NBER WORKING PAPER SERIES

AN EXPLORATION OF TREND-CYCLE DECOMPOSITION METHODOLOGIES IN SIMULATED DATA

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Working Paper 26750
http://www.nber.org/papers/w26750

NATIONAL BUREAU OF ECONOMIC RESEARCH
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February 2020

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NBER Working Paper No. 26750
February 2020
JEL No. E32

ABSTRACT

This paper uses simulations to explore the properties of the HP filter of Hodrick and Prescott (1997), the BK filter of Baxter and King (1999), and the H filter of Hamilton (2018) that are designed to decompose a univariate time series into trend and cyclical components. Each simulated time series approximates the natural logarithms of U.S. Real GDP, and they are a random walk, an ARIMA model, two unobserved components models, and models with slowly changing nonstationary stochastic trends and definitive cyclical components. In basic time series, the H filter dominates the HP and BK filters in more closely characterizing the underlying framework, but in more complex models, the reverse is true.

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An online appendix is available at http://www.nber.org/data-appendix/w26750

1 Introduction

The purpose of this paper is to explore how well or poorly three univariate methodologies that seek to decompose a time series into a growth or trend component and a cyclical component perform in controlled simulation environments. Such results should provide guidance to those seeking to decompose actual economic time series. The three filtering methodologies are the HP filter of Hodrick and Prescott (1997), the BK filter of Baxter and King (1999), and the H filter of Hamilton (2018).

Each of the simulated time series is calibrated to the natural logarithm of quarterly U.S. Real GDP from 1947 to 2019, and each series has approximately the same sample mean and standard deviation as the continuously compounded rate of growth of real GDP.[1] I examine six series beginning with a random walk and an ARIMA model, in which there is no distinction between trend and cycle. I then examine two unobserved components models with well-defined time series trends and cycles. Finally, I consider two models with slowly changing nonstationary trends and definitive cyclical components. In the models without definitive cyclical components, I adopt the trend and cycle definitions in Hamilton (2018), which are defined below.

The findings indicate that the HP filter and the BK filter are quite similar and do relatively poorly at identifying cyclical components in straightforward time series environments in which the first-differenced series is stationary. The H filter does much better in these simple environments. As the time series become more complex, the performance of the HP and BK filters more closely characterize the underlying cyclical frameworks than the H filter. This is true even when the simulations are from well-defined unobserved components models with constant parameters. Thus, which methodology one might use depends on one's priors about the nature of the series.

2 Background Motivation

Economists have long sought to understand the nature of business cycles. In 1978, the stylized facts of business cycles utilized the NBER's methodology as exemplified in Burns and Mitchell (1946). That approach required the investigator's judgment to identify peaks and troughs in a time series and then to compare specific cycles and reference cycles. It was not well suited to the computer age.

Hodrick and Prescott (1997) sought to develop an alternative to the NBER's methodology that was designed to be easily applied to a variety of time series.[2] They wrote (p.1):

> The purpose of this paper is to document some features of aggregate economic fluctuations sometimes referred to as business cycles. The investigation uses quarterly date from the postwar

[1] All remaining references in the paper to GDP will refer to the natural logarithm of quarterly U.S. Real GDP.

[2] The original paper, Hodrick and Prescott (1980), was started in 1978. The paper was submitted to the *American Economic Review* and was rejected by a referee who advocated use of Bayesian smoothness priors. At the fall 1980 Carnegie-Rochester Conference Prescott discussed the paper with William Dewald, who was the editor of the *Journal of Money, Credit, and Banking* (JMCB). Dewald solicited the paper for the JMCB with the promise that it would be reviewed by a top referee, who revealed himself in his referee's report to be Milton Friedman. Friedman argued that the paper should not be published because it was "measurement without theory," a reference to Koopmans (1947) criticism of Burns and Mitchell. Dewald felt compelled to accept Friedman's recommendation. In 1995, after the paper had become one of the most widely cited unpublished papers in economics, Stephen Cecchetti, the editor of the JMCB, approached the authors about publishing the paper in the JMCB without the knowledge that it had been previously rejected. The authors accepted the invitation with the restriction that they would only revise the paper to update its tables with current data and that they would not be required to address any of the controversies about the paper that had arisen over the years. All future references to the paper in this paper will be to the published version.

> U.S. economy. The fluctuations studied are those that are too rapid to be accounted for by slowly changing demographic and technological factors and changes in stocks of capital that produce secular growth in output per capita.

Hodrick and Prescott (1997) sought to develop stylized facts that could guide the development of equilibrium models of the business cycle. They stated (p.2):

> This study should be viewed as documenting some systematic deviations from the restrictions upon observations implied by neoclassical growth theory. Our statistical approach does not utilize standard time series analysis. Our prior knowledge concerning the processes generating the data is not of the variety that permits us to specify a probability model as required for application of that analysis. We proceed in a more cautious manner that requires only knowledge that can be supported by economic theory. The maintained hypothesis, based on growth theory considerations, is that the growth component of aggregate economic time series varies smoothly over time.

Section 3 of this paper develops the formalities of the HP filter and discusses some of the criticisms that have arisen over the years.

An alternative econometric approach to decomposing a time series into growth and cyclical components is the unobserved components model. Early examples of this class of models were developed by Nelson and Plosser (1982), Harvey (1985), Watson (1986), and Clark (1987). These formal econometric approaches take a stand on the time series aspects of the growth component and the cyclical component as well the correlations of their innovations. Section 4 presents the Clark (1987) model. Because of the complexity of estimating such unobserved components models, especially in simulation environments in which the model is clearly false, I do not include unobserved components models in the simulations. On the other hand, I do use the Clark (1987) model estimated over the full sample as one of the simulated time series.

Baxter and King (1999) provide another way of isolating the business cycle component of a time series using filtering methods based on frequency domain analysis. Their analysis is motivated by reasoning similar to Hodrick and Prescott (1997). They wrote (p.575):

> Contemporary students of the business cycle still face the same basic issue as Burns and Mitchell (1946) did fifty years ago: How should one isolate the cyclical component of an economic time series? In particular, how should one separate business-cycle elements from slowly evolving secular trends and rapidly varying seasonal or irregular components?

Their solution involves defining what one means by the business cycle and transforming or filtering the macroeconomic series to coincide with this definition. They wrote (p.575):

> Technically, we develop approximate band-pass filters that are constrained to produce stationary outcomes when applied to growing time series. For the empirical applications in this paper, we adopt the definition of business cycles suggested by the procedures and findings of NBER researchers like Burns and Mitchell. Burns and Mitchell specified that business cycles were cyclical components of no less than six quarters (eighteen months) in duration, and they found that U.S. business cycles typically last fewer than 32 quarters (eight years).

Their solution is a band-pass filter. I discuss the BK filter in Section 5.

Hamilton (2018) offers the most recent alternative methodology for decomposing a time series into growth and cyclical components. As the title of his paper suggests, Hamilton (2018) first examines several perceived flaws in the Hodrick and Prescott (1997) approach. Hamilton (2018) then states (p. 836):

> Here I suggest an alternative concept of what we might mean by the cyclical component of a possibly nonstationary series: How different is the value at $t + h$ from the value that we would have expected to see based on its behavior through date t? This concept of the cyclical component has several attractive features. First, as Den Haan (2000) noted, the forecast error is stationary for a wide class of nonstationary processes. Second, the primary reason that we would be wrong in predicting the value of most macro and financial variables at a horizon of $h = 8$ quarters ahead is cyclical factors such as whether a recession occurs over the next two years and the timing of recovery from any downturn.

Section 6 presents the H filter, which Hamilton (2018) notes is related to the Beveridge and Nelson (1981) decomposition of a time series into a stochastic trend that is a random walk and a stationary cyclical component. I consequently discuss the Beveridge and Nelson (1981) decomposition and how it differs from the Hamilton (2018) decomposition.

Section 7 examines the Morley, Nelson, and Zivot (2003) ARIMA model of GDP and their critique of the implications of unobserved components models for the relative importance of the unit root process in the Beveridge and Nelson (1981) decomposition. I also use this ARIMA model as one of the simulated time series.

Section 8 considers the variance ratio analysis of Cochrane (1988). Examining long-horizon variance ratios is an alternative way to analyze the importance of the unit root component in a time series.

Section 9 provides statistical analysis of the actual GDP series, and section 10 provides the simulation analysis. I explore six different time series of increasing in complexity. While the H filter dominates the HP and BK filters in simple environments, the reverse is true when the simulated models contain growth components that vary slowly over time.

Section 11 considers another possible trend series: potential GDP as calculated by the Congressional Budget Office. After demonstrating that this trend changes significantly over time because it is substantively revised as new information arrives, I find that the volatility of the cyclical component derived using the most recent version of potential GDP is in between the volatilities of the HP and H cycles.

Section 12 provides concluding remarks. A series of Appendixes provides some technical details, and an Online Appendix contains results of the simulations for a sample size that is double the size considered in the paper. These simulations suggest that none of the implications of the analysis is changed in this larger sample size.

3 The Hodrick and Prescott (1997) Filter

The HP filter decomposes a time series, y_t, into a growth or trend component, g_t, and a cyclical component, c_t:

$$y_t = g_t + c_t. \tag{1}$$

The decomposition constructs the sequence of growth components to minimize the sum of the squared deviations of the actual series from the growth component subject to a penalty for changes in the rate of growth. The specific functional form is

$$\min_{\{g_t\}_{t=-1}^{T}} \sum_{t=1}^{T} (y_t - g_t)^2 + \lambda[(g_t - g_{t-1}) - (g_{t-1} - g_{t-2})]^2. \quad (2)$$

The cyclical component is then found by subtracting the growth component from the actual value.

The Appendix provides a discussion of the solution of the HP filtering problem. Here, I merely note that the growth component and consequently the cyclical component are filtered versions of the y_t series. That is, they are two-sided weighted averages of the raw data. Hamilton (2018) writes the solution for the vector of growth components as $g^* = A^* y$, where y is the vector of observations on y_t and the rows of A^* represent the filter weights for the trend components. Hence, from equation (1), the rows of $I - A^*$ represent the weights for the cyclical components.

The weights for the cyclical HP filter associated with an infinite sample are given in the Appendix and are presented in Figure 1 along with the weights from the BK filter that will be discussed below. The cyclical component is approximately equal to the current value of the series minus a declining weighted average of five years of quarterly values on either side of the current quarter. Away from the end points of the series, the cyclical weights in a sample of 289 observations differ from the infinite sample weights by less than 0.001.

In deciding on the size of λ that controls the magnitude of the penalty for changes in the rate of growth, Hodrick and Prescott (1997) described a probability model that they suggested might provide some intuition for its appropriate value. They wrote (p.4):

> The following probability model is useful for bringing to bear prior knowledge in the selection of the smoothing parameter λ. If the cyclical components and the second differences of the growth components were independently and identically distributed, normal variables with means zero and variances σ_1^2 and σ_2^2 (which they are not), the conditional expectation of g_t, given the observations, would be the solution to the program (2) when $\sqrt{\lambda} = \sigma_1/\sigma_2$ ... Our prior view is that a 5 percent cyclical component is moderately large as is a one-eighth of 1 percent change in the growth rate in a quarter. This led us to select $\sqrt{\lambda} = 5/(1/8) = 40$ or $\lambda = 1600$.

It is important to note that Hodrick and Prescott (1997) explicitly argued that this specific probability model was false, and they were sensitive to the fact that the results could depend on the choice of λ. Therefore, they noted that the standard deviation and the autocorrelations of the cyclical component of real GNP (p. 4-5),

> ... change little if λ is reduced by a factor of four to 400 or increased by a factor of four to 6400.

Examination of Table 1 in Hodrick and Prescott (1997) confirms that statement.

3.1 Some Criticisms of the Hodrick and Prescott (1997) Filter

One of the first criticisms of the HP filter was that the only formal probability model for which the decomposition is optimal is the explicit probability model discussed above regarding the choice of λ, which as mentioned, Hodrick and Prescott (1997) noted was explicitly false. Hamilton (2018) presents a version of this criticism in his Proposition 1 proving that if a researcher sought to estimate the growth component by minimizing the expected squared difference between the true growth component and a linear combination of y, that is

$$\min_{a_t} E(g_t - a_t' y), \tag{3}$$

the solution for the vector of estimated growth components, $\tilde{g}$, would be

$$\tilde{g} = \tilde{A}y. \tag{4}$$

Then, if the true model of the growth component were

$$g_t = 2g_{t-1} - g_{t-2} + \nu_t, \tag{5}$$

where ν_t is a white noise uncorrelated with c_t, and if the cyclical component is also a white noise, then $\tilde{A}$ would converge to the HP trend filter weights, A^*. Hamilton (2018) then notes that if this situation were true, the person applying the Hodrick and Prescott (1997) filter would be "unhappy" because the resulting cyclical series would be white noise.

From a statistical perspective, there is nothing wrong with this argument, and indeed, this probability model is the one described by Hodrick and Prescott (1997) in their discussion of the choice of λ. But, they also explicitly stated that they did not believe that knowledge of the nature of the growth component was sufficient to permit the application of a specific probability model.

A second criticism concerns the ability of the HP filtered data to induce cycles where none exist. Just as Nelson and Kang (1981) warned that linearly detrending a time series that is a random walk induces spurious periodicity that is a function of the length of the sample, Cogley and Nason (1995) demonstrated that applying the HP filter to a series that is a random walk imposes complex dynamic properties on the growth and cyclical components that obviously are not present.

As in any two-sided filter, it is also the case that observations near the beginning and the end of the sample have different weights in their growth components than those far from either end, which further distorts the dynamics of the cyclical component. It is certainly worthwhile keeping these criticisms in mind when applying any filtering technique.

4 The Clark (1987) Unobserved Components Model

If one thinks that they have sufficient information to formulate a statistical model of the growth and cyclical components, estimation of an unobserved components model is possible.[3] Here, I examine the Clark (1987) model it does not assume that GDP is stationary after first differencing, or $I(1)$.[4] Below, I provide maximum likelihood estimates from the full sample of the coefficients with their standard errors in parentheses.[5]

Clark's (1987) unobserved components model postulates that GDP is the sum of a stochastic trend and a stochastic cycle as in equation (1), and the change in the trend is modeled with a time-varying conditional

[3] Of course, one could also estimate a multivariate model of the underlying causes of growth and cycles, but I leave this important issue to future discussions.

[4] A series that is stationary after taking d differences is said to be integrated of order d or an $I(d)$ series.

[5] The Appendix compares the estimation in Clark (1987) with estimates of the model from my data using the same sample period as Clark (1987), and the differences are minor. In the full sample estimation, all of the changes in parameters are smaller than one of their respective standard errors.

mean, d_{t-1}, and a homoskedastic innovation. If the lag operator, L, is defined by $L^k y_t = y_{t-k}$, and the first difference operator is $\Delta = (1 - L)$, the second equation of the model is

$$\Delta g_t = d_{t-1} + \underset{(0.114)}{0.545}\, w_t, \tag{6}$$

where w_t is $N(0, 1)$. The conditional mean of the change in the trend is postulated to be a random walk whose innovation has a relatively small estimated standard deviation:

$$d_t = d_{t-1} + \underset{(0.016)}{0.021}\, u_t, \tag{7}$$

where u_t is $N(0, 1)$. The cyclical component is modeled as an ARIMA(2,0,0):

$$c_t = \underset{(0.122)}{1.510}\, c_{t-1} - \underset{(0.127)}{0.565}\, c_{t-2} + \underset{(0.126)}{0.603}\, v_t, \tag{8}$$

where v_t is $N(0, 1)$. The three innovations, u_t, w_t, and v_t, are mutually uncorrelated.

After estimating the model, the cyclical component of the model can be inferred from the application of Kalman smoothing, and Figure 2 presents the implied unobserved components cycle (the dashed line) along with the HP cycle (the solid line). The correlation of the two series is 0.786.

