run interest rate risk. When output is below potential, investors in these models want to borrow, pushing up interest rates when marginal utility is high. This makes bonds risky assets with a high risk premium. In fact, the long-term government bond is the riskiest asset in economies with transitory risk (Alvarez and Jermann, 2005). To keep government debt risk-free ($\beta^D = 0$), the government has to offset the interest rate risk in long-run debt strips by producing safer surpluses in the near future. This dramatically shortens the horizon over which it can insure households. Effectively, we have replaced the large long-run output risk premium in the permanent-shock model with a large interest rate risk premium in the transitory-shock model. A detailed discussion is in Appendix D. In sum, all results about the trade-off at long and short horizons go through.

8 Conclusion

There are limits to the government's ability to make risk-free promises. The exposure of tax rates to economic shocks must be engineered judiciously to keep the debt risk-free and bondholders insulated from those same shocks. When bondholders are insulated,

[3] Rather, these models have mean-reverting processes for productivity and government spending (e.g., Chari, Christiano, and Kehoe, 1994; Debortoli, Nunes, and Yared, 2017; Bhandari, Evans, Golosov, and Sargent, 2017).

taxpayers must bear the aggregate risk. There is no scope for insurance of both taxpayers and bondholders over long horizons in the presence of priced shocks to output, be they permanent or transitory in nature. We characterize the extent of short-run insurance that the government can provide to households through fiscal policy in the wake of an adverse macro-economic shock in a model with realistic asset prices and debt policies. We find that insisting on risk-free debt (protecting bondholders) severely restricts the tax and surplus process. The more debt there is outstanding, the more output risk must be borne by taxpayers. Global demand for U.S. safe assets and the associated revenue stream from convenience yields alleviates the trade-off between insuring taxpayers and bondholders somewhat, but only in the short run. The only way the government can ultimately provide insurance to taxpayers over all horizons while keeping the debt risk-free is by saving rather than borrowing.

When we impose the restriction of risk-free debt, together with plausibly debt dynamics, the implied surplus dynamics are at odds with the data. Since surpluses in the data do not behave like those predicted by a model of risk-free debt, we conclude that the government may in fact not be manufacturing risk-free debt. This conclusion is borne out by the experience of COVID-19 when the value of the portfolio of outstanding U.S. Treasurys lost 26% between March 2020 and October 2023.

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A Proofs

A.1 Proof of Proposition 1

Proof. From the intertemporal budget constraint, we have:

$$D_t = P_t^T - P_t^G.$$

The returns on the debt, tax claim, and spending claim are defined as:

$$R_{t+1}^D = \frac{D_{t+1} + S_{t+1}}{D_t},$$
$$R_{t+1}^T = \frac{P_{t+1}^T + T_{t+1}}{P_t^T},$$
$$R_{t+1}^G = \frac{P_{t+1}^G + G_{t+1}}{P_t^G}.$$

It follows immediately that:

$$R_{t+1}^D = \frac{P_{t+1}^T - P_{t+1}^G + T_{t+1} - G_{t+1}}{D_t} = \frac{P_t^T}{D_t} R_{t+1}^T - \frac{P_t^G}{D_t} R_{t+1}^G.$$

Taking conditional covariances with minus the SDF and dividing by the conditional variance of the SDF, we get:

$$\frac{Cov_t\left(-M_{t+1}, R_{t+1}^D\right)}{Var_t(M_{t+1})} = \frac{P_t^T}{D_t} \frac{Cov_t\left(-M_{t+1}, R_{t+1}^T\right)}{Var_t(M_{t+1})} - \frac{P_t^G}{D_t} \frac{Cov_t\left(-M_{t+1}, R_{t+1}^G\right)}{Var_t(M_{t+1})}.$$

This implies that:

$$\beta_t^D = \frac{P_t^T}{D_t} \beta_t^T - \frac{P_t^G}{D_t} \beta_t^G.$$

Rearranging:

$$\beta_t^T = \frac{D_t}{D_t + P_t^G} \beta_t^D + \frac{P_t^G}{D_t + P_t^D} \beta_t^G.$$

where we used that $P_t^T = D_t + P_t^G$. From the Euler equation, the following relationship between risk and expected return holds for $j = D, T, G, Y$:

$$\mathbb{E}_t\left[R_{t+1}^j - R_t^f\right] = \frac{-Cov_t\left(M_{t+1}, R_{t+1}^j\right)}{Var_t(M_{t+1})} \frac{Var_t(M_{t+1})}{\mathbb{E}_t[M_{t+1}]} = \beta_t^j \frac{Var_t(M_{t+1})}{\mathbb{E}_t[M_{t+1}]}.$$

Therefore, we also get the relationship between the risk premia:

$$\mathbb{E}_t\left[R_{t+1}^D - R_t^f\right] = \frac{P_t^T}{D_t} \mathbb{E}_t\left[R_{t+1}^T - R_t^f\right] - \frac{P_t^G}{D_t} \mathbb{E}_t\left[R_{t+1}^G - R_t^f\right].$$

Rearranging:

$$\mathbb{E}_t\left[R_{t+1}^T - R_t^f\right] = \frac{P_t^G}{D_t + P_t^G} \mathbb{E}_t\left[R_{t+1}^G - R_t^f\right] + \frac{D_t}{D_t + P_t^G} \mathbb{E}_t\left[R_{t+1}^D - R_t^f\right].$$

Imposing that debt is risk-free amounts to $\beta_t^D = 0$ and $\mathbb{E}_t\left[R_{t+1}^D - R_t^f\right] = 0$. We immediately get:

$$\beta_t^T = \frac{P_t^G}{D_t + P_t^D} \beta_t^G,$$

and

$$\mathbb{E}_t\left[R_{t+1}^T - R_t^f\right] = \frac{P_t^G}{D_t + P_t^G} \mathbb{E}_t\left[R_{t+1}^G - R_t^f\right],$$

which proves the proposition and the following corollary. □

A.2 Proof of Proposition 2

Proof. To verify the expression, we first conjecture that the price of the k-period output strip is $\mathbb{E}_t[M_{t,t+k}Y_{t+k}] = \xi_k Y_t$, for $k \geq 0$. It follows immediately that $\xi_0 = 1$. To verify the conjecture note that the k-period output strip at time t becomes a $k-1$ period output strip in period $t+1$. It must satisfy the pricing relationship:

$$\begin{aligned}
\xi_k Y_t &= \mathbb{E}_t[M_{t,t+k}Y_{t+k}] = \mathbb{E}_t[M_{t,t+1}\mathbb{E}_{t+1}[M_{t+1,t+k}Y_{t+k}]] = \mathbb{E}_t[M_{t,t+1}\xi_{k-1}Y_{t+1}] \\
&= \mathbb{E}_t\left[\exp\left(-\rho - \frac{1}{2}\gamma^2 + \mu + (\sigma-\gamma)\varepsilon_{t+1}\right)\right]\xi_{k-1}Y_t, \\
&= \exp\left(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\sigma-\gamma)^2\right)\xi_{k-1}Y_t, \\
&= \exp\left(\mu - \rho + \frac{1}{2}\sigma^2 - \gamma\sigma\right)\xi_{k-1}Y_t,
\end{aligned}$$

which verifies the conjecture and implies the recursion:

$$\xi_k = \xi_{k-1}\exp\left(\mu - \rho + \frac{1}{2}\sigma^2 - \gamma\sigma\right) = \xi_{k-1}\xi_1.$$

Similarly, we define a k-period surplus strip as a claim to S_{t+k}, with price given by $\mathbb{E}_t[M_{t,t+k}S_{t+k}] = \chi_k Y_t$. The pricing of the first surplus strip is given by the following expression:

$$\begin{aligned}
\mathbb{E}_t[M_{t,t+1}S_{t+1}] &= \mathbb{E}_t[M_{t,t+1}\{-dY_{t+1}(1-\exp(-\mu+\rho-\sigma\varepsilon_{t+1}))\}] \\
&= -d\mathbb{E}_t[M_{t,t+1}Y_{t+1}] + d\mathbb{E}_t[M_{t,t+1}Y_{t+1}\exp(-\mu+\rho-\sigma\varepsilon_{t+1})], \\
&= -dY_t\mathbb{E}_t\left[\exp\left(\mu-\rho-\frac{1}{2}\gamma^2+(\sigma-\gamma)\varepsilon_{t+1}\right)\right] + dY_t\mathbb{E}_t\left[\exp\left(-\frac{1}{2}\gamma^2-\gamma)\varepsilon_{t+1}\right)\right], \\
&= \left[1-\exp\left(\mu-\rho+\frac{1}{2}\sigma^2-\gamma\sigma\right)\right]dY_t,
\end{aligned}$$

where the first equality uses the definition of the surplus implied by the government budget constraint. This implies:

$$\chi_1 = \left[1-\exp\left(\mu-\rho+\frac{1}{2}\sigma^2-\gamma\sigma\right)\right]d = (1-\xi_1)\,d.$$

Then, similarly, the pricing of the kth surplus strip is given by:

$$\chi_k Y_t = \mathbb{E}_t[M_{t,t+k}S_{t+k}] = \mathbb{E}_t[M_{t,t+1}\chi_{k-1}Y_{t+1}] = \chi_{k-1}\exp\left(\mu-\rho+\frac{1}{2}\sigma^2-\gamma\sigma\right)Y_t.$$

This verifies the conjecture and implies the recursion:

$$\chi_k = \chi_{k-1}\exp\left(\mu-\rho+\frac{1}{2}\sigma^2-\gamma\sigma\right) = \chi_{k-1}\xi_1,$$

starting from the expression for χ_1 given above. This calculation implies that we cannot simply price the surplus strips off the risk-free yield curve, even though the entire debt is risk-free. The proper discount rate contains a risk premium term $\gamma\sigma$.

The price of the surplus claim, which is the sum of the prices of all the surplus strips, is given by:

$$\mathbb{E}_t\left[\sum_{k=1}^{\infty}M_{t,t+k}S_{t+k}\right] = \sum_{k=1}^{\infty}\chi_k Y_t = \chi_1(1+\xi_1+\xi_1^2+\ldots)Y_t = \frac{1-\xi_1}{1-\xi_1}dY_t = dY_t,$$

which proves the proposition. □

A.3 Proof of Proposition 3

Proof. Since government spending is a constant fraction x of output, the price of the spending strip is proportional to the price of the output strip. From the previous proposition, this immediately implies: $\mathbb{E}_t\left[M_{t,t+k}G_{t+k}\right] = x\xi_k Y_t$, for $k \geq 0$. It follows that the price of the spending claim, which is the sum of the prices of all the spending strips, is given by:

$$P_t^G = \mathbb{E}_t\left[\sum_{k=1}^{\infty} M_{t,t+k}G_{t+k}\right] = \sum_{k=1}^{\infty} x\xi_k Y_t = x\xi_1(1+\xi_1+\xi_1^2+\ldots)Y_t = x\frac{\xi_1}{1-\xi_1}Y_t.$$

From the budget constraint, we have: $D_t = P_t^T - P_t^G$. It immediately follows that:

$$P_t^T = \mathbb{E}_t\left[\sum_{k=1}^{\infty} M_{t,t+k}T_{t+k}\right] = \mathbb{E}_t\left[\sum_{k=1}^{\infty} M_{t,t+k}\left(S_{t+k}+G_{t+k}\right)\right] = D_t + P_t^G = dY_t + x\frac{\xi_1}{1-\xi_1}Y_t = \left(d + x\frac{\xi_1}{1-\xi_1}\right)Y_t.$$

This proves the first part of the proposition.

