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FORWARD GUIDANCE:
IS IT USEFUL AWAY FROM THE LOWER BOUND?

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ABSTRACT

During the recent economic crisis, when nominal interest rates were at their effective lower bounds, central banks used forward guidance announcements about future policy rates to conduct their monetary policy. Many policymakers believe that forward guidance will remain in use after the end of the crisis; however, there is uncertainty about its effectiveness. In this paper, we study the impact of forward guidance in a stylized new Keynesian economy away from the effective lower bound on nominal interest rates. Using closed-form solutions, we show that the impact of forward guidance on the economy depends critically on a specific monetary policy rule, ranging from non-existing to immediate and unrealistically large, the so-called forward guidance puzzle. We show that the size of the smallest root (or eigenvalue) captures model dynamics better than the underlying parameters. We argue that the puzzle occurs under very special empirically implausible and socially sub-optimal monetary policy rules, whereas empirically relevant Taylor rules lead to sensible implications.

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1 Introduction

Forward guidance (FG) refers to central banks' announcements regarding future policy rates. During Great Recession and its aftermath, FG was widely used by central banks as an unconventional monetary policy tool when nominal interest rates were constrained at their effective lower bounds (ELB). For example, in August 2011, the Fed signalled an intention to maintain nominal interest rates at zero at least until the middle of 2013. Blinder, Ehrmann, Haan and Jansen (2017) surveyed 55 heads of the central banks, and they found that during this period, about a half of the central banks adopted a FG policy of some type. In the survey, policymakers believe that FG will continue to be used as a monetary policy tool after the end of the economic crisis, however, they have doubts about its effectiveness, in particular in the absence of active ELB. In fact, the Fed started using FG much earlier, in 2003-2004, when the ELB was not binding; see Carlson, Eggertsson and Mertens (2008), and Plosser (2013). The goal of this paper is to make progress in understanding whether or not FG is a potentially useful monetary policy tool when the ELB does not bind policy.

We consider a stylized new Keynesian economy that is away from the ELB, and we assume that the monetary authority uses a Taylor-style rule. There are different definitions of forward guidance in the related literature. For example, Bean (2013), then deputy governor of the Bank of England, states that FG "is intended primarily to clarify our reaction function," a description of how the policy interest rate will react to economic variables, or a monetary policy rule. According to this definition, FG is simply an announcement that monetary policy follows a policy rule now and in the future. Alternatively, Reifschneider and Williams (2000), and Woodford (2013) define FG as an announced deviation from the policy rule; they consider such a deviation in the context of the ELB.

In this paper, we define FG as a deviation from policy rule, but not necessarily in the context of the ELB. In particular, we consider FG that takes the form of an anticipated, one-time interest-rate shock. We restrict attention to conventional equilibria that do not explode in the future – forward stable equilibria. We distinguish three cases, depending on how responsive central banks are to variables in the rule conducting their monetary policy. To illustrate these three cases, suppose that a Taylor rule contains just a feedback to inflation.[1]

First, if the inflation coefficient in the Taylor rule is smaller than one, then the model has one stable (smaller-than-one) root and one unstable (larger-than-one) root and there is a multiplicity of equilibria. This case is studied in Werning (2015) and Cochrane (2017a) under the assumption that the inflation coefficient is effectively zero.[2] In this paper, we establish a general class of Taylor rules that leads to indeterminacy of equilibria, and

[1]Under more general Taylor rules, considered in this paper, there is a feedback to inflation, expected inflation and output gap, and the responsiveness of monetary policy depends on all the feedback coefficients.

[2]Bianchi and Nicoló (2017) proposed an approach for dealing with indeterminacy in the model in which the response to inflation in the Taylor rule is not zero but sufficiently small.

we show that large differences between multiple equilibria are a generic property of the model in which the monetary policy is not sufficiently responsive to insure the equilibrium uniqueness.

Second, when the inflation coefficient in the Taylor rule reaches unity, the stable root becomes unstable (specifically, it also reaches unity), and the equilibrium becomes unique. This case exhibits the so-called *forward guidance puzzle* – a counterintuitive implication of the model that the central bank's announcements about future interest rates have immediate and unrealistically large effects on the economy; see Carlstrom, Fuerst and Paustian (2015), Del Negro, Giannoni and Patterson (2012, 2015), and McKay, Nakamura and Steinsson (2016). More generally, we show that the FG puzzle is a very special edge-of-the-knife case in which one of the two characteristic roots is unity.

Finally, when the inflation coefficient exceeds one, then both roots remain unstable (either both roots are real and larger than one or they are complex with the amplitude larger than one), the equilibrium remains unique and the FG puzzle disappears. The resulting model has the following common-sense predictions: (i) the effect of FG is not excessively large if policy announcements refer to a distant future; (ii) such effect decreases with the horizon of a policy announcement (i.e., the further away in the future is anticipated interest-rate shock, the smaller is the effect of this shock on today's economy); (iii) FG can have a detrimental effect on output, at least during some periods, or it can lead to cyclical fluctuations of a decreasing amplitude.[3]

Why do not we have the FG puzzle when the monetary policy is sufficiently responsive? Technically, the assumptions in Del Negro et al. (2012, 2015) and McKay et al. (2016) imply that interest-rate shocks affect the current output without discounting, so that the effect of today's shock on current output is as strong as the one that happens in a million of years. We show analytically that the introduction of more realistic and plausible Taylor rules restores discounting, so that the effect of distant future shocks on today's economy is practically non-existent.[4]

In a Taylor rule that contains just a feedback to inflation, the inflation coefficient exceeding unity is sufficient to ensure local determinacy. In the more general policy rule of the paper, the conditions for two unstable eigenvalues are more complex. An important point of the paper is to show that namely these roots (or eigenvalues) capture model dynamics better than underlying parameters.

The main body of our analysis relies on closed-form solutions. Depending on specific parameterization of Taylor rule, the model can have four different types of solutions that corresponds to four different types of characteristic roots: i) one stable and one unstable real root, ii) two distinct unstable real roots, iii) two repeated unstable real roots and iv) two unstable complex roots.[5] To the best of our knowledge, we are the first to derive

[3]We argue that prediction (ii) is related to turnpike theorem established for the neoclassical optimal growth model; see Maliar, Maliar, Taylor and Tsener (2015) for a discussion.

[4]McKay et al. (2016) argue that the introduction of idiosyncratic uncertainty and borrowing constraints restores discounting in the Euler equation and helps resolve the FG puzzle.

[5]In the present paper, we analyze a discrete-time version of the model. Maliar (2018) shows how to

closed-form solutions for the cases of repeated real and complex roots. Complex roots are not an exotic peculiarity but plausible case: we show that the most stylized Taylor rules with both inflation and output gaps can lead to complex roots. The solutions we construct are applicable not only to the deterministic, perfect-foresight version of the model but also to the stochastic model with random disturbances, including persistent government-spending shocks, interest-rate shocks, etc. The stochastic part of our analysis is related to a method of undetermined coefficients, described in Taylor (1986).

We use numerical simulation to explore the properties of equilibria under more general assumptions when closed-form solutions are infeasible. We find that the results we establish analytically are robust to the model's modifications. They hold both in a fully non-linear version of the new Keynesian model, as well as in the model augmented to include capital. Moreover, our findings are robust to the solution methods used, specifically, we find that both Fair and Taylor's (1983) extended path method and Maliar et al.'s (2015) extended function path method lead to similar results.

There is a variety of alternative ways of resolving the FG puzzle. In particular, Del Negro, Giannoni and Patterson (2015) constructed a perpetual-youth version of Smets and Wouters (2007) model in which the presence of cohorts results in heavier discounting of the future and as a result, to smaller effects of the central bank's announcements on current aggregate variables. McKay et al. (2016) introduced idiosyncratic household risk and borrowing constraints. Husted, Rogers and Sun (2017) modified their model to allow for monetary policy uncertainty (they assume that there is only 50 percent chance that the central bank will implement the anticipated policy rate change) and find that this modification leads to a smaller response in output. Kaplan, Moll and Violante (2017) showed that a heterogeneous-agent version of the standard new Keynesian model generates smaller effects of FG on output than a representative-agent counterpart of their model. Gabaix (2017) assumed that agents are not fully rational to unusual events and they do not have perfect foresight of the future, etc.[6]

Unlike the above literature, we do not attempt to modify the baseline new Keynesian model to resolve the FG puzzle. Instead, we argue that the policy rules, that lead to the FG puzzle, are extreme and implausible. In particular, Del Negro et al. (2015) generate the FG puzzle by assuming that prices are fixed. In turn, McKay et al. (2016) assume that the central bank sets the current nominal interest rate to completely accommodate anticipated inflation. That is, if inflation expectations were, for example, 300 percent, the central bank would simply set the nominal interest rate to 302 percent to ensure a 2 percent real interest rate. However, high inflation is costly for the economy, and the actual monetary authorities will fight inflationary expectations instead of accommodating them. Inflation stabilization is the key prescription of the Taylor rule used by actual

construct parallel closed-form solutions for the continuous-time model.

[6] Other papers that study the effectiveness of FG include Levin, López-Salido, Nelson and Yun (2010), Werning (2012), Den Haan (2013), Carlstrom et al. (2015), Chung (2015), Bundick and Smith (2016), Keen, Richter and Throckmorton (2016), Galí (2017), Walsh (2017), Hagedorn, Luo, Manovskii and Mitman (2018), among others.

central banks and is in line with the optimal monetary policy.[7] An active response of the monetary authority prevents the FG puzzle from happening: the effect of future shocks on today's variables quickly fades away when the shocks are translated farther in the future. We conclude that what produces the FG puzzle is not the new Keynesian model itself but an unrealistic and suboptimal way of modeling the monetary authority's responses.[8]

Finally, our closed-form solutions help us understand and compare the effects of FG on economies with and without active ZLB. Interestingly, FG was primarily motivated by active ZLB, however, the literature first discovered and analyzed the FG puzzle in models without active ZLB; see Del Negro et al. (2012, 2015) and McKay et al. (2016). In those papers, the models have unique solutions in which the impact of FG on the economy is unrealistically large. In turn, the literature that focuses on active ZLB finds multiplicity of equilibria; see Carlstrom et al. (2015), Werning (2015) and Cochrane (2017a). In these papers, the effect of FG on the economy can range from extremely large to nonexistent depending on which equilibrium is selected. The present paper connects these two streams of the literature: we show that the backward explosion away from ZLB is similar to the one established under the ZLB scenario in Cochrane (2017a). In either of the two cases, the effectiveness of FG is fully determined by the size of the smaller eigenvalue, which is a single sufficient statistics for capturing the role of all model's ingredients and parameters in the backward explosion. The theorems we establish make it possible to see up-front whether or not a specific parameterization of the new Keynesian model leads to the FG puzzle under both ZLB and no-ZLB scenarios.

The rest of the paper is as follows: In Section 2, we derive closed-form solutions in the stylized new Keynesian model. In Section 3, we explore the model's implications under several alternative coefficients of the monetary rule. In Section 4, we characterize the optimal monetary policy by solving the Ramsey problem. In Section 5, we show a collection of experiments which illustrate the robustness of our analysis to the introduction of uncertainty, nonlinearity, and capital; finally, in Section 6, we conclude.

2 Stylized new Keynesian model

In this section, we outline a standard three-equation linear new Keynesian model, express it as a second-order difference equation, derive a closed-form solution and characterize some of its properties.

[7] Woodford (2001) shows that the optimal Ramsey rule closely resembles the empirically relevant Taylor rule with both inflation and output gap and with standard calibration of its coefficients.

[8] There are papers that document FG puzzle in medium-scale models parameterized by empirically plausible Taylor rules, see, e.g., the FRBNY DSGE model of Del Negro et al. (2015). Our analysis does not allow us to say what features produce FG puzzle in other studies but allows us to affirm that such features are not present in the stylized new Keynesian model if empirically plausible Taylor rule is used.

2.1 The model

We consider the standard linearized new Keynesian model that consists of, respectively, an IS equation and Phillips curve, expressed in deviations from the steady state

$$x_t = E_t[x_{t+1}] - \sigma(i_t - i_t^n - E_t[\pi_{t+1}]), \tag{1}$$

$$\pi_t = \beta E_t[\pi_{t+1}] + \kappa(x_t + g_t), \tag{2}$$

where x_t is an output gap; π_t is inflation; i_t is a nominal interest rate; i_t^n is a natural rate of interest; g_t is a disturbance; κ is a slope of the Phillips curve; and σ is an intertemporal elasticity of substitution.[9] The Phillips-curve shifter g_t can be interpreted as a direct marginal cost increase due to, for example, capital destruction, technical regress, government spendings (up to a scaling factor); see a discussion in Cochrane (2017a).

For the third equation, we assume that the nominal interest rate i_t is determined by a stylized Taylor rule

$$i_t = i_t^* + \phi_\pi \pi_t + \phi_{E\pi} E_t[\pi_{t+1}] + \phi_y x_t + \varepsilon_t, \tag{3}$$

where $\{i_t^*\}$ is a desired interest rate path; $\phi_\pi \geq 0$, $\phi_{E\pi} \geq 0$ and $\phi_y \geq 0$ are constant coefficients; and ε_t is a disturbance that may include both anticipated and unanticipated shocks. The rule studied in Taylor (1993) corresponds to $\phi_\pi = 1.5$, $\phi_y = 0.5$, $\phi_{E\pi} = 0$ and $i_t^* = 1$.[10]

2.2 Second-order difference equation

We next derive key regions to the parameters and eigenvalues of this system. It will be convenient to re-write the model (1)–(3) as a second-order difference equation. We substitute i_t from (3) into (1), use (2) to express x_t and x_{t+1} and substitute them into (1) to obtain

$$E_t[\pi_{t+2}] + bE_t[\pi_{t+1}] + c\pi_t = -z_t, \tag{4}$$

where $b \equiv -1 - \frac{1}{\beta} - \sigma\phi_y - \frac{\sigma\kappa(1-\phi_{E\pi})}{\beta}$; $c \equiv \frac{(1+\sigma\phi_y)}{\beta} + \frac{\sigma\kappa\phi_\pi}{\beta}$; z_t includes all exogenous variables, g_t, i_t^*, ε_t, i_t^n,

$$z_t \equiv \frac{\kappa}{\beta}\left[g_{t+1} - g_t\left(1 + \sigma\phi_y\right) + \sigma\left(i_t^* + \varepsilon_t - i_t^n\right)\right]. \tag{5}$$

Let us first construct a solution for the economy with perfect foresight by eliminating the expectation operator in (4) and later, we will show how to generalize the solution to the case of uncertainty. Below, we establish some properties a homogenous equation $\pi_{t+2} + b\pi_{t+1} + c\pi_t = 0$ that corresponds to (4).

[9] In Section 6 and Appendix A, we describe a fully nonlinear model whose linearized version corresponds to the model (1), (2) under some further restrictions. In particular, the slope of the Phillips curve is $\kappa = \frac{(1-\beta\theta)(1-\theta)}{\theta}(1+\vartheta)$, where β is a discount factor; θ is a share of not reoptimizing firms; ϑ is a parameter of the utility function $u(C_t, L_t) = \frac{C_t^{1-1/\sigma}-1}{1-1/\sigma} - \frac{L_t^{1+\vartheta}-1}{1+\vartheta}$; and $\sigma \to 1$.

[10] Our analysis abstracts from the issues of commitment, discretion and time inconsistency. These issues are studied, for example, in Walsh (2017).

Theorem 1 *The roots* $m_{1,2} = \frac{-b \pm \sqrt{b^2-4c}}{2}$ *to characteristic equation* $m^2 + bm + c = 0$ *satisfy:*

Case	*Restrictions on eigenvalues*	*Restrictions on parameters*
i)	*either* $\lvert m_1\rvert > 1, \lvert m_2\rvert < 1$ *or* $\lvert m_1\rvert < 1, \lvert m_2\rvert > 1$	$\phi_{E\pi} < \phi^1_{E\pi}$ *and* $\phi_{E\pi} > \phi^4_{E\pi}$
ii)	$\lvert m_1\rvert \geq 1,\ \lvert m_2\rvert \geq 1$	$\phi^1_{E\pi} \leq \phi_{E\pi} < \phi^2_{E\pi}$ *and* $\phi^3_{E\pi} \leq \phi_{E\pi} < \phi^4_{E\pi}$
iii)	$m_1 = m_2 = m$ *with* $\lvert m\rvert > 1$	$\phi_{E\pi} = \phi^2_{E\pi}$ *and* $\phi_{E\pi} = \phi^3_{E\pi}$
iv)	$m_{1,2} = \mu \pm \eta\iota$ *with* $r \equiv \sqrt{\mu^2+\eta^2} > 1$	$\phi^2_{E\pi} < \phi_{E\pi} < \phi^3_{E\pi}$

where $\phi^1_{E\pi} = 1 - \phi_\pi - \frac{(1-\beta)\phi_y}{\kappa}$, $\phi^2_{E\pi} = \phi^1_{E\pi} + \frac{1}{\sigma\kappa}\left(\sqrt{1+\phi_\pi\sigma\kappa+\sigma\phi_y} - \sqrt{\beta}\right)^2$, $\phi^3_{E\pi} = \phi^1_{E\pi} + \frac{1}{\sigma\kappa}\left(\sqrt{1+\phi_\pi\sigma\kappa+\sigma\phi_y} + \sqrt{\beta}\right)^2$ *and* $\phi^4_{E\pi} = \phi^1_{E\pi} + \frac{2}{\sigma\kappa}\left(1+\sigma\phi_y+\sigma\kappa\phi_\pi+\beta\right)$.

Proof. See Appendix B. ■

Thus, if the roots are real and distinct, there are two possibilities: either one root is stable and the other root is unstable (case i) or both roots are unstable (case ii). If the roots are either real and repeated (case iii) or complex (case iv), they are always unstable. Thus, it is never the case that both roots are stable in the considered area of the parameter space.

We have further results for the economy in which the rule (3) contains only actual but not future inflation.