This finding is similar to that of Harvey and Jaeger (1993) who advocate estimation of formal statistical unobserved components models of the trend and cyclical components in which the trend is an ARIMA(0,2,1) and the cycle is an ARIMA(2,0,1). They note (p. 232),

> We argue that there are series where it is unreasonable to assume a smooth trend *a priori* and therefore the question whether or not σ_η^2 is set to zero is an empirical one.

Here σ_η^2 in their notation refers to the variance of the innovation in equation (7). Although the cyclical component of GDP estimated by Harvey and Jaeger (1993) is quite similar to the Hodrick and Prescott (1997) series, they find that this is not true of the Australian real GDP series.

One might argue that allowing the conditional mean of the rate of growth of GDP to be a random walk with a normal innovation is absurd since doing so allows the expected, and therefore realized, rate of growth of GDP to explode over time. Nevertheless, such an assumption can be defended as the appropriate small sample estimation technique. Clark (1987) defends the assumption on a priori grounds (p. 800):

> In addition, it seemed inappropriate to assume a constant growth rate in advance, given the decline of U.S. productivity growth in the 1970s and reduction of labor force growth in the 1980s, both of which should have shifted the underlying growth of output.

Figure 3 presents the smoothed estimation of the conditional mean of the rate of growth from estimation of the Clark (1987) model on the full sample. One sees the slowly declining growth rate from the almost 0.95% per quarter in the 1950's to the 0.53% per quarter after the financial crisis of 2008 and the great recession.

The Appendix demonstrates that the univariate time series process for y_t implied by the Clark (1987) model is an ARIMA(2,2,3),

$$\phi(L)\Delta^2 y_t = \theta(L)\epsilon_t, \tag{9}$$

where $\phi(L) = 1 - \phi_1 L - \phi_2 L^2$, and $\theta(L) = 1 - \theta_1 L - \theta_2 L^2 - \theta_3 L^3$.

To assess whether one could use standard unit root tests to demonstrate the necessity to second difference the y_t series to induce stationarity, I conducted 5,000 simulations with 289 observations from the estimated Clark (1987) model, and I estimated augmented Dickey and Fuller (1979) (ADF) tests of the form

$$\Delta y_t = c + \phi \Delta y_{t-1} + \beta_1 \Delta^2 y_{t-1} + \beta_2 \Delta^2 y_{t-2} + \beta_3 \Delta^2 y_{t-3} + \beta_4 \Delta^2 y_{t-4} + \epsilon_t. \tag{10}$$

Failure to reject the null hypothesis $\phi = 1$ would indicate that second differencing is appropriate, yet in **all** of the simulations, the value of the test statistic rejected this null hypothesis at the .05 marginal level of significance in favor of the stationary alternative. The mean value of the ADF test statistics in the simulations was -7.073.[6] Thus, the ADF tests strongly reject the presence of a unit root in the rate of growth of the simulated GDP series even though a unit root is actually present and the underlying y_t series is truly $I(2)$. In the actual data, the ADF test statistic is -8.427, and its nominal p-value is 0.001. Given the findings of the simulations, the size of the test is clearly not correct for this type of economic time series.

Morley et al. (2003) note that to achieve identification in unobserved components models, including the Clark (1987) model, the models are typically estimated under the assumption that the innovations are uncorrelated. The parameters of the corresponding ARIMA model in equation (9) are consequently constrained. I return to this issue below in discussing the implications of various models for the importance of the unit root component in GDP.

5 The Baxter and King (1999) Filter

Baxter and King (1999) use a frequency domain methodology to estimate the cyclical component of a time series.[7] The Appendix contains a more formal discussion of these methods. Here, I merely note that the Baxter and King (1999) band-pass methodology is a symmetric, two-sided filter such that the cyclical component is a weighted sum of the underlying data,

$$c_t = \sum_{k=-K}^{K} w_k y_{t-k}, \tag{11}$$

where the weights sum to zero. For quarterly observations Baxter and King (1999) recommend the BP(6,32) band-pass filter that seeks to capture variation in the series with periodicity between 6 and 32 quarters, and they set $K = 12$. The weights, which are given in their Table 4, are plotted in Figure 1, and are reproduced in the Appendix for the reader's convenience.[8] The Appendix in Baxter and King (1999) demonstrates that any symmetric, two-sided filter, whose weights sum to zero, contains a backward difference, $(1 - L)$, and a forward difference, $(1 - L^{-1})$. The BK filter consequently produces a stationary series if the underlying series is $I(2)$.

While Hodrick and Prescott (1997) and Baxter and King (1999) approach the problem of defining the

[6] The .05 critical value of the ADF test with more than 250 observations is -2.88. The percentage of the simulations that rejected the null hypothesis of a unit root was the same with up to seven lags on the right-hand side of equation (10), and it was also the same when I doubled the sample size.

[7] Stock and Watson (1999) use the BK filter to discuss the cyclical components of 71 economic time series.

[8] See the Appendix for additional discussion of low-pass and band-pass filters. When discussing the simulations below, I estimate the trend associated with the Baxter and King (1999) methodology using a low-pass filter that keeps fluctuations with a periodicity greater than 32 quarters. Christiano and Fitzgerald (2003) provide an alternative band-pass filter that uses different weights for different time periods and is not symmetric. I also discuss this approach in the Appendix.

cyclical component of a series from quite different directions, both approaches filter the data, albeit with somewhat different weights as demonstrated in Figure 1. Nevertheless, the estimated cyclical components of GDP are remarkably similar as demonstrated in Figure 4 in which the BK cycle is the dashed line and the HP cycle is the solid line. The correlation between the two definitions of the cyclical component of GDP is 0.963. Baxter and King (1999) note that this finding of similar cycles from the two methods does not hold true for all series as the HP filter allows more high frequency variation into the cyclical component, and some series, such as inflation, have significant high frequency variation.

6 The Hamilton (2018) Filter

Hamilton (2018) develops a decomposition of a time series into trend and cyclical components by regressing the h-period-ahead value of the series, y_{t+h}, on a constant, the current value of the series, y_t, and $(p-1)$ additional lags, $y_{t-1}, ..., y_{t-p+1}$. In practice, for quarterly macroeconomic time series, Hamilton (2018) suggests $h = 8$ and $p = 4$, as in

$$y_{t+8} = \beta_0 + \beta_1 y_t + \beta_2 y_{t-1} + \beta_3 y_{t-2} + \beta_4 y_{t-3} + \nu_{t+8}. \tag{12}$$

The trend component at time $t+8$, denoted g_{t+8}, is defined to be the fitted value from this regression,

$$g_{t+8} = \widehat{\beta_0} + \widehat{\beta_1} y_t + \widehat{\beta_2} y_{t-1} + \widehat{\beta_3} y_{t-2} + \widehat{\beta_4} y_{t-3}. \tag{13}$$

The cyclical component at time $t+8$, denoted c_{t+8}, is defined to be the deviation of the realized value of y_{t+8} from the estimated trend,

$$c_{t+8} = y_{t+8} - g_{t+8}. \tag{14}$$

Figure 5 presents the H cycle as the dashed line and the HP cycle as the solid line. The correlation of the two series is 0.731, but the H cycle is more than twice as volatile as the HP cycle.

Hamilton (2018) notes that if $\Delta^d y_t$ is stationary and $p > d$, the regression in equation (12) will use d of the coefficients to induce stationarity in the residuals, because otherwise the sum of the squared residuals would explode, and the remaining $p+1-d$ coefficients will reflect the regression of a stationary representation of the future process onto the stationary lagged values of the process. For example, if Δy_t is stationary, the sum of the slope coefficients in equation (12) would converge to one as the sample size gets large, and the regression in equation (12) would be equivalent in large samples to running the following regression,

$$y_{t+8} - y_t = \delta_0 + \delta_1 \Delta y_t + \delta_2 \Delta y_{t-1} + \delta_3 \Delta y_{t-2} + \nu_{t+8}. \tag{15}$$

Hamilton (2018) notes that if predictability is low, the left-hand side of equation (15) suggests that his cyclical fluctuations would be quite similar to those found from simply taking the 8-*th* difference of the data. That is, the cyclical component would be defined as

$$c^D_{t+8} = y_{t+8} - y_t, \tag{16}$$

which imposes a unit root but without the adjustment for the constant term or the forecasts from current and lagged $y_t's$ that arise in the definition of c_{t+8}. Hamilton (2018) finds that these two cyclical components, c_{t+8} and c^D_{t+8}, do indeed look very similar, although c_{t+8} is zero mean by construction and c^D_{t+8} contains the average growth rate over a two-year period.

To see the importance of this growth rate, Figure 6 plots the difference between the H cycle and 8-*th* differenced GDP. Because the average growth rate in the first half of the sample is greater than the average growth rate in the second half, the difference is negative prior to 1984 and positive thereafter.

6.1 The Beveridge and Nelson (1981) Decomposition

The Hamilton (2018) methodology is related to the Beveridge and Nelson (1981) decomposition of a time series, y_t, into the sum of a stochastic trend, z_t, that is a random walk with drift, and a serially correlated stationary process, c_t,

$$y_t = z_t + c_t. \tag{17}$$

This decomposition assumes that y_t is $I(1)$. From the Wold representation of a stationary time series, Δy_t can be expressed as an unconditional mean, which I assume to be the only deterministic component, plus possibly infinite-order moving average process,

$$\Delta y_t = \mu + A(L)\epsilon_t, \tag{18}$$

where $A(L) = \sum_{j=0}^{\infty} a_j L^j$, $a_0 = 1$ and $\sum_{j=0}^{\infty} a_j^2 < \infty$. In the discussion that follows, it is useful to define the notation

$$A_k = \sum_{j=k}^{\infty} a_j, \quad k = 0, 1, 2, ... \tag{19}$$

Beveridge and Nelson (1981) define the stochastic trend in y_t to be the long-run forecast of y_{t+h} based on time t information. That is,

$$z_t = \lim_{h\to\infty} \lim_{p\to\infty} E(y_{t+h}|y_t, y_{t-1}, ..., y_{t-p}). \tag{20}$$

With this definition, it is straightforward to see that z_t evolves as a random walk with constant drift, and as shown below, the response of z_t to an innovation in ϵ_t is A_0. Thus,

$$z_t = \mu + z_{t-1} + A_0\epsilon_t, \tag{21}$$

in which case the cyclical component must be

$$c_t = -A_1\epsilon_t - A_2\epsilon_{t-1} - A_3\epsilon_{t-2} - ... \tag{22}$$

6.1.1 On the relative importance of the random walk component

To see that A_0 represents the response of z_t to an innovation in ϵ_t, recognize that the a_j coefficients give the impulse responses of the expected growth rate of Δy_{t+j} in the future to a shock to the process today, that is, $a_j = E_t\left(\frac{d\Delta y_{t+j}}{d\epsilon_t}\right)$. Then, use the fact that the level of future output is $y_{t+j} = y_{t-1} + \Delta y_t + \Delta y_{t+1} + ...\Delta y_{t+j}$.

Consequently, the impulse response of the expected level of future output at $t+j$ to an innovation at t is

$$E_t\left(\frac{dy_{t+j}}{d\epsilon_t}\right) = 1 + a_1 + ... + a_j. \tag{23}$$

Thus, A_0 represents the long-run impact of a shock to the expected level of future output:

$$\lim_{j\to\infty} E_t\left(\frac{dy_{t+j}}{d\epsilon_t}\right) = \sum_{j=0}^{\infty} a_j = A_0. \tag{24}$$

If $A_0 = 1$, an innovation to the process has a permanent effect on the long run prediction, which is consistent with y_t being a random walk with drift. If $A_0 > 1$, the expected increase in the long-run level of the process is larger than the current innovation, and if $A_0 < 1$, the expected increase in the long-run level of the process is smaller than the innovation.

6.1.2 How Hamilton (1994) Differs from Beveridge and Nelson (1981)

The Hamilton (2018) decomposition into trend and cycle contains two changes relative to the Beveridge and Nelson (1981) decomposition. First, rather than forecasting the $\lim_{h\to\infty} E_t(y_{t+h})$, Hamilton (2018) considers $E_t(y_{t+8})$. The second change is the timing of the trend. The Beveridge and Nelson (1981) decomposition defines the trend at time t to be $\lim_{h\to\infty} E_t(y_{t+h})$, but the Hamilton (1994) decomposition defines the trend at time t to be $E_{t-8}(y_t)$. Also, whereas the Beveridge and Nelson (1981) requires the underlying time series to be $I(1)$, the Hamilton (1994) decomposition is well defined as long as the series is $I(d)$ and $d \leq p$, where p is the number of right-hand-side variables in equation (12).

6.2 Potential Problems with the Hamilton (2018) Methodology

The stationary representation in equation (15) highlights two potential problems with the Hamilton (2018) methodology. First, the approach is designed for classical time series environments, and it may not work well when samples are small relative to the changes taking place in the underlying economic process or when the changes in the underlying trend do not allow for straightforward time series models.

A second problem arises even in large samples if the projection of y_{t+8} onto y_t and the three lagged y_{t-j}'s does not capture all of the forecasting information in the past history of the process.[9]

To understand this criticism, suppose the y_t series is $I(1)$ and consider the Beveridge and Nelson (1981) decomposition in equation (17) and the Wold decomposition in equation (18). Taking the expectation at time t of equation (17) at $t+8$ gives

$$E_t(y_{t+8}) = E_t(z_{t+8}) + E_t(c_{t+8}). \tag{25}$$

From equation (21)

$$E_t(z_{t+8}) = 8\mu + z_t, \tag{26}$$

and from equation (22)

$$E_t(c_{t+8}) = -A_9\epsilon_t - A_{10}\epsilon_{t-1} - A_{11}\epsilon_{t-2} - ... \tag{27}$$

[9]See Schüler (2018) for analysis of the properties of the H filter in the frequency domain and critiques of its properties.

Substituting into equation (25) gives

$$E_t(y_{t+8}) = 8\mu + z_t - A_9\epsilon_t - A_{10}\epsilon_{t-1} - A_{11}\epsilon_{t-2} - ... \tag{28}$$

or after substituting for $z_t = y_t - c_t$, we have

$$E_t(y_{t+8}) = 8\mu + y_t + (A_1 - A_9)\epsilon_t + (A_2 - A_{10})\epsilon_{t-1} + (A_3 - A_{11})\epsilon_{t-2} - ... \tag{29}$$

Hamilton (2018) identifies the trend component of a series as the forecast from the regression in equation (12), and while the lagged y_{t-j} terms will be correlated with the lagged ϵ_{t-j} terms in equation (29), the lagged y_{t-j} terms may not capture all of the dynamic effects.

7 The Morley, Nelson, and Zivot (2003) Model

Those who view many macroeconomic time series as $I(1)$ often estimate ARIMA models, which provide an easy way to estimate the value of A_0. For example, Morley et al. (2003) model GDP as an ARIMA(2,1,2),

$$\phi(L)\Delta y_t = \theta(L)\epsilon_t, \tag{30}$$

where $\phi(L)$ and $\theta(L)$ are second-order polynomials in the lag operator. In ARIMA models, the long-run impulse response can be estimated parametrically by

$$A_0 = \phi(1)^{-1}\theta(1). \tag{31}$$

The Appendix compares estimation of the Morley et al. (2003) model with my data set for their sample period showing that the results are similar but not identical.[10] Estimating this model for my full sample period gives

$$\Delta y_t = \underset{(0.070)}{0.320} + \underset{(0.143)}{1.271}\Delta y_{t-1} - \underset{(0.093)}{0.682}\Delta y_{t-2} + \epsilon_t - \underset{(0.146)}{0.979}\epsilon_{t-1} + \underset{(0.086)}{0.540}\epsilon_{t-2}. \tag{32}$$

I will use this model as one of the simulation economies.

