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# PRUDENTIAL MONETARY POLICY

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## ABSTRACT

Should monetary policymakers raise interest rates during a boom to rein in financial excesses? We theoretically investigate this question using an aggregate demand model with asset price booms and financial speculation. In our model, monetary policy affects financial stability through its impact on asset prices. Our main result shows that, when macroprudential policy is imperfect, there are conditions under which small doses of prudential monetary policy (PMP) can provide financial stability benefits that are equivalent to tightening leverage limits. PMP reduces asset prices during the boom, which softens the asset price crash when the economy transitions into a recession. This mitigates the recession because higher asset prices support leveraged, high-valuation investors' balance sheets. The policy is most effective when the recession is more likely and leverage limits are neither too tight nor too slack. With shadow banks, whether PMP "gets in all the cracks" or not depends on the constraints faced by shadow banks.



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# 1. Introduction

Should monetary policy have a *prudential* dimension? That is, should policymakers raise interest rates, or delay a cut, to rein in financial excesses during a boom? This question has occupied the minds of central bankers and monetary policy researchers for decades. At present, there are two dominant views. The *fully-separable* view contends that monetary policy should focus exclusively on its traditional mandate while delegating financial stability concerns to macroprudential policy (see, e.g., Weidmann (2018); Svensson (2018)). The *non-separable* view argues that, in practice, macroprudential policy might be insufficient for dealing with financial excesses since its tools are limited and inflexible (see, e.g., Stein (2014); Gourio et al. (2018)). This debate has spawned a growing literature investigating the costs and benefits of *prudential monetary policy (PMP)*. In this paper, we provide a new rationale for PMP, and we describe a set of circumstances under which PMP is as effective as macroprudential policy. This equivalence is useful since, as highlighted by the non-separable view, monetary policy in practice is significantly nimbler than macroprudential policy when responding to cyclical fluctuations.[1]

PMP has obvious costs: it slows down the economy and leads to inefficient factor utilization during the boom. The benefits are less well understood. One of the main arguments for PMP is the *asset price channel:* monetary policy can mitigate the asset price boom and therefore make the subsequent crash smaller and less costly (see, e.g., Borio (2014); Adrian and Liang (2018)). This view is supported by evidence that monetary policy has a sizable, nearly immediate impact on asset prices. Despite its potential importance, there is little formal analysis on how the asset price channel of PMP works and whether (or when) it improves social welfare. We fill this gap by developing an aggregate demand model with asset price booms and speculation.

In our model, the economy starts in a boom with high asset valuations—high prices relative to short-run productivity and asset payoffs. There is a chance that asset valuations will decline without a change in short-run productivity—consistent with a large finance literature on "time-varying risk premia" (see, e.g., Campbell (2014)). The decline in asset valuations lowers aggregate demand (due to a consumption wealth effect). Monetary policy cuts interest rates to stimulate demand but it might be constrained. We work with the zero lower bound constraint, but our mechanism applies for other frictions that prevent policymakers from cutting interest rates during recessions. With *interest rate frictions*, the economy experiences a *risk-centric demand recession*—a recession associated with a decline in risky asset valuations (see Caballero and Simsek (forthcoming)).

Our focus is on the boom preceding the recession, which features not only high asset valuations but also *financial speculation*—investors with heterogeneous valuations trading risky

[1] Adrian et al. (2017) summarize the results from a tabletop exercise conducted by the Federal Reserve that "aimed at confronting Federal Reserve Bank presidents with a plausible, albeit hypothetical, macro-financial scenario that would lend itself to macroprudential considerations...From among the various tools considered, tabletop participants found many of the prudential tools less attractive due to implementation lags and limited scope of application...Monetary policy came more quickly to the fore as a financial stability tool than might have been thought before the exercise."

financial assets. We focus on speculation among investors with heterogeneous beliefs (optimists and pessimists), but similar insights apply if speculation is driven by other forces such as heterogeneous risk tolerances (e.g., banks and households). Financial speculation matters because optimists' wealth share is a key state variable for the economy. In particular, when optimists have more wealth in the recession, they push up asset prices and aggregate demand, softening the recession. However, individual optimists who take on leverage during the boom (and pessimists who lend to them) do not internalize the welfare effects of optimists' wealth losses during the recession. This externality motivates policy interventions in the boom. Macroprudential policy is in theory the ideal tool for disciplining optimists' risk taking, but in practice it is imperfect. Our main result shows when PMP can effectively reduce optimists' risk exposure.

To illustrate this result, we introduce some notation and relations (we provide microfoundations in the main text). Specifically, let $s = 1$ and $s = 2$ denote the *boom* and the *recession* states, respectively. The economy is set in continuous time and transitions from the boom state to the recession state according to a Poisson process. Let $\alpha_s$ and $Q_s$ denote optimists' wealth share and the price of capital (asset price) in state $s$, respectively. In the recession state $s = 2$, the asset price (and thus aggregate demand) is an increasing function of optimists' wealth share:

$$Q_2 = Q_2(\alpha_2). \tag{1}$$

In the boom state $s = 1$, optimists choose an above-average leverage ratio, $\omega_1^o > 1$. Therefore, if there is a transition to the recession state, their wealth share declines. Specifically,

$$\frac{\alpha_2}{\alpha_1} = 1 - (\omega_1^o - 1)\left(\frac{Q_1}{Q_2} - 1\right), \tag{2}$$

where $Q_1/Q_2 > 1$ captures the magnitude of the price decline after the transition. Note that this equation also describes an increasing relation between optimists' wealth share, $\alpha_2$, and the asset price in the recession, $Q_2$ (since $\omega_1^o > 1$ ). Given a boom wealth share $\alpha_1$, the equilibrium pair $(\alpha_2, Q_2)$ corresponds to the intersection of two increasing relations (1) and (2), similar to Kiyotaki and Moore (1997). Figure 1 provides a graphical representation of these relations.

As a benchmark, suppose the monetary authority sets interest rates in the boom to achieve asset prices and aggregate demand consistent with potential output, $Q_1 = Q^*$. In the recession, monetary policy is constrained, so asset prices and aggregate demand fall short of potential output, $Q_2 < Q^*$. A larger wealth share for optimists, $\alpha_2$, increases asset prices and aggregate demand and softens the recession. This effect is an *aggregate demand externality,* which provides a rationale for prudential policies that improve optimists' wealth share in the recession, $\alpha_2$.

Eq. (2) suggests that there are two prudential channels policymakers can use to increase $\alpha_2$. First consider macroprudential policy that reduces optimists' leverage ratio, $\omega_1^o$. This policy increases $\alpha_2$ by reducing optimists' exposure to a given asset price decline, $Q^*/Q_2$. Second, suppose instead that optimists' leverage ratio is fixed, $\omega_1^o = \overline{\omega}_1^o$, and consider PMP that lowers

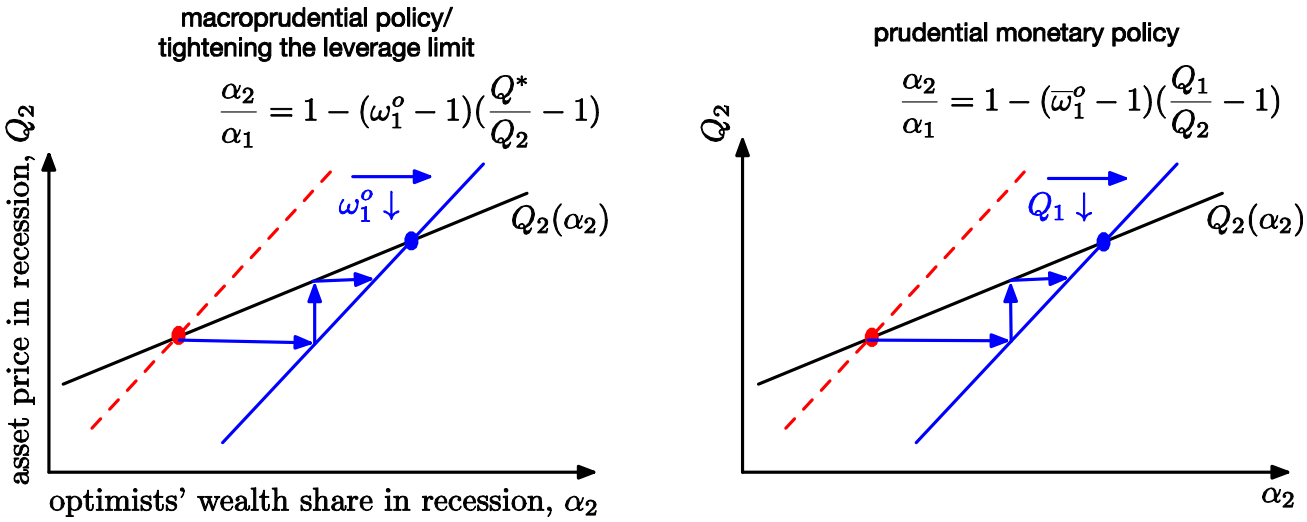


Figure 1: Graphical illustration of the relations that determine optimists' wealth share and the asset price in recession, $(\alpha_2, Q_2)$. The left (resp. right) panel illustrates the effect of macroprudential policy (resp. PMP).

the asset price during the boom, $Q_1 < Q^*$ (by increasing the interest rate). This policy increases $\alpha_2$ by decreasing the size of the asset price decline, $Q_1/Q_2$, for a given level of optimists' exposure. Figure 1 shows that these two policies can achieve the same allocations, illustrating the logic behind our main result.

PMP has two drawbacks relative to macroprudential policy. First, optimists' leverage ratio has to be at least somewhat constrained, $\omega_1^o = \overline{\omega}_1^o$. In practice, this is likely to be the case for a variety of reasons—due to financial frictions, self-imposed limits, or binding macroprudential policy. Nonetheless, our model provides a cautionary note for environments in which optimists' constraints are loose. In particular, in the extreme case where optimists are fully unconstrained, their leverage ratio adjusts to completely undo the prudential effects of monetary policy. That is, once $\omega_1^o$ adjusts, $\alpha_2$ does *not* depend on $Q_1$. The intuition is that, since optimists perceive smaller risks from transitioning to a recession, they increase their leverage ratio. This result illustrates that PMP is more effective when optimists face tighter leverage limits.

