STOCK PRICES UNDER TIME-VARYING DIVIDEND RISK:
AN EXACT SOLUTION IN AN INFINITE-HORIZON
GENERAL EQUILIBRIUM MODEL

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## ABSTRACT

The effects on asset prices of changes in risk are studied in a general
equilibrium model in which the conditional risk evolves stochastically over
time.  The savings decisions of consumers take account of the fact that
conditional risk is a serially correlated random variable.  By restricting the
specification of consumers' preferences and the stochastic specification of
dividends, it is possible to obtain an exact solution for the prices of the
aggregate stock and riskless one-period bonds.  An increase in the conditional
risk reduces the stock price if and only if the elasticity marginal utility is
less than one.

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The relation between risk and the prices of individual stocks has long been the subject of both theoretical and empirical research in financial economics. Recently, however, interest has arisen in the relation between risk and the pricing of the aggregate, or market, portfolio. In its barest form the question is: What is the effect of an increase in risk on the price of the market portfolio? In a recent paper, Pindyck (1984) examines monthly returns for the New York Stock Exchange Index from the CRSP tape and concludes that a substantial portion of the fall in the stock market during the 1970's was due to an increase in risk. However, Pindyck's finding that increased risk was a substantial contributor to the stock market decline has been challenged on both theoretical and empirical grounds. Poterba and Summers (1986) have argued that the effect of increased risk on stock prices depends on the persistence of the increased risk. They used daily data to estimate the variance of stock returns and then estimated the degree of persistence in their constructed series for the variance. They found that changes in variance were not persistent enough to account for the magnitude of the observed decline in the stock market.

The empirical studies conducted by Pindyck (1984) and Poterba and Summers (1986) were partial equilibrium in nature. In particular, neither study examined the effect of an increase in risk on the equilibrium riskless rate of return. In an insightful paper, Barsky (1986) applies the Lucas (1978) asset pricing model to examine the effects of an increase in the variance of dividends. He finds that an increase in the variance of dividends is likely to increase the risk premium on equities vis-a-vis riskless assets as argued by Pindyck, but would also tend to reduce the riskless rate of return as portfolio holders attempt to substitute away from riskier equities toward riskless assets. The net effect of an increase in risk on the required rate

of return on equities depends on whether the increase in the risk premium is greater or less than the decrease in the riskless rate of return. Barsky analyzes several parametric examples in which he can determine conditions for stock prices to rise unambiguously or to fall unambiguously in response to an increase in risk. Barsky's major finding may be summarized as follows: If investors are not very averse to intertemporal substitution, then an increase in the variability of dividends reduces stock prices; however, if investors' utility functions display more aversion to intertemporal substitution than is displayed by logarithmic utility, then an increase in dividend variability will increase stock prices.

Barsky's analysis is appropriate for comparing stock prices in two economies that are identical in every respect except that the variance of dividends is higher in one economy than in the other. However, this analysis is, strictly speaking, not appropriate for analyzing the time-series behavior of stock prices in an economy with a time-varying volatility of dividends. A more appropriate framework would be to include time-varying volatilities in the specification of the model and to examine equilibrium price behavior in a market in which consumers understand that volatility can change over time. I pursue this approach in this paper.

Though motivated by empirical analyses of stock prices and risk, this paper is not intended to provide an explanation of any particular episode of stock market behavior. Its intended contribution, which is theoretical, is twofold. First, it provides explicit solutions for the prices of riskless bonds and the aggregate stock portfolio in a competitive general equilibrium model in which the conditional mean and conditional variance of the underlying dividend process evolve stochastically over time. By including variation in the conditional distribution of dividends in the model specification, one can

meaningfully ask what is the equilibrium response of any particular asset price to a change in the conditional distribution of dividends. The specific application of this model to analyze the response of asset prices to an increase in risk is the second intended contribution of the paper.

The strategy followed in this paper is to parameterize a Lucas (1978) asset pricing model and then to derive an exact solution for the stock price as a function of the appropriate state variables. The price function is a solution to a functional equation and in general, functional equations do not have closed form solutions. In order to obtain exact solutions, it is often necessary to restrict preferences and/or technology quite a bit. For example, Michener (1982) restricted preferences to be logarithmic and obtained a closed-form solution for stock prices. Alternatively, the dividend process could be assumed to be i.i.d. and this restriction could be exploited to obtain a solution for stock prices. However, neither of these restrictions is appropriate here. Because many of Barsky's results depend precisely on whether the elasticity of marginal utility is greater or less than one, logarithmic preferences, which have a unitary elasticity of marginal utility, would not be rich enough to capture the scope of Barsky's findings. As for i.i.d. dividends, I have relaxed both the independence assumption and the assumption that dividends are identically distributed over time. Relaxing the assumption of identically (conditionally) distributed dividends is an absolute prerequisite for the study of the joint time-series behavior of stock prices and the riskiness of dividends in a rational expectations model. I chose to relax the assumption of independently distributed dividends not because of the exigencies of logical consistency but because of the overwhelming evidence in the data that dividends are highly serially correlated. Furthermore, as I

show in section VI, relaxation of the i.i.d. assumption is necessary if we are
to model the imperfect correlation of stock prices and bond prices.

Although the basic model of asset pricing used in this paper is
essentially the Lucas (1978) model of asset pricing, the development of a
general equilibrium model with stochastically-varying conditional volatility
is new.[1] The behavior of asset prices under stochastically-varying
conditional risk has recently been examined in general equilibrium models by
Gennotte and Marsh (1987) and Giovannini (1987).[2] Gennotte and Marsh develop
a continuous-time model of a nonmonetary economy that is similar in many
respects to the model presented below. They specify dividends to have a
constant instantaneous expected growth rate and allow the variance of the
growth rate of dividends to vary stochastically. This specification differs
from the stochastic specification presented below and thus the equilibrium
price function derived by Gennotte and Marsh differs from the equilibrium
price function that I derive. Nevertheless, Gennotte and Marsh's substantive
results about the effects of risk on stock prices are qualitatively the same
as my results, and are consistent with those presented by Barsky.

