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ABSTRACT

For a firm that cannot raise external funds, cash on hand serves as precautionary saving. We derive a closed-form expression for the target level of cash on hand in the presence of persistent cash flows. Contrary to conventional wisdom, a mean-preserving increase in the volatility of cash flow can decrease this target. Over the set of admissible parameter values the average impact of volatility on the target is zero. Endogenous selection, reflecting termination of firms that run out of cash, leads to a positive average impact of volatility on the target level of cash, consistent with empirical findings.

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How much cash should a financially-constrained firm hold to enable it to survive a period of negative cash flows? How does that level of cash depend on the variability of net cash flows from operations? These questions were first addressed by Miller and Orr (1966) who analyzed the cash management problem of a firm that faces stochastic cash flows.[1] This classic contribution derived a simple closed-form expression that shows that the optimal average level of cash holdings is an increasing function of the variability of exogenous cash flows from operations. In the half century since Miller and Orr, models of the precautionary demand for cash by firms have become more sophisticated, and are often solved numerically rather than analytically. Most analyses support the original Miller-Orr finding that an increase in the volatility of exogenous cash flows will increase optimal cash holdings. In their review of the dynamic corporate finance literature, Strebulaev and Whited (2011) explain "The central intuition behind the results in this figure is that any model feature that raises the probability of needing external finance in the future also raises the shadow value of cash and thus optimal cash levels....Both variance and highly serially correlated shocks increase the probability of needing external funds for investment, so average cash balances are increasing in both of these parameters." (pp. 105-107)

The conventional finding that precautionary cash holding is an increasing function of the volatility of cash flow was initially derived in a model in which exogenous cash flows from operations are serially uncorrelated. However, as we show in this paper, if exogenous cash flows are positively serially correlated, the optimal level of cash holding can be a decreasing function of the volatility of exogenous cash flows from operations. To demonstrate this result, we model cash flows from operations as a two-state Markov process in continuous time, with regimes of positive cash flows (Regime H) alternating with regimes of negative cash flows (Regime L) at random times governed by Poisson arrivals. Negative cash flows could arise from wage payments, costs of materials, or fixed costs of operation, that together exceed revenues in Regime L.

Our focus in this paper is on a financially-constrained firm for which it is costly to raise external funds. To put the financial constraint in its starkest form, we assume that it is prohibitively costly to raise external funds so cash flow from operations and any interest on cash held by the firm are the only sources of cash. The role of cash is to function as precautionary saving that enables the firm to make payments required when it is in Regime

[1]Miller and Orr did not focus on avoiding termination during a period of negative cash flow. They focussed on minimizing costly transactions between cash and other assets offering a higher rate of return in the face of stochastic inflows and outflows of cash.

L facing negative cash flow. If the firm holds zero cash at any time when it is in Regime L, it is forced to terminate because it cannot make the payments required in Regime L. The cost of termination is that the shareholders lose the continuation value of the firm associated with future cash flows.[2] Termination resulting from complete depletion of cash holdings while in Regime L is the costly stochastic event that motivates the firm to hold cash. However, shareholders do not accumulate cash in the firm without bound because their rate of time preference exceeds the rate of return on cash, which provides an incentive for the firm to pay dividends to shareholders. The optimal payout policy–equivalently optimal cash management–reflects the tension between the incentive to retain cash in the firm as a precaution to avoid termination and the incentive to pay out cash as dividends to impatient shareholders.

Our specification of the firm and its decision problem is parsimonious. It consists only of an exogenous two-state Markov process for cash flow, the rate of time preference of shareholders, and the rate of return on cash, which is set to zero beginning in Section 4. The only decision that the firm makes is the payout decision, specifically, how much cash to pay out as dividends and how much to retain within the firm as cash on hand. The optimal payout policy is to pay zero dividends whenever cash flow is negative or whenever cash flow is positive and cash on hand is less than an optimally determined value X^*. Thus, in Regime L, the firm draws down its cash on hand to make required payments, and in Regime H, the firm accumulates cash until cash on hand reaches X^*. When the stock of accumulated cash in Regime H reaches X^*, the firm begins to pay dividends at a rate equal to its net cash flow from operations, thereby maintaining cash on hand constant and equal to X^*.[3] The decision problem of the firm is simply to determine the optimal value of the target level X^*.

The parsimony of the model is designed to focus on the precautionary motive for holding cash and to facilitate derivation of a closed-form solution for X^*. After deriving that solution, we use it analyze the impact on X^* of various parameters of the model. Most notable is the analysis of the impact of cash flow volatility on X^*. In particular, we analyze a mean-preserving increase in the variance of the unconditional distribution of the Markov process for cash flow by increasing the absolute values of the cash flows in both regimes; that is, we increase the value of cash flow when it is positive (Regime H) and decrease its value when it is negative (Regime L). Unlike conventional wisdom, which is that a mean-

[2]For simplicity, we assume that the firm has zero salvage value when it terminates.

[3]If the firm earns interest at rate r on cash, then its total cash flow and optimal dividends in Regime H will be $rX^* + cash\ flow\ from\ operations$ when $X = X^*$. If $r = 0$, total cash flow is simply cash flow from operations.

preserving spread increases the level of precautionary cash holdings, we find that the optimal level of cash holding is not monotonic in volatility. Moreover, we find that there is a critical value of volatility such that optimal X^* is positive for volatility below the critical value and is zero for volatility above the critical value. That is, contrary to conventional wisdom, comparing volatilities on either side of the critical value, the higher values of volatility are associated with lower values of X^*.

A mean-preserving spread of the distribution of exogenous cash flows hastens the time when a firm with a given amount of current cash on hand will run out of cash if it remains in Regime L. The hastening of termination would seem to increase the optimal level of X^*. However, the mean-preserving spread on cash flow makes each dollar of cash less effective at avoiding termination and thus can reduce the optimal value of X^*. Cash on hand allows the firm to "buy time" in Regime L while waiting for a Poisson arrival of Regime H. Each dollar of cash buys $\frac{1}{|\phi^L|}$ units of time, where $\phi^L < 0$ is the cash flow in Regime L, so $\left|\phi^L\right|$ is the rate at which cash is depleted in Regime L. By increasing $\left|\phi^L\right|$, a mean-preserving spread on the distribution of cash flow reduces the efficacy of each dollar of cash as a precaution against termination. As we show analytically, this reduction in the efficacy of cash as a precaution can outweigh the hastening of termination; in that case, a mean-preserving spread in the distribution of cash flow will reduce X^*.

In our framework with persistent cash flows, the arrival of Regime L means that cash flows in the immediate future will be negative and will remain negative until the next arrival of Regime H. A mean-preserving spread that reduces ϕ^L, equivalently, increases $\left|\phi^L\right|$, not only reduces cash flow at the instant when Regime L arrives, but it reduces the expected cash flow as long as the firm remains in Regime L. It is this reduction in expected *future* cash flow, which is a consequence of the persistence of regimes, that reduces the efficacy of cash as a precaution against termination.

The main focus of this paper is on the optimal value of X^*, but in the process of deriving the optimal value of X^*, we develop an expression for the value of the firm when its level of cash on hand equals the target level X^*. In the absence of financial constraints, the value of the firm for risk-neutral shareholders would simply be the value of cash on hand plus the conditional expected present value of the stream of cash flows over the infinite future.[4] But with the financial constraints that we impose, the firm will eventually encounter a sufficiently

[4]We specify the stochastic process for exogenous cash flow so that the conditional expected present value of the stream of cash flows over the infinite future is always positive, even in Regime L, so the firm would never choose to terminate.

long Regime L that it runs out of cash and is forced to terminate. The value of the firm excludes all cash flows after the stochastic termination date. Nevertheless, the value of the firm can be calculated as the expected present value of "distorted" cash flows over the infinite future, where the distortion takes a simple form in this two-state world. The distorted cash flows are simply equal to the actual cash flows multiplied by the marginal value of cash on hand evaluated with cash on hand equal to X^*. We show that the marginal value of cash on hand equals one in Regime H and equals a simple ratio of parameters, greater than one, in Regime L. Therefore, the expected present value of the stream of distorted cash flow places a greater weight on negative cash flows than on positive cash flows, and this change in weighting appropriately reduces the expected present value of the cash flows by just enough to account for the inevitable termination of the firm.

We derive an expression for X^* that holds globally in the admissible region of parameter space. Analyzing this expression in the neighborhood of parameter space where the marginal value of cash on hand is close to one reveals some helpful insights. We show that in this neighborhood, the optimal value of X^* is approximately equal to the expected present value of the value of the firm holding cash X^* when the next Regime L arrives. By choosing to hold cash X^*, the shareholders forego current dividends of X^* to preserve the option for the firm to continue operation when the next Regime L arrives. As we show, a mean-preserving spread on the unconditional distribution of exogenous cash flows reduces the value of the firm in Regime L, and hence reduces the optimal value of X^*. Thus, in this neighborhood, the greater volatility unambiguously reduces X^*, which is contrary to the conventional result.

The optimal target level of cash, X^*, is a non-monotonic function of mean-preserving increases in the volatility of cash flow. The only reason for the firm to hold cash, rather than pay dividends to impatient shareholders, is to allow the firm to survive for at least a while during an episode of negative cash flow in Regime L. There are two distinct cases in which this incentive to hold cash is so small that X^* is close to zero. In one case, which holds for sufficiently small volatility, the cash flow in Regime L is close enough to the positive mean that it is almost zero. In this case, even a tiny amount of cash will allow the firm to survive for an arbitrarily long period of time, so X^* is close to zero. In the other case, which holds for sufficiently large volatility, the cash flow in Regime L is so far below the positive unconditional mean that in Regime L the conditional expected present value of cash flows from operations over the firm's remaining lifetime, plus the value of X^*, is negative. In this case, the firm will want to terminate as soon as Regime L arrives, so there is no incentive to hold cash, and $X^* = 0$. For intermediate values of volatility of cash flow between these two distinct cases, the optimal target level of cash is positive. Thus, as volatility increases in a mean-preserving way from low values, to intermediate values, to high values, the optimal target level of cash starts at zero, increases to positive values, and then falls to zero. That is, X^* is a non-monotonic function of the unconditional variance for a given unconditional mean of cash flows. The (local) impact of a mean-preserving increase in the volatility of cash flows can be written as X^*_θ, which is the partial derivative of X^* with respect to the unconditional coefficient of variation θ, holding constant the unconditional mean of cash flow. The non-monotonicity of X^* in the volatility θ implies that X^*_θ is positive for some values of θ and negative for some values of θ. Remarkably, however, since X^* is zero both for very low values of θ, say $\theta \leq \theta_A$, and for very high values of θ, say $\theta \geq \theta_B$, the integral $\int_{\theta_A}^{\theta_B} X^*_\theta d\theta = 0$. Thus, the average (local) impact of θ on X^* is zero, if θ is uniformly distributed across firms for a given unconditional mean of cash flows.

Endogenous terminations reduce the population of high-volatility firms more rapidly than the population of low-volatilty firms of the same age. Thus, if θ is uniformly distributed across firms at the time of birth in Regime H with zero cash on hand, heterogeneous endogenous terminations tilt the population toward low-volatility firms, for which $X^*_\theta > 0$. Thus, the cross-sectional average value of X^*_θ is positive, consistent with empirical findings. This explanation suggests caution in interpreting a positive coefficient on volatility in regressions of cash on volatility. Even though our model can be consistent with such a positive coefficient, as just described, our comparative statics results show that for any given firm, a sufficiently large increase in volatility will reduce, rather than increase, the optimal target level of cash.

Throughout this paper, we focus on the level of precautionary saving measured in dollars. Specifically, X^* is measured in dollars. However, precautionary saving can alternatively be measured in units of time. Specifically, how long will a precautionary holding of X^* dollars of cash enable a firm to survive a continuous episode in Regime L, when cash flows out of the firm at rate $-\phi^L > 0$? The answer to this question is $\frac{X^*}{-\phi^L}$. Remarkably, we show that this alternative measure of precautionary saving is, in our numerical illustration, a monotonically decreasing function of the coefficent of variation, θ. This finding is notable for two reasons. First, the monotonic relationship between $\frac{X^*}{-\phi^L}$ and θ stands in sharp contrast to the relationship between X^* and θ, which is necessarily non-monotonic over the range from θ_A to θ_B. Second, the finding that precautionary saving measured by $\frac{X^*}{-\phi^L}$ is a decreasing function of θ is the opposite of the conventional wisdom that precautionary saving is an increasing function of volatility.

Brief review of the literature

The original analysis of business precautionary demand for cash by Miller and Orr (1966) was conducted in a model in which cash flows are serially independent. The serial independence facilitated the derivation of a simple closed-form solution for the optimal target level of cash, X^*, in our notation. In addition, the serial independence allowed for the unambiguous finding that X^* is an increasing function of the volatility of cash flows. Since that original contribution, the literature has introduced various additional features of the firm's problem, such as taxation, capital investment, opportunities for the costly issuance of debt and equity, and serial correlation of cash flows. Bolton et al. (2011) examine investment and cash holdings in a firm facing serially uncorrelated shocks. Their brief discussion of a mean-preserving spread is limited to describing the concavity of the value function and does not address the impact on the optimal level of cash holdings. Also in the framework of serially uncorrelated shocks, Hugonnier et al. (2014) examine optimal cash holdings and the bottom figure in Panel B of their Figure 6 shows that the payout thresholds of cash (the analogues of our X^*) are monotonically increasing in volatility.

Some existing studies in the literature include serial correlation. With a notable exception discussed in the next paragraph, these papers conclude that optimal cash holdings are an increasing function of cash flow volatility. For instance, the review of dynamic corporate finance by Strebulaev and Whited cited above allows for positive serial correlation of cash flow and concludes that optimal cash holding is an increasing function of volatility. The analysis in that review is a slightly simplified form of the model in Riddick and Whited (2009), which also finds a positive association of optimal cash holding and cash flow volatility in a framework with positively serially correlated cash flow. The upper right panel of Figure 2 in Riddick and Whited shows that the ratio of the stock of cash to assets is an increasing function of volatility.

Décamps et al. (2016) models cash flow shocks as the sum of temporary and permanent shocks, unlike our model, which features a persistent but stationary cash flow process. Their paper shows that an increase in the volatility of the temporary component increases the optimal cash balance. An increase in the volatility of the permanent component also increases optimal cash holding if the permanent and temporary components are negatively correlated. Both of these findings are consistent with the literature discussed above. However, Décamps et al. (2016) observes that if the temporary and permanent shocks are sufficiently positively correlated, then optimal cash holdings could fall when the volatility of the permanent shock increases. This observation is based on numerical calculations.[5]

A key feature of our model is that the target level of cash on hand is a non-monotonic function of the volatility of cash flow. Hartman-Glaser et al. (2019) also find that the target level of cash on hand is non-monotonic in volatility (see Figure 1 in that paper). However, this non-monotonicity is a consequence of the agency costs in a dynamic contracting framework. When these agency costs are arbitrarily small, the target level of cash is monotonically increasing in volatility, as in conventional result.

Opler et al. (1999) find empirically that firms in industries with higher volalitilty of cash flow tend to hold larger amounts of cash. However, Han and Qiu (2007) refine this empirical finding by emphasizing the distinction between firms that are financially constrained and firms that not financially constrained. They present a model in which firms with higher volatility of cash flow hold higher amounts of cash if and only if they are financially constrained; for firms that are not financially constrained, there is no relationship between cash flow volatility and holdings of cash. Their empirical analysis confirms these different relationships for constrained and unconstrained firms. These findings suggest that the finding of a positive relationship between volatility and cash on hand is evidence of financial constraints. However, our analysis of precautionary cash holdings in the presence of a hard financial constraint shows that the higher cash flow volatility can lead to either a higher or lower optimal level of cash on hand. Indeed the relationship between cash flow volatility and cash on hand is non-monotonic.

Relative to the literature described above, our paper makes three major contributions. First, we derive a closed-form solution for optimal cash holdings, which, to our knowledge, is the first such closed-form solution in the presence of persistent shocks to cash flow. This closed-form solution facilitates the analytic description and interpretation of the behavior of optimal cash holding. Second, we show analytically that an increase in cash flow volatility can lead to a decrease in optimal cash holdings. This finding is counter to the conventional result. Our analytic result provides an intuitive explanation. Third, we show that endogenous selection arising from heterogeneous survival probabilities of firms can rationalize a positive cross-sectional average impact of volatility on optimal cash holdings, consistent with empirical findings. To our knowledge, this selection mechanism does not appear in the

[5] The formal mathematical analysis in Décamps et al. (2016) merely concludes that the sign of the relationship between cash flow volatility and optimal cash holdings is necessarily positive if the permanent and temporary components of cash flow are negatively correlated; Décamps et al. (2016) concludes that the sign could be positive or negative if the permanent and temporary components of cash flow are positively correlated.

existing literature.

1 The Firm's Decision Problem

Consider a firm that operates in continuous time and has an exogenous stochastic stream of net cash flow from operations ϕ_t at time t. The cash flow ϕ_t evolves according to a two-state Markov regime-switching process. Specifically, in Regime H, cash flow is $\phi_t = \phi^H > 0$, and in Regime L, cash flow is $\phi_t = \phi^L < 0$. The negative cash flows in Regime L can arise from unavoidable costs, such as the costs of maintaining a fixed capital stock, paying wages, or making purchases from suppliers, that exceed revenues.

The transitions between Regimes H and L are Poisson events with possibly different arrival intensities. When $\phi_t = \phi^H$, cash flow remains equal to ϕ^H until the regime switches (from H to L) and ϕ changes to ϕ^L. The instantaneous probability of switching to Regime L from Regime H is $\mu^L > 0$, and the instantaneous probability of switching to Regime H from Regime L *is* $\mu^H > 0$.

Shareholders discount future cash flows at rate $\rho > 0$.

Definition 1 *Define the roundtrip discount factor* $\Gamma \equiv \frac{\mu^H}{\rho+\mu^H}\frac{\mu^L}{\rho+\mu^L} < 1$.

The roundtrip discount factor Γ is the expected present value, discounted at rate ρ, of one dollar at future time t^*, which is the first time that the firm returns to its current regime after leaving the current regime.[6]

Definition 2 *Define the myopic value of a regime as the expected present value of operating profit* ϕ_t *over the remaining duration of the current regime. The myopic value of Regime* H *is* $\Phi^H \equiv \frac{\phi^H}{\rho+\mu^L} > 0$ *and the myopic value of Regime* L *is* $\Phi^L \equiv \frac{\phi^L}{\rho+\mu^H} < 0$.[7]

Even though shareholders have rational expectations that extend indefinitely far into the future, these myopic values have an important role in characterizing many features of the firm's decision problem. For now, the myopic values and the roundtrip discount factor help provide simple expressions for the conditional expectations of the present value of potential

[6] Suppose that Regime j prevails at time 0, continues to prevail until $t_1 > 0$, when the other regime $(-j)$ arrives; the next arrival of Regime j is at $t^* > t_1$. Then $\Gamma \equiv E\left\{e^{-\rho t^*}\right\} = \int_0^\infty \mu^{(-j)} e^{-\left(\rho+\mu^{(-j)}\right)t_1} \int_{t_1}^\infty \mu^{(j)} e^{-\left(\rho+\mu^{(j)}\right)(t^*-t_1)} dt^* dt_1 = \frac{\mu^{(j)}}{\rho+\mu^{(j)}}\frac{\mu^{(-j)}}{\rho+\mu^{(-j)}} = \frac{\mu^H}{\rho+\mu^H}\frac{\mu^L}{\rho+\mu^L}$.