Second, even when monetary policy achieves the same prudential objectives as macroprudential policy, it is more costly because it lowers asset prices during the boom, $Q_1 < Q^*$. This lowers aggregate demand and reduces factor utilization below the efficient level. However, in a neighborhood of the price level that ensures efficient factor utilization ($Q^*$), these negative welfare effects are second order. On the other hand, the beneficial effects of softening the recession are first order. We formalize this insight and establish that (when optimists are subject to some leverage limit) the first-order welfare effects of PMP are *exactly the same* as the effects of tightening the leverage limit directly. Put differently, *for small policy changes, PMP is as effective as macroprudential policy.* PMP increases unemployment in a booming economy, which

has negligible costs, and reduces unemployment during a recession, which has sizeable benefits.[2]

We also characterize the optimal monetary policy in our environment and establish two comparative statics results. First, the planner utilizes PMP more when the leverage limit (or macroprudential policy) is at an intermediate level. Intuitively, when the limit is too loose, PMP is not worthwhile because it requires a large decline in $Q_1$ to push optimists against their constraints. Naturally, when the limit is already too tight, further tightening via PMP is not beneficial. These two extreme cases illustrate that macroprudential policy and PMP can be *complements* as well as *substitutes*. Second, as expected, the planner utilizes PMP more when she perceives a greater probability of transitioning into a risk-centric recession. This result highlights that the planner should calibrate PMP by monitoring the *tail risk in asset prices*—the chance of a sufficiently large decline that would trigger a demand recession.

Finally, one of the main practical concerns with prudential policies is the presence of "shadow banks"—lightly regulated high-valuation agents who can circumvent regulatory constraints. Stein (2013) noted that in these environments PMP might have an advantage over macroprudential policy "because it gets in all of the cracks." We extend our analysis to consider shadow banks—optimists who are not subject to regulatory leverage limits. We find that whether PMP is more effective than macroprudential policy depends on the nature of the leverage limits faced by shadow banks. Even if shadow banks circumvent the regulatory leverage limit, they might still be constrained due to financial frictions or self-imposed limits. In this case, PMP is indeed more effective than macroprudential policy, because it improves the financial stability of all banks whereas macroprudential policy stabilizes only the regulated banks. We also analyze the other extreme case in which shadow banks are fully unconstrained. In this case, PMP can still replicate the financial stability benefits of macroprudential policy; however, both policies are weaker than when there are no shadow banks. The policies are weaker because of general equilibrium feedbacks: shadow banks (when unconstrained) respond to the stabilizing benefits of either policy by increasing their leverage and risk taking.

*Literature review.* Our paper is part of a large literature that investigates the effect of monetary policy on financial stability. Adrian and Liang (2018) provide a recent survey (see also Smets (2014)). As they note, easy monetary policy can generate financial vulnerabilities by fueling credit growth, exacerbating the maturity mismatch of financial intermediaries, and inflating asset prices. Our paper focuses on the asset-price channel, which is underexplored.

One strand of the literature emphasizes that loose monetary policy can reduce risk premia during the boom by exacerbating the "reach for yield" (see, e.g., Rajan (2006); Maddaloni and

[2] This discussion illustrates how our main result may apply beyond our particular model of risk-centric recessions. For example, suppose the recession features no interest rate frictions, but there are *financial frictions* and *fire-sale prices* that increase in *experts'* wealth share. Suppose experts take on leverage during the boom to increase the size of their investments (as in Lorenzoni (2008)). The analogues of Eqs. (1) and (2) apply in this setting. Hence, as long as experts' leverage is constrained, PMP would improve experts' balance sheets in the recession and increase welfare. In this alternative setup, the policy would increase welfare by mitigating *fire-sale externalities*, whereas in our model PMP internalizes aggregate demand externalities.

Peydró (2011); Borio and Zhu (2012); Morris and Shin (2014); Lian et al. (2018); Acharya and Naqvi (2018)). In our model, monetary policy does not directly affect the risk premium—it affects asset prices mainly through the traditional discount rate channel. Nonetheless, we find a role for PMP because the reduction in asset prices during the boom softens the asset price crash after a transition to recession. Our channel is stronger (and it operates through the same key equations) if loose monetary policy reduces the risk premium during the boom (see, e.g., Bernanke and Kuttner (2005); Adrian and Shin (2010); Hanson and Stein (2015); Gertler and Karadi (2015); Gilchrist et al. (2015) for empirical evidence and Adrian and Duarte (2016) for a model in which monetary policy affects the risk premium).

Our paper complements the literature emphasizing the credit channel. A number of papers show that monetary policy can affect financial stability by influencing credit growth or leverage. Woodford (2012) articulates this channel using a New Keynesian framework (that builds upon Curdia and Woodford (2010)) in which loose monetary policy increases the leverage of financial institutions (or borrowers), which in turn increases the probability of a crisis (by assumption). A growing empirical literature has documented that rapid credit growth is indeed associated with more frequent and more severe financial crises (e.g., Borio and Drehmann (2009); Jordà et al. (2013)). Recent work uses the empirical estimates from this literature to calibrate Woodford-style models and quantify the costs and benefits of PMP. Svensson (2017); IMF (2015) argue that the costs of this policy exceed the benefits, whereas Gourio et al. (2018); Adrian and Liang (2018) find mixed effects. We show that monetary policy can also affect financial stability by influencing asset prices during the boom. Moreover, our model does not require a financial crisis: there are benefits if the economy transitions into a plain-vanilla recession (associated with a decline in asset prices). Hence, quantitative analyses that rely purely on the credit channel and crises likely underestimate the potential benefits of PMP.

It is instructive to contrast our analysis with Svensson (2017), who works with a different setup and reaches different conclusions. Svensson (2017) *assumes* that weaker aggregate demand in the boom translates into weaker aggregate demand if the economy transitions into recession. This assumption implies that since PMP weakens aggregate demand in the boom, it also weakens aggregate demand in the recession. However, while one can imagine some mechanisms by which weakness in the boom persists into the recession (such as habit formation or backward-looking inflation expectations), it is clearly not generic. For example, it does not follow in a model like ours where PMP weakens asset prices in the boom but *strengthens asset prices and aggregate demand in the recession.* Likewise, when aggregate demand has features of a stock variable, weaker aggregate demand in the boom naturally strengthens aggregate demand in the recession.[3]

In our model, PMP causes an output gap during the boom, which generates a second-order

[3] This is the case in standard models that feature diminishing returns to spending. For example, Rognlie et al. (2018) shows how weaker investment in the boom creates pent-up investment demand in the recession (when capital has diminishing returns). Berger et al. (2018) show that, with fixed-rate mortgages, higher interest rates in the boom shift household refinancing from the boom to the recession. Mian et al. (2019) establish similar results when borrowers and savers differ in their marginal propensities to save out of permanent income.

welfare loss (for small changes in policy), and mitigates the output gap during the recession, which generates a first-order welfare gain. Kocherlakota (2014) and Stein (2014) derive similar insights by assuming that the Fed uses a quadratic loss function to penalize deviations of unemployment from its target. They show that targeting financial stability fits naturally into the Fed's dual mandate. Our model provides a microfoundation for their key assumption that accommodative monetary policy exacerbates financial vulnerability.

Our paper is part of a growing theoretical literature that analyzes the interactions between macroprudential and monetary policies in environments with aggregate demand externalities (e.g., Korinek and Simsek (2016)).[4] This literature typically concludes that financial stability issues are best addressed with macroprudential policy. We assume macroprudential policy is constrained, and we find a role for monetary policy that interacts with macroprudential policy. We also investigate the asset price channel, whereas Korinek and Simsek (2016) and Farhi and Werning (2016) focus on credit. Rognlie et al. (2018) analyze investment and show that incorporating this ingredient would strengthen our main result. When alternative policies are imperfect, PMP can be used to reduce investment during the boom, which increases investment and asset prices in the recession (see also Footnote 3).

Finally, although our mechanism is more general, our model is related to a growing literature on belief disagreements and speculation (e.g., Scheinkman and Xiong (2003); Fostel and Geanakoplos (2008); Geanakoplos (2010); Simsek (2013a,b); Cao (2017); Borovička (2020); Bailey et al. (2019); Iachan et al. (2015); Heimer and Simsek (2019); Bigio and Zilberman (2019)). Similar to Caballero and Simsek (forthcoming), we analyze speculation when aggregate demand can influence output due to interest rate rigidities. We depart from our earlier work by assuming that financial markets are incomplete due to exogenous leverage limits (see Remark 4). This assumption ensures that monetary policy affects the extent of speculation.

In Section 2 we introduce the basic environment and provide a partial characterization of the equilibrium. In Section 3, we characterize the equilibrium in the recession state and illustrate the aggregate demand externalities that motivate policy interventions. In Section 4, we characterize the equilibrium in the boom state and establish our main result that shows monetary policy can replicate the financial stability effects of macroprudential policy. In Section 5, we introduce our welfare criterion and show that PMP achieves the same welfare effects as macroprudential policy. In Section 6, we characterize the optimal PMP in our setting and establish its comparative statics. In Section 7, we add "shadow banks" to our framework and analysis. Section 8 concludes and is followed by several appendices that contain omitted derivations and proofs.

[4] Several papers analyze the interaction of macroprudential and monetary policies but focus on other frictions (e.g., Stein (2012); Collard et al. (2017); Martinez-Miera and Repullo (2019); Allen et al. (2019); Acharya and Plantin (2019)). A vast literature theoretically investigates macroprudential policy but doesn't focus on nominal rigidities or monetary policy (see, e.g., Dávila and Korinek (2017) and the references therein).