The return on the tax claim can be stated as:

$$R_{t+1}^T = \frac{P_{t+1}^T + T_{t+1}}{P_t^T} = \frac{\left(d + x\frac{\xi_1}{1-\xi_1}\right)Y_{t+1} + \left[x - d\left(1-\exp\left(-\mu+\rho-\sigma\varepsilon_{t+1}\right)\right)\right]Y_{t+1}}{\left(d + x\frac{\xi_1}{1-\xi_1}\right)Y_t} = \frac{x\frac{1}{1-\xi_1}}{d + x\frac{\xi_1}{1-\xi_1}}\frac{Y_{t+1}}{Y_t} + \frac{d\exp(\rho)}{d + x\frac{\xi_1}{1-\xi_1}}.$$

Similarly, the return on the spending claim can be stated as:

$$R_{t+1}^G = \frac{P_{t+1}^G + G_{t+1}}{P_t^G} = \frac{x\frac{\xi_1}{1-\xi_1}Y_{t+1} + xY_{t+1}}{x\frac{\xi_1}{1-\xi_1}Y_t} = \frac{x\frac{1}{1-\xi_1}Y_{t+1}}{x\frac{\xi_1}{1-\xi_1}Y_t} = \frac{1}{\xi_1}\frac{Y_{t+1}}{Y_t}.$$

Armed with these expressions, we get the following expression for the covariance:

$$cov\left(R_{t+1}^T, M_{t,t+1}\right) = \frac{x\frac{1}{1-\xi_1}}{d + x\frac{\xi_1}{1-\xi_1}}cov\left(\frac{Y_{t+1}}{Y_t}, M_{t,t+1}\right)$$

and

$$cov\left(R_{t+1}^G, M_{t,t+1}\right) = \frac{1}{\xi_1}cov\left(\frac{Y_{t+1}}{Y_t}, M_{t,t+1}\right)$$

which implies:

$$cov\left(R_{t+1}^T, M_{t,t+1}\right) = \frac{x\frac{\xi_1}{1-\xi_1}}{d + x\frac{\xi_1}{1-\xi_1}}cov\left(R_{t+1}^G, M_{t,t+1}\right).$$

From the Euler equation, the following relationship between risk and expected return holds for $j = T, G, Y$:

$$\mathbb{E}_t\left[R_{t+1}^j - R_t^f\right] = \frac{-Cov_t\left(M_{t+1}, R_{t+1}^j\right)}{Var_t(M_{t+1})}\frac{Var_t(M_{t+1})}{\mathbb{E}_t(M_{t+1})}.$$

The same relationship between the covariances follows for expected returns:

$$\mathbb{E}_t\left[R_{t+1}^T - R_t^f\right] = \frac{x\frac{\xi_1}{1-\xi_1}}{d + x\frac{\xi_1}{1-\xi_1}}\mathbb{E}_t\left[R_{t+1}^G - R_t^f\right].$$

This proves the second part of the proposition. □

A.4 Proof of Proposition 4

Case of Stationary Debt/Output Ratio

Proof. We recall that the debt/output ratio dynamics are given by

$$\log d_{t+1} = \phi_0 + \phi_1 \log d_t + \phi_2 \log d_{t-1} - \lambda \varepsilon_{t+1} - \frac{1}{2}\lambda^2.$$

Solve for the price of the one-period debt strip:

$$\begin{aligned}
\mathbb{E}_t[M_{t,t+1}D_{t+1}] &= \mathbb{E}_t[M_{t,t+1}Y_{t+1}d_{t+1}] \\
&= \exp\left(\mu - \rho + \phi_0 + \phi_1 \log d_t + \phi_2 \log d_{t-1}\right) \mathbb{E}_t\left[\exp\left(-\gamma\varepsilon_{t+1} - \frac{1}{2}\gamma^2 - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2 + \sigma\varepsilon_{t+1}\right)\right] Y_t, \\
&= \exp\left(\mu - \rho + \frac{1}{2}\sigma^2 - \gamma\sigma + \phi_0 + \phi_1 \log d_t + \phi_2 \log d_{t-1} + \lambda(\gamma - \sigma)\right) Y_t, \\
&= \exp\left(\kappa_1 + \phi_1 \log d_t + \phi_2 \log d_{t-1}\right) Y_t, \\
&= \exp\left(\kappa_1 + \psi_{1,1} \log d_t + \psi_{2,1} \log d_{t-1}\right) Y_t
\end{aligned}$$

where $\kappa_1 = \mu - \rho + \frac{1}{2}\sigma^2 - \gamma\sigma + \phi_0 + \lambda(\gamma - \sigma)$ and $\psi_{1,0} = 1, \psi_{1,1} = \phi_1, \psi_{1,j} = \phi_1\psi_{1,j-1} + \phi_2\psi_{1,j-2}$ and $\psi_{2,0} = 0, \psi_{2,1} = \phi_2, \psi_{2,j} = \phi_1\psi_{2,j-1} + \phi_2\psi_{2,j-2}$, for $j \geq 2$. Next, we price the debt strip two periods hence:

$$\begin{aligned}
\mathbb{E}_t[M_{t,t+2}D_{t+2}] &= \mathbb{E}_t\left[M_{t,t+1}\mathbb{E}_{t+1}\left[M_{t+1,t+2}D_{t+2}\right]\right], \\
&= \mathbb{E}_t\left[M_{t,t+1}\exp(\kappa_1 + \phi_1 \log d_{t+1} + \phi_2 \log d_t)Y_{t+1}\right], \\
&= \mathbb{E}_t\left[M_{t,t+1}\exp(\kappa_1 + (\phi_1^2 + \phi_2)\log d_t + \phi_1\phi_2 \log d_{t-1} + \phi_1\phi_0 - \phi_1\lambda\varepsilon_{t+1} - \frac{1}{2}\phi_1\lambda^2 + \mu + \sigma\varepsilon_{t+1})\right] Y_t \\
&= \exp\left(\kappa_1 + (\phi_1^2 + \phi_2)\log d_t + \phi_1\phi_2 \log d_{t-1} + \phi_1\phi_0 + \mu - \rho\right) \mathbb{E}_t\left[\exp\left(-\gamma\varepsilon_{t+1} - \frac{1}{2}\gamma^2 - \phi_1\lambda\varepsilon_{t+1} - \frac{1}{2}\phi_1\lambda^2 + \sigma\varepsilon_{t+1}\right)\right] Y_t \\
&= \exp(\kappa_1 + \kappa_2)\exp((\phi_1^2 + \phi_2)\log d_t + \phi_1\phi_2 \log d_{t-1})Y_t, \\
&= \exp(\kappa_1 + \kappa_2)\exp(\psi_{1,2} \log d_t + \psi_{2,2} \log d_{t-1})Y_t.
\end{aligned}$$

where $\kappa_2 = \mu - \rho + \frac{1}{2}\sigma^2 - \gamma\sigma + \phi_1\phi_0 + \phi_1\lambda(\gamma - \sigma) + \frac{1}{2}\lambda^2(\phi_1^2 - \phi_1)$. The debt strip price three periods hence is given by:

$$\begin{aligned}
\mathbb{E}_t[M_{t,t+3}D_{t+3}] &= \mathbb{E}_t\left[M_{t,t+1}\mathbb{E}_{t+1}\left[M_{t+1,t+3}D_{t+3}\right]\right], \\
&= \mathbb{E}_t\left[M_{t,t+1}\exp(\kappa_1 + \kappa_2)\exp(\psi_{1,2} \log d_{t+1} + \psi_{2,2} \log d_t)Y_{t+1}\right], \\
&= \mathbb{E}_t\left[M_{t,t+1}\exp(\kappa_1 + \kappa_2 + (\phi_1\psi_{1,2} + \psi_{2,2})\log d_t + \psi_{1,2}\phi_2 \log d_{t-1} + \psi_{1,2}\phi_0 - \psi_{1,2}\lambda\varepsilon_{t+1} - \frac{1}{2}\psi_{1,2}\lambda^2 + \mu + \sigma\varepsilon_{t+1})\right] Y_t \\
&= \exp\left(\kappa_1 + \kappa_2\right)\exp\left((\phi_1\psi_{1,2} + \psi_{2,2})\log d_t + \psi_{1,2}\phi_2 \log d_{t-1} + \psi_{1,2}\phi_0 + \mu - \rho\right) \mathbb{E}_t\left[\exp\left(-\gamma\varepsilon_{t+1} - \frac{1}{2}\gamma^2 - \psi_{1,2}\lambda\varepsilon_{t+1} - \frac{1}{2}\psi_{1,2}\lambda^2 + \sigma\varepsilon_{t+1}\right)\right] Y_t \\
&= \exp(\kappa_1 + \kappa_2 + \kappa_3)\exp\left(\psi_{1,3} \log d_t + \psi_{2,3} \log d_{t-1}\right) Y_t.
\end{aligned}$$

where $\kappa_3 = \mu - \rho + \frac{1}{2}\sigma^2 - \gamma\sigma + \psi_{1,2}\phi_0 + \psi_{1,2}\lambda(\gamma - \sigma) + \frac{1}{2}\lambda^2(\psi_{1,2}^2 - \psi_{1,2})$. The debt strip price j periods hence is given by:

$$\mathbb{E}_t[M_{t,t+j}D_{t+j}] = \exp\left(\sum_{k=1}^{j} \kappa_k\right) \exp\left(\psi_{1,j} \log d_t + \psi_{2,j} \log d_{t-1}\right) Y_t,$$

Note that

$$\sum_{k=1}^{j} \kappa_k = \left(\mu - \rho + \frac{1}{2}\sigma^2 - \gamma\sigma\right) j + C$$

where

$$C = \sum_{k=1}^{j} \left\{\phi_0\psi_{1,k-1} + \lambda(\gamma - \sigma)\psi_{1,k-1} + \frac{1}{2}\lambda^2(\psi_{1,k-1}^2 - \psi_{1,k-1})\right\}.$$

In the limit, as the horizon of the debt strip goes to infinity

$$\lim_{j\to\infty} \mathbb{E}_t[M_{t,t+j}D_{t+j}] = 0 \quad \iff \mu - \rho + \frac{1}{2}\sigma^2 - \gamma\sigma < 0$$

The necessary and sufficient condition for the TVC to be satisfied is a high enough output risk premium, or equivalently, a price/dividend ratio for the one-period output strip that is less than one. To see this, note that the variable C converges to a constant since $\lim_{j\to\infty}\psi_{1,j} = 0$ when the process for debt/output is stationary, i.e., when both roots of the characteristic equation $1 - \phi_1 z - \phi_2 z^2 = 0$ lie outside the unit circle. Under that condition, $\lim_{j\to\infty}\exp(\sum_{k=1}^{j}\kappa_k) = 0$. Note that the condition for the TVC condition to be satisfied does not depend on ϕ_1, ϕ_2, or λ. □

Case of Random Walk

Proof. Now, assume $\phi_1 = 1$ and $\phi_2 = 0$. Then $\psi_{1,j} = 1$ and $\psi_{2,j} = 0, \forall j$. It follows that $\kappa_j = \mu - \rho + \frac{1}{2}\sigma^2 - \gamma\sigma + \phi_0 + \lambda(\gamma - \sigma), \forall j$, in the proof above for the stationary case. The TVC is

$$\lim_{j\to\infty} \mathbb{E}_t[M_{t,t+j}D_{t+j}] = \lim_{j\to\infty} \exp(\sum_{k=1}^{j}\kappa_k)\exp(\log d_t)Y_t,$$

which is 0 if and only if

$$\mu - \rho + \frac{1}{2}\sigma^2 - \gamma\sigma + \phi_0 + \lambda(\gamma - \sigma) < 0.$$

When we set $\phi_0 = 0$, so that the debt/output ratio follows a random walk without drift, the result follows. □

A.5 Proof of Proposition 5

Proof. The dynamics of debt/output ratio is given by

$$\log d_t - \bar{d} = \phi_1(\log d_{t-1} - \bar{d}) + \phi_2(\log d_{t-2} - \bar{d}) - \lambda\varepsilon_t.$$

Conjecture that

$$\log d_{t+j} - \bar{d} = \psi_{1,j}(\log d_t - \bar{d}) + \psi_{2,j}(\log d_{t-1} - \bar{d}) - \lambda\sum_{k=0}^{j-1}\psi_{1,k}\varepsilon_{t+j-k},$$

where $\psi_{1,0} = 1, \psi_{1,1} = \phi_1, \psi_{1,j} = \phi_1\psi_{1,j-1} + \phi_2\psi_{1,j-2}$ and $\psi_{2,0} = 0, \psi_{2,1} = \phi_2, \psi_{2,j} = \phi_1\psi_{2,j-1} + \phi_2\psi_{2,j-2}$, for $j \geq 2$.

Verify the conjecture by plugging in the conjecture into the period $t + j + 1$ debt dynamics:

$$\begin{aligned}\log d_{t+j+1} - \bar{d} &= \phi_1(\log d_{t+j} - \bar{d}) + \phi_2(\log d_{t+j-1} - \bar{d}) - \lambda\varepsilon_{t+j+1} \\ &= (\phi_1\psi_{1,j} + \phi_2\psi_{1,j-1})(\log d_t - \bar{d}) + (\phi_1\psi_{2,j} + \phi_2\psi_{2,j-1})(\log d_{t-1} - \bar{d}) \\ &\quad - \lambda\left(\varepsilon_{t+j+1} + \phi_1\psi_{1,0}\varepsilon_{t+j} + \sum_{k=2}^{j}(\phi_1\psi_{1,k-1} + \phi_2\psi_{1,k-2})\varepsilon_{t+j+1-k}\right) \\ &= \psi_{1,j+1}(\log d_t - \bar{d}) + \psi_{2,j+1}(\log d_{t-1} - \bar{d}) - \lambda\sum_{k=0}^{j}\psi_{1,k}\varepsilon_{t+j+1-k},\end{aligned}$$

which verifies our conjecture.