[7] The myopic value of the firm in a Regime H that prevails from time 0 to a random date $t_1 > 0$ is $\Phi^H = E\left\{\int_0^{t_1} \phi^H e^{-\rho t} dt\right\} = \phi^H E\left\{\frac{1-e^{-\rho t_1}}{\rho}\right\} = \phi^H \left[\frac{1}{\rho}\left(1 - \frac{\mu^L}{\rho+\mu^L}\right)\right] = \frac{\phi^H}{\rho+\mu^L}$. Similarly, $\Phi^L = \frac{\phi^L}{\rho+\mu^H}$.

operating profits over the infinite future, ignoring the fact that the firm will eventually be forced to terminate in some Regime L when it runs out of cash. Specifically,[8]

$$E\left\{\int_t^{\infty}\phi_s e^{-\rho(s-t)}ds|\phi_t=\phi^L\right\} = \frac{1}{1-\Gamma}\left(\Phi^L+\frac{\mu^H}{\rho+\mu^H}\Phi^H\right) \tag{1a}$$

$$E\left\{\int_t^{\infty}\phi_s e^{-\rho(s-t)}ds|\phi_t=\phi^H\right\} = \frac{1}{1-\Gamma}\left(\Phi^H+\frac{\mu^L}{\rho+\mu^L}\Phi^L\right). \tag{1b}$$

To ensure the expected present value of cash flows is always positive, even conditioning on $\phi_t=\phi^L<0$ as in equation (1a), we assume that

$$-\Phi^L<\frac{\mu^H}{\rho+\mu^H}\Phi^H<\Phi^H, \tag{2}$$

which implies that the myopic value of the losses in a Regime L is smaller than the myopic value of positive cash flow in a Regime H.[9] This condition implies that, in the absence of any financial constraints, the continuation value of the firm would always be positive. Therefore, the firm would never choose to terminate unless it is forced to do so.

The only decision facing the firm is the payout decision, that is, how much cash to retain and how much to distribute to shareholders as dividends. The firm holds a stock of cash on hand, $X_t\geq 0$, and earns an interest rate $0\leq r<\rho$ on this cash.[10] Cash on hand enables the firm to pay cash outflows, $-\phi^L>0$, required in Regime L. To sharpen the financial constraint, we assume that the shareholders of the firm cannot inject any new funds into the firm to make these required payments. Therefore, any required cash payments must be paid from X_t. If $X_t=0$ and $\phi_t<0$, then the firm fails to make required payments and hence immediately and permanently terminates with zero salvage value.

The shareholders of the firm are risk neutral. Their objective is to maximize the expected present value of dividends received from the firm, discounted at rate ρ. The assumption that shareholders cannot inject any additional funds into the firm implies that dividends cannot be negative. In principle, dividends can be paid as a finite flow per unit of time or

[8] $E\left\{\int_t^{\infty}\phi_s e^{-\rho(s-t)}ds|\phi_t=\phi^L\right\}=\int_t^{\infty}\phi^L e^{-\left(\rho+\mu^H\right)(s-t)}ds+\int_t^{\infty}\mu_H e^{-\left(\rho+\mu^H\right)t_H}dt_H\int_{t_H}^{\infty}\phi^H e^{-\left(\rho+\mu^L\right)(s-t)}ds$ $+\Gamma E\left\{\int_t^{\infty}\phi_s e^{-\rho(s-t)}ds|\phi_t=\phi^L\right\}=\Phi^L+\frac{\mu^H}{\rho+\mu^H}\Phi^H+\Gamma E\left\{\int_t^{\infty}\phi_s e^{-\rho(s-t)}ds|\phi_t=\phi^L\right\}$, which is the conditional expectation in equation (1a). A similar derivation leads to the conditional expectation in equation (1b).

[9] The first inequality in equation (2) can be expressed in terms of ϕ^L and ϕ^H as $\left(\rho+\mu^L\right)\phi^L+\mu^H\phi^H>0$.

[10] One rationale for $\rho>r$ is that the firm is subject to an exogenous catastrophic shock that terminates the firm. If this shock is a Poisson shock with arrival intensity $\gamma>0$, then the effective discount rate of shareholders is $\rho=\rho^*+\gamma$, where ρ^* is the pure rate of time preference. In this case, if r is a riskless rate equal to the pure rate of time preference, ρ^*, then ρ exceeds r. An alternative rationale given by Bolton, Chen, and Wang (2011) is based on agency costs associated with free cash flow.

as lump sums at discretely-spaced points of time. However, a firm that has been in Regime H at some point in the past, and has followed an optimal payout policy since that time, will never find it optimal to pay a lump-sum dividend at the current time. We will call such a firm an "ongoing firm" and confine attention to ongoing firms in this paper.[11]

The equality of sources and uses of funds for an ongoing firm is given by

$$\dot{X}_t + D_t = rX_t + \phi_t, \tag{3}$$

where the sources of funds are the interest receipts on cash, rX_t, and the net cash flow from operations, ϕ_t; the uses of funds are to accumulate cash at rate $\dot{X}_t$ and to pay dividends at rate $D_t \geq 0$.

Let

$$V_t \equiv E_t \left\{ \int_t^{\tau} D_s e^{-\rho(s-t)} ds \right\} \tag{4}$$

be the conditional expected present value of the flow of dividends from time t until the endogenous time $\tau \geq t$ when the firm runs out of cash while facing a negative cash flow from operations. Thus, τ is the time at which the firm is forced to terminate because it cannot make required payments when $\phi_t < 0$. Formally,

$$\tau \equiv \min \left\{ s \geq t : X_s = 0 \text{ and } \phi_s < 0 \right\}. \tag{5}$$

Because the firm cannot borrow, nor raise additional funds by issuing equity or paying negative dividends, it must terminate at time τ.

Let $V^H(X_t)$ and $V^L(X_t)$ be the maximized expected present values, in Regimes H and L, respectively, of the flows of dividends from time t until the termination time τ. The Hamilton-Jacoby-Bellman (HJB) equation during Regime H is

$$\rho V^H = D + \left(rX + \phi^H - D \right) V_X^H + \mu^L \left(V^L - V^H \right), \tag{6}$$

which along with the complementary slackness condition

$$\left(V_X^H - 1 \right) D = 0, \tag{7}$$

implies

$$\rho V^H = \left(rX + \phi^H \right) V_X^H + \mu^L \left(V^L - V^H \right). \tag{8}$$

[11] A firm that does not have a history of optimal payout (and thus is not an ongoing firm, as defined here) may find itself in a position in which it has so much cash on hand that it is optimal to pay an immediate one-time lump-sum dividend.

Similarly, during Regime L, the HJB equation is

$$\rho V^L = \left(rX + \phi^L\right) V_X^L + \mu^H \left(V^H - V^L\right). \tag{9}$$

The term on the left hand side of equation (8) is the required return on V^H per unit of time and the two terms on the right hand side of this equation comprise the expected return per unit of time. The first term on the right hand side of equation (8), $\left(rX_t + \phi^H\right) V_X^H$, is the product of the net inflow of cash before dividends and the marginal valuation of cash on hand, V_X^H. The second term on the right hand side of equation (8) is the change in the firm's value if the regime switches to L from H, multiplied by μ^L, the instantaneous probability of such a switch. The interpretation of the HJB equation in Regime L in equation (9) is symmetric.

The ODEs in equations (8) and (9) must satisfy the following boundary conditions

$$V^L(0) = 0 \tag{10}$$

$$\begin{aligned} V_X^H(X^*) &= 1, \text{ if } X^* > 0 && \text{(11a)} \\ V_X^H(0) &\leq 1, \text{ if } X^* = 0 && \text{(11b)} \end{aligned}$$

and

$$V_{XX}^H(X^*) = 0, \tag{12}$$

where X^* is the optimal value of cash on hand, X, that triggers the payment of dividends in Regime H. The boundary condition in equation (10) states that if $\phi_t = \phi^L < 0$ and the firm has zero cash on hand, then the value of the firm is zero, because it must terminate immediately.

The boundary condition in equation (11a) states that in Regime H if $X = X^* > 0$, then an extra dollar of cash on hand is worth a dollar to shareholders. That is, shareholders are indifferent about whether to retain an additional dollar of cash within the firm or to pay out that dollar as a current dividend. If $X = X^* > 0$ in Regime H, then the firm pays out dividends at rate $rX^* + \phi^H$ to keep X equal to $X^* > 0$. Alternatively, if $X^* = 0$, then $V_X^H(0) \leq 1$ (equation 11b) and the firm always pays dividends in Regime H because a dollar is worth at least as much in the hands of shareholders as it is worth as cash on hand within the firm.

The marginal valuation of cash within the firm, $V_X^H(X)$, attains its minimum, which is one, when $X = X^*$. The boundary condition in equation (12) is the first-order condition for this minimization.

We will use the ODEs in equations (8) and (9) and the boundary conditions in equations (10) - (12) to derive the value functions in Regimes H and L and the optimal target level of cash on hand, X^*. Before proceeding to a more complete analysis, we present $V^H(X)$ for the case in which $X^* = 0$.

Proposition 1 *If $X^* = 0$, then for all $X \geq 0$, $V^H(X) = \Phi^H + X$, so $V_X^H(X) = 1$ and $V_{XX}^H(X) = 0$, where $\Phi^H \equiv \frac{\phi^H}{\rho+\mu^L}$ is the myopic value of Regime H.*

When $X^* = 0$, an optimal dividend policy is to pay out any cash on hand, $X > 0$, immediately as dividends and then pay dividends at a rate equal to the cash flow $\phi^H > 0$ for the remainder of the current Regime H. When Regime L arrives, the firm has no cash on hand and terminates immediately. Thus, the expected present value of the immediate dividend X and the expected future dividends during the current Regime H, Φ^H, is $V^H(X) = \Phi^H + X$. That is, the value function $V^H(X)$ is a linear function of $X \geq 0$ with slope equal to one. The intercept of this function is the myopic value of Regime H, $\Phi^H \equiv \frac{\phi^H}{\rho+\mu^L}$.

2 Marginal Value of Cash on Hand

The marginal value of cash on hand when $X = 0$ is a key determinant of whether the level of cash, X^*, that triggers dividends is positive or zero. The firm will not pay dividends when the marginal value of cash on hand exceeds one, that is, when an additional dollar of cash inside the firm is worth more to shareholders than an additional dollar of dividends. To compute the marginal value of a dollar of cash on hand in Regime L, differentiate equation (8) with respect to X and rearrange to obtain

$$V_X^L = \frac{1}{\mu^L}\left[\left(\rho - r + \mu^L\right)V_X^H - \left(rX + \phi^H\right)V_{XX}^H\right]. \tag{13}$$

Evaluate equation (13) at $X = X^*$ and use the boundary conditions in equations (11a) and (12) to obtain

$$V_X^L(X^*) = 1 + \frac{\rho - r}{\mu^L} > 1, \text{ if } X^* > 0. \tag{14}$$

An ongoing firm that has been pursuing optimal payout policy will always have $X \leq X^*$ because it will not accumulate additional X beyond X^* in Regime H. When Regime L arrives, the firm will have $X \leq X^*$, and since profits from operations, ϕ^L, are negative and[12] $rX^* \leq -\phi^L$, the source of funds in equation (3), $rX^* + \phi^L$, is less than or equal to zero; hence X cannot increase in Regime L. The concavity[13] of $V^L(X)$, together with equation (14), implies that $V_X^L(X) \geq V_X^L(X^*) > 1$ for all $X \leq X^*$, so an ongoing firm with $X^* > 0$ prefers to retain earnings rather than pay dividends whenever it is in Regime L. The argument in this paragraph proves the following proposition.[14]

Proposition 2 *If $X^* > 0$, then $V_X^L(X^*) = 1 + \frac{\rho - r}{\mu^L} > 1$, which implies that an ongoing firm never pays dividends in Regime L.*

The optimal target value of cash on hand in Regime H, X^*, can be either positive or zero, depending on the configuration of values of the fundamental parameters ρ, r, ϕ^H, ϕ^L, μ^H, and μ^L. The following definition provides a function of these fundamental parameters that determines whether X^* is zero or positive.

Definition 3 *Define* $\Lambda \equiv \Lambda\left(\rho, r, \phi^H, \phi^L, \mu^H, \mu^L\right) \equiv \frac{\phi^H}{\rho+\mu^L} \frac{\mu^H}{-\phi^L} \frac{\mu^L}{\rho - r + \mu^L}$.

Proposition 3 $X^* > 0 \Longleftrightarrow V_{XX}^H(0) < 0 \Longleftrightarrow \Lambda > 1$.

Proposition 3 states that if $\Lambda > 1$, then $X^* > 0$ so that when an ongoing firm is in Regime H, it accumulates cash until $X = X^*$. If $\Lambda \leq 1$, then $X^* = 0$ and the firm never accumulates cash; it always pays dividends ϕ^H in Regime H and then terminates when Regime L arrives. The locus of parameter values for which $\Lambda = 1$ is the border between the region of parameter space where $X^* = 0$ and the region where $X^* > 0$. The following corollary states that the marginal value of cash on hand, $V_X^H(0)$, equals Λ on this border.

Corollary 1 *If $\Lambda = 1$, then $X^* = 0$ and $V_X^H(0) = \Lambda$.*

[12] If rX^* were greater than $-\phi^L > 0$, then $rX^* + \phi_t$ would always be positive and once the firm reaches $X_t = X^*$, it would be able to (1) pay positive dividends at every point in time, even if Regime L persists forever, and (2) still maintain a cushion of cash on hand forever. Since the discount rate of shareholders, ρ, exceeds the riskless interest rate, r, shareholders would prefer to have the superfluous cushion of cash, $X - X^*$, paid out as dividends.

[13] Appendix C proves that $V^H(X)$ and $V^L(X)$ are both concave.

[14] Proposition 2 does not address $V_X^L(X^*)$ in the case in which $X^* = 0$ because an ongoing firm would terminate immediately upon entering Regime L if $X^* = 0$.

To understand why $V_X^H(0) = \Lambda$, suppose that $\Lambda = 1$ so that $X^* = 0$, which means that the optimal value of cash on hand is zero. Consider the following small deviation from this optimal policy. Suppose that the firm chooses a target level of cash equal to an infinitesimal $\varepsilon > 0$. Therefore, when Regime H prevails, the firm retains profits until $X = \varepsilon$, and then pays dividends at rate ϕ^H for the remainder of the current Regime H, which we label H_1. Regime H_1 is followed by Regime L_1, during which the firm's cash on hand shrinks at rate $-\phi^L > 0$. The firm will be forced to terminate during Regime L_1 unless Regime L_1 turns out to last for a period of time less than $\frac{\varepsilon}{-\phi^L}$. Since $\frac{\varepsilon}{-\phi^L}$ is arbitrarily small, the probability that Regime L_1 lasts for a period of time less than $\frac{\varepsilon}{-\phi^L}$ is approximately $\mu^H \frac{\varepsilon}{-\phi^L}$. In this unlikely event, the firm will transition to Regime H_2 holding an amount of cash on hand less than ε, and can expect to receive profits with present value Φ^H during that regime.[15] When Regime H_2 ends, it is followed by Regime L_2. The conditional probability that the firm will survive to the end of Regime L_2 is less than or equal to $\mu^H \frac{\varepsilon}{-\phi^L}$. The joint probability that the firm survives both Regimes L_1 and L_2 is negligible (of order ε^2), so we can ignore any profits received after Regime H_2. Therefore, the marginal value of cash on hand in Regime H, $V_X^H(0)$, is the product of three terms: (1) $\frac{\mu^L}{\rho+\mu^L}$ is the expected present value, as of time 0, of a dollar at time $t_1 > 0$ when the current Regime H ends; (2) $\frac{\mu^H}{-\phi^L}\varepsilon$ is the probability of surviving until Regime H_2 once the firm enters Regime L_1; and (3) $\Phi^H \equiv \frac{\phi^H}{\rho+\mu^L}$, which is the myopic value of Regime H_2. As ε approaches zero, this product approaches $\Lambda\varepsilon$, so the marginal value of cash on hand, ε, is Λ.

3 Local Comparative Statics

This section presents local comparative statics concerning the effects of changes in the primitive parameters ρ, r, ϕ^H, ϕ^L, μ^H, and μ^L on the optimal target level of cash X^*. The comparative statics are local in the sense that we confine attention to combinations of parameter values for which $\Lambda - 1$ is in a positive neighborhood of zero. Recall from Corollary 1 that when $\Lambda - 1 = 0$, the optimal target level of cash on hand, X^*, equals zero.

Proposition 4 *Starting from a parameter configuration for which* $\Lambda \equiv \Lambda\left(\rho, r, \phi^H, \phi^L, \mu^H, \mu^L\right) = 1$*, and hence* $X^* = 0$*, the following changes in parameters increase* X^* *to a positive value:*

1. *a decrease in* ρ

[15] During Regime H_2, the firm will use some of this profits to rebuild its cash on hand to ε, but this amount of profits will be less than ε, which is infinitesimal.

2. *an increase in* r

3. *an increase in* μ^H

4. *an increase in* μ^L *if* $\mu^L < \rho\sqrt{1-\frac{r}{\rho}}$

5. *a decrease in* μ^L *if* $\mu^L > \rho\sqrt{1-\frac{r}{\rho}}$

6. *an increase in* ϕ^H

7. *an increase in* ϕ^L.

Starting from a parameter configuration for which $\Lambda = 1$ so that $V_X^H(0) = 1$ and $X^* = 0$, any change that increases Λ will increase $V_X^H(0)$ to a value greater than one, and hence will increase X^* to a positive value. A decrease in ρ increases $V_X^H(0)$, both by increasing $\frac{\mu^L}{\rho+\mu^L}$, the expected present value of a dollar at the time when the next Regime L arrives, and by increasing the myopic value of Regime H, $\Phi^H \equiv \frac{\phi^H}{\rho+\mu^L}$. Therefore, a decrease in ρ increases X^* (Statement 1 of Proposition 4). An increase in r, which is the rate of return earned on cash on hand, X, leads to a positive optimal target level of cash on hand, X^* (Statement 2). An increase in μ^H increases the probability $\frac{\mu^H}{-\phi^L}\varepsilon$ that a firm that enters Regime L with $X = \varepsilon$ will emerge from Regime L with cash on hand and hence can continue to operate in the next Regime H. Therefore, an increase in μ^H increases X^* (Statement 3).

An increase in μ^L has two opposing effects on the marginal valuation of cash. On the one hand, an increase in μ^L reduces the expected time until the arrival of the next Regime L and hence the next Regime H, thereby increasing the marginal valuation of cash and increasing X^*. On the other hand, an increase in μ^L reduces the myopic valuation $\Phi^H \equiv \frac{\phi^H}{\rho+\mu^L}$ and thus reduces X^*. If $\mu^L < \rho\sqrt{1-\frac{r}{\rho}}$, then the first effect dominates (Statement 4), but if $\mu^L > \rho\sqrt{1-\frac{r}{\rho}}$, then the second effect dominates (Statement 5).