## 2. Environment and equilibrium

In this section we introduce our general dynamic environment. We then provide a definition and a partial characterization of the equilibrium. In subsequent sections we further analyze this equilibrium under different assumptions about monetary policy.

**Potential output and risk premium shocks.** The economy is set in infinite continuous time, $t \in [0, \infty)$, with a single consumption good and a single factor of production, capital. Let $k_{t,s}$ denote the capital stock at time $t$ in the aggregate state $s \in S$.

The rate of capital utilization is endogenous and denoted by $\eta_{t,s} \in [0, 1]$. When utilized at this rate, $k_{t,s}$ units of capital produce

$$A\eta_{t,s}k_{t,s} \tag{3}$$

units of the consumption good. The capital stock follows the process

$$\frac{dk_{t,s}/dt}{k_{t,s}} = g_s - \delta\left(\eta_{t,s}\right). \tag{4}$$

The depreciation function $\delta\left(\eta_{t,s}\right)$ is weakly increasing and weakly convex (and strictly so above some range $\eta_{t,s} \geq \underline{\eta}$). Hence, utilizing capital at a higher rate allows the economy to produce more current output at the cost of faster depreciation and slower output growth. Without nominal rigidities, there is an optimal level of capital utilization denoted by $\eta^*$, which we characterize later. With nominal rigidities, the economy may operate below this level of utilization, $\eta_{t,s} \leq \eta^*$, which we refer to as a demand recession.

The expected growth rate of capital (before depreciation) is $g_s$, which is an exogenous parameter. The states, $s \in S$, differ only in terms of $g_s$. For simplicity, we assume there are three states, $s \in \{1, 2, 3\}$. The economy starts in state $s = 1$. While in states $s \in \{1, 2\}$, the economy transitions into state $s' \equiv s + 1$ according to a Poisson process that we describe below. Once the economy reaches $s = 3$, it stays there forever.

Fluctuations in the expected growth rate, $g_s$, affect our analysis mainly by generating fluctuations in asset valuations unrelated to short-run productivity. Hence, we view these fluctuations as a convenient modeling device to capture "time-varying risk premia" driven by beliefs, risks, or risk attitudes.[5] We assume the parameters satisfy $g_2 < \min(g_1, g_3)$: the economy starts with high valuations, transitions to lower valuations, and transitions back to higher valuations (see Figure 2). Under appropriate assumptions, we will establish that the decline in asset prices in state 2 will trigger a demand recession. Accordingly, we refer to states 1, 2, and 3 as "the boom," "the recession," and "the recovery," respectively. For analytical tractability, we focus on a single business cycle. We assume the economy transitions between states according to

[5] In Caballero and Simsek (forthcoming), we formalize this intuition by showing that (in a two period model) changes in $g_s$ generate the same effect on asset prices and current economic activity as changes in risk aversion and (perceived or real) risk. A large literature documents that time-varying risk premia are a pervasive phenomenon in financial markets (see Cochrane (2011); Shiller (2014); Campbell (2014) for recent reviews).

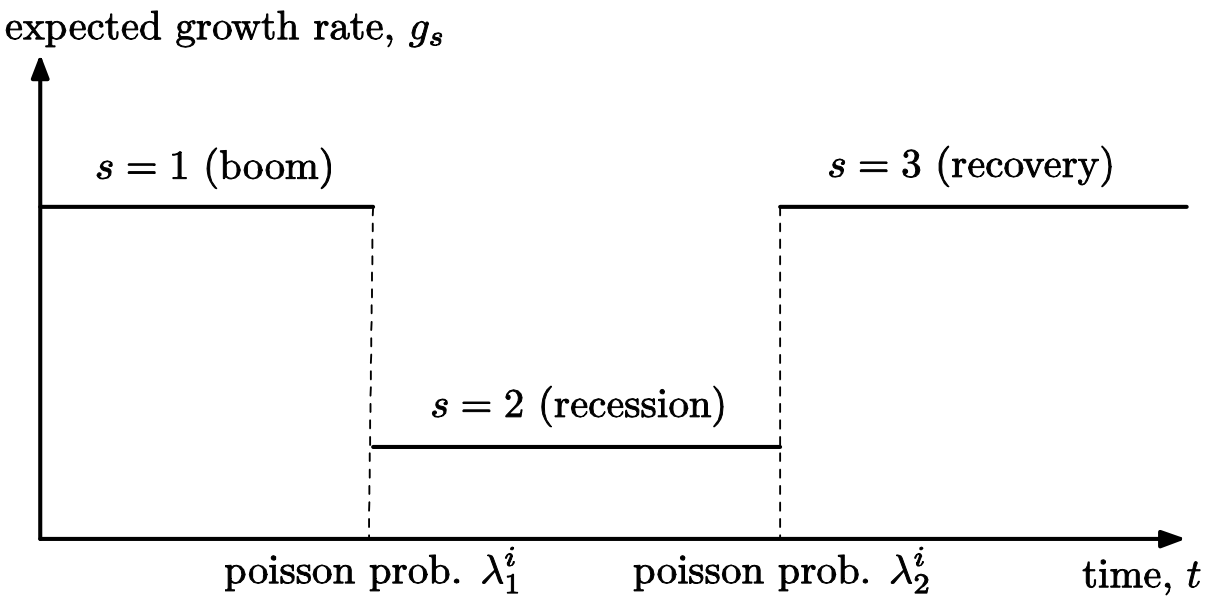


Figure 2: The timeline of events.

exogenous shocks (see the concluding section for whether PMP could endogenously trigger the recession).

**Transition probabilities and belief disagreements.** We let $\lambda_s^i > 0$ denote investor $i$'s belief about the Poisson transition probability from state $s$ into state $s' = s + 1$. Since state $s = 3$ is an absorbing state, $\lambda_3^i = 0$ for each $i$. For the remaining states, we assume there are two types of investors, $i \in \{o, p\}$. Type $o$ investors are "optimists," and type $p$ investors are "pessimists." We assume investors' beliefs satisfy:

**Assumption 1.** $\lambda_1^o < \lambda_1^p$ and $\lambda_2^o > \lambda_2^p$.

When the economy is in the boom state $s = 1$, optimists assign a smaller transition probability to the recession state $s = 2$. When the economy is in the recession state, they assign a greater transition probability to the recovery state $s = 3$.

The key role of "optimists" is that they value risky assets more than "pessimists" *in both states* so that: (i) during the boom, they will take on leverage, and (ii) during the recession, they will increase risky asset prices. Hence, we view disagreements as a convenient modeling device to capture *persistent* heterogeneous asset valuations. In particular, we can also think of "optimists" as banks (or institutional investors) that are more risk tolerant and less Knightian than households or pension funds ("pessimists").

**Menu of financial assets.** There are two types of financial assets. First, there is a market portfolio that represents a claim on all output (see Remark 1 for how to interpret the market portfolio in our context). We let $Q_{t,s}k_{t,s}$ denote the price of the market portfolio, so $Q_{t,s}$ is the

price per unit of capital. We let $r_{t,s}$ denote the instantaneous expected return on the market portfolio conditional on no transition. Second, there is a risk-free asset in zero net supply. We denote its instantaneous return by $r^f_{t,s}$.

In Caballero and Simsek (forthcoming), we allow for Arrow-Debreu securities that enable investors to trade the transition risk. Here, we assume financial markets are incomplete and thus investors speculate by adjusting their position on the market portfolio, i.e., changing their leverage ratio (see also Remark 4).

**Market portfolio price and return.** Absent state transitions, the price of capital $Q_{t,s}$ follows an endogenous, deterministic path. Using Eq. (4), the growth rate of the price of the market portfolio is given by

$$\frac{d\left(Q_{t,s}k_{t,s}\right)/dt}{Q_{t,s}k_{t,s}} = g_s - \delta\left(\eta_{t,s}\right) + \frac{\dot{Q}_{t,s}}{Q_{t,s}},$$

where we use the notation $\dot{X} \equiv dX/dt$. Consequently, the return of the market portfolio absent state transitions can be written as

$$r_{t,s} = \frac{y_{t,s}}{Q_{t,s}k_{t,s}} + g_s - \delta\left(\eta_{t,s}\right) + \frac{\dot{Q}_{t,s}}{Q_{t,s}}. \tag{5}$$

Here, $y_{t,s}$ denotes the endogenous level of output at time $t$. The first term captures the "dividend yield" component of return. The second term captures the capital gain conditional on no transition, which reflects the expected (net) growth of capital and its price.

**Portfolio choice.** Investors are identical except for their beliefs about state transitions, $\lambda^i_s$. They continuously make consumption and portfolio allocation decisions. At any time $t$ and state $s$, investor $i$ has some financial wealth denoted by $a^i_{t,s}$. She chooses her consumption rate, $c^i_{t,s}$, and the fraction of her wealth to allocate to the market portfolio, $\omega^i_{t,s}$. She invests the residual fraction, $1-\omega^i_{t,s}$, in the risk-free asset.

Note that $\omega^i_{t,s}$ also captures the investors' *leverage ratio*. We impose a leverage limit in the boom state $s=1$:

$$\omega^i_{t,1} \leq \overline{\omega}_{t,1}, \tag{6}$$

where $\overline{\omega}_{t,1} \geq 1$ (to ensure market clearing). We allow for $\overline{\omega}_{t,1} = \infty$, in which case the leverage limit does not bind. Our main result applies when the leverage limit may bind (see Remark 2 for how to interpret the leverage limit in our context). We assume there is no leverage limit in the remaining states—this does not play an important role beyond simplifying the analysis.