Theorem 2 *Assume* $\phi_{E\pi} = 0$*. Then, the roots* $m_{1,2} = \frac{-b \pm \sqrt{b^2-4c}}{2}$ *to characteristic equation* $m^2 + bm + c = 0$ *satisfy:*

Case	*Restrictions on eigenvalues*	*Restrictions on parameters*
i)	*either* $\lvert m_1\rvert > 1, \lvert m_2\rvert < 1$ *or* $\lvert m_1\rvert < 1, \lvert m_2\rvert > 1$	$\phi_\pi < \phi^1_\pi$
ii)	$\lvert m_1\rvert \geq 1,\ \lvert m_2\rvert \geq 1$	$\phi^1_\pi \leq \phi_\pi < \phi^2_\pi$
iii)	$m_1 = m_2 = m$ *with* $\lvert m\rvert > 1$	$\phi_\pi = \phi^2_\pi$
iv)	$m_{1,2} = \mu \pm \eta\iota$ *with* $r \equiv \sqrt{\mu^2+\eta^2} > 1$	$\phi_\pi > \phi^2_\pi$

where $\phi^1_\pi = 1 - \frac{(1-\beta)\phi_y}{\kappa}$ *and* $\phi^2_\pi = \phi^1_\pi + \frac{\beta}{4\sigma\kappa}\left(1 - \frac{1}{\beta} - \sigma\phi_y - \frac{\sigma\kappa}{\beta}\right)^2$.

Proof. *See Appendix B.* ■

In other words, when the response of the monetary authority to inflation $\phi_\pi < \phi^1_\pi$ is weak, we have one stable and one unstable root; when the response to inflation becomes stronger, both roots become unstable $\phi^1_\pi \leq \phi_\pi < \phi^2_\pi$; when the response to inflation reaches the threshold level ϕ^2_π, the roots become repeated and unstable and finally; when $\phi_\pi > \phi^2_\pi$, the roots are complex and unstable. Complex roots are not a theoretical peculiarity but the most typical case: the stylized parameterization of the Taylor rule leads to complex roots, such as $\phi_\pi = 2$ and $\phi_y = 0.5$ used in Coibion et al. (2012).

Woodford (2001) calculated the boundary ϕ_π^1 of Theorem 2. Cochrane (2011) derived stability conditions under several Taylor rules with leads and lags, including $i_t = \phi_\pi \pi_t$, $i_t = \phi_{E\pi} E_t [\pi_{t+1}]$, and $i_t = \phi_\pi \pi_t + \phi_y x_t$ (see his Appendix B, Section E). Our Theorems 1 and 2 provide sharper results by establishing boundaries that separate different types of roots, namely, distinct real roots, repeated real roots and complex roots. These theorems are a useful step in constructing closed-form solutions since different types of roots lead to different types of solutions.

2.3 Closed-form solution

We now show closed-form solutions to the model (1)–(3) under four possible cases of characteristic roots established in Theorem 1. To the best of our knowledge, closed-form solutions for the cases ii)–iv) have not been derived in the literature yet.

Theorem 3 *The solution to the new Keynesian model (4) for cases i)-iv) in Theorem 1 is given by:*
i). For two distinct real roots such that root m_1 is unstable and root m_2 is stable, i.e., $|m_1| \geq 1$ and $|m_2| < 1$, we have

$$\pi_t = C_1 m_1^t + C_2 m_2^t + \frac{1}{m_1 - m_2} E_t \left[\sum_{s=t}^{\infty} m_1^{t-1-s} z_s + \sum_{s=-\infty}^{t-1} m_2^{t-1-s} z_s \right]; \qquad (6)$$

if $|m_1| < 1$ and $|m_2| \geq 1$, we flip the subscripts.
ii) For two distinct real roots $m_1 \neq m_2$ that are unstable $|m_1| \geq 1$, $|m_2| \geq 1$, we have

$$\pi_t = C_1 m_1^t + C_2 m_2^t + \frac{1}{m_1 - m_2} E_t \left[\sum_{s=t}^{\infty} m_1^{t-1-s} z_s - \sum_{s=t}^{\infty} m_2^{t-1-s} z_s \right]; \qquad (7)$$

iii). For two repeated real roots $m_1 = m_2 = m$ that are unstable $|m| > 1$, we have:

$$\pi_t = (C_1 + C_2 t) m^t + \frac{1}{m} E_t \left[(t-1) \sum_{s=t}^{\infty} m^{t-1-s} z_s - \sum_{s=t}^{\infty} s m^{t-1-s} z_s \right]; \qquad (8)$$

iv). For complex roots $m_{1,2} = \mu \pm \eta\iota$ that are unstable $r \equiv \sqrt{\mu^2 + \eta^2} \geq 1$, we have:

$$\pi_t = C_1 r^t \cos(\theta t) + C_2 r^t \sin(\theta t) + \frac{1}{\eta} E_t \left[\sum_{s=t}^{\infty} r^{t-1-s} \sin(\theta (t-1-s)) z_s \right]. \qquad (9)$$

where $\theta \equiv \arctan\left(\frac{\eta}{\mu}\right)$.
In i)–iv), C_1, C_2 in (6)-(9) are arbitrary constants.

Proof. The solution to (4) is given by the sum of a general solution to homogeneous equation $\pi_{t+2} + b\pi_{t+1} + c\pi_t = 0$ and a particular solution satisfying the non-homogeneous equation (4). Homogeneous second-order difference equations with constant coefficients are well studied in the field of differential equations; the solutions to such equations are the parts of expressions (6)–(9) that contain integration constants C_1 and C_2. In contrast, there is no general approach that can deliver particular solutions to the studied non-homogenous equations. One possible approach is a version of the "trial and error" method that parameterizes a particular solution by a linear combination of two terms in the solution to the homogeneous equations and identifies the coefficients in the combination to satisfy the given non-homogeneous equation – this is the approach we used in the paper.[11] The resulting particular solutions to non-homogeneous equations are the remaining parts of (6)–(9) (those that do not contain C_1 and C_2). The fact that the constructed particular solutions satisfy the non-homogeneous equation can be verified directly, by substituting them into (4). Finally, the solutions derived for the deterministic economy also hold for the version of the model with uncertainty after the introduction of the operator of conditional expectation; this fact follows by the law of iterative expectations. ■

Two observations are in order: First, the output gap can be found from

$$\kappa x_t = \pi_t - \beta E_t \pi_{t+1}.$$

For example, for case i), we have

$$\kappa x_t = C_1 m_1^t (1 - \beta m_1) + C_2 m_2^t (1 - \beta m_2) + \frac{1}{m_1 - m_2} E_t \left[m_1^{-1} z_t + (1-\beta) \sum_{s=t+1}^{\infty} m_1^{t-1-s} z_s - \beta m_2^{-1} z_t + (1-\beta) \sum_{s=-\infty}^{t-1} m_2^{t-1-s} z_s \right].$$

Second, in Theorem 3 we abstract from the possibility of sunspots. In particular, in case i), if sunspot solutions are taken into account, formula (6) becomes

$$\pi_t = \sum_{s=-\infty}^{t} C_{2s} m_2^{t-s} + \frac{1}{m_1 - m_2} E_t \left[\sum_{s=t}^{\infty} m_1^{t-1-s} z_s + \sum_{s=-\infty}^{t-1} m_2^{t-1-s} z_s \right].$$

where $\{C_{2s}\}$ is any sequence of unpredictable random variables with $E_{t-1} C_{2t} = 0$ (not just $C_2 m_2^t$). (Here, we set C_1 – a constant associated with an unstable eigenvalue m_1 – to zero

[11] To construct the particular solutions in the cases i) and ii), we can also use the approach of Cochrane (2017a) of decomposing the second-order difference equation into two first-order difference equations, however, this approach does not directly apply to cases iii) and iv).

to get a solution which is stable forward). That is, there is not just time-0 indeterminacy but sunspots may emerge at any date.[12]

In the main text, we focus on the deterministic case in which the sequence of shocks $\{z_t, z_{t+1}, ...\}$ is announced at t. Since the agents have perfect foresight about future shocks, the expectation operator can be omitted. We can also extend deterministic solutions to the case of uncertainty, including temporary and permanent shocks, anticipated and unanticipated shocks, as well as mixtures of deterministic trends and stochastic shocks. Another method that allows us to deal with uncertainty is proposed in Taylor (1986). Taylor's (1986) method does not specify how to construct a solution to the deterministic model like those we obtain in Theorem 3; see Appendix C for a comparison.

The particular solutions in (6)–(9) are forward stable (nonexplosive) by construction. A comparison of cases i) and ii) reveals one regularity about the construction of forward stable particular solutions: if a root m_i is unstable, i.e., $m_i > 1$, we use a particular solution that is forward looking $\sum_{s=t}^{\infty} m_i^{t-1-s} z_s$, while if it is stable, i.e., $m_i < 1$, we use the one that is backward looking $-\sum_{s=-\infty}^{t-1} m_i^{t-1-s} z_s$. To make the entire solution stable, the solution to the homogeneous equation must also be forward stable. In (7)–(9), this requires us to set the integration constants at $C_1 = 0$ and $C_2 = 0$. However, in case i), stability is consistent with any integration constant C_2 on the stable root m_2. Therefore, the stable solution is unique in cases ii)-iv), and it is indeterminate in case i).

The constructed closed-form solutions are convenient for applications. They allow us to analytically construct the path for $\{\pi_t\}$ for a given sequence of disturbances $\{z_t\}$. The corresponding solution for the output gap $x_t = \frac{1}{\kappa}(\pi_t - \beta\pi_{t+1}) - g_t$ follows from the Phillips curve (2). We use the constructed solutions to produce the results in Section 3.

Cochrane (2017b) obtained closed-form solutions for a discrete-time version of the new Keynesian model in which one root is stable (negative) and the other root us unstable (positive); see his equation (88). Cochrane (2017a) showed a parallel result for a continuous-time versions of the model; see his equation (6). In addition to that case, our Theorem 3 shows closed-form solutions for three other cases, namely, two distinct unstable real roots, repeated real roots and complex roots.[13] Maliar (2018) constructs the corresponding solutions in the continuous-time version of the model (1)–(3).

[12] The same point holds for the other solutions but there sunspots lead to explosive equilibria which we typically rule out.

[13] Cochrane (2017a) considers a model with two equations (the IS and Phillips curves) and two unknowns $\{x_t, \pi_t\}$, and he closes the model by specifying the interest rate path $\{i_t\}$ directly. Cochrane (2017a) argues that it is possible to reverse-engeneer a Taylor rule that is consistent with the constructed path, namely, given the path for $\{x_t, \pi_t, i_t\}$, it is possible to back up the implied i_t^*, ϕ_π, $\phi_{E\pi}$ and ϕ_y in (3). The underlying assumption behind this construction is that the agents do not realize that the interest rate follows the Taylor rule but believe that it follows the given path. In terms of our analysis, this approach is equivalent to setting ϕ_π, $\phi_{E\pi}$ and ϕ_y to zero.

3 Effectiveness of FG

In this section, we use the constructed closed-form solutions to explore the response of the economy to FG – an anticipated shock to the policy rule – under several alternative parameterizations of the Taylor rule. We discuss cases ii), iv) and i) in Sections 3.1, 3.2 and 3.3, respectively (to save on space, we omit the edge-of-the-knife case iii)). We consider case i) in the last place because it is usually associated with the analysis of ZLB, and the current paper is not concerned with zero-bound cases.

Among the cases established in Theorems 1-3, cases i) and ii) are the most studied ones in the context of FG; see, e.g., Werning (2015), McKay et al. (2016), Cochrane (2017a), and Husted et al. (2017). Case iv) is less explored but it is also plausible. In particular, it encompasses the values of ϕ_π and ϕ_y that are widely discussed in the literature to be the most empirically plausible, such as $\phi_\pi = 2$ and $\phi_y = 0.5$ (see Coibion et al., 2012). The condition that $\phi_\pi > 1$ (and $\phi_y = 0$) is referred to as the *Taylor Principle.* Recent Monetary Policy Reports of the Fed focus on this case: "Policy rules can incorporate key principles of good monetary policy. One key principle is ... . A third key principle is that, to stabilize inflation, the policy rate should be adjusted by more than one-for-one in response to persistent increases or decreases in inflation." (see the report from July 7, 2017, February 23, 2018, July 17, 2018).

3.1 Accommodating inflation: the FG puzzle

Under case ii) of Theorems 1–3, we attain the turning point when the smallest root m_2 reaches a unit size. There are many different combinations of the parameters $\phi_{E\pi}$, ϕ_π and ϕ_y that lead to a unit root. The most well-known case is *a forward guidance puzzle* – an observation that in a stylized new Keynesian model, output and inflation react excessively and unrealistically to central bank's announcements about future interest rates changes; see Del Negro et al. (2012, 2015) and McKay et al. (2016). To illustrate the FG puzzle, in line with McKay et al. (2016), we assume that the interest rate is determined by the Taylor rule with $i_t = i_t^n + \pi_{t+1} + \varepsilon_t$ (this is a specific case of the monetary policy rule (3) under $\phi_{E\pi} \searrow 1$, $\phi_\pi = 0$ and $\phi_y = 0$). The corresponding roots are $m_1 = \frac{1}{\beta}$ and $m_2 \searrow 1$, meaning that we are in case ii) of Theorems 1-2. Let us assume that all shocks are zero, except of the shock at T which is equal to $z_T = \frac{\kappa\sigma\varepsilon}{\beta}$. By (7), for $t \leq T$, the solution is given by

$$\pi_t = \frac{1}{1/\beta - 1}\left[\sum_{s=t}^{\infty}\left(\frac{1}{\beta}\right)^{t-s-1} z_s - \sum_{s=t}^{\infty} z_s\right] = \frac{\kappa\sigma\varepsilon}{1-\beta}\left[\beta^{T-t+1} - 1\right], \tag{10}$$

and $\pi_t = 0$ for $t > T$. From the Phillips curve (1), we have $x_t = \frac{1}{\kappa}(\pi_t - \beta\pi_{t+1}) = -\sigma\varepsilon$ for $t \leq T$ and $x_t = 0$ for $t > T$. This means that, a shock that will happen in any remote period T has the same effect on current output x_t as the one that happens at present. The impact of future shocks on inflation is even more dramatic: the further away the shock is in the future, the larger is its effect on today's inflation, as (10) shows. Note however that inflation is not explosive backward but rather converges to a limit, given by $\lim_{t \to -\infty} \pi_t = -\frac{\beta}{1-\beta} z_T = -\frac{\kappa\sigma\varepsilon}{1-\beta}$.

Figure 1 illustrates the FG puzzle graphically. The figure plots the output gap, inflation and nominal interest rate in response to a one-percent negative shock to the nominal interest rate that happens in the 30th quarter, $T = 30$. To produce this and all subsequent solutions, we parameterize the model by $\kappa = 0.11$, $\sigma = 1$ and $\beta = 0.99$.[14]

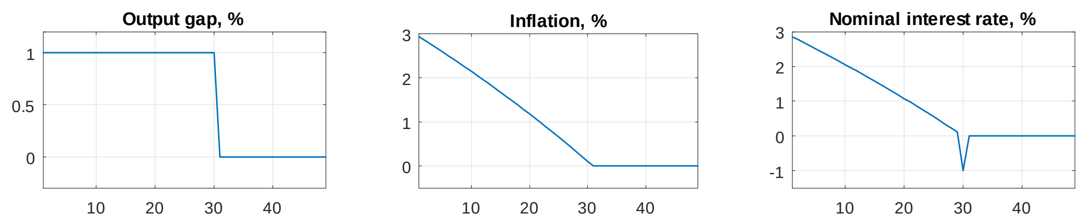

Figure 1. Forward guidance: Taylor rule with expected inflation, $\phi_{E\pi} \searrow 1$.

The output gap goes up immediately by one percent in response to a distant future shock. The inflation goes up by about three percent, and then gradually decreases and reaches the steady state level in period 30. We should point out that there will be the same magical effects of promised future fiscal stimuli and capital destruction. This is because a positive monetary policy shock acts similarly to a disturbance to government spendings and capital destruction.

Why are the future shocks so powerful? As was pointed out by Del Negro et al. (2012, 2015) and McKay et al. (2016), this happens because the effect of future shocks on output is not discounted. To see the point, let us apply forward recursion to the IS curve (1) by imposing a forward stability condition $\lim_{s \to \infty} x_{t+s} = 0$ (a steady-state value) and let us assume again the monetary policy rule $i_t = i_t^n + \pi_{t+1} + \varepsilon_t$. We get

$$x_t = -\sigma \sum_{s=t}^{\infty} (i_s - i_s^n - \pi_{s+1}) = -\sigma \sum_{s=t}^{\infty} \varepsilon_s. \tag{11}$$

(In the case of a single T-period shock, (11) leads to the same expression as our closed-form solution $x_t = -\sigma\varepsilon$). In formula (11), there is no discounting and all shocks have the same effect on output.

To study the robustness of the FG puzzle, let us consider a more general Taylor rule (3) with expected inflation such that $\phi_{E\pi} > 1$; for example, in Figure 2, we illustrate the case $\phi_{E\pi} = 3$, $\phi_\pi = 0$ and $\phi_y = 0$. Then, we are in case iv) of Theorems 1 and 2 with

[14] The value of $\kappa = 0.11$ corresponds to a fraction of non-reoptimizing firms $\theta = 0.83$ and a utility-function parameter $\vartheta = 2.09$; see our footnote 8 for the formula of κ.

unstable complex roots. In response to the shock in period 30, the model's variables start fluctuating from period 0 and on, up to period 30.

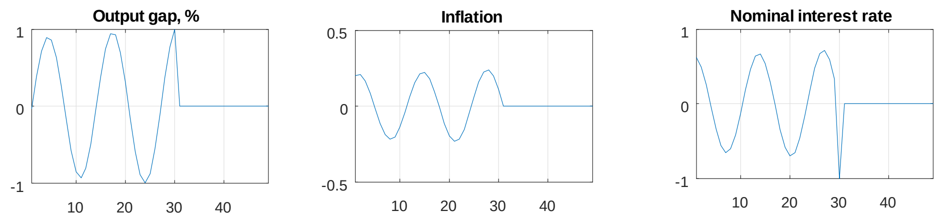

Figure 2. Forward guidance: Taylor rule with expected inflation, $\phi_{E\pi} = 3$.

There is a gradual decay but it is very slow, since we are close to unit root $|r| \approx 1.005$ under our benchmark calibration. While this oscillating case is discomforting, we draw attention that it is produced by a meaningful positive (although too strong) response of the interest rate to expected inflation; we cannot rule it out as a less appealing case of negative Taylor-rule coefficients, also leading to oscillations.