An increase in ϕ^H increases the dividend that will be paid per unit of time during Regime H when $X = X^*$ and thus increases the value of being able to emerge from Regime L with cash on hand to enter the next Regime H. Therefore, an increase in ϕ^H increases the marginal valuation of a unit of cash and increases X^* (Statement 6). An increase in ϕ^L, that is, a reduction in $\left|\phi^L\right|$ reduces the rate at which cash on hand is depleted during Regime L and thus increases the probability $\mu^H\frac{\varepsilon}{-\phi^L}$ that the firm emerges from the next Regime L with cash on hand to enter the next Regime H. Therefore, the marginal valuation of a unit of cash increases and hence X^* increases (Statement 7).

3.1 Mean-Preserving Change in Variance

The only motivation for the firm to hold cash in this model is that cash on hand is a form of precautionary saving that mitigates the chance that the firm will be forced to terminate by running out of cash while in Regime L. Typically, in models of corporate precautionary saving, a mean-preserving spread of the distribution of a stochastic variable leads to an increase in precautionary saving. However, in the model presented here, a mean-preserving spread of the unconditional distribution of cash flow, ϕ_t, can lead to a *decrease* in X^*, which is the optimal target level of precautionary saving in Regime H. Indeed, in the local comparative statics in this section, a mean-preserving spread in ϕ necessarily decreases precautionary saving.[16] This finding differs from the typical finding because ϕ_t is positively serially correlated in the current model but is serially uncorrelated in typical models.

Proposition 5 *Starting from a parameter configuration for which $\Lambda = 1$ and hence $X^* = 0$, a mean-preserving change in $\phi^H > 0$ and $\phi^L < 0$ that decreases the unconditional variance of ϕ increases X^* to a positive value.*

A mean-preserving decrease in the variance of the unconditional distribution of ϕ increases ϕ^L and decreases ϕ^H while keeping $\mu^L\phi^L + \mu^H\phi^H$ unchanged for given values of μ^L and μ^H. The increase in ϕ^L, which decreases $-\phi^L > 0$, slows the depletion of cash during Regime L and thus lengthens the window of time that the firm can survive in Regime L before exhausting a given amount of cash balances. Thus an increase in ϕ^L increases the probability that the firm will emerge from Regime L to a subsequent Regime H, which increases $V_X^H(0)$ and X^* (Statement 7 of Proposition 4). Working in the opposite direction, a decrease in ϕ^H decreases the myopic value of Regime H, $\Phi^H \equiv \frac{\phi^H}{\rho+\mu^L}$, which decreases the value of emerging from Regime L. Therefore, a decrease in ϕ^H decreases $V_X^H(0)$ and X^* (Statement 6 of Proposition 4). To see which of these opposing effects is dominant, observe from Definition 3 that Λ can be written as

$$\Lambda = \frac{\mu^L}{\rho+\mu^L}\frac{\mu^L}{\rho-r+\mu^L}\frac{\mu^H\phi^H}{\mu^H\phi^H - M}, \tag{15}$$

where $0 < M \equiv \mu^L\phi^L + \mu^H\phi^H < \mu^H\phi^H$ is unchanged by a mean-preserving change in ϕ^H and ϕ^L for given values of μ^L and μ^H. Starting from an initial parameter configuration in which

[16] Formally, Proposition 5 analyzes a mean-preserving *decrease* in variance, which increases Λ and hence increases X^*. A mean-preserving spread, that is, a mean-preserving increase in variance, would reduce Λ and hence X^* would remain unchanged and equal to zero.

$\Lambda = 1$, a mean-preserving decrease in ϕ^H increases the ratio $\frac{\mu^H \phi^H}{\mu^H \phi^H - M}$ and hence increases Λ to a value greater than one, thereby increasing X^* to a positive number.

Precautionary saving in the current framework is induced by the desire to avoid termination with its consequent loss of future dividends. A reduction in the variance of ϕ increases $\phi^L < 0$, which reduces the speed at which cash is depleted during Regime L, and thus increases the probability that a firm that enters Regime L with a given amount of cash on hand will avoid running out of cash before the next Regime H arrives. This effect dominates the opposing effect associated with a decrease in ϕ^H, and thus a mean-preserving decrease in variance increases the marginal value of a unit of cash on hand and increases X^*. Here is where the serial correlation becomes important: Once the firm enters Regime L, it will remain in Regime L for a period of time until the next Poisson arrival of a regime change. While the firm remains in Regime L, it persistently loses cash on hand at the rate $-\phi^L$ until it runs out of cash or the next Regime H arrives, whichever comes first. As shown above, a firm that enters Regime L with small $\varepsilon > 0$ of cash on hand has probability $\mu^H \frac{\varepsilon}{-\phi^L}$ of surviving until the next Regime H with some cash on hand. A reduction in the loss $-\phi^L > 0$ increases this probability and thus increases the marginal valuation of cash on hand. This calculation depends on the fact that ϕ_t is serially correlated so that a change in ϕ^L has a substantive effect on the probability of reaching the next Regime H.

4 Zero interest earned on cash held by the firm: a closed-form solution

For the remainder of this paper, assume that the interest rate earned on cash held by the firm, r, equals zero. This assumption allows derivation of a closed-form solution to the system of ODEs in equations (8) and (9) and boundary conditions in equations (10) - (12).[17] Setting $r = 0$ in the ODEs in equations (8) and (9) yields a system of first-order linear constant-coefficient homogeneous ordinary differential equations

$$\begin{bmatrix} V_X^H \\ V_X^L \end{bmatrix} = A \begin{bmatrix} V^H \\ V^L \end{bmatrix} \tag{16}$$

[17]If $\rho > r > 0$, the valuation functions $V^H(X)$ and $V^L(X)$ that solve the ODEs in equations (8) and (9) are linear combinations of confluent hypergeometric functions.

where

$$A \equiv \left[\begin{array}{cc} \frac{1}{\Phi^H} & -\frac{\mu^L}{\rho+\mu^L}\frac{1}{\Phi^H} \\ -\frac{\mu^H}{\rho+\mu^H}\frac{1}{\Phi^L} & \frac{1}{\Phi^L} \end{array} \right]. \tag{17}$$

It is straightforward to solve the system of ODEs in equation (16). The details of the solution procedure are presented in Appendix A, which shows that the general solution of the system of ODEs is

$$\left[\begin{array}{c} V^H(X) \\ V^L(X) \end{array} \right] = c_1 \left[\begin{array}{c} 1 \\ \frac{\rho+\mu^L}{\mu^L}\left(1-\omega_1\Phi^H\right) \end{array} \right] e^{\omega_1(X-X^*)} + c_2 \left[\begin{array}{c} 1 \\ \frac{\rho+\mu^L}{\mu^L}\left(1-\omega_2\Phi^H\right) \end{array} \right] e^{\omega_2(X-X^*)} \tag{18}$$

where $\omega_1 < 0 < \omega_2$ are the eigenvalues[18] of A and the undetermined constants c_1, c_2, and X^* are pinned down by the boundary conditions (10), (11a), and (12) as shown in Appendix A. That appendix also shows that $V^H(X)$ and $V^L(X)$ both have positive third derivatives with respect to X for $0 \le X \le X^*$. The literature (Leland (1968), Sandmo (1970), Zeldes (1989), Kimball (1990), Carroll and Kimball (1996)) on household precautionary saving emphasizes that a positive third derivative of the utility function implies that a mean-preserving spread of household income increases precautionary saving. However, in the problem of the firm we analyze in this paper, a mean-preserving spread in cash flow can decrease precautionary saving despite the fact that the third derivative of the value function with respect to X is positive.[19]

Definition 4 *Define* $\Omega \equiv \frac{\Phi^L+\Gamma\Phi^H}{1-\Gamma-(\Phi^L+\Gamma\Phi^H)\omega_2}$.

As we will show, the parameter Ω facilitates a simple expression for X^*. It is also a key determinant of whether X^* is positive or zero, and it provides an upper bound on X^* when it is positive.

Lemma 1 $1-\Gamma-\left(\Phi^L+\Gamma\Phi^H\right)\omega_i = -\omega_i^2\left(1-\omega_j\Phi^H\right)\Phi^H\Phi^L > 0$, *for* $i \neq j$.

Lemma 1 implies that the denominator of Ω is positive, which implies that the sign of Ω is the same as the sign of the numerator, $\Phi^L+\Gamma\Phi^H$.

Lemma 2 *If* $r = 0$, *then* $sign(\Omega) = sign\left(\Gamma\Phi^H+\Phi^L\right) = sign(\Lambda-1)$.

[18]Lemma 4 in Appendix B presents four useful properties of the eigenvalues.

[19]In the context of a household saving problem, Huggett and Vidon (2002) points out that optimal saving can fall in response to an increase in earnings risk, even with a positive third derivative of the utility function.

The optimal value of X^* is given by the following proposition.

Proposition 6 *Assume that* $r = 0$.

1. *If* $\Gamma\Phi^H + \Phi^L \geq 0$, *then* $X^* \equiv \frac{1}{\omega_2 - \omega_1} \ln\left(1 + (\omega_2 - \omega_1)\Omega\right) \geq 0$.

2. *If* $\Gamma\Phi^H + \Phi^L \leq 0$,*then* $X^* \equiv 0$.

The following corollary provides an upper bound on X^*. In Proposition 10 we show that this upper bound becomes very tight when $\Gamma\Phi^H + \Phi^L > 0$ is close to zero.

Corollary 2 $X^* \leq \max\{0, \Omega\}$.

The only reason for the firm to hold cash is to be able to continue operation when Regime L arrives. Of course, for the firm to want to continue operation in Regime L, the conditional expected present value of ϕ_t over the infinite future must be non-negative even when the current regime is L. The assumption in equation (2), which we rewrite as $\Phi^L + \frac{\mu^H}{\rho+\mu^H}\Phi^H > 0$, ensures that $E\left\{\int_t^\infty \phi_s e^{-\rho(s-t)} ds | \phi_t = \phi^L\right\} > 0$. However, even with equation (2), the benefit from continuing operation may not be sufficiently strong to induce the firm to want to accumulate cash on hand when it is in Regime H. Lemma 2 provides the stronger condition, $\Phi^L + \Gamma\Phi^H > 0$, that induces the firm to accumulate cash on hand, that is, to have $X^* > 0$. This stronger condition can be interpreted in terms of the expected present value of flows over an infinite future, but in this case, it is the expected present value of a "distorted" cash flow process $\widetilde{\phi}_t$, where

$$\widetilde{\phi}_t \equiv \left\{ \begin{array}{c} \phi^H \text{ if } \phi_t = \phi^H \\ \frac{\rho+\mu^L}{\mu^L}\phi^L \text{ if } \phi_t = \phi^L \end{array} \right\}. \tag{19}$$

The distorted process magnifies the negative cash flows in Regime L by the factor $\frac{\rho+\mu^L}{\mu^L} > 1$, which equals $V_X^L(X^*)$ when $\Phi^L + \Gamma\Phi^H \geq 0$ (see Proposition 2 and set $r = 0$).

It is straightforward to show that

$$E\left\{\int_{t_1}^\infty \widetilde{\phi}_s e^{-\rho(s-t)} ds | \phi_t = \phi^H\right\} = \frac{\Phi^L + \Gamma\Phi^H}{1-\Gamma}, \tag{20}$$

where $t_1 = \min\left\{s > t : \phi_s = \phi^L\right\}$. That is, $\frac{\Phi^L+\Gamma\Phi^H}{1-\Gamma}$ equals the expected value, conditional on being in Regime H, of the present value of the distorted process $\widetilde{\phi}_t$ over the infinite future beginning at time t_1 when the next Regime L arrives.

The following proposition uses the distorted process $\widetilde{\phi}_t$ to evaluate the value function at $X = X^*$ in both regimes.

Proposition 7 *If $r = 0$ and $\Phi^L + \Gamma\Phi^H \geq 0$, then*

1. $V^H(X^*) = \frac{1}{1-\Gamma}\left(\Phi^H + \Phi^L\right) = E\left\{\int_t^\infty \widetilde{\phi}_s e^{-\rho(s-t)} ds | \phi_t = \phi^H\right\}$

2. $V^L(X^*) = \frac{\rho+\mu^L}{\mu^L}\frac{1}{1-\Gamma}\left(\Phi^L + \Gamma\Phi^H\right) = E\left\{\int_t^\infty \widetilde{\phi}_s e^{-\rho(s-t)} ds | \phi_t = \phi^L\right\}$.

Since $\widetilde{\phi}_t \leq \phi_t$, for all t, with strict inequality when $\phi_t = \phi^L$, the valuation $V^H(X^*)$ is less than the expected present value of the (non-distorted) cash flows conditional on currently being in Regime H shown in equation (1b). Similarly, $V^L(X^*)$ is less than the expected present value of the (non-distorted) cash flows over the infinite future conditional on currently being in Regime L shown in equation (1a). In effect, the magnification factor $\frac{\rho+\mu^L}{\mu^L}$ appropriately magnifies the negative cash flows in Regime L to reduce the expected present value of cash flows over the infinite future to account for the eventuality that the firm will terminate.

5 The Effect of Volatility on the Target Level of Cash on Hand

In this section we analyze the impact on X^* of a mean-preserving increase in the volatility of profitability. It will be convenient to use the coefficient of variation to summarize the mean and standard deviation of the unconditional distribution of cash flows. Define $\theta \equiv \frac{\sigma}{m}$ as the coefficient of variation where m and σ are the mean and standard deviation, respectively, of the unconditional distribution of cash flows. A mean-preserving spread, which is an increase in σ holding m fixed, can be represented as an increase in θ holding m fixed.

Definition 5 *Define three special values of θ, the coefficient of variation of the unconditional distribution of cash flows*

1. $\theta_A \equiv \sqrt{\frac{\mu^L}{\mu^H}}$

2. $\theta_C \equiv \left(1 + \frac{\mu^H+\mu^L}{\rho}\right)\sqrt{\frac{\mu^L}{\mu^H}}$

3. $\theta_B \equiv (1-\lambda)\theta_A + \lambda\theta_C$, *where* $\lambda \equiv \left(2 + \frac{\rho}{\mu^L}\right)^{-1} < \frac{1}{2}$.

The following lemma describes the importance of θ_A and θ_C.

Lemma 3

1. *$\phi^L < 0$ if and only if $\theta > \theta_A$*

2. *$\Phi^L + \frac{\mu^H}{\rho+\mu^H}\Phi^H > 0$ if and only if $\theta < \theta_C$*

Admissible values of the coefficient of variation satisfy both the restriction $\phi^L < 0$ (Statement 1 of Lemma 3) and the restriction in equation (2) that the conditional expected present value of the infinite stream of potential future cash flows is positive (Statement 2 of Lemma 3). Therefore, the admissible values of the coefficient of variation satisfy

$$\sqrt{\frac{\mu^L}{\mu^H}} \equiv \theta_A < \theta < \theta_C \equiv \left(1 + \frac{\mu^H + \mu^L}{\rho}\right)\sqrt{\frac{\mu^L}{\mu^H}}. \tag{21}$$

Lemma 2 leads to the following proposition.

Proposition 8 *Assume that $r = 0$. Then for admissible values of the coefficient of variation θ, $\theta_A < \theta < \theta_C$,*

1. *$X^* > 0$ if $\theta < \theta_B$*

2. *$X^* = 0$ if $\theta \geq \theta_B$.*

Remarkably, the bounds on the coefficient of variation, θ_A and θ_C, in equation (21), as well as the critical value of the coefficient of variation, θ_B, in Proposition 8 depend only on $\frac{\mu^H}{\rho}$ and $\frac{\mu^L}{\rho}$.

We interpret Proposition 8 in the context of an example illustrated in Figure 1, which shows the coefficient of variation $\theta \equiv \frac{\sigma}{m}$ on the horizontal axis and X^* on the vertical axis, for the case in which $r = 0$, $\rho = 0.05$, $\mu^L = 0.10$, $\mu^H = 0.40$ and the mean of the unconditional distribution of ϕ is 1. At point A in Figure 1, the coefficient of variation is $\theta_A \equiv \sqrt{\frac{\mu^L}{\mu^H}}$, so that $\phi^L = 0$. The neighborhood located immediately to the right of point A will be called Neighborhood A.[20] At point C, the coefficient of variation equals $\theta_C \equiv \left(1 + \frac{\mu^H+\mu^L}{\rho}\right)\sqrt{\frac{\mu^L}{\mu^H}}$, which is the upper bound on its admissible values in equation (21).[21] Point B is located at the critical value of the coefficient of variation $\theta_B \equiv (1-\lambda)\theta_A + \lambda\theta_C$ identified in Proposition

[20]For points along the dashed line to the left of point A, $\phi^L > 0$. For these (inadmissible) points, cash flow is never negative, and hence there is no reason for the firm to hold cash, so $X^* = 0$.

[21]For (inadmissible) points along the dashed line to the right of Point C, the conditional expectation of cash flows, $E\left\{\int_t^\infty \phi_s e^{-\rho(s-t)} ds | \phi_t = \phi^L\right\}$, is negative and the firm would want to terminate in Regime L, even if it could pay negative dividends. Therefore, there is no incentive to hold cash and hence $X^* = 0$.

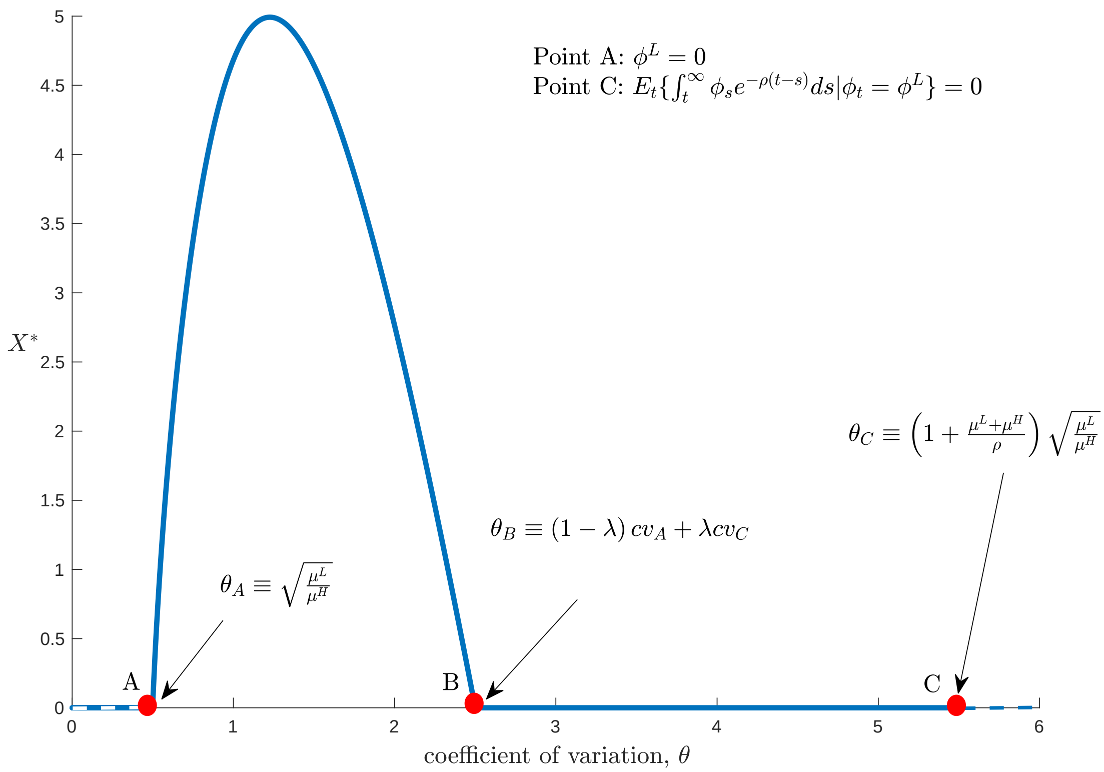

Figure 1: Optimal target level of cash X^* as a function of the coefficient of variation of cash flows.