For analytical tractability, we assume investors have log utility. The investors' problem (at

time $t$ and state $s$) can then be written as

$$V_{t,s}^{i}\left(a_{t,s}^{i}\right)=\max_{\left[c_{\tilde{t},\tilde{s}},\omega_{\tilde{t},\tilde{s}}^{i}\right]_{\tilde{t}\geq t,\tilde{s}}}E_{t,s}^{i}\left[\int_{t}^{\infty}e^{-\rho\tilde{t}}\log c_{\tilde{t},\tilde{s}}^{i}d\tilde{t}\right] \tag{7}$$

$$\text{s.t.}\begin{cases} da_{t,s}^{i}=\left(a_{t,s}^{i}\left(r_{t,s}^{f}+\omega_{t,s}^{i}\left(r_{t,s}-r_{t,s}^{f}\right)\right)-c_{t,s}\right)dt & \text{absent transition,}\\ a_{t,s'}^{i}=a_{t,s}^{i}\left(1+\omega_{t,s}^{i}\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s}}\right) & \text{if there is a transition to state } s'\neq s \end{cases} \tag{8}$$

and $\omega_{t,1}^{i}\leq\overline{\omega}_{t,1}$.

Here, $E_{t,s}^{i}\left[\cdot\right]$ denotes the expectation operator corresponding to investor $i$'s beliefs for state transition probabilities.

**Equilibrium in asset markets.** Asset markets clear when the total wealth held by investors is equal to the value of the market portfolio both before and after the portfolio allocation decisions:

$$a_{t,s}^{o}+a_{t,s}^{p}=\omega_{t,s}^{o}a_{t,s}^{o}+\ \omega_{t,s}^{p}a_{t,s}^{p}=Q_{t,s}k_{t,s}. \tag{9}$$

When the conditions in (9) are satisfied, the market clearing condition for the risk-free asset (which is in zero net supply) holds.

**Nominal rigidities and equilibrium in goods markets.** The supply side of our model features nominal rigidities similar to the New Keynesian model. There is a continuum of competitive production firms that own the capital stock and produce the final good. For simplicity, these production firms have pre-set nominal prices that never change (see Remark 3 for how partial price flexibility would affect our analysis). Firms choose their capital utilization rate, $\eta_{t,s}$, to maximize their market value subject to demand constraints. They take into account both that greater $\eta_{t,s}$ increases production according to Eq. (3) and that it leads to faster capital depreciation according to Eq. (4).

First consider the benchmark case without price rigidities. In this case, firms solve

$$\max_{\eta_{t,s}}\eta_{t,s}Ak_{t,s}-\delta\left(\eta_{t,s}\right)Q_{t,s}k_{t,s}. \tag{10}$$

The optimality condition is given by

$$\delta'\left(\eta_{t,s}\right)Q_{t,s}=A. \tag{11}$$

That is, the frictionless level of utilization ensures that the marginal depreciation rate is equal to the marginal product of capital.

Next consider the case with price rigidities. Firms solve problem (10) with the additional constraint that their output is determined by aggregate demand. As in the New Keynesian model, firms optimally meet this demand as long as their price exceeds their marginal cost. In

a symmetric environment, the real price per unit of consumption good is one for all firms, and each firm's marginal cost is given by $\frac{\delta'(\eta_{t,s})Q_{t,s}}{A}$. Therefore, firms' optimality condition is[6]

$$y_{t,s} = \eta_{t,s} A k_{t,s} = c^o_{t,s} + c^p_{t,s} \qquad \text{as long as } \delta'\left(\eta_{t,s}\right) Q_{t,s} \leq A. \tag{12}$$

**Interest rate rigidity and monetary policy.** Our assumption that production firms do not change their prices implies that the aggregate nominal price level is fixed. The real risk-free interest rate, then, is equal to the nominal risk-free interest rate, which is determined by monetary policy. We assume there is a lower bound on the nominal interest rate, which we set as zero for convenience: $r^f_{t,s} \geq 0$.

We model monetary policy as a sequence of interest rates, $\left\{r^f_{t,s}\right\}_{t,s}$, and implied levels of factor utilization and asset price levels, $\left\{\eta_{t,s}, Q_{t,s}\right\}_{t,s}$, chosen subject to the zero lower bound constraint. Absent price rigidities, factor utilization and asset price levels satisfy condition (11). Therefore, we define the *conventional output-stabilization policy* as

$$r^f_{t,s} = \max\left(0, r^{f*}_{t,s}\right) \text{ for each } s, \tag{13}$$

where $r^{f*}_{t,s}$ ("rstar") is recursively defined as the instantaneous interest rate that obtains when condition (11) holds and the planner follows the output-stabilization policy in (13) at all future times and states.

Our goal is to understand whether the planner might want to use (pre-announced) monetary policy for prudential purposes in the boom state. In particular, we assume the planner follows the conventional output-stabilization policy in (13) for the recession and the recovery states $s \in \{2, 3\}$, but she might deviate from this rule in the boom state $s = 1$. For now, we allow the planner to choose an arbitrary path, $\left\{r^f_{t,1}, Q_{t,1}, \eta_{t,1}\right\}_t$, that is consistent with the equilibrium conditions. We specify the monetary policy further in Section 4.2 and define the equilibrium below.

**Definition 1.** The equilibrium is a collection of processes for allocations, prices, and returns such that capital evolves according to Eq. (4), its instantaneous return is given by Eq. (5), investors maximize their expected utility subject to a leverage limit in the boom state (cf. problem 7), asset markets clear (cf. Eq. (9)), goods markets clear (cf. Eq. (12)), and the monetary authority follows the conventional output-stabilization policy in states $s \in \{2, 3\}$ [cf. Eq. (13)] and chooses a feasible path $\left\{r^f_{t,1}, Q_{t,1}, \eta_{t,1}\right\}_t$ in state $s = 1$.

**Remark 1** (Interpreting the market portfolio)**.** In a richer environment with multiple assets, the market portfolio would capture the wealth-weighted average of all risky assets. In particular, our analysis can accommodate multiple asset classes such as bonds, housing, and equities—subject

[6] If instead the marginal cost exceeded the price, $\frac{\delta'(0)Q_{t,s}}{A} > 1$, firms would choose $\eta_{t,s} = 0$ and produce $y_{t,s} = 0$. This case does not emerge in equilibrium.

to their own price fluctuations—as long as these assets are held in equal relative proportion by "optimists" (banks) and "pessimists." In practice, banks are typically more exposed to bond prices, as well as to house prices, but relatively less exposed to stock prices. Since our goal is to analyze how PMP affects bank balance sheets, we interpret the market portfolio as a weighted average of all risky assets *according to their weights in banks' portfolios* (see the concluding section for further discussion).

**Remark 2** (Interpreting the leverage limit)**.** We view the leverage limit as capturing a variety of unmodeled (and relevant) features that would make banks' leverage ratio (to some extent) exogenous to the aggregate risk that we focus on. First, the leverage limit can capture a regulatory constraint. Second, the limit can capture a market-imposed leverage constraint due to financial frictions such as moral hazard, adverse selection, lenders' uncertainty or their desire for safety. Third, the limit can be self-imposed: specifically, it can capture banks' leverage choice in a richer environment that features multiple risks, e.g., sectoral risks in addition to the aggregate risk that we model. In such an environment, as long as financial markets are incomplete, banks' leverage choice would be determined by a combination of risks and therefore would respond relatively less to changes in aggregate risk.

**Remark 3** (Partial price flexibility)**.** While our assumption of fixed nominal prices is extreme, partial price flexibility would not substantially change our qualitative conclusions. In fact, partial flexibility would typically *amplify* the recession: it would lead to nominal price deflation during the recession that would *exacerbate* the lower bound in (13) (see Caballero and Simsek (forthcoming) for further discussion). Partial flexibility would play a more important role for our analysis in the boom: PMP in the boom can generate price deflation (in addition to negative output gaps). In the concluding section, we discuss how PMP can be adjusted in practice to reduce its side effects on inflation.

We next provide a generally applicable partial characterization of the equilibrium. In subsequent sections, we use this characterization to describe the equilibrium in the different states and policy regimes.

### 2.1. Equilibrium in the goods market

We first state the following result for the equilibrium in the goods market.

**Lemma 1.** *The equilibrium features the* output-asset price relation*:*

$$y_{t,s}/k_{t,s} = A\eta_{t,s} = \rho Q_{t,s}. \tag{14}$$

*The return of the market portfolio (absent transition) is*

$$r_{t,s} = \rho + g_s - \delta\left(\eta_{t,s}\right) + \frac{\dot{Q}_{t,s}}{Q_{t,s}} \quad \textit{where } \eta_{t,s} = \frac{Q_{t,s}}{Q^*}\eta^*. \tag{15}$$

*Here, $\eta^*$ and $Q^*$ denote the efficient level of the capital utilization and the asset price characterized, respectively, by*

$$\delta'(\eta^*)\,\eta^* = \rho, \tag{16}$$

$$Q^* = \frac{A\eta^*}{\rho}. \tag{17}$$

Eq. (14) establishes a tight relation between output and asset prices. This relation follows because, in view of log utility, each investor optimally consumes a fraction of her wealth each period. Therefore, aggregate spending is a fraction of aggregate wealth. Since output is determined by spending, aggregate output is also a fraction of aggregate wealth. Combining this with Eq. (5) implies the equilibrium return on the market portfolio is given by Eq. (15).

Eq. (16) describes the efficient factor utilization without nominal rigidities. It follows from Eqs. (11) and (14). Note that the efficient factor utilization is constant across states.

Eq. (17) describes the efficient asset price and it follows from Eq. (14). This is the asset price level such that the associated aggregate demand leads to efficient capital utilization (and ensures actual output is exactly equal to potential output). When $Q_{t,s} < Q^*$, we have $\eta_{t,s} < \eta^*$: capital is utilized below its efficient level, which we interpret as a demand recession. Therefore, the traditional monetary policy in (13) can be replaced by

$$Q_{t,s} = Q^*, r^f_{t,s} \geq 0 \text{ or } Q_{t,s} < Q^*, r^f_{t,s} = 0. \tag{18}$$

Monetary policy can equivalently be thought of as stabilizing the price per capital.