Whereas the papers discussed above examine the behavior of asset prices
in nonmonetary economies, Giovannini (1987) analyzes asset prices in a
monetary economy. In particular, he introduces money into a Lucas asset
pricing model by specifying a cash-in-advance constraint. This extension
allows him to analyze nominal as well as real returns and to study the role of
liquidity in determining asset prices. There are two important differences
between Giovannini's results and the results that I derive below. First, in
analyzing the effects of increased risk on the price of stocks, Giovannini's
results are diametrically opposite the results in this paper (and also
diametrically opposite Barsky's (1986) results). He explains this difference

-5-

as a consequence of the role of liquidity and the nature of the cash-in-
advance constraint.  Second, Giovannini does not formally model persistence of
the volatility parameter of the conditional distribution of dividends.
Nevertheless, he concludes that the more persistent are the changes in
volatility, the smaller will be the effect of such changes on stock prices.
By contrast, I explicitly model the persistence of the volatility of the
conditional distribution of dividends and demonstrate unambiguously that the
more persistent are the changes in volatility, the greater will be the effect
of such changes on stock prices.

In the first two sections of the paper, I present a simple Lucas (1978)
asset pricing model.  The behavior of the individual consumer is modeled in
section I and equilibrium is analyzed in section II.  In section III, I
present the stochastic specification of dividends.  The dividends are
conditionally lognormal and the two parameters of the conditional distribution
of dividends are specified to evolve stochastically over time.  Having fully
specified the model in the first three sections, I derive an exact solution
for the stock price as a function of the contemporaneous dividend and the
contemporaneous values of the distributional parameters in Section IV.  Then I
use the price function in Section V to examine the effects on the equilibrium
stock price of changes in the expected growth rate and riskiness of
dividends.  In section VI, I analyze the behavior of bond prices.  I use the
explicit solutions for the prices of stocks and bonds to analyze the behavior
of the risk premium on stocks in section VII.  Concluding remarks are
presented in Section VIII.

I.  Individual Consumer Behavior

In this section I analyze the behavior of a representative infinitely-
lived consumer who allocates his resources in each period between consumption,

1.1.11

$c_t$, and saving in the form of stocks. A share of stock pays a dividend $y_t > 0$ in period t and has an (ex-dividend) price of $p_t$ at the end of period t. Let $k_t$ be the number of shares of stock the consumer holds at the end of period t, and suppose that all of the consumer's disposable resources come from his holding of stock. Therefore, the budget constraint of the representative consumer is

$$(p_t + y_t)k_{t-1} = c_t + p_t k_t \ . \tag{1}$$

The objective function of the representative consumer at time t is

$$E_t\{ \sum_{j=0}^{\infty} \beta^j u(c_{t+j})\} \ , \tag{2}$$

where $0 < \beta < 1$, $E_t\{ \}$ denotes the expectation conditional on the information set at time t, and u( ) is strictly increasing and strictly concave. In particular, suppose that marginal utility is isoelastic so that $u(c) = \frac{c^{1-\alpha} - 1}{1 - \alpha}$ with $\alpha > 0$ and $u'(c) = c^{-\alpha}$. Note that $\alpha$ is the coefficient of relative risk aversion.

The representative consumer maximizes the objective function (2) subject to the budget constraint in (1). The first-order necessary condition for this maximization problem is well-known to be

$$p_t u'(c_t) = \beta E_t\{(p_{t+1} + y_{t+1})u'(c_{t+1})\} \ . \tag{3}$$

The left hand side of (3) is the loss in utility from reducing consumption in period t by $p_t$ units in order to buy an additional share of stock. If this stock is held for one period and is sold ex-dividend in period t+1, then consumption in period t+1 can be increased by $p_{t+1} + y_{t+1}$. The right-hand side of (3) is the expected discounted increase in utility associated with this additional consumption in period t+1. Optimality requires that the

reduction in utility in period t be equal to the expected discounted increase in utility in period t+1.

## II. Equilibrium

It is convenient to adopt the normalization that the number of shares of stock is equal to the number of consumers. The only good in the economy is the non-storable dividend $y_t$, and all shares of stock pay identical dividends. Equilibrium in the goods market requires that aggregate consumption is equal to the aggregate dividend. Since there is a representative consumer, equilibrium requires that $c_t = y_t$. Using this equilibrium condition we can evaluate the consumer's first-order condition in equilibrium as

$$p_t u'(y_t) = \beta E_t\{(p_{t+1} + y_{t+1})u'(y_{t+1})\} . \tag{4}$$

Under the assumption of isoelastic marginal utility, equation (4) becomes

$$p_t y_t^{-\alpha} = \beta E_t\{(p_{t+1} + y_{t+1})y_{t+1}^{-\alpha}\} . \tag{5}$$

The next step is to derive a price function which expresses the equilibrium stock price $p_t$ as a function of the current dividend $y_t$ as well as any other state variables relevant for calculating the conditional expectation in (5). An explicit solution for the price function will, of course, depend on the distributional assumptions on the dividend process. I present these assumptions in the next section.