8. The neighborhood immediately to the left of point B will be called Neighborhood B. For values of the coefficient of variation less than θ_B, that is, to the left of Point B, $X^* > 0$; for values of the coefficient of variation greater than θ_B, that is, to the right of Point B, $X^* = 0$. Point B is located a fraction $\lambda \equiv \left(2 + \frac{\rho}{\mu^L}\right)^{-1} < 0.5$ of the distance between Points A and C. In this particular example, $\lambda = \left(2 + \frac{0.05}{0.10}\right)^{-1} = 0.4$, so point B is located 40% of the distance from point A toward point C. Thus, for 60% of the interval of admissible values of the coefficient of variation in equation (21), the optimal holding of cash is zero. As we have pointed out, X^* is not monotonic in the coefficient of variation in the interval from point A to point B. X^* is increasing in the coefficient of variation in Neighborhood A, and X^* is decreasing in the coefficient of variation in Neighborhood B. For admissible values of the coefficient of variation, the highest values, to the right of point B, are associated with the lowest, namely zero, X^*.

5.1 Neighborhoods of Parameter Space Where X^* Is Small

Proposition 6 provides a closed-form expression for X^* that holds globally for all admissible parameter configurations. This subsection analyzes $X^* > 0$ in two neighborhoods of parameter space where X^* is close to zero. To see why X^* is close to zero in Neighborhood A, suppose that the firm has cash on hand equal to $\varepsilon > 0$ when Regime L arrives. While Regime L continues to prevail, the firm loses cash at rate $-\phi^L > 0$ so the firm will be able to survive, without running out of cash, in Regime L for a period of time as long as $\frac{\varepsilon}{-\phi^L}$. In Neighborhood A, ϕ^L is arbitarily close to zero so the firm could avoid running out of cash for an arbitrarily long period of time, even if the cash on hand at the beginning of Regime L, ε, is arbitrarily small. As Proposition 9 below states, X^* goes to zero as ϕ^L goes to zero, but X^* goes to zero more slowly than ϕ^L goes to zero, so the ratio $\frac{X^*}{-\phi^L}$ becomes arbitrarily large.

X^* measures precautionary saving in terms of dollars. The ratio $\frac{X^*}{-\phi^L}$ is an alternative measure of precautionary saving expressed in units of time rather than dollars. Specifically, $\frac{X^*}{-\phi^L}$ is the length of time that the firm can spend continuously in Regime L before it runs out of cash. In contrast to X^*, which is non-monotonic in θ (Figure 1), the ratio $\frac{X^*}{-\phi^L}$ is monotonic in θ. As illustrated in Figure 2, $\frac{X^*}{-\phi^L}$, is a monotonically *decreasing* function of θ. This monotonicity is in the opposite direction of the conventional result. That is, a mean-preserving spread reduces, rather than increases, this alternative measure of precautionary saving. Although we do not prove that $\frac{X^*}{-\phi^L}$ is downward sloping over the entire domain of the coefficient of variation, we prove that it is monotonically decreasing in both Neighborhood A and Neighborhood B. This ratio is also of interest because, as we show in Proposition 13 in Section 6, it is a key determinant of life expectancy of the firm.

Proposition 9 *(Neighborhood A) Assume that* $r = 0$, $\Phi^L + \Gamma\Phi^H > 0$ *and that* $\lim_{\phi^L \nearrow 0} \Phi^H > 0$ *is finite.*[22] *For given* μ^L *and* μ^H, $\lim_{\phi^L \nearrow 0} X^* = 0$ *and* $\lim_{\phi^L \nearrow 0} \frac{X^*}{-\phi^L} = \infty$.

Now consider Neighborhood B, which is the set of parameter configurations for which $\Lambda - 1 > 0$ is arbitrarily small. We have already examined (in Section 3) the behavior of X^* in Neighborhood B. Lemma 2 implies that when $r = 0$, this neighborhood can equivalently be described as the neighborhood where $\Phi^L + \Gamma\Phi^H > 0$ is arbitrarily small.

[22] In particular, this proposition applies when $\Phi^H > 0$ remains fixed when ϕ^L changes, and when $\phi^H > 0$ changes to maintain the unconditional mean $m \equiv \frac{\mu^L\phi^L + \mu^H\phi^H}{\mu^L + \mu^H} > 0$ unchanged when ϕ^L changes.

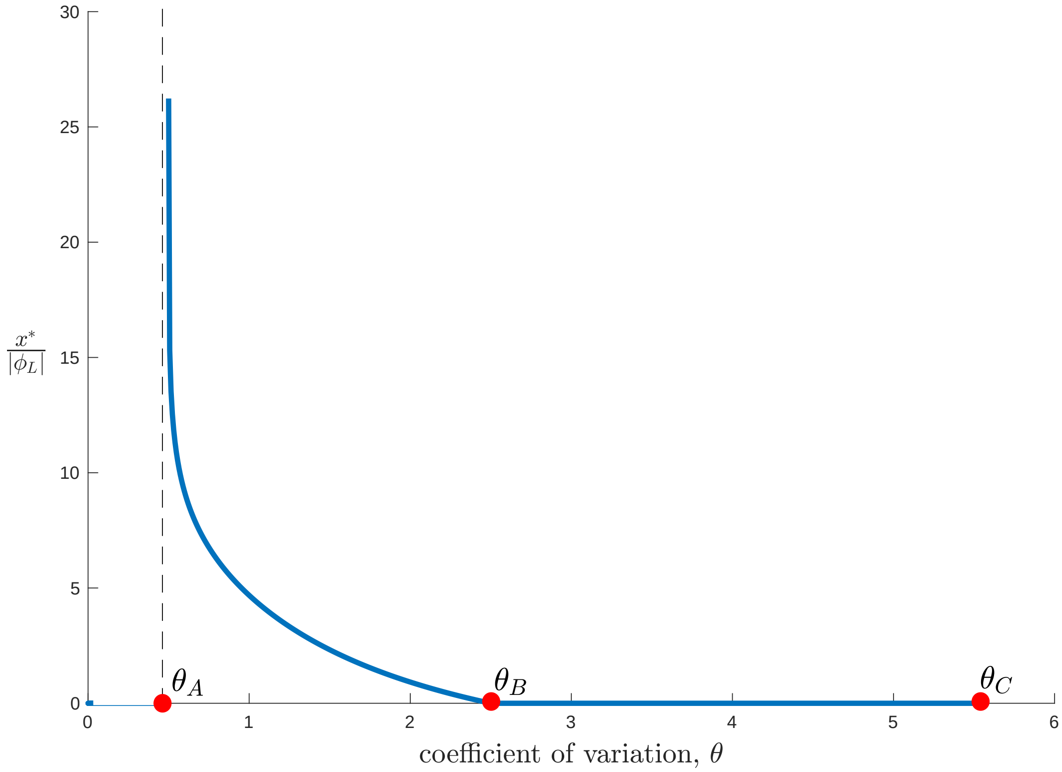

Figure 2: Ratio of optimal target level of cash, X^*, to the flow of losses, $|\phi_L|$, as a function of the coefficient of variation of cash flows.

Proposition 10 *Assume that $r = 0$. Consider any $\Phi^L < 0$ and $\Phi^H > 0$ bounded away from zero for which $\Phi^L + \Gamma\Phi^H > 0$ is arbitrarily close to zero (Neighborhood B). Then $X^* \simeq \Omega \simeq \frac{\Phi^L + \Gamma\Phi^H}{1-\Gamma} = \frac{\mu^L}{\rho+\mu^L} V^L(X^*)$.*

Proposition 10 implies that for a parameter configuration in Neighborhood B the value of X^* is approximately equal to $\frac{\mu^L}{\rho+\mu^L} V^L(X^*)$, which is the expected present value of the continuation value of the firm at the time that the next Regime L arrives. By foregoing dividends for a while in Regime H to allow cash on hand to reach X^*, shareholders incur an investment cost of X^* to enable the firm to enter the next Regime L with cash on hand equal to X^*. When the firm enters Regime L with cash on hand X^*, the value of the firm will be $V^L(X^*)$, which has expected present value $\frac{\mu^L}{\rho+\mu^L} V^L(X^*)$. Proposition 10 states that this expected present value is approximately equal to the investment cost X^* that reflects foregone dividends in Regime H.

The approximate equality of X^* and $\frac{\mu^L}{\rho+\mu^L}V^L(X^*)$, which was derived from the closed-form solution for X^*, holds when $r=0$. It is straightforward to show that if r is positive and less than ρ, then $X^* \approx \frac{\mu^L}{\rho-r+\mu^L}V^L(X^*)$. This more general approximation follows directly from $\lim_{X\to 0}\frac{XV_X^L(X)}{V^L(X)}=1$,[23] and Proposition 2, which states that $V_X^L(X^*)=\frac{\rho-r+\mu^L}{\mu^L}$.

5.2 A Mean-Preserving Spread of the Unconditional Distribution of Cash Flows

In this subsection we analyze the impact of a mean-preserving spread of the unconditional distribution of cash flows for given values of the transition intensities μ^H and μ^L. In particular, we focus on increases in ϕ^H and decreases in ϕ^L, for given values of μ^H and μ^L, that do not change the unconditional mean $\frac{\mu^H\phi^H+\mu^L\phi^L}{\mu^L+\mu^H}$. For such mean-preserving spreads,[24]

$$\frac{d\Phi^L}{d\Phi^H}\mid_{MPS} = -\frac{\mu^H\left(\rho+\mu^L\right)}{\mu^L\left(\rho+\mu^H\right)} \lesseqgtr -1 \text{ as } \mu^H \gtreqless \mu^L, \tag{22}$$

where the notation $\frac{d\Phi^L}{d\Phi^H}\mid_{MPS}$ indicates the change in Φ^L in response to a change in Φ^H that maintains the unconditional mean of ϕ unchanged. Equation (22) and Proposition 7 imply the following proposition.

Proposition 11 *If $r=0$ and $\Phi^L+\Gamma\Phi^H \geq 0$, then*

1. $\frac{dV^L(X^*)}{d\Phi^H}\mid_{MPS} = \frac{1}{1-\Gamma}\frac{\mu^H}{\rho+\mu^H}\left[1-\left(\frac{\rho+\mu^L}{\mu^L}\right)^2\right] < 0.$

2. $\frac{dV^H(X^*)}{d\Phi^H}\mid_{MPS} = \frac{1}{1-\Gamma}\left(\frac{\rho}{\rho+\mu^H}\right)\left(1-\frac{\mu^H}{\mu^L}\right) \lesseqgtr 0$ *as* $\mu^H \gtreqless \mu^L$.

[23] Evaluate the HJB in equation (9) at $X=0$ and use the boundary condition $V^L(0)=0$ to obtain $V_X^L(0)=-\frac{\mu^H}{\phi^L}V^H(0)$, which is finite, since $V^H(0)$ is finite. Differentiate the HJB in equation (9) with respect to X and evaluate the resulting equation at $X=0$ to obtain $\left(1-\frac{r}{\rho+\mu^H}\right)V_X^L(0)=\Phi^L V_{XX}^L(0)+\frac{\mu^H}{\rho+\mu^H}V_X^H(0)$. To show that $V_{XX}^L(0)$ is finite, it suffices to show that $V_X^H(0)$ is finite. Evaluate the HJB in equation (8) at $X=0$ and use the boundary condition $V^L(0)=0$ to obtain $V^H(0)=\Phi^H V_X^H(0)$, so $V_X^H(0)$ is finite and hence $V_{XX}^L(0)$ is finite. Since $V_X^L(0)$ and $V_{XX}^L(0)$ are finite $\lim_{X\to 0}\frac{XV_X^L(X)}{V^L(X)}=\frac{\lim_{X\to 0}V_X^L(X)+\lim_{X\to 0}XV_{XX}^L(X)}{\lim_{X\to 0}V_X^L(X)}=\frac{\lim_{X\to 0}V_X^L(X)}{\lim_{X\to 0}V_X^L(X)}=1$.

[24] Use $\phi^H=\left(\rho+\mu^L\right)\Phi^H$ and $\phi^L=\left(\rho+\mu^H\right)\Phi^L$, so that the unconditional mean of cash flow is $\frac{\mu^H}{\mu^L+\mu^H}\left(\rho+\mu^L\right)\Phi^H+\frac{\mu^L}{\mu^L+\mu^H}\left(\rho+\mu^H\right)\Phi^L$. Totally differentiate this expression for the unconditional mean with respect to Φ^H and Φ^L, while keeping the unconditional mean unchanged, to obtain $\mu^H\left(\rho+\mu^L\right)d\Phi^H+\mu^L\left(\rho+\mu^H\right)d\Phi^L=0$, which immediately implies equation (22) in the text.

A mean-preserving spread reduces ϕ^L and increases ϕ^H. If the firm is currently in Regime L, the reduction in ϕ^L, which leads to a magnified reduction in the distorted flow $\widetilde{\phi}^L \equiv \frac{\rho+\mu^L}{\mu^L}\phi^L$, hits the firm immediately and dominates the impact of the increase in ϕ^H that is received later. Therefore, a mean-preserving spread reduces $V^L(X^*)$ (Statement 1 of Proposition 11).[25] This statement facilitates an alternative interpretation of the impact on X^* of a mean-preserving spread for parameter values in Neighborhood B. In that neighborhood, the optimal target level of cash on hand, X^*, is approximately equal to $\frac{\mu^L}{\rho+\mu^L}V^L(X^*)$. Statement 1 of Proposition 11 is that a mean-preserving spread of the distribution of cash flow reduces $V^L(X^*)$. Therefore, in Neighborhood B, a mean-preserving spread of the unconditional distribution of cash flows reduces X^*. This result is a special case of the local result in subsection 3.1 with $r=0$ and does not apply for all admissible combinations of the parameters ρ, ϕ^L, ϕ^H, μ^L, and μ^H.

In Neighborhood A, which is the negative neighborhood of $\phi^L=0$, a mean-preserving spread of the distribution cash flows increases X^*. Proposition 9 implies that in Neighborhood A, a mean-preserving increase in $\phi^L<0$ accompanied by a decrease in ϕ^H reduces X^* and increases $\frac{X^*}{-\phi^L}$. Such mean-preserving changes move ϕ^L and ϕ^H closer together and thus reduce the variance of the unconditional distribution. Therefore, a mean-preserving spread, which increases the variance of the unconditional distribution, has the opposite effect. That is, in Neighborhood A, a mean-preserving spread increases X^* and reduces $\frac{X^*}{-\phi^L}$.

To summarize, a mean-preserving spread increases X^* in Neighborhood A but reduces X^* in Neighborhood B.

A mean-preserving spread of the unconditional distribution of cash flow for given values of μ^H and μ^L is an increase in θ holding m fixed. We express this impact formally as $X^{*\prime}(\theta)$. The conventional finding, as summarized by Strebulaev and Whited (2011), for example, is $X^{*\prime}(\theta)>0$. However, we have shown that $X^{*\prime}(\theta)$ is negative in Neighborhood B. Moreover, this counter-conventional finding is not limited to that neighborhood. The following proposition illustrates that $X^{*\prime}(\theta)<0$ is pervasive enough in parameter space that the average value of $X^{*\prime}(\theta)$ is zero.

Proposition 12 *For $r=0$ and $\varepsilon>0$, $\lim_{\varepsilon\to 0}\frac{1}{\theta_B-\theta_A-2\varepsilon}\int_{\theta_A+\varepsilon}^{\theta_B-\varepsilon}X^{*\prime}(\theta)\,d\theta=0$.*

[25] However, if the firm is currently in Regime H, the valuation $V^H(X^*)$ will decrease or increase depending on whether μ^H is larger or smaller than μ^L. On one hand, the increase in ϕ^H increases the cash flows in the immediate future before the reduction in ϕ^L reduces future cash flows. This effect would tend to increase $V^H(X^*)$. On the other hand, the reduction in ϕ^L is magnified by the factor $\frac{\rho+\mu^L}{\mu^L}>1$, which tends to reduce $V^H(X^*)$. If $\mu^H<\mu^L$, the first effect dominates, and $V^H(X^*)$ increases; if $\mu^H>\mu^L$, the second effect dominates, and $V^H(X^*)$ decreases (Statement 2 of Proposition 11).

Proposition 12 implies that averaging over the interval (θ_A, θ_B), the average impact of volatility on $X^*(\theta)$ is zero. For the remaining admissible values of the coefficient of variation, that is, for $\theta \in [\theta_B, \theta_C)$, $X^*(\theta) \equiv 0$, so trivially, $X^{*\prime}(\theta) \equiv 0$.

6 Endogenous Cross-Section of Surviving Firms

The empirical literature typically finds that increased volatility increases X^*. We interpret this finding as a positive value of the cross-sectional average impact of θ on X^*, that is, for $\varepsilon > 0$

$$\lim_{\varepsilon \to 0} \int_{\theta_A+\varepsilon}^{\theta_B-\varepsilon} X^{*\prime}(\theta)\, dG(\theta) > 0,$$

where $G(\theta)$ is the cross-sectional distribution of θ across firms. Our model provides an endogenous mechanism that reconciles the zero value of average $X^{*\prime}(\theta)$ in Proposition 12 and the positive value of average $X^{*\prime}(\theta)$ found empirically. This endogenous mechanism operates through differential survival probabilities of firms with different volatilities. Specifically, firms with highly volatile cash flows tend to run out of cash, and thus are forced to terminate, sooner than firms with less volatile cash flows. As time proceeds, the cross-sectional distribution of volatility among the survivors in a given cohort of firms shifts toward low-volatility firms, which are the firms for which $X^{*\prime}(\theta) > 0$. This endogenous shifting of the distribution toward firms with $X^{*\prime}(\theta) > 0$ increases the cross-sectional average impact, $X^{*\prime}(\theta)$, to a positive number, consistent with empirical findings.

The life expectancy of a firm born at, say, time 0 in Regime H with zero cash on hand is $E\{\tau\}$, the expectation of the termination date τ defined in equation (5). The following proposition provides a closed-form expression for life expectancy.

Proposition 13 *Assume that $r = 0$. The life expectancy of a firm with zero cash on hand in Regime H is* $E\{\tau\} = \frac{1}{\mu^L}\left[1 + \frac{\phi^H}{m}\left(e^{-\left(\mu^L+\mu^H\right)\frac{m}{\phi^H}\frac{X^*}{\phi^L}} - 1\right)\right]$.

For parameter configurations in Neighborhood B, which is the positive neighborhood of $\Phi^L + \Gamma\Phi^H = 0$, the following corollary provides a simple expression for the life expectancy $E\{\tau\}$.

Corollary 3 *Assume that $r = 0$. In Neighborhood B, which is the positive neighborhood of* $\Phi^L + \Gamma\Phi^H = 0$, $E\{\tau\} \simeq \frac{1}{\mu^L} + \left(1 + \frac{\mu^H}{\mu^L}\right)\frac{X^*}{-\phi^L}$.