Using the static budget constraint at time $t + j$, the surplus/output ratio s_{t+j} is given by:

$$s_{t+j} = \frac{S_{t+j}}{Y_{t+j}} = d_{t+j-1}\exp(\rho - \mu - \sigma\varepsilon_{t+j}) - d_{t+j}.$$

Using the above expressions for d_{t+j} and d_{t+j-1}, we can write the surplus/output ratio s_{t+j} as follows:

$$\begin{aligned} s_{t+j} = \exp &\left(\bar{d} + \psi_{1,j-1}(\log d_t - \bar{d}) + \psi_{2,j-1}(\log d_{t-1} - \bar{d}) - \lambda \sum_{k=0}^{j-2} \psi_{1,k}\varepsilon_{t+j-1-k} + \rho - \mu - \sigma\varepsilon_{t+j} \right) \\ - \exp &\left(\bar{d} + \psi_{1,j}(\log d_t - \bar{d}) + \psi_{2,j}(\log d_{t-1} - \bar{d}) - \lambda \sum_{k=0}^{j-1} \psi_{1,k}\varepsilon_{t+j-k} \right) \end{aligned}$$

The conditional expectation is

$$\begin{aligned} \mathbb{E}[s_{t+j}] = \exp &\left(\bar{d} + \psi_{1,j-1}(\log d_t - \bar{d}) + \psi_{2,j-1}(\log d_{t-1} - \bar{d}) + \frac{1}{2}\lambda^2 \sum_{k=0}^{j-2} \psi_{1,k}^2 + \rho - \mu + \frac{1}{2}\sigma^2 \right) \\ - \exp &\left(\bar{d} + \psi_{1,j}(\log d_t - \bar{d}) + \psi_{2,j}(\log d_{t-1} - \bar{d}) + \frac{1}{2}\lambda^2 \sum_{k=0}^{j-1} \psi_{1,k}^2 \right), \end{aligned}$$

which can be restated as

$$\begin{aligned} \mathbb{E}[s_{t+j}] = \exp &\left(\bar{d} + \psi_{1,j-1}(\log d_t - \bar{d}) + \psi_{2,j-1}(\log d_{t-1} - \bar{d}) + \frac{1}{2}\lambda^2 \sum_{k=0}^{j-2} \psi_{1,k}^2 \right) \\ \times &\left[\exp(\rho - \mu + \frac{1}{2}\sigma^2) - \exp\left((\psi_{1,j} - \psi_{1,j-1})(\log d_t - \bar{d}) + (\psi_{2,j} - \psi_{2,j-1})(\log d_{t-1} - \bar{d}) + \frac{1}{2}\lambda^2 \psi_{1,j-1}^2 \right) \right]. \end{aligned}$$

The derivative w.r.t. $(\log d_t - \bar{d})$ is given by:

$$\frac{\partial \mathbb{E}[s_{t+j}]}{\partial(\log d_t - \bar{d})} = \psi_{1,j-1}\mathbb{E}[s_{t+j}] - (\psi_{1,j} - \psi_{1,j-1})\mathbb{E}[d_{t+j-1}].$$

□

A.6 Proof of Proposition 6

Proof. We assume that prior to the shock at time $t+1$, debt is at its long-run mean: $\log d_t - \bar{d} = 0$ and $\log d_{t-1} - \bar{d} = 0$. We also assume that the risk-free rate satisfies: $\rho = \mu$. When the log of the debt/output process follows an $AR(2)$, the surplus/output ratio is then given by:

$$s_{t+j} = \exp\left(\bar{d} - \lambda \sum_{k=0}^{j-2} \psi_{1,k}\varepsilon_{t+j-1-k} - \sigma\varepsilon_{t+j} \right) - \exp\left(\bar{d} - \lambda \sum_{k=0}^{j-1} \psi_{1,k}\varepsilon_{t+j-k} \right)$$

For $j = 1$, we obtain:

$$s_{t+1} = \exp(\bar{d} - \sigma\varepsilon_{t+1}) - \exp(\bar{d} - \lambda\varepsilon_{t+1}).$$

The derivative of the surplus/output ratio at $t+1$ w.r.t. the output growth shock ε_{t+1}, evaluated at $\varepsilon_{t+1} = 0$, is given by:

$$\frac{\partial s_{t+1}}{\partial \varepsilon_{t+1}} = (\lambda - \sigma)\exp(\bar{d}).$$

This expression is positive if and only if $\lambda > \sigma$.

For $j = 2$, we obtain:

$$s_{t+2} = \exp\left(\bar{d} - \sigma\varepsilon_{t+2} - \lambda\varepsilon_{t+1}\right) - \exp\left(\bar{d} - \lambda\varepsilon_{t+2} - \lambda\psi_{1,1}\varepsilon_{t+1}\right)$$

The derivative of the surplus/output ratio at $t+2$ w.r.t. the output growth shock ε_{t+1}, evaluated at $\varepsilon_{t+1} = \varepsilon_{t+2} = 0$, is given by:

$$\frac{\partial s_{t+2}}{\partial \varepsilon_{t+1}} = \lambda(\psi_{1,1} - 1)\exp(\bar{d}).$$

This is positive if and only if $\psi_{1,1} = \phi_1 > 1$.

For $j = 3$, we obtain:

$$s_{t+3} = \exp\left(\bar{d} - \sigma\varepsilon_{t+3} - \lambda\varepsilon_{t+2} - \lambda\psi_{1,1}\varepsilon_{t+1}\right) - \exp\left(\bar{d} - \lambda\varepsilon_{t+3} - \lambda\psi_{1,1}\varepsilon_{t+2} - \lambda\psi_{1,2}\varepsilon_{t+1}\right)$$

The derivative of the surplus/output ratio at $t+3$ w.r.t. the output growth shock ε_{t+1}, evaluated at $\varepsilon_{t+1} = \varepsilon_{t+2} = \varepsilon_{t+3} = 0$, is given by:

$$\frac{\partial s_{t+3}}{\partial \varepsilon_{t+1}} = \lambda(\psi_{1,2} - \psi_{1,1})\exp(\bar{d}).$$

This is positive if and only if $\psi_{1,2} > \psi_{1,1}$. Equivalently, if $\phi_1^2 + \phi_2 > \phi_1$.

For $j > 3$, we obtain:

$$s_{t+j} = \exp\left(\bar{d} - \lambda\sum_{k=0}^{j-2}\psi_{1,k}\varepsilon_{t+j-1-k} - \sigma\varepsilon_{t+j}\right) - \exp\left(\bar{d} - \lambda\sum_{k=0}^{j-1}\psi_{1,k}\varepsilon_{t+j-k}\right)$$

The derivative of the surplus/output ratio at $t+j$ w.r.t. the output growth shock ε_{t+1}, evaluated at $\varepsilon_{t+k} = 0$, for $k = 1, \cdots, j$, is given by:

$$\frac{\partial s_{t+j}}{\partial \varepsilon_{t+1}} = \lambda(\psi_{1,j-1} - \psi_{1,j-2})\exp(\bar{d}).$$

This is positive if and only if $\psi_{1,j-1} > \psi_{1,j-2}$. Equivalently, if $\phi_1\psi_{1,j-2} + \phi_2\psi_{1,j-3} > \psi_{1,j-2}$. □

A.7 Proof of Proposition 7

Proof. Starting from the definition of debt at time $t+1$:

$$D_{t+1} = \mathbb{E}_{t+1}\left[\sum_{j=1}^{\infty} M_{t+1,t+1+j}S_{t+1+j}\right],$$

and the definition of the return on debt:

$$\begin{aligned} D_t R_{t+1}^D &= D_{t+1} + S_{t+1}, \\ &= \mathbb{E}_{t+1}\left[\sum_{j=1}^{\infty} M_{t+1,t+j}S_{t+j}\right], \\ &= \mathbb{E}_{t+1}\left[\sum_{j=1}^{h} M_{t+1,t+j}S_{t+j}\right] + \mathbb{E}_{t+1}\left[M_{t+1,t+h}D_{t+h}\right]. \end{aligned}$$

We can take the conditional covariances with minus the SDF on both sides:

$$D_t cov_t\left(-M_{t+1}, R_{t+1}^D\right) = cov_t\left(-M_{t+1}, \mathbb{E}_{t+1}\left[\sum_{j=1}^{h} M_{t+1,t+j}S_{t+j}\right]\right) + cov_t\left(-M_{t+1}, \mathbb{E}_{t+1}[M_{t+1,t+h}D_{t+h}]\right).$$

Dividing through by $D_t Var_t[M_{t+1}]$ and using the definitions of the debt return beta and the surplus cash-flow beta, we obtain:

$$\beta_t^D = \beta_t^{S,CF}(h) + \frac{cov_t\left(-M_{t+1}, \mathbb{E}_{t+1}\left[M_{t+1,t+h}D_{t+h}\right]\right)}{D_t Var_t[M_{t+1}]}.$$

If debt is risk-free, $\beta_t^D = 0$, and we obtain the first part of the proposition:

$$\beta_t^{S,CF}(h) = \frac{cov_t\left(M_{t+1}, \mathbb{E}_{t+1}[M_{t+1,t+h}D_{t+h}]\right)}{D_t Var_t[M_{t+1}]}.$$

The expression can be rewritten

$$\beta_t^{S,CF}(h) = \frac{\mathbb{E}_t[M_{t+1}]}{D_t Var_t[M_{t+1}]} \frac{cov_t\left(M_{t+1}, \mathbb{E}_{t+1}[M_{t+1,t+h}D_{t+h}]\right)}{\mathbb{E}_t[M_{t+1}]}.$$

Recall from appendix A.4 that:

$$\mathbb{E}_t\left[M_{t,t+h}D_{t+h}\right] = \exp\left(\sum_{k=1}^{h}\kappa_k + \psi_{1,h}\log d_t + \psi_{2,h}\log d_{t-1}\right)Y_t,$$

Shifting forward one period in time from t to $t+1$ and reducing the horizon from h to $h+1$, the above expression implies:

$$\begin{aligned}
\mathbb{E}_{t+1}\left[M_{t+1,t+h}D_{t+h}\right] &= \exp\left(\sum_{k=1}^{h-1}\kappa_k + \psi_{1,h-1}\log d_{t+1} + \psi_{2,h-1}\log d_t\right)Y_{t+1}, \\
&= \exp\left(\sum_{k=1}^{h-1}\kappa_k + (\psi_{1,h-1}\phi_1 + \psi_{2,h-1})\log d_t + \psi_{1,h-1}\phi_2\log d_{t-1} + \psi_{1,h-1}\phi_0 + \mu\right)Y_t \\
&\times\exp\left(-\lambda\psi_{1,h-1}\varepsilon_{t+1} - 0.5\lambda^2\psi_{1,h-1} + \sigma\varepsilon_{t+1}\right)
\end{aligned}$$

The covariance term can be broken into two components:

$$\frac{cov_t\left(M_{t+1}, \mathbb{E}_{t+1}[M_{t+1,t+h}D_{t+h}]\right)}{\mathbb{E}_t[M_{t+1}]} = \frac{\mathbb{E}_t\left[M_{t+1}\mathbb{E}_{t+1}[M_{t+1,t+h}D_{t+h}]\right]}{\mathbb{E}_t[M_{t+1}]} - \mathbb{E}_t\left[\mathbb{E}_{t+1}[M_{t+1,t+h}D_{t+h}]\right].$$

The numerator of the first term is:

$$\begin{aligned}
&= \exp\left(\sum_{k=1}^{h-1}\kappa_k + (\psi_{1,h-1}\phi_1 + \psi_{2,h-1})\log d_t + \psi_{1,h-1}\phi_2\log d_{t-1} + \psi_{1,h-1}\phi_0 + \mu - \rho\right)Y_t \\
&\times\mathbb{E}_t\left[\exp\left(-\gamma\varepsilon_{t+1} - 0.5\gamma^2 - \lambda\psi_{1,h-1}\varepsilon_{t+1} - 0.5\lambda^2\psi_{1,h-1} + \sigma\varepsilon_{t+1}\right)\right]
\end{aligned}$$

The conditional expectation in the second line works out to be:

$$\exp\left(-\gamma\sigma + 0.5\sigma^2 + \lambda\psi_{1,h-1}(\gamma-\sigma) + 0.5\lambda^2(\psi_{1,h-1}^2 - \psi_{1,h-1})\right)$$

Thus, the numerator of the first term becomes:

$$\exp\left(\sum_{k=1}^{h}\kappa_k + \psi_{1,h}\log d_t + \psi_{2,h}\log d_{t-1}\right)Y_t = \mathbb{E}_t\left[M_{t,t+h}D_{t+h}\right]$$

where we used the definition

$$\kappa_h = \mu - \rho + 0.5\sigma^2 - \gamma\sigma + \psi_{1,h-1}\phi_0 + \lambda\psi_{1,h-1}(\gamma-\sigma) + 0.5\lambda^2(\psi_{1,h-1}^2 - \psi_{1,h-1}).$$

Hence, we obtain:

$$\begin{aligned}
\frac{cov_t\left(M_{t+1}, \mathbb{E}_{t+1}[M_{t+1,t+h}D_{t+h}]\right)}{\mathbb{E}_t[M_{t+1}]} &= \frac{\mathbb{E}_t\left[M_{t,t+h}D_{t+h}\right]}{\mathbb{E}_t[M_{t+1}]} - \mathbb{E}_t\left[M_{t+1,t+h}D_{t+h}\right], \\
&= \mathbb{E}_t\left[M_{t+1,t+h}D_{t+h}\right]\left(\frac{\mathbb{E}_t\left[M_{t,t+h}D_{t+h}\right]}{\mathbb{E}_t[M_{t+1}]\mathbb{E}_t\left[M_{t+1,t+h}D_{t+h}\right]} - 1\right)
\end{aligned}$$

where we used the law of iterated expectations on the second term in the first line. Finally, we solve for

$$\frac{\mathbb{E}_t\left[M_{t,t+h}D_{t+h}\right]}{\mathbb{E}_t[M_{t+1}]\mathbb{E}_t\left[M_{t+1,t+h}D_{t+h}\right]} = \frac{\mathbb{E}_t\left[M_{t+1}M_{t+1,t+h}D_{t+h}\right]}{\mathbb{E}_t[M_{t+1}]\mathbb{E}_t\left[M_{t+1,t+h}D_{t+h}\right]} = \exp\left(\gamma(\lambda\psi_{1,h-1} - \sigma)\right)$$