## III. Stochastic Specification of Dividends

Suppose that the dividend process is conditionally lognormal with a serially correlated conditional mean and a serially correlated conditional coefficient of variation. More precisely, suppose that, conditional on the information set available in period t, $\ln y_{t+1}$ is $N(m_t, s_t^2)$. Define $\mu_t$ and

1.1.11

$v_t$ to be the conditional mean and conditional coefficient of variation, respectively, of $y_{t+1}$, and observe that

$$\mu_t \equiv E_t(y_{t+1}) = \exp(m_t + \frac{1}{2} s_t^2) > 0 \qquad (6a)$$

$$v_t^2 \equiv \frac{Var_t(y_{t+1})}{\mu_t^2} = \exp(s_t^2) - 1 \geq 0 . \qquad (6b)$$

The conditional mean and conditional coefficient of variation of dividends, $\mu_t$ and $v_t$, will vary over time. They are in the information set at time t and are relevant state variables for the determination of the equilibrium stock price in period t because consumers who are making saving decisions in period t must forecast the dividend and stock price in period t+1. Although I will specify the stochastic behavior of $\mu_t$ and $v_t$, it is often convenient to use an alternative 2-parameter representation of the distribution. Define $\omega_t$ and $\theta_t$ to be

$$\omega_t \equiv \mu_t^{1-\alpha} = \exp\left[(1 - \alpha)(m_t + \frac{1}{2} s_t^2)\right] > 0 \qquad (7a)$$

$$\theta_t \equiv \left[1 + v_t^2\right]^{\alpha(\alpha-1)/2} = \exp\left[\alpha(\alpha - 1)s_t^2/2\right] > 0 . \qquad (7b)$$

and observe the following useful relation

$$E_t\left\{y_{t+1}^{1-\alpha}\right\} = \omega_t\theta_t . \qquad (8)$$

The parameter $\omega_t$ is monotonically related to the conditional mean $\mu_t$. In particular, if the relative risk aversion parameter $\alpha < 1$, then $\omega_t$ is an increasing function of $\mu_t$; if $\alpha > 1$, then $\omega_t$ is a decreasing function of $\mu_t$. Note that for logarithmic preferences ($\alpha = 1$), $\omega_t = 1$ regardless of the value of $\mu_t$.

The parameter $\theta_t$ is monotonically related to the coefficient of variation of the distribution of dividends. If $\alpha < 1$ then $\theta_t$ is a decreasing function of the coefficient of variation, but if $\alpha > 1$ then $\theta_t$ is an increasing function of the coefficient of variation. Under logarithmic preferences $\theta_t$ is identically equal to 1.

Before solving the functional equation (5) to obtain a price function, I must first specify the stochastic behavior of $\mu_t$ and $\nu_t$, which implies the stochastic behavior of $\omega_t$ and $\theta_t$. The dynamic evolution of the distributional parameters $\mu_{t+1}$ and $\nu_{t+1}$ proceeds in two steps. In the first step, it is determined whether $\mu_{t+1}$ and/or $\nu_{t+1}$ change from their respective values in period t. In the second step, new values of $\mu_{t+1}$ and/or $\nu_{t+1}$ are drawn, if it was determined in the first step that either of these distributional parameters will change between period t and period t+1. Observe from the definitions of $\omega_t$ and $\theta_t$ in (7a, b) that $\omega$ changes if and only if $\mu$ changes and that $\theta$ changes if and only if $\nu$ changes.

To be more specific about the stochastic evolution of the conditional distribution parameters, let $\rho_\omega$ be the probability that $\omega_{t+1}$ remains equal to $\omega_t$, which is equal to the probability that $\mu_{t+1}$ remains equal to $\mu_t$; let $\rho_\theta$ be the probability that $\theta_{t+1}$ remains equal to $\theta_t$, which is equal to the probability that $\nu_{t+1}$ remains equal to $\nu_t$. Let g be the probability that both $\omega_{t+1}$ and $\theta_{t+1}$ (or equivalently $\mu_{t+1}$ and $\nu_{t+1}$) remain unchanged. If the dates at which $\omega_t$ (and hence $\mu_t$) changes are independent of the dates at which $\theta_t$ (and hence $\nu_t$) changes, then $g = \rho_\omega \rho_\theta$. If there is a positive relation between the dates at which $\omega_t$ and $\theta_t$ change, then $g > \rho_\omega \rho_\theta$. In any case, the probability g must satisfy[3]

$$0 \leq g \leq \min (\rho_\omega, \rho_\theta) \tag{9}$$

The probability that both $\omega_t$ and $\theta_t$ change is equal to $1 - \rho_\omega - \rho_\theta + g$.

Now consider the second step in the stochastic evolution of the distributional parameters. If a new value of $\mu$ is drawn, it is drawn from a continuous distribution with density function $f_\mu(\mu)$; if a new value of $\nu$ is drawn, it is drawn from a continuous distribution with density function $f_\nu(\nu)$. If new values of both $\mu$ and $\nu$ are drawn in the same period, the drawings are independent.[4]

Although the stochastic specification is stated in terms of assumptions about the distributions of $\mu_t$ and $\nu_t$, the solution of the functional equation for the stock price function can conveniently be written as a function of $\omega$ and $\theta$. Define $\bar{\omega}$ and $\bar{\theta}$ to be the unconditional means of $\omega_t$ and $\theta_t$, respectively. More formally,

$$\bar{\omega} \equiv \int_0^\infty \mu^{1-\alpha} f_\mu(\mu) d\mu \quad \text{and} \quad \bar{\theta} \equiv \int_0^\infty \left[ 1 + \nu^2 \right]^{\alpha(\alpha-1)/2} f_\nu(\nu) d\nu \ .$$

The assumptions on the stochastic evolution of the distributional parameters imply that

$$E_t(\omega_{t+1}) = \rho_\omega \omega_t + (1 - \rho_\omega)\bar{\omega} \ ; \quad 0 \le \rho_\omega \le 1 \tag{10a}$$

$$E_t(\theta_{t+1}) = \rho_\theta \theta_t + (1 - \rho_\theta)\bar{\theta} \ ; \quad 0 \le \rho_\theta \le 1 \tag{10b}$$

In addition, the assumed independence of the realizations of $\mu_t$ and $\nu_t$ implies, along with (10a, b), that

$$\text{Cov}_t(\omega_{t+1}, \theta_{t+1}) = (g - \rho_\theta \rho_\omega)(\omega_t - \bar{\omega})(\theta_t - \bar{\theta}) \tag{11}$$

Note that if the dates at which $\omega_t$ and $\theta_t$ change are independent, then $g = \rho_\theta \rho_\omega$, and hence the conditional covariance in (11) is equal to zero. If

there is a positive correlation between the dates at which these two
parameters change, then $g > \rho_\theta \rho_\omega$.