An interesting question is: How different would the results be if the monetary rule (3) depended on actual rather than expected inflation, i.e., $\phi_{E\pi} = 0$, $\phi_{\pi} = 1$ and $\phi_y = 0$, which implies $i_t = i_t^n + \pi_t + \varepsilon_t$? We analyze this case in Appendix D. We find that again we have a unit root $m_2 = 1$ but in that case, only inflation explodes backward but not output. Finally, we find that increasing ϕ_{π} leads to a more rapid decay of cyclical fluctuations in inflation. Thus, we conclude that the FG puzzle disappears, when $\phi_{E\pi}$ and ϕ_{π} increase although backward stabilization can be very slow.

3.2 Plausible Taylor rules: sensible FG effects

In this section, we show that the introduction of the output gap in the Taylor rule helps us insure backward stability in a sense that neither inflation nor the output gap react by large amounts to the announcement of the future interest-rate change. Therefore, the FG puzzle is not observed in that case.

Taylor rule with the output gap and anticipated inflation. Let us consider the rule (3) with anticipated inflation $\phi_{E\pi} > 0$ and the output gap $\phi_y > 0$. Substituting the Taylor rule (3) into the IS curve (1), we obtain $x_t = \frac{1}{1+\sigma\phi_y}\left[E_t x_{t+1} - \sigma\left(\phi_{E_\pi} - 1\right) E_t \pi_{t+1} - \sigma\varepsilon_t\right]$. By making recursive substitution and by imposing $\lim\limits_{s\to\infty} x_{t+s} = 0$, we get

$$x_t = -\left(\phi_{E_\pi} - 1\right)\widetilde{\beta}\sigma \sum_{s=t}^{\infty} \widetilde{\beta}^{s-t} E_s \pi_{s+1} - \widetilde{\beta}\sigma \sum_{s=t}^{\infty} \widetilde{\beta}^{s-t} \varepsilon_s, \tag{12}$$

where $\widetilde{\beta} \equiv \frac{1}{1+\sigma\phi_y}$ is an effective discount factor. Since $\phi_y > 0$, the effect of future shocks on today's output is discounted at the rate $\widetilde{\beta} < 1$, unlike under the FG puzzle (11). In particular, in the benchmark case $\phi_{E\pi} \searrow 1$, we obtain $x_t = -\widetilde{\beta}\sigma \sum_{s=t}^{\infty} \widetilde{\beta}^{s-t} \varepsilon_s$, which is identical to the FG-puzzle formula (11) (up to the multiplicative term) except that now we have discounting. Note that discounting is present for any $\phi_y > 0$, except of the limiting case $\phi_y = 0$, which corresponds to the FG puzzle.

Taylor rule with the output gap and actual inflation. Alternatively, we can consider the Taylor rule with actual inflation $\phi_\pi > 0$ and the output gap $\phi_y > 0$. Substituting the t-period Taylor rule (3) into (1) and imposing (2), we get $x_t = \frac{\beta}{\beta+\sigma\kappa+\sigma\beta\phi_y} x_{t+1} - \frac{\beta\sigma}{\beta+\sigma\kappa+\sigma\beta\phi_y}\left(\phi_\pi \pi_t - \frac{\pi_t}{\beta} + \varepsilon_t\right)$. Again, by using a forward recursive substitution of the future output gaps and by imposing $\lim_{s\to\infty} x_{t+s} = 0$, we obtain

$$x_t = -\left(\phi_\pi - \frac{1}{\beta}\right)\overline{\beta}\sigma \sum_{s=t}^{\infty} \overline{\beta}^{s-t} \pi_s - \overline{\beta}\sigma \sum_{s=t}^{\infty} \overline{\beta}^{s-t} \varepsilon_s, \tag{13}$$

where $\overline{\beta} \equiv \frac{\beta}{\beta+\sigma\kappa+\sigma\beta\phi_y} < 1$ is an effective discount factor. Like in the previous case, we have $\overline{\beta} < 1$, so that the effect of the shock ε_s on today's output is discounted at the rate $\overline{\beta}$. Furthermore, in a special case $\phi_\pi = \frac{1}{\beta}$, we obtain $x_t = -\overline{\beta}\sigma \sum_{s=t}^{\infty} \overline{\beta}^{s-t} \varepsilon_s$, which is again identical to the FG-puzzle formula (11) except for the presence of discounting (and the multiplicative term $\overline{\beta}$). Interestingly, discounting does not disappear even if $\phi_y = 0$ since we have $\overline{\beta} = \frac{\beta}{\beta+\sigma\kappa} < 1$ (as long as $\kappa > 0$). However, a larger output gap in the Taylor rule (3) makes discounting stronger.

A comparison of the Taylor rules with anticipated and actual inflation. In Figure 3, we compare the results under two Taylor rules (3) that both contain an identical output gap coefficient $\phi_y = 0.5$ but one rule contains just expected inflation $\phi_{E\pi} = 2$ (see a red line), while the other contains just actual inflation $\phi_\pi = 2$ (see a blue line). These are the conventional values used in the related literature, see, e.g., Taylor (1993), and Coibion et al. (2012). We used the constructed closed form solutions, to generate the series for the output gap, inflation and the interest rate for these two cases.

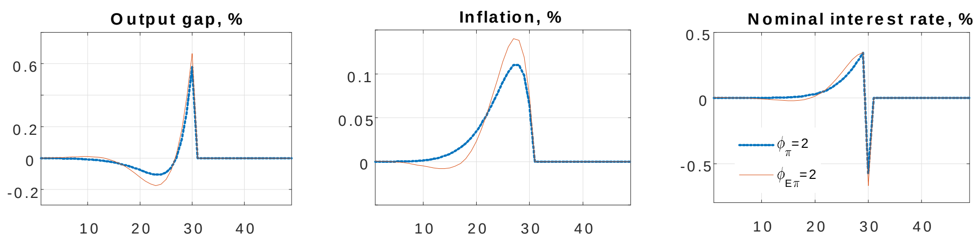

Figure 3. Forward guidance: Taylor rules with the output gap $\phi_y = 0.5$: a comparison of anticipated inflation , $\phi_{E\pi} = 2$, and actual inflation, $\phi_\pi = 2$.

In the figure, the two Taylor rule's parameterizations lead to similar qualitative behavior of the model's variables. This similarity can be understood by looking at our decompositions (12) and (13) which imply that the formulas for x_t are virtually identical although the effective discount factor may be different. Note that the FG puzzle is not observed in these cases: the effect of a distant shock on today's variables is negligible, i.e., inflation and output are backward stable. In sum, the plausible Taylor rule insures both backward stability and equilibrium uniqueness.

Finally, it turns out that for our calibration of the other model's parameters (see Section 3.1), the considered parameterization of the Taylor rule is close to the one that separates the cases ii) and iv) of Theorems 1-3. In particular, $\phi_{E\pi} = 2$ or $\phi_\pi = 2$ leads to complex roots, while $\phi_{E\pi} = 1.5$ or $\phi_\pi = 1.5$ leads to real roots. Qualitatively, dynamics in both cases look very similar; we show the case of the real roots in Figure D3 of Appendix D. The latter case is distinguished in Taylor (1993) as the most plausible one.

Can contemporaneous shocks mess up with FG shocks? Our key finding is that the FG puzzle depends on the Taylor-rule parameterization. It is obtained under the assumption of given contemporaneous policy shocks. An important question is: What would happen to effects of future (FG) shocks if we take into account contemporaneous shocks? In particular, if our result is true in a broad context, it should be also true that contemporaneous impulse responses are barely affected (or at the very least they are affected much less than the responses to future shocks) by changing the coefficients of the Taylor rule. To answer, we compare contemporaneous impulse responses for two parameterizations: $\phi_{E\pi} = 1$, $\phi_\pi = 0$, $\phi_y = 0$ and $\phi_{E\pi} = 0$, $\phi_\pi = 1.5$, $\phi_y = 0.5$ (our empirically plausible parameterization); both of these parameterizations lead to case ii) in the paper. We find that the differences in responses to a 1% contemporaneous shock are very small between these two parameterizations. Therefore, our findings that the puzzle depends on the monetary policy rule are still valid if contemporaneous shocks are included into the analysis.

3.3 Choosing the interest rate path directly: indeterminate effects of FG

We finally consider a version of the model that has one stable root and one unstable root, and as a result, has multiple (forward) stable solutions; see case i) of Theorems 1–3. An example of such model is the one in which the coefficients in the Taylor rule (3) are all equal to zero, i.e., $\phi_\pi = 0$, $\phi_y = 0$, $\phi_{E\pi} = 0$, so that the monetary authority does not follow any rule but choose the sequence of interest rates $i_0, i_1, ...$ directly.

By Theorem 3, case i), under $|m_1| \geq 1$ and $|m_2| < 1$, a collection of stable solutions is obtained from (6) by setting C_1 – a constant associated with an unstable eigenvalue m_1 – to zero. An impulse-response function takes the form

$$\pi_t = C_2 m_2^t + \frac{1}{m_1 - m_2} E_t \left[\sum_{s=t}^{\infty} m_1^{t-1-s} z_s + \sum_{s=0}^{t-1} m_2^{t-1-s} z_s \right], \tag{14}$$

where C_2 is an arbitrary constant. In particular, for a single anticipated shock, $\varepsilon_t = 0$ for $t \neq T$, and $\varepsilon_T = \varepsilon$, we have $z_t = 0$ for $t \neq T$, and $z_T = \frac{\kappa\sigma\varepsilon}{\beta}$. Substituting the latter result into the solution (14), we get the following impulse responses:

$$t \leq T,\ \pi_t = C_2 m_2^t + \frac{\kappa\sigma\varepsilon}{\beta(m_1 - m_2)} m_1^{t-1-T}; \tag{15}$$

$$t > T,\ \pi_t = C_2 m_2^t + \frac{\kappa\sigma\varepsilon}{\beta(m_1 - m_2)} m_2^{t-1-T}. \tag{16}$$

That is, the economy is driven by a forward-looking component $\sum_{s=t}^{\infty} m_1^{t-1-s} z_s$ before the shock occurs and it is driven by a backward-looking component $\sum_{s=-\infty}^{t-1} m_2^{t-1-s} z_s$ afterwords. Since stability is consistent with any C_2, we can choose it in an arbitrary manner.

A combination $|m_1| \geq 1$ and $|m_2| < 1$ is related to the zero-bound case on which much of the FG literature focuses; see. e.g., Cochrane (2017a). The impulse response to a central bank's promise to hold the interest rate at zero will take the form (15), (16). These solutions generically converge going forward, meaning that they diverge going backward: announcements of policy deviations further in the future have larger effects today. This result, which extends to output, is the classic "forward guidance puzzle" of the zero-bound literature, and it extends to any passive policy regime, e.g., $\phi_\pi < 0$, $\phi_y = 0$, $\phi_{E\pi} = 0$. Here, the FG puzzle is a smooth function of m_2.

For our model, we plot the impulse responses (15), (16) for $T = 30$ in Figure 4. For our benchmark calibration of κ, σ and β, described in Section 3.1, the eigenvalues are $m_1 = 1.4052$ and $m_2 = 0.7188$. We display two stable equilibria: in one of them, we chose

$C_2 = 0$, and in the other, we chose C_2 such that initial inflation is zero, i.e., $\pi_0 = 0$.

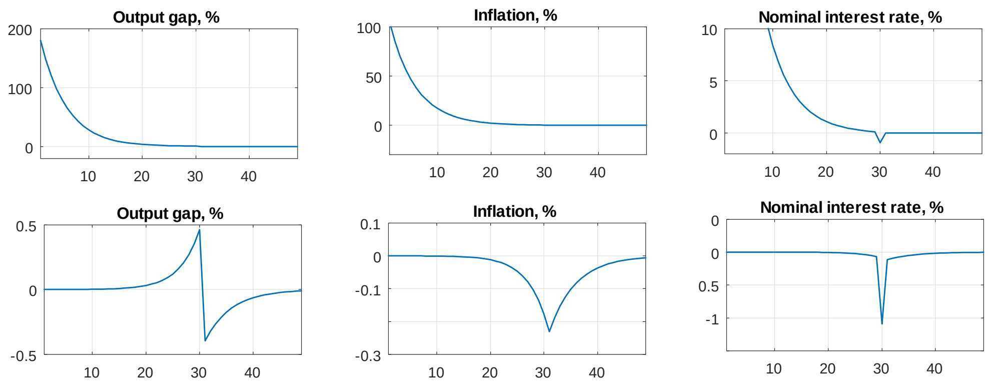

Figure 4. Multiple equilibria under $\phi_{E\pi} = 0$, $\phi_\pi = 0$, $\phi_y = 0$: equlibria with $C_2 = 0$ (top panels) and with $\pi_0 = 0$ (bottom panels)

Cochrane (2017a) argued that if the monetary authority follows a discretionary policy of choosing the interest rate path, the effect of FG on the economy depends critically on the equilibrium selection (i.e., on the integration constant C_2), ranging from extremely large to practically non-existent. In the figure, a possible impact of FG on output ranges from a drastic immediate increase in output (equilibrium with $C_2 = 0$) to an essentially absent immediate increase in output (equilibrium with $\pi_0 = 0$). Similarly, the effect on inflation is dramatic in the former case, and it is very mild in the latter case. The first panel is a classical FG puzzle response. Thus, large differences between multiple equilibria emphasized by Cochrane (2017a) are not limited to his setup with a zero response to inflation but are a generic property of the model in which the monetary policy is not sufficiently responsive to insure the equilibrium uniqueness.

The zero-bound literature reasons that some choices of C_2 make more sense than the others. To illustrate, suppose there is a transitory IS disturbance that lasts from period 0 to T (e.g., the real interest rate happens to be negative in this interval of time). Werning (2012) selected an equilibrium with $\pi_{T+1} = 0$, arguing that people will expect it because of forward-looking optimality. This choice of inflation implies a specific value of C_2 and $\pi_0 \neq 0$. The resulting solution generically explodes backwards and therefore, implies the forward-guidance puzzle. Cochrane (2017a) proposed to resolve the puzzle by setting $\pi_0 = 0$. Similar to the case of the bottom graph in Figure 4, the responses converge going backward. To justify, he noted that $\pi_0 < 0$ represents an unexpected deflation that induces an increase in the value of government debt which requires fiscal tightening to pay off. Absent such fiscal policy, we have $\pi_0 = 0$, which, as argued above, resolves the puzzle.

3.4 Backward stability and turnpike theorem

All our solutions are constructed in a way that makes them forward stable. Whether they are also backward stable or not depends on a specific parameterization of the Taylor rule. The FG puzzle in Figure 1 is an example of backward unstable (explosive) solution and so are the examples shown in Figure 4 (upper panel) and Figure 2 (see also Figure D1 in Appendix D). In turn, the solutions in Figure 4 (bottom panel) and Figure 3 are backward stable (see also Figure D2 in Appendix D). Backward stability can be related to the so-called *turnpike theorem* which is well-known for the neoclassical growth model; see Brock (1971) and McKenzie (1976). The turnpike theorem states that a backward-looking trajectory of the finite-horizon growth model is situated arbitrary close to that of the corresponding infinite-horizon model in initial periods, provided that time horizon is sufficiently large; see Maliar, Maliar, Taylor and Tsener (2015) for a formulation of the turnpike theorem and discussion.

An important role in turnpike analysis is played by the rate of backward convergence. As an illustration, in our model, we vary the terminal condition under two Taylor rules (3): one rule has just actual inflation $\phi_\pi = 3$ and the other rule has just expected inflation $\phi_{E\pi} = 3$.

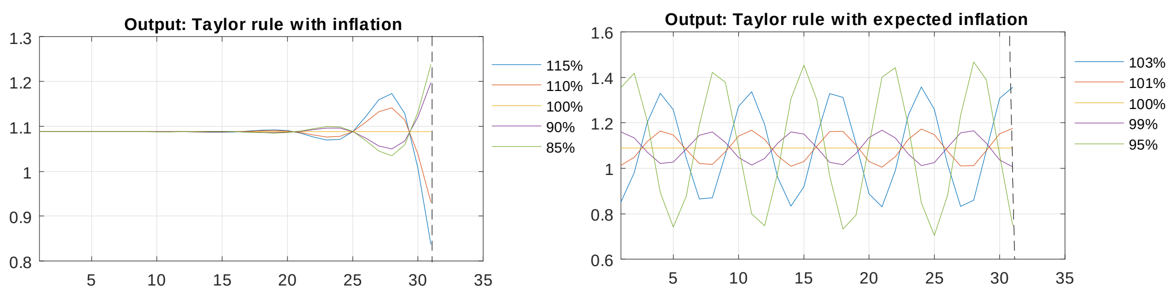

Figure 5. The effect of variations in the terminal condition under the Taylor rules with $\phi_\pi = 3$ and $\phi_{E\pi} = 3$.

In both models, the roots are complex, so by case iv) of Theorem 3, the solutions are backward stable and thus, satisfy the turnpike theorem. While the model with actual inflation in Figure 5 shows a rapid backward convergence pattern, the model with expected inflation appears to have a non-convergent cyclical pattern. In the latter model, the root is close to unity: as a result, the fluctuations decay so slowly that they appear to be nonvanishing (we had similar tendency in Figure 2).

It is easy to check that the backward explosive case i) of Theorems 1-3 and the FG-puzzle case violate the turnpike theorem. The corresponding figures would look like Figure 4 (upper panel) and Figure 1, respectively. (in effect, variations in the terminal condition are similar to future shocks).

Let us consider the FG-puzzle case which violates the turnpike theorem. Our analysis suggests that the monetary authority's rule that leads to the FG-puzzle case is suboptimal. Indeed, high inflation is costly for the economy, and yet the monetary authority adjusts interest rate to fully fit the inflation expectation $i_t = i_t^n + \pi_{t+1} + \varepsilon_t$. For example, if the people expect 300 percent inflation, the monetary authority will do nothing but set 302 percent nominal interest rate to guarantee the real interest rate of 2 percent. Such "passive" monetary authority response is the reason for why the FG-puzzle solution in Figure 1 explodes backward.