Putting it all together, we have

$$\beta_t^{S,CF}(h) = \frac{\mathbb{E}_t[M_{t+1}]}{D_t Var_t[M_{t+1}]} \mathbb{E}_t\left[M_{t+1,t+h} d_{t+h} Y_{t+h}\right]\left(\exp\left(\gamma(\lambda\psi_{1,h-1} - \sigma)\right) - 1\right)$$

which proves the second part of the proposition. □

A.8 Proof of Corollary 2

Proof. We recall the dynamics for the government spending/output ratio:

$$x_{t+1} = \exp\left(\varphi_0^g + \varphi_1^g \log x_t - b_g \varepsilon_{t+1} - \frac{1}{2} b_g^2\right).$$

The cash-flow beta for the cumulative spending process is defined as:

$$\beta_t^{G,CF}(h) = -\frac{cov_t\left(M_{t+1}, \mathbb{E}_{t+1}[\sum_{j=1}^h M_{t+1,t+j} G_{t+j}]\right)}{D_t Var_t[M_{t+1}]}.$$

We first solve for the price of a one-period spending strip, recalling that $G_{t+1} = x_{t+1} Y_{t+1}$:

$$\begin{aligned}
\mathbb{E}_t\left[M_{t,t+1} G_{t+1}\right] &= \mathbb{E}_t\left[\exp\left(\mu - \rho - \frac{1}{2}\gamma^2 + (\sigma - \gamma)\varepsilon_{t+1} + \varphi_0^g + \varphi_1^g \log x_t - b_g \varepsilon_{t+1} - \frac{1}{2} b_g^2\right)\right] Y_t, \\
&= \exp\left(\mu - \rho + \frac{1}{2}\sigma^2 - \gamma\sigma + \varphi_0^g + b_g(\gamma - \sigma)\right) \exp\left(\varphi_1^g \log x_t\right) Y_t, \\
&= \exp\left(\zeta_1 + \varphi_1^g \log x_t\right) Y_t,
\end{aligned}$$

where $\zeta_1 = \mu - \rho + \frac{1}{2}\sigma^2 - \gamma\sigma + \varphi_0^g + b_g(\gamma - \sigma)$.

By the same token, the price of the two-period spending strip can be derived from the price of the one-period strip next year:

$$\begin{aligned}
\mathbb{E}_t[M_{t,t+2} G_{t+2}] &= \mathbb{E}_t\left[M_{t,t+1} \mathbb{E}_{t+1}\left[M_{t+1,t+2} G_{t+2}\right]\right], \\
&= \mathbb{E}_t\left[M_{t,t+1} \exp(\zeta_1 + \varphi_1^g \log x_{t+1}) Y_{t+1}\right], \\
&= \mathbb{E}_t\left[M_{t,t+1} \exp(\zeta_1 + \varphi_1^g \varphi_0^g + (\varphi_1^g)^2 \log x_t - \varphi_1^g b_g \varepsilon_{t+1} - \frac{1}{2} b_g^2 \varphi_1^g + \mu + \sigma\varepsilon_{t+1})\right] Y_t \\
&= \exp(\zeta_1 + \zeta_2) \exp\left((\varphi_1^g)^2 \log x_t\right) Y_t.
\end{aligned}$$

where $\zeta_2 = \mu - \rho + \frac{1}{2}\sigma^2 - \gamma\sigma + \varphi_1^g \varphi_0^g + \varphi_1^g b_g(\gamma - \sigma) + \frac{1}{2} b_g^2((\varphi_1^g)^2 - \varphi_1^g)$.

The debt strip price h periods hence is given by:

$$\mathbb{E}_t[M_{t,t+h} G_{t+h}] = \exp\left(\sum_{k=1}^h \zeta_k\right) \exp\left((\varphi_1^g)^h \log x_t\right) Y_t,$$

where

$$\sum_{k=1}^h \zeta_k = \left(\mu - \rho + \frac{1}{2}\sigma^2 - \gamma\sigma\right) h + \sum_{k=1}^h \left\{(\varphi_1^g)^{h-1} \varphi_0^g + b_g(\gamma - \sigma)(\varphi_1^g)^{h-1} + \frac{1}{2} b_g^2 \left(((\varphi_1^g)^{h-1})^2 - (\varphi_1^g)^{h-1}\right)\right\}.$$

Now we can return to the covariance:

$$\begin{aligned}
cov_t(M_{t+1}, \mathbb{E}_{t+1}[M_{t+1,t+j} G_{t+j}]) &= \mathbb{E}_t[M_{t,t+1} \mathbb{E}_{t+1}[M_{t+1,t+j} G_{t+j}]] - \mathbb{E}_t[M_{t,t+1}] \mathbb{E}_t[\mathbb{E}_{t+1}[M_{t+1,t+j} G_{t+j}]], \\
&= \mathbb{E}_t[M_{t,t+1}] \mathbb{E}_t[M_{t+1,t+j} x_{t+j} Y_{t+j}] \left(\frac{\mathbb{E}_t[M_{t,t+1} \mathbb{E}_{t+1}[M_{t+1,t+j} x_{t+j} Y_{t+j}]]}{\mathbb{E}_t[M_{t,t+1}] \mathbb{E}_t[M_{t+1,t+j} x_{t+j} Y_{t+j}]} - 1\right), \\
&= \mathbb{E}_t[M_{t,t+1}] \mathbb{E}_t[M_{t+1,t+j} x_{t+j} Y_{t+j}] \left(\exp\left\{\gamma\left((\varphi_1^g)^{h-1} b_g - \sigma\right)\right\} - 1\right)
\end{aligned}$$

since

$$\mathbb{E}_{t+1}[M_{t+1,t+j}x_{t+j}Y_{t+j}] = \exp\left(\sum_{k=1}^{h-1}\zeta_k\right)\exp\left((\varphi_1^g)^{h-1}\log x_{t+1}\right)Y_{t+1}$$
$$= \exp\left(\sum_{k=1}^{h-1}\zeta_k + \mu + (\varphi_1^g)^{h-1}\varphi_0^g - \frac{1}{2}b_g^2(\varphi_1^g)^{h-1}\right)\exp\left((\varphi_1^g)^h\log x_t\right)Y_t\exp\left(-(\varphi_1^g)^{h-1}b_g\varepsilon_{t+1} + \sigma\varepsilon_{t+1}\right)$$

which has a conditional covariance with the innovation in M_{t+1}, $-\gamma\varepsilon_{t+1}$, of $\gamma\left((\varphi_1^g)^{h-1}b_g - \sigma\right)$.

Summing across all spending strips, we obtain the required result:

$$\beta_t^{G,CF}(h) = -\sum_{j=1}^{h}\frac{\mathbb{E}_t[M_{t+1}]}{D_t var_t[M_{t+1}]}\mathbb{E}_t[M_{t+1,t+j}x_{t+j}Y_{t+j}](\exp\left\{\gamma((\varphi_1^g)^{j-1}b_g - \sigma)\right\} - 1).$$

The second part of the proposition, $\beta_t^{T,CF}(h) = \beta_t^{S,CF}(h) + \beta_t^{G,CF}(h)$ follow immediately from the definition of the primary surplus: $S_t = T_t - G_t$.

□

B Statistical Tests for Model-Implied vs. Realized Surpluses

This appendix provides statistical tests for the null hypothesis that risk-free debt model's implied expected surpluses are statistically different from the realized surpluses. We also test the null that the risk-free debt model's implied expected surpluses are statistically different from the CBO-predicted surpluses. We remind the reader that the CBO projections are made under current law. They do not make assumptions on future fiscal adjustments, but rather compute what likely future surpluses will be given existing tax and spending policies. Historically, their projections have been too rosy since the assumption that spending increases or tax cuts that are set to expire will indeed expire have been belied by politicians' actions. We consider two different exercises.

B.1 Hypothesis Testing

First, we consider a simple regression of the realized surplus/output ratios in year $t+j$ on the model's implied expected surplus/output ratios for that year, based on the information available in year t. The regression equation is given by:

$$s_{t+j} = \alpha + \beta \mathbb{E}_t[s_{t+j}] + \varepsilon_{t,t+j}.$$

The sample includes all horizons j from 1 to 10 and all years t from 1947 until 2022. If the risk-free debt model-implied surplus expectation is an unbiased predictor of the realized surplus, we expect $\alpha = 0$ and $\beta = 1$.

We report the results in Table B.1. The first panel considers the full sample from 1947 onwards. The regression intercept which captures the average difference between the annual realized and model-implied surplus/output ratios is -0.43%. We can reject the null hypothesis that this difference is zero at the 1% level. We also find that the slope coefficient β is negative and statistically significantly different from 1. Over the full sample, the model's projected surpluses move in the opposite direction as the realized surpluses, strongly refuting the plausibility of the risk-free debt model's implied surplus dynamics.

Panel B considers the post-2008 sample. In this subsample, the average gap between realized surplus and model-implied projected surplus is much larger at -4.14%. We resoundingly reject the null hypothesis of equality of means with a p-value less than 1%, despite the relatively short sample. The slope coefficient β is again far away from 1, which implies that the model's projected surpluses fail to capture the variation in realized surpluses.

Table B.1: Regression of Implied Surpluses on Realized Surpluses

Panel A: Full Sample, Realized Surplus					
Data Surplus	Model Surplus	Intercept	p-value	Slope	p-value
-0.27	-0.51	-0.43	0.00	-0.30	0.00
Panel B: Post-2008 Sample, Realized Surplus					
Data Surplus	Model Surplus	Intercept	p-value	Slope	p-value
-3.66	1.48	-4.14	0.00	0.33	0.05
Panel C: Post-2008 Sample, CBO Surplus Projections					
CBO Surplus	Model Surplus	Intercept	p-value	Slope	p-value
-1.15	1.73	-0.89	0.00	-0.15	0.01

Notes: Panel A and B: Regression of the realized surplus/output ratios in year $t+j$ on the model's implied expected surplus/output ratios for that year, based on the information available in year t. The regression equation is given by: $s_{t+j} = \alpha + \beta \mathbb{E}_t[s_{t+j}] + \varepsilon_{t,t+j}$. The sample includes all horizons j from 1 to 10 and all years t from 1947 until 2022 (pooled regression). The p-value in column (4) tests the null that realized and predicted surpluses are equal. The p-value in column (6) tests the null that the slope coefficient is equal to 1. Panel C: Regression of the CBO surplus projections on the model-implied expected surpluses in the post-2008 subsample.

Finally, in Panel C, we regress the CBO surplus projections on the risk-free model-implied expected surpluses in the post-2008 subsample. The regression intercept implies that model-implied surpluses are again systematically higher than the CBO projections by 0.89% per year with p-value below 1%. We also note that the CBO systematically overpredicted realized surpluses over this period by 2.51% per year (-1.15% versus -3.66%). Again, the risk-free model-implied surpluses fail to capture the variation in the CBO projections.

In sum, the difference between the surpluses implied by the risk-free government debt model and both the realized and the CBO-projected surpluses are economically and statistically large.

B.2 Simulation-Based Standard Errors

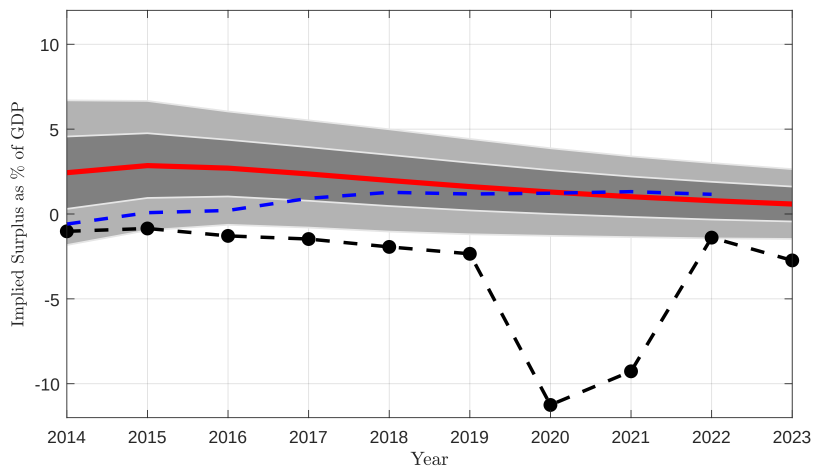
Our second exercise to test whether the risk-free model's implied expected surpluses are statistically different from the realized surpluses is a simulation-based exercise. For year t in the sample, we use the AR(2) model to simulate 10,000 paths of debt and model-implied surpluses for the next 10 years. We then compute the 1-standard-deviation and 2-standard-deviation confidence intervals for the model's implied expected surpluses. We assess where in the risk-free model's confidence intervals the realized surpluses as well as the CBO surplus projections fall.