The implied long-run impulse response coefficient from this estimation is

$$A_0 = \phi(1)^{-1}\theta(1) = (1 - 0.979 + 0.540)/(1 - 1.271 + 0.682) = 1.364, \tag{33}$$

which is also quite similar to the value found by Morley et al. (2003) of 1.276.

As discussed above, if one believes that GDP is $I(1)$ with constant ARIMA parameters, finding $A_0 > 1$ implies that the Beveridge and Nelson (1981) random walk component of the series contributes more to the innovation variance of the rate of growth of y_t than the corresponding cyclical component because the innovation to the cyclical component is perfectly negatively correlated with the innovation to the stochastic trend. Thus, in the full sample, if $A_0 = 1.364$ the implied standard deviation of the innovation to the stochastic trend would be 1.364 times the standard deviation of the innovation to the rate of growth, and the standard deviation of the cyclical component must correspondingly be 0.364 times the standard deviation

[10] The differences in the estimated parameters appear to be minor and could be due to revisions in the GDP data or to the convergence criteria used in the non-linear estimation.

of the innovation to the rate of growth. The next subsection provides a non-parametric way of examining this conclusion.

8 The Cochrane (1988) Variance Ratio

As just noted, the innovation variance of the random walk component is $A_0^2\sigma_\epsilon^2$. Cochrane (1988) suggests a third way of thinking about the importance of A_0. Because the spectral density of Δy_t at frequency ω is $S_{\Delta y}(e^{-i\omega}) = |A_0(e^{-i\omega})|^2\sigma_\epsilon^2$, the innovation variance of the random walk component is equal to the spectral density of Δy_t at frequency zero, that is

$$\sigma_\Delta^2 z = A_0^2\sigma_\epsilon^2 = S_{\Delta y}(e^{-i0}). \tag{34}$$

Thus, Cochrane (1988) argues that for any $I(1)$ series (p. 905),

> ...we can break it into permanent (random walk) and temporary (stationary) components, we can examine the response of long-term forecasts to an innovation, or we can examine the spectral density at frequency zero of its first differences. All three interpretations allow us to think of the permanence of the fluctuations of a series as a continuous phenomenon rather than a discrete choice.

In the Beveridge and Nelson (1981) decomposition, the stochastic trend and the cycle are perfectly correlated. Cochrane (1988) credits Watson (1986) as the first to prove that the Beveridge and Nelson (1981) decomposition is the only trend-cycle decomposition that is guaranteed to exist, but in Cochrane's (1988) Fact 2, he demonstrates that if an alternative trend-cycle decomposition exists, with arbitrary correlation between the innovations to the stochastic trend and the cycle, the innovation standard deviation of the trend is the same as that of the Beveridge and Nelson (1981) decomposition.

Rather than assess the importance of the random walk component for the long-run forecast by developing a parametric estimate of A_0 from an ARIMA model, Cochrane (1988) develops non-parametric estimates using variances of long-run differences. Because the long-run difference is sum of short-run differences, $y_{t+k} - y_t = \Delta y_{t+k} + \Delta y_{t+k-1} + ... + \Delta y_{t+1}$, the variance of the $k-$period difference times $1/k$ is

$$\sigma_k^2 = \frac{1}{k}V(y_{t+k} - y_t) = \left(1 + 2\sum_{j=1}^{k-1}\frac{(k-j)}{k}\rho_j\right)\sigma_{\Delta y}^2 \tag{35}$$

where ρ_j is the $j-$th order autocorrelation of Δy_t. Cochrane (1988) recognizes that

$$\lim_{k\to\infty}\sigma_k^2 = S_{\Delta y}(e^{-i0}). \tag{36}$$

Thus, since $A_0^2\sigma_\epsilon^2 = S_{\Delta y}(e^{-i0})$, this non-parametric approach provides an alternative way of estimating the innovation variance of the random walk component.

To obtain a lower bound estimate of A_0, Campbell and Mankiw (1987) recognize that the predictable fraction of Δy_t is

$$R^2 = 1 - \frac{\sigma_\epsilon^2}{\sigma_{\Delta y}^2}. \tag{37}$$

They then divide the variance of the long-run difference by k times the variance of Δy_t to get the variance ratio, V_k, recognizing that

$$\lim_{k\to\infty} V_k = A_0^2 \frac{\sigma_\epsilon^2}{\sigma_{\Delta y}^2}. \tag{38}$$

Hence, Campbell and Mankiw (1987) note that

$$A_0 = \sqrt{\frac{V_k}{(1-R^2)}}. \tag{39}$$

Hence, $\sqrt{V_k}$, for a sufficiently large k, provides a lower bound estimate of A_0 because $(1-R^2)$ is a fraction.

In the simulations I report $\sqrt{V_k}$ as a diagnostic statistic on the importance of the random walk component. It is important to remember, though, that the discussion in the current and last sections presumes that the time series is $I(1)$. As noted above, in dealing with macroeconomic data that have underlying trends that change over time, that assumption may be problematic.

9 Statistics for U.S. Real GDP

This section provides statistics for GDP for a sample period from 1947:1 to 2019:1. Table 1 presents two sets of statistics. The columns in Panel A labeled "H", "HP", and "BK" refer to the respective filtering methods of Hamilton (2018), Hodrick and Prescott (1997), and Baxter and King (1999), and the statistics are the standard deviations of the cyclical component and the standard deviations of the changes in the trend. The values in the tables have units of percentage points per quarter. The standard deviations of the HP and BK cycles are 1.550 and 1.490, respectively, while the standard deviation of the H cycle is more than twice those values at 3.295. The standard deviations of the changes in the HP and BK trends are correspondingly quite close at 0.261 and 0.270, while the standard deviation of the change in the H trend is almost four times as large at 1.022. Since the average growth rate of GDP over the full sample is 0.774 per quarter, the standard deviation of the change in the H trend is larger than the average growth rate.

Panel B of Table 1 reports the square roots of the variance ratios. The estimated lower bounds on A_0 from the square roots of the variance ratios are only slightly greater than the parametric estimate of 1.364 from the ARIMA(2,1,2) model in Section 7.

These results, in which the parametric and non-parametric estimates of A_0 are quite similar and imply a large unit root component in GDP, contrast to the results in Cochrane (1988) who examines annual data on the log of real GNP per capita from 1869 to 1986 and finds non-parametric estimates of A_0 that are substantively less than one, while the parametric approaches imply values of A_0 that are substantively greater than one.

Cochrane (1988) reconciles the different implications of the two methodologies by noting that the variance ratio focuses exclusively on the spectral density at frequency zero, while the maximum likelihood estimation of an ARIMA model focuses on all frequencies and will trade off doing less well at low frequencies if it can do better at higher frequencies. Such is not the case here.[11]

[11]The Online Appendix reproduces Table 1 using data on quarterly GDP per capita. The standard deviations of the cycles and the changes in the trends are quite similar to those in Table 1 here, but the square roots of the variance ratios decline monotonically from 1.328 for $k = 10$ to 1.056 for $k = 80$. The Online Appendix also reports estimation of the Morley et al. (2003) model for GDP per capita finding similar coefficients to those reported here and an estimate of $A_0 = 1.290$.

Morley et al. (2003) note that estimation of unobserved components models in which the data are assumed to be $I(1)$ often imply cyclical components whose innovation variance is larger than that of the random walk component, that is estimates of $A_0 < 1$. They reconcile these findings with the opposite conclusion from estimation of ARIMA models that find estimates of $A_0 > 1$ by noting that the unobserved components model imply constraints on the ARIMA representation that arise from the assumption in the unobserved components models that the innovations in the cyclical and growth components are uncorrelated. They note that these constraint are generally rejected by the data.

The next section considers simulations of time series to determine both which of the trend-cycle decomposition methods works the best in particular situations and whether the conclusions of the ARIMA representations and the variance ratio analysis should be trusted.

10 Analysis of Simulations

This section presents a series of tables that examine the mean values of several statistics from 5,000 simulations of six different time series. The statistics are designed to provide information on the performance of the filtering methodologies of Hodrick and Prescott (1997), Baxter and King (1999), and Hamilton (2018) in different time series environments. The simulated time series all have 289 observations, the same length as the GDP data examined above, and they generally have the same sample mean and standard deviation as the rate of growth of GDP.[12]

10.1 The Representative Statistics

In addition to the unconditional standard deviations of the cyclical components and the changes in trend reported above, each Panel A of the Tables also includes the correlations of the cyclical components from the respective filters with the simulated cyclical components, the correlations of the changes in the filtered trends with the changes in simulated trends, and the root mean squared errors (RMSE) of the cyclical components.[13]

Each Panel B in the following Tables includes two regression diagnostics that examine the relative importance of the cyclical components of either the HP or BK filter and the H filter in explaining the simulated cyclical components. The first regression diagnostic is the estimated slope coefficient in

$$c_t - c_t^j = \alpha + \delta(c_t^H - c_t^j) + \epsilon_t, \tag{40}$$

where c_t represents the simulated cycle, c_t^H represent the H cycle, and c_t^j is either the HP or BK cycle. The second diagnostic is the adjusted R^2 from this regression. An estimate of δ equal to zero indicates that the HP or BK cycle explains the simulated cycle with no additional explanatory power from the H cycle, while an estimated δ equal to one indicates the opposite, that the H cycle explains the simulated cycle with no

[12] The Online Appendix reproduces the same tables but with 578 observations. The means of the statistics are little changed with the longer sample.

[13] If the simulated cycle is c_t and the filtered cycle is $\hat{c}_t$, the RMSE is

$$RMSE = \sqrt{(1/T)\sum_{t=1}^{T}(c_t - \hat{c}_t)^2}.$$

additional explanatory power from the HP or BK cycle. The adjusted R^2 similarly indicates the additional explanatory power of the H cycle in the presence of the HP or BK cycle.

Finally, each Panel C of the following Tables includes the square roots of the variance ratios. While variance ratios are designed to inform the debate about the importance of the variance of a unit root component in GDP relative to the variance of the cyclical component in time series, I examine these statistics in environments that are $I(1)$ and not $I(1)$.

The first two tables examine two $I(1)$ time series with constant unconditional moments in which there are no distinct growth and cyclical components. These models are a random walk with drift and an ARIMA(2,1,2) model, calibrated as in the Morley et al. (2003) model estimated above.

Next, I introduce time series containing distinct growth and cyclical components. The first of these models has a serially correlated growth component that has a constant unconditional mean. The second analyzes the Clark (1987) model, as estimated above, in which the conditional mean rate of growth is itself a random walk.

Finally, the last two time series allow for changes in the unconditional mean rate of growth. One can think of these models as capturing realizations of low-probability regime changes in the first case or simply as slow-downs in the rate of growth in the second case. These final two series do not fall into classic time series models.

10.2 Defining the Growth and Cyclical Components

Before presenting the results of the simulations, I briefly discuss the different interpretations of trends and cycles in the various simulations. In the first two models, separately defined growth and cyclical components do not exist, whereas in the last four models, such components are present. When these components are present, I take the goal of the HP and BK filters to be to isolate these cyclical components. This is a different objective than what Hamilton (2018) defines as the cycle.

When no well-defined growth and cyclical components are present and when they can be calculated, I define the trend and cycle to be modifications of the Hamilton (2018) definitions. As in equation (13), Hamilton (2018) defines the trend to be the projection of y_{t+8} onto the information set $\{1, y_t, y_{t-1}, y_{t-2}, y_{t-3}, \}$. In the first four models, I define the modified Hamilton (2018) trend to be $g^H_{t+8} = E_t(y_{t+8})$ and the corresponding cycle to be $c^H_{t+8} = y_{t+8} - g^H_{t+8}$. These definitions substitute the full information set at time t rather than simply the regression of y_{t+8} onto current and lagged values of y_t as I want to see how well the empirical methodolgies capture the true $E_t(y_{t+8})$ and not simply examine the small sample properties of the Hamilton (2018) regression. While it is straightforward to analytically calculate these values for simple models like the random walk, the Appendix describes how this is done for the more complicated ARIMA(2,1,2) model and the two unobserved components models. For the two unobserved components models, in which there are well-defined growth and cyclical components, I allow for both the model definition of the cycle as well as the modified Hamilton definition just discussed. In the last two simulations in which there is no well defined population value of $E_t(y_{t+8})$, the only cyclical component is the one that is being simulated.

Thus, for the random walk model and the ARIMA model, one should expect the HP filter and BK filter to be dominated by the H filter as the HP filter and BK filters are not designed to produce an estimate of $c^H_{t+8} = y_{t+8} - E_t(y_{t+8})$. Similarly, in the last four simulations, when the goal is to isolate the true cycle, we should expect the H filter to be dominated by the HP and BK filters. This is indeed what I find. Perhaps the most interesting cases are the unobserved components models in which both concepts of trend are present.

10.3 A Random Walk Model

The first simulated time series is a random walk with drift. I use this model because Hamilton (2018) states (p. 833):

> The presumption by users of the HP filter is that it offers a reasonable approach to detrending for a range of commonly encountered economic time series. The leading example of a time-series process for which we would want to be particularly convinced of the procedure's appropriateness would be a random walk.

I am not sure why Hamilton (2018) thinks that a random walk is the 'leading example' of a time series that one would want to detrend with the HP filter, because such a series does not share the slowly moving growth component that Hodrick and Prescott (1997) argue characterizes the macroeconomic series that they sought to detrend. Nevertheless, it is useful to start the simulations with a random walk to demonstrate the pros and cons of the various decomposition methods.

The simulations involve 289 observations of

$$y_t = 0.779 + y_{t-1} + 0.933\epsilon_t, \tag{41}$$

where ϵ_t is independently and identically distributed $N(0, 1)$. The drift and standard deviation of the process correspond to the sample mean and standard deviation of the quarterly continuously compounded rate of growth of GDP over the full sample.

Table 2 presents the results of the simulations. As discussed in the previous subsection, without a clear distinction between growth and cyclical components, I adopt the Hamilton (2018) definitions modified to be based on the full information set. The columns in Panel A labeled "H", "HP", and "BK" refer to the mean values of the row statistics from the 5,000 simulations of the respective filtering methods of Hamilton (2018), Hodrick and Prescott (1997), and Baxter and King (1999). The column labeled "In Sim" contains the sample means of the realized values of the row statistics. Panel B presents the mean values of the slope coefficients and R^2's in regression (40). Panel C presents the mean values from the 5,000 simulations of the square roots of the variance ratios.

In this case, as expected, the H filter performs quite well. The mean value of the standard deviation of the H cycle is 2.539, which is the same as the sample mean in the In Sim column. The mean value of the standard deviation of the change in the H trend is 0.893 compared to the In Sim value of 0.932. The correlation of the H cycle with the simulated cycle is 0.979, and the correlation of the change in the H trend with the change in the simulated trend is 0.972. Because the data are in natural logarithms, the RMSE of 0.632 represents a difference of 0.632%.

The statistics for the HP and BK cycles and changes in the trend are much farther from the true values. The average standard deviations of the HP and BK cycles are only 1.190 and 1.073, respectively; and the standard deviations of the changes in trends are only 0.182 for HP and 0.188 for BK. The correlations of the HP and BK cycles with the simulated cycle are only 0.660 and 0.632, and the correlations of the changes in the HP and BK trends with the changes in the simulated trend are 0.144 and 0.149. The RMSE's of the two methods, at 2.057 and 2.128, are also more than three times the H value.

The regression diagnostics also indicate that the H cycle clearly dominants the HP or BK cycles as the means of the slope coefficients are 0.999 for HP and 1.002 for BK, and the means of the R^2's are 0.932 for HP and 0.940 for BK. Clearly, if the leading economic time series that macroeconomists wanted to detrend were truly random walks with constant drifts, the H filter would dominate the HP and BK filters.

Now, consider the Variance Ratio analysis in Panel C of Table 2. With the degrees of freedom adjustment advocated by Cochrane (1988), the results almost perfectly line up with what is expected in the case of a random walk. The means across the simulations of the square roots of the variance ratios range from 1.000 for $k = 10$ to 1.003 for $k = 80$. Thus, in this case we would correctly conclude that a shock to the process is permanent.