In contrast, the models that satisfy the turnpike theorem agree with our intuition and a common sense. In such models, the future commitments have a sizable effect on today's variables only if such commitments refer to relatively near future. The farther away the commitments are advanced in the future, the less impact they have at present. That is, the effect of FG will increase as the economy approaches the period of the shock (or the terminal condition). These implications hold for the neoclassical growth model, as well as for the new Keynesian models under the plausible Taylor rules (e.g., $\phi_\pi = 1.5$ and $\phi_y = 0.5$). Moreover, Woodford (2001) shows that such a Taylor rule provides a good approximation to the Ramsey policy rule, which suggests that the optimal policy rules are also backward stable and do not lead to the FG puzzle.[15] However, if the monetary authority happens to follow some arbitrary and suboptimal policy rules such as accommodation of high inflationary expectations, it would not be surprising to see dramatic consequences in line with those predicted by the FG-puzzle example. There seems to be a paradox here: on one hand, an optimal policy rule implies a large response of the interest rate to inflation (e.g., $\phi_\pi = 1.5$). On the other hand, a lower ϕ_π (e.g., $\phi_\pi \leq 1$) gives the central bank more power to affect the economy by open mouth policy.

4 Robustness of our findings

Up to now, we have studied a linearized version of the basic new Keynesian model that admits closed-form solutions. More general versions of the model do not admit closed-form solutions, so we resort to numerical analysis. The model with FG is non-stationary, the optimal decision rules change from one period to another, driven by anticipatory effects, and the conventional numerical methods that construct time-invariant value and policy functions are not applicable. We use two methods that are designed to solve such models: an extended path method of Fair and Taylor (1983) and an extended function path method of Maliar et al. (2015, 2017). In both methods, we impose (forward) stability: for the former method, we assume that in the terminal period, the economy arrives in the steady state, while for the latter method, we assume that the economy asymptotically converges to stationary. Thus, all the equilibria in our simulations are forward stable equilibria by construction. We first ask whether or not our findings are robust to the introduction

[15] The analysis of Woodford (2001) requires some additional assumptions; see Woodford (2003) for a discussion and generalizations.

of nonlinearity; we then introduce uncertainty in the nonlinear model by assuming six exogenous shocks; and we finally augment the nonlinear model to include capital.

4.1 Nonlinearity

In this section, we consider a nonlinear version of the basic new Keynesian model. The difference between global and local determinacy might show up in the behavior of the real economy in times of extreme inflation (not just in ZLB). Specifically, we consider a nonlinear new Keynesian model analyzed in Maliar and Maliar (2015).[16] The economy is populated by households, final-good firms, intermediate-good firms, monetary authority and government. In particular, the monetary authority follows a Taylor rule

$$R_t \equiv R_* \left(\frac{R_{t-1}}{R_*}\right)^{\mu} \left[\left(\frac{E_t \pi_{t+1}}{\pi_{tar}}\right)^{\phi_{E\pi}} \left(\frac{\pi_t}{\pi_{tar}}\right)^{\phi_\pi} \left(\frac{Y_t}{Y_{N,t}}\right)^{\phi_y}\right]^{1-\mu} \exp\left(\eta_{R,t}\right), \tag{17}$$

where R_t is the gross nominal interest rate at t; R_* is the steady state level of nominal interest rate; π_{tar} is the target inflation; $Y_{N,t}$ is the natural level of output; and $\eta_{R,t}$ is a monetary shock following the standard first-order autoregressive process. In addition to the three variables in our baseline linearized model of Section 2, the nonlinear model has consumption, labor, price dispersion and the supplementary variables S and F following from the profit maximizing conditions of monopolistic firms; see Appendix A for the model's description, the list of the first-order, the calibration and solution procedures, as well as a list of linearized equations. [17]

As an example, in Figure 6, we compare the effect of FG on linear and nonlinear solutions under the Taylor rule (17) parameterized by $\phi_{E\pi} = 0$, $\phi_\pi = 1/\beta$ and $\phi_y = 0$ (we assume $\mu = 0$). The anticipated shock here (as well as in the rest of Section 5) happens

[16] It would be interesting to explore in details eigenvalues and global behavior of the FRBNY model, studied in Del Negro et al. (2015). The last paper just reported few impulse responses without exploring alternative model's parameterizations.

[17] The linearized version of the model does not correspond exactly to the three-equation model studied before, e.g., the former includes government spendings and the endogenous natural level of output, the presence of which does not lead to the three-equation model; see Appendix A.

at $T = 20$.

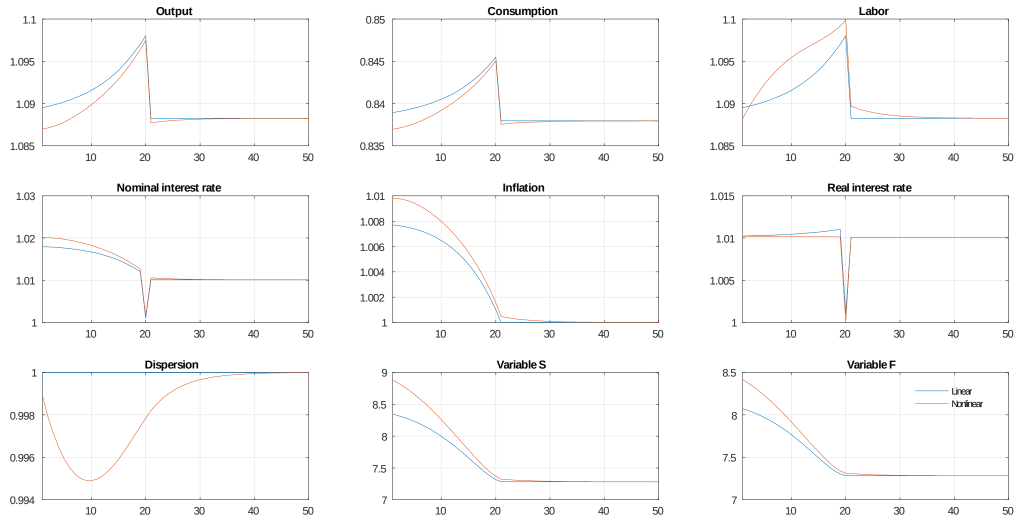

Figure 6. A comparison of linear and nonlinear solutions under $\phi_{E\pi} = 0$, $\phi_\pi = 1/\beta$ and $\phi_y = 0$.

We can see some qualitative differences between the linear and nonlinear solutions. For example, in the initial period, the nonlinear model predicts that output goes down, while the linear model predicts that it goes up. However, quantitatively, these differences are not very significant.

We explore a number of other parameterizations and obtain similar results. For example, for the Taylor rule with the output gap ($\phi_y = 0.5$) and persistence in the interest rate ($\mu = 0.82$), the difference between the linearized and nonlinear solutions is minimal, independently of whether the rule is parameterized by expected inflation or actual inflation . Moreover, similar to the linearized model (see Figure 3), we find that the nonlinear solutions are practically indistinguishable in two cases of $\phi_{E\pi} = 3$ and $\phi_\pi = 0$ and $\phi_\pi = 3$ and $\phi_{E\pi} = 0$.

4.2 Multiple sources of uncertainty

We next study how the introduction of more general sources of uncertainty affects the model's predictions about the effectiveness of FG. As described in Appendix A, we introduce six different shocks into the nonlinear model. As an example, in Figure 7, we introduce uncertainty in the nonlinear model which exhibited the FG puzzle, i.e., we parameterize the Taylor rule (17) by $\phi_{E\pi} \searrow 1$, $\phi_{\pi} = 0$; $\phi_{y} = 0$ and $\mu = 0$).

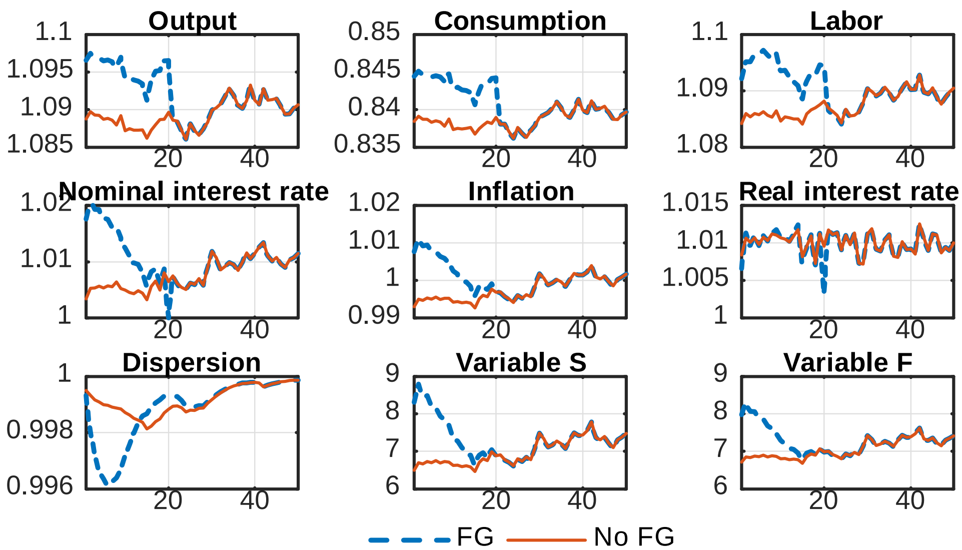

Figure 7. Forward guidance in the nonlinear stochastic model $\phi_{E\pi} \searrow 1$, $\phi_{\pi} = 0$ and $\phi_{y} = 0$.

The time series in Figure 7 look very similar to the FG puzzle dynamics in the corresponding deterministic model of Section 3.1. The output and inflation jump up in the initial period.

We also analyze the model with the Taylor rule that includes actual inflation $\phi_{E\pi} = 0$, $\phi_{\pi} \searrow 1$, $\phi_{y} = 0$ and $\mu = 0$ (not reported). The effect of the FG in the model with uncertainty is very similar to the one in the deterministic version of the model analyzed in Appendix D; see Figure D1. In our experiments, those parameterizations of the Taylor rule that led to backward stable solutions in the deterministic model also result in backward stable solutions in the model with uncertainty. Overall, we conclude that the introduction of uncertainty does not significantly affect the predictions of the model about the FG effectiveness.

4.3 Capital

We finally study how the introduction of capital into the basic new Keynesian model affects the model's implications about the effectiveness of FG; see Appendix E for a description of such a model. In Figure 8, we show the non-linear solution under the Taylor rule (17) parameterized by either expected inflation $\phi_{E\pi} = 2$, $\phi_{\pi} = 0$ or actual inflation $\phi_{E\pi} = 0$,

$\phi_\pi = 2$ and the remaining coefficients are set at $\phi_y = 0.5$ and $\mu = 0.82$.

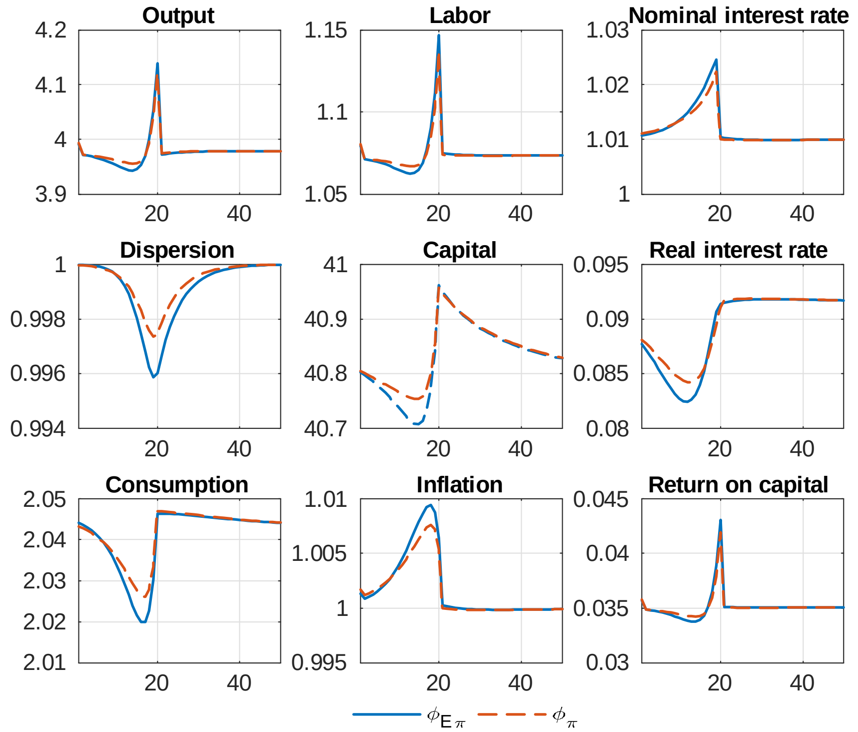

Figure 8. Model with capital: Taylor rules with expected inflation $\phi_{E\pi} = 2$, versus actual inflation $\phi_\pi = 2$ under $\phi_y = 0.5$ and $\mu = 0.82$.

With capital and the lagged nominal interest rate in the Taylor rule, the model has now two additional endogenous state variables. In this model, output responds not only to labor-input but also to capital-input changes. These differences do not affect qualitative implications of the model about the FG effectiveness: the series in Figure 8 are backward stable and qualitatively similar to those in the baseline three equation model in Figure 3.

We perform a number of sensitivity experiments by varying the parameters in the Taylor rule (17). Some of these experiments are provided in Appendix E. In particular, in Figures E1-E3, we show the same experiments for the model with capital as those shown for the basic linear model in Figures 2, D1, D2. We do observe some qualitative differences between the model with capital and the basic linearized model in Section 3 but our most important finding remains unchanged: the Taylor rule with a weak response to inflation, e.g., $\phi_{E\pi} \searrow 1$ or $\phi_\pi \searrow 1$, produces backward explosive dynamics with some version of the FG puzzle, while a more responsive monetary policy (for example, with larger values of ϕ_π or with the inclusion of the output gap ϕ_y) eliminates the puzzle.

5 FG puzzle with and without ZLB

FG became well-known as a policy instrument during the period of an active zero lower bound. Central banks used FG because they had a limited ability to conduct conventional monetary policy. However, FG was first analyzed theoretically in models with no zero lower bound, and the FG puzzle was also discovered in such models; see Del Negro et al (2012, 2015) and McKay et al. (2016).

To be more specific, Del Negro et al. (2015) highlight an excessive power of FG in the context of FRBNY DSGE model. Their theoretical analysis assumes an operating Taylor principle and no zero lower bound (i.e., no max operator in the Taylor rule), and they look for a "...solution to the log-linear approximation of the model's equilibrium conditions around the deterministic steady state". To explain why the FG puzzle holds in their model, Del Negro et al (2015) offer the following explanation (see their p. 4): "...forward guidance puzzle results from the interaction of many features of DSGE models. These include the excess sensitivity of consumption to interest rate changes and the front-loading associated with the New Keynesian Phillips curve, which have long been criticized for being counterfactual. In addition, we stress the excessive response of consumption to interest rate changes far in the future implied by the standard consumption Euler equation, emphasized also in a recent paper by McKay et al. (2015). As we explain in Section 3.3, what is novel about the forward guidance policy experiment is that it compounds all these implications, bringing to the fore the limitations of typical medium scale DSGE models used for policy analysis." Thus, their analysis suggests that the considered medium-scale DGSE model is not sufficiently large and realistic to resolve the FG puzzle.

Similarly, the paper of McKay et al. (2016) contains an extensive discussion of various ingredients needed for resolution of the FG puzzle. Their baseline analysis also abstracts from the zero lower bound, namely, they make the following assumption about the policy: "Suppose for simplicity that the monetary policy of the central bank is given by an exogenous rule for the real interest rate where the real interest rate tracks the natural real rate with some error: $r_t = i_t - E_t\pi_{t+1} = r_t^n + \epsilon_{t,t-j}$ ...Suppose...the monetary authority announces that the real interest rate will be lower by 1 percent for a single quarter five years in the future, but maintained at the natural real rate of interest in all other quarters..." This policy is assumed in both the standard and modified models extended to include incomplete markets (see, e.g., pages 3134, 3135, 3137 and 3146). Interestingly, in the lower-bound scenario, their incomplete-market model entirely falls apart, producing backward explosions even larger than the standard model; see their Section C.

On the other hand, Campbell et al. (2016) find that the FG puzzle is not a generic feature of medium-scale NK models. Their medium-scale NK model estimated with US data differs from that in Del Negro et al. (2015) in several dimensions, and it produces realistic responses for empirically plausible interest-rate pegs. See also more recent literature that focuses on resolving the FG puzzle, including Husted et al. (2017), Kaplan et al. (2017) and Gabaix (2017).

In contrast, the papers of Carlstrom et al. (2015), Werning (2015) and Cochrane

(2017a) study the effects of FG by explicitly focusing on a lower bound scenario. In particular, Carlstrom et al. (2015) assumes that after the ZLB episode is over, the economy follows the standard Taylor rule with empirically plausible values of coefficients $\phi_\pi = 1.5$ and $\phi_y = 0.5$. The solution is worked out backward by starting from the steady state, by proceeding backward up to the period when the lower bound begins to bind, and by constructing the solution up to the initial period. In the three equation model, Carlstrom et al. (2015) explain backward explosiveness (i.e., the FG puzzle) by looking at eigenvalue but this is an unstable eigenvalue rather than a stable one (as follows from their explanation on page 233, integration constant on the unstable eigenvalue is not set to zero). Thus, that paper highlights a different mechanism for generating explosive behavior than FG puzzle compared to the literature that analyzes forward stable solutions. In the medium-scale DSGE models of Christiano et al. (2005) and Smets and Wouters (2007), Carlstrom et al. (2015) find diverse effects of FG – from none to very large – depending on the number of FG periods and on the presence of inflation indexation. These effects seem to follow a very complex, sometimes "nonsensical" structure.

In turn, Werning (2015) and Cochrane (2017a) show that backward explosion is still present even if we restrict attention to forward-stable solutions. In particular, Cochrane (2017a) demonstrates that the stylized new Keynesian model has multiplicity of equilibria during the ZLB episodes and that it may or may not lead to a backward explosion (the FG puzzle) depending on the equilibrium selection via the choice of integration constant on stable eigenvalue. The effectiveness of FG and the magnitude of backward explosion depend on the size of the smaller eigenvalue. In particular, there is no FG puzzle if the monetary authority manages to coordinate on equilibrium which involves no backward explosion.