The expression for $E\{\tau\}$ in Corollary 3 can be viewed as the sum of three components. The first component, $\frac{1}{\mu^L}$, is the expected time until the current Regime H ends and the next Regime L arrives. Since X^* is in a neighborhood of zero, the firm will, with very high probability, accumulate cash equal to X^* in the current Regime H and arrive in Regime L with cash on hand equal to X^*. The second component, $\frac{X^*}{-\phi^L}$, is the length of uninterrupted time in Regime L needed to completely exhaust X^* and thus force the firm to terminate. If Regime L were to last indefinitely, for instance, because of a zero value for μ^H, the life expectancy, $E\{\tau\}$, would simply be the sum of the first two components, $\frac{1}{\mu^L}$ and $\frac{X^*}{-\phi^L}$. However, with $\mu^H > 0$, there is a chance that a new Regime H arrives before X is completely exhausted in Regime L. In that case, life expectancy would exceed the sum of the first two components by $\frac{\mu^H}{\mu^L}\frac{X^*}{-\phi^L}$, which is an extension of life expectancy reflecting the possibility of escaping from Regime L before exhausting cash on hand.

In Neighborhood A, $\frac{X^*}{-\phi^L}$ is arbitrarily large (Proposition 9) so the life expectancy, $E\{\tau\}$, in Proposition 13 is also arbitrarily large. To illustrate why this life expectancy is arbitrarily large, consider a firm with a parameter configuration in Neighborhood A that starts at time 0 in Regime H with zero cash on hand. The firm follows the optimal payout policy so it retains all operating profits as cash on hand until its cash on hand reaches $X = X^*$. Let t_1 be the time at which the first Regime L after time 0 arrives. Use the identity $E\{\tau\}$ $= \Pr\left\{t_1 < \frac{X^*}{\phi^H}\right\} \times E\left\{\tau | t_1 < \frac{X^*}{\phi^H}\right\} + \Pr\left\{t_1 \geq \frac{X^*}{\phi^H}\right\} \times E\left\{\tau | t_1 \geq \frac{X^*}{\phi^H}\right\}$ and the facts that $\Pr\left\{t_1 < \frac{X^*}{\phi^H}\right\} \geq 0$ and $E\left\{\tau | t_1 < \frac{X^*}{\phi^H}\right\} \geq 0$ to obtain

$$E\{\tau\} \geq \Pr\left\{t_1 \geq \frac{X^*}{\phi^H}\right\} \times E\left\{\tau | t_1 \geq \frac{X^*}{\phi^H}\right\} \geq \exp\left(-\mu^L \frac{X^*}{\phi^H}\right) \times \left(\frac{X^*}{\phi^H} + \frac{X^*}{-\phi^L}\right). \quad (23)$$

The second inequality in equation (23) uses the fact that the distribution of the arrival time t_1 is exponential so $\Pr\left\{t_1 \geq \frac{X^*}{\phi^H}\right\} = \exp\left(-\mu^L \frac{X^*}{\phi^H}\right)$ and the fact that if $t_1 \geq \frac{X^*}{\phi^H}$, then the firm will have cash on hand X^* when Regime L arrives at time t_1, and this amount of cash on hand guarantees that the firm will continue to survive for a period of time $\frac{X^*}{-\phi^L}$ after $t_1 \geq \frac{X^*}{\phi^H}$. Since $\lim_{\phi^L \nearrow 0} X^* = 0, \lim_{\phi^L \nearrow 0} \frac{X^*}{-\phi^L} = \infty$, and $\lim_{\phi^L \nearrow 0} \exp\left(-\mu^L \frac{X^*}{\phi^H}\right) = 1$, we have $\lim_{\phi^L \nearrow 0} E\{\tau\} = \lim_{\phi^L \to 0} \frac{X^*}{-\phi^L} = \infty$.

Figure 3 illustrates the impact of endogenous survival on the cross-sectional distribution of volatility. Underlying this figure is the assumption that at each point in time, a mass of firms (normalized to one) is born in Regime H, each with zero cash on hand. These newborn firms have identical values of shareholders' rate of time preference, ρ, and transition

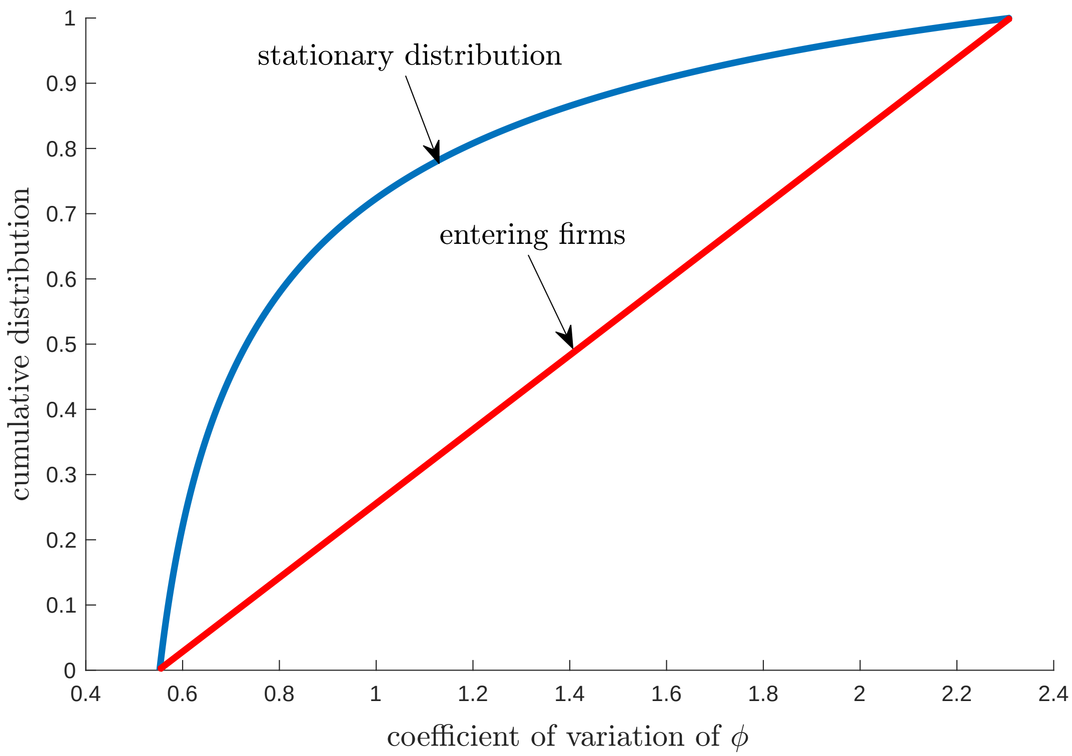

Figure 3: Cumulative distribution function of entering firms and stationary distribution function.

intensities, μ^H and μ^L. They have heterogeneous unconditional coefficients of variation of cash flows, θ, distributed uniformly on the open set of admissible values (θ_A, θ_C), where θ_A and θ_C are the coefficients of variation at points A and C, respectively, in Figure 1. Let's confine attention to firms for which $X^* > 0$ so they don't terminate immediately when the first Regime L arrives. Therefore, we confine attention to firms for which $\theta \in (\theta_A, \theta_B)$, where $\theta_B \equiv (1 - \lambda)\theta_A + \lambda\theta_C$ is the critical value of the coefficient of variation in Proposition 8 and at point B in Figure 1. For each value of $\theta \in (\theta_A, \theta_B)$, there is a continuum of firms that face idiosyncratic regime changes over the spans of their lives until they terminate. As time passes, the endogenous termination of firms that run out of cash in Regime L eliminates more firms with high θ than with low θ.

Figure 3 shows the c.d.f $G(\theta)$ of the cross-sectional distribution of θ. Since $G(\theta)$ is strictly concave in Figure 3, the density function, $g(\theta) \equiv G'(\theta)$, is decreasing in θ, that is, $g'(\theta) < 0$.

Proposition 14 *Assume that $r = 0$, $g'(\theta) < 0$, and $X^*(\theta) > 0$ is concave in θ for $\theta \in (\theta_A, \theta_B)$. Then $\lim_{\varepsilon>0\to 0} \int_{\theta_A+\varepsilon}^{\theta_B-\varepsilon} X^{*\prime}(\theta)\, g(\theta)\, d\theta > 0$.*

This example, in which the cross-sectional density function $g(\theta)$ is decreasing and the function $X^*(\theta)$ is concave, illustrates the need for caution in interpreting regression findings of an average positive impact of volatility on cash holdings. The positive impact need not imply that a high-volatility firm will have a higher X^* than an otherwise-identical firm with lower volatility. The positive average impact of volatility on X^* could simply reflect the endogenous selection effect resulting from the faster termination of high-volatility firms.

7 Conclusion

Firms undertake precautionary saving to protect themselves against unfavorable events that can arise randomly. The conventional result is that an increase in the variance of random events will lead to an increase in the optimal amount of precautionary saving. We have re-examined that conventional result in a simple model of a firm that faces positively serially correlated cash flows. Our analysis has at least three contributions. First, we provide a closed-form solution for the optimal target level of cash, X^*, held by a risk-neutral firm in an environment with serially correlated cash flows. We use this closed-form solution for X^* to provide an interpretation of the magnitude of X^*. Second, and most substantively, we show that a mean-preserving increase in the variance of cash flows can actually reduce X^*—the opposite of the conventional result. We provide an analytical, yet intuitive, explanation the role of serial correlation in the counter-conventional result. Third, we show that cross-sectional heterogeneity in the coefficient of variation, θ, leads to cross-sectional variation in life expectancy of firms, which tilts the cross-sectional distribution of θ in a sample surviving firms toward firms with with low θ. This endogenous selection effect can cause the average impact of volatility on X^* in a cross section of surviving firms to be positive, consistent with empirical findings.

To focus on precautionary saving in a tractable framework, we assume that the firm has no access to external funds, either from borrowing or from issuing of equity. We consider a stochastic process for cash flow that has only two possible realizations. One possible realization of cash flow is positive and the other possible realization is negative. The transitions between these realizations are governed by Poisson processes. When cash flow is positive, the firm accumulates cash until its cash on hand reaches an optimally-chosen level X^*; any positive cash flows received when the stock of cash on hand equals X^* are then paid out as dividends. When cash flow is negative, the firm draws down its cash and is forced to terminate if cash flow remains negative when the stock of cash on hand is zero.

In order for the firm to want to accumulate any cash, it is necessary that even when the current cash flow is negative, the conditional expected present value of the infinite stream of cash flows is positive. Otherwise the firm would choose to terminate when cash flow is negative even without being forced to do so. However, this necessary condition is not sufficient for X^* to be positive. The stronger condition that is sufficient for X^* to be positive is that the expected present value of "distorted" cash flows is positive, where the distorted cash flows magnify the negative cash flows but do not magnify positive cash flows.

Our model has six parameters: the two possible realizations of cash flow, two transition probabilities between these values, shareholders' rate of time preference, and the rate of return on cash. We first show that X^* is positive if and only if a particular function of these parameters exceeds zero. To derive a closed-form solution for X^*, we then confine attention to the focal case in which the rate of return on cash equals zero. The closed-form solution holds for all admissible parameter values. We then show that in a particular neighborhood of parameter space (Neighborhood B), the optimal target level of cash when cash flow is positive, X^*, is approximately equal to the expected present value of $V^L(X^*)$, the value of the firm when it enters Regime L with cash on hand equal to X^*. In this case, the optimal level of cash on hand reflects the equality of the marginal cost of accumulating cash (foregone dividends) and the marginal benefit of accumulating cash (the value of sustaining the firm when cash flows turn negative).

The major substantive contribution of this paper is the deconstruction of the conventional result that a mean-preserving spread of stochastic flows increases X^*. We focus on the coefficient of variation of the unconditional distribution of cash flows, recognizing that a mean-preserving spread on this distribution can be viewed as an increase in the coefficient of variation holding the mean constant. In this context, the conventional result is that X^* is an increasing function of the coefficient of variation of cash flows. We demonstrate analytically that X^* is not monotonic in the coefficient of variation. Moreover, there is a critical value of the coefficient of variation (θ_B) such that X^* is positive for values of the coefficient of variation below the critical value, but X^* is zero for values of the coefficient of variation above the critical value. That is, the higher values of the coefficient of variation are associated with lower X^*—the opposite of the conventional result. In addition, we show that the marginal impact on X^* of an increase in volatility averages to zero over the entire parameter space. We then show that if the population of firms consists of the surviving firms from all previously-born cohorts of firms, the population will be weighted more towards firms for which the impact of volatility on X* is positive. This selection mechanism can reconcile the results of this paper with the findings of the empirical literature.

The counter-conventional result is so striking that it demands an explanation. For values of the coefficient of variation slightly below the critical value described above, the marginal value of the first dollar of cash on hand is greater than one, so it is optimal retain that dollar inside the firm. That is, X^* is positive. A mean-preserving spread that increases the coefficient of variation to its critical value reduces the marginal value of the first dollar of cash on hand to one, and the firm might as well pay that dollar as dividends; the optimal value of X^* is zero. To see why an increase in the coefficient of variation reduces the marginal value of the first dollar of cash on hand, observe that a mean-preserving increase in variance increases the positive level of cash flow and reduces the negative value of cash flow. The reduction in the negative value of cash flow increases the rate at which the firm draws down its cash when facing a persistent negative cash flow. This more rapid draw-down of cash reduces the window of time that a given dollar of cash will allow the firm to survive when facing a persistent episode of negative cash flow. Therefore, the marginal benefit of holding a dollar of cash to stave off termination is reduced, making a dollar of cash less effective as a precaution against termination, thereby reducing X^*. Though the formal analysis was conducted with a parsimonious parametric model, the underlying intuition about the reduced efficacy of a dollar of cash in staving off termination is quite general.

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Appendix

A Solution of the System of ODEs in Equation (16)

The solution to the system of ODEs in equation (16) can be expressed in terms of the eigenvalues and eigenvectors of the 2×2 matrix A defined in equation (17). The eigenvalues, ω_1 and ω_2 are the roots of the following characteristic equation

$$q(\omega) \equiv \omega^2 - (trA)\,\omega + \det A = 0. \tag{A.1}$$

where $trA = \frac{1}{\Phi^H} + \frac{1}{\Phi^L}$ and $\det A = \frac{1}{\Phi^H}\frac{1}{\Phi^L}(1-\Gamma)$. It is straightforward to verify that[26] $\left[\begin{array}{cc} 1 & \frac{\rho+\mu^L}{\mu^L}\left(1-\omega_i\Phi^H\right)\end{array}\right]'$ is an eigenvector of A with eigenvalue ω_i, $i=1,2$.

The general solution to a two-equation system of constant-coefficient homogeneous first-order linear ODEs is a linear combination of the product of $e^{\omega_i X}$ and the eigenvector corresponding to the eigenvalue ω_i, $i=1,2$, so

$$\left[\begin{array}{c} V^H(X) \\ V^L(X)\end{array}\right] = c_1\left[\begin{array}{c} 1 \\ \frac{\rho+\mu^L}{\mu^L}\left(1-\omega_1\Phi^H\right)\end{array}\right] e^{\omega_1(X-X^*)} + c_2\left[\begin{array}{c} 1 \\ \frac{\rho+\mu^L}{\mu^L}\left(1-\omega_2\Phi^H\right)\end{array}\right] e^{\omega_2(X-X^*)} \tag{A.2}$$

Differentiate the expression for $V^H(X)$ in equation (A.2) twice with respect to X, evaluate $V^H_X(X)$ and $V^H_{XX}(X)$ at $X=X^*$, and use the boundary conditions in equations (11a) and (12) to obtain

$$V^H_X(X^*) = c_1\omega_1 + c_2\omega_2 = 1 \tag{A.3}$$

and

$$V^H_{XX}(X^*) = c_1\omega_1^2 + c_2\omega_2^2 = 0. \tag{A.4}$$

Equations (A.3) and (A.4) are two linear equations in the constants c_1 and c_2. Equation (A.4) implies

$$c_1\omega_1^2 = -c_2\omega_2^2, \tag{A.5}$$

[26]The first element of $A\left[\begin{array}{cc} 1 & \frac{\rho+\mu^L}{\mu^L}\left(1-\omega_i\Phi^H\right)\end{array}\right]'$ is $\frac{1}{\Phi^H} - \frac{\mu^L}{\rho+\mu^L}\frac{1}{\Phi^H}\frac{\rho+\mu^L}{\mu^L}\left(1-\omega_i\Phi^H\right) = \omega_i$. The second element of $A\left[\begin{array}{cc} 1 & \frac{\rho+\mu^L}{\mu^L}\left(1-\omega_i\Phi^H\right)\end{array}\right]'$ is $-\frac{\mu^H}{\rho+\mu^H}\frac{1}{\Phi^L} + \frac{1}{\Phi^L}\frac{\rho+\mu^L}{\mu^L}\left(1-\omega_i\Phi^H\right) = \frac{\rho+\mu^L}{\mu^L}\left(-\Gamma\frac{1}{\Phi^L} + \frac{1}{\Phi^L}\left(1-\omega_i\Phi^H\right)\right)$ $= \frac{\rho+\mu^L}{\mu^L}\Phi^H\left(-\Gamma\frac{1}{\Phi^L}\frac{1}{\Phi^H} + \frac{1}{\Phi^L}\left(\frac{1}{\Phi^H}-\omega_i\right)\right) = \frac{\rho+\mu^L}{\mu^L}\Phi^H\left((1-\Gamma)\frac{1}{\Phi^L}\frac{1}{\Phi^H} - \frac{1}{\Phi^L}\omega_i\right) = \frac{\rho+\mu^L}{\mu^L}\Phi^H\left(\det A - \frac{1}{\Phi^L}\omega_i\right)$ $= \frac{\rho+\mu^L}{\mu^L}\Phi^H\left(-\omega_i^2 + (trA)\,\omega_i - \frac{1}{\Phi^L}\omega_i\right) = \omega_i\frac{\rho+\mu^L}{\mu^L}\Phi^H\left(-\omega_i + trA - \frac{1}{\Phi^L}\right) = \omega_i\frac{\rho+\mu^L}{\mu^L}\Phi^H\left(-\omega_i + \frac{1}{\Phi^H}\right) =$ $\omega_i\frac{\rho+\mu^L}{\mu^L}\left(1-\omega_i\Phi^H\right)$.

which, along with equation (A.3), implies

$$c_1 = \frac{1}{\omega_2 - \omega_1} \frac{\omega_2}{\omega_1} < 0 \tag{A.6}$$

and

$$c_2 = \frac{1}{\omega_1 - \omega_2} \frac{\omega_1}{\omega_2} > 0. \tag{A.7}$$

Evaluate $V^L(X)$ in equation (A.2) at $X = 0$, and then use the boundary condition $V^L(0) = 0$ from equation (10) to obtain

$$V^L(0) = c_1 \frac{\rho + \mu^L}{\mu^L} \left(1 - \omega_1 \Phi^H\right) e^{-\omega_1 X^*} + c_2 \frac{\rho + \mu^L}{\mu^L} \left(1 - \omega_2 \Phi^H\right) e^{-\omega_2 X^*} = 0. \tag{A.8}$$

Rearrange equation (A.8) using $c_1 \omega_1^2 = -c_2 \omega_2^2$ from equation (A.5) to obtain

$$e^{(\omega_2 - \omega_1) X^*} = Z \equiv \frac{\omega_1^2}{\omega_2^2} \frac{1 - \omega_2 \Phi^H}{1 - \omega_1 \Phi^H} > 0 \tag{A.9}$$

which implies

$$X^* = \max \left[\frac{1}{\omega_2 - \omega_1} \ln Z, 0 \right]. \tag{A.10}$$

The fact that $Z > 0$ in equation (A.9) follows from Statement 1 of Lemma 4. Use Lemma 1, which implies

$$\omega_1^2 \left(1 - \omega_2 \Phi^H\right) = \frac{-1}{\Phi^H \Phi^L} \left[1 - \Gamma - \left(\Phi^L + \Gamma \Phi^H\right) \omega_1\right]$$

and

$$\omega_2^2 \left(1 - \omega_1 \Phi^H\right) = \frac{-1}{\Phi^H \Phi^L} \left[1 - \Gamma - \left(\Phi^L + \Gamma \Phi^H\right) \omega_2\right],$$

to rewrite the definition of Z in equation (A.9) as

$$Z \equiv \frac{1 - \Gamma - \left(\Phi^L + \Gamma \Phi^H\right) \omega_1}{1 - \Gamma - \left(\Phi^L + \Gamma \Phi^H\right) \omega_2}. \tag{A.11}$$

Add and substract $\left(\Phi^L + \Gamma \Phi^H\right) \omega_2$ in the numerator of Z in equation (A.11) and rearrange to obtain

$$Z = 1 + \frac{\Phi^L + \Gamma \Phi^H}{1 - \Gamma - \left(\Phi^L + \Gamma \Phi^H\right) \omega_2} \left(\omega_2 - \omega_1\right). \tag{A.12}$$

Equations (A.10) and (A.12) together imply Proposition 6.