Panel A of Figure B.1 presents the results using $t = 2013$ as an example. The red line represents the risk-free model's average expected implied surpluses over the next ten years from 2014 until 2023 across the 10,000 simulations. The shaded areas represent the 1-standard-deviation and 2-standard-deviation confidence intervals. The black dashed line represents the realized surpluses, and the blue dashed line represents the CBO surplus projections as of 2013. The model-implied surpluses are consistently higher than the realized surpluses. The data fall outside the 95% confidence interval of the risk-free debt model. The CBO projections in 2013 are also a highly unlikely draw from the risk-free model-implied surplus distribution and remain below the model-implied mean for the first five years of the projection window. As noted earlier, the CBO projections turned out to be more optimistic than the realized surpluses.

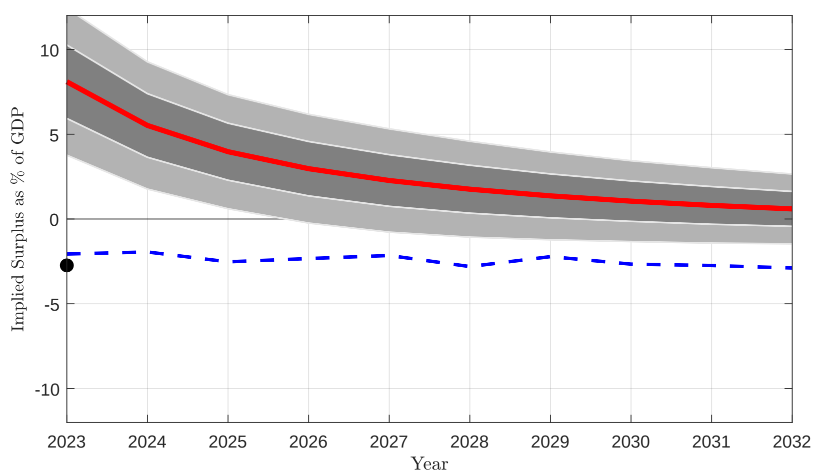
Figure B.1(b) presents the results using $t = 2022$, our last observation from the CBO. We only have one data point for the realized surplus for 2023, which lies far outside the 95%-CI of the model. The CBO projections are all outside of that CI as well, showing statistically significant differences.

Figure B.1: Model's Implied Surpluses and Realized Surpluses

Panel A: Using Information as of 2013



Panel B: Using Information as of 2022



B.3 Model-Implied Surpluses with Time-Varying Growth and Rate Expectations

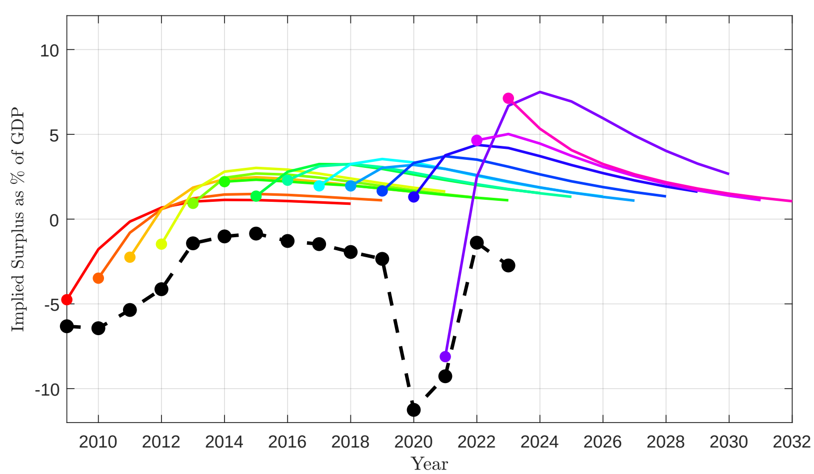
Finally, we consider an extension of our exercise which allows for time-varying growth and rate expectations. Returning to Proposition 5, when we construct the model-implied surplus expectations

$$\mathbb{E}_t[s_{t+j}] = \mathbb{E}_t[d_{t+j-1}]\left[\exp(\rho(t) - \mu(t) + \sigma^2/2) - \exp(\log \mathbb{E}_t[d_{t+j}] - \log \mathbb{E}_t[d_{t+j-1}])\right],$$

we use CBO projections of the nominal 10-year Treasury yields and the nominal GDP growth rates to construct the steady-state $\rho(t) - \mu(t)$ in the model. Specifically, if we are in period t and construct the model-implied expectation of surplus in $t+j$, we use the CBO's time-t projections of the future path of growth rates and interest rates between period $t+1$ to $t+j$, which we annualize, as the proxy for $\rho(t) - \mu(t)$ in the expression above.

Figure B.2 reports the model-implied surpluses for the post-GFC sample. Consistent with the baseline result reported in the main text, which assumes a constant value for $\rho - \mu$, the model-implied surpluses remain consistently higher than the realized surpluses.

Figure B.2: Model's Implied Surpluses and Realized Surpluses, with Time-Varying Growth and Rate Expectations



C Model With Rare Disasters

C.1 Setup

Let Y_t denote the aggregate endowment and let $\lambda_t = Y_t/Y_{t-1}$ denote its growth rate. The growth rate $\lambda_t = \lambda(z_t)$ depends on the aggregate shock, which is i.i.d. over time. We use lowercase symbols to denote logs. To derive the condition for dynamic efficiency without assuming log-normality, we use the cumulant-generating function defined as $K(s) = \log \mathbb{E}[\exp(s \log \lambda_{t+1})]$. Using κ_n to denote the n^{th} cumulant of $\log \lambda_{t+1}$, the cumulant-generating function $K(s)$ can also be expressed as $K(s) = \sum_{n=1}^{\infty} \kappa_n s^n/n!$. We use $g = \kappa_1 = \mathbb{E}[\log \lambda_{t+1}]$ to denote the expected log aggregate consumption growth rate. We use r^C_{t+1} to denote the return on a claim to aggregate consumption and r to denote the risk-free rate.

We adopt the version of rare disasters of Backus, Chernov, and Martin (2011) in which aggregate endowment growth consists of a standard Gaussian component w and a jump component u:

$$\log \lambda_{t+1} = w_{t+1} + u_{t+1}.$$

The first component w is normally distributed as $N(\mu, \sigma^2)$: $w_{t+1} = \mu + \sigma \varepsilon_{t+1}$. The second component is a Poisson mixture of normals. The number of jumps j takes on non-negative integer values with probabilities $e^{-\varpi}\varpi^j/j!$. The parameter ϖ, the jump intensity, is the mean of j. Each jump triggers a draw from a normal distribution with mean θ and variance δ^2 for the domestic agent. Conditional on the number of jumps j, the domestic jump component is normally distributed as $u_t|j \sim N(j\theta, j\delta^2)$. If ϖ is small, the jump model is well approximated by a Bernoulli mixture of normals. If ϖ is large, multiple jumps can occur frequently. This functional form is known as the Merton (1976) model. In the macro-finance literature, this specification has been applied by Bates (1988), Naik and Lee (1990), Backus et al. (2011), and Martin (2013).

C.2 Calibration

Table C.1 reports our calibration. We aim to target the same GDP and risk premium moments as in our baseline model with disaster risk. We choose the coefficient of risk aversion α of 11 to match the annual GDP risk premium (in logs) of 3.00%. The real risk-free rate is constant and equal to the sample average of 5-year nominal Treasury yield minus inflation.

Table C.1: Disaster Model Calibration for U.S.

α	11.14	coefficient of relative risk aversion
r	1.76%	real risk-free rate
μ	3.2%	mean of growth rate of Gaussian component of output
σ	1.0%	std. of growth rate of Gaussian component of output
θ	-19.66%	mean of jump component of output
δ	8%	std. of jump component of output
ϖ	1%	arrival intensity of jump
g	3%	mean of log growth rate ($\kappa_1 = \mu + \varpi\theta$)
$sd(\log \lambda)$	2.35%	standard deviation of log output growth
rp	3.00%	log GDP risk premium

To assess whether the economy is dynamically efficient, we compare the discount rate on a claim to aggregate output, given by the sum of the risk-free rate plus the unlevered equity premium (in logs), to the expected growth rate of the economy. Under our calibration, the log output risk premium given by:

$$rp = \mathbb{E}_t\left[r^C_{t+1} - r\right] = g + K(-\alpha) - K(1-\alpha) = 3.00\%. \tag{C.1}$$

After adding 3.00% to the risk-free rate, we find that this economy is dynamically efficient, i.e., $r + rp > g$, despite its low risk-free rate $r < g$.

C.3 Fiscal Dynamics with Constant Debt/Output Ratio

We start by considering the simplest case of constant spending/output and debt/output ratio policies.

Assumption 5. *(a) The government commits to a constant spending/output ratio $x = G_t/Y_t$. (b) The government commits to a constant debt/output ratio $d = D_t/Y_t$.*

Under Assumption 5, the government budget constraint implies a counter-cyclical process for tax revenue-to-GDP ratio (the tax rate τ_t):

$$\frac{T_t}{Y_t} = \frac{G_t}{Y_t} - \frac{D_t}{Y_t} + R^f_{t-1}\frac{D_{t-1}}{Y_t} = x + d\left(\exp(r - \log\lambda_t) - 1\right). \tag{C.2}$$

To keep the debt risk-free, the government must make the tax revenue claim counter-cyclical: $\partial(T_t/Y_t)/\partial\lambda_t < 0$. When the growth rate of output is low ($\lambda_t < 0$), tax revenue needs to increase as a fraction of GDP. Tax rates must rise in recessions. The magnitude of the counter-cyclicality of taxes is increasing in the debt-to-GDP ratio d. Similarly, the primary surplus/output ratio:

$$s_t = \frac{S_t}{Y_t} = \frac{T_t - G_t}{Y_t} = d\left(\exp(r - \log\lambda_t) - 1\right) \tag{C.3}$$

is counter-cyclical: $\partial s_t/\partial\lambda_t < 0$. In periods in which the growth shocks are negative enough ($\exp(r - \log\lambda_t) > 1$), the government must run a primary surplus.

Under a constant debt/output ratio, the TVC for government debt is:

$$\lim_{h\to\infty}\mathbb{E}_t\left[M_{t,t+h}D_{t+h}\right] = \lim_{h\to\infty}\exp\left\{-h(r + rp - g)\right\} d\cdot Y_t, \tag{C.4}$$

$$= \lim_{h\to\infty}\exp\left\{h(-r - K(-\alpha) + K(1-\alpha))\right\} d\cdot Y_t, \tag{C.5}$$

where the unlevered consumption risk premium rp is given by Eqn. (C.1). The TVC is satisfied if and only if $r + rp > g$. The textbook condition $r < g$ is neither necessary nor sufficient for a TVC violation. A necessary and sufficient condition for the TVC to be satisfied is that there is enough permanent, priced risk in output. The output risk premium rp must be high enough. This condition is easily satisfied for the disaster calibration.

Next, we turn to valuing the debt as the expected present-discounted value of future surpluses using the pricing kernel.

Proposition 10. *Under Assumption 5, if the TVC holds and the primary surplus satisfies (C.3), the government debt value, which is the sum of the values of the surplus strips, is a constant fraction of output:*

$$D_t = \mathbb{E}_t\left[\sum_{k=1}^{\infty} M_{t,t+k}S_{t+k}\right] = dY_t. \tag{C.6}$$

The proof is in Appendix C.6.1

Next, we turn to the baseline result characterizing the expected return and the return beta of the tax claim.

Proposition 11. *Under Assumption 5, if the TVC holds*
(a) The ex-dividend values of the spending and revenue claims are given by:

$$P^G_t = x\frac{q_1}{1-q_1}Y_t, \quad P^T_t = \left(d + x\frac{q_1}{1-q_1}\right)Y_t, \tag{C.7}$$

where $\log q_1 = -r - K(-\alpha) + K(1-\alpha)$ *is the log price/dividend ratio of a one-period output strip, a claim to output next year.*
(b) The risk premia and betas on the tax, spending, and consumption claims satisfy:

$$\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] = \frac{x\frac{q_1}{1-q_1}}{d + x\frac{q_1}{1-q_1}}\mathbb{E}_t\left[R^G_{t+1} - R^f_t\right] < \mathbb{E}_t\left[R^G_{t+1} - R^f_t\right] = \mathbb{E}_t\left[R^Y_{t+1} - R^f_t\right] \tag{C.8}$$

$$\beta^T = \frac{x\frac{q_1}{1-q_1}}{d + x\frac{q_1}{1-q_1}}\beta^G < \beta^G = \beta^Y. \tag{C.9}$$

The proof is in Appendix C.6.2.

The expected return on a claim to output is given by $\mathbb{E}_t\left[R^Y_{t+1}\right] = \exp(r + K(1) + K(-\alpha) - K(1-\alpha))$. Hence, the (log of the multiplicative) output risk premium in levels is equal to $RP = K(1) + K(-\alpha) - K(1-\alpha)$. Note that $rp = \kappa_1 - K(1-\alpha) + K(-\alpha)$ is the log risk premium on the output claim. The difference with RP is $L(\lambda) = K(1) - \kappa_1$, a Jensen inequality term measuring the entropy of the growth rate. In the log-normal case, this Jensen term is $\sigma^2/2$. Since government spending is a constant fraction of output, the risk premium on the spending claim equals that of the output claim: $\mathbb{E}[R^G_{t+1} - R^f_t] = \mathbb{E}[R^Y_{t+1} - R^f_t]$. The beta of the spending claim equals the beta of the output claim: $\beta^G = \beta^Y > 0$.