The stochastic properties in (10a, b) and (11) provide enough structure
to solve the functional equation (5) to obtain the exact price function.
However, the properties in (10a, b) and (11) are stated in terms of $\omega$ and $\theta$
which involve the preference parameter $\alpha$ as well as the parameters of the
distribution of dividends. Since the preference parameter $\alpha$ is
time-invariant, the fact that (10a, b) and (11) depend on the parameter $\alpha$
should not affect the qualitative conclusions about the time-series co-
movement of the distributional parameters and the stock price. However, if
one wanted to compare stock prices across two economies with identical
evolutions of $\mu_t$ and $\nu_t$, but with different degrees of risk aversion, then it
would be necessary to state the stochastic specification solely in terms
of $\mu_t$ and $\nu_t$. Below I present an example of a stochastic specification stated
solely in terms of $\mu$ and $\nu$ that is consistent with the assumptions in (10a, b)
and (11). Those readers who are content with the stochastic specification in
(10a, b) and (11) can proceed to Section IV without loss of continuity.

Let $f_\mu(\mu)$ have the following lognormal specification

$$f_\mu(\mu) = \frac{1}{\mu S} \frac{1}{\sqrt{2\pi}} \exp\left[-(\ln \mu - M)^2/2S^2\right] \text{ for } \mu > 0$$

so that the unconditional mean value of $\mu$ is $\exp[M + 0.5S^2]$. Under this
distributional assumption it can be shown that

$$\bar{\omega} = E(\mu^{1-\alpha}) = \exp\left[(1 - \alpha)M + \tfrac{1}{2}(1 - \alpha)^2 S^2\right] \tag{12}$$

Let $f_\nu(\nu) = 2\gamma\nu(1 + \nu^2)^{-(1+\gamma)}$ where $\gamma > 1$. In this case, $1 + \nu^2$ has a Pareto
distribution with an unconditional mean value of $\gamma/(\gamma - 1)$. Under the
additional assumption that $\gamma > \alpha(\alpha - 1)/2$, it can be shown that

$$\bar{\theta} = \frac{\gamma}{\gamma - \alpha(\alpha - 1)/2} \tag{13}$$

Equations (7a) and (12) could be substituted into (10a) to obtain an expression for $E_t(\omega_{t+1})$ directly in terms of $\mu_t$, the parameters of the density function $f_\mu(\mu)$, and the preference parameter $\alpha$. Similarly, equations (7b) and (13) could be substituted into (10b) to obtain an expression for $E_t(\theta_{t+1})$ directly in terms of $\nu_t$, the parameters of the density function $f_\nu(\nu)$, and the preference parameter $\alpha$. However, it is more convenient to solve the functional equation in terms of $\omega_t$ and $\theta_t$.

## IV. An Exact Price Function

In this section I present an exact solution to the functional equation in (5) when the dividend process is as described in Section III. The functional equation (5) can be rewritten using (8) to obtain

$$p_t y_t^{-\alpha} = \beta E_t\{p_{t+1} y_{t+1}^{-\alpha}\} + \beta \omega_t \theta_t . \tag{14}$$

Suppose, as will be verified below, that the stock price $p_t$ is given by

$$p_t = p(y_t, \omega_t, \theta_t) \equiv [a + b\omega_t\theta_t + d\omega_t + e\theta_t]y_t^{\alpha} . \tag{15}$$

To verify that (15) is a solution to the functional equation (5), observe that if (15) correctly describes the stock price, then

$$E_t\{p_{t+1} y_{t+1}^{-\alpha}\} = a + bE_t\{\omega_{t+1}\theta_{t+1}\} + dE_t\{\omega_{t+1}\} + eE_t\{\theta_{t+1}\} . \tag{16}$$

The conditional expectation $E_t\{\omega_{t+1}\theta_{t+1}\}$ can be calculated using (11) and the fact that $E_t\{\omega_{t+1}\theta_{t+1}\} = E_t\{\omega_{t+1}\}E_t\{\theta_{t+1}\} + \text{Cov}_t\{\omega_{t+1}\theta_{t+1}\}$ to obtain

$$E_t\{\omega_{t+1}\theta_{t+1}\} = (1 - \rho_\omega - \rho_\theta + g)\overline{\omega\theta} + (\rho_\omega - g)\overline{\theta}\omega_t + (\rho_\theta - g)\overline{\omega}\theta_t + g\omega_t\theta_t \tag{17}$$

Substituting (17) into (16) and then substituting the resulting expression

into (14), it can be verified, using the method of undetermined coefficients, that (15) is a solution to the functional equation where[5]

$$b = \frac{\beta}{1 - g\beta} > 0 \qquad (18a)$$

$$d = \frac{b\beta\bar{\theta}(\rho_\omega - g)}{1 - \rho_\omega\beta} \geq 0 \quad (18b)$$

$$e = \frac{b\beta\bar{\omega}(\rho_\theta - g)}{1 - \rho_\theta\beta} \geq 0 \quad (18c)$$

$$a = \frac{\beta}{1 - \beta} \left[ b(1 - \rho_\omega - \rho_\theta + g)\overline{\omega\theta} + d(1 - \rho_\omega)\bar{\omega} + e(1 - \rho_\theta)\bar{\theta} \right] \geq 0 \qquad (18d)$$

Thus, (15) is an exact price function which relates the equilibrium stock price in each period to contemporaneous values of $y_t$, $\omega_t$ and $\theta_t$.[6]

V.  The Relation between the Stock Price and the Conditional Distribution of Dividends

In this section I use the exact price function (15) to analyze the contemporaneous relation between the distributional parameters of dividends and the stock price. First, I will derive the relation between the parameters $\omega$ and $\theta$ and the stock price. Then, I will use these results to examine the relation between the parameters $\mu$ and $\nu$ and the stock price.