10.4 An ARIMA Model

The next simulations utilize the ARIMA(2,1,2) model of Morley et al. (2003) estimated on the full sample as in equation (32). As in the previous subsection, without a clear distinction between growth and cyclical components, I adopt the modified Hamilton (2018) definitions of the trend and cycle based on the full information set.

Notice how well the H filter tracks the simulated cycle and the change in the simulated trend. The standard deviation of the H cycle is 3.238, only slightly smaller than the In Sim value of 3.306. The standard deviation of the change in the H trend is 1.084, which is only slightly above the In Sim value of 1.075. The correlations of the H cycle with the true cycle, 0.977, and of the change in the H trend with the change in the true trend, 0.944, are also quite high. The RMSE of the H cycle remains relatively low at 0.837, and the slope coefficients in the regression diagnostic have means of 0.986 for HP and 0.989 for BK indicating once again essentially no role for the HP or BK cycles in explaining the true cycle in the presence of the H cycle.

These findings might perhaps be anticipated because the simulated process is a well defined $I(1)$ time series process with constant parameters. With 289 observations and enough lags as regressors, the estimated parameters in the H filter forecasting regression should converge to be relatively close to the true model. The presence of MA terms does cause problems in this regard, but once one forecasts out eight quarters, this concern becomes relatively minor.

The HP and BK filters again show relatively poor ability to replicate the underlying modified Hamilton (2018) trend and cycle decomposition. The means of the standard deviations of the HP and BK cycles are only about half of the In Sim value, and the means of the standard deviations of the changes in the HP and BK trends are well below the In Sim value. The correlations of the estimated HP and BK cyclical components with the true cycle are 0.685 and 0.676, respectively. The correlations of the estimated changes in trend with the true changes in trend are also quite low, and the RMSE's remain substantively elevated relative to the H filter.

Consistent with the estimated results from the actual U.S. data, the mean values of the square roots of the variance ratios are essentially 1.26 indicating that the unit root component of the process is dominant compared to the implied cyclical component in the Beveridge and Nelson (1981) decomposition.

10.5 An Model with a Constant Unconditional Mean Change in Trend

For the third set of simulations I examine a model in which the simulated time series, y_t, is the sum of a stochastic trend, g_t, and a stochastic cycle, c_t, as in equation (1). In this case there are well-defined trend and cycle components to be identified by the decomposition techniques.

The stochastic trend is modeled as an ARIMA(1,1,0), and the change in the trend is highly serially correlated:

$$\Delta g_t = 0.07786 + 0.900\Delta g_{t-1} + w_t. \tag{42}$$

The innovations, w_t, are drawn from a $N(0, 0.002)$. The cyclical component is modeled as an ARIMA(2,0,0),

$$c_t = 1.25c_{t-1} - 0.45c_{t-2} + v_t, \tag{43}$$

and the innovations, v_t are drawn from a $N(0, 0.6385)$. The innovations in the trend and cycle are uncorrelated.

The unconditional mean of the change in the trend is 0.7786, which is the same as the sample mean of the quarterly growth rate of GDP. The standard deviation of Δy_t also matches the actual standard deviation of the rate of growth of GDP, which is 0.933. The unconditional standard deviation of the cyclical component of 1.766 is large relative to the unconditional standard deviation of the change in the change in the trend of 0.103. The ratio of these standard deviations is $1.766/0.103 = 17.146$.

Table 4 presents the results of the simulations. While the means of the standard deviations of the HP and BK cycles of 1.423 and 1.364 understate the target of 1.766 by 19% and 23%, respectively, the mean of the standard deviations of the H cycles of 2.419 overstates the target by 37%.

The mean of the simulated standard deviations of the changes in the trend of 0.098 is closely matched by the means of the standard deviations of the changes in the HP and BK trends of 0.104 and 0.115, respectively. In contrast, the mean of the simulated standard deviations of the changes in the H trend is 0.797, which is 7.74 times larger than the target value.

The means of the correlations of the implied cycles with the true cycles are 0.741 for the H cycle and 0.887 and 0.857 for HP and BK cycles, respectively. Thus, this correlation metric also favors the latter methodologies. Similarly, the mean of the correlations of the estimated changes in the H trend with the changes in the simulated trend of 0.041 is much lower than the comparable statistic of 0.543 for the HP filter and 0.495 for the BK filter.

The RMSE of the H cycle of 1.638 is also substantively larger than those of the HP and BK methods, which are 0.840 and 0.936, respectively. Finally, the means of the slope coefficients of 0.087 for HP and 0.150 for BK and the R^2's 0.041 for HP and 0.082 for BK clearly indicate that in the presence of the HP or BK cycles, the H cycle has very little ability to explain the simulated cycle.

The Appendix demonstrates that the stationary univariate time series representation of the simulated data in this subsection is a constrained version of an ARIMA(3,1,2). With 289 observations, one might have thought that the formal time series method of the H filter would be able to isolate the separate trend and cyclical components, but doing so requires convergence of the estimates to the true parameters. There are three free parameters in the true model and four regressors in the H filter, but recall that the sum of the coefficients in the regression converges to one in this case. Thus, one conjecture why the methodology fails is because the MA terms are not orthogonal to the included regressors, and thus the OLS estimates cannot converge to the true parameters. Nevertheless, there is also a difference between the simulated trend and $E_t(y_{t+8})$, which is what Hamilton (2018) seeks to estimate.

When I calculate the alternative definition of the trend, $E_t(y_{t+8})$ within the model, and I measure the alternative cycle as $y_{t+8} - E_t(y_{t+8})$, I find that the average correlation of this definition of the cycle with the cycle in the model, c_t, is 0.962. It is perhaps unsurprising then that the averages of the correlations of this alternative cycle with the HP and BK cycles are larger, with values of 0.854 and 0.825, respectively, than the average correlation of this alternative cycle with the H cycle, which is 0.744. Similarly, the average RMSE of the alternative cycle minus the H cycle of 1.632 is larger than the corresponding values of 0.974 for the HP cycle and 1.059 for the BK cycle.

The average values of the square roots of the variance ratios in these simulations decline from 0.915 for $k = 10$ to 0.563 for $k = 80$, which is consistent with the fact that in these simulations, the unit root in the trend does not having a dominant influence on the evolution of the y_t process. Because the square roots of the variance ratios start below one and decline with increases in k, this artificial process, while matching some characteristics of the actual data, does not match the variance ratios estimated from actual GDP.

10.6 An Unobserved Components Model

This section analyzes simulations of the Clark (1987) model with parameter values set at the values estimated from the full sample as in equations (6) to (8). These simulated data are stationary after taking second differences. The Appendix demonstrates that this unobserved components model is a constrained version of an ARIMA(2,2,3) with constant coefficients.

Once again, it is in this sense that Hamilton (2018) argues that his methodology should do well. He explicitly addresses this issue on (p.836):

> Suppose that the *d-th* difference of y_t is stationary for some d. For example, $d = 2$ would mean that the growth rate is nonstationary, but the change in the growth rate is stationary.

He then states Proposition 4 and notes (p.837):

> Proposition 4 establishes that if we estimate an OLS regression of y_{t+h} on a constant and the $p = 4$ most recent values of y as of date t,
>
> $$y_{t+h} = \beta_0 + \beta_1 y_t + \beta_2 y_{t-1} + \beta_3 y_{t-2} + \beta_4 y_{t-3} + v_{t+h},$$
>
> the residuals,
>
> $$\hat{v}_{t+h} = y_{t+h} - \hat{\beta}_0 - \hat{\beta}_1 y_t - \hat{\beta}_2 y_{t-1} - \hat{\beta}_3 y_{t-2} - \hat{\beta}_4 y_{t-3},$$
>
> offer a reasonable way to construct the transient component for a broad class of underlying processes. The series is stationary provided that fourth differences of y_t are stationary,

Table 5 presents the results of the simulations. Once again, the means of the standard deviations of the HP and BK cycles of 1.631 and 1.557 understate the target by 39% and 42%, respectively; while the mean of the standard deviations of the H cycles of 3.445 overstates the target by 29%. Although the change in the trend is not a stationary variable in these simulations, I continue to use the sample standard deviation of the change in trend as a metric to compare the detrending methods. The mean of the simulated sample standard deviations of the changes in the trend is 0.564, and the means of the standard deviations of the changes in HP and BK trends of 0.234 and 0.246 understate that value by half, while the mean of the simulated standard deviations of the changes in the H trend of 1.238 is more than double that sample value.

The correlations of the HP and BK cycles with the true cycle of 0.651 and 0.676 are higher than the analogous correlation for the H method of 0.510. This is also true of the correlations of the filtered changes in trends with the changes in the simulated trend although here none of the three methods is particularly highly correlated with the changes in the simulated trend.

The RMSE's of the HP and BK cycles of 2.130 and 2.089, respectively, are also better than the RMSE of the H cycle of 3.168. Finally, the slope coefficient of 0.080 for the HP cycle and 0.084 for the BK cycle and the R^2's of 0.032 for HP and 0.034 for BK demonstrate that the H cycle adds little to the ability of the HP or BK cycle to explain the simulated cycles.

When I calculate the alternative definition of the trend, $E_t(y_{t+8})$, within the model, and I measure the alternative cycle as $y_{t+8} - E_t(y_{t+8})$, I find that the average correlation of this definition of the cycle with the true cycle in the model is 0.767. This lower correlation than in the ARIMA simulations means that there is more of a difference between the two concepts of the cycle, and I now find that the correlation of the H cycle with alternative cycle of 0.831 is larger than the correlation of the alternative cycle with the HP and BK cycles of 0.762. The average RMSE of the alternative cycle minus the H cycle of 1.962 is slightly smaller than the corresponding values of 2.020 for the HP cycle and 2.046 for the BK cycle.

The means of the square roots of the variance ratios range from 1.382 for $k = 10$ to 1.433 for $k = 80$. It is interesting that these statistics match the sample statistics from the actual data quite well even though the variance ratios are not well-defined statistics because the underlying data are $I(2)$. This is one sense in which the Clark (1987) model has good small sample statistics.

10.7 Models with Changing Unconditional Mean Growth

The final two simulated series retain the trend plus cycle model of the previous subsection, but in this subsection, changes in the trend also reflect the declining rate of growth observed in the sample. Because the cyclical components are the same in these two simulations, and the simulated average statistics are also quite similar, I discuss the two models together.

The first model reflects what one might think of as ex post regime changes in the growth process, but without a probability model associated with such changes, while the mean growth rate in the second model contains a linear trend reflecting the ex post slowly declining rate of growth observed in the data. In each case, the change in the stochastic trend is also serially correlated and the sample mean of the growth component coincides with the sample mean of the growth rate from the actual data.[14]

The first model of the change in the trend is

$$\Delta g_t = \mu_i + 0.900\Delta g_{t-1} + \nu_t, \tag{44}$$

and the innovations, ν_t, are drawn from a $N(0, 0.002)$. The intercepts, μ_i, take on three different values, $\mu_1 = 0.1$ for the first third of the sample, $\mu_2 = 0.07786$ for the second third of the sample, and $\mu_3 = 0.0554$ for the last third of the sample. With these values, the average growth rate of the simulated series matches the sample mean of the data while also allowing for the observed slowdown in the growth rate of GDP. The unconditional standard deviation of the change in the trend is 0.103 within each regime, but there is no unconditional value for the full process because of the changing unconditional means. If there were

[14]The absence of a formal time series model for the trend in the first case also implies that it is impossible to calculate the alternative Hamilton (2018) definition of the trend.

a constant unconditional mean, the model would match the standard deviation of the data, and with the changes in unconditional means, the average of the standard deviations of the rate of growth of y_t increases only to 0.951, which is just slightly larger than the 0.933 value in the data.

The second model of the change in the trend is

$$\Delta g_t = 0.07786 + 0.04428(145 - t)/289 + 0.900\Delta g_{t-1} + \nu_t, \tag{45}$$

where t runs from 1 to 289, and the innovations, ν_t, are again drawn from a $N(0, 0.002)$. Here, the sample mean rate of growth during the 289 observations coincides with the sample mean rate of growth of GDP, but the rate of growth declines slowly over time.

In both cases, the cyclical component is modeled as an ARIMA(2,0,0),

$$c_t = 1.25c_{t-1} - 0.45c_{t-2} + \epsilon_t, \tag{46}$$

and the innovations, ϵ_t, are drawn from a $N(0, 0.6385)$. The innovations to the cyclical component are uncorrelated with the innovations to the trend component.

The results of the two simulations are presented in Tables 6 and 7, which have the same format as the previous tables. Because the results are very similar, in discussing them I simply present the results from the second simulations in parenthesis after the results from the first simulations. While the means of the standard deviations of the HP cycles of 1.426 (1.423) and the BK cycles of 1.367 (1.362) understate the target of 1.766 by 18% (18%) and 22% (22%), the mean of the standard deviations of the H cycles of 2.448 (2.408) overstates the target by substantially more, 41% (38%).

The mean of the simulated standard deviations of the changes in the trend is 0.205 (0.152), which is essentially matched by the means of the standard deviations of the changes in HP and BK trends of 0.206 (0.156) and 0.207 (0.160), respectively. In contrast, the mean of the simulated standard deviations of the changes in the H trend is 0.801 (0.792), almost (more than) four times larger than the true value.

The means of the correlations of the implied cycles with the true cycles are 0.731 (0.744) for the H cycle, and 0.885 (0.887) and 0.855 (0.857) for the HP and BK cycles, respectively. The latter approaches similarly dominate the mean of the correlations of the estimated changes in trend with the true changes in trend as the H trend produces 0.202 (0.133), which is much lower than the comparable statistic of 0.885 (0.800) for the HP trend and 0.851 (0.750) for the BK trend. The means of the RMSE's of the H and BK cycles of 0.845 (0.840) and 0.945 (0.940) are substantively less than the corresponding values of 1.682 (1.623) for the H cycle. Finally, the slope coefficients for the HP regression of 0.081 (0.091) and the BK values of 0.139 (0.156) as well as the R^2's of 0.043 (0.045) for HP and 0.078 (0.086) for BK again indicate that the H filter adds little to the ability of either the HP filter or the BK filter to identify the cycle in the underlying data.

Although the variance ratios for these processes are not well-defined statistics, their sample values can be computed. In both sets of simulations, one sees variances ratios that increase with k, from 1.081 (0.987) for $k = 10$ to 1.673 (1.152) for $k = 80$. Since there is no stationary unit root component in the underlying trend, the variance ratios give a false reading of the importance of the trend relative to the importance of the cyclical component in the variation of the underlying y_t series.

11 Potential GDP as a Trend

The Congressional Budget Office (CBO) produces a potential GDP series that is often used to produce the GDP Gap measured as the log difference between actual and potential GDP. In this section, I first describe the process that the CBO uses in constructing potential GDP and then demonstrate that the revisions of the process are substantive and imply relatively large changes in trend growth. Finally, I examine the correlations of the GDP Gap, calculated with the latest revision in the potential GDP series, with the H and HP cycles.[15]

The CBO calculates a potential GDP series twice a year and typically publishes them in January and August. Shackleton (2018) describes the process that the CBO uses. He states (p.2-3):

> CBO builds its estimate of aggregate potential output from estimates of the potential output in several different sectors of the overall economy. Those estimates, in turn, depend on estimates of the potential inputs to sectoral production processes as well as their potential productivity.
>
> CBO's approach to identifying underlying productive capacity is to focus on fundamental determinants of supply rather than on fluctuations in aggregate demand. That approach is based on the notion that the economy has an underlying but unobserved trend path along which output, employment, and investment could develop without triggering inflationary instability or recession.