C.4 Fiscal Dynamics with Counter-cyclical Debt/Output Ratios

We allow the government to vary the debt/output ratio counter-cyclically. We consider a flexible class of $AR(P)$ processes for the debt/output ratio.

Assumption 6. *The government commits to a policy for the debt/output ratio $d_t = D_t/Y_t$ given by:*

$$\log d_t = \phi_0 + \sum_{p=1}^{P} \phi_p \log d_{t-p} - \varphi \log \lambda_t - K(-\varphi),$$

where $\varphi > 0$ so that the debt-output ratio increases in response to a negative output shock $\log \lambda_t$.

We include the constant $-K(-\varphi)$ so that $\mathbb{E}[\exp(-\varphi \log \lambda_t - K(-\varphi))] = 1$ and the last two terms do not affect the unconditional mean of the debt/output ratio.

The results of Proposition 11 continue to hold with d_t replacing d.

How persistent can the debt/output ratio be without violating TVC?

Proposition 12. *Under Assumption 6 with $P = 2$, if the debt/output ratio is stationary, then the TVC condition for government debt is satisfied if and only if $-r - K(-\alpha) + K(1 - \alpha) < 0$.*

The proof is in Appendix C.6.3. The latter condition is satisfied whenever the price-dividend ratio of a claim to next period's output, q_1, is less than one. That is, when investors are willing to pay less than Y_t today for a claim to Y_{t+1}. This requires the discount rate to exceed the growth rate: $r + rp > g$.

We can compute the impulse-response function (IRF) of the surplus with respect to an output growth shock in closed-form when the government issues risk-free debt. These moments are particularly powerful because they do not depend on the properties of the pricing kernel. We start from the expression for the surplus/output ratio in period $t + j$ for $j \geq 1$:

$$s_{t+j} = \frac{S_{t+j}}{Y_{t+j}} = d_{t+j-1} \exp(r - \log \lambda_{t+j}) - d_{t+j}$$

which follows directly from the government's static budget constraint. Prior to the shock, the debt/output ratio is at its long-run mean: $d_t = \bar{d}$.

Proposition 13. *Under Assumption 6 with $P = 2$, if the TVC is satisfied, then the IRF of the surplus/output ratio is given by:*

$$\frac{\partial s_{t+j}}{\partial \log \lambda_{t+1}} = \begin{cases} -\bar{d} \exp\left(r + \frac{K(-\varphi)}{\varphi}\right) + \varphi \bar{d}, & \text{for } j = 1, \\ -\bar{d}\varphi \exp\left(r + \frac{K(-\varphi)}{\varphi}\right) + \varphi \psi_1 \bar{d}, & \text{for } j = 2, \\ -\bar{d}\varphi\psi_{j-2} \exp\left(r + \frac{K(-\varphi)}{\varphi}\right) + \varphi \psi_{j-1} \bar{d}, & \text{for } j \geq 3, \end{cases}$$

where ψ_j denote the coefficients in the autocorrelation function: $\psi_1 = \phi_1$, $\psi_2 = \phi_2 + \phi_1\psi_1$, and $\psi_j = \phi_1\psi_{j-1} + \phi_2\psi_{j-2}$ for $j > 2$, and where we evaluate the derivatives at $\log \lambda_t = \frac{-K(-\varphi)}{\varphi}$, $\forall t$.

The proof is in Appendix C.6.4.

C.5 Insurance Trade-off over Short Horizons

How much consumption smoothing can the government achieve for households by issuing more debt to pay for net transfers in response to bad shocks? When debt is risk-free, no insurance can be provided over long horizons. Over shorter horizons, the government can provide some insurance by backloading some of the aggregate risk. We provide a summary statistic to quantify the insurance provision over each horizon.

It is useful to define the beta of a claim to the debt outstanding H periods from now as:

$$\beta_t^{Dstrip}(H) = \frac{cov_t\left(-M_{t+1}, \mathbb{E}_{t+1}\left[M_{t+1,t+H} D_{t+H}\right]\right)}{D_t var_t(M_{t+1})}. \tag{C.10}$$

This debt strip is an asset that pays the market value of the debt at time $t + H$, D_{t+H}, as its cash flow; $\beta_t^{Dstrip}(H)$ measures the riskiness of that asset. The following proposition connects the cash-flow beta of cumulative surpluses to the beta of the debt strip in (C.10).

Proposition 14. *The return beta of debt equals the cash-flow beta of the discounted surpluses over H periods plus the return beta of debt outstanding H periods from now: $\beta_t^D = \beta_t^{S,CF}(H) + \beta_t^{Dstrip}(H)$. When debt is risk-free today ($\beta_t^D = 0$), then the cash-flow beta of surpluses is determined by minus the return beta of debt outstanding H periods from now: $\beta_t^{S,CF}(H) = -\beta_t^{Dstrip}(H)$.*

The proof is in Appendix C.6.5.

The general Proposition 14 specializes to the following result under an AR(2) process for the debt/output ratio:

Proposition 15. *Under Assumption 3 with $P = 2$, when debt is risk-free, the cash-flow beta of the discounted surpluses over the next H periods is given by:*

$$\beta_t^{S,CF}(H) = \frac{\mathbb{E}_t[M_{t+1}]\mathbb{E}_t[\mathbb{E}_{t+1}[M_{t+1,t+H}D_{t+H}]]}{D_t var_t(M_{t+1})} \times (\exp(K(1-\alpha-\varphi\zeta_{1,H-1}) - K(-\alpha) - K(1-\varphi\zeta_{1,H-1})) - 1)$$

where $\zeta_{1,j}$ and $\zeta_{2,j}$ are defined recursively as $\zeta_{1,j} = \zeta_{1,j-1}\phi_1 + \zeta_{2,j-1}$ and $\zeta_{2,j} = \zeta_{1,j-1}\phi_2$.

The proof is in Appendix C.6.6. The sign of $\beta_t^{S,CF}(H)$ is determined by the sign of $K(1-\alpha-\varphi\zeta_{1,H-1}) - K(-\alpha) - K(1-\varphi\zeta_{1,H-1})$. As $H \to \infty$, this expression converges to minus the output risk premium $-RP$ where $RP = -K(1-\alpha) + K(-\alpha) + K(1)$. In the long run, only output risk is left because debt is cointegrated with output. In the short-run, the risk properties of the surpluses depend on the parameters that govern the riskiness of the debt issuance process.

Just as we studied the cash-flow betas for the cumulative surplus process $\beta_t^{S,CF}(H)$, we can compute cash-flow betas for the cumulative tax revenue process $\beta_t^{T,CF}(H)$ and spending process $\beta_t^{G,CF}(H)$. The following corollary shows how to compute them in the AR(2) case.

Corollary 3. *Under Assumption 3 with $P = 2$, when debt is risk-free, the cash flow beta of cumulative spending and tax revenues satisfy:*

$$\beta_t^{G,CF}(H) = -\sum_{k=1}^{H} \frac{\mathbb{E}_t[M_{t+1}]}{D_t var_t[M_{t+1}]}\mathbb{E}_t[\mathbb{E}_{t+1}[M_{t+1,t+k}G_{t+k}]] \times (\exp(K(1-\alpha-b_g(\phi_1^g)^{k-1}) - K(-\alpha) - K(1-b_g(\phi_1^g)^{k-1})) - 1).$$

$$\beta_t^{T,CF}(H) = \beta_t^{G,CF}(H) + \beta_t^{S,CF}(H).$$

The proof is in Appendix C.6.7. The properties of the $\beta_t^{G,CF}(H)$ depend on the persistence and cyclicality of the exogenous spending/GDP process, which we detail in equation (C.11). The properties of $\beta_t^{T,CF}(H)$ depend on the risk properties of both the surplus and the spending claim.

To make the model's implications for tax revenues as comparable to the data as possible, we posit a more realistic process for spending/output than the one we have worked with hitherto. Specifically, we assume that the government commits to a policy for the spending/output ratio $x_t = G_t/Y_t$ given by:

$$\log x_t = \phi_0^g + \phi_1^g \log x_{t-1} - b_g \log \lambda_t - K(-b_g). \tag{C.11}$$

Figure C.1 reports the risk premia across horizons using the same debt process in the disaster model. Compared to the baseline model, the risk premia are larger in absolute magnitudes due to the disaster risk. However, the extent to which the government can provide insurance to taxpayers remains very limited. The risk premium on the surplus strip has to be negative starting in year 4; the cumulative surplus claim risk premium begins its decline in year 4. The latter turns negative in year 13.

C.6 Proofs Disaster Model

C.6.1 Proof of Proposition 10

Proof. To verify the expression, first conjecture the pricing of the output strip is $\mathbb{E}_t[M_{t,t+k}Y_{t+k}] = \xi_k Y_t$, for $k \geq 0$. Then $\xi_0 = 1$ and

$$\xi_k Y_t = \mathbb{E}_t[M_{t,t+k}Y_{t+k}] = \mathbb{E}_t[M_{t,t+1}\xi_{k-1}Y_{t+1}] = \exp(-r - K(-\alpha) + K(1-\alpha))\xi_{k-1}Y_t,$$
$$\xi_k Y_t = \exp(-r - K(-\alpha) + K(1-\alpha))\xi_{k-1}Y_t,$$

which verifies the conjecture and implies $\xi_k = \xi_{k-1}\exp(-r - K(-\alpha) + K(1-\alpha))$. Similarly, we define a k-period surplus strip as a claim to S_{t+k}, with price given by $\mathbb{E}_t[M_{t,t+k}S_{t+k}] = \chi_k Y_t$. The pricing of the first surplus strip is given by the following expression:

$$\mathbb{E}_t[M_{t,t+1}S_{t+1}] = \mathbb{E}_t\left[M_{t,t+1}\{-dY_{t+1}\left(1 - R_t^f \exp[-(\log\lambda_{t+1})]\right)\}\right] = -d\mathbb{E}_t[M_{t,t+1}Y_{t+1}] + dY_tR_t^f\mathbb{E}_t[M_{t,t+1}],$$
$$= [1 - \exp(-r - K(-\alpha) + K(1-\alpha))]\, dY_t,$$

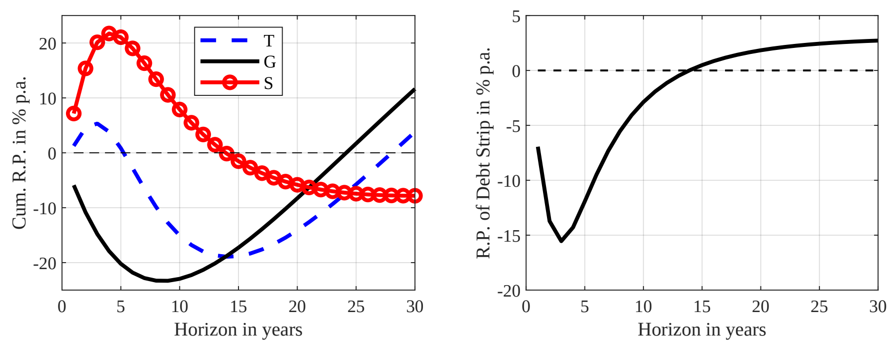
where the first equality uses the definition of the surplus and the government budget constraint.
This implies $\chi_1 = [1 - \exp(-r - K(-\alpha) + K(1-\alpha))]\, d$. Then, similarly, the pricing of the kth surplus strip is given by:

$$\chi_k Y_t = \mathbb{E}_t[M_{t,t+k}S_{t+k}] = \mathbb{E}_t[M_{t,t+1}\mathbb{E}_{t+1}[M_{t+1,t+k}S_{t+k}]] = \mathbb{E}_t[M_{t,t+1}\chi_{k-1}Y_{t+1}] = \chi_{k-1}\exp(-r - K(-\alpha) + K(1-\alpha))Y_t.$$

Figure C.1: Risk Premia Across Horizons

The figure plots the risk premium of cumulative discounted cash flows, $\beta_t^{i,CF}(h) \times \frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]}$, in the left panel against the horizon h, for $i \in \{S, G, T\}$. The right panel plots the risk premium on the debt strips, $\beta_t^{Dstrip}(h) \times \frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]}$, with debt strip beta given by (C.10). The parameters are given in Table C.1, and the risk premia are evaluated at the long-run mean log spending/output and log debt/output ratios.