Repeated differentiation of equation (A.2) with respect to X leads to the following expressions for $V_{(j)}^{H}(X)$ and $V_{(j)}^{L}(X)$, which are the j-th order derivatives of $V^{H}(X)$ and $V^{L}(X)$, respectively, with respect to X.

$$\begin{bmatrix} V_{(j)}^{H}(X) \\ V_{(j)}^{L}(X) \end{bmatrix} = c_1\omega_1^j \begin{bmatrix} 1 \\ \frac{\rho+\mu^L}{\mu^L}\left(1-\omega_1\Phi^H\right) \end{bmatrix} e^{\omega_1(X-X^*)} + c_2\omega_2^j \begin{bmatrix} 1 \\ \frac{\rho+\mu^L}{\mu^L}\left(1-\omega_2\Phi^H\right) \end{bmatrix} e^{\omega_2(X-X^*)}. \quad \text{(A.13)}$$

Since $c_1 < 0$, $c_2 > 0$ (equations A.6 and A.7) and $\omega_1 < 0 < \omega_2 < \frac{1}{\Phi^H}$ (Statement 1 of Lemma 4), it follows that $V_{(j)}^{H}(X) > 0$ and $V_{(j)}^{L}(X) > 0$ for $j = 1, 3, 5, ...$ In particular, the third derivatives of the value functions $V^{H}(X)$ and $V^{L}(X)$ both positive.

Since $V_{XXX}^{H}(X) > 0$, it follows that $V_{XX}^{H}(X) < V_{XX}^{H}(X^*) = 0$ for $0 \leq X < X^*$. Therefore, $V^{H}(X)$ is strictly concave for $0 \leq X < X^*$. Use the two rows of equation (A.13) to obtain

$$V_{(j)}^{L}(X) = \frac{\rho+\mu^L}{\mu^L}\left[V_{(j)}^{H}(X) - \omega_1\Phi^H c_1\omega_1^j e^{\omega_1(X-X^*)} - \omega_2\Phi^H c_2\omega_2^j e^{\omega_2(X-X^*)}\right]. \quad \text{(A.14)}$$

Evaluate equation (A.14) for $j = 2$ at $X = X^*$, and use $V_{XX}^{H}(X^*) = 0$ to obtain

$$V_{XX}^{L}(X^*) = -\Phi^H \frac{\rho+\mu^L}{\mu^L}\left(\omega_1^3 c_1 + \omega_2^3 c_2\right) < 0. \quad \text{(A.15)}$$

Since $V_{XXX}^{L}(X) > 0$, we have $V_{XX}^{L}(X) < V_{XX}^{L}(X^*) < 0$ for $0 \leq X \leq X^*$. Therefore, $V^{L}(X)$ is strictly concave for $0 \leq X \leq X^*$.

B Proofs

Proof of Proposition 1. If $X^* = 0$, the firm maintains a zero balance of cash on hand. If, for some reason, the firm is holding cash, $X > 0$, it immediately pays this entire amount to its shareholders as dividends and then, for the remainder of the current Regime H, pays all net inflows of cash from operations, $\phi^H > 0$, as dividends as soon as they arrive. When the current Regime H ends and the next Regime L arrives, the firm terminates. Therefore, the expected present value of dividends is X plus the expected present value of cash flows from operations over the duration of Regime H. That is, $V^{H}(X)$ equals X plus the myopic value of Regime H in Definition 2, $\Phi^H \equiv \frac{\phi^H}{\rho+\mu^L}$. Therefore,

$$V^{H}(X) = \Phi^H + X, \quad \text{(B.1)}$$

which implies $V_{X}^{H}(X) = 1$ and $V_{XX}^{H}(X) = 0$. ■

Proof of Proposition 2. See paragraph immediately preceding Proposition 2 in the main text. ■

Proof of Proposition 3. Since $V_{XX}^{H}(X^*) = 0$, it follows that if $V_{XX}^{H}(0) < 0$, then $X^* \neq 0$, so $X^* > 0$. Also, if $X^* > 0$, then $V_{XX}^{H}(X) < 0$ for $0 \leq X < X^*$, so $V_{XX}^{H}(0) < 0$. Now evaluate the ODEs in equations (8) and (9) at $X = 0$ and use $V^L(0) = 0$ from equation (10) to obtain $V_X^H(0) = \frac{\rho+\mu^L}{\phi^H} V^H(0)$ and $V^H(0) = \frac{-\phi^L}{\mu^H} V_X^L(0)$, respectively, so that $V_X^H(0) = \frac{\rho+\mu^L}{\mu^H} \frac{-\phi^L}{\phi^H} V_X^L(0)$. Evaluate equation (13) at $X = 0$ to obtain $V_X^L(0) = \frac{1}{\mu^L}\left[\left(\rho - r + \mu^L\right) V_X^H(0) - \phi^H V_{XX}^H(0)\right]$, which can be substituted into the previous equation to obtain $V_X^H(0) = \frac{\rho+\mu^L}{\mu^H} \frac{-\phi^L}{\phi^H} \frac{1}{\mu^L}\left[\left(\rho - r + \mu^L\right) V_X^H(0) - \phi^H V_{XX}^H(0)\right]$. Rearrange the equation to obtain $\left(\frac{\phi^H}{\rho+\mu^L} \frac{\mu^H}{-\phi^L} \frac{\mu^L}{\rho-r+\mu^L} - 1\right) V_X^H(0) = -\frac{\phi^H}{\rho-r+\mu^L} V_{XX}^H(0)$. Therefore, since $V_X^H(0) > 0$ and $-\frac{\phi^H}{\rho-r+\mu^L} < 0$, it follows that $V_{XX}^H(0) < 0$ if and only if $\Lambda \equiv \frac{\phi^H}{\rho+\mu^L} \frac{\mu^H}{-\phi^L} \frac{\mu^L}{\rho-r+\mu^L} > 1$. ■

Proof of Corollary 1. Proposition 3 implies that if $\Lambda \equiv \frac{\phi^H}{\rho+\mu^L} \frac{\mu^H}{-\phi^L} \frac{\mu^L}{\rho-r+\mu^L} = 1$, then $X^* = 0$. Since $X^* = 0$, Proposition 1 implies that $V_X^H(0) = 1 = \Lambda$. ■

Proof of Proposition 4. Starting from $\Lambda\left(\rho, r, \phi^H, \phi^L, \mu^H, \mu^L\right) \equiv \frac{\phi^H}{\rho+\mu^L} \frac{\mu^H}{-\phi^L} \frac{\mu^L}{\rho-r+\mu^L} = 1$, any change in a parameter that increases Λ will increase X^* to a positive value. Inspection of the definition of $\Lambda\left(\rho, r, \phi^H, \phi^L, \mu^H, \mu^L\right)$ immediately reveals that the following changes individually increase Λ and hence increase X^* to a positive value: a decrease in ρ (Statement 1); an increase in r (Statement 2); an increase in μ^H (Statement 3); an increase in ϕ^H (Statement 6); an increase in ϕ^L since ϕ^L is negative (Statement 7).

To determine the impact of a small change in μ^L, differentiate $\Lambda\left(\rho, r, \phi^H, \phi^L, \mu^H, \mu^L\right) \equiv \frac{\phi^H}{\rho+\mu^L} \frac{\mu^H}{-\phi^L} \frac{\mu^L}{\rho-r+\mu^L}$ with respect to μ^L to obtain $\frac{\partial\Lambda}{\partial\mu^L} = -\frac{\Lambda}{\rho+\mu^L} + \frac{\Lambda}{\mu^L} - \frac{\Lambda}{\rho-r+\mu^L} = \frac{\Lambda}{\mu^L(\rho+\mu^L)(\rho-r+\mu^L)}\left[-\mu^L\left(\rho - r + \mu^L\right) + \left(\rho + \mu^L\right)\left(\rho - r + \mu^L\right) - \mu^L\left(\rho + \mu^L\right)\right]$. Since $\frac{\Lambda}{\mu^L(\rho+\mu^L)(\rho-r+\mu^L)} > 0$, $sign\left(\frac{\partial\Lambda}{\partial\mu^L}\right) = sign\left(\mu^L r + \left(\rho + \mu^L\right)\left(\rho - r + \mu^L\right) - 2\mu^L\left(\rho + \mu^L\right)\right) = sign\left(-\rho r + \left(\rho - \mu^L\right)\left(\rho + \mu^L\right)\right)$ $= sign\left(-\rho r + \rho^2 - \left(\mu^L\right)^2\right) = sign\left(\left(\rho - r\right)\rho - \left(\mu^L\right)^2\right) = sign\left(\sqrt{\left(\rho - r\right)\rho} - \mu^L\right) =$ $sign\left(\rho\sqrt{\left(1 - \frac{r}{\rho}\right)} - \mu^L\right)$. Therefore, if $\mu^L < \sqrt{\left(\rho - r\right)\rho}$, then $\frac{\partial\Lambda}{\partial\mu^L} > 0$ so an increase in μ^L increases X^* to a positive value (Statement 4). Alternatively, if $\mu^L > \sqrt{\left(\rho - r\right)\rho}$, then $\frac{\partial\Lambda}{\partial\mu^L} < 0$ and a decrease in μ^L increases X^* to a positive value (Statement 5). ■

Proof of Proposition 5. Rewrite $\Lambda \equiv \frac{\phi^H}{\rho+\mu^L} \frac{\mu^H}{-\phi^L} \frac{\mu^L}{\rho-r+\mu^L}$ in Definition 3 as $\Lambda = \frac{\mu^L}{\rho+\mu^L} \frac{\mu^L}{\rho-r+\mu^L} \frac{\mu^H\phi^H}{-\mu^L\phi^L} = \frac{\mu^L}{\rho+\mu^L} \frac{\mu^L}{\rho-r+\mu^L} \frac{\mu^H\phi^H}{\mu^H\phi^H-M}$, where $0 < M \equiv \mu^L\phi^L + \mu^H\phi^H < \mu^H\phi^H$ is unchanged by a mean-preserving change in ϕ^H and ϕ^L, for given μ^H and μ^L. Starting from an initial parameter configuration in which $\Lambda = 1$, a mean-preserving decrease in ϕ^H and increase in ϕ^L decreases $\mu^H\phi^H$ while maintaining M unchanged which increases the ratio $\frac{\mu^H\phi^H}{\mu^H\phi^H-M}$ and hence increases

Λ to a value greater than one. Therefore, X^* increases to a positive number. ■

Proof of Lemma 1. Use the characteristic equation $q(\omega) = \omega^2 - (trA)\,\omega + \det A = 0$ to obtain $\omega_i^2 = (trA)\,\omega_i - \det A$, where ω_i, $i = 1,2$ is a root of the characteristic equation. Therefore, $\omega_i^2 \left(1 - \omega_j \Phi^H\right) = \omega_i^2 - \omega_i^2 \omega_j \Phi^H = (trA)\,\omega_i - \det A - \omega_i \Phi^H \det A$, which can be rewritten as $\omega_i^2 \left(1 - \omega_j \Phi^H\right) = \left(\frac{1}{\Phi^H} + \frac{1}{\Phi^L}\right)\omega_i - \frac{1}{\Phi^H}\frac{1}{\Phi^L}(1-\Gamma) - \omega_i \Phi^H \frac{1}{\Phi^H}\frac{1}{\Phi^L}(1-\Gamma) = \frac{1}{\Phi^H}\frac{1}{\Phi^L}\left[\left(\Phi^H + \Phi^L\right)\omega_i - (1-\Gamma) - \omega_i \Phi^H (1-\Gamma)\right] = \frac{1}{\Phi^H}\frac{1}{\Phi^L}\left[\left(\Phi^L + \Gamma\Phi^H\right)\omega_i - (1-\Gamma)\right]$. Multiply both sides of this equation by $-\Phi^H\Phi^L$ to obtain $1-\Gamma-\left(\Phi^L + \Gamma\Phi^H\right)\omega_i = -\Phi^H\Phi^L\omega_i^2\left(1-\omega_j\Phi^H\right) > 0$, where the inequality follows from $\Phi^H > 0$, $\Phi^L < 0$, and $\omega_1 < \omega_2 < \frac{1}{\Phi^H}$. ■

Proof of Lemma 2. Lemma 1 implies that the denominator of $\Omega \equiv \frac{\Phi^L + \Gamma\Phi^H}{1-\Gamma-(\Phi^L+\Gamma\Phi^H)\omega_2}$ is positive, so $sign(\Omega) = sign\left(\Phi^L + \Gamma\Phi^H\right)$. To prove the final equality in this lemma, use the definition of Λ to obtain $\Lambda - 1 = \left(\frac{\phi^H}{\rho+\mu^L}\frac{\mu^H}{\rho+\mu^H}\frac{\mu^L}{\rho+\mu^L} + \frac{\phi^L}{\rho+\mu^H}\right)\frac{\rho+\mu^H}{-\phi^L} = \left(\Phi^H \frac{\mu^H}{\rho+\mu^H}\frac{\mu^L}{\rho+\mu^L} + \Phi^L\right)\frac{\rho+\mu^H}{-\phi^L}$. Then use the definition of $\Gamma \equiv \frac{\mu^H}{\rho+\mu^H}\frac{\mu^L}{\rho+\mu^L}$ to obtain $\Lambda - 1 = \left(\Phi^H\Gamma + \Phi^L\right)\frac{\rho+\mu^H}{-\phi^L}$ so $sign(\Lambda - 1) = sign\left(\Phi^H\Gamma + \Phi^L\right)$ since $\frac{\rho+\mu^H}{-\phi^L} > 0$. ■

Proof of Proposition 6. See Appendix A. ■

Proof of Corollary 2. First consider the case in which $\Omega > 0$. Lemma 2 and Statement 1 of Proposition 6 together imply that $X^* - \Omega = \frac{1}{\omega_2-\omega_1}\ln\left(1 + (\omega_2-\omega_1)\Omega\right) - \Omega = \frac{1}{\omega_2-\omega_1}\left[\ln\left(1 + (\omega_2-\omega_1)\Omega\right) - (\omega_2-\omega_1)\Omega\right] = \frac{1}{\omega_2-\omega_1}h(z)$, where $h(z) \equiv \ln(1+z) - z$ and $z \equiv (\omega_2-\omega_1)\Omega > 0$. Observe that $h(0) = 0$ and $h'(z) = \frac{-z}{1+z} < 0$ for $z > 0$. Therefore, $h(z) < 0$ for $z > 0$, so $X^* - \Omega < 0$. Therefore, $X^* < \Omega = \max\{0, \Omega\}$. Now consider $\Omega \le 0$, which implies $\Gamma\Phi^H + \Phi^L \le 0$. Statement 2 of Proposition 6 implies that $X^* = 0 \le \max\{0, \Omega\}$. ■

Lemma 4 *The eigenvalues ω_1 and ω_2 have the following properties*

1. $\frac{1}{\Phi^L} < \omega_1 < 0 < (1-\Gamma)\frac{1}{\Phi^H} < \omega_2 < \frac{1}{\Phi^H}$
2. $\omega_1 + \omega_2 = trA = \frac{1}{\Phi^H} + \frac{1}{\Phi^L} < 0$
3. $\omega_1\omega_2 = \det A = \frac{1}{\Phi^H}\frac{1}{\Phi^L}(1-\Gamma) < 0$
4. $\frac{1}{\omega_1} + \frac{1}{\omega_2} = \frac{1}{1-\Gamma}\left(\Phi^H + \Phi^L\right) > 0.$

Proof of Lemma 4. The characteristic equation associated with the matrix A in equation (17) is $q(\omega) \equiv \omega^2 - (trA)\,\omega + \det A$, where $trA = \frac{1}{\Phi^H} + \frac{1}{\Phi^L} < 0$, where the inequality follows from equation (2), and $\det A = \frac{1}{\Phi^H}\frac{1}{\Phi^L}(1-\Gamma) < 0$. Since $q''(\omega) = 2 > 0$, the characteristic polynomial $q(\omega)$ is convex. Since $q\left(\frac{1}{\Phi^L}\right) = \left(\frac{1}{\Phi^L}\right)^2 - \left(\frac{1}{\Phi^H} + \frac{1}{\Phi^L}\right)\left(\frac{1}{\Phi^L}\right) + \frac{1}{\Phi^H}\frac{1}{\Phi^L}(1-\Gamma) = -\frac{1}{\Phi^H}\frac{1}{\Phi^L}\Gamma > 0$, $q(\omega_1) = 0$, $q(0) = \det A < 0$, $q\left((1-\Gamma)\frac{1}{\Phi^H}\right) = \left((1-\Gamma)\frac{1}{\Phi^H}\right)^2 - \left(\frac{1}{\Phi^H} + \frac{1}{\Phi^L}\right)(1-\Gamma)\frac{1}{\Phi^H} +$