## 2.2. Equilibrium in asset markets

The asset market equilibrium depends on investors' optimal leverage ratio, $\omega^i_{t,s}$, which we characterize next. To this end, we define investors' relative wealth shares as

$$\alpha^i_{t,s} \equiv \frac{a^i_{t,s}}{Q_{t,s}k_{t,s}} \text{ for } i \in \{o,p\}. \tag{19}$$

The wealth shares sum to one, $\alpha^o_{t,s} + \alpha^p_{t,s} = 1$ [cf. Eq. (9)]. The budget constraints in (7) imply that investors' leverage ratio, $\omega^i_{t,s}$, is closely related to their wealth share *after transition*, $\alpha^i_{t,s'}$:

$$\frac{\alpha^i_{t,s'}}{\alpha^i_{t,s}} - 1 = \left(\omega^i_{t,s} - 1\right)\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}}. \tag{20}$$

When the transition decreases the asset price, $Q_{t,s'} < Q_{t,s}$, an investor's wealth share decreases after the transition, $\alpha^i_{t,s'} < \alpha^i_{t,s}$, if and only if she has above-average leverage, $\omega^i_{t,s} > 1$. Moreover, we can think of investors as effectively choosing a target wealth share after transition, $\alpha^i_{t,s'}$, and adjusting their leverage ratio, $\omega^i_{t,s}$, to achieve this target.

**Lemma 2.** *Investors choose a wealth share after transition, $\alpha^i_{t,s'}$, that satisfies*

$$r_{t,s} + \lambda^i_s \frac{\alpha^i_{t,s}}{\alpha^i_{t,s'}} \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}} \geq r^f_{t,s} \quad \text{with equality if } \omega^i_{t,s} < \overline{\omega}_{t,s}. \tag{21}$$

*The asset market is in equilibrium ((9) holds) when*

$$\alpha^o_{t,s'} + \alpha^p_{t,s'} = 1. \tag{22}$$

*In equilibrium, if there is no transition, investors' wealth share evolves according to*

$$\frac{\dot{\alpha}^i_{t,s}}{\alpha^i_{t,s}} = \lambda^p_s \frac{\alpha^p_{t,s}}{\alpha^p_{t,s'}} \left(1 - \frac{\alpha^i_{t,s'}}{\alpha^i_{t,s}}\right). \tag{23}$$

Eq. (21) is investors' portfolio optimality condition [cf. problem (7)]. As long as the investor is unconstrained, she adjusts her leverage ratio until the risk-adjusted expected return on capital is equal to the risk-free rate. The risk-adjusted return captures aggregate price changes $\left(\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s'}}\right)$ as well as the adjustment of marginal utility relative to other investors if there is a transition $\left(\frac{\alpha^i_{t,s}}{\alpha^i_{t,s'}}\right)$. For the equilibria we analyze, the leverage limit never binds for pessimists. Consequently, the optimality condition (21) always holds as equality for pessimists, but it might apply as inequality for optimists.

Eq. (22) recasts asset market clearing in terms of investors' wealth shares. Together with Eq. (21), this equation characterizes the equilibrium asset price $Q_{t,s}$.

Eq. (23) describes investors' wealth shares if there is no state transition. It follows from investors' budget constraints (7) and the asset market equilibrium conditions. Pessimists' beliefs (superscript $p$) appear in this expression because the optimality condition (21) always holds as equality for them, so we can use their beliefs to price assets. In equilibrium, there is a risk-reward trade-off. If an investor chooses $\alpha^i_{t,s'} < \alpha^i_{t,s}$ (resp. $\alpha^i_{t,s'} > \alpha^i_{t,s}$) so that her wealth share decreases (resp. increases) after a state transition, then she also has $\dot{\alpha}^i_{t,s} > 0$ (resp. $\dot{\alpha}^i_{t,s} < 0$) so her wealth share grows (resp. shrinks) if there is no state transition.

We end this section with the following result that characterizes these equations further for the case in which the leverage limit doesn't bind. To state the result, we define the wealth-weighted average of the transition probability as

$$\overline{\lambda}_{t,s} = \alpha_{t,s}\lambda^o_{t,s} + (1 - \alpha_{t,s})\,\lambda^o_{t,s}.$$

**Lemma 3.** *Suppose optimists are unconstrained, $\omega^o_{t,1} < \overline{\omega}_{t,1}$. In equilibrium, investors' wealth*

*shares evolve according to*

$$\frac{\alpha^i_{t,s'}}{\alpha^i_{t,s}} = \frac{\lambda^i_s}{\overline{\lambda}_{t,s}} \text{ if there is a state change to } s', \tag{24}$$

$$\frac{\dot{\alpha}^i_{t,s}}{\alpha^i_{t,s}} = -\left(\lambda^i_s - \overline{\lambda}_{t,s}\right) \text{ if there is no state change.} \tag{25}$$

*The equilibrium asset price satisfies* the risk balance condition

$$r_{t,s} + \overline{\lambda}_{t,s}\left(1 - \frac{Q_{t,s}}{Q_{t,s'}}\right) = r^f_{t,s}. \tag{26}$$

When the leverage limit doesn't bind, Eq. (21) implies $\lambda^o_s \frac{\alpha^o_{t,s}}{\alpha^o_{t,s'}} = \lambda^p_s \frac{\alpha^p_{t,s}}{\alpha^p_{t,s'}}$: investors trade the market portfolio until they equate their perceived probabilities multiplied by marginal utilities. The results follow from combining this observation with Lemma 2.

Eqs. $(24-25)$ show that, when type $i$ investors perceive a below-average transition probability, $\lambda^i_s < \overline{\lambda}_{t,s}$, their wealth share decreases after transition but drifts upward absent transition. Conversely, when investors assign above-average transition probability, their wealth share increases after transition but drifts downward absent transition. Eq. (26) says the asset price is determined *as if there is a "representative investor" that has the wealth-weighted average belief.*

**Remark 4** (Role of market incompleteness due to the leverage limit)**.** Lemmas 2 and 3 clarify the differences between this model and the one in Caballero and Simsek (forthcoming). The equations in Lemma 3 are the same as their counterparts in Caballero and Simsek (forthcoming), where we allow investors to trade transition risks via Arrow-Debreu securities. The intuition is that, as long as the leverage limit does not bind, the market portfolio and the risk-free asset are sufficient to dynamically complete the market. The main difference in this setting is that the leverage limit *can* bind, in which case the wealth-share dynamics are different than in Caballero and Simsek (forthcoming) and are characterized by the equations in Lemma 2.

## 3. The recession and aggregate demand externalities

We next characterize the equilibrium starting in the recession state and illustrate the aggregate demand externalities that motivate prudential policies. For the rest of the paper, with a slight abuse of notation, we often drop the superscript $o$ from optimists' wealth share:

$$\alpha_{t,s} \equiv \alpha^o_{t,s}.$$

Pessimists' wealth share is the complement of this expression, $\alpha^p_{t,s} = 1 - \alpha_{t,s}$. We will describe the remaining variables as functions of optimists' wealth share, so this convention will simplify the notation. The following result describes the equilibrium starting in the recession state.

**Proposition 1.** *Suppose parameters satisfy Assumption A1 in the appendix. The recovery state $s = 3$ features positive interest rates and efficient asset prices, $r^f_{t,3} > 0, Q_{t,3} = Q^*$, whereas the recession state $s = 2$ features zero interest rates and inefficiently low asset prices, $r^f_{t,2} = 0, Q_{t,2} < Q^*$. The asset price in the recession is a function of optimists' wealth share, $Q_{t,2} = Q_2(\alpha_{t,2})$, that satisfies*

$$Q_2(\alpha) < Q^* \text{ and } \frac{dQ_2(\alpha)}{d\alpha} > 0 \text{ for each } \alpha \in (0,1). \tag{27}$$

*Since $\eta_{t,s} = \frac{Q_{t,s}}{Q^*}\eta^*$ [cf. Lemma 1], greater $\alpha$ in the recession improves improves factor utilization and output as well as asset prices.*

The results for the recovery state are straightforward since there is no further transition and no speculation. To show the results for the recession state, note that Lemma 3 applies since there is no leverage limit in state $s = 2$. Then, combining the risk balance condition (26) with Eq. (15), we obtain

$$\rho + g_2 - \delta\left(\frac{Q_{t,2}}{Q^*}\eta^*\right) + \frac{\dot{Q}_{t,2}}{Q_{t,2}} + \overline{\lambda}_{t,2}\left(1 - \frac{Q_{t,2}}{Q^*}\right) = r^f_{t,2}. \tag{28}$$

This equation together with the goods market equilibrium condition (18) determines the path of interest rates and asset prices, $\left\{r^f_{t,2}, Q_{t,2}\right\}$. To see how, first consider the case in which the interest rate constraint does not bind. In this case, we have $Q_{t,2} = Q^*$ and Eq. (28) yields the output-stabilizing interest rate: $r^{f,*}_{t,2} = \rho + g_2 - \delta(\eta^*)$.

Under Assumption A1, $g_2$ is sufficiently low that $r^{f,*}_{t,2} < 0$. In this case, we instead have $r^f_{t,2} = 0$ and Eq. (28) yields an equation for the asset price, $Q_{t,2}$. In the appendix, we solve this equation (together with investors' wealth dynamics) and establish Eq. (27).

Intuitively, a reduction in $g_2$ exerts downward pressure on asset prices due to low expected output growth. Monetary policy responds by lowering the risk-free interest rate, $r^f_{t,2}$, and keeps asset prices at the efficient level, $Q_{t,2} = Q^*$. However, if monetary policy is constrained, then condition (28) requires $Q_{t,2}$ to fall below $Q^*$. This asset price decline increases the expected return of the market portfolio. The equilibrium obtains when "the representative investor" (who has the wealth-weighted average belief) is indifferent between holding the market portfolio and the risk-free asset. However, the decline in $Q_{t,2}$ also lowers aggregate spending and triggers a demand recession.

Importantly, the asset price is higher when optimists have a greater wealth share in the recession, $\alpha_{t,2}$. In this case, since "the representative investor" is more optimistic (greater $\overline{\lambda}_{t,2}$), a smaller decline in $Q_{t,2}$ is sufficient to make her indifferent. This leads to higher asset prices, which in turn improves spending and mitigates the recession. Figure 3 illustrates the solution for a particular parameterization (described in Appendix A.6).