There are six sectors in the CBO model: non-farm business, farm, household, non-profit, federal government, and state and local government. The modeling involves linear regressions with business cycle factors, but they also estimate sequences of linear trends that allow for changes in the growth rate of the economy. Shackleton (2018) (p. 10) describes how this process can lead to large changes in potential GDP over time:

> Particularly significant changes in CBO's estimates of potential output can occur after the economy is determined by NBER to have reached a new business cycle peak, an event that usually leads the agency to change the period over which it estimates various trends. For example, according to the data available in early 2007, the United States was in the midst of a business cycle that had begun in the first quarter of 2001 but had not yet peaked; the last full peak-to-peak business cycle had begun in the third quarter of 1990 and ended in early 2001. As a result, the historical trends used to project future potential series in 2007 began in the third quarter of 1990. After NBER determined (in 2010) that a peak had occurred in the final quarter of 2007, CBO introduced new trends that began at the peak in the first quarter of 2001 and that were distinct from the trends estimated for the 1990–2001 business cycle. Consequently, the projected growth rates of potential series were no longer strongly influenced by actual growth rates during the 1990–2001 cycle and were more strongly influenced by growth rates that occurred after the 2001 peak.

To demonstrate how the CBO's procedure leads to change in potential GDP as additional data accumulate, Figure 7 displays a sequence of the percentage deviations of updated, subsequent forecasts of potential GDP relative to the first forecast made in 2007. The five subsequent forecasts begin with the second potential GDP series released in 2007 and end with the second release of 2009. Notice that they decline systematically over time. Figure 7 demonstrates two features of the data. The potential output forecast in 2009 for 2015 is

[15] Given the similarity of the HP and BK cyclical components, I only report results for the former.

substantively below what was previously forecast in early 2007 by 4.5%, but also, the historical values prior to 2007 are also revised in 2009 to be 1% below what was previously considered to be potential. Clearly, although the CBO uses a multivariate economic approach to developing the trend of the economy, there are substantial revisions to this trend as the economy evolves.

To examine the relation of the GDP Gap to other measures of the business cycle, I construct the GDP Gap using the most recent version of the CBO's potential GDP. Then, the question to be addressed is how different is this estimate of the business cycle from the cyclical components of the various filtering methods discussed above.

Figure 8 plots the HP cycle as the solid line, the H cycle as the dashed line, and the GDP Gap as the dotted line. The standard deviation of the GDP Gap is 2.405 which is 0.890 below the standard deviation of the H cycle and 0.855 above the standard deviation of the HP cycle. The correlations of the GDP Gap with the two cycles are 0.680 for the H cycle and 0.746 for the HP cycle. The RMSE of the HP cycle compared to the GDP Gap is 1.715 which is substantively smaller than the RMSE of the H cycle compared to GDP Gap of 2.848. Thus, along this dimension, the HP filter is better than the H filter. Similarly, in the regression diagnostic of equation (40), the estimate of δ is 0.215 (s.e. 0.085), and the adjusted R^2 is 0.102 indicating only marginal additional explanatory power of the H cycle for the GDP Gap once the HP cycle is taken into account.

While the strong correlations of the GDP Gap with the HP cycle and the H cycle are apparent in Figure 8, close examination of the Figure also suggests that for the GDP Gap, negative values outweigh positive ones. In contrast to the HP and H cycles, which are zero mean by construction, the sample mean of the GDP Gap is actually negative. During the sample, potential GDP overstates actual GDP by 0.616%..

This is not an intentional feature of the CBO's construction of potential GDP. Shackleton (2015) begins his discussion of the possible reasons why the CBO's potential output systematically exceeds realized GDP by stating (p. 4):

> The causes of the average shortfall of output relative to its potential during recent business cycles, and during the past half-century as a whole, are not entirely clear.

After considering several sources, he concludes that the CBO has not systematically overstated potential GDP. Nevertheless, in forecasting exercises the CBO realizes its forecasts of potential GDP have typically exceeded realizations by 0.5% over the last 50 years, and it now builds such a discrepancy into its forecasts of actual GDP. This seems like an odd methodology.

12 Conclusions

Since Burns and Mitchell (1946), economists have examined alternative methods of decomposing macroeconomic time series into growth and cyclical components with the goal of providing advice to those who seek to develop stylized facts to guide the development of macroeconomic models of growth and business cycles. By examining simulations of time series approximately calibrated to actual GDP, this article sheds light on the desirability of using the alternative methods proposed by Hodrick and Prescott (1997), Baxter and King (1999), and Hamilton (2018), as well as providing some additional insight into the debate about the relative importance of unit root components in the Beveridge and Nelson (1981) decomposition.

The Hamilton (1994) method is grounded in formal time series analysis and provides time series definitions of the trend at time $t + 8$ as $E_t(y_{t+8})$ and the cycle as $y_{t+8} - E_t(y_{t+8})$. The simulations in this paper demonstrate that the Hamilton (1994) methodology works almost perfectly for simple time series models, such as a random walk or an ARIMA(2,1,2) model with constant parameters and 289 observations. In contrast, the filtering methods of Hodrick and Prescott (1997) and Baxter and King (1999), which are not designed with forecasting in mind, seem flawed in such situations in their ability to isolate this type of time series cycle.

Nevertheless, as the simulated time series models become more complex, the performance of the H filter deteriorates, and those of the HP and BK filters methods improve dramatically. In the two unobserved components models that have underlying cyclical component to be identified in addition to the Hamilton (1994) cycle, the HP and BK filters perform better than the H filter.

For the final simulations, in which the time series are composed of distinct growth and cyclical components and in which the growth component varies slowly over time but does not have a well defined time series representation implying the inability to calculate the Hamilton (1994) cycle, the cycles from HP and BK filters are much closer to the underlying cycle than the H filter.

Consequently, the most desirable approach to decomposing a time series into growth and cyclical components and hence the advice that one would give to someone that wants to detrend a series to focus on cyclical fluctuations clearly depends on the underlying model that one has in mind. For GDP, if one thinks that growth is caused by slowly moving changes in demographics, like population growth and changes in rates of labor force participation, as well as slowly moving changes in the productivity of capital and labor, then the filtering methods of Hodrick and Prescott (1997) and Baxter and King (1999) seem like the superior way to model the cyclical component. To my way of thinking, the title of the Hamilton (2018) paper is clearly too strong.

The simulations of the paper also address the issue of the relative importance of the nonstationary stochastic trend or growth component compared to the stationary cyclical component in generating fluctuations in GDP. The simulations demonstrate that it is relatively easy to generate a series that is not $I(1)$, that looks like GDP, and in which the growth component does not dominate the cyclical component in contrast to the conclusions of typical Beveridge and Nelson (1981) decompositions.

Of course, developing simultaneous multivariate statistical models of the growth component and the cyclical component based on the underlying economics that drive growth and business cycles remains the gold standard to which research should strive.[16] In thinking about these economic models, though, it is useful to have stylized facts about the variability and covariability of macroeconomic time series over the business cycle. In pursuing such research in the future, I intend to use the methods of Hodrick and Prescott (1997) and Baxter and King (1999).

[16] An example of such an approach is Fernald et al. (2017) who take the unemployment rate as a measure of the cycle and cyclically adjust variables using Okun's law. They then use a growth accounting analysis on cyclically adjusted series to examine the slow recovery of output following the Great Recession concluding that the perceived shortfall in GDP growth primarily reflected two factors: slow growth of total factor productivity, and a decline in labor force participation. They note that both of these factors reflect forces largely unrelated to the financial crisis and resulting recession and that both forces were in play before the recession.

References

Baxter, M. and R. G. King (1999). Measuring business cycles: approximate band-pass filters for economic time series. *Review of Economics and Statistics 81*(4), 575–593.

Beveridge, S. and C. R. Nelson (1981). A new approach to decomposition of economic time series into permanent and transitory components with particular attention to measurement of the 'business cycle'. *Journal of Monetary Economics 7*(2), 151–174.

Burns, A. F. and W. C. Mitchell (1946). *Measuring Business Cycles* NBER, New York.

Campbell, J. Y. and N. G. Mankiw (1987). Are output fluctuations transitory? *Quarterly Journal of Economics 102*(4), 857–880.

Christiano, L. J. and T. J. Fitzgerald (2003). The band pass filter. *International Economic Review 44*(2), 435–465.

Clark, P. K. (1987). The cyclical component of us economic activity. *Quarterly Journal of Economics 102*(4), 797–814.

Cochrane, J. H. (1988). How big is the random walk in gnp? *Journal of Political Economy 96*(5), 893–920.

Cogley, T. and J. M. Nason (1995). Effects of the hodrick-prescott filter on trend and difference stationary time series: Implications for business cycle research. *Journal of Economic Dynamics and Control 19*(1-2), 253–278.

Den Haan, W. J. (2000). The comovement between output and prices. *Journal of Monetary Economics 46*(1), 3–30.

Dickey, D. A. and W. A. Fuller (1979). Distribution of the estimators for autoregressive time series with a unit root. *Journal of the American Statistical Association 74*(366a), 427–431.

Fernald, J. G., R. E. Hall, J. H. Stock, and M. W. Watson (2017). The disappointing recovery of output after 2009. *Brookings Papers on Economic Activity*, 1–59.

Hamilton, J. D. (1994). *Time Series Analysis.* Princeton University Press, Princeton, NJ.

Hamilton, J. D. (2018). Why you should never use the Hodrick-Prescott filter. *Review of Economics and Statistics 100*(5), 831–843.

Harvey, A. C. (1985). Trends and cycles in macroeconomic time series. *Journal of Business & Economic Statistics 3*(3), 216–227.

Harvey, A. C. and A. Jaeger (1993). Detrending, stylized facts and the business cycle. *Journal of Applied Econometrics 8*(3), 231–247.

Hodrick, R. J. and E. C. Prescott (1980). Postwar US business cycles: an empirical investigation. Carnegie-Mellon University Working Paper.

Hodrick, R. J. and E. C. Prescott (1997). Postwar US business cycles: an empirical investigation. *Journal of Money, Credit, and Banking*, 1–16.

King, R. G. and S. Rebelo (1993). Low frequency filtering and real business cycles. *Journal of Economic Dynamics and Control 17*, 207–231.

Koopmans, T. C. (1947). Measurement without theory. *Review of Economics and Statistics 29*(3), 161–172.

Morley, J. C., C. R. Nelson, and E. Zivot (2003). Why are the Beveridge-Nelson and unobserved-components decompositions of gdp so different? *Review of Economics and Statistics 85*(2), 235–243.

Nelson, C. R. and H. Kang (1981). Spurious periodicity in inappropriately detrended time series. *Econometrica*, 741–751.

Nelson, C. R. and C. R. Plosser (1982). Trends and random walks in macroeconmic time series: some evidence and implications. *Journal of Monetary Economics 10*(2), 139–162.

Schüler, Y. S. (2018). On the cyclical properties of Hamilton's regression filter. Deutsche Bundesbank Discussion Paper No. 03/2018.

Shackleton, R. (2015). Why CBO projects that actual output will be below potential output on average. Congressional Budget Office, Report 49890.

Shackleton, R. (2018). Estimating and projecting potential output using CBO's forecasting growth model. Congressional Budget Office, Working Paper 3.

Stock, J. H. and M. W. Watson (1999). Business cycle fluctuations in US macroeconomic time series. *Handbook of Macroeconomics 1*, 3–64.

Watson, M. W. (1986). Univariate detrending methods with stochastic trends. *Journal of Monetary Economics 18*(1), 49–75.

A Solution of the Hodrick and Prescott (1997) Model

A closed-form solution for the trend in the Hodrick and Prescott (1997) model can be obtained by defining some matrix notation. Let T be the sample size, and define $\tilde{T} = T + 2$. Define the T dimensional vector of observations by $y = (y_T, y_{T-1}, ..., y_2, y_1)'$ and the $\tilde{T}$ dimensional vector of growth components $g = (g_T, g_{T-1}, ..., g_0, g_{-1})'$. Define the $(T \times \tilde{T})$ matrix $H = [I_{T\times T}\ 0_{T\times 2}]$, and define the $(T \times \tilde{T})$ matrix

$$Q = \begin{bmatrix} 1 & -2 & 1 & 0 & ... & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & ... & 0 \\ . & . & 0 & 0 & 0 & . & ... & . \\ . & . & . & . & . & . & ... & . \\ . & . & . & . & . & . & ... & . \\ 0 & 0 & ... & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & ... & 0 & 1 & -2 & 1 \end{bmatrix}.$$

The objective function in equation (3) can therefore be written as

$$\min_g (y - H'g)'(y - H'g) + \lambda(Qg)'(Qg), \tag{47}$$

and the first-order conditions with respect to g are

$$-2H'y + 2H'Hg + 2\lambda Q'Qg = 0. \tag{48}$$

Hence, the solution for the growth component is

$$g = (H'H + \lambda Q'Q)^{-1}H'y, \tag{49}$$

in which case A^* in the text equals $(H'H + \lambda Q'Q)^{-1}H'$. While Hodrick and Prescott (1997) were aware of this direct solution, it involves inversion of a $(\tilde{T} \times \tilde{T})$ matrix, which they viewed as potentially problematic given the memory capacity of computers in 1978. The state space nature of the problem offered the simpler Kalman filtering and smoothing solutions that only involve inversions of (2×2) matrixes.

What the solution for g in equation (49) demonstrates is that the data are being filtered. Hodrick and Prescott (1997) note that the solution implies

$$g_t = \sum_{j=1}^{T} w_{jt}^T y_j \tag{50}$$

where the w_{jt}^T are the elements of the t-th row from the bottom of the A^* matrix. They also note that if the sample were infinite, the solution would be

$$g_t = \sum_{j=-\infty}^{\infty} w_j^{\infty} y_{t+j}, \tag{51}$$

where

$$w_j^{\infty} = 0.8941^j(0.056168\cos(0.11168j) + 0.055833\sin(0.11168j)), \tag{52}$$

for $j \geq 0$ and $w_j^{\infty} = w_{-j}^{\infty}$ for $j < 0$. See also King and Rebelo (1993) and Hamilton (2018) for additional discussion of the properties of the solution to the HP filtering problem.

B The Baxter and King (1999) Band-Pass Filter

In the frequency domain, a time series is characterized by sine waves of frequencies between $-\pi$ and π. The periodicity of a time series corresponds to the number of units of time over which fluctuations occur. Periodicity, p, is related to frequency, ω, by $p = 2\pi/\omega$. For business cycle analysis, Baxter and King (1999) sought to isolate fluctuations with periodicities between 6 and 32 quarters.

The basic building block of the band-pass filter is the low-pass filter - a filter that retains only slowly moving components of the data. An ideal, symmetric, low-pass filter passes only frequencies $-\underline{\omega} \leqslant 0 \leqslant \underline{\omega}$. Baxter and King (1999) derive the properties of such filters demonstrating that in the time domain, the filter, as a function of the lag operator, would be

$$b(L) = \sum_{h=-\infty}^{\infty} b_h L^h \tag{53}$$

where the b_h coefficients are found as

$$b_h = \frac{1}{2\pi} \int_{-\pi}^{\pi} \beta(\omega) e^{i\omega h} d\omega, \tag{54}$$

and $\beta(\omega) = 1$ for $|\omega| \leq \underline{\omega}$ and $\beta(\omega) = 0$ for $|\omega| > \underline{\omega}$. Solving the integral gives the optimal weights $b_0 = \underline{\omega}/\pi$ and $b_h = \sin(h\underline{\omega})/h\pi$ for $h = 1, 2, ...$ Baxter and King (1999) also derive that truncating the optimal weights at $h = K$ gives the optimal two-side filter with $(2K + 1)$ weights. Because the truncated weights do not sum to unity, the actual weights are $a_h = b_h + \theta$ where

$$\theta = \frac{1 - \sum_{h=-K}^{h=K} b_h}{(2K + 1)}. \tag{55}$$

Taking the difference between two low-pass filters with different cutoffs, $\underline{\omega}_l$ and $\underline{\omega}_h$, gives the optimal band-pass filter. The weights for the BP(6,32) filter are $w_0 = 0.2777, w_1 = 0.2204, w_2 = 0.0838, w_3 = -0.0521, w_4 = -0.1184, w_5 = -0.1012, w_6 = -0.0422, w_7 = 0.0016, w_8 = 0.0015, w_9 = -0.0279, w_10 = -0.0501, w_{11} = -0.0423, w_{12} = -0.0119$.