Let $p_y$, $p_\omega$ and $p_\theta$ denote the partial derivatives of the price function $p(y_t, \omega_t, \theta_t)$ with respect to $y_t$, $\omega_t$, and $\theta_t$, respectively. It follows immediately from (15) that at time t, these partial derivatives are

$$p_y = \alpha p_t / y_t \qquad (19a)$$

$$p_\omega = \left[ b\theta_t + d \right] y_t^\alpha \qquad (19b)$$

$$p_\theta = [b\omega_t + e]y_t^\alpha \tag{19c}$$

To determine the signs of these partial derivatives, recall that I have assumed that $0 \le \rho_\omega \le 1$ and $0 \le \rho_\theta \le 1$. In addition, the assumption that dividends are positive implies that $\omega > 0$. Finally, since the coefficient of variation must be nonnegative, it follows that $\theta > 0$. Therefore, inspection of (18a, b, c) reveals that the coefficient b is positive and the coefficients d and e are non-negative. Hence, the partial derivatives in (19a, b, c) are all positive.

To analyze the response of stock prices to changes in the conditional distribution parameters $\mu_t$ and $\nu_t$, equations (7a, b) can be used to express the stock price as a function of $y_t$, $\mu_t$ and $\nu_t$, $p_t = \tilde{p}(y_t, \mu_t, \nu_t)$. Using (7a, b) and the price function $p_t = p(y_t, \omega_t, \theta_t)$ yields

$$\frac{\partial \tilde{p}_t}{\partial \mu_t} = (1 - \alpha)\{p_\omega \omega_t/\mu_t\} \tag{20a}$$

$$\frac{\partial \tilde{p}_t}{\partial \nu_t} = -(1 - \alpha)\{p_\theta \alpha \theta_t \frac{\nu_t}{1 + \nu_t^2}\} \tag{20b}$$

Since $\alpha$, $p_\omega$, $p_\theta$, $\omega_t$, $\theta_t$, $\nu_t$ and $\mu_t$ are all positive, it follows that the terms in curly brackets in (20a, b) are positive. Therefore, it follows immediately that $\frac{\partial \tilde{p}_t}{\partial \mu_t}$ and $\frac{\partial \tilde{p}_t}{\partial \nu_t}$ are of opposite sign (except for the case of logarithmic utility when they are both equal to zero). In particular, if $\alpha < 1$, then an increase in the conditional expected dividend increases the current stock price, but an increase in the conditional coefficient of variation reduces the current stock price. Alternatively, if $\alpha > 1$, then the current stock price is negatively related to the conditional mean dividend, but the current stock price is positively related to the conditional coefficient of variation.

The analysis above provides some formal underpinning to the argument by Poterba and Summers (1986) that the magnitude of the effect on stock prices of increased dividend volatility is an increasing function of the persistence of the increase in volatility. To examine the role of the persistence of volatility, $\rho_\theta$, in determining the effect on stock price of a change in volatility, it is necessary to examine the effect of an increase in $\rho_\theta$ on the coefficients b and e of the price function. In examining these coefficients, I must specify what happens to g, the probability that neither $\omega_t$ nor $\theta_t$ changes, when I consider a change in $\rho_\theta$. I will first examine two simple special cases and then I will state a more general result.

First, suppose that g is held fixed when $\rho_\theta$ is increased. In this case, b is unaffected but e is increased by the increase in $\rho_\theta$. It follows from (19c) that the magnitude of $p_\theta$ is increased when $\rho_\theta$ is increased.

Alternatively, suppose that the dates at which $\omega$ and $\theta$ change are uncorrelated so that $g = \rho_\theta \rho_\omega$. Substituting $\rho_\theta \rho_\omega$ for g in the expressions for b and e in (18a, c) implies that both b and e are non-decreasing in $\rho_\theta$ and at least one of these coefficients is strictly increasing in $\rho_\theta$. Once again, the magnitude of $p_\theta$ is increased when $\rho_\theta$ is increased.

More generally suppose that $0 \leq \dfrac{dg}{d\rho_\theta} \leq 1$. This assumption implies that when the persistence of $\theta$ is increased, the probability that both $\theta$ and $\omega$ remain unchanged cannot decrease; also the probability that $\theta$ remains unchanged and $\omega$ changes cannot decrease. This assumption includes the two special cases presented above. Under this assumption, b and e are, once again, non-decreasing in $\rho_\theta$ and at least one of these coefficients is increasing in $\rho_\theta$. Thus, in this more general case, it remains true that the magnitude of $p_\theta$ is increasing in $\rho_\theta$. Therefore, (20b) implies that the magnitude of $\partial \tilde{p}/\partial \nu_t$ is increasing in $\rho_\theta$.

1.1.11

Now consider the relation of these results to those in Poterba and
Summers (1986) and Giovannini (1987). The results presented above are
consistent with the claim by Poterba and Summers that the magnitude of the
stock price response to a change in volatility is an increasing function of
the persistence of volatility. However, the Poterba-Summers analysis was
partial equilibrium and predicted an unambiguous negative relation between
dividend riskiness and stock prices. The general equilibrium framework
employed here shows that the effect on stock prices can be in either direction
but that the magnitude of the effect is an increasing function of $\rho_\theta$. As for
Giovannini's (1987) results about the role of persistence, the results here
are in direct contradiction to his conclusion that the more persistent is the
change in volatility, the smaller will be its effect on stock prices. Indeed,
because of his timing assumptions and the binding cash-in-advance constraint,
he finds that stock prices are invariant to permanent changes in volatility.
This seemingly counterintuitive result appears to depend on the nature and
timing of transactions in Giovannini's cash-in-advance structure.