As we formalize in Section 5, the positive relationship between optimists' wealth and output in the recession is an *aggregate demand externality.* In particular, individual optimists that take on leverage during the boom (and pessimists that lend to them) do not internalize the effects of

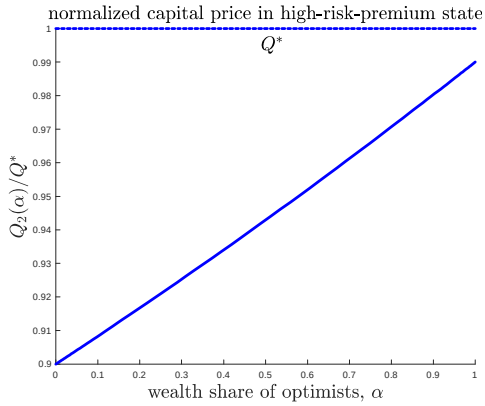


Figure 3: Equilibrium price of capital in the recession state $s = 2$. The dotted line illustrates the frictionless price level, $Q^*$.

their financial decisions on output in the recession. This motivates policy interventions in the boom designed to increase optimists' wealth share in the recession, which we turn to next.

# 4. The boom and prudential policies

In this section, we establish our main result that shows monetary policy can replicate the prudential benefits of tightening an existing leverage limit. To this end, we characterize the equilibrium in the boom state, $s = 1$, focusing on optimists' wealth share after transition to recession, $\alpha_{t,2}$ (which will be the key input to our welfare analysis in Section 5). We start by analyzing a benchmark without PMP and show that macroprudential policy that tightens the leverage limit increases $\alpha_{t,2}$. We then introduce our main ingredient and allow monetary policy to be used for prudential purposes. In that context, we first establish a negative result: when there is no leverage limit, PMP is useless because optimists endogenously change their risk taking to undo the prudential benefits. We then consider the case with a leverage limit and establish our main result.

## 4.1. Benchmark without prudential monetary policy

First suppose there is no PMP: that is, monetary policy follows the conventional output-stabilization policy in (13) in state $s = 1$. The next result characterizes the equilibrium and illustrates the effect of macroprudential policy. To state the result, suppose the leverage limit in the boom state can be written as a function of optimists' wealth share, $\overline{\omega}_{t,1} = \overline{\omega}_1(\alpha_{t,1})$. This ensures $\alpha_{t,1}$ is the only state variable. We denote the equilibrium variables as functions of optimists' wealth share and the leverage limit *function*: $\alpha_{t,2} = \alpha_2(\alpha, \overline{\omega}_1(\cdot))$ denotes optimists' wealth share *after* transition when their current wealth share is $\alpha_{t,1} = \alpha$ and the leverage limit is described by $\overline{\omega}_{t,1} = \overline{\omega}_1(\alpha_{t,1})$ for each $t$. We use the notation $\alpha_2(\alpha, \infty)$ to denote the equilibrium when there is no leverage limit: $\overline{\omega}_1(\alpha) = \infty$ for each $\alpha$.

**Proposition 2.** *Suppose A1–A3 hold and there is no PMP: the planner follows the traditional monetary policy in* (13)*. Then, the boom state features positive interest rates,* $r^f_{t,1} = r^f_1(\alpha, \overline{\omega}_1) > 0$*, and efficient asset prices,* $Q_{t,1} = Q^*$*.*

*Suppose first there is no leverage limit,* $\overline{\omega}_1 = \infty$*. Optimists have an above-average leverage ratio,* $\omega^o_1(\alpha, \infty) > 1$*. If there is transition to recession, optimists' wealth share shrinks:*

$$\alpha_2(\alpha, \infty) = \alpha \frac{\lambda^o_1}{\overline{\lambda}_1(\alpha)} < \alpha. \tag{29}$$

*If there is no transition, optimists' wealth share grows:* $d(\dot{\alpha}_{t,1}/\alpha_{t,1}) = (1-\alpha)(\lambda^p_1 - \lambda^o_1) > 0$*.*

*Now suppose there is a leverage limit. Consider* $\alpha \in (0,1)$ *such that the leverage limit binds,* $\overline{\omega}_1(\alpha) \leq \omega^o_1(\alpha, \infty)$*. Decreasing the leverage limit (via macroprudential policy) increases optimists' wealth share after transition to recession:* $\frac{d\alpha_2(\alpha,\overline{\omega}_1)}{d(-\overline{\omega}_1(\alpha))} > 0$*. It also slows down the growth rate of optimists' wealth share if there is no transition,* $\frac{d(\dot{\alpha}_{t,1}/\alpha_{t,1})}{d(-\overline{\omega}_1(\alpha))} < 0$*.*

The first part of Proposition 2 shows that, under appropriate parametric conditions (Assumption A2), the equilibrium in the boom state without PMP features an efficient price level.

The second part considers the case in which there is no leverage limit. Optimists take on above-average leverage. Consequently, their wealth share declines if there is a transition to recession—as this is associated with a price decline—but grows if there is no transition. These wealth-share dynamics follow from Lemma 3.

The last part of the proposition shows that macroprudential policy that tightens optimists' leverage limit improves optimists' wealth-share after transition, $\alpha_{t,2}$. Hence, macroprudential policy achieves prudential benefits.[7] To see how, consider the characterization of equilibrium when the leverage limit binds. Eq. (20) implies

$$\begin{aligned} \frac{\alpha_2(\alpha, \overline{\omega}_1)}{\alpha} &= 1 - (\overline{\omega}_1(\alpha) - 1)\left[\frac{Q_1}{Q_2} - 1\right], \\ \text{where } Q_1 &= Q^* \quad \text{and} \quad Q_2 = Q_2(\alpha_2(\alpha, \overline{\omega}_1)). \end{aligned} \tag{30}$$

The first line is the microfounded version of Eq. (2) from the introduction. The second line features the microfounded version of Eq. (1). As illustrated by Figure 1 in the introduction, the equilibrium can be visualized as the intersection of two increasing relations. Under appropriate regularity conditions (Assumption A3), there is a unique solution that satisfies $\alpha_2(\alpha, \overline{\omega}_1) \in [\alpha_2(\alpha, \infty), \alpha]$. Moreover, tightening the leverage limit $\overline{\omega}_1(\alpha)$ shifts Eq. (30) upward: that is, it increases optimists' wealth share, $\alpha_2(\alpha, \overline{\omega}_1)$, given a price level in the recession $Q_2 < Q^*$. This creates a virtuous cycle that results in a higher optimists' wealth share and higher asset prices, $\alpha_2$ and $Q_2$ (see the left panel of Figure 1).

[7] On the other hand, the result also illustrates potential *dynamic* costs of macroprudential policy. A tighter limit slows down the growth of optimists' wealth share if the recession is not realized. We return to this point in Section 6.

### 4.2. Prudential monetary policy

We now assume that macroprudential policy is inflexible: the planner cannot change the existing leverage limits. We ask whether PMP can achieve similar prudential benefits. Formally, suppose the planner does not follow the rule in (13) in state $s = 1$ but instead sets the interest rate to target an asset price level that might be lower than the efficient level, $Q_{t,1} \leq Q^*$. We assume the planner's price target can be written as a function of optimists' wealth share:

$$Q_{t,1} = Q_1(\alpha_{t,1}) \leq Q^*.$$

We denote the equilibrium variables as functions of the PMP *function* (in addition to the earlier variables): $\alpha_2(\alpha, \overline{\omega}_1(\cdot), Q_1(\cdot))$ denotes optimists' wealth share after transition, when monetary policy is described by $Q_{t,1} = Q_1(\alpha_{t,1})$ for each $t$.[8]

#### 4.2.1. PMP without a leverage limit

First consider the case without a leverage limit, $\overline{\omega}_1 = \infty$. In this case, we establish a *negative* result: PMP does not achieve its prudential objectives.

**Proposition 3.** *Suppose Assumptions A1–A3 hold. Suppose there is some PMP, $Q_1(\cdot)$, but no leverage limit, $\overline{\omega}_1 = \infty$. Optimists' wealth share dynamics are the same same as in the benchmark without prudential policy. In particular, PMP does not affect optimists' wealth share after transition to recession, $\alpha_{t,2}$.*

This result says that, absent a leverage limit PMP, does not provide prudential benefits. To see why, note that Lemma 3 applies in this case. As long as the leverage limit does not bind, investors' portfolio optimality condition (21) implies $\lambda_1^o \frac{\alpha_{t,1}}{\alpha_{t,2}} = \lambda_1^p \frac{1-\alpha_{t,1}}{1-\alpha_{t,2}}$ *regardless of asset prices or returns.* This implies $\alpha_{t,2} = \alpha_2(\alpha, \infty) = \alpha \frac{\lambda_1^o}{\lambda_1(\alpha)}$. Intuitively, as the planner changes the asset price $Q_1$ (and asset returns), investors retrade the asset to equate (as before) their perceived probabilities multiplied by marginal utilities.

It is also instructive to consider *how* investors trade to undo the effects of PMP. Note that optimists' leverage ratio is determined by the following analogue of Eq. (30):

$$\frac{\alpha_2(\alpha, \infty)}{\alpha} = 1 - (\omega_1^o(\alpha, \infty, Q_1) - 1)\left[\frac{Q_1(\alpha)}{Q_2(\alpha_2(\alpha, \infty))} - 1\right].$$

A decline in $Q_1(\alpha)$ results in a smaller price drop after transition (the term inside the brackets). Therefore, optimists' wealth share after transition ($\alpha_2$) remains unchanged because they *increase* their leverage ratio, $\omega_1^o(\alpha, \infty, Q_1) > \omega_1^o(\alpha, \infty)$. Optimists increase their leverage because they perceive the transition to recession as less risky but they desire the same portfolio risk as before.