AR(2) Debt/Output ($\phi_1 = 1.38, \phi_2 = -0.45$)



Note that this calculation also implies that we cannot simply price these strips off the risk-free yield curve, even though the entire debt is risk-free. The solution is given by:

$$\chi_1 = d\left[1 - \exp(-r - K(-\alpha) + K(1-\alpha))\right]; \chi_k = \chi_{k-1}\exp(-r - K(-\alpha) + K(1-\alpha)),$$

which implies that $\mathbb{E}_t \sum_{k=1}^{\infty}[M_{t,t+k}S_{t+k}] = \sum_{k=1}^{\infty}\chi_k Y_t = \chi_1(1 + q_1 + q_1^2 + \ldots)Y_t = \frac{1-q_1}{1-q_1} dY_t = dY_t$, where $q_1 = \exp(-r - K(-\alpha) + K(1-\alpha))$. □

C.6.2 Proof of Proposition 11

Proof. From the one-period government budget constraint, we obtain that: $\frac{T_t}{Y_t} = x - d\left(1 - R^f \frac{Y_{t-1}}{Y_t}\right)$. The return on the tax claim can be stated as:

$$R_{t+1}^T = \frac{P_{t+1}^T + T_{t+1}}{P_t^T} = \frac{(d + x\frac{q_1}{1-q_1})Y_{t+1} + (x - d\left(1 - R^f \frac{Y_t}{Y_{t+1}}\right))Y_{t+1}}{(d + x\frac{q_1}{1-q_1})Y_t} = \frac{x\frac{1}{1-q_1}Y_{t+1}}{(d + x\frac{q_1}{1-q_1})Y_t} + \frac{dR^f}{(d + x\frac{q_1}{1-q_1})}.$$

Similarly, the return on the spending claim can be stated as:

$$R_{t+1}^G = \frac{P_{t+1}^G + G_{t+1}}{P_t^G} = \frac{x\frac{q_1}{1-q_1}Y_{t+1} + xY_{t+1}}{x\frac{q_1}{1-q_1}Y_t} = \frac{x\frac{1}{1-q_1}Y_{t+1}}{x\frac{q_1}{1-q_1}Y_t}.$$

Armed with these expressions, we get the following expression for the covariance: $cov(R_{t+1}^T, M_{t,t+1}) = \frac{x\frac{q_1}{1-q_1}}{(d + x\frac{q_1}{1-q_1})} cov(R_{t+1}^G, M_{t,t+1})$, which also translates to $\mathbb{E}_t\left[R_{t+1}^T - R^f\right] = \frac{x\frac{q_1}{1-q_1}}{d + x\frac{q_1}{1-q_1}} \mathbb{E}_t\left[R_{t+1}^C - R^f\right]$. □

C.6.3 Proof of Proposition 12

Proof. The debt dynamics are described by the following AR(2) process:

$$\log d_t = \phi_0 + \phi_1 \log d_{t-1} + \phi_2 \log d_{t-2} - \varphi \log \lambda_t - K(-\varphi).$$

Let $\log \bar{d}$ denote the long-run mean, s.t.

$$\log \bar{d} = \phi_0 + \phi_1 \log \bar{d} + \phi_2 \log \bar{d} - \varphi \log \lambda_t - K(-\varphi).$$

Then,

$$\log \bar{d} = \frac{1}{1 - \phi_1 - \phi_2}(\phi_0 - \varphi \mathbb{E}[\log \lambda_t] - K(-\varphi)).$$

Also,this implies that:

$$\log D_t = \log d_t + \log Y_t = \phi_0 + \phi_1 \log d_{t-1} + \phi_2 \log d_{t-2} + \log Y_{t-1} + (1-\varphi) \log \lambda_t - K(-\varphi).$$

and we can price a claim to future debt:

$$\begin{aligned}\mathbb{E}_t[M_{t,t+1}D_{t+1}] &= \mathbb{E}_t[M_{t,t+1}Y_{t+1}d_{t+1}] \\ &= \exp(\phi_1 \log d_t + \phi_2 \log d_{t-1} + \phi_0 - r + K(1-\alpha-\varphi) - K(-\alpha) - K(-\varphi))Y_t.\end{aligned}$$

Let

$$\xi_1 = \phi_0 - r + K(1-\alpha-\varphi) - K(-\alpha) - K(-\varphi).$$

Then $\mathbb{E}_t[M_{t,t+1}D_{t+1}] = \exp(\phi_1 \log d_t + \phi_2 \log d_{t-1} + \xi_1)Y_t$.

Then, assume

$$\mathbb{E}_t[M_{t,t+j}D_{t+j}] = \exp(\zeta_{1,j} \log d_t + \zeta_{2,j} \log d_{t-1} + \sum_{k=1}^{j} \xi_k)Y_t.$$

Next, by induction,

$$\begin{aligned}\mathbb{E}_t[M_{t,t+j}D_{t+j}] &= \mathbb{E}_t[M_{t,t+1}\mathbb{E}_{t+1}[M_{t+1,t+j}D_{t+j}]] \\ &= \mathbb{E}_t[M_{t,t+1}\exp(\zeta_{1,j-1}\log d_{t+1} + \zeta_{2,j-1}\log d_t + \sum_{k=1}^{j-1}\xi_k)Y_{t+1}] \\ &= \mathbb{E}_t[\exp(-r - K(-\alpha) + (1-\alpha-\zeta_{1,j-1}\varphi)\log\lambda_{t+1} + \zeta_{1,j-1}(\phi_0 + \phi_1\log d_t + \phi_2 \log d_{t-1} - K(-\varphi)) + \zeta_{2,j-1}\log d_t + \sum_{k=1}^{j-1}\xi_k)Y_t] \\ &= \mathbb{E}_t[\exp(-r - K(-\alpha) + K(1-\alpha-\zeta_{1,j-1}\varphi) + \zeta_{1,j-1}(\phi_0 + \phi_1\log d_t + \phi_2 \log d_{t-1} - K(-\varphi)) + \zeta_{2,j-1}\log d_t + \sum_{k=1}^{j-1}\xi_k)Y_t] \\ &= \exp((\zeta_{1,j-1}\phi_1 + \zeta_{2,j-1})\log d_t + \zeta_{1,j-1}\phi_2 \log d_{t-1} + \sum_{k=1}^{j}\xi_k)Y_t.\end{aligned}$$

with

$$\begin{aligned}\xi_j &= -r - K(-\alpha) + K(1-\alpha-\zeta_{1,j-1}\varphi) + \zeta_{1,j-1}(\phi_0 - K(-\varphi)), \\ \zeta_{1,j} &= \zeta_{1,j-1}\phi_1 + \zeta_{2,j-1}, \\ \zeta_{2,j} &= \zeta_{1,j-1}\phi_2.\end{aligned}$$

For large enough j, ξ_j converges to the following expression: $-r - K(-\alpha) + K(1-\alpha)$.

So the TVC can be expressed as follows:

$$\lim_{T\to\infty} \mathbb{E}_t\left[M_{t,t+T}D_{t+T}\right] = \lim_{T\to\infty} \exp(\zeta_{1,T}\log d_t + \zeta_{2,T}\log d_{t-1} + \sum_{k=1}^{T}\xi_k)Y_t = 0,$$

which is satisfied iff $-r - K(-\alpha) + K(1-\alpha) < 0$. □

C.6.4 Proof of Proposition 13

Proof. The surplus/output ratio is given by:

$$s_t = \frac{T_t - G_t}{Y_t} = d_{t-1} \exp(r - \log \lambda_t) - d_t.$$

We use ψ_j to denote the infinite MA representation of the debt/output process:

$$\log d_t = \log \bar{d} - \sum_{j=0}^{\infty} \psi_j (\varphi \log \lambda_{t-j} + K(-\varphi)),$$

where $\psi_1 = \phi_1$, $\psi_2 = \phi_2 + \phi_1 \psi_1$, and $\psi_j = \phi_1 \psi_{j-1} + \phi_2 \psi_{j-2}$ for $j > 2$.

We evaluate the derivatives at $\log \lambda_t = -K(-\varphi)/\varphi$ for all t, such that $\varphi \log \lambda_t + K(-\varphi) = 0$. Then, we can simplify the surplus and its derivative as follows:

$$\begin{aligned} s_{t+1} &= \exp(\log \bar{d} - \sum_{j=0}^{\infty} \psi_j (\varphi \log \lambda_{t-j} + K(-\varphi))) \exp(r - \log \lambda_{t+1}) \\ &- \exp(\log \bar{d} - \sum_{j=0}^{\infty} \psi_j (\varphi \log \lambda_{t-j+1} + K(-\varphi))) \end{aligned}$$

and

$$\frac{\partial s_{t+1}}{\partial \log \lambda_{t+1}} = -\bar{d} \exp\left(r + \frac{K(-\varphi)}{\varphi}\right) + \varphi \bar{d}.$$

Similarly,

$$\begin{aligned} s_{t+2} &= \exp(\log \bar{d} - \sum_{j=0}^{\infty} \psi_j (\varphi \log \lambda_{t-j+1} + K(-\varphi))) \exp(r - \log \lambda_{t+2}) \\ &- \exp(\log \bar{d} - \sum_{j=0}^{\infty} \psi_j (\varphi \log \lambda_{t-j+2} + K(-\varphi))) \end{aligned}$$

and

$$\frac{\partial s_{t+2}}{\partial \log \lambda_{t+1}} = -\varphi \bar{d} \exp\left(r + \frac{K(-\varphi)}{\varphi}\right) + \psi_1 \varphi \bar{d}.$$

Similarly,

$$\frac{\partial s_{t+j}}{\partial \log \lambda_{t+1}} = -\psi_{j-2} \varphi \bar{d} \exp\left(r + \frac{K(-\varphi)}{\varphi}\right) + \psi_{j-1} \varphi \bar{d}.$$

□

C.6.5 Proof of Proposition 14

Proof. We start from the return equation and take expectations.

$$D_t \mathbb{E}_{t+1}[R^D_{t+1}] = \mathbb{E}_{t+1}[\sum_{j=1}^{h} M_{t+1,t+j} S_{t+j}] + \mathbb{E}_{t+1}[M_{t+1,t+h} D_{t+h}].$$

We obtain the following result:

$$D_t Cov_t(-M_{t+1}, \mathbb{E}_{t+1}[R^D_{t+1}]) = Cov_t(-M_{t+1}, \mathbb{E}_{t+1}[\sum_{j=1}^{h} M_{t+1,t+j} S_{t+j}]) + Cov_t(-M_{t+1}, \mathbb{E}_{t+1}[M_{t+1,t+h} D_{t+h}]).$$

After dividing both sides by $D_t Var_t[M_{t+1}]$, we obtain the debt return beta on the left-hand side, the surplus cash-flow beta as the first term on the right-hand side, and the debt strip beta as the second term on the right-hand side. □

C.6.6 Proof of Proposition 15

Proof. Since

$$M_{t+1} = \exp(\log \beta - \alpha \log \lambda_{t+1}),$$

and

$$d_{t+1}Y_{t+1} = \exp(\phi_0 + \phi_1 \log d_t + \phi_2 \log d_{t-1} + \log Y_t + (1-\varphi)\log \lambda_{t+1} - K(-\varphi)),$$

we have

$$\begin{aligned} cov_t(M_{t+1}, S_{t+1}) &= cov_t(M_{t+1}, -d_{t+1}Y_{t+1}) \\ &= -\mathbb{E}_t[M_{t+1}d_{t+1}Y_{t+1}] + \mathbb{E}_t[M_{t+1}]\mathbb{E}_t[d_{t+1}Y_{t+1}] \\ &= -\mathbb{E}_t[M_{t+1}]\mathbb{E}_t[d_{t+1}Y_{t+1}](\exp(K(1-\alpha-\varphi) - K(-\alpha) - K(1-\varphi)) - 1) \end{aligned}$$

Similarly, since

$$\begin{aligned} \mathbb{E}_{t+1}[M_{t+1,t+j}D_{t+j}], &= \exp(\zeta_{1,j-1}\log d_{t+1} + \zeta_{1,j-2}\phi_2 \log d_t + \sum_{k=1}^{j-1}\xi_k)Y_{t+1}, \\ &= \exp((1-\varphi\zeta_{1,j-1})\log \lambda_{t+1} + \ldots), \end{aligned}$$

where the omitted terms do not depend on λ_{t+1}, then, we have

$$\begin{aligned} cov_t(M_{t+1}, \mathbb{E}_{t+1}[\sum_{k=1}^{j} M_{t+1,t+k}S_{t+k}]) &= cov_t(M_{t+1}, -\mathbb{E}_{t+1}[M_{t+1,t+j}D_{t+j}]) \\ &= -\mathbb{E}_t[M_{t+1}\mathbb{E}_{t+1}[M_{t+1,t+j}D_{t+j}]] + \mathbb{E}_t[M_{t+1}]\mathbb{E}_t[\mathbb{E}_{t+1}[M_{t+1,t+j}D_{t+j}]] \\ &= -\mathbb{E}_t[M_{t+1}]\mathbb{E}_t[\mathbb{E}_{t+1}[M_{t+1,t+j}D_{t+j}]] \\ &\times (\exp(K(1-\alpha-\varphi\zeta_{1,j-1}) - K(-\alpha) - K(1-\varphi\zeta_{1,j-1})) - 1) \end{aligned}$$

Recall

$$\begin{aligned} \beta_t^{S,CF}(h) &= -\frac{cov_t(M_{t+1}, \mathbb{E}_{t+1}[\sum_{k=1}^{h} M_{t+1,t+k}S_{t+k}])}{D_t var_t(M_{t+1})} \\ &= \frac{\mathbb{E}_t[M_{t+1}]\mathbb{E}_t[\mathbb{E}_{t+1}[M_{t+1,t+h}D_{t+h}]]}{D_t var_t(M_{t+1})} \\ &\times (\exp(K(1-\alpha-\varphi\zeta_{1,h-1}) - K(-\alpha) - K(1-\varphi\zeta_{1,h-1})) - 1) \end{aligned}$$