## VI. Bond Prices

In this section I derive the price of one-period riskless bonds and
examine the co-movement of bond prices and stock prices. Let $q_t$ be the price
in period t of a one-period riskless bond that yields one unit of output in
period t+1. A consumer who is considering buying an additional bond in period
t would have to reduce $c_t$ by $q_t$ units, suffering a utility loss of $q_t u'(c_t)$ in
period t. The reward to purchasing the bond is that consumption in period t+1
could be increased by one unit which increases expected discounted utility
by $\beta E_t \{ u'(c_{t+1}) \}$. Equating the current loss in utility with the future gain
in utility yields the bond price

1.1.11

$$q_t = \beta E_t\{u'(c_{t+1})\}/u'(c_t) \tag{21}$$

Recalling that in equilibrium $c_t = y_t$, the bond price equation in (21) can be written as

$$q_t = \beta E_t\{u'(y_{t+1})\}/u'(y_t) \tag{22}$$

To compare the behavior of stock and bond prices, divide (4) by (22) to obtain

$$p_t/q_t = \frac{E_t\{(p_{t+1} + y_{t+1})u'(y_{t+1})\}}{E_t\{u'(y_{t+1})\}} \tag{23}$$

Observe that if $y_t$ were i.i.d., then the conditional expectations in the numerator and denominator of the right-hand side of (23) would each be constant. Hence $p_t$ and $q_t$ would be perfectly correlated. However, by relaxing the assumption that $y_t$ is i.i.d., I have allowed for stock prices and bond prices to be imperfectly correlated.

To calculate the equilibrium price of a riskless bond $q_t$, recall that with an isoelastic utility function, $u'(y_t) = y_t^{-\alpha}$. Therefore, equation (21) can be rewritten as

$$q_t = \beta E_t\{y_{t+1}^{-\alpha}\}/y_t^{-\alpha} \tag{24}$$

Under the distributional assumptions in section III, it can be shown that

$$E_t\{y_{t+1}^{-\alpha}\} = \exp[-\alpha m_t + \tfrac{1}{2}\alpha^2 s_t^2] \tag{25}$$

Using the definitions of $\mu_t$ and $v_t^2$ in (6a, b), (25) can be rewritten as

$$E_t\{y_{t+1}^{-\alpha}\} = \mu_t^{-\alpha}[1 + v_t^2]^{\frac{\alpha(1+\alpha)}{2}} \tag{26}$$

To express the bond price as a function of $y_t$, $\mu_t$ and $v_t^2$, substitute (26) into (24) to obtain

1.1.11

$$q_t = q(y_t, \mu_t, \nu_t^2) \equiv \beta \left(\frac{y_t}{\mu_t}\right)^\alpha \left[1 + \nu_t^2\right]^{\frac{\alpha(1+\alpha)}{2}} \tag{27}$$

It follows from inspection of (27) that $\partial q_t/\partial y_t > 0$, $\partial q_t/\partial \mu_t < 0$ and $\partial q_t/\partial \nu_t^2 > 0$. In words, a reduction in the expected growth rate of dividends $(\mu_t/y_t)$ and an increase in the riskiness of dividends each cause bond prices to rise. Note that, as with stock prices, the responses of bond prices to increases in $\mu_t$ and $\nu_t^2$ are in opposite directions. If $\alpha < 1$, an increase in $\nu_t^2$ (or a reduction in $\mu_t$) causes a reduction in the price of stocks but an increase in the price of bonds. However, if $\alpha > 1$, then stock and bond prices both rise in response to an increase in $\nu_t$ or a reduction in $\mu_t$.

## VII. The Risk Premium on Stocks

I have analyzed the effects of dividend riskiness on the price of stocks and on the price of one-period riskless bonds. This analysis was motivated by Pindyck's empirical study of the effect of risk on stock prices. French, Schwert and Stambaugh (1986) also studied the empirical relation between risk and stock market behavior, but instead of focusing directly on stock prices, they analyzed the risk premium on stocks relative to the riskless rate of return. They found "reliable evidence of a positive relation between the expected risk premium on common stocks and the predictable level of volatility." (p. 20) In addition, they found that the excess return on stocks relative to bonds is negatively related to the contemporaneous innovation in stock price volatility. Their explanation of this finding is that an unexpected increase in volatility leads to an upward revision in forecasts of future volatility. In response to this increase in future volatility, the stock price falls, and it is this fall in the stock price that accounts for a low realized rate of return. Their argument presumes that an increase in risk

leads to a reduction in stock prices. This presumption is consistent with the general equilibrium model presented above if $\alpha < 1$.

In this section, I derive an expression for the ex ante risk premium on stocks and examine the relation between this risk premium and dividend volatility. Under logarithmic utility, the ex ante risk premium is an increasing function of dividend volatility. However, if $\alpha \neq 1$, then for some allowable, though implausible, parameter values, the risk premium is a decreasing function of risk.

Let $R_{t+1}$ be the gross return on stocks between period t and period t+1 so that

$$R_{t+1} = (p_{t+1} + y_{t+1})/p_t \tag{28}$$

Let $R_{t+1}^F = 1/q_t$ be the gross rate of return on a one-period risk-free real discount bond. It follows from (27) that

$$R_{t+1}^F = \beta^{-1} \mu_t^\alpha y_t^{-\alpha} (1 + \nu_t^2)^{\frac{-\alpha(1+\alpha)}{2}} \tag{29}$$

The (ex ante) risk premium on stocks is $E_t\{R_{t+1} - R_{t+1}^F\}$ and the (realized) excess return on stocks is $R_{t+1} - R_{t+1}^F$.

Before deriving the ex ante risk premium, I will introduce an additional simplifying assumption. Specifically, I will assume that, conditional on information available at time t, the parameter vector $(\omega_{t+1}, \theta_{t+1})$ is independent of the dividend $y_{t+1}$. Under this assumption it is convenient to define the following function of the distributional parameters $\omega_t$ and $\theta_t$,

$$p_t^* \equiv a + b\omega_t\theta_t + d\omega_t + e\theta_t \tag{30}$$

1.1.11

Observe from (15) that

$$p_t = p_t^* y_t^\alpha \qquad (31)$$

Using (31), the functional equation in (14) can be rewritten as

$$p_t^* = \beta E_t\{p_{t+1}^*\} + \beta \omega_t \theta_t \qquad (32)$$

and the gross rate of return on stocks can be written as

$$R_{t+1} = \frac{p_{t+1}^* y_{t+1}^\alpha + y_{t+1}}{p_t^* y_t^\alpha} \qquad (33)$$