The contribution of the present paper is to show that for the baseline three-equation new Keynesian model, the whole issue of FG is about the size of a smaller eigenvalue. This is true both for the economy facing active ZLB and the one away from the ZLB constraint. The backward explosion away from ZLB is similar to the one established in Cochrane (2017a) under the lower-bound scenario. Our analytical results make it possible to cleanly see whether the new Keynesian model leads to the FG puzzle or not under a given parametrization. The size of the smallest eigenvalue is a sufficient statistics to capture the role of all the ingredients and parameters in the backward explosion.

Do we resolve FG puzzle? In a sense, yes we do, namely, we demonstrate analytically that the baseline three equation new Keynesian model parameterized by a plausible Taylor rule does not produce a backward explosion. We show that similar quantitative results hold for the nonlinear version of the baseline new Keynesian model and the version of the model with capital. We conclude that baseline new Keynesian model does not have the FG puzzle as long as plausible Taylor rules are used.

In the broader literature on the FG puzzle, our paper may be interpreted as follows: Previous work has shown that various (behavioral) frictions can dampen the economy's response to rate cuts in the far future. The present paper establishes that such dampening is already implied by plausible Taylor rules. However, this particular source of dampening might not be enough to break the FG puzzle in all models, so additional frictions might be needed. In particular, there are models that produce excessively large responses to the Fed announcement under empirically plausible Taylor rules, e.g., Del Negro et al. (2012, 2015), McKay et al. (2016) and Carlstrom et al. (2015). Presumably, such models have additional ingredients that accentuate the backward explosion. Our analysis does not provide a basis to determine which ingredients of these models lead to FG puzzle. But we can affirm that the FG puzzle is not present in the stylized new Keynesian model under plausible Taylor rules, as well as in the version of this model with capital; this is true for both linear and nonlinear analysis.

There is another sense in which we do not resolve the FG puzzle, namely, nothing in our analysis prevents monetary authority from taking advantage of the open-mouth policy by using the rules that lead to backward explosion, e.g., $i_t = i_t^n + \pi_{t+1} + \varepsilon_t$. However, we remind that rules like that are suboptimal. One of the fundamental principals of monetary theory is that the central bank aims to maximize social welfare. If the central bank does something else, the models in this paper predict a variety of empirically implausible outcomes, such as the FG puzzle, or even large cyclical fluctuations. The fact that suboptimal monetary policies lead to counterintuitive and counterfactual implications is not entirely surprising: one would expect equally puzzling implications from the models in which consumers do not maximize utilities or firms do not maximize profits. A reasonable approach would be for the monetary authority to aim at attaining an optimal outcome by implementing a Taylor-style rule that approximates Ramsey policy; and such an approach does not lead to the FG puzzle.

6 Conclusion

In this paper we examine the impact of FG—defined as an announced and anticipated deviation of the central bank's policy interest rate from a policy rule, and we showed that the impact depends crucially on the form of the policy rule. We concentrate on the "FG puzzle," in which such an announced deviation of the policy interest rate from the policy rule, even in the far future, has huge, even implausible, effects on inflation or output today.

We use a three-equation new Keynesian model with an Euler equation, a price adjustment equation, and a policy rule. We delineate four regions of the parameter space for this model and prove analytically that these regions correspond to (i) one stable and one unstable root for which there are multiple stable solutions and the FG puzzle may arise, (ii) one unit root and one unstable root for which there is a unique stable solution and the FG puzzle arises, and (iii and iv) two unstable roots for which there is a unique stable solution and the FG puzzle does not arise. In region (i)-(iii) the roots are real, and in region (iv) they are complex.

We show that coefficients of the policy rule—the responsiveness of the policy interest rate to output, inflation, and expected inflation—are the key determinants of each region. We argue that the policy rules that generate regions (iii) and (iv), where there is no

FG puzzle, constitutes far better and more empirically realistic monetary policy. First, these regions include the parameters of the original Taylor (1993) rule in which the Taylor Principle that the response to inflation is greater than one holds. Second, the region that includes the Taylor principle accords well with the fully optimal monetary policy rule for this model, which is derived analytically from a Ramsey-type intertemporal optimization problem. Third, the Fed argues in its recent monetary policy reports that a key monetary principle is that "the policy rate should be adjusted by more than one-for-one" in response to inflation, which also generates the regions with no FG puzzle. Fourth, empirical results show that monetary policy works better when the policy rule parameters are in this region.

We show that these results are robust by considering nonlinear models and solving them numerically, and by adding capital and labor to the model. We consider alternative solution techniques and a variety of terms in the policy rule. We also show that the models without the FG puzzle correspond to well-known turnpike properties of growth models.

While FG became popular at the time of binding ZLB during and after the Great Recession, it was used before that time by the Fed and other central banks, and surveys show that there is interest in FG at central banks in the future. Indeed, out of 55 heads of central banks, surveyed in Blinder et al. (2016), none said that FG should be discontinued after the crisis; 59% and 12.8% think that it is a potential instrument in the same and modified form, respectively. Among monetary policymakers, there is a significant voice calling for FG in normal times. Bernanke (2017) argues that FG can be useful before the next recession hits, by noting that "... when ZLB looms, rate cuts should be aggressive ... Forward guidance, of the Odyssean variety, would come next ... . Relative to earlier experience, I would expect a much earlier adoption of state-contingent, quantitative commitments to hold rates low." The former Fed's chair, Yellen (2018) has a similar opinion by arguing that " the FOMC should seriously consider pursuing a lower-for-longer or makeup strategy for setting short rates when the zero lower bound binds and should articulate its intention to do so before the next zero lower bound episode". Mester (2014) views FG as a device that in normal time "conveys to the public how policy is likely to respond to changes in economic conditions"; Coeuré (2018) also supports the usefulness of FG "beyond the timing to lift-off", etc. The results in this paper indicate that the best FG is for central banks to base policy decisions on rules in which there is no FG puzzle, as well as to indicate that they will base policy decisions in the future.

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Appendix A. The nonlinear new Keynesian model

In this section, we describe the basic new Keynesian model that leads to the three-equation model (1), (2) and (3) studied in the main text.

Households. The representative household solves

$$\max_{\{C_t,L_t,B_t\}_{t=0,\dots,\infty}} E_0\sum_{t=0}^{\infty}\beta^t \exp\left(\eta_{u,t}\right)\left[\frac{C_t^{1-\sigma}-1}{1-\sigma}-\exp\left(\eta_{L,t}\right)\frac{L_t^{1+\vartheta}-1}{1+\vartheta}\right] \tag{18}$$

$$\text{s.t. } P_tC_t+\frac{B_t}{\exp\left(\eta_{B,t}\right)R_t}+T_t=B_{t-1}+W_tL_t+\Pi_t, \tag{19}$$

where $\left(B_0,\eta_{u,0},\eta_{L,0},\eta_{B,0}\right)$ is given; C_t, L_t, and B_t are consumption, labor and nominal bond holdings, respectively; P_t, W_t and R_t are the commodity price, nominal wage and (gross) nominal interest rate, respectively; $\eta_{u,t}$ and $\eta_{L,t}$ are exogenous preference shocks to the overall momentary utility and disutility of labor, respectively; $\eta_{B,t}$ is an exogenous premium in the return to bonds; T_t is lump-sum taxes; Π_t is the profit of intermediate-good firms; $\beta\in(0,1)$ is the discount factor; $\sigma>0$ and $\vartheta>0$ are the utility-function parameters. The shocks follow standard AR(1) processes with means zero and constant standard deviations.

Final-good firms. Perfectly competitive final-good firms produce final goods using intermediate goods. A final-good firm buys $Y_t(i)$ of an intermediate good $i\in[0,1]$ at price $P_t(i)$ and sells Y_t of the final good at price P_t in a perfectly competitive market. The profit-maximization problem is

$$\max_{Y_t(i)} \quad P_tY_t-\int_0^1 P_t(i)\,Y_t(i)\,di \tag{20}$$

$$\text{s.t. } Y_t=\left(\int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}}\,di\right)^{\frac{\varepsilon}{\varepsilon-1}}, \tag{21}$$

where (21) is a Dixit-Stiglitz aggregator function with $\varepsilon\geq 1$.

Intermediate-good firms. Monopolistic intermediate-good firms produce intermediate goods using labor and are subject to sticky prices. The firm i produces the intermediate good i. To choose labor in each period t, the firm i minimizes the nominal total cost, TC (net of government subsidy v),

$$\min_{L_t(i)} \quad \mathrm{TC}\left(Y_t(i)\right)=(1-v)\,W_tL_t(i) \tag{22}$$

$$\text{s.t. } Y_t(i)=\exp\left(\eta_{a,t}\right)L_t(i), \tag{23}$$

$$\eta_{a,t+1}=\rho_a\eta_{a,t}+\epsilon_{a,t+1}, \qquad \epsilon_{a,t+1}\sim\mathcal{N}\left(0,\sigma_a^2\right), \tag{24}$$

where $L_t(i)$ is the labor input; $\exp(\eta_{a,t})$ is the productivity level such that $\eta_{a,t}$ follows the standard AR(1) process. The firms are subject to Calvo-type price setting: a fraction $1-\theta$ of the firms sets prices optimally, $P_t(i) = \widetilde{P}_t$, for $i \in [0,1]$, and the fraction θ is not allowed to change the price and maintains the same price as in the previous period, $P_t(i) = P_{t-1}(i)$, for $i \in [0,1]$. A reoptimizing firm $i \in [0,1]$ maximizes the current value of profit over the time when $\widetilde{P}_t$ remains effective,

$$\max_{\widetilde{P}_t} \sum_{j=0}^{\infty} \beta^j \theta^j E_t \left\{ \Lambda_{t+j} \left[\widetilde{P}_t Y_{t+j}(i) - P_{t+j} \mathrm{mc}_{t+j} Y_{t+j}(i) \right] \right\} \tag{25}$$

$$\text{s.t. } Y_t(i) = Y_t \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon}, \tag{26}$$

where (26) is the demand for an intermediate good i (follows from the first-order condition of (20), (21)); Λ_{t+j} is the Lagrange multiplier on the household's budget constraint (19); mc_{t+j} is the real marginal cost of output at time $t+j$ (which is identical across the firms).

Government. Government finances a stochastic stream of public consumption by levying lump-sum taxes and by issuing nominal debt. The government budget constraint is

$$T_t + \frac{B_t}{\exp(\eta_{B,t}) R_t} = P_t \frac{\overline{G} Y_t}{\exp(\eta_{G,t})} + B_{t-1} + v W_t L_t, \tag{27}$$

where $\overline{G}$ is the steady-state share of government spending in output; $v W_t L_t$ is the subsidy to the intermediate-good firms; $\eta_{G,t}$ is a government-spending shock, that follows the standard AR(1) process.

Monetary authority. The monetary authority follows a Taylor rule

$$R_t \equiv R_* \left(\frac{R_{t-1}}{R_*} \right)^{\mu} \left[\left(\frac{E_t \pi_{t+1}}{\pi_{tar}} \right)^{\phi_{E\pi}} \left(\frac{\pi_t}{\pi_{tar}} \right)^{\phi_\pi} \left(\frac{Y_t}{Y_{N,t}} \right)^{\phi_y} \right]^{1-\mu} \exp(\eta_{R,t}), \tag{28}$$

where R_t is the gross nominal interest rate at t; R_* is the steady state level of nominal interest rate; π_{tar} is the target inflation; $Y_{N,t}$ is the natural level of output; and $\eta_{R,t}$ is a monetary shock following the standard AR(1) process.

Natural level of output. The natural level of output $Y_{N,t}$ is the level of output in an otherwise identical economy but without distortions. It is a solution to the following planner's problem

$$\max_{\{C_t, L_t\}_{t=0,\ldots,\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \exp(\eta_{u,t}) \left[\frac{C_t^{1-\sigma} - 1}{1-\sigma} - \exp(\eta_{L,t}) \frac{L_t^{1+\vartheta} - 1}{1+\vartheta} \right] \tag{29}$$

$$\text{s.t. } C_t = \exp(\eta_{a,t}) L_t - G_t, \tag{30}$$

where $G_t \equiv \frac{\overline{G} Y_t}{\exp(\eta_{G,t})}$ is given, and $\eta_{u,t+1}$, $\eta_{L,t+1}$, $\eta_{a,t+1}$, and $\eta_{G,t}$ follow the same processes as in the nonoptimal economy. The FOCs of the problem (29), (30) imply that $Y_{N,t}$ depends only on exogenous shocks (see equation (39) below).

Equilibrium conditions. We summarize the equilibrium conditions below:

$$S_t = \frac{\exp\left(\eta_{u,t} + \eta_{L,t}\right)}{\exp\left(\eta_{a,t}\right)} L_t^{\vartheta} Y_t + \beta \theta E_t \left\{\pi_{t+1}^{\varepsilon} S_{t+1}\right\}, \quad (31)$$

$$F_t = \exp\left(\eta_{u,t}\right) C_t^{-\sigma} Y_t + \beta \theta E_t \left\{\pi_{t+1}^{\varepsilon-1} F_{t+1}\right\}, \quad (32)$$

$$C_t^{-\sigma} = \frac{\beta \exp\left(\eta_{B,t}\right) R_t}{\exp\left(\eta_{u,t}\right)} E_t \left[\frac{C_{t+1}^{-\sigma} \exp\left(\eta_{u,t+1}\right)}{\pi_{t+1}}\right], \quad (33)$$

$$\frac{S_t}{F_t} = \left[\frac{1 - \theta \pi_t^{\varepsilon-1}}{1-\theta}\right]^{\frac{1}{1-\varepsilon}}, \quad (34)$$

$$\Delta_t = \left[(1-\theta)\left[\frac{1 - \theta \pi_t^{\varepsilon-1}}{1-\theta}\right]^{\frac{\varepsilon}{\varepsilon-1}} + \theta \frac{\pi_t^{\varepsilon}}{\Delta_{t-1}}\right]^{-1}, \quad (35)$$

$$Y_t = \exp\left(\eta_{a,t}\right) L_t \Delta_t, \quad (36)$$

$$C_t = \left(1 - \frac{\overline{G}}{\exp\left(\eta_{G,t}\right)}\right) Y_t, \quad (37)$$

$$R_t = R_* \left(\frac{R_{t-1}}{R_*}\right)^{\mu} \left[\left(\frac{E_t \pi_{t+1}}{\pi_{tar}}\right)^{\phi_{E\pi}} \left(\frac{\pi_t}{\pi_{tar}}\right)^{\phi_{\pi}} \left(\frac{Y_t}{Y_{N,t}}\right)^{\phi_y}\right]^{1-\mu} \exp\left(\eta_{R,t}\right), \quad (38)$$

and $Y_{N,t}$ is given by

$$Y_{N,t} = \left[\frac{\exp\left(\eta_{a,t}\right)^{(\sigma+\vartheta)(1-\sigma)}}{\left[1 - \frac{\overline{G}}{\exp\left(\eta_{G,t}\right)}\right]^{\sigma} \exp\left(\eta_{L,t}\right)}\right]^{\frac{1}{\vartheta+\sigma}}. \quad (39)$$

Here, the variables S_t and F_t are introduced for a compact representation of the profit-maximization condition of the intermediate-good firm and are defined recursively; $\pi_{t+1} \equiv \frac{P_{t+1}}{P_t}$ is the gross inflation rate between t and $t+1$; Δ_t is a measure of price dispersion across firms (also referred to as efficiency distortion). To get condition (31), we impose $\frac{\varepsilon}{\varepsilon-1}(1-v) = 1$, which ensures that the model admits a deterministic steady state (this assumption is commonly used in the related literature; see, e.g., Christiano et al. 2005). An interior equilibrium is described by 8 equilibrium conditions (31)–(38), and 6 processes for exogenous shocks. The system of equations must be solved with respect to 8 unknowns $\{C_t, Y_t, L_t, \pi_t, \Delta_t, R_t, S_t, F_t\}$. There are 2 endogenous and 6 exogenous state variables, , $\{\Delta_{t-1}, R_{t-1}\}$, and $\{\eta_{u,t}, \eta_{L,t}, \eta_{B,t}, \eta_{a,t}, \eta_{R,t}, \eta_{G,t}\}$, respectively.