[8] We use the same notation as in the previous section to denote the equilibrium in the benchmark in which the planner follows the conventional output-stabilization policy: e.g., $\alpha_2(\alpha, \overline{\omega}_1)$ denotes the equilibrium when monetary policy is described by $Q_{t,1} = Q^*$ for each $t$.

### 4.2.2. PMP with a leverage limit

The previous discussion suggests that PMP *can* affect investors' equilibrium exposures if optimists are constrained by *some* leverage limit. We next present our main result, which formalizes this intuition and shows that monetary policy can replicate the prudential effects of tightening the leverage limit.

**Proposition 4.** *Suppose Assumptions A1-A3 hold. Consider the benchmark equilibrium without PMP, $Q_1(\cdot) = Q^*$. Fix a level $\alpha \in (0,1)$ that is associated with some leverage limit, $\overline{\omega}_1(\alpha) < \infty$ (that might or might not bind). Consider an alternative leverage limit $\tilde{\omega}_1(\cdot)$ that agrees with $\overline{\omega}_1(\cdot)$ everywhere except for $\alpha$ and that satisfies $\tilde{\omega}_1(\alpha) < \min\left(\overline{\omega}_1(\alpha), \omega_1^o(\alpha, \infty)\right)$, and a PMP $\tilde{Q}_1(\cdot)$ that agrees with $Q_1(\cdot)$ everywhere except for $\alpha$. Then:*

*(i) There exists $\tilde{Q}_1(\alpha) < Q^*$ such that the PMP (with the original leverage limit) generates the same effect on optimists' wealth share after transition as the alternative leverage limit (without PMP):*

$$\alpha_2\left(\alpha, \overline{\omega}_1, \tilde{Q}_1\right) = \alpha_2\left(\alpha, \tilde{\omega}_1\right).$$

*Targeting a lower effective limit requires targeting a lower asset price, $\frac{\partial \tilde{Q}_1(\alpha)}{\partial \tilde{\omega}_1(\alpha)} > 0$.*

*(ii) PMP requires setting a higher interest rate than the benchmark without policy:*

$$r_1^f\left(\alpha, \overline{\omega}_1, \tilde{Q}_1\right) > r_1^f\left(\alpha, \overline{\omega}_1\right).$$

*Targeting a lower effective limit requires setting a higher interest rate, $\frac{\partial r_1^f\left(\alpha, \overline{\omega}_1, \tilde{Q}_1\right)}{\partial \tilde{\omega}_1(\alpha)} < 0$.*

The first part of Proposition 4 shows that monetary policy can replicate the prudential effects of tightening the leverage limit that we established in Proposition 2. To see the intuition, consider a situation in which there is a limit that binds for optimists so that $\omega_1^o(\alpha, \overline{\omega}_1, Q_1) = \overline{\omega}_1(\alpha)$. Then,

$$\frac{\alpha_2(\alpha, \overline{\omega}_1, Q_1)}{\alpha} = 1 - (\overline{\omega}_1(\alpha) - 1)\left[\frac{Q_1(\alpha)}{Q_2(\alpha_2(\alpha, \overline{\omega}_1, Q_1))} - 1\right]. \tag{31}$$

Recall that tightening the leverage limit mitigates optimists' wealth decline by reducing their leverage ratio, $\tilde{\omega}_1 < \omega_1$. For a given asset price $Q_2$, monetary policy can achieve the same wealth decline for optimists at the leverage limit, $\omega_1 = \overline{\omega}_1$, by reducing the asset price decline, $\frac{\tilde{Q}_1}{Q_2} < \frac{Q^*}{Q_2}$. Unlike in Proposition 3, optimists cannot undo the effects because the policy pushes their leverage ratio against the limit, $\omega_1 = \overline{\omega}_1$. Note also that, once the policy increases the price level in the recession, $Q_2$, it generates a similar virtuous cycle as a policy that directly tightens the leverage limit (see the right panel of Figure 1).

The second part of Proposition 3 shows that PMP requires raising the interest rate above the conventional policy benchmark with output stabilization. As expected, targeting a lower asset price requires a higher interest rate.

Proposition 4 is essentially static: it considers a policy change at a particular instant while leaving the policy at other times unchanged. This is useful for illustrating how PMP works, but it does not impact the *dynamic* equilibrium. We next establish a dynamic version of this result, which we use for numerical illustration as well as for welfare analysis. To state the result, we parameterize the leverage limit function, $\overline{\omega}(\alpha, l)$ where $l \in L \subset \mathbb{R}_+$, and lower levels of $l$ correspond to a tighter leverage limit, $\frac{\partial \overline{\omega}(\alpha,l)}{\partial l} > 0$ for $\alpha \in (0,1)$. An example is the simple leverage limit function

$$\overline{\omega}_1(\alpha, l) = l \text{ with } l \in L = (1, \infty). \tag{32}$$

Here, the leverage limit doesn't depend on $\alpha$ and a lower $l$ corresponds to a tighter limit for all $\alpha$. Whenever we parameterize the leverage limit function, we simplify the notation by denoting the corresponding equilibrium variables with $\alpha_2(\alpha, l, Q_1)$ (as opposed to $\alpha_2(\alpha, \overline{\omega}_1(\cdot, l), Q_1)$).

**Proposition 5.** *Consider the setup of Proposition 4. Suppose there is a leverage limit function, $\overline{\omega}_1(\alpha, l)$, parameterized so that lower levels of $l$ correspond to a tighter limit. For each $\tilde{l} < l$ sufficiently close to $l$, there exists a PMP, denoted by $Q_1\left(\cdot, \tilde{l}\right)$, such that optimists' wealth share after transition is the same as when the leverage limit is given by $\overline{\omega}_1\left(\alpha, \tilde{l}\right)$ without PMP:*

$$\alpha_2\left(\alpha, l, Q_1\left(\cdot, \tilde{l}\right)\right) = \alpha_2\left(\alpha, \tilde{l}\right) \text{ for each } \alpha \in (0,1).$$

**Numerical illustration.** We next illustrate PMP with a numerical example. Optimists' and pessimists' beliefs about the probability of a transition to recession are given by $\lambda_1^o = 0.09 < \lambda_1^p = 0.9$. The remaining parameters are as described in Appendix A.6. Suppose optimists are subject to the simple leverage limit function in (32). We set the current limit so that it barely binds when optimists have half of the wealth share: $l = \omega_1^o(0.5, \infty) = 9.03$. The planner would like to tighten this limit by a quarter, $\tilde{l} = 0.75l = 6.77$, but she cannot control the leverage limit directly. Instead, the planner implements the replicating prudential policy, $Q_1\left(\alpha, \tilde{l}\right)$.

Figure 4 plots the equilibrium functions for three different policy specifications over the range $\alpha \in [0.4, 0.9]$. The red dashed lines correspond to the case with the current leverage limit $l$ but no prudential policy of any kind. The black dash-dotted lines correspond to tightening the leverage limit directly, $\tilde{l} = 0.75l$. Finally, the blue solid lines correspond to implementing this tightening via PMP, $Q_1\left(\alpha, \tilde{l}\right)$.

The top left panel illustrates optimists' leverage ratio as a function of their wealth share for each specification. Optimists have an above-average leverage ratio. The current leverage limit restricts optimists' leverage ratio only slightly (not visible in the figure). The proposed tightening would restrict their leverage ratio considerably more. PMP *raises* optimists' leverage ratio (over the range $\alpha > 0.5$) as it pushes them against the leverage limit.

The top middle panel illustrates optimists' wealth share after transition normalized by their current wealth share, $\alpha_2(\alpha)/\alpha$. Optimists' wealth share declines after transition, $\alpha_2(\alpha)/\alpha < 1$. PMP replicates the effect of tightening the leverage limit and therefore increases optimists'

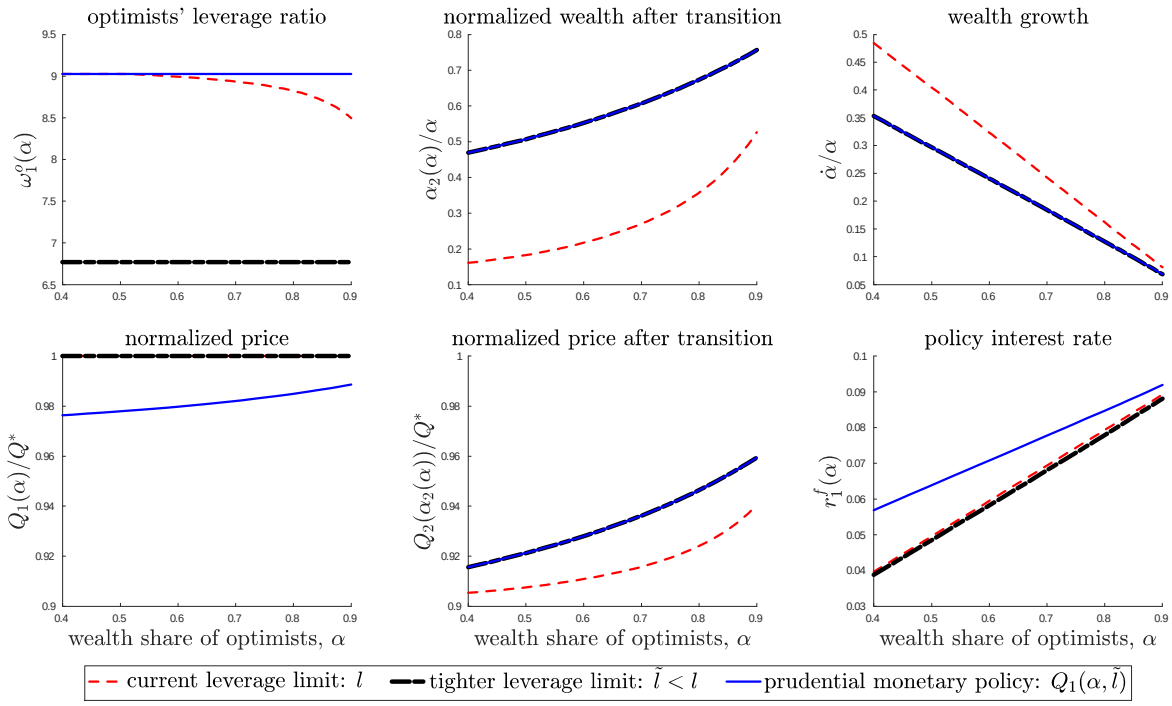


Figure 4: Equilibrium functions in the boom state $s = 1$ for different specifications of the leverage limit and PMP.

wealth share after transition. The top right panel illustrates that this effective tightening slows down the growth of optimists' wealth share if there is no transition.