Because the BK filter breaks a time series into three sets of frequencies, I do not define their trend to be the difference between the realization of a series and its business cycle component. I define their trend to be fluctuations with a periodicity greater than or equal to 32 quarters and hence a frequency such that $|\omega| \leq 2\pi/32$. I also set the truncation level at $K = 12$ as they do in their preferred cyclical band-pass filter.

Christiano and Fitzgerald (2003) develop an alternative band-pass filter that solves a slightly different problem than Baxter and King (1999). Let x_t represent the raw data, whose estimated drift, defined as $(x_T - x_1)/(T - 1)$, has been removed. Then, the goal of Christiano and Fitzgerald (2003) is to decompose x_t into a component y_t that has power only at business cycle frequencies and a component $\tilde{x}_t$ that has power only at non-business-cycle frequencies. If the time series were infinite, the solution would use equations (53) and (54), with $y_t = b(L)x_t$ where $b_0 = (\overline{\omega} - \underline{\omega})/\pi$ and $b_h = (\sin(h\overline{\omega}) - \sin(h\underline{\omega}))/h\pi$. Since the data are finite, this filter is infeasible, and Christiano and Fitzgerald (2003) propose choosing the feasible solutions, $\hat{y}_t$, to minimize the T different problems

$$E[(y_t - \hat{y}_t)^2 | x] \tag{56}$$

where x is the vector of T observations. The solutions to equation (56), which use all observations, are

$$\hat{y}_t = \sum_{j=-f}^{p} \hat{b}_j^{p,f} x_{t-j} \tag{57}$$

where $f = T - t$ and $p = t - 1$.

By expressing the T problems in equation (56) in the frequency domain and using the definition of the variance, the problems become

$$\min_{\hat{b}_j^{p,f}, j=-f\ldots p} \int_{-\pi}^{\pi} |b(e^{-i\omega}) - \hat{b}^{p,f}(e^{-i\omega})|^2 f_x(\omega) d\omega \tag{58}$$

where$f_x(\omega)$ is the spectral density of x_t and $\hat{b}^{p,f}(L) = \sum_{j=-f}^{p} \hat{b}_j^{p,f} L^j$.

Christiano and Fitzgerald (2003) stress that the presence of $f_x(\omega)$ in equation (58) indicates that the solutions to the minimization problems depend on the properties of the time series representation of the underlying data, which have to be estimated, in contrast to the weights in the ideal band pass filter that do not. They also note that the solution uses T different filters, one for each date, that the filters are not stationary with respect to t, and that the filters weight past and future observations on x_t asymmetrically. After conducting some extensive analysis, Christiano and Fitzgerald (2003) conclude that both non-stationarity and asymmetry of the filters are useful but non-stationarity is relatively more important, that the degree of non-stationarity and asymmetry in optimal finite filters is small, and that the gain from using the true spectral density versus assuming the series is a random walk is minimal.

Upon implementing the CF filter with the random walk assumption for GDP, the correlations of the CF cycle with the HP and BK cycles are 0.88 and 0.91, respectively, and the standard deviation of the CF cycle is 1.508 versus 1.550 for the HP cycle and 1.490 for the BK cycle. Given these values, I choose not to report additional statistics associated with the CF filter.

C Estimation of the Clark (1987) Model

For the period 1947:1 to 1985:4 the reported estimates from Clark (1987) are

$$\Delta g_t = d_{t-1} + 0.64 w_t, \tag{59}$$

$$d_t = d_{t-1} + 0.01 u_t, \tag{60}$$

$$c_t = 1.53 c_{t-1} - 0.59 c_{t-2} + 0.74 v_t. \tag{61}$$

When I estimate this model on my data set for the same sample, I find

$$\Delta g_t = d_{t-1} + \underset{(0.219)}{0.674} w_t, \tag{62}$$

$$d_t = d_{t-1} + \underset{(0.027)}{0.014} u_t, \tag{63}$$

$$c_t = \underset{(0.186)}{1.490} c_{t-1} - \underset{(0.192)}{0.571} c_{t-2} + \underset{(0.237)}{0.746} v_t, \tag{64}$$

The differences in the estimated parameters appear to be minor and could be due to revisions in the GDP data or to the convergence criteria used in the non-linear estimation.

D Estimation of the Morley, Nelson, and Zivot (2003) Model

For the period 1947:I to 1998:II the reported estimates from Morley et al. (2003) are

$$\Delta y_t = \underset{(0.086)}{0.378} + \underset{(0.152)}{1.342}\Delta y_{t-1} - \underset{(0.173)}{0.706}\Delta y_{t-2} + \epsilon_t - \underset{(0.196)}{1.054}\epsilon_{t-1} + \underset{(0.225)}{0.519}\epsilon_{t-2}. \tag{65}$$

In ARIMA models, the long-run impulse response can be found by

$$A_0 = \phi(1)^{-1}\theta(1) = (1 - 1.1054 + 0.519)/(1 - 1.342 + 0.706) = 1.276. \tag{66}$$

When I estimate this model on my data set for the same sample, I find

$$\Delta y_t = \underset{(0.074)}{0.372} + \underset{(0.116)}{1.342}\Delta y_{t-1} - \underset{(0.084)}{0.770}\Delta y_{t-2} + \epsilon_t - \underset{(0.127)}{1.086}\epsilon_{t-1} + \underset{(0.099)}{0.621}\epsilon_{t-2}. \tag{67}$$

The estimated parameters are reasonably similar, and the implied long-run impulse response for my data is also quite similar,

$$A_0 = \phi(1)^{-1}\theta(1) = (1 - 1.1086 + 0.621)/(1 - 1.342 + 0.770) = 1.250. \tag{68}$$

The differences in the estimated parameters could be due to revisions in the GDP data or to the convergence criteria used in the non-linear estimation.

E The Cochrane (1988) Long Difference Variance Estimator

The Cochrane (1988) estimator of $(1/k)$ times the variance of the $k-$difference in y_t is

$$\sigma_k^2 = \frac{T}{k(T-k)(T-k+1)} \sum_{j=k}^{T} (y_j - y_{j-k} - (k/T)(y_T - y_0))^2. \tag{69}$$

The degrees of freedom adjustment is important for the small sample properties of the estimator.

F ARIMA Versions of Trend and Cycle Models

This section derives the univariate ARIMA versions of models with different specifications of growth components or trends and cyclic components.

F.1 The Changing Trend Model with Constant Unconditional Mean

The model in which y_t is composed of a growth component with constant unconditional mean growth, but serially correlated conditional mean growth, and a cyclical component is summarized by three equations. The first is the decomposition in equation (1). The second equation postulates an ARIMA(1,1,0) for the growth component,

$$\Delta g_t = \mu + \rho \Delta g_{t-1} + w_t. \tag{70}$$

The third equation postulates an ARIMA(2,0,0) for the cyclical component,

$$c_t = \phi_1 c_{t-1} + \phi_2 c_{t-2} + v_t. \tag{71}$$

The innovations are independently and identically distributed and are uncorrelated with each other. To relate this model to a univariate ARIMA model for y_t, first define the two AR polynomials in the lag operator, $\rho(L) = 1 - \rho_1 L$, and $\phi(L) = 1 - \phi_1 L - \phi_2 L^2$. Then, take the first difference of equation (1) and substitute for Δg_t and Δc_t in terms of their innovations,

$$\Delta y_t = \Delta g_t + \Delta c_t = \rho(1)^{-1}\mu + \rho(L)^{-1} w_t + \phi(L)^{-1} \Delta v_t. \tag{72}$$

Multiplying through by $\rho(L)\phi(L)$ gives

$$\rho(L)\phi(L)\Delta y_t = \phi(1)\mu + \phi(L) w_t + \rho(L)\Delta v_t. \tag{73}$$

The stochastic part of the right-hand side of equation (73) can be shown to be a second order moving average process, $\theta(L)\epsilon_t = 1 + \theta_1 L + \theta_2 L^2 \epsilon_t$ in which $\epsilon_t = w_t + u_t$, and the parameters are functions of the ρ and ϕ parameters and the variances of the two fundamental innovations. The left-hand side of equation (73) is an AR(3) in the first difference of y_t, and the polynomial in the lag operator is $\psi(L) = 1 - \psi_1 L - \psi_2 L^2 - \psi_3 L^3 = \phi(L)\rho(L)$ where the ψ parameters are functions of the ρ and ϕ parameters. Hence, the y_t process is an ARIMA(3,1,2),

$$\psi(L)\Delta y_t = \phi(1)\mu + \theta(L)\epsilon_t. \tag{74}$$

F.2 The Clark (1987) Unobserved Components Model

The Clark (1987) unobserved components model, in which the change in trend growth is nonstationary, is summarized by four equations. The first equation is again the decomposition in equation (1). The second equation specifies that the change in the growth component has a time varying conditional mean,

$$\Delta g_t = d_{t-1} + w_t. \tag{75}$$

The third equation specifies that the conditional mean follows a random walk,

$$d_t = d_{t-1} + u_t. \tag{76}$$

The fourth equation specifies that the cyclical component is an AR(2):

$$c_t = \phi_1 c_{t-1} + \phi_2 c_{t-2} + v_t. \tag{77}$$

To relate this model to a univariate ARIMA model, first define the AR polynomial in the lag operator, $\phi(L) = 1 - \phi_1 L - \phi_2 L^2$. From equations (75) and (76) note that taking the second difference of the trend produces a stationary series,

$$\Delta^2 g_t = u_{t-1} + \Delta w_t. \tag{78}$$

Thus, by applying the second difference operator to equation (1) and substituting from equations (77) and (78), one finds

$$\Delta^2 y_t = \Delta^2 g_t + \Delta^2 c_t = u_{t-1} + \Delta w_t + \phi(L)^{-1}\Delta^2 v_t \tag{79}$$

Multiplying through by $\phi(L)$ gives

$$\phi(L)\Delta^2 y_t = \phi(L)u_{t-1} + \phi(L)\Delta w_t + \Delta^2 v_t. \tag{80}$$

The right-hand side of equation (80) can be shown to be a third-order moving average process, $\theta(L)\epsilon_t = (1 + \theta_1 L + \theta_2 L^2 + \theta_3 L^3)\epsilon_t$ in which $\epsilon_t = w_t + v_t$, and the parameters are functions of the ϕ parameters and the variances of the three innovations. Hence, the y_t process is an ARIMA(2,2,3),

$$\phi(L)\Delta^2 y_t = \theta(L)\epsilon_t. \tag{81}$$

G Deriving the Modified Hamilton (2018) Trend and Cycle

This section derives what I refer to as the modified Hamilton (2018) trend and cycle in the ARIMA model and in the two unobserved components model. I take the modified H trend to be $g^H_{t+8} = E_t(y_{t+8})$ and the modified H cycle to be $c^H_{t+8} = y_{t+8} - g_{t+8}$. These definitions substitute the conditional expectation of y_{t+8} based on the full information set at time t rather than the regression forecast of y_{t+8} based on the reduced information set, $\{y_t, y_{t-1}, y_{t-2}, y_{t-3}\}$, because I want to see how well the empirical analysis captures the true conditional mean and not simply to examine the small sample properties of the regression.

G.1 The Modified H Trend and H Cycle in the ARIMA Model

Let y_t be characterized by an ARIMA(2,1,2) model as in Morley et al. (2003). Then,

$$\Phi(L)\Delta y_t = \mu + \Theta(L)\epsilon_t \tag{82}$$

where $\Phi(L)$ and $\Theta(L)$ are second-order polynomials in the lag operator. Multiplying through by $\Phi(L)^{-1}$ gives the Wold representation

$$\Delta y_t = \mu_0 + a(L)\epsilon_t, \tag{83}$$

where $\mu_0 = \Phi(1)^{-1}\mu$,

$$a(L) = \Phi(L)^{-1}\Theta(L) = \sum_{j=0}^{\infty} a_j L^j, \tag{84}$$

and $a_0 = 1$.

Now, to find $E_t(y_{t+8})$ use the fact that $y_{t+8} - y_t = \Delta y_{t+8} + \Delta y_{t+7} + ... + \Delta y_{t+1}$, and substitute from equation (83) to find

$$y_{t+8} = y_t + 8\mu + a(L)(\epsilon_{t+8} + \epsilon_{t+7} + ... + \epsilon_{t+1}). \tag{85}$$

Recall that A_k represents the infinite sum of the a_j coefficients beginning at $j = k$. Then, with a bit of algebra using equation (85) one finds

$$y_{t+8} = y_t + 8\mu + \sum_{j=1}^{8}(A_0 - A_{9-j})\epsilon_{t+j} + \sum_{j=0}^{\infty}(A_{j+1} - A_{j+9})\,\epsilon_{t-j}. \tag{86}$$

To calculate the model's cycle at time $t+8$ in the simulations of the ARIMA(2,1,2) model, I use Matlab's "impulse" command to generate the a_j's from the model's known parameters, and I calculate the modified H cycle using equation (86) to be the unanticipated part of y_{t+8}

$$c_{t+8} = y_{t+8} - E_t(y_{t+8}) = \sum_{j=1}^{8}(A_0 - A_{9-j})\epsilon_{t+j}. \tag{87}$$

The trend is consequently found as $g_{t+8} = y_{t+8} - c_{t+8}$.

G.2 The Modified H Trend and H Cycle in the First Unobserved Components Model

In addition to equation (1), this unobserved components model has two additional equations:

$$\Delta g_t = \mu + \rho\Delta g_{t-1} + \nu_t \tag{88}$$

$$c_t = \phi_1 c_{t-1} + \phi_2 c_{t-1} + \epsilon_t. \tag{89}$$

The modified H trend is

$$g^H_{t+8} = E_t(y_{t+8}) = E_t(g_{t+8}) + E_t(c_{t+8}). \tag{90}$$

From equation (88), it can be demonstrated that

$$E_t(g_{t+8}) = g_t + \left(\mu\sum_{j=1}^{8}(1-\rho^j)(1-\rho)^{-1}\right) + \rho(1-\rho^8)(1-\rho)^{-1}\Delta g_t. \tag{91}$$

To derive $E_t(c_{t+8})$ write equation (89) in first-order companion form:

$$\begin{bmatrix} c_t \\ c_{t-1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c_{t-1} \\ c_{t-2} \end{bmatrix} + \begin{bmatrix} \epsilon_t \\ 0 \end{bmatrix}. \tag{92}$$

Define $e_1 = [1,0]'$. Then,

$$E_t(c_{t+8}) = e_1'\begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix}^8\begin{bmatrix} c_t \\ c_{t-1} \end{bmatrix}. \tag{93}$$

Plugging equations (91) and (93) into equation (90) gives g^H_{t+8}, and $c^H_{t+8} = y_{t+8} - g^H_{t+8}$.

G.3 The Modified H Trend and H Cycle in the Clark (1987) Model

In addition to equations (1) and (89) the two other equations of the Clark (1987) unobserved components model are

$$\Delta g_t = d_{t-1} + w_t \tag{94}$$

and

$$\Delta d_t = u_t \tag{95}$$

It is straightforward to see that

$$E_t(g_{t+8}) = g_t + E_t(\Delta g_{t+1} + ... + \Delta g_{t+8}) = g_t + E_t(d_t + d_{t+1} + ...d_{t+7}) = g_t + 8d_t \tag{96}$$

because $E_t(d_{t+j}) = d_t$. Since $E_t(c_{t+8})$ is the same as in the previous subsection, g^H_{t+8} can be found from equations (93) and (96).