$\frac{1}{\Phi^H}\frac{1}{\Phi^L}(1-\Gamma) = -\Gamma(1-\Gamma)\left(\frac{1}{\Phi^H}\right)^2 < 0$, $q(\omega_2) = 0$, and $q\left(\frac{1}{\Phi^H}\right) = \left(\frac{1}{\Phi^H}\right)^2 - \left(\frac{1}{\Phi^H} + \frac{1}{\Phi^L}\right)\left(\frac{1}{\Phi^H}\right) + \frac{1}{\Phi^H}\frac{1}{\Phi^L}(1-\Gamma) = -\frac{1}{\Phi^H}\frac{1}{\Phi^L}\Gamma > 0$, it follows that $\frac{1}{\Phi^L} < \omega_1 < 0 < \omega_2 < \frac{1}{\Phi^H}$ (Statement 1). The sum of the roots of $q(\omega) = 0$ is the negative of the coefficient on the linear term in ω. Therefore, $\omega_1 + \omega_2 = trA = \frac{1}{\Phi^H} + \frac{1}{\Phi^L} < 0$ (Statement 2). The product of the roots is the constant term in $q(\omega)$, which is $\det A = \frac{1}{\Phi^H}\frac{1}{\Phi^L}(1-\Gamma) < 0$ (Statement 3). Finally, $\frac{1}{\omega_1} + \frac{1}{\omega_2} = \frac{\omega_1+\omega_2}{\omega_1\omega_2} = \frac{trA}{\det A} = \frac{\Phi^H+\Phi^L}{1-\Gamma}$ (Statement 4). ■

Proof of Proposition 7. Assume that $\Phi^L + \Gamma\Phi^H \geq 0$. Evaluate the first row of equation (18) at $X = X^*$ to obtain $V^H(X^*) = c_1 + c_2$. Use the expressions for c_1 and c_2 from equations (A.6) and (A.7), respectively, to obtain $V^H(X^*) = \frac{1}{\omega_2-\omega_1}\left(\frac{\omega_2}{\omega_1} - \frac{\omega_1}{\omega_2}\right) = \frac{\omega_2^2-\omega_1^2}{\omega_2-\omega_1}\frac{1}{\omega_1\omega_2}$ $= \frac{\omega_1+\omega_2}{\omega_1\omega_2} = \frac{1}{\omega_1} + \frac{1}{\omega_2}$. Use Statement 4 of Lemma 4, $\frac{1}{\omega_1} + \frac{1}{\omega_2} = \frac{1}{1-\Gamma}\left(\Phi^H + V_L^H\right) > 0$, to obtain $V^H(X^*) = \frac{1}{1-\Gamma}\left(\Phi^H + V_L^H\right)$. Rewrite the ODE in equation (8), setting $r = 0$, as $\left(\rho + \mu^L\right)V^H(X) = \phi^H V_X^H(X) + \mu^L V^L(X)$. Evaluate this ODE at $X = X^*$, use the boundary condition $V_X^H(X^*) = 1$, and divide both sides of the equation by $\rho + \mu^L$ using the definition of the myopic value $\Phi^H \equiv \frac{\phi^H}{\rho+\mu^L}$ to obtain $V^H(X^*) = \Phi^H + \frac{\mu^L}{\rho+\mu^L}V^L(X^*)$. Therefore, $V^L(X^*) = \frac{\rho+\mu^L}{\mu^L}\left(V^H(X^*) - \Phi^H\right) = \frac{\rho+\mu^L}{\mu^L}\left(\frac{1}{1-\Gamma}\left(\Phi^H + \Phi^L\right) - \Phi^H\right) = \frac{\rho+\mu^L}{\mu^L}\frac{1}{1-\Gamma}\left(\Gamma\Phi^H + \Phi^L\right) = \frac{1}{1-\Gamma}\left(\frac{\mu^H}{\rho+\mu^H}\Phi^H + \frac{\rho+\mu^L}{\mu^L}\Phi^L\right) < \frac{1}{1-\Gamma}\left(\Phi^H + \Phi^L\right) = V^H(X^*)$. ■

Proof of Proposition 9. The eigenvalues ω_1 and ω_2 satisfy

$$\omega^2 - T\omega + \det(A) = 0 \tag{B.2}$$

where

$$T \equiv tr(A) = \frac{1}{\Phi^L} + \frac{1}{\Phi^H} < 0 \tag{B.3}$$

and

$$\det(A) = \frac{1}{\Phi^L}\frac{1}{\Phi^H}(1-\Gamma) < 0. \tag{B.4}$$

Observe that

$$\lim_{\Phi^L \nearrow 0} T = -\infty \tag{B.5}$$

and

$$\lim_{\Phi^L \nearrow 0} \Phi^L T = 1.$$

The eigenvalues are the roots of the characteristic equation (B.2)

$$\omega_i = \frac{1}{2}\left(T \pm \sqrt{T^2 - 4\det(A)}\right), \tag{B.6}$$

which can be rewritten as

$$\omega_i = \frac{1}{2}T\left(1 \mp \sqrt{1 - 4\frac{1}{T}\frac{\det(A)}{T}}\right). \tag{B.7}$$

Define

$$Y \equiv \frac{T}{\det(A)} = \frac{\frac{1}{\Phi^L} + \frac{1}{\Phi^H}}{\frac{1}{\Phi^L}\frac{1}{\Phi^H}(1-\Gamma)} = \frac{1}{1-\Gamma}\left(\Phi^H + \Phi^L\right) > 0. \tag{B.8}$$

Equation (B.8) implies that

$$\lim_{\Phi^L \nearrow 0} Y = \frac{1}{1-\Gamma}\lim_{\Phi^L \nearrow 0} \Phi^H > 0 \tag{B.9}$$

is finite. Substitute equation (B.8) into equation (B.7) to obtain

$$\omega_i = \frac{1}{2}T\left(1 \mp \sqrt{1 - 4\frac{1}{T}\frac{1}{Y}}\right) \tag{B.10}$$

so

$$\omega_2 - \omega_1 = -T\sqrt{1 - 4\frac{1}{T}\frac{1}{Y}} \tag{B.11}$$

and

$$\omega_2 = \frac{1}{2}\left[(\omega_1 + \omega_2) + \omega_2 - \omega_1\right] = \frac{1}{2}\left[T + (\omega_2 - \omega_1)\right]. \tag{B.12}$$

Observe that

$$\lim_{\Phi^L \nearrow 0} \sqrt{1 - 4\frac{1}{T}\frac{1}{Y}} = 1. \tag{B.13}$$

Therefore,

$$\lim_{\Phi^L \nearrow 0} (\omega_2 - \omega_1) = \lim_{\Phi^L \nearrow 0} -T\sqrt{1 - 4\frac{1}{T}\frac{1}{Y}} = -\lim_{\Phi^L \nearrow 0} T = \infty \tag{B.14}$$

and

$$\lim_{\Phi^L \nearrow 0} -\Phi^L(\omega_2 - \omega_1) = \lim_{\Phi^L \nearrow 0} \Phi^L T\sqrt{1 - 4\frac{1}{T}\frac{1}{Y}} = \lim_{\Phi^L \nearrow 0} \Phi^L T = 1. \tag{B.15}$$

Recall the characteristic equation $q(\omega) \equiv \omega^2 - T\omega + \det(A) = 0$, which can be rewritten as

$$q(\omega) \equiv \omega^2 - \omega\left(\frac{1}{\Phi^L} + \frac{1}{\Phi^H}\right) + \frac{1}{\Phi^L}\frac{1}{\Phi^H}(1-\Gamma) = 0. \tag{B.16}$$

Multiply both sides of equation (B.16) by $\Phi^L\Phi^H$ to obtain

$$\Phi^L\Phi^H\omega^2 - \omega\left(\Phi^H + \Phi^L\right) + 1 - \Gamma = 0. \tag{B.17}$$

Consider the limit as $\Phi^L \nearrow 0$. In this limit, the characteristic equation is $-\omega\Phi^H + 1 - \Gamma = 0$, so

$$\lim_{\Phi^L \nearrow 0} \omega_2 = \frac{1-\Gamma}{\lim_{\Phi^L \nearrow 0} \Phi^H} > 0 \tag{B.18}$$

is finite. Use equation (B.18) and the definition of Ω to obtain

$$\begin{aligned} \lim_{\Phi^L \nearrow 0} \Omega &= \lim_{\Phi^L \nearrow 0} \frac{\Phi^L + \Gamma\Phi^H}{1-\Gamma-(\Phi^L+\Gamma\Phi^H)\,\omega_2} = \frac{\Gamma \lim_{\Phi^L \nearrow 0} \Phi^H}{1-\Gamma-\Gamma\lim_{\Phi^L \nearrow 0}\Phi^H \frac{1-\Gamma}{\lim_{\Phi^L \nearrow 0}\Phi^H}} \\ &= \frac{\Gamma}{(1-\Gamma)^2} \lim_{\Phi^L \nearrow 0} \Phi^H > 0, \end{aligned} \tag{B.19}$$

which is finite. Recall that

$$X^* = \frac{1}{\omega_2 - \omega_1} \ln\left(1 + \Omega\left(\omega_2 - \omega_1\right)\right). \tag{B.20}$$

Since $\lim_{\Phi^L \nearrow 0} \Omega$ is finite, Ω is finite for all Φ^L in a negative neighborhood of zero, so $\Omega \le \overline{\Omega}$, where $\overline{\Omega}$ is finite, for all Φ^L in a negative neighborhood of zero. Therefore,

$$X^* \le \frac{1}{\omega_2 - \omega_1} \ln\left(1 + \overline{\Omega}\left(\omega_2 - \omega_1\right)\right). \tag{B.21}$$

Since $\lim_{\Phi^L \nearrow 0}(\omega_2 - \omega_1) = \infty$ and $\lim_{z\to\infty} \frac{1}{z}\ln(1+az) = 0$ for $a > 0$, we have

$$\lim_{\Phi^L \nearrow 0} X^* = 0. \tag{B.22}$$

Now consider the ratio of X^* to the operating loss in Regime L, $-\phi^L$,

$$\frac{X^*}{-\phi^L} = \frac{X^*}{(\rho+\mu^H)(-\Phi^L)} = \frac{1}{(\rho+\mu^H)(-\Phi^L)(\omega_2-\omega_1)} \ln\left(1+\Omega\left(\omega_2-\omega_1\right)\right). \tag{B.23}$$

Since $\lim_{\Phi^L \nearrow 0}\left(-\Phi^L\right)(\omega_2 - \omega_1) = 1$, $\lim_{\Phi^L \nearrow 0} \Omega = \frac{\Gamma}{(1-\Gamma)^2} \lim_{\Phi^L \nearrow 0} \Phi^H > 0$, and $\lim_{\Phi^L \nearrow 0}(\omega_2 - \omega_1) = \infty$, equation (B.23) implies that $\lim_{\Phi^L \nearrow 0} \frac{X^*}{-\Phi^L} = \infty$. ■

Proof of Proposition 10. 1. ($\Omega \simeq \frac{\Phi^L+\Gamma\Phi^H}{1-\Gamma}$) Consider a sequence of parameters such that in the limit Φ^L and Φ^H are bounded away from zero and $\Phi^L + \Gamma\Phi^H > 0$ is arbitrarily close to 0. Since Φ^L and Φ^H (as well as $\Gamma < 1$) are bounded away from zero, the eigenvalues ω_1 and ω_2 are bounded above and below. In particular, since ω_2 is bounded above, $\left(\Phi^L + \Gamma\Phi^H\right)\omega_2$ approaches zero, so $\frac{1-\Gamma}{1-\Gamma-(\Phi^L+\Gamma\Phi^H)\omega_2}$ approaches 1, and hence $\Omega \equiv$

$\frac{\Phi^L+\Gamma\Phi^H}{1-\Gamma-(\Phi^L+\Gamma\Phi^H)\omega_2} = \frac{\Phi^L+\Gamma\Phi^H}{1-\Gamma}\frac{1-\Gamma}{1-\Gamma-(\Phi^L+\Gamma\Phi^H)\omega_2}$ approaches $\frac{\Phi^L+\Gamma\Phi^H}{1-\Gamma} > 0$, which is arbitrarily small. 2. $(X^* \simeq \Omega)$ Since $\omega_2 - \omega_1 > 0$ is bounded from above, $\frac{\Phi^L+\Gamma\Phi^H}{1-\Gamma}(\omega_2 - \omega_1)$ approaches 0, and hence from part 1, $(\omega_2 - \omega_1)\Omega$ approaches zero so $X^* = \frac{1}{\omega_2-\omega_1}\ln(1+(\omega_2-\omega_1)\Omega) \simeq \Omega$. 3. $(\frac{\Phi^L+\Gamma\Phi^H}{1-\Gamma} \simeq \frac{\mu^L}{\rho+\mu^L}V^L(X^*))$ This statement is simply equation (20). ■

Proof of Proposition 11. Differentiate $V^L(X^*) = \frac{\rho+\mu^L}{\mu^L}\frac{1}{1-\Gamma}\left(\Phi^L + \Gamma\Phi^H\right)$ in Statement 2 of Proposition 7 to obtain $\frac{dV^L(X^*)}{d\Phi^H}\mid_{MPS} = \frac{\rho+\mu^L}{\mu^L}\frac{1}{1-\Gamma}\left(\frac{d\Phi^L}{d\Phi^H}\mid_{MPS} + \Gamma\right)$ and then use equation (22) to obtain $\frac{dV^L(X^*)}{d\Phi^H}\mid_{MPS} = \frac{\rho+\mu^L}{\mu^L}\frac{1}{1-\Gamma}\left(-\frac{\mu^H\left(\rho+\mu^L\right)}{\mu^L(\rho+\mu^H)} + \frac{\mu^H}{\rho+\mu^H}\frac{\mu^L}{\rho+\mu^L}\right) = \frac{1}{1-\Gamma}\frac{\mu^H}{\rho+\mu^H}\left[1-\left(\frac{\rho+\mu^L}{\mu^L}\right)^2\right]$ (Statement 1). Differentiate $V^H(X^*) = \frac{1}{1-\Gamma}\left(\Phi^H + \Phi^L\right)$ in Statement 1 of Proposition 7 to obtain $\frac{dV^H(X^*)}{d\Phi^H}\mid_{MPS} = \frac{1}{1-\Gamma}\left(1-\frac{\mu^H\left(\rho+\mu^L\right)}{\mu^L(\rho+\mu^H)}\right) = \frac{1}{1-\Gamma}\frac{\rho}{\rho+\mu^H}\left(1-\frac{\mu^H}{\mu^L}\right)$ (Statement 2). ■

Proof of Lemma 3. Express ϕ^H and ϕ^L in terms of the mean of the unconditional distribution, $m \equiv \frac{\mu^H\phi^H+\mu^L\phi^L}{\mu^H+\mu^L}$, and the standard deviation of the unconditional distribution, $\sigma \equiv \sqrt{\frac{\mu^H(\phi^H-m)^2+\mu^L(\phi^L-m)^2}{\mu^H+\mu^L}}$, as $\phi^H = m+\sigma\sqrt{\frac{\mu^L}{\mu^H}}$ and $\phi^L = m-\sigma\sqrt{\frac{\mu^H}{\mu^L}}$. Statement 1 follows immediately from the expression for ϕ^L that $\phi^L < 0$ if and only if $m < \sigma\sqrt{\frac{\mu^H}{\mu^L}}$, which is equivalent to $\theta = \frac{\sigma}{m} > \sqrt{\frac{\mu^L}{\mu^H}} \equiv \theta_A$. To prove statement 2, observe that $\Phi^L + \frac{\mu^H}{\rho+\mu^H}\Phi^H > 0$ if and only if $\frac{\phi^L}{\rho+\mu^H} + \frac{\mu^H}{\rho+\mu^H}\frac{\phi^H}{\rho+\mu^L} > 0$ if and only if $m-\sigma\sqrt{\frac{\mu^H}{\mu^L}} + \frac{\mu^H}{\rho+\mu^L}\left(m+\sigma\sqrt{\frac{\mu^L}{\mu^H}}\right) > 0$ if and only if $\frac{\rho+\mu^L+\mu^H}{\rho+\mu^L}m > \sigma\sqrt{\frac{\mu^H}{\mu^L}} - \frac{\mu^H}{\rho+\mu^L}\sigma\sqrt{\frac{\mu^L}{\mu^H}} = \left(1-\frac{\mu^L}{\rho+\mu^L}\right)\sigma\sqrt{\frac{\mu^H}{\mu^L}} = \frac{\rho}{\rho+\mu^L}\sigma\sqrt{\frac{\mu^H}{\mu^L}}$ if and only $\theta_C \equiv \left(1+\frac{\mu^L+\mu^H}{\rho}\right)\sqrt{\frac{\mu^L}{\mu^H}} > \frac{\sigma}{m}$. ■

Proof of Proposition 8. First consider values of the coefficient of variation less than or equal to $\sqrt{\frac{\mu^L}{\mu^H}}$, so that $\sigma \le m\sqrt{\frac{\mu^L}{\mu^H}}$, which implies $\phi^L = m-\sigma\sqrt{\frac{\mu^H}{\mu^L}} \ge 0$. In this case, the cash flow from operations is always non-negative, so there is no need for the firm to hold any cash; that is, $X^* = 0$ if the coefficient of variation is less than or equal to $\sqrt{\frac{\mu^L}{\mu^H}}$. Now consider admissible values of the coefficent of variation greater than or equal to $\sqrt{\frac{\mu^L}{\mu^H}}$. Lemma 2 states that the sign of Ω is the same as the sign of $\Gamma\Phi^H + \Phi^L$. Therefore X^*, which has the same sign as Ω, will be positive if and only if $\Gamma\Phi^H > -\Phi^L$, which can be written as $\left(\frac{\mu^L}{\rho+\mu^L}\frac{\mu^H}{\rho+\mu^H}\right)\frac{\phi^H}{\rho+\mu^L} > -\frac{\phi^L}{\rho+\mu^H}$ and simplified to $\mu^L\mu^H\phi^H > -\left(\rho+\mu^L\right)^2\phi^L$. Use $\phi^H = m+\sigma\sqrt{\frac{\mu^L}{\mu^H}}$ and $\phi^L = m-\sigma\sqrt{\frac{\mu^H}{\mu^L}}$ to rewrite this condition as $\mu^L\mu^H\left(m+\sigma\frac{\mu^L}{\mu^H}\sqrt{\frac{\mu^H}{\mu^L}}\right) > \left(\rho+\mu^L\right)^2\left(-m+\sigma\sqrt{\frac{\mu^H}{\mu^L}}\right)$. Divide both sides of this expression by $m > 0$ and rearrange to obtain $\mu^L\mu^H + \left(\rho+\mu^L\right)^2 > \left(\left(\rho+\mu^L\right)^2 - \left(\mu^L\right)^2\right)\frac{\sigma}{m}\sqrt{\frac{\mu^H}{\mu^L}}$, which implies $\mu^L\mu^H + \rho^2 + 2\rho\mu^L + \left(\mu^L\right)^2 > \left(\rho^2 + 2\rho\mu^L\right)\frac{\sigma}{m}\sqrt{\frac{\mu^H}{\mu^L}}$. Divide both sides of this condition by ρ^2 to obtain

$\frac{\mu^L}{\rho}\frac{\mu^H}{\rho}+1+2\frac{\mu^L}{\rho}+\left(\frac{\mu^L}{\rho}\right)^2>\left(1+2\frac{\mu^L}{\rho}\right)\frac{\sigma}{m}\sqrt{\frac{\mu^H}{\mu^L}}$. Now divide both sides of the condition by $\left(1+2\frac{\mu^L}{\rho}\right)$ to obtain $\frac{\frac{\mu^H}{\rho}+\frac{\mu^L}{\rho}}{\frac{\rho}{\mu^L}+2}+1>\frac{\sigma}{m}\sqrt{\frac{\mu^H}{\mu^L}}$ which implies $\frac{\sigma}{m}\sqrt{\frac{\mu^H}{\mu^L}}<1+\left(2+\frac{\rho}{\mu^L}\right)^{-1}\left(\frac{\mu^H}{\rho}+\frac{\mu^L}{\rho}\right)$. Thus, $X^*>0$ if and only if $\frac{\sigma}{m}<\left[1+\left(2+\frac{\rho}{\mu^L}\right)^{-1}\left(\frac{\mu^H}{\rho}+\frac{\mu^L}{\rho}\right)\right]\sqrt{\frac{\mu^L}{\mu^H}}$. ■

Proof of Proposition 12. Since $X^*(\theta)$ is differentable with respect to θ on the interval (θ_A,θ_B), $\lim_{\varepsilon>0\to 0}\int_{\theta_A+\varepsilon}^{\theta_B-\varepsilon}X^{*\prime}(\theta)\,d\theta=\lim_{\varepsilon>0\to 0}\left[X^*(\theta_B-\varepsilon)-X^*(\theta_A+\varepsilon)\right]=0$, where the final equality follows from $\lim_{\varepsilon>0\to 0}X^*(\theta_A+\varepsilon)=0$ and $\lim_{\varepsilon>0\to 0}X^*(\theta_B-\varepsilon)=0$. ■

Proof of Proposition 13.