Linearized equilibrium conditions. Below, we provide linearized versions of the equilibrium conditions (31)–(39):

$$\begin{aligned} S_t - S_* &= L_*^{\vartheta} Y_* (\eta_{u,t} + \eta_{L,t} - \eta_{a,t}) - \vartheta L_*^{\vartheta-1} (L - L_*) Y_* \\ &\quad + L_*^{\vartheta} (Y_t - Y_*) + \beta\theta S_* \varepsilon \pi_*^{\varepsilon-1} E_t(\pi_{t+1} - \pi_*) + \beta\theta\varepsilon\pi_*^{\varepsilon-1}\pi_*^{\varepsilon} E_t (S_{t+1} - S_*), \end{aligned}$$

$$\begin{aligned} F_t - F_* &= \eta_{u,t} C_*^{-\sigma} Y_* + (-\sigma) C_*^{-\sigma-1} Y_* (C_t - C_*) + C_*^{-\sigma} (Y_t - Y_*) \\ &\quad + \beta\theta(\varepsilon - 1)\pi_*^{\varepsilon-2} F_* E_t (\pi_{t+1} - \pi_*) + \beta\theta\varepsilon\pi_*^{\varepsilon-1} E_t(F_{t+1} - F_*), \end{aligned}$$

$$\begin{aligned} -\sigma C_*^{-\sigma-1}(C_t - C_*) &= \beta R_* \frac{C_*^{-\sigma}}{\pi_*}(\eta_{B,t} - \eta_{u,t}) + \beta \frac{C_*^{-\sigma}}{\pi_*}(R_t - R_*) \\ &\quad + \beta R_* (-\sigma) \frac{C_*^{-\sigma-1}}{\pi_*} E_t(C_{t+1} - C_*) - \beta R_* \frac{C_*^{-\sigma}}{\pi_*^2} E_t(\pi_{t+1} - \pi_*) \\ &\quad + \beta R_* (-\sigma) \frac{C_*^{-\sigma}}{\pi_*} \rho_{\eta_u} \eta_{u,t}, \end{aligned}$$

$$\begin{aligned} \frac{1}{F_*}(S_t - S_*) + \frac{S_*}{F_*^2}(F_t - F_*) &= \\ &= \left(\frac{1 - \theta\pi_*^{\varepsilon-1}}{(1-\theta)}\right)^{\frac{1}{1-\varepsilon}-1} \frac{\theta}{1-\theta} \pi_*^{\varepsilon-2} (\pi_t - \pi_*), \end{aligned}$$

$$\begin{aligned} \Delta_t - \Delta_* &= \left[\frac{(1-\theta)\left(1 - \theta\pi_*^{\varepsilon-1}\right)}{1-\theta}\right]^{\frac{\varepsilon}{\varepsilon-1}} + \theta \frac{\pi_*^{\varepsilon}}{\Delta_*^{-2}} \left[\frac{1 - \theta\pi_*^{\varepsilon-1}}{1-\theta}\right]^{\frac{\varepsilon}{\varepsilon-1}-1} \varepsilon\theta\pi_*^{\varepsilon-2} \\ &\quad - \theta \frac{\varepsilon}{\Delta_*} \pi_*^{\varepsilon-1} (\pi_t - \pi_*) + \theta \frac{\pi_*^{\varepsilon}}{\Delta_*^2} (\Delta_{t-1} - \Delta_*), \end{aligned}$$

$$Y_t - Y_* = \eta_{a,t} L_* \Delta_* + (L_t - L_*)\Delta_* + (\Delta_t - \Delta_*) L_*,$$

$$C_t - C_* = \frac{1 - \overline{G}}{Y_*} \eta_{G,t} + (1 - \overline{G})(Y_t - Y_*),$$

$$\begin{aligned} R_t - R_* &= R_* \eta_{R,t} + \mu(R_{t-1} - R_*) + (1-\mu)\phi_\pi \frac{R_*}{\pi_*}(\pi_t - \pi_*) + \\ &\quad (1-\mu)\phi_{E\pi} \frac{R_*}{\pi_*} E_t(\pi_{t+1} - \pi_*) + (1-\mu)\phi_y \frac{R_*}{Y_*}(Y_t - Y_*) \end{aligned}$$

$$
\begin{aligned}
Y_{N,t} - Y_{N_*} &= \frac{1}{\vartheta+\sigma}(1-\overline{G})^{\frac{-\sigma}{\vartheta+\sigma}-1}\left[\frac{1+\vartheta}{1-\overline{G}}\right]^{\sigma}\eta_{a,t} \\
&\quad +\sigma(1-\overline{G})^{-\sigma-1}\eta_{G,t} + \eta_{L,t}(1-\overline{G})^{-\sigma}.
\end{aligned}
$$

Under the assumptions of no government spendings, no shocks, and no endogenous natural level of output, the above nine linearized equilibrium conditions can be reduced to three equations (1)–(3) used in the main text; see, e.g., Galí (2008).

Calibration procedure. We assume $\sigma = 1$ and $\vartheta = 2.09$ in the utility function (18); $\mu = 0.82$ in the Taylor rule (38); $\varepsilon = 4.45$ in the production function of the final-good firm (21); $\theta = 0.83$ (the fraction of the intermediate-good firms affected by price stickiness); $\overline{G} = 0.23$ in the government budget constraint (27). We set the discount factor at $\beta = 0.99$. To parameterize the Taylor rule (38), we use the steady-state interest rate $R_* = \frac{\pi_*}{\beta}$, and the target inflation, $\pi_* = 1$ (a zero net inflation target). In a stochastic version of the model, we calibrate the parameters in the processes for shocks as follows: In the AR(1) processes for shocks, we assume the autocorrelation coefficients, $\rho_u = 0.92$, $\rho_G = 0.95$, $\rho_L = 0.25$, $\rho_a = 0.95$, $\rho_B = 0.22$, $\rho_R = 0.15$, and the standard deviations of shocks $\sigma_u = 0.054\%$, $\sigma_G = 0.038\%$, $\sigma_L = 0.018\%$, $\sigma_a = 0.045\%$, $\sigma_B = 0.023\%$ and $\sigma_R = 0.028\%$ (these values come from Del Negro et al., 2007, and from Smets and Wouters, 2007).

In the three-equation model, we use similar parameter values, namely, the slope of the Phillips curve $\kappa = \frac{(1-\beta\theta)(1-\theta)}{\theta}(1+\vartheta)$ is computed under the same values of the parameters β, θ, ϑ; the coefficient of relative risk aversion is $\sigma = 1$.

Solution procedure. In Section 6, we solve linear and nonlinear versions of the model by using extended path (EP) method by Fair and Taylor (1983). The model starts in the steady state, in particular, we assume $R_* = \frac{\pi_*}{\beta}$. In the initial period, the monetary authority announces that at $t = 30$, the nominal interest rate will go down by 1%. We then construct the path for the model's variables to satisfy the model's equations. We solve the model for 50 periods, and we extend the path to 150 periods.

Appendix B. Proofs of Theorems 1 and 2

In this section, we prove Theorems 1 and 2 that establish the regions for the parameters $\phi_\pi \geq 0$, $\phi_{E\pi} \geq 0$ and $\phi_y \geq 0$ corresponding to different types of characteristic roots in the model (1), (2) and (3).

Proof to Theorem 1. The roots to the characteristic equation $m^2 + bm + c\pi_t = 0$ are given by

$$m_1 = \frac{-b + \sqrt{b^2 - 4c}}{2}, \tag{40}$$

$$m_2 = \frac{-b - \sqrt{b^2 - 4c}}{2}, \tag{41}$$

where $b \equiv -1 - \frac{1}{\beta} - \sigma\phi_y - \frac{\sigma\kappa(1-\phi_{E\pi})}{\beta}$, and $c \equiv \frac{\left(1+\sigma\phi_y\right)}{\beta} + \frac{\sigma\kappa\phi_\pi}{\beta}$. It is useful to note that $c > 1$.

We start by showing statement ii) of Theorem 1.

Two unstable real roots. To have $|m_1| \geq 1$, $|m_2| \geq 1$, we must have one of the following cases:

$$a) \begin{bmatrix} m_1 \leq -1 \\ m_2 \geq 1 \end{bmatrix}, \quad b) \begin{bmatrix} m_1 \geq 1 \\ m_2 \leq -1 \end{bmatrix}, \quad c) \begin{bmatrix} m_1 \geq 1 \\ m_2 \geq 1 \end{bmatrix}, \quad d) \begin{bmatrix} m_1 \leq -1 \\ m_2 \leq -1 \end{bmatrix}.$$

Let us first rule out *Cases a) and b)* by showing that there are no parameters values that satisfy both restrictions.

Case a) By construction, we have $m_2 < m_1$, so this case is impossible.

Case b) Since $m_1 \geq 1$, we have $\sqrt{b^2 - 4c} \geq 2 + b$.

Since $m_2 \leq -1$, we have $-\sqrt{b^2 - 4c} \leq -2 + b$ which is equivalent $\sqrt{b^2 - 4c} \geq 2 - b$.

i) If $b \geq 0$, then $\sqrt{b^2 - 4c} \geq 2 + b$ implies $\sqrt{b^2 - 4c} \geq 2 - b$, so we only need to insure the former inequality:

$$b^2 - 4c \geq (2 + b)^2 \Rightarrow -4c \geq 4 + 4b, \text{ impossible since } c > 0.$$

ii) If $b < 0$, then $\sqrt{b^2 - 4c} > 2 - b$ implies $\sqrt{b^2 - 4c} > 2 + b$, so again, we only need to insure the former inequality:

$$b^2 - 4c > (2 - b)^2 \Rightarrow -4c > 4 - 4b, \text{ impossible since } c > 0.$$

By combining i) and ii), we conclude that the case b) is impossible.

Case c) Since $m_2 \geq 1$ implies $m_1 \geq 1$, we only need to insure $m_2 \geq 1$, i.e., $\frac{-b-\sqrt{b^2-4c}}{2} \geq 1$. This implies

$$-\sqrt{b^2 - 4c} \geq 2 + b. \tag{42}$$

Since the root is real, we must have $b^2 - 4c \geq 0$. This implies two possibilities: if $b > 0$, we must have $b > 2\sqrt{c}$ and if $b < 0$, we must have $-b > 2\sqrt{c}$. However, the former possibility violates (42), so we are left with $b \leq -2\sqrt{c}$, which leads to boundary value $\phi^2_{E\pi}$:

$$\begin{aligned}
\left(-\frac{1}{\beta} - \left(1+\sigma\phi_y\right) - \frac{\sigma\kappa\left(1-\phi_{E\pi}\right)}{\beta}\right) &\leq -2\sqrt{\frac{\left(1+\sigma\phi_y\right)}{\beta} + \frac{\sigma\kappa\phi_\pi}{\beta}} \\
\phi_{E\pi} &\leq 1 + \frac{1}{\sigma\kappa}\left[1 + \left(1+\sigma\phi_y\right)\beta - 2\sqrt{\left(1+\sigma\phi_y+\sigma\kappa\phi_\pi\right)\beta}\right] \\
\phi_{E\pi} \leq 1 - \phi_\pi - \frac{(1-\beta)\,\phi_y}{\kappa} &+ \frac{1}{\sigma\kappa}\left[1 + \left(1+\sigma\phi_y\right)\beta + \phi_\pi + \frac{(1-\beta)\,\phi_y}{\kappa} - 2\sqrt{\left(1+\sigma\phi_y+\sigma\kappa\phi_\pi\right)\beta}\right] \\
\phi_{E\pi} \leq 1 - \phi_\pi - \frac{(1-\beta)\,\phi_y}{\kappa} &+ \frac{1}{\sigma\kappa}\left[1 + \sigma\kappa\phi_\pi + \sigma\phi_y + \beta - 2\sqrt{\left(1+\sigma\phi_y+\sigma\kappa\phi_\pi\right)\beta}\right] \equiv \phi^2_{E\pi}.
\end{aligned} \tag{43}$$

Furthermore, we re-write (42) as

$$\begin{aligned}
\sqrt{b^2-4c} &\leq -2-b, \\
\left(\sqrt{b^2-4c}\right)^2 &\leq (-2-b)^2, \\
-4c &\leq 4+4b.
\end{aligned}$$

The latter inequality implies $c + 1 \geq -b$, which leads to the boundary value $\phi^1_{E\pi}$:

$$\begin{aligned}
1 + \frac{\left(1+\sigma\phi_y\right)}{\beta} + \frac{\sigma\kappa\phi_\pi}{\beta} &\geq 1 + \frac{1}{\beta} + \sigma\phi_y + \frac{\sigma\kappa\left(1-\phi_{E\pi}\right)}{\beta} \\
\phi^1_{E\pi} \equiv 1 - \phi_\pi + \frac{\beta\phi_y}{\kappa}\left(1-\frac{1}{\beta}\right) &\leq \phi_{E\pi}.
\end{aligned}$$

Finally, we consider *Case d)*. The analysis of this case is similar to *Case c)*. Since $m_1 \leq -1$ implies $m_2 \leq -1$, we only need to insure $m_1 \leq -1$, i.e., $\frac{-b+\sqrt{b^2-4c}}{2} \leq -1$. This implies

$$\sqrt{b^2-4c} \leq -2+b. \tag{44}$$

Since the root is real, we must have $b^2 - 4c \geq 0$. This implies two possibilities: if $b > 0$, we must have $b > 2\sqrt{c}$ and if $b < 0$, we must have $-b > 2\sqrt{c}$. However, the latter possibility violates (44), so we are left with $b \geq 2\sqrt{c}$, which leads to the boundary value $\phi^3_{E\pi}$:

$$\left(-\frac{1}{\beta} - \left(1+\sigma\phi_y\right) - \frac{\sigma\kappa\left(1-\phi_{E\pi}\right)}{\beta}\right) \geq 2\sqrt{\frac{\left(1+\sigma\phi_y\right)}{\beta} + \frac{\sigma\kappa\phi_\pi}{\beta}}$$

$$\phi_{E\pi} \geq 1 + \frac{1}{\sigma\kappa}\left[1 + \left(1+\sigma\phi_y\right)\beta + 2\sqrt{\left(1+\sigma\phi_y+\sigma\kappa\phi_\pi\right)\beta}\right]$$

$$\phi_{E\pi} \geq 1 - \phi_\pi - \frac{\left(1-\beta\right)\phi_y}{\kappa} + \frac{1}{\sigma\kappa}\left[1 + \left(1+\sigma\phi_y\right)\beta + \phi_\pi + \frac{\left(1-\beta\right)\phi_y}{\kappa} + 2\sqrt{\left(1+\sigma\phi_y+\sigma\kappa\phi_\pi\right)\beta}\right]$$

$$\phi_{E\pi} \geq 1 - \phi_\pi - \frac{\left(1-\beta\right)\phi_y}{\kappa} + \frac{1}{\sigma\kappa}\left[1 + \sigma\kappa\phi_\pi + \sigma\phi_y + \beta + 2\sqrt{\left(1+\sigma\phi_y+\sigma\kappa\phi_\pi\right)\beta}\right] \equiv \phi^3_{E\pi}. \tag{45}$$

Furthermore, we re-write (44) as

$$\begin{aligned} \left(\sqrt{b^2-4c}\right)^2 &\leq \left(-2+b\right)^2 \\ -4c &\leq 4-4b. \end{aligned}$$

This implies $c+1 \geq b$, which leads to the boundary value $\phi^4_{E\pi}$:

$$\begin{aligned} \frac{\left(1+\sigma\phi_y\right)}{\beta} + \frac{\sigma\kappa\phi_\pi}{\beta} + 1 &\geq -1 - \frac{1}{\beta} - \sigma\phi_y - \frac{\sigma\kappa\left(1-\phi_{E\pi}\right)}{\beta} \\ \phi_{E\pi} &\leq \frac{\left(1+\sigma\phi_y\right) + \sigma\kappa\phi_\pi + \beta + \beta + 1 + \beta\sigma\phi_y + \sigma\kappa}{\sigma\kappa} \\ \phi_{E\pi} &\leq 1 - \phi_\pi - \frac{\left(1-\beta\right)\phi_y}{\kappa} \\ &\quad + \frac{\left(1+\sigma\phi_y\right) + \sigma\kappa\phi_\pi + \beta + \beta + 1 + \beta\sigma\phi_y + \sigma\kappa}{\sigma\kappa} - 1 + \phi_\pi + \frac{\left(1-\beta\right)\phi_y}{\kappa} \\ \phi_{E\pi} &\leq 1 - \phi_\pi - \frac{\left(1-\beta\right)\phi_y}{\kappa} + \frac{2\left(1+\sigma\phi_y+\sigma\kappa\phi_\pi+\beta\right)}{\sigma\kappa} \equiv \phi^4_{E\pi}. \end{aligned}$$

We next show statement iii) of Theorem 1.

Two repeated real roots. To have repeated real roots, it must be that $b^2 - 4c = 0$. There are two possible solutions $b = 2\sqrt{c}$ and $b = -2\sqrt{c}$. By using the results (43) and (45) obtained for *Cases c) and d)* of the statement ii), we obtain that the corresponding parameterizations are $\phi_{E\pi} = \phi^2_{E\pi}$ and $\phi_{E\pi} = \phi^3_{E\pi}$.

To see that the resulting root $m = -\frac{b}{2}$ is unstable, notice that $b = 2\sqrt{c}$ and $b = -2\sqrt{c}$ imply $m = -\sqrt{c}$ and $m = \sqrt{c}$, respectively. Since $c > 1$, we conclude that $|m| > 1$.

We now show statement iv) of Theorem 1.

Complex roots. For complex roots, we must have $b^2 - 4c < 0$, which implies $-2\sqrt{c} < b < 2\sqrt{c}$. Again, based on the results (43) and (45) obtained for *Cases c) and d)* of the statement ii), we obtain that the corresponding parameter range is $\phi^2_{E\pi} < \phi_{E\pi} < \phi^3_{E\pi}$. To see that the complex root $m_{1,2} = \mu \pm \eta\iota$ is unstable, we compute $r \equiv \sqrt{\mu^2 + \eta^2} = \sqrt{\left(\frac{-b}{2}\right)^2 + \left(\frac{\sqrt{4c-b^2}}{2}\right)^2} = \sqrt{c} > 1$.

We finally show statement i) of Theorem 1.

One stable and one unstable real roots. To analyze this case, we actually show that there are no parameter values for which we have two stable roots, i.e., $|m_1| < 1$ and $|m_2| < 1$. Indeed, the existence of two stable roots implies that $-1 < m_1 < 1$ and $-1 < m_2 < 1$, i.e.,

$$
\begin{aligned}
-1 &< \frac{-b + \sqrt{b^2 - 4c}}{2} < 1, \\
-1 &< \frac{-b - \sqrt{b^2 - 4c}}{2} < 1.
\end{aligned}
$$

If $-1 < \frac{-b-\sqrt{b^2-4c}}{2}$, then we have $-1 < \frac{-b+\sqrt{b^2-4c}}{2}$ and if $\frac{-b+\sqrt{b^2-4c}}{2} < 1$, then we have $\frac{-b-\sqrt{b^2-4c}}{2} < 1$. So, we must check only the following two conditions:

$$
\begin{gathered}
-1 < \frac{-b - \sqrt{b^2 - 4c}}{2}, \\
\frac{-b + \sqrt{b^2 - 4c}}{2} < 1.
\end{gathered}
$$

These conditions can be, respectively, re-written as

$$\sqrt{b^2 - 4c} < 2 - b, \tag{46}$$

$$\sqrt{b^2 - 4c} < 2 + b. \tag{47}$$

Since the roots are real, we have $b^2 > 4c$ which means that either $b > \sqrt{4c}$ or $b < -\sqrt{4c}$. Since $c > 1$, these last two inequalities imply that either $b > 2$ or $b < -2$. But then the restrictions (46) and (47) cannot be satisfied simultaneously: if $b > 2$, the right side of (46) is negative and if $b < -2$, the right side of (47) is negative. Since the roots are real and we discarded the possibility of two stable roots, we conclude that we must have one stable and one unstable root, except of those cases when two roots are unstable and when the roots are complex, i.e., everywhere except of the range $\phi^1_{E\pi} \leq \phi_{E\pi} \leq \phi^4_{E\pi}$. This completes the proof of the statement i) of Theorem 1. ■

Proof to Theorem 2. The roots to the characteristic equation $m^2 + bm + c\pi_t = 0$ are again given by (40) and (41), where under the assumption $\phi_{E\pi} = 0$, we have $b \equiv -1 - \frac{1}{\beta} - \sigma\phi_y - \frac{\sigma\kappa}{\beta}$, and $c \equiv \frac{(1+\sigma\phi_y)}{\beta} + \frac{\sigma\kappa\phi_\pi}{\beta}$. It is useful to note that $b < 0$ and $c > 1$.