The assumption that, conditional on information available at time t,
$(\omega_{t+1}, \theta_{t+1})$ is independent of $y_{t+1}$ implies that $p_{t+1}^*$ is conditionally
independent of $y_{t+1}^\alpha$. Therefore, the conditional expectation of $R_{t+1}$ is

$$E_t\{R_{t+1}\} = \frac{E_t\{p_{t+1}^*\}E_t\{y_{t+1}^\alpha\} + \mu_t}{p_t^* y_t^\alpha} \qquad (34)$$

The right-hand side of (34) involves the conditional expectations of
$p_{t+1}^*$ and $y_t^\alpha$. The conditional expectation of $p_{t+1}^*$ can be obtained immediately
from (32) as

$$E_t\{p_{t+1}^*\} = \beta^{-1} p_t^* - \omega_t \theta_t \qquad (35)$$

The conditional expectation of $y_{t+1}^\alpha$, which follows from the conditional
lognormality of $y_{t+1}$, is

$$E_t\{y_{t+1}^\alpha\} = \mu_t^\alpha \theta_t \qquad (36)$$

Substituting (35) and (36) into (34) yields

$$E_t\{R_{t+1}\} = [\beta^{-1} \mu_t^\alpha \theta_t + (1 - \theta_t^2) \frac{\mu_t}{p_t^*}] y_t^{-\alpha} \qquad (37)$$

1.1.11

The ex ante risk premium on stocks is calculated by subtracting the riskless rate in (29) from the expected rate of return in (37) to obtain

$$E_t\{R_{t+1} - R_{t+1}^F\} = h(\mu_t, \nu_t)y_t^{-\alpha} \tag{38a}$$

where $\qquad h(\mu_t, \nu_t) \equiv \beta^{-1}\mu_t^{\alpha}[\theta_t - (1 + \nu_t^2)^{-\alpha(\alpha+1)/2}] + \dfrac{\mu_t}{p_t^*}(1 - \theta_t^2) \tag{38b}$

It can be shown that $h(\mu_t, \nu_t)$ is positive if and only if $\nu_t > 0$. Thus, the ex ante risk premium on stocks is positive if and only if dividends are risky. Furthermore, inspection of (38a) reveals that the ex ante risk premium is a declining function of the most recent dividend, $y_t$.

A natural question to study at this point is whether the ex ante risk premium is an increasing function of risk. More formally, the question is whether $h(\mu_t, \nu_t)$ is an increasing function of the coefficient of variation $\nu_t$. The answer turns out to be that $h(\mu_t, \nu_t)$ is not a monotonic function of $\nu_t$, in general. In order to shed some light on the behavior of the risk premium I will first examine the case of logarithmic utility which yields a crisp set of results. Then I will present some numerical results for the case in which $\alpha = 6$. These results will demonstrate that the risk premium can decline when risk increases.

First consider the case with logarithmic utility, i.e., $\alpha = 1$. Recall from (7a, b) that under logarithmic utility $\omega_t \equiv \theta_t \equiv 1$. In this case, $p_t^*$ is equal to $a + b + d + e$. Using the expressions for a, b, d, and e in (18a - d) yields

$$p_t^* \equiv \frac{\beta}{1 - \beta} \qquad \text{if } \alpha = 1 . \tag{39}$$

Substituting (39) into the expression for $h(\mu_t, \nu_t)$ in (38b), and using the expression for the ex ante risk premium in (38a), yields

1.1.11

$$E_t\{R_{t+1} - R_{t+1}^F\} = \beta^{-1}(\mu_t/y_t) \frac{\nu_t^2}{1 + \nu_t^2} \qquad \text{if } \alpha = 1 \qquad (40)$$

It follows from (40) that under logarithmic utility, the expected risk premium is an increasing function of the ex ante volatility of dividends $\nu_t$, if the exogenous stochastic variables $\mu_t$ and $y_t$ are held fixed.

In the case with logarithmic utility it turns out that the volatility of the rate of return on stocks is equal to the volatility of the dividend. Using the expression for $p_t^*$ in (39), it follows immediately from (36) that

$$p_t = \frac{\beta}{1 - \beta} y_t \qquad (41)$$

Recalling the definition of the (gross) rate of return on stocks in (28), it follows from (41) that

$$R_{t+1} = \beta^{-1} y_{t+1}/y_t \qquad (42)$$

The ex ante rate of return on stocks is $\beta^{-1}\mu_t/y_t$ so that the squared coefficient of variation of returns is

$$\text{Var}(R_{t+1})/[E_t\{R_{t+1}\}]^2 = \nu_t^2 \qquad (43)$$

Thus, the volatility of returns is the same as the volatility of dividends under logarithmic utility. Equation (40) is consistent with the positive relation between the expected risk premium and the expected volatility of returns found by French, Schwert and Stambaugh (1986). However, it is not consistent with their finding of a negative relation between the innovation in volatility and the ex post risk premium. Under logarithmic utility, there is no relation between ex post returns and the innovation in volatility because the stock price does not change in response to a change in riskiness. However, if $\alpha < 1$, then the stock price will fall in response to an unanticipated increase in risk as found by French, Schwert and Stambaugh.[7]

Under logarithmic utility, the ex ante risk premium is an increasing function of the coefficient of variation $\nu_t$. However, as a matter of economic theory, this result does not hold for all isoelastic utility functions. Consider the following example in which the distributional parameters $\omega_t$ and $\theta_t$ are i.i.d. over time ($\rho_\omega = \rho_\theta = g = 0$): $\alpha = 6.0$, $\beta = 0.95$, $y_t = 4.0$, $\mu_t = 0.5$, $\bar{\omega} = 0.62$, $\bar{\theta} = 2.00$. Now consider two different levels of risk: $\nu_t^2 = 0.20$ and $\nu_t^2 = 0.30$. The ex ante risk premium, $E_t\{R_{t+1} - R_{t+1}^F\}$, falls from $.2983 \times 10^{-5}$ to $.2976 \times 10^{-5}$ when $\nu_t^2$ is increased from 0.20 to 0.30.