This also implies

$$\begin{aligned} RP_t^{Dstrip}(h) &= \frac{cov_t\left(M_{t+1}, \frac{(\mathbb{E}_{t+1}-\mathbb{E}_t)M_{t+1,t+h}D_{t+h}}{\mathbb{E}_t[M_{t+1,t+h}D_{t+h}]}\right)}{var_t(M_{t+1})}\frac{var_t(M_{t+1})}{\mathbb{E}_t[M_{t+1}]} \\ &= 1 - \exp(K(1-\alpha-\varphi\zeta_{1,h-1}) - K(-\alpha) - K(1-\varphi\zeta_{1,h-1})). \end{aligned}$$

□

C.6.7 Proof of Corollary 3

Proof. If spending/GDP ratio x follows

$$\log x_t = \phi_0^g + \phi_1^g \log x_{t-1} - b_g \log \lambda_t - K(-b_g).$$

then

$$\begin{aligned} cov_t(M_{t+1}, \mathbb{E}_{t+1}[M_{t+1,t+j}x_{t+j}Y_{t+j}]) &= \mathbb{E}_t[M_{t+1}\mathbb{E}_{t+1}[M_{t+1,t+j}x_{t+j}Y_{t+j}]] - \mathbb{E}_t[M_{t+1}]\mathbb{E}_t[\mathbb{E}_{t+1}[M_{t+1,t+j}x_{t+j}Y_{t+j}]] \\ &= \mathbb{E}_t[M_{t+1}]\mathbb{E}_t[\mathbb{E}_{t+1}[M_{t+1,t+j}x_{t+j}Y_{t+j}]] \\ &\times (\exp(K(1-\alpha-b_g(\phi_1^g)^{j-1}) - K(-\alpha) - K(1-b_g(\phi_1^g)^{j-1})) - 1) \end{aligned}$$

and

$$
\begin{aligned}
\beta_t^{G,CF}(h) &= -\frac{cov_t(M_{t+1}, \mathbb{E}_{t+1}[\sum_{k=1}^{h} M_{t+1,t+k} G_{t+k}])}{D_t var_t(M_{t+1})} \\
&= -\frac{\mathbb{E}_t[M_{t+1}]}{D_t var_t(M_{t+1})} \sum_{k=1}^{h} \mathbb{E}_t[\mathbb{E}_{t+1}[M_{t+1,t+k} x_{t+k} Y_{t+k}]] \\
&\quad \times (\exp(K(1-\alpha-b_g(\phi_1^g)^{k-1}) - K(-\alpha) - K(1-b_g(\phi_1^g)^{k-1})) - 1)
\end{aligned}
$$

□

D Insurance Trade-off with Transitory Risk

We study the insurance trade-off in a model where output only experiences transitory shocks. The SDF is given by:

$$
\log M_{t+1} = \log\beta - \alpha\Delta\log Y_{t+1},
$$

and the log aggregate endowment is given by

$$
\log Y_{t+1} = \rho_c \log Y_t + \log\lambda_{t+1}, \tag{D.1}
$$

where $\log\lambda_{t+1}$ is distributed i.i.d. over time. Transitory output (or productivity) risk is the standard assumption in macro-economic models, as well as in models of optimal fiscal policy. For an example, see Bhandari et al. (2017, pp. 653), which features a mean-reverting process for productivity growth and government spending. See Chari et al. (1994); Debortoli et al. (2017) for other examples. Our qualitative results would go through if we used the equilibrium SDF implied by these models.

While not the focus in those literatures, models with only transitory risk have unappealing asset pricing properties. Specifically, in models with only transitory shocks, the market price of aggregate risk is typically low. The modern asset pricing literature has consistently found that permanent cash-flow shocks—shocks to the growth rate, rather than to the level of output—receive a high price of risk in the market. Substantial, permanent priced risk is necessary to explain the high equity risk premium. Models without priced permanent risk imply an unrealistic amount of long-run interest rate risk. Long-term bonds are the riskiest assets in economies with only temporary risk (Backus, Chernov, and Zin, 2014). This is counter-factual as the expected return on the stock market exceeds the expected return on a long-term bond in the data.

That said, we investigate the trade-off faced by the government in this textbook economy with only transitory risk. We find that, in order to keep the debt risk-free in the presence of this interest rate risk, the government needs to deliver an even safer surplus process than in our benchmark model with only permanent risk. As a result, interest rate risk reduces the scope for insurance of households, and the trade-off between insuring households and arbitrageurs is even steeper than in the benchmark model.

We calibrate the endowment shock process $\log\lambda_{t+1}$ in the same way as in the disaster economy of Appendix C. We use $K(s)$ to denote the cumulant generating function of $\log\lambda_{t+1}$. It follows that the log of the real risk-free rate is:

$$
r_t = -\log\beta - K(-\alpha) - \alpha(1-\rho_c)\log Y_t.
$$

The price of an output strip $q_t^1 = \mathbb{E}_t[M_{t+1}Y_{t+1}]$ is given by:

$$
\begin{aligned}
\log q_t^1 &= \log\beta + \log\mathbb{E}[\exp((1-\alpha)\log\lambda_{t+1})] + (1-\alpha)\rho_c\log Y_t + \alpha\log Y_t \\
&= \log\beta + K(1-\alpha) + (\alpha + (1-\alpha)\rho_c)\log Y_t.
\end{aligned}
$$

To price the debt strip, we consider the same AR(2) debt/GDP dynamics as in the main text. Then,

$$
\begin{aligned}
&\mathbb{E}_t[M_{t,t+1}D_{t+1}] = \mathbb{E}_t[M_{t,t+1}Y_{t+1}d_{t+1}] \\
&= \mathbb{E}_t[\exp(\log\beta - (\alpha-1)(\rho_c-1)\log Y_t + \phi_0 + \phi_1\log d_t + \phi_2\log d_{t-1} + (1-\alpha-\varphi)\log\lambda_{t+1} - K(-\varphi))]Y_t \\
&= \exp(\phi_1\log d_t + \phi_2\log d_{t-1} + (\alpha-1)(1-\rho_c)\log Y_t + \xi_1)Y_t
\end{aligned}
$$

where $\xi_1 = \phi_0 + \log\beta + K(1-\alpha-\varphi) - K(-\varphi)$.

We conjecture and verify that for all $j > 0$, $\mathbb{E}_t[M_{t,t+j}D_{t+j}] = \exp(\zeta_{1,j}\log d_t + \zeta_{2,j}\log d_{t-1} + \omega_j\log Y_t + \sum_{k=1}^{j}\xi_k)Y_t$. By induction, the coefficients are

$$
\begin{aligned}
\xi_j &= \log\beta + K(1-\alpha-\zeta_{1,j-1}\varphi + \omega_{j-1}) + \zeta_{1,j-1}(\phi_0 - K(-\varphi)) \\
\zeta_{1,j} &= \zeta_{1,j-1}\phi_1 + \zeta_{2,j-1} \\
\zeta_{2,j} &= \zeta_{1,j-1}\phi_2
\end{aligned}
$$

$$\omega_j = \omega_{j-1}\rho_c + (\alpha - 1)(1 - \rho_c) = (\alpha - 1)(1 - \rho_c^j)$$

Since

$$\mathbb{E}_{t+1}[M_{t+1,t+j}D_{t+j}] = \exp(\zeta_{1,j-1}\log d_{t+1} + \zeta_{2,j-1}\log d_t + \omega_{j-1}\log Y_{t+1} + \sum_{k=1}^{j-1}\xi_k)Y_{t+1}$$
$$= \exp((1 - \varphi\zeta_{1,j-1} + \omega_{j-1})\log \lambda_{t+1} + ...)$$

we have

$$cov_t(M_{t+1}, \mathbb{E}_{t+1}[\sum_{k=1}^{j} M_{t+1,t+k}S_{t+k}]) = cov_t(M_{t+1}, -\mathbb{E}_{t+1}[M_{t+1,t+j}D_{t+j}])$$
$$= -\mathbb{E}_t[M_{t+1}\mathbb{E}_{t+1}[M_{t+1,t+j}D_{t+j}]] + \mathbb{E}_t[M_{t+1}]\mathbb{E}_t[\mathbb{E}_{t+1}[M_{t+1,t+j}D_{t+j}]]$$
$$= -\mathbb{E}_t[M_{t+1}]\mathbb{E}_t[\mathbb{E}_{t+1}[M_{t+1,t+j}D_{t+j}]](\exp(K(1-\alpha-\varphi\zeta_{1,j-1}+\omega_{j-1}) - K(-\alpha) - K(1-\varphi\zeta_{1,j-1}+\omega_{j-1})) - 1)$$

Therefore, the cash-flow beta of the cumulative discounted surplus process over horizon H is given by:

$$\beta_t^{S,CF}(H) = -\frac{cov_t(M_{t+1}, \mathbb{E}_{t+1}[\sum_{k=1}^{H} M_{t+1,t+k}S_{t+k}])}{D_t var_t(M_{t+1})}$$
$$= \frac{\mathbb{E}_t[M_{t+1}]\mathbb{E}_t[\mathbb{E}_{t+1}[M_{t+1,t+H}D_{t+H}]]}{D_t var_t(M_{t+1})}(\exp(K(1-\alpha-\varphi\zeta_{1,H-1}+\omega_{H-1}) - K(-\alpha) - K(1-\varphi\zeta_{1,H-1}+\omega_{H-1})) - 1)$$

where

$$\mathbb{E}_t[M_{t+1}] = \mathbb{E}_t[\exp(\log\beta - \alpha\log\lambda_{t+1} - \alpha(\rho_c - 1)\log Y_t)]$$
$$= \exp\left(\log\beta + K(-\alpha) + \alpha(1-\rho_c)\log Y_t\right) = \exp\left(-r_t\right).$$

Figure D.1: Risk Premia Across Horizons with Transitory Output Risk

The figure plots the risk premium of cumulative discounted cash flows, $\beta_t^{i,CF}(H) \times \frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]}$, in the left panel against the horizon H. The right panel plots minus the risk premium on the debt strips: $RP_t^{Dstrip}(H)$. The parameters are given in Table C.1, and the risk premia are evaluated at the long-run mean log spending/output and log debt/output ratios. The aggregate output process satisfies Eq. (D.1), and we adjust the growth parameter μ so that the unconditional average growth is zero in this model.

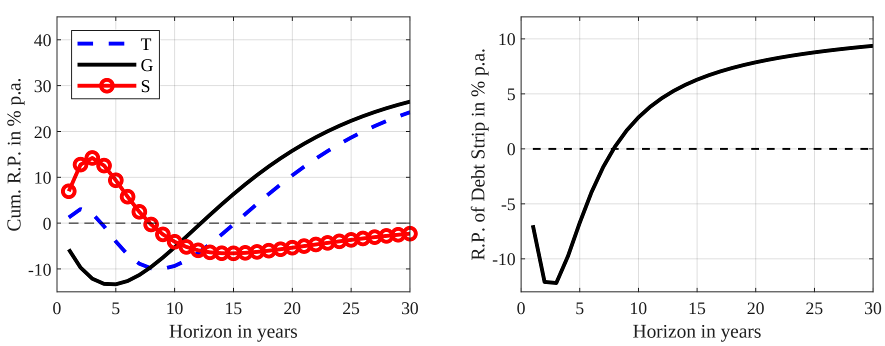


Figure D.1 plots the risk premia across horizons for the model with only transitory output risk. The red line is for the surplus process and plots the product of the cash-flow beta $\beta_t^{S,CF}(H)$ and the market price of risk. To keep debt risk-free, the government has to offset the interest rate risk by generating safe surpluses, or equivalently, risky taxpayer liabilities. Surprisingly, even when there are no permanent shocks to output and the pricing kernel, the government cannot insure taxpayers over longer horizons. In fact, the trade-off worsens. Because the output innovations are transitory and the debt/output ratio is stationary, the output component of the risk premium converges to zero as the horizon increases. The interest rate risk does not converge to zero, and explains the entire long-run debt strip risk premium, plotted in the right panel, which is large and positive. Recall that the long-term bond is the riskiest asset in an economy with only transitory risk. The large and positive debt strip beta implies a negative cumulative surplus cash-flow beta, which indicates the inability of the government to provide insurance to taxpayers over longer horizons. Some insurance is possible over short horizons, just like in the benchmark model with the same AR(2) dynamics for the debt/output ratio but with permanent rather than transitory output risk.

This result does not hinge on the specific pricing kernel we use. In the absence of arbitrage opportunities, if the pricing kernel is not subject to permanent innovations, the zero-coupon bond with the longest maturity will always earn the highest expected log return given by the entropy of the pricing kernel. Hence, we know that this interest rate risk premium exceeds the log risk premium of the consumption strip.

In sum, while the transitory nature of output risk allows for insurance of taxpayers in the short-run, this is more than offset by the rising interest rate risk which accumulates with the horizon. Compared to the permanent risk case, we have replaced long-run output risk with more long-run interest rate risk. The main results of the paper go through for the case of transitory output risk as well.