We start by showing statement ii) of Theorem 2.

Two unstable real roots. To have $|m_1| \geq 1,$ $|m_2| \geq 1$, we must have one of the following cases:

$$a) \begin{bmatrix} m_1 \leq -1 \\ m_2 \geq 1 \end{bmatrix}, \qquad b) \begin{bmatrix} m_1 \geq 1 \\ m_2 \leq -1 \end{bmatrix}, \qquad c) \begin{bmatrix} m_1 \geq 1 \\ m_2 \geq 1 \end{bmatrix}, \qquad d) \begin{bmatrix} m_1 \leq -1 \\ m_2 \leq -1 \end{bmatrix}.$$

Cases a) and b) The results of Theorem 1 apply here as well, so we conclude that these two cases are impossible.

Case c) Since $m_2 \geq 1$ implies $m_1 \geq 1$, we only need to insure $m_2 \geq 1$, i.e., $\frac{-b-\sqrt{b^2-4c}}{2} \geq 1$. Like in *Case c)* of Theorem 1, this implies that $b \leq -2\sqrt{c}$ and results in the corresponding boundary value ϕ_π^2:

$$\begin{aligned} \left(-\frac{1}{\beta} - \left(1+\sigma\phi_y\right) - \frac{\sigma\kappa}{\beta}\right) &\leq -2\sqrt{\frac{\left(1+\sigma\phi_y\right)}{\beta} + \frac{\sigma\kappa\phi_\pi}{\beta}} \\ \frac{1}{\beta} + \left(1+\sigma\phi_y\right) + \frac{\sigma\kappa}{\beta} &\geq 2\sqrt{\frac{\left(1+\sigma\phi_y\right)}{\beta} + \frac{\sigma\kappa\phi_\pi}{\beta}} \\ \phi_\pi &\leq 1 - \frac{\left(1-\beta\right)\phi_y}{\kappa} + \frac{\beta}{4\sigma\kappa}\left\{1 - \frac{1}{\beta} - \sigma\phi_y - \frac{\sigma\kappa}{\beta}\right\}^2 \equiv \phi_\pi^2 \end{aligned} \tag{48}$$

The boundary value ϕ_π^1 follows by (42). Like in Case 3 of Theorem 1, we have $c+1 \geq -b$ and consequently, we obtain

$$\begin{aligned} 1 + \frac{\left(1+\sigma\phi_y\right)}{\beta} + \frac{\sigma\kappa\phi_\pi}{\beta} &\geq 1 + \frac{1}{\beta} + \sigma\phi_y + \frac{\sigma\kappa}{\beta} \\ \phi_\pi^1 \equiv 1 - \frac{\left(1-\beta\right)\phi_y}{\kappa} &\leq \phi_\pi. \end{aligned} \tag{49}$$

Finally, we consider *Case d).* Following the reasoning of the corresponding case of Theorem 1, we conjecture that we must have $b \geq 2\sqrt{c}$. But this is not possible since by definition, we have $b < 0$ and $c > 1$. Thus, unlike Theorem 1, here we do not have boundary values analogous to $\phi_{E\pi}^3$ and $\phi_{E\pi}^4$ in Theorem 2.

We next show statement iii) of Theorem 2.

Two repeated real roots. To have repeated real roots, it must be that $b^2 - 4c = 0$. There are two possible solutions $b = 2\sqrt{c}$ and $b = -2\sqrt{c}$. But in the present case, we have $b < 0$, only the latter root is possible. Using the results (48) obtained for *Case c)* of the statement ii), we obtain that the corresponding parameterization is $\phi_\pi = \phi_\pi^2$. Again, to see that the resulting root $m = -\frac{b}{2}$ is unstable, notice that $b = -2\sqrt{c}$ imply $m = \sqrt{c}$, respectively. Since $c > 1$, we conclude that $|m| > 1$.

We now by show statement iv) of Theorem 2.

Complex roots. For complex roots, we must have $b^2 - 4c < 0$, which implies $-2\sqrt{c} < b < 2\sqrt{c}$. The argument of *Case d)* of the present proof rules out the possibility of $b \geq 2\sqrt{c}$ so that the second inequality always holds. Therefore, the roots are complex whenever $-2\sqrt{c} < b$, which together with the result (48) obtained for *Case c)* of the statement ii) leads us to $\phi_\pi > \phi_\pi^2$. Like in Theorem 1, the complex root $m_{1,2} = \mu \pm \eta\iota$ is unstable since $r \equiv \sqrt{\mu^2 + \eta^2} = \sqrt{\left(\frac{-b}{2}\right)^2 + \left(\frac{\sqrt{4c-b^2}}{2}\right)^2} = \sqrt{c} > 1$.

We finally show statement i) of Theorem 1.

One stable and one unstable real roots. The arguments of Theorem 1 apply to this case as well as, so that we conclude that there are no parameter values for which we have two stable roots, i.e., $|m_1| < 1$ and $|m_2| < 1$. Since the roots are real and we discarded the possibility of two stable roots, we conclude that we must have one stable and one unstable root, except when two roots are unstable and when the roots are complex which yields the range $\phi_\pi < \phi_\pi^1$ and completes the proof of the statement. ■

Appendix C. Method of undetermined coefficients for linear stochastic models

In the economy with stochastic shocks, we should construct conditional expectations for future shocks. Our closed-form solutions provide a convenient way of modeling a variety of uncertainty scenarios, including temporary and permanent shocks, anticipated and unanticipated shocks, as well as mixtures of deterministic trends and stochastic shocks. Since the characteristic roots in (6)–(9) are non-random, the expectation operator can be brought inside the summations, for example, $E_t\left[\sum_{s=t}^{\infty} m_1^{t-1-s} z_s\right] = \sum_{s=t}^{\infty} m_1^{t-1-s} E_t\left[z_s\right]$, so that effectively, we need to compute $E_t\left[z_s\right]$ for $s \geq t$.

As an illustration, let us assume that z_t follows a first-order autoregressive process $z_{t+1} = \rho z_t + \varepsilon_{t+1}$, in which case, we have $E_t\left[z_s\right] = \rho^{s-t} z_t$ for $s \geq t$. Furthermore, let us assume that the roots are complex, i.e., case iv). Then, the solution (9) can be re-written as

$$\pi_t = C_1 r^t \cos\left(\theta t\right) + C_2 r^t \sin\left(\theta t\right) + \frac{z_t}{\eta\rho}\left[\sum_{s=t}^{\infty}\left(\frac{r}{\rho}\right)^{t-1-s} \sin\left(\theta\left(t-1-s\right)\right)\right]. \tag{50}$$

To simulate stochastic time-series solution, we draw a sequence of shocks for z_t, find π_t from (50) and compute x_t from (2). Similar formulas are easy to show for the remaining cases (6)–(8); in those cases, the roots m_i can be adjusted to ρ by $\frac{m_i}{\rho}$ and the term $\frac{z_t}{\rho}$ can be taken out of the summation. Examples of time-series solutions to the stochastic versions of the model are shown in Section 6.

To ensure stationarity in the stochastic case, we need to impose the same restrictions on the homogeneous solutions as those necessary for forward stability in the deterministic case. In particular, we obtain a unique stationary solution in cases (7)–(9) by setting $C_1 = 0$ and $C_2 = 0$, and there is a multiplicity of stationary solutions in case (6) since any C_2 is consistent with stationarity.

Closed-form solutions to new Keynesian models with uncertainty are studied in Taylor (1986) by using a method of undetermined coefficients – an analytical technique that reduces a stochastic difference equation to the deterministic one. In contrast, our present analysis proceeds in the opposite direction: we first constructed a solution to the deterministic model, and we then generalized it to the stochastic case. Below, we show that both approaches lead to the same stochastic solution under a general linear process for shock z_t. Taylor's (1986) method does not specify how to construct a solution to the deterministic model like those we obtain in Theorem 3, which is our main contribution. Finally, Cochrane (2017b) constructs related solutions for the stochastic version of the model in which one root is stable and the other root us unstable, which corresponds to our case i).

We assume that z_t follows a general linear process with a representation

$$z_s = \sum_{j=0}^{\infty} \vartheta_j \varepsilon_{s-j}, \tag{51}$$

where ϑ_j is a sequence of parameters, and ε_t is a serially uncorrelated random variable with zero mean. The process (51) includes important types of policy shocks as special cases, in particular, it allows us to distinguish between temporary and permanent shocks, as well as anticipated and unanticipated shocks; see Taylor (1986) for a discussion. We first construct a closed-form solution of our Theorem 3 under (51), and we next show that the method of undetermined coefficients of Taylor (1986) leads to the same solution. We omit the homogeneous solution because it is the same in the deterministic and stochastic models, and we concentrate on particular solutions.

Closed-form solutions of Theorem 3. As an example, consider the closed-form solution (9) of Theorem 3 for the model with complex roots, which under assumption (51) becomes

$$\pi_t = \frac{1}{\eta} E_t \left[\sum_{s=t}^{\infty} h_{t-1-s} \sum_{j=0}^{\infty} \theta_j \varepsilon_{s-j} \right],$$

where $h_{t-1-s} \equiv r^{t-1-s} \sin(\theta(t-1-s))$ is compact notation. The latter expression can be written as

$$\pi_t = \frac{1}{\eta} E_t \left[h_{-1} \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j} + h_0 \sum_{j=1}^{\infty} \theta_j \varepsilon_{t+1-j} + h_1 \sum_{j=0}^{\infty} \theta_j \varepsilon_{t+2-j} + ... \right].$$

Since $E_t[\varepsilon_{t+\tau-j}] = 0$ for any $\tau \geq 0$, we can compute expectation with the following sequence of steps:

$$\pi_t = \frac{1}{\eta}\left[h_{-1}\sum_{j=0}^{\infty}\theta_j\varepsilon_{t-j} + h_0\sum_{j=1}^{\infty}\theta_j\varepsilon_{t+1-j} + h_1\sum_{j=2}^{\infty}\theta_j\varepsilon_{t+2-j} + ...\right]$$
$$= \frac{1}{\eta}\left[h_{-1}\sum_{j=0}^{\infty}\theta_j\varepsilon_{t-j} + h_0\sum_{j=0}^{\infty}\theta_{j+1}\varepsilon_{t-j} + h_1\sum_{j=0}^{\infty}\theta_{j+2}\varepsilon_{t-j} + ...\right]$$
$$= \frac{1}{\eta}\sum_{j=0}^{\infty}\varepsilon_{t-j}\left[h_{-1}\theta_j + h_0\theta_{j+1} + h_1\theta_{j+2} + ...\right] = \frac{1}{\eta}\sum_{j=0}^{\infty}\varepsilon_{t-j}\sum_{s=j}^{\infty}h_{j-1-s}\theta_s. \quad (52)$$

Method of undetermined coefficients. The method of undetermined coefficients described in Taylor (1986) requires us to guess that a solution for π_t has the same kind of representation as (51), specifically,

$$\pi_t = \sum_{j=0}^{\infty}\varepsilon_{t-j}\gamma_j, \quad (53)$$

where γ_j is a sequence of unknown coefficients. By taking into account that $E_t[\gamma_0\varepsilon_{t+1}] = 0$, we obtain $E_t[\pi_{t+1}] = \sum_{j=1}^{\infty}\gamma_j\varepsilon_{t-j+1}$ and $E_{t+1}[\pi_{t+2}] = \sum_{j=2}^{\infty}\gamma_j\varepsilon_{t-j+2}$. Therefore, we can re-write (4) as

$$\sum_{j=2}^{\infty}\gamma_j\varepsilon_{t-j+2} + b\sum_{j=1}^{\infty}\gamma_j\varepsilon_{t-j+1} + c\sum_{j=0}^{\infty}\gamma_j\varepsilon_{t-j} = -\sum_{j=0}^{\infty}\vartheta_j\varepsilon_{t-j}. \quad (54)$$

Equating the coefficients on both sides of the equality (54) gives us a set of restrictions

$$\gamma_{j+2} + b\gamma_{j+1} + c\gamma_j = -\vartheta_j, \qquad j = 0, 1, ... \quad (55)$$

This is a deterministic difference equation with a forcing variable ϑ_j. It has the same structure as a stochastic difference equation and it is identical up to notation to the deterministic version of the equation (4) studied in the main text. Therefore, the coefficients of the stochastic equation (55) can be described by formulas (6)-(9) of Theorem 3, again up to notation. For example, Theorem 3, case (9) implies that the sequence of the coefficients (55) in the model with complex roots is given by $\gamma_j = \frac{1}{\eta}\left[\sum_{s=j}^{\infty}h_{j-1-s}\vartheta_s\right]$, which together with (53) implies the same solution for π_t as (52). The equivalence for the remaining cases (6)-(9) can be shown similarly.

Appendix D. Supplementary results for Section 3

In this section, we present some supplementary results for Section 3.

D1. Taylor rule with actual inflation: a weaker version of the FG puzzle

We now consider a different version of the monetary policy rule (3), namely, we assume that it contains actual inflation instead of expected inflation. To begin with, let the coefficients in (3) be $\phi_{E\pi} = 0$, $\phi_\pi = 1$ and $\phi_y = 0$. Here, we again have case ii) of Theorems 1-3 with two roots $m_1 = \frac{1+\sigma k}{\beta} > 1$ and $m_2 \searrow 1$, so that the solution is given by (10)

$$\pi_t = \frac{1}{\frac{1+\sigma k}{\beta} - 1}\left[\sum_{s=t}^{\infty}\left(\frac{1+\sigma k}{\beta}\right)^{t-s-1} z_s - \sum_{s=t}^{\infty} z_s\right] = \frac{\kappa\sigma\varepsilon}{1-\beta+\sigma k}\left[\left(\frac{\beta}{1+\sigma k}\right)^{T-t+1} - 1\right], \tag{56}$$

where the last expression corresponds to the case of a single shock $z_T = \frac{\kappa\sigma\varepsilon}{\beta}$. From the Phillips curve, the corresponding solution for output is $x_t = \frac{-\sigma\varepsilon}{1-\beta+\sigma k}\left(1 - \beta + \frac{\beta\sigma k}{1+\sigma k}\left(\frac{\beta}{1+\sigma k}\right)^{T-t}\right)$. If the shock is distant, i.e., $T \gg t$, so that $\left(\frac{\beta}{1+\sigma k}\right)^{T-t} \approx 0$, the future shock increases output initially to $x_0 \approx \frac{-\sigma\varepsilon}{1-\beta+\sigma k}(1-\beta) \approx -0.08\sigma\varepsilon$ and continues to raise it to reach $x_T = \frac{-\sigma\varepsilon}{1-\beta+\sigma k}\left(1 - \beta + \frac{\beta\sigma k}{1+\sigma k}\right) \approx 0.9\sigma\varepsilon$ at T (assuming $\kappa = 0.11$, $\sigma = 1$ and $\beta = 0.99$). This case is illustrated in Figure D1.

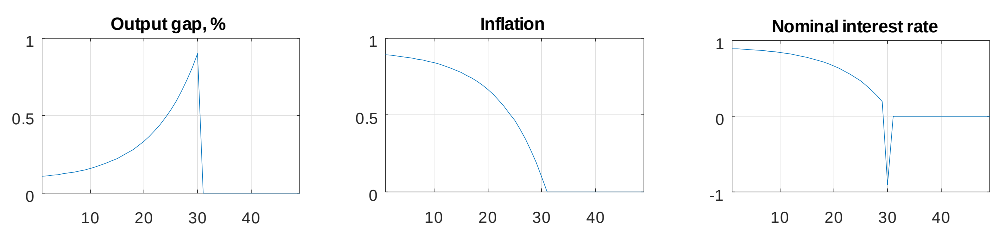

Figure D1. Forward guidance: Taylor rule with actual inflation, $\phi_\pi = 1$.

Thus, we observe a weaker version of the FG puzzle in which only the initial response of inflation to anticipated interest rate shock is large but the initial response of output is modest (it is about 8 percent of what we had in the FG puzzle case (11)). As the economy approaches the period 30, the impact of the shock on output gradually increases to reach 90 percent of the FG puzzle effect (11).

The effect of future shocks on output is dampened because we now have discounting in the IS curve; see formula (13) under $\phi_y = 0$. In fact, as the value of ϕ_π increases, discounting becomes sufficient to dampen the effect of FG on inflation as well, so that the FG puzzle is not observed any longer even for inflation. For example, in Figure D2, we show the solution under the Taylor rule (3) with $\phi_{E\pi} = 0$, $\phi_\pi = 3$ and $\phi_y = 0$.

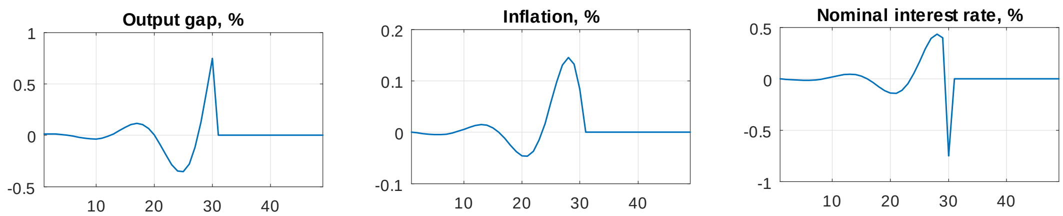

Figure D2. Forward guidance: Taylor rule with actual inflation, $\phi_\pi = 3$.

D2. Taylor rule with actual inflation and output gap: the case of the real roots

In Section 3.3, we compare the solution under two Taylor rules (3) with output gap: one contains the actual inflation and the other contains the expected inflation. In the former case, the model is parameterized by $\phi_y = 0.5$ and $\phi_{E\pi} = 2$, and in the latter case, it is parameterized by $\phi_y = 0.5$ and $\phi_\pi = 2$. These parameterizations lead to complex roots, which correspond to case iv) of Theorems 1-3.