The moment-generating function of τ is defined as

$$f(X_t;\lambda)\equiv E_t\left(e^{-\lambda(\tau-t)}\right).$$

Since $e^{-\lambda t}f(X_t;\lambda)$ is a martingale, the function $f(X_t;\lambda)$ satisfies the same ODEs as the value function $V(X_t)$ with ρ replaced by λ. It is subject to the boundary conditions:

$$f_X^H(X^*)=0, \tag{B.24}$$
$$f^L(0)=1. \tag{B.25}$$

The boundary condition (B.24) indicates that X is reflected at X^* (so that $f^H(X^*+\varepsilon)=f^H(X^*)$ for any $\varepsilon>0$, and accordingly $f_X^H(X^*)=0$).The boundary condition evaluated at $X_t=0$ is $f^L(0;\lambda)=E_t\left(e^{-\lambda(t-t)}\right)=1$.

Letting $\Phi^{H,\lambda},\Phi^{L,\lambda},\omega_1^{(\lambda)},\omega_2^{(\lambda)}$ denote counterparts of $\Phi^H,\Phi^H,\omega_1,\omega_2$ with ρ replaced by λ, the general solution to $f(X)$ is given by

$$\begin{bmatrix} f^H(X) \\ f^L(X)\end{bmatrix}=c_1^{(\lambda)}\begin{bmatrix} 1 \\ \frac{\lambda+\mu^L}{\mu^L}\left(1-\omega_1^{(\lambda)}\Phi^{H,\lambda}\right)\end{bmatrix}e^{\omega_1^{(\lambda)}(X-X^*)}+c_2^{(\lambda)}\begin{bmatrix} 1 \\ \frac{\lambda+\mu^L}{\mu^L}\left(1-\omega_2^{(\lambda)}\Phi^{H,\lambda}\right)\end{bmatrix}e^{\omega_2^{(\lambda)}(X-X^*)},$$

where X^* denotes the target level of cash of the firm.

Imposing the two boundary conditions (B.24) and (B.25) gives

$$c_2^{(\lambda)}=-c_1^{(\lambda)}\frac{\omega_1^{(\lambda)}}{\omega_2^{(\lambda)}}$$

and

$$c_1^{(\lambda)} = \frac{1}{\left[\frac{\lambda+\mu^L}{\mu^L}\left(1-\omega_1^{(\lambda)}\Phi^{H,\lambda}\right)\right]e^{-\omega_1^{(\lambda)}X^*} - \frac{\omega_1^{(\lambda)}}{\omega_2^{(\lambda)}}\left[\frac{\lambda+\mu^L}{\mu^L}\left(1-\omega_2^{(\lambda)}\Phi^{H,\lambda}\right)\right]e^{-\omega_2^{(\lambda)}X^*}}.$$

Accordingly, the moment generating function for a newly born firm, which has zero assets and finds itself in Regime H is given (after a few simplifications)

$$f^H(0;\lambda) = \frac{\mu^L}{\lambda+\mu^L}\frac{\omega_2^{(\lambda)} - \omega_1^{(\lambda)}e^{-\left(\omega_2^{(\lambda)}-\omega_1^{(\lambda)}\right)X^*}}{\omega_2^{(\lambda)}\left(1-\omega_1^{(\lambda)}\Phi^{H,\lambda}\right) - \omega_1^{(\lambda)}\left(1-\omega_2^{(\lambda)}\Phi^{H,\lambda}\right)e^{-\left(\omega_2^{(\lambda)}-\omega_1^{(\lambda)}\right)X^*}} \tag{B.26}$$

The life expectancy of a newly born firm, which has zero assets at birth, is therefore

$$E(\tau) = -\frac{df^H(0;0)}{d\lambda}. \tag{B.27}$$

To calculate the right hand side of equation (B.27), define $A^{(\lambda)}$ as in equation (17), but with λ replacing ρ. Similarly let $\Gamma^{(\lambda)}$ be the counterpart of Γ, but with λ replacing ρ. We note that $\Gamma^{(0)} = 1$, and accodrdingly $\det\left(A^{(0)}\right) = 0$. Hence, the roots $\omega_1^{(0)}$ and $\omega_2^{(0)}$ in equation (B.26) are given by

$$\omega_1^{(0)} = \frac{1}{\Phi^{H,0}} + \frac{1}{\Phi^{L,0}} = \frac{\phi^L\mu^L + \phi^H\mu^H}{\phi^L\phi^H}, \quad \omega_2^{(0)} = 0.$$

The implicit function theorem applied to equation (A.1) gives

$$\begin{aligned}
\frac{d\omega_2^{(0)}}{d\lambda}|_{\lambda=0} &= \frac{\frac{d\det\left(A^{(0)}\right)}{d\lambda}}{tr\left(A^{(0)}\right)} = \frac{-\frac{1}{\Phi^{H,0}\Phi^{L,0}}\frac{d\Gamma^{(0)}}{d\lambda}}{\frac{\phi^L\mu^L+\phi^H\mu^H}{\phi^L\phi^H}} = \frac{\frac{\mu^H\mu^L}{\phi^H\phi^L}\left(\frac{1}{\mu^H}+\frac{1}{\mu^L}\right)}{\frac{\phi^L\mu^L+\phi^H\mu^H}{\phi^L\phi^H}} \\
&= \frac{\mu^H+\mu^L}{\phi^L\mu^L+\phi^H\mu^H}.
\end{aligned}$$

Next, define

$$\begin{aligned}
B_1(\lambda) &\equiv \omega_2^{(\lambda)} - \omega_1^{(\lambda)}e^{-\left(\omega_2^{(\lambda)}-\omega_1^{(\lambda)}\right)X^*}, \\
B_2(\lambda) &\equiv \omega_2^{(\lambda)}\left(1-\omega_1^{(\lambda)}\Phi^{H,\lambda}\right) - \omega_1^{(\lambda)}\left(1-\omega_2^{(\lambda)}\Phi^{H,\lambda}\right)e^{-\left(\omega_2^{(\lambda)}-\omega_1^{(\lambda)}\right)X^*},
\end{aligned}$$

so that $f^H(0;\lambda)$ can be written more compactly as

$$f^H(0;\lambda) = \frac{\mu^L}{\lambda+\mu^L}\frac{B_1(\lambda)}{B_2(\lambda)}.$$

Since $\omega_2^{(0)} = 0$, we have that $B_1(0) = B_2(0) = -\omega_1^{(0)} e^{\omega_1^{(0)} X^*}$. Accordingly,

$$\frac{d}{d\lambda}\left(f^H(0;0)\right) = -\frac{1}{\mu^L} + \frac{1}{B_2(0)}\left(\frac{dB_1(0)}{d\lambda} - \frac{dB_2(0)}{d\lambda}\right). \tag{B.28}$$

Note that $B_1(\lambda) - B_2(\lambda) = \omega_2^{(\lambda)}\omega_1^{(\lambda)}\Phi^{H,\lambda}\left(1 - e^{-\left(\omega_2^{(\lambda)} - \omega_1^{(\lambda)}\right)X^*}\right)$ so that using $\omega_2^{(0)} = 0$ gives

$$\frac{dB_1(0)}{d\lambda} - \frac{dB_2(0)}{d\lambda} = \frac{d\omega_2^{(0)}}{d\lambda}\omega_1^{(0)}\Phi^{H,0}\left(1 - e^{\omega_1^{(0)} X^*}\right) \tag{B.29}$$

Combining $B_1(0) = B_2(0) = -\omega_1^{(0)} e^{\omega_1^{(0)} X^*}$ with (B.28) and (B.29) leads to

$$\begin{aligned}
\frac{d}{d\lambda}\left(f^H(0;0)\right) &= -\frac{1}{\mu^L} - \frac{\omega_1^{(0)}\Phi^{H,0}\frac{d\omega_2^{(0)}}{d\lambda}\left[1 - e^{\omega_1^{(0)} X^*}\right]}{\omega_1^{(0)} e^{\omega_1^{(0)} X^*}} \\
&= -\frac{1}{\mu^L} - \Phi^{H,0}\frac{d\omega_2^{(0)}}{d\lambda}\left[e^{-\omega_1^{(0)} X^*} - 1\right] \\
&= -\frac{1}{\mu^L} - \frac{\phi^H}{\mu^L}\frac{\mu^H + \mu^L}{\phi^L\mu^L + \phi^H\mu^H}\left[e^{-\frac{\phi^L\mu^L + \phi^H\mu^H}{\phi^L\phi^H}X^*} - 1\right].
\end{aligned}$$

The equation immediately above, along with equation (B.27), leads to the life expectancy in Proposition 13. ∎

Proof of Proposition 14. Let θ_0 be such that $X^{*\prime}(\theta) \geq 0$ if $\theta < \theta_0$ and $X^{*\prime}(\theta) \leq 0$ if $\theta > \theta_0$. Then $\int_{\theta_A+\varepsilon}^{\theta_B-\varepsilon} X^{*\prime}(\theta)\, g(\theta)\, d\theta = \int_{\theta_A+\varepsilon}^{\theta_0} X^{*\prime}(\theta)\, g(\theta)\, d\theta + \int_{\theta_0}^{\theta_B-\varepsilon} X^{*\prime}(\theta)\, g(\theta)\, d\theta >$ $\int_{\theta_A+\varepsilon}^{\theta_0} X^{*\prime}(\theta)\, g(\theta_0)\, d\theta + \int_{\theta_0}^{\theta_B-\varepsilon} X^{*\prime}(\theta)\, g(\theta_0)\, d\theta = g(\theta_0)\int_{\theta_A+\varepsilon}^{\theta_B-\varepsilon} X^{*\prime}(\theta)\, d\theta =$
$= g(\theta_0)\left[X^*(\theta_B - \varepsilon) - X^*(\theta_A + \varepsilon)\right]$. Since $\lim_{\varepsilon>0\to 0} X^*(\theta_B - \varepsilon) = 0 = \lim_{\varepsilon>0\to 0} X^*(\theta_A + \varepsilon)$, we have $\lim_{\varepsilon>0\to 0}\int_{\theta_A+\varepsilon}^{\theta_B-\varepsilon} X^{*\prime}(\theta)\, g(\theta)\, d\theta > 0$.] ∎

C Concavity of the Value Function

In this appendix we prove that the value functions $V^H(X)$ and $V^L(X)$ are strictly concave for $0 \leq X < X^*$, where X^* is the lowest positive value of X such that $V_X^H(X) \leq 1$. We confine attention to the case in which $V_X^H(0) > 1$, since if $V_X^H(0) \leq 1$, the target level of cash on hand in Regime H is zero and the set of X that satisfy $0 \leq X < X^*$ is empty.) The HJBs in equations (8) and (9) are repeated below for convenience

$$\left(\phi^H + rX\right)V_X^H + \mu^L V^L - \left(\rho + \mu^L\right)V^H = 0, \tag{C.1}$$

$$\left(\phi^L + rX\right)V_X^L + \mu^H V^H - \left(\rho + \mu^H\right)V^L = 0. \tag{C.2}$$

First, we prove that $V_{XX}^{H}(X) < 0$ for all X such that $0 \leq X < X^{*}$. Suppose otherwise. Then there exists some non-negative $X_1 < X^{*}$ such that $V_{XX}^{H}(X_1) = 0$. Differentiating (C.1) gives

$$\left(\phi^{H} + rX\right) V_{XX}^{H} + \mu^{L} V_{X}^{L} - \left(\rho + \mu^{L} - r\right) V_{X}^{H} = 0. \tag{C.3}$$

Therefore,

$$V_{X}^{L}(X_1) = \frac{\rho + \mu^{L} - r}{\mu^{L}} V_{X}^{H}(X_1) - \frac{\phi^{H} + rX_1}{\mu^{L}} V_{XX}^{H}(X_1). \tag{C.4}$$

Differentiating (C.2) and using (C.4) and $V_{XX}^{H}(X_1) = 0$ gives after some re-arrangement

$$\frac{\phi^{L} + rX_1}{\mu^{H}} V_{XX}^{L}(X_1) - \left[\left(1 + \frac{\rho - r}{\mu^{H}}\right)\left(1 + \frac{\rho - r}{\mu^{L}}\right) - 1\right] V_{X}^{H}(X_1) = 0. \tag{C.5}$$

Since $V_{X}^{H}(X_1) > 0$, $\rho - r > 0$, and (using footnote 12) $\phi^{L} + rX < \phi^{L} + rX^{*} \leq 0$, it follows that that $V_{XX}^{L}(X_1) < 0$. Differentiating (C.3) gives

$$\left(\phi^{H} + rX_1\right) V_{XXX}^{H}(X_1) + \mu^{L} V_{XX}^{L}(X_1) - \left(\rho + \mu^{L} - 2r\right) V_{XX}^{H}(X_1) = 0. \tag{C.6}$$

Since $V_{XX}^{H}(X_1) = 0$, $\phi^{H} + rX_1 \geq \phi^{H} > 0$, and $V_{XX}^{L}(X_1) < 0$, it follows that

$$V_{XXX}^{H}(X_1) = \frac{-\mu^{L} V_{XX}^{L}(X_1)}{\phi^{H} + rX_1} > 0. \tag{C.7}$$

Hence $V_{X}^{H}(X)$ attains a local minimum at X_1. If there were a local maximum of $V_{X}^{H}(X)$ anywhere in $[X_1, X^{*}]$, then $V_{XX}^{H}(X)$ would be zero and $V_{XXX}^{H}(X)$ would be negative at that value of X. However, we have shown that if $V_{XX}^{H}(X) = 0$, then $V_{XXX}^{H}(X) > 0$, so no such local maximum exists. Therefore, $V_{X}^{H}(X)$ is increasing over the interval $[X_1, X^{*}]$, so $V_{X}^{H}(X_1) \leq V_{X}^{H}(X^{*}) \leq 1$, which contradicts fact that X^{*} is (by definition) the smallest value of X such that $V_{X}^{H}(X_1) \leq 1$. Therefore, $V_{XX}^{H}(X) < 0$ for all X such that $0 \leq X < X^{*}$.

Now we prove that $V_{XX}^{L}(X) < 0$ for all X such that $0 \leq X < X^{*}$. Suppose, contrary to what is to be proved, that $V_{XX}^{L}(X_2) \geq 0$ at some $0 \leq X_2 < X^{*}$. Differentiating (C.2) and rearranging gives

$$V_{X}^{L}(X_2) = \frac{\mu^{H}}{\rho + \mu^{H} - r} V_{X}^{H}(X_2) + \frac{\phi^{L} + rX_2}{\rho + \mu^{H} - r} V_{XX}^{L}(X_2). \tag{C.8}$$

In addition, for that level of X_2, differentiating (C.1) gives

$$\left(\phi^{H} + rX_2\right) V_{XX}^{H}(X_2) + \mu^{L} V_{X}^{L}(X_2) - \left(\rho + \mu^{L} - r\right) V_{X}^{H}(X_2) = 0. \tag{C.9}$$

Substitute the expression for $V_X^L(X_2)$ from equation (C.8) into equation (C.9) and multiply both sides by $\frac{\rho+\mu^H-r}{\mu^L\mu^H}$ to obtain

$$\left[\begin{array}{c}\frac{\rho+\mu^H-r}{\mu^L\mu^H}\left(\phi^H+rX_2\right)V_{XX}^H\left(X_2\right)\\-\left[\left(1+\frac{\rho-r}{\mu^H}\right)\left(1+\frac{\rho-r}{\mu^L}\right)-1\right]V_X^H\left(X_2\right)\end{array}\right]=-\frac{\phi^L+rX_2}{\mu^H}V_{XX}^L\left(X_2\right)\geq 0, \tag{C.10}$$

where the inequality follows from $\phi^L+rX_2\leq 0$ and $V_{XX}^L(X_2)\geq 0$. Since $V_X^H(X_2)>0$, and as we proved earlier $V_{XX}^H(X)<0$, it must be the case that

$$\frac{\rho+\mu^H-r}{\mu^L\mu^H}\left(\phi^H+rX_2\right)V_{XX}^H\left(X_2\right)-\left[\left(1+\frac{\rho-r}{\mu^H}\right)\left(1+\frac{\rho-r}{\mu^L}\right)-1\right]V_X^H\left(X_2\right)<0, \tag{C.11}$$

which contradicts equation (C.10). Therefore, $V_{XX}^L(X)<0$ for all X such that $0\leq X<X^*$.

D The stationary distribution of volatility

Let $N(\theta)$ denote the steady-state mass of firms with a given coefficient of variation of cash flows equal to θ, for $\theta_A<\theta<\theta_C$. Firms have potentially different values of θ, but all firms have identical values of μ^H, μ^L, ρ, and identical unconditional mean cash flow $m=\frac{\mu^H\phi^H+\mu^L\phi^L}{\mu^H+\mu^L}>0$. At each point of time, a new cohort of firms is born with zero cash on hand and a given value of θ. At birth, the mass of entering forms is normalized to one for each θ; therefore, the distribution of θ across firms is uniform at birth.

To calculate the stationary distribution of θ across firms, $N(\theta)$, let $G(T-s;\theta)$ denote the fraction of firms that were born at time s that are still alive at time T. Assuming that at each point in time there is a continuum of identical firms born with a given coefficient of variation, θ, the law of large numbers along with the assumption that the transitions between high and low regimes are idiosyncratic implies that

$$G(T-s;\theta)=\int_{T-s}^{\infty}g(\tau;\theta)\,d\tau,$$

where $g(\tau;\theta)$ is the distribution of the termination time for a firm born with zero assets. The steady-state mass of firms in existence is therefore $N(\theta)=\int_{-\infty}^{T}G(T-s;\theta)\,ds=\int_0^\infty G(u;\theta)\,du$, where the second equality used the substitution $u=T-s$. Since $G'(u;\theta)=-g(u;\theta)$

$$N(\theta)=\int_0^\infty G(u;\theta)\,du=-uG(u;\theta)\,|_0^\infty+\int_0^\infty ug(u;\theta)\,du=\int_0^\infty \tau g(\tau;\theta)\,d\tau=E(\tau;\theta).$$

Therefore, the stationary distribution of θ across firms is given by $\frac{N(\theta)}{\int_{\theta_L}^{\theta_H} N(\theta) d\theta}$.