I offer the following remarks in an attempt to shed some light on this finding. First, with a large degree of risk aversion and with a high degree of risk, the gross riskless rate becomes almost zero (in this example, $R_{t+1}^F = .8728 \times 10^{-7}$ when $\nu_t^2 = 0.20$). Indeed, the riskless rate becomes only a small fraction of the risk premium (in this case, $R_{t+1}^F / E_t\{R_{t+1} - R_{t+1}^F\} = 0.0293$ when $\nu_t^2 = 0.20$). Thus, in response to an increase in $\nu_t^2$, most of the movement in the risk premium is due to a change in the ex ante rate of return on stocks, $E_t\{R_{t+1}\}$, rather than to a change in the riskless rate. As shown in Section V, an increase in $\nu_t^2$ causes the stock price, $p_t$, to increase if $\alpha > 1$ which tends to decrease the expected rate of return, $E_t\{R_{t+1}\}$.

Although I have shown that there are allowable values of the parameters under which the ex ante risk premium is a decreasing function of the degree of risk, I do not claim that such a negative relation holds for empirically relevant parameter values. I presented these results simply to establish that the standard assumptions on the utility function are not sufficient to establish unambiguously that the risk premium is an increasing function of the degree of risk.

VIII.  Conclusion

In this paper I have developed a simple stochastic general equilibrium model to analyze stock and bond prices.  The methodological contribution of the paper is the derivation of exact solutions for asset prices in a rational expectations model with a time-varying distribution of dividends.  The derived pricing equations allow us to examine the joint time series behavior of stock prices, bond prices and distributional parameters.  Including the time-varying distribution in the general equilibrium asset pricing model is important if we are to try to relate historical changes in stock and bond prices to changes in dividend risk.

The substantive conclusions of the paper provide some theoretical support for aspects of previous partial equilibrium empirical studies by Poterba and Summers (1986) and French, Schwert and Stambaugh (1986).  In assessing Pindyck's (1984) argument that the decline in the stock market in the 1970's was substantially due to increased risk, Poterba and Summers argued that to explain the large magnitude of the decline in stock prices, the increase in volatility would have to be more persistent than indicated by their estimates.  The analysis above is consistent with their argument that the magnitude of the response of the stock price to an increase in riskiness is an increasing function of the persistence in riskiness.  However, the general equilibrium analysis is consistent with their supposition about the direction of the effect only when the coefficient of relative risk aversion ($\alpha$) is less than one.

French, Schwert and Stambaugh analyzed the behavior of the risk premium on common stocks.  The general equilibrium analysis above is, for plausible parameter values, generally supportive of their finding of a positive relation between the expected risk premium and predicted volatility.  However, their

explanation of their finding of a negative relation between the innovation in volatility and the realized risk premium relies on a presumed negative relation between expected volatility and stock prices. This presumed relation holds in the general equilibrium model above only if $\alpha < 1$.

The substantive findings of this paper support and extend Barsky's (1986) finding that an increase in riskiness can either increase or decrease stock prices depending on the curvature of the utility function. Barsky's model is a two-period model and the model presented above is infinite-horizon. The advantage of the multi-period model relative to the two-period model is that it allows the distributional parameters to evolve stochastically over time. We can then examine the joint time series behavior of riskiness and stock prices rather than simply rely on the comparative statics analysis that is possible in a two-period model.

Footnotes

[1]The two-state versions of the Lucas (1978) model examined by LeRoy and LaCivita (1981) and Mehra and Prescott (1985) allow for the conditional coefficient of variation to vary stochastically over time. However, these models do not allow the conditional mean and conditional coefficient of variation to vary independently over time.

[2]More recently, Flood (1987) and Hodrick (1987) have extended the discrete-time models developed in this paper and in Giovannini (1987) to analyze asset prices in open-economy models.

[3]In addition, the probability g must satisfy $g \geq \rho_\omega + \rho_\theta - 1$.

[4]The responses of the stock price to changes in $\mu_t$ and $\nu_t$ would be unaffected if I relax the assumption that $\mu_t$ and $\nu_t$ are independent. See footnote 2.

[5]If I relax the assumption that $\nu_t$ and $\mu_t$ are independent when both are drawn, then $\omega_t$ and $\theta_t$ will be correlated. Let $\delta$ be the correlation of $\omega_t$ and $\theta_t$ when both $\mu_t$ and $\nu_t$ are drawn. In this case, b, d, and e are unchanged and $a = \frac{\beta}{1 - \beta} \{b(1 - \rho_\omega - \rho_\theta + g)(\overline{\omega\theta} + \delta) + d(1 - \rho_\omega)\overline{\omega} + e(1 - \rho_\theta)\overline{\theta}\}$.

[6]I have ignored explosive solutions for stock prices. For example, let $z_t$ be a stochastic process that is independent of $y_t$, $\omega_t$ and $\theta_t$ at all leads and lags. Suppose that $E_t\{z_{t+1}\} = \beta^{-1}z_t$. Now define $P_t \equiv p_t + \delta z_t$ where $p_t$ is the stock price in (15). It is easily verified that $P_t$ satisfies (14) and is a potential candidate solution for the stock price. However, unless $\delta = 0$, the conditional expectation $E_t\{P_{t+j}\} = E_t\{p_{t+j}\} + \delta E_t\{z_{t+j}\} = E_t\{p_{t+j}\} + \delta\beta^{-j}z_t$ grows without bound as j grows without bound. I impose the restriction $\delta = 0$ to rule out this explosive behavior.

[7]Strictly speaking, $\nu_t$ is the riskiness of dividends and French, Schwert and Stambaugh examined the riskiness of returns. Using the coefficient of

variation as a measure of volatility, dividend volatility is identical to rate of return volatility in the special case of logarithmic utility ($\alpha = 1$), but this identity is not generally true if $\alpha \neq 1$.

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