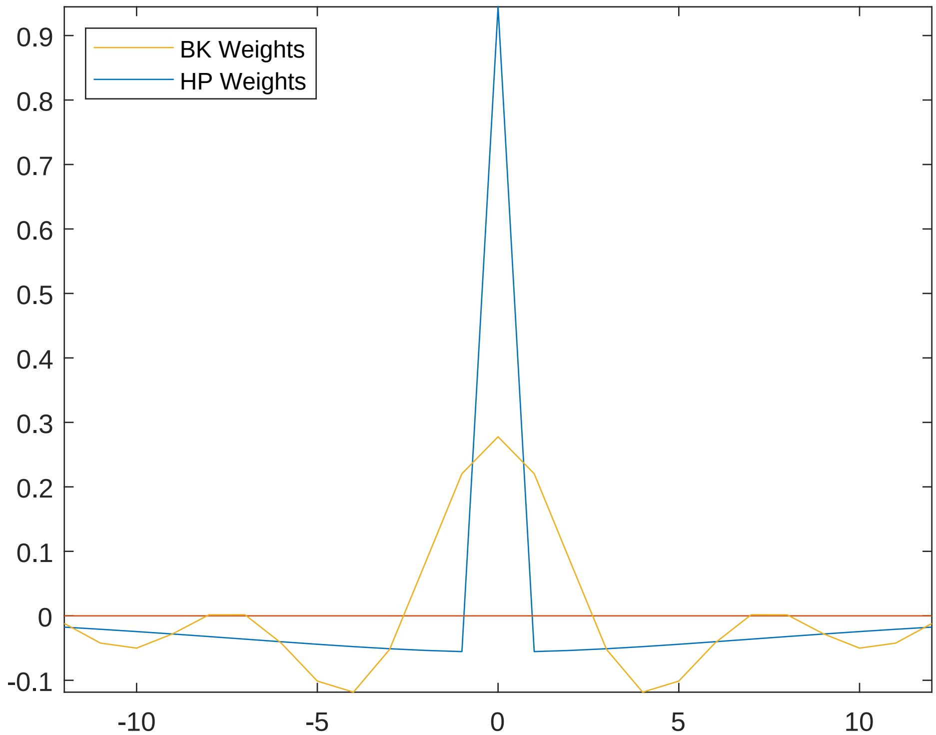

Figure 1: Hodrick and Prescott (1997) and Baxter and King (1999) Cycle Weights

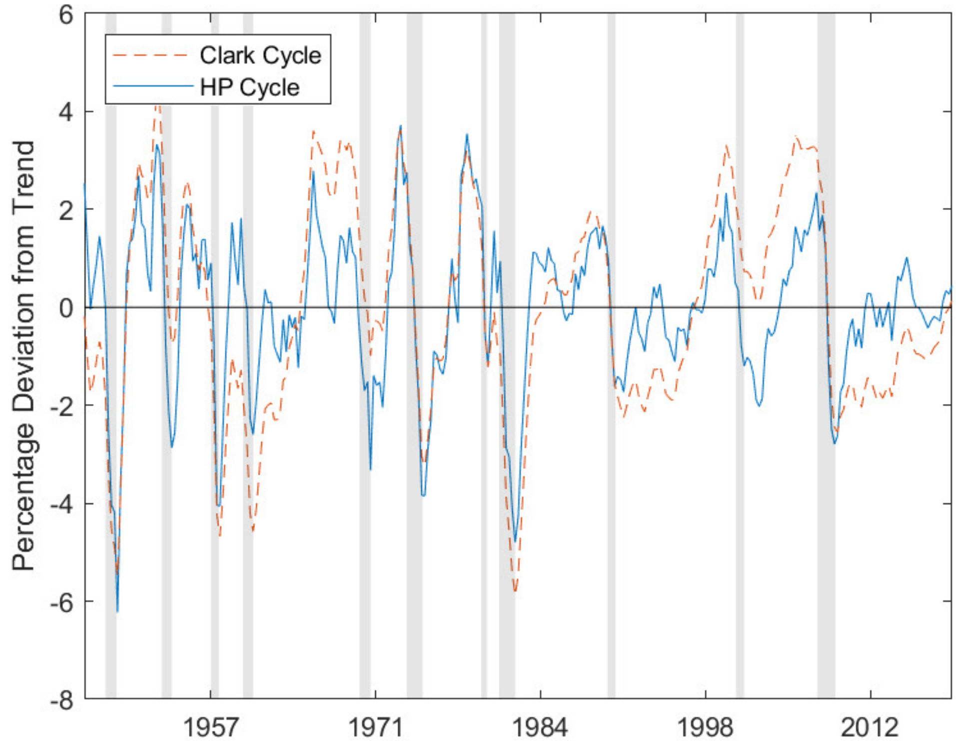

Figure 2: Clark (1987) and Hodrick and Prescott (1997) Cycles

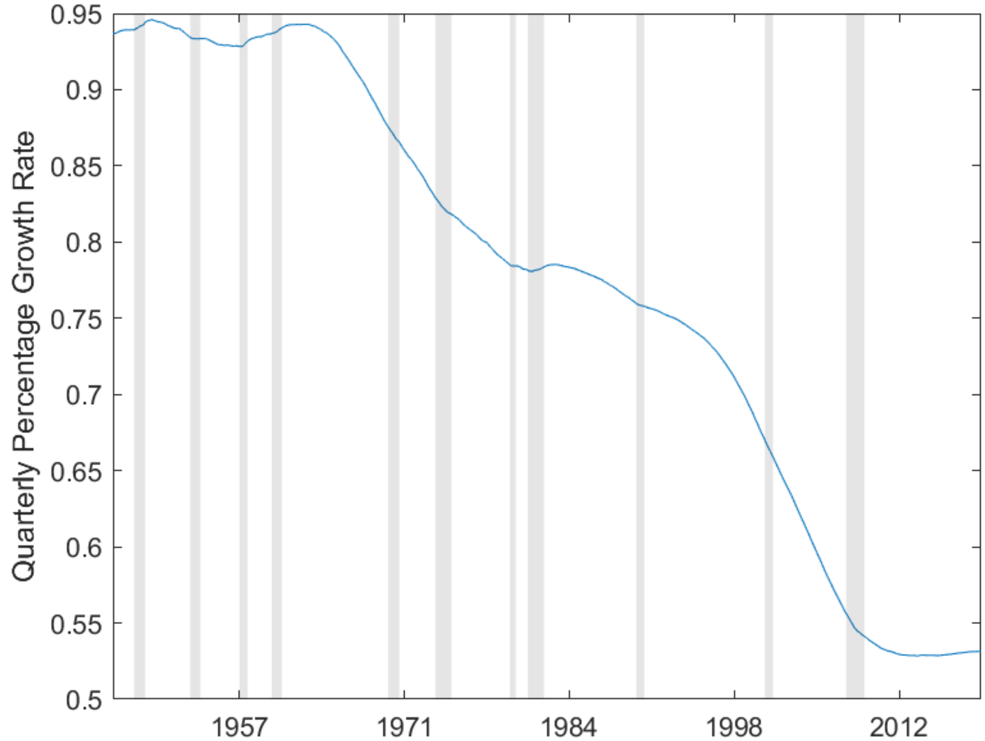

Figure 3: Conditional Mean of Change in Trend in the Clark (1987) Model

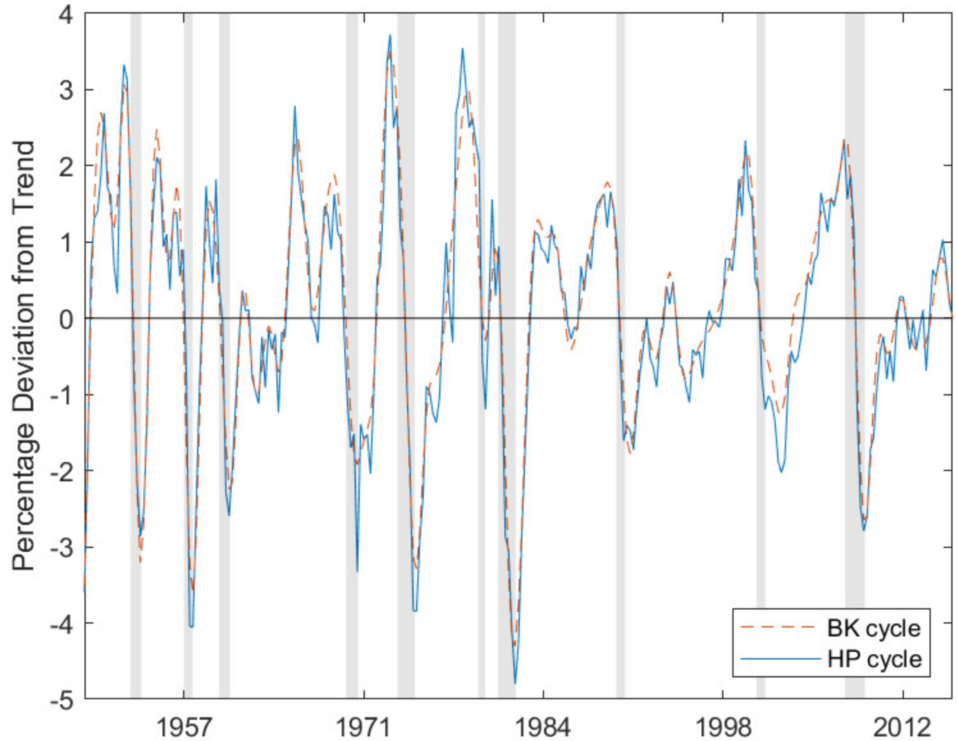

Figure 4: Baxter and King (1999) and Hodrick and Prescott (1997) Cycles

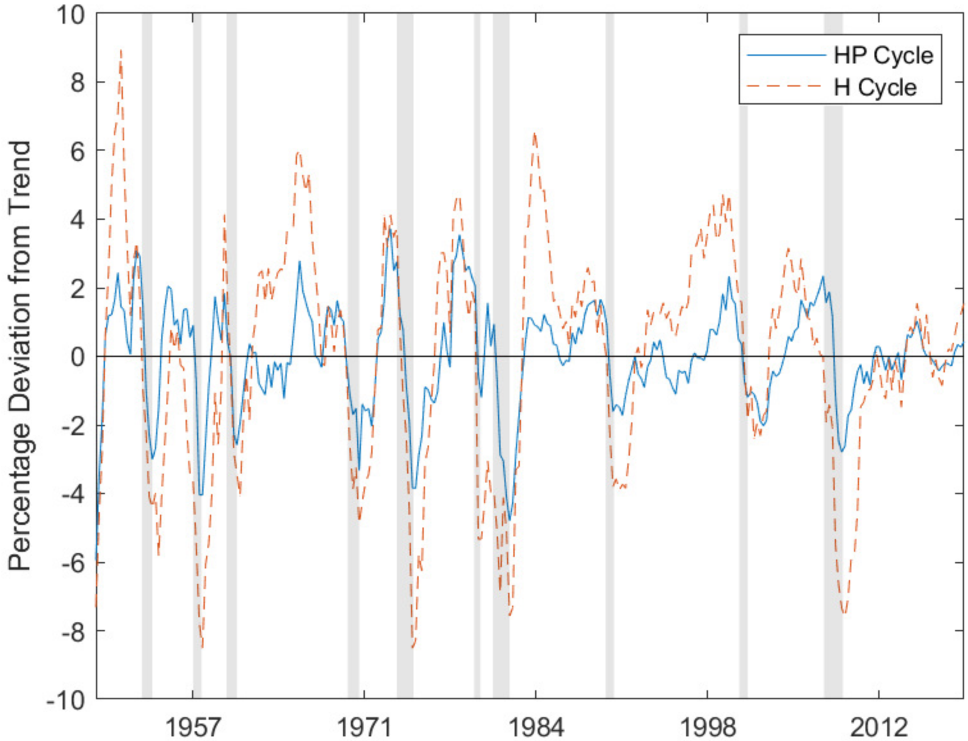

Figure 5: Hamilton (2018) and Hodrick and Prescott (1997) Cycles

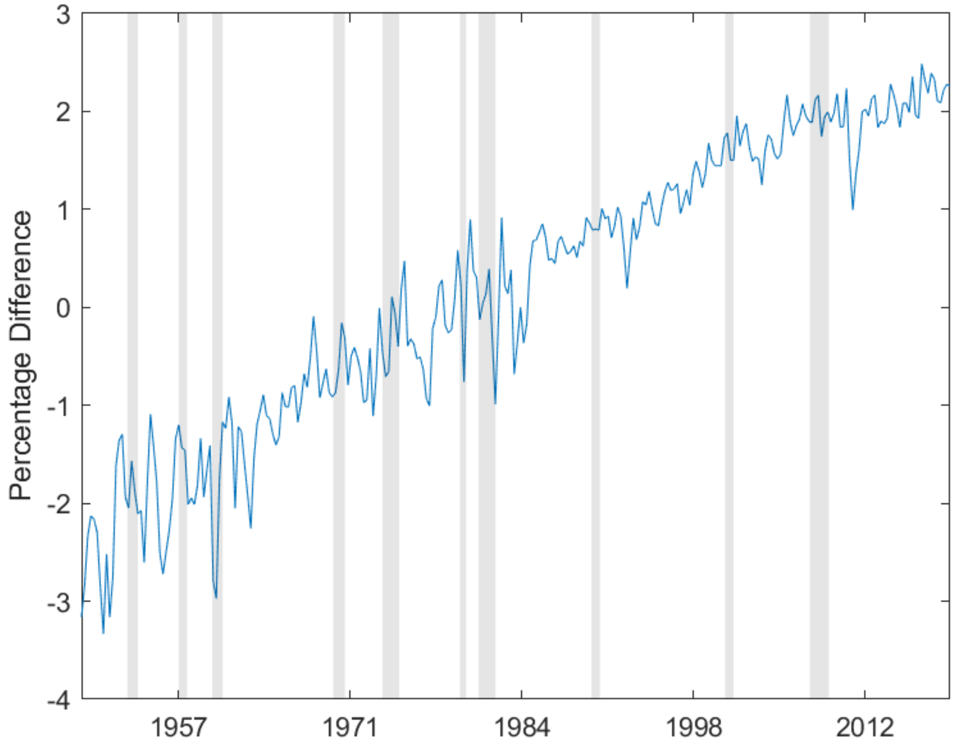

Figure 6: Hamilton (2018) Cycle minus 8-th Difference GDP

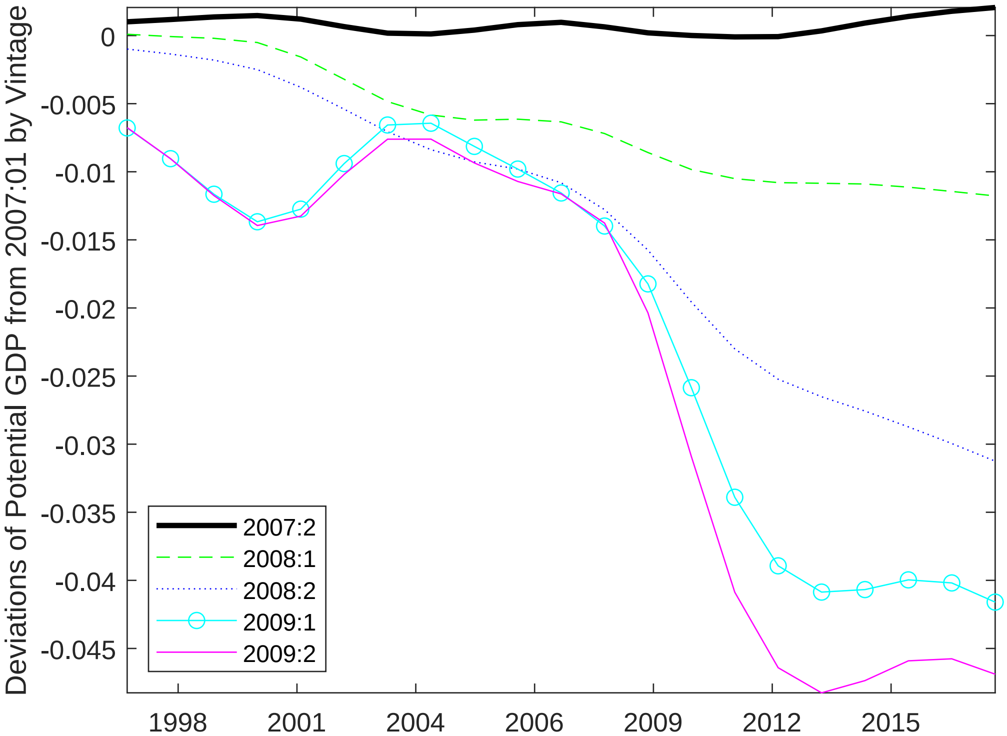

Figure 7: Deviations of Future Potential GDP's from Vintage 2007:01

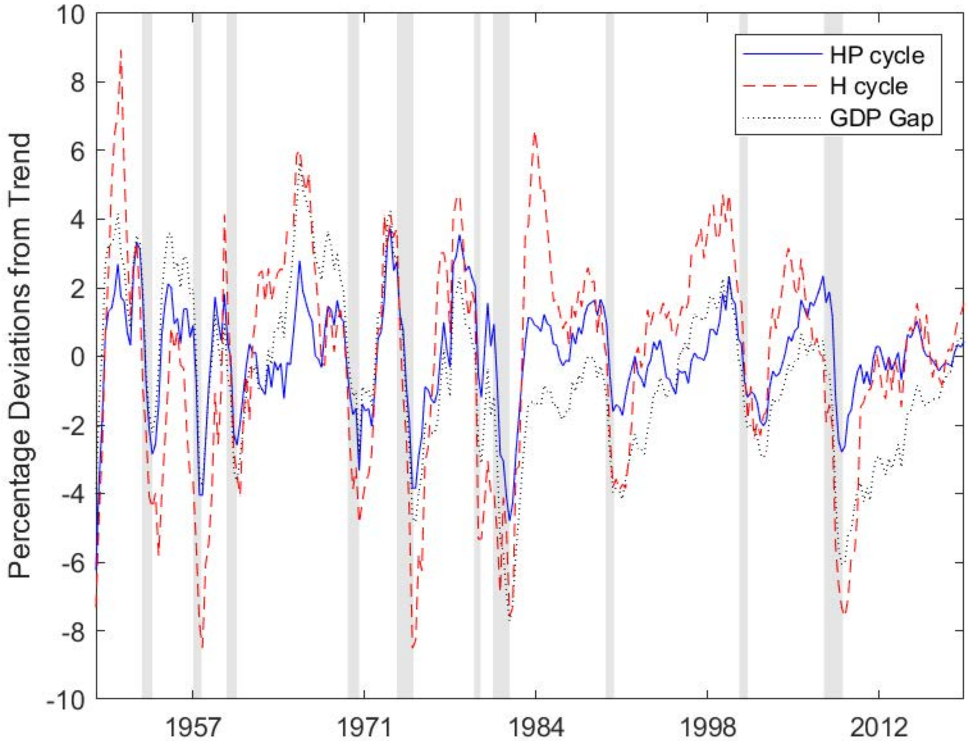

Figure 8: GDP Gap with Hamilton (2018) and Hodrick and Prescott (1997) Cycles

Table 1: Statistics on U.S. Real GDP

The table presents two sets of statistics on GDP for a sample period from 1947:1 to 2019:1. In Panel A the columns labeled "H", "HP", and "BK" refer to the respective methods of decomposing a time series into trend and cycle of Hamilton (2018), Hodrick and Prescott (1997), and Baxter and King (1999), and the statistics are the standard deviations of the cyclical components and the standard deviations of the changes in the trends. Panel B reports the square roots of the Variance Ratios.

Panel A: Standard Deviations

	H	HP	BK
Standard Deviation of Cycle	3.295	1.550	1.490
Standard Deviation of ΔTrend	1.022	0.261	0.270

Panel B: Square Roots of Variance Ratios

k	10	20	30	40	50	60	70	80
$\sqrt{V_k}$	1.381	1.392	1.424	1.455	1.446	1.463	1.480	1.410

Table 2: Simulated Statistics from a Random Walk

The table presents sample means of statistics from 5,000 simulations of length 289 of a random walk with drift calibrated such that the drift and the standard deviation coincide, respectively, with the sample mean and sample standard deviation of the rate of growth of GDP. In Panel A the columns labeled “H”, “HP”, and “BK” refer to the respective methods of decomposing a time series into trend and cycle of Hamilton (2018), Hodrick and Prescott (1997), and Baxter and King (1999). The column labeled “In Sim” is the sample mean of the realized values of the row statistic in the simulations. Panel B presents the slope coefficient and R^2 in the regression of the simulated cycle minus the HP or BK cycle on a constant and the H cycle minus the HP or BK cycle. Panel C presents the square roots of the Variance Ratios.

Panel A: Means of Standard Deviations, Correlations, and RMSEs

	H	HP	BK	In Sim
Standard Deviation of Cycle	2.539	1.190	1.073	2.539
Standard Deviation of ΔTrend	0.893	0.182	0.188	0.932
Correlation of Cycles	0.979	0.660	0.632	
Correlation of ΔTrends	0.972	0.144	0.149	
RMSE of Cycles	0.632	2.057	2.128	

Panel B: Means of Regression Diagnostics

	HP	BK
Slope Coefficient	0.999	1.002
R^2	0.932	0.940

Panel C: Means of Square Roots of Variance Ratios

k	10	20	30	40	50	60	70	80
$\sqrt{V_k}$	1.000	1.000	1.000	1.000	1.000	1.002	1.003	1.003

Table 3: Simulated Statistics from an ARIMA model

The table presents sample means of statistics from 5,000 simulations of length 289 from the ARIMA(2,1,2) model of Morley et al. (2003) estimated with GDP data for the full sample. The model is

$$\Delta y_t = 0.320 + 1.271\Delta y_{t-1} - 0.682\Delta y_{t-2} + \epsilon_t - 0.979\epsilon_{t-1} + 0.540\epsilon_{t-2},$$

and ϵ_t is distributed $N(0, 0.7236)$. In Panel A the columns labeled "H", "HP", and "BK" refer to the respective methods of decomposing a time series into trend and cycle of Hamilton (2018), Hodrick and Prescott (1997), and Baxter and King (1999). The column labeled "In Sim" is the sample mean of the realized values of the row statistic in the simulations. Panel B presents the slope coefficient and R^2 in the regression of the simulated cycle minus the HP or BK cycle on a constant and the H cycle minus the HP or BK cycle. Panel C presents the square roots of the Variance Ratios.

Panel A: Means of Standard Deviations, Correlations, and RMSEs

	H	HP	BK	In Sim
Standard Deviation of Cycle	3.238	1.525	1.472	3.306
Standard Deviation of ΔTrend	1.084	0.228	0.236	1.075
Correlation of Cycles	0.977	0.685	0.676	
Correlation of ΔTrends	0.944	0.151	0.159	
RMSE of Cycles	0.837	2.569	2.603	

Panel B: Means of Regression Diagnostics

	HP	BK
Slope Coefficient	0.986	0.989
R^2	0.921	0.927

Panel C: Means of Square Roots of Variance Ratios

k	10	20	30	40	50	60	70	80
$\sqrt{V_k}$	1.273	1.268	1.265	1.263	1.262	1.263	1.263	1.264

Table 4: Simulated Statistics from a Model with Constant Unconditional Mean Change in Trend

The table presents sample means of statistics from 5,000 simulations of length 289 in which the simulated time series is the sum of a stochastic trend and a stochastic cycle,

$$y_t = g_t + c_t$$

The trend is modeled as an ARIMA(1,1,0). The change in the trend is

$$\Delta g_t = 0.07786 + 0.900\Delta g_{t-1} + \nu_t,$$

and the innovations are drawn from a $N(0, 0.002)$. The cyclical component is modeled as an ARIMA(2,0,0)

$$c_t = 1.25c_{t-1} - 0.45c_{t-2} + \epsilon_t,$$

and the innovations are drawn from a $N(0, 0.6385)$. The innovations in the trend and cycle are uncorrelated. In Panel A the columns labeled "H", "HP", and "BK" refer to the respective methods of decomposing a time series into trend and cycle of Hamilton (2018), Hodrick and Prescott (1997), and Baxter and King (1999). The column labeled "In Sim" is the sample mean of the realized values of the row statistic in the simulations. Panel B presents the slope coefficient and R^2 in the regression of the simulated cycle minus the HP or BK cycle on a constant and the H cycle minus the HP or BK cycle. Panel C presents the square roots of the Variance Ratios.

Panel A: Means of Standard Deviations, Correlations, and RMSEs

	H	HP	BK	In Sim
Standard Deviation of Cycle	2.419	1.423	1.364	1.742
Standard Deviation of ΔTrend	0.797	0.104	0.115	0.098