Interestingly, a relatively small change in parameterization produces a switch to the real roots, which is the case ii) or iii) of Theorem 1-3. As an example, we show the solutions for slightly different parameterizations, namely, $\phi_y = 0.5$ and $\phi_{E\pi} = 1.5$ and $\phi_y = 0.5$ and $\phi_\pi = 1.5$.

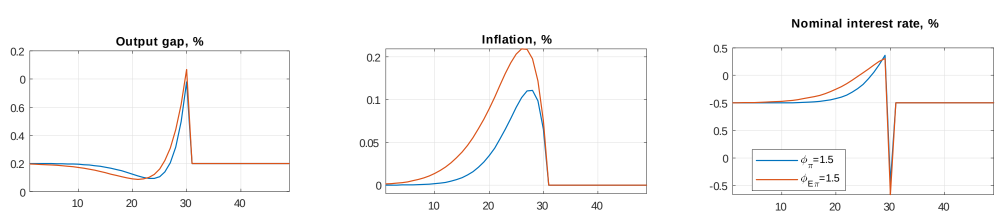

Figure D3. Forward guidance: Taylor rule with output gap $\phi_y = 0.5$ and expected inflation $\phi_{E\pi} = 1.5$ vesus actual inflation $\phi_\pi = 1.5$.

Qualitatively, the solutions shown in Figure D3 are very similar to those reported in Figure 3 in the main text. Quantitatively, the difference in inflation between the two solutions is somewhat larger under $\phi_{E\pi} = 1.5$ and $\phi_{\pi} = 1.5$ than under $\phi_{E\pi} = 2$ and $\phi_{\pi} = 2$ reported in the main text.

Appendix E. New Keynesian model with capital

In this section, we extend the basic new Keynesian model described in Appendix A to include capital and provide additional sensitivity experiments.

E1. The model

We formulate the new Keynesian model with capital.

Households. A household j solves:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \exp\left(\eta_{u,t}\right) \left[\frac{C_t\left(j\right)^{1-\sigma} - 1}{1-\sigma} - \exp\left(\eta_{L,t}\right) \frac{L_t\left(j\right)^{1+\varphi} - 1}{1+\varphi} \right]$$

$$\text{s.t. } P_t C_t\left(j\right) + P_t \exp\left(\eta_{I,t}\right) I_t\left(j\right) + \frac{B_t\left(j\right)}{\exp\left(\eta_{B,t}\right) R_t} + T_t\left(j\right) = \tag{57}$$

$$B_{t-1}\left(j\right) + R_t^k K_{t-1}\left(j\right) + W_t L_t\left(j\right) + \Pi_t\left(j\right)$$

$$K_t\left(j\right) = \left(1-\delta\right) K_{t-1}\left(j\right) + I_t\left(j\right)$$

where $\beta \in (0,1)$ is the discount factor; σ and φ are the utility-function parameters; $\delta \in (0,1]$ is the depreciation rate of capital.

$\eta_{u,t}$ and $\eta_{L,t}$ are exogenous preference shocks: the former scales the overall momentary utility and the latter affects the marginal disutility of labor; $C_t\left(j\right)$, $L_t\left(j\right)$, $I_t\left(j\right)$, $K_{t-1}\left(j\right)$ and $B_t\left(j\right)$ are consumption, labor, investment, capital stock and nominal bonds holdings, respectively; P_t, W_t, R_t^k and R_t are the commodity price, nominal wage, (gross) nominal interest rate on capital and (gross) nominal interest rate on bonds, respectively; $\eta_{B,t}$ is an exogenous premium in the return to bonds (might reflect inefficiency in the financial sector, e.g., a risk premium required by households to hold a one-period bond); $\eta_{I,t}$ is an exogenous capital-embodied technology shock; $T_t\left(j\right)$ is lump-sum taxes; $\Pi_t\left(j\right)$ is the profit of intermediate-good producers. The exogenous shocks follow the following exogenous stochastic processes:

$$\eta_{u,t} = \rho^u \eta_{u,t-1} + \varepsilon_{u,t}, \qquad \varepsilon_{u,t} \sim \mathcal{N}\left(0, \sigma_u^2\right)$$

$$\eta_{L,t} = \rho^L \eta_{L,t-1} + \varepsilon_{L,t}, \qquad \varepsilon_{L,t} \sim \mathcal{N}\left(0, \sigma_L^2\right)$$

$$\eta_{B,t} = \rho^B \eta_{B,t-1} + \varepsilon_{B,t}, \qquad \varepsilon_{B,t} \sim \mathcal{N}\left(0, \sigma_B^2\right)$$

$$\eta_{I,t} = \rho^I \eta_{I,t-1} + \varepsilon_{I,t}, \qquad \varepsilon_{I,t} \sim \mathcal{N}\left(0, \sigma_I^2\right)$$

Final-good producers. They are the same as in the model without capital.

Intermediate-good producers. Technology of a producer i is

$$Y_t(i) = \exp\left(\eta_{a,t}\right) K_{t-1}(i)^{\alpha} L_t(i)^{1-\alpha}$$

where $L_t(i)$ is labor; $K_{t-1}(i)$ is capital; $\exp\left(\eta_{a,t}\right)$ is the productivity level that follows the exogenous stochastic process

$$\eta_{a,t} = \rho^a \eta_{a,t-1} + \varepsilon_{a,t}, \qquad \varepsilon_{a,t} \sim \mathcal{N}\left(0, \sigma_a^2\right)$$

Total cost (net of government subsidy v_t) in nominal terms:

$$\min_{K_{t-1}(i), L_t(i)} TC\left(Y_t(i)\right) = \left\{(1-v_t)\left(R_t^k K_{t-1}(i) + W_t L_t(i)\right)\right\} \tag{58}$$

$$\text{s.t. } Y_t(i) = \exp\left(\eta_{a,t}\right) K_{t-1}(i)^{\alpha} L_t(i)^{1-\alpha} \tag{59}$$

where W_t is the nominal wage rate.

A reoptimizing firm solves the same problem as in the model without capital.

Monetary authority. The monetary authority follows the same Taylor rule (28) as in the model without capital.

Natural level of output. The natural output level $Y_{N,t}$ can be determined from

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \exp\left(\eta_{u,t}\right)\left[\frac{(C_t)^{1-\sigma}-1}{1-\sigma} - \exp\left(\eta_{L,t}\right)\frac{(L_t)^{1+\varphi}-1}{1+\varphi}\right]$$

$$\text{s.t. } C_t^* + \exp\left(\eta_{I,t}\right)\left[K_t - (1-\delta)K_{t-1}\right] = \exp\left(\eta_{a,t}\right)(K_{t-1})^{\alpha}(L_t)^{1-\alpha} - G_t \tag{60}$$

with $G_t = \left(1 - \frac{1}{\exp\left(\eta_{G,t}\right)}\right) Y_{N,t}$.

Summary of equilibrium conditions.

$$S_t = \frac{\exp\left(\eta_{u,t}+\eta_{L,t}\right)}{\exp\left(\eta_{a,t}\right)} C_t^{-\sigma} Y_t \frac{r_t^k}{\alpha}\left[\frac{K_{t-1}}{L_t}\right]^{1-\alpha} + \beta\theta E_t\left\{\pi_{t+1}^{\varepsilon} S_{t+1}\right\}, \tag{61}$$

$$F_t = \exp\left(\eta_{u,t}\right) C_t^{-\sigma} Y_t + \beta\theta E_t\left\{\pi_{t+1}^{\varepsilon-1} F_{t+1}\right\}, \tag{62}$$

$$\frac{S_t}{F_t} = \left[\frac{1-\theta\pi_t^{\varepsilon-1}}{1-\theta}\right]^{\frac{1}{1-\varepsilon}}, \tag{63}$$

$$\Delta_t = \left[(1-\theta)\left[\frac{1-\theta\pi_t^{\varepsilon-1}}{1-\theta}\right]^{\frac{\varepsilon}{\varepsilon-1}} + \theta\frac{\pi_t^{\varepsilon}}{\Delta_{t-1}}\right]^{-1}, \tag{64}$$

$$\exp\left(\eta_{u,t}\right)C_t^{-\sigma}=\beta\exp\left(\eta_{B,t}\right)R_tE_t\left\{\frac{\exp\left(\eta_{u,t+1}\right)C_{t+1}^{-\sigma}}{\pi_{t+1}}\right\}, \tag{65}$$

$$\exp\left(\eta_{u,t}+\eta_{I,t}\right)C_t^{-\sigma}=\beta E_t\left\{\exp\left(\eta_{u,t+1}\right)C_{t+1}^{-\sigma}\left[\exp\left(\eta_{I,t+1}\right)(1-\delta)+r_{t+1}^k\right]\right\}, \tag{66}$$

$$r_t^k=\frac{\alpha}{1-\alpha}\exp\left(\eta_{L,t}\right)L_t^{1+\varphi}C_t^{\sigma}K_{t-1}^{-1}, \tag{67}$$

$$Y_t=\exp\left(\eta_{a,t}\right)K_{t-1}^{\alpha}L_t^{1-\alpha}\Delta_t, \tag{68}$$

$$C_t+\exp\left(\eta_{I,t}\right)\left[K_t-(1-\delta)K_{t-1}\right]=\left(1-\frac{\overline{G}}{\exp\left(\eta_{G,t}\right)}\right)Y_t, \tag{69}$$

where r_t^k is the marginal productivity of capital. There are 10 equations and 10 unknowns (C_t, L_t, K_t, Y_t, π_t, Δ_t, R_t, r_t^k, S_t, F_t). There are 7 exogenous shocks ($\eta_{a,t}$, $\eta_{u,t}$, $\eta_{L,t}$, $\eta_{B,t}$, $\eta_{R,t}$, $\eta_{G,t}$, $\eta_{I,t+1}$) and 3 endogenous state variables ($K_{t-1}, \Delta_{t-1}, R_{t-1}$).

E2. Additional numerical results for the new Keynesian model with capital

We now report the sensitivity experiments that complement the numerical results of Section 5.3. In our first sensitivity experiment, we assume the Taylor rule (28) has just actual inflation $\phi_\pi \searrow 1$. Recall that in the model without capital, this parameterization led to a version of the FG puzzle when just inflation reacts immediately, while the output gap increases gradually; see e.g., Figures D1 in Appendix D.

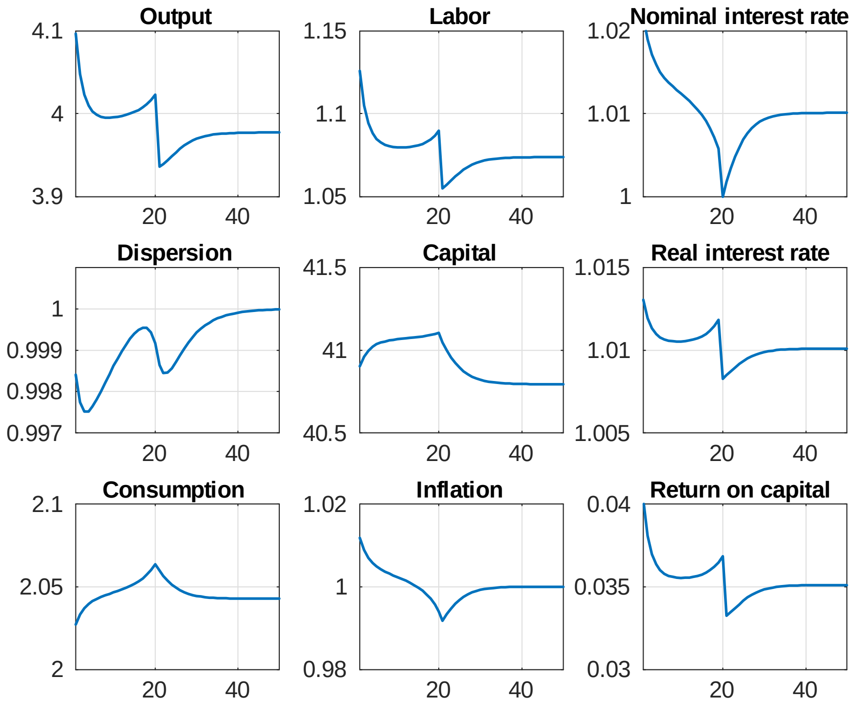

Figure E1. Model with capital: Taylor rule with actual inflation, $\phi_\pi = 1$, $\phi_{E\pi} = 0$, $\phi_y = 0$.

We observe some qualitative differences between the dynamics of the models with and without capital. As we can see from Figure E1, there are immediate effects of FG on both output and inflation, while consumption goes up slowly. Hence, we observe a stronger version of the FG puzzle than in the model without capital under this specific parameterization.

However, when ϕ_π in (28) increases, the FG puzzle disappears as it does in the model without capital. In Figure E2, we show the model with capital in which $\phi_\pi = 3$.

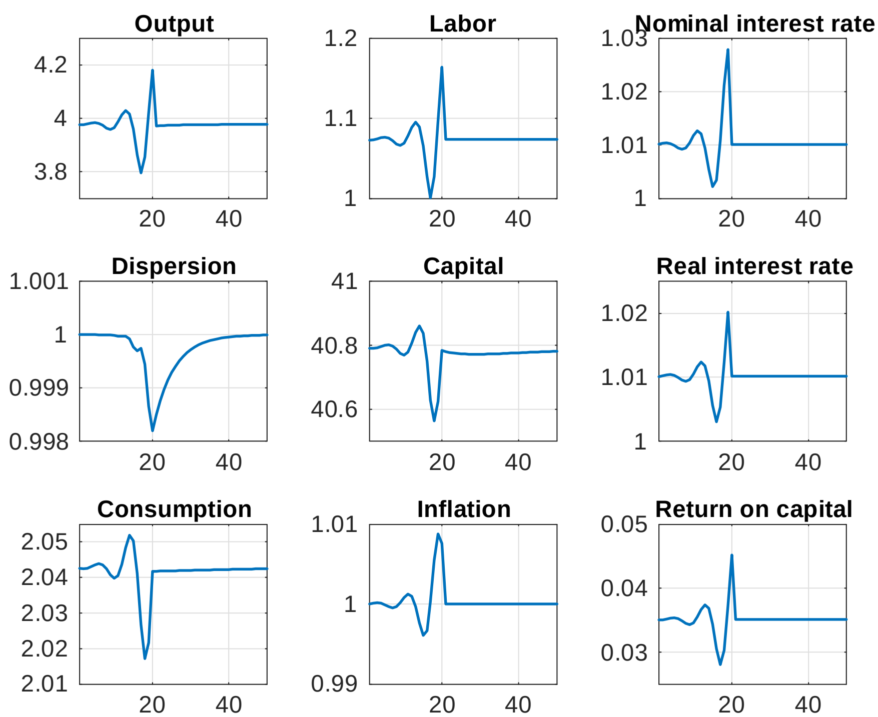

Figure E2. Model with capital: Taylor rule with actual inflation, $\phi_\pi = 3$, $\phi_{E\pi} = 0$, $\phi_y = 0$.

This case is qualitatively similar to the one reported for the baseline linear model in Figure D2.

Finally, in Figure E3, we consider the FG puzzle scenario, i.e., we assume Taylor rule (28) with just expected inflation $\phi_{E\pi} \searrow 1$.

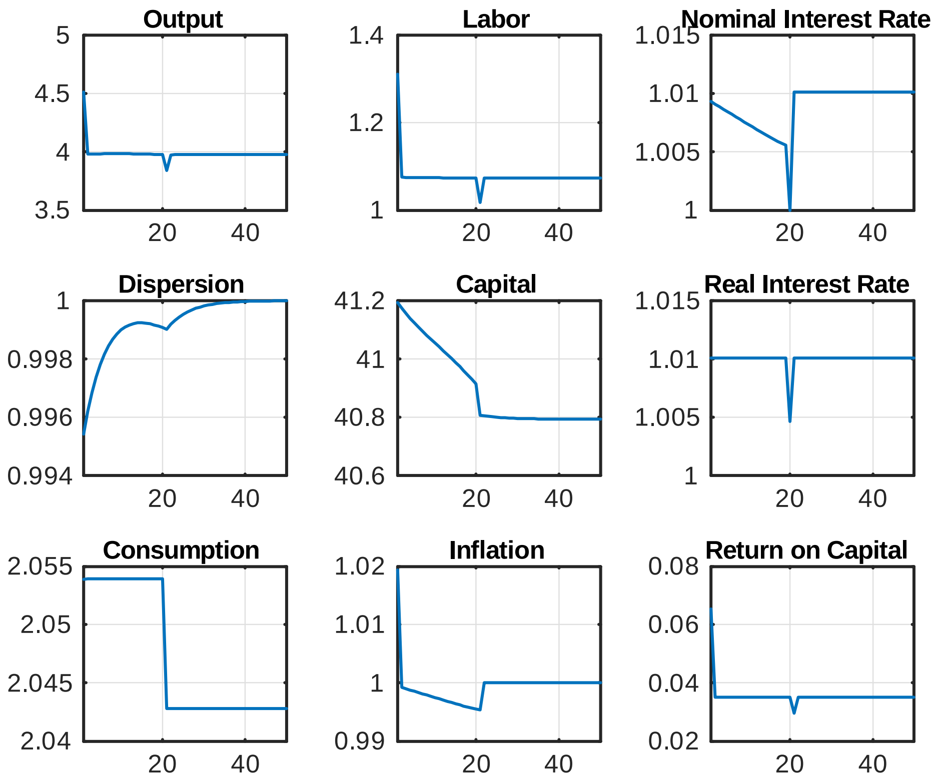

Figure E3. Model with capital: Taylor rule with expected inflation, $\phi_{E\pi} = 1$.

Unlike in the case of $\phi_\pi = 1$, FG is not very effective under $\phi_{E\pi} = 1$. As we can see from Figure E3, labor jumps up immediately in the first period and then it goes down in the second period, in spite of the smooth behavior of capital. The behavior of output here drastically differs from the one in the model without capital, namely, output jumps in the first period but goes back to the original level in the second period and remains there until the shock happens. Therefore, in terms of output, FG has no long-term effect but only a brief initial effect. It is surprising that consumption does not mimic output but behaves as in the basic model without capital, i.e., it jumps immediately and stays high until the shock happens. Hence, the consumption pattern looks resembles the FG puzzle in the baseline model. Thus, some version of the FG puzzle is observed in this case as well.