Correlation of Cycles	0.741	0.887	0.857	
Correlation of ΔTrends	0.041	0.543	0.495	
RMSE of Cycles	1.638	0.840	0.936	

Panel B: Means of Regression Diagnostics

	HP	BK
Slope Coefficient	0.087	0.150
R^2	0.041	0.082

Panel C: Means of Square Roots of Variance Ratios

k	10	20	30	40	50	60	70	80
$\sqrt{V_k}$	0.915	0.714	0.647	0.612	0.593	0.579	0.570	0.563

Table 5: Simulated Statistics from the Clark (1987) Unobserved Components Model

The table presents sample means of statistics from 5,000 simulations of length 289 generate from the Clark (1987) model estimated on the full sample. The simulated time series is the sum of a stochastic trend and a stochastic cycle,

$$y_t = g_t + c_t.$$

The change in the trend has a conditional mean,

$$\Delta g_t = d_{t-1} + 0.545 w_t,$$

and the conditional mean is a random walk with a relatively small standard deviation of its innovation,

$$d_t = d_{t-1} + 0.021 u_t.$$

The cyclical component is modeled as an ARIMA(2,0,0),

$$c_t = 1.510 c_{t-1} - 0.565 c_{t-2} + 0.603 v_t,$$

and the three innovations, u_t, w_t, and v_t, are standard normal random variables. The three innovations are uncorrelated. In Panel A the columns labeled "H", "HP", and "BK" refer to the respective methods of decomposing a time series into trend and cycle of Hamilton (2018), Hodrick and Prescott (1997), and Baxter and King (1999). The column labeled "In Sim" is the sample mean of the realized values of the row statistic in the simulations. Panel B presents the slope coefficient and R^2 in the regression of the simulated cycle minus the HP or BK cycle on a constant and the H cycle minus the HP or BK cycle. Panel C presents the square roots of the Variance Ratios.

Panel A: Means of Standard Deviations, Correlations, and RMSEs

	H	HP	BK	In Sim
Standard Deviation of Cycle	3.445	1.631	1.557	2.667
Standard Deviation of ΔTrend	1.238	0.234	0.246	0.564
Correlation of Cycles	0.510	0.651	0.676	
Correlation of ΔTrends	0.028	0.251	0.234	
RMSE of Cycles	3.168	2.130	2.089	

Panel B: Means of Regression Diagnostics

	HP	BK
Slope Coefficient	0.080	0.084
R^2	0.032	0.034

Panel C: Means of Square Roots of Variance Ratios

k	10	20	30	40	50	60	70	80
$\sqrt{V_k}$	1.382	1.306	1.288	1.305	1.337	1.374	1.410	1.443

Table 6: Simulated Statistics from a Model with a Changing Unconditional Mean Change in Trend

The table presents sample means of statistics from 5,000 simulations of length 289 in which the simulated time series is the sum of a stochastic trend and a stochastic cycle,

$$y_t = g_t + c_t.$$

The trend is modeled as an ARIMA(1,1,0) with three different intercepts, $\mu_1 = 0.1$ for the first third of the simulated data, $\mu_2 = 0.07786$ for the second third, and $\mu_3 = 0.0554$ for the final third. The change in the trend is

$$\Delta g_t = \mu_i + 0.900\Delta g_{t-1} + \nu_t,$$

and the innovations are drawn from a $N(0, 0.002)$. The cyclical component is modeled as an ARIMA(2,0,0)

$$c_t = 1.25c_{t-1} - 0.45c_{t-2} + \epsilon_t,$$

and the innovations are drawn from a $N(0, 0.6385)$. The innovations in the trend and cycle are uncorrelated. In Panel A the columns labeled "H", "HP", and "BK" refer to the respective methods of decomposing a time series into trend and cycle of Hamilton (2018), Hodrick and Prescott (1997), and Baxter and King (1999). The column labeled "In Sim" is the sample mean of the realized values of the row statistic in the simulations. Panel B presents the slope coefficient and R^2 in the regression of the simulated cycle minus the HP or BK cycle on a constant and the H cycle minus the HP or BK cycle. Panel C presents the square roots of the Variance Ratios.

Panel A: Means of Standard Deviations, Correlations, and RMSEs

	H	HP	BK	In Sim
Standard Deviation of Cycle	2.448	1.426	1.367	1.742
Standard Deviation of ΔTrend	0.801	0.206	0.207	0.205
Correlation of Cycles	0.731	0.885	0.855	
Correlation of ΔTrends	0.202	0.878	0.851	
RMSE of Cycles	1.682	0.845	0.945	

Panel B: Means of Regression Diagnostics

	HP	BK
Slope Coefficient	0.081	0.139
R^2	0.043	0.078

Panel C: Means of Square Roots of Variance Ratios

k	10	20	30	40	50	60	70	80
$\sqrt{V_k}$	1.081	1.101	1.216	1.334	1.442	1.536	1.614	1.673

Table 7: Simulated Statistics from a Model with a Slowly Changing Unconditional Mean Change in Trend

The table presents sample means of statistics from 5,000 simulations of length 289 in which the simulated time series is the sum of a stochastic trend and a stochastic cycle,

$$y_t = g_t + c_t$$

The trend is modeled as an ARIMA(1,1,0) with a changing unconditional mean. The change in the trend is

$$\Delta g_t = 0.07786 + 0.04428(145 - t)/289 + 0.900\Delta g_{t-1} + \nu_t,$$

where t ranges from 1 to 289, and the innovations are drawn from a $N(0, 0.002)$. The cyclical component is modeled as an ARIMA(2,0,0)

$$c_t = 1.25c_{t-1} - 0.45c_{t-2} + \epsilon_t,$$

and the innovations are drawn from a $N(0, 0.6385)$. The innovations in the trend and cycle are uncorrelated. In Panel A the columns labeled "H", "HP", and "BK" refer to the respective methods of decomposing a time series into trend and cycle of Hamilton (2018), Hodrick and Prescott (1997), and Baxter and King (1999). The column labeled "In Sim" is the sample mean of the realized values of the row statistic in the simulations. Panel B presents the slope coefficient and R^2 in the regression of the simulated cycle minus the HP or BK cycle on a constant and the H cycle minus the HP or BK cycle. Panel C presents the square roots of the Variance Ratios.

Panel A: Means of Standard Deviations, Correlations, and RMSEs

	H	HP	BK	In Sim
Standard Deviation of Cycle	2.408	1.423	1.362	1.742
Standard Deviation of ΔTrend	0.792	0.156	0.160	0.152
Correlation of Cycles	0.744	0.887	0.857	
Correlation of ΔTrends	0.133	0.800	0.750	
RMSE of Cycles	1.623	0.840	0.940	

Panel B: Means of Regression Diagnostics

	HP	BK
Slope Coefficient	0.091	0.156
R^2	0.045	0.086

Panel C: Means of Square Roots of Variance Ratios

k	10	20	30	40	50	60	70	80
$\sqrt{V_k}$	0.987	0.899	0.931	0.980	1.030	1.076	1.117	1.152