The bottom left panel illustrates the equilibrium asset price in the boom state normalized by the efficient level. The leverage limit (its current level or hypothetical tightening) leaves the asset price equal to its efficient level. In contrast, PMP reduces the asset price by around 2%. This relatively small decline is able to replicate the effects of a large reduction in optimists' leverage ratio because optimists' initial leverage ratio is high. With high and constrained leverage, small changes in asset prices have large effects on optimists' balance sheets [cf. (31)].

The bottom middle panel illustrates the price after a transition to recession normalized by the efficient level. PMP increases the asset price during the recession. We can gain intuition for this result by comparing this panel with the bottom left panel. By lowering the asset price during the boom, PMP reduces the asset price *decline* after a transition to recession. This smaller decline supports optimists' balance sheets and thus improves the asset price *level* during the recession by around 2%.

The bottom right panel illustrates the equilibrium interest rate. The leverage limit reduces the policy interest rate because it reduces optimists' effective asset demand. In contrast, PMP increases the policy interest rate (by less than 2 percentage points). This reduces the asset price, as illustrated by the bottom left panel, which results in a smaller asset price decline when there is a transition to recession.

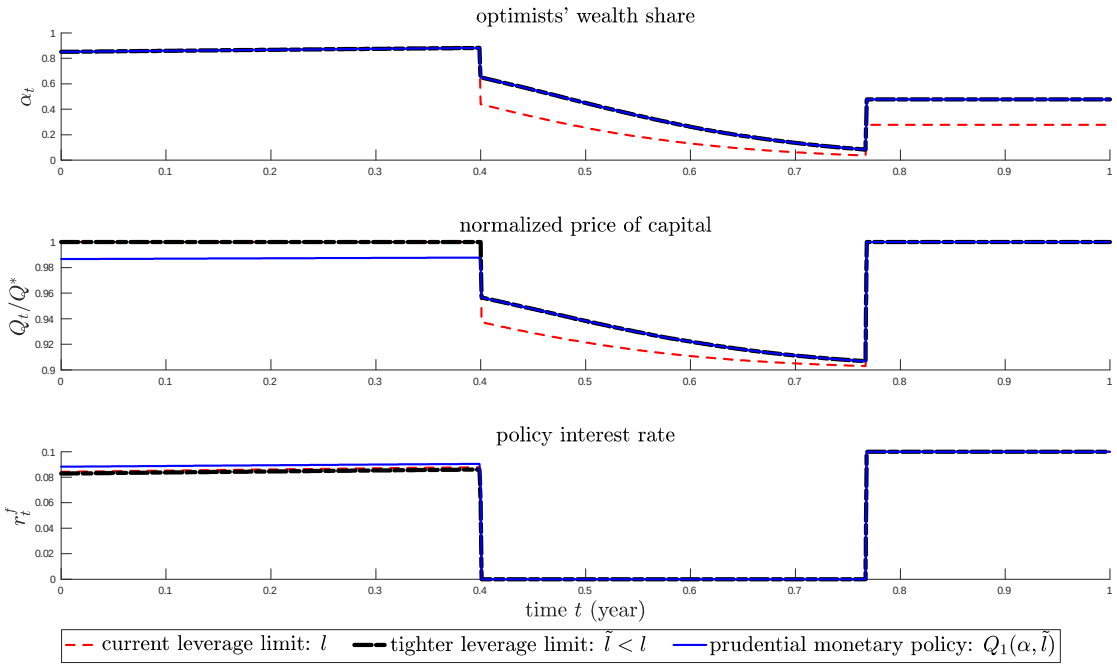


Figure 5: Simulation of the equilibrium path starting with $a_0 = 0.85$ and $s = 1$ for different specifications of the leverage limit and PMP.

Figure 5 simulates the equilibrium variables over time (for each policy specification) for a particular initial wealth share for optimists, $\alpha_0$, and a particular realization of uncertainty. We take $\alpha_0 = 0.85$ and consider a path in which the economy transitions into the recession at $t = 0.4$ (other choices lead to qualitatively similar effects). The plots illustrate that PMP raises the asset price in the recession at the cost of reducing the asset price in the boom.

## 5. Welfare equivalence of prudential policies

We have so far established that PMP can achieve similar prudential benefits as macroprudential policy. However, compared to macroprudential policy, PMP is associated with some costs because it reduces asset prices and output during the boom [cf. Figures 4 and 5]. In this section, we develop a formal welfare criterion and evaluate both the costs and benefits of PMP. We show that, for small changes, PMP achieves *the same* welfare effects as macroprudential policy.

We evaluate welfare with the *gap value function* that we first introduced in Caballero and Simsek (forthcoming). This function captures the welfare loss due to demand recessions. It is an exact version of the approximate quadratic loss function typically used in the New Keynesian literature. We first introduce our welfare criterion and discuss its properties.

**A gap value criterion.** We assume the planner has her own beliefs (which can be different from optimists' and pessimists' beliefs) and let the superscript $pl$ denote the planner's beliefs. For

$i \in \{o, p\}$, we use $V_{t,s}^{i,pl}\left(a_{t,s}^{i}\right)$ to denote type $i$ investors' equilibrium value calculated according to the planner's beliefs. In view of log utility, the value function takes the form

$$V_{t,s}^{i,pl}\left(a_{t,s}^{i}\right) = \frac{\log\left(a_{t,s}^{i}/Q_{t,s}\right)}{\rho} + v_{t,s}^{i,pl}.$$

The normalized value function $v_{t,s}^{i,pl}$ captures the value when the investor holds one unit of the capital stock (or wealth, $a_{t,s}^{i} = Q_{t,s}$). We further decompose this term as

$$v_{t,s}^{i,pl} = v_{t,s}^{i*,pl} + w_{t,s}^{pl}. \tag{33}$$

The *frictionless value function* $v_{t,s}^{i*,pl}$ obtains in a counterfactual economy where the evolution of wealth shares are left unchanged but asset prices are equal to the frictionless level, $Q_{t,s} = Q^*$ for each $t, s$. This captures all determinants of welfare (including the benefits/costs from speculation) except for suboptimal factor utilization. The residual term, $w_{t,s}^{pl}$, corresponds to the *gap value function* that captures the losses from suboptimal utilization.

Specifically, in the appendix we show that the gap value function solves the following differential equation:

$$\begin{aligned} \rho w_{t,s}^{pl} - \frac{\partial w_{t,s}^{pl}}{\partial t} &= W\left(Q_{t,s}\right) + \lambda_{s}^{pl}\left(w_{t,s'}^{pl} - w_{t,s}^{pl}\right), \\ \text{where } W\left(Q_{t,s}\right) &\equiv \log\frac{Q_{t,s}}{Q^*} - \frac{1}{\rho}\left(\delta\left(\frac{Q_{t,s}}{Q^*}\eta^*\right) - \delta\left(\eta^*\right)\right). \end{aligned} \tag{34}$$

Here, $W\left(Q_{t,s}\right)$ is strictly concave with a maximum, $Q_{t,s} = Q^*$, and maximum value, $W\left(Q^*\right) = 0$ (cf. Eq. (16)). This function captures the instantaneous welfare losses when the asset price (and factor utilization) deviates from its efficient level, $Q_{t,s} \neq Q^*$. Therefore, the gap value $w_{t,s}^{pl}$ corresponds to the present discounted value of expected welfare losses due to demand recessions.

Our welfare criterion posits that the planner maximizes the gap value evaluated with her own belief, $w_{t,s}^{pl}$. In contrast, the standard Pareto criterion would focus on the total value function evaluated with each investor's own belief (i.e., $v_{t,s}^{i,i}$). In our context, the gap value criterion has several advantages. First, it sidesteps questions about whether speculation increases or reduces welfare (see Brunnermeier et al. (2014) for further discussion). Second, it aligns with the mandates of monetary policy in practice: the planner exclusively focuses on minimizing output gaps relative to a frictionless benchmark (similar to Kocherlakota (2014) and Stein (2014)). Third, the gap value does not depend on $i \in \{o, p\}$: it is *the same for optimists and pessimists* [cf. Eq. (34)]. Therefore, the gap value does not require Pareto weights. Intuitively, an improvement in aggregate demand benefits all investors by the same (proportional) amount.[9]

Following Brunnermeier et al. (2014), we also assume the planner's beliefs are in the convex

[9] In our setup, investors have *heterogeneous wealth share dynamics* (driven by their *heterogeneous leverage*). The resulting heterogeneous welfare effects are captured by the first-best value function, $v_{t,s}^{i*,pl}$. Given their wealth share dynamics, all investors are affected equally by a change in the path of asset prices and output.

hull of optimists' and pessimists' beliefs: $\lambda_1^{pl} \in [\lambda_1^o, \lambda_1^p]$ and $\lambda_2^{pl} \in [\lambda_2^p, \lambda_2^o]$. Our results are qualitatively robust to the choice of the planner's beliefs in these sets. As before, the planner's gap value can be written as a function of optimists' wealth share, $w_{t,s}^{pl} = w_s^{pl}(\alpha_{t,s})$.

**Aggregate demand externalities.** We next characterize the gap value function in the recession and formalize the aggregate demand externalities that motivate policy intervention.

**Lemma 4.** *Consider the setup of Proposition 5 (without PMP). In the recession state, the gap value function satisfies*

$$w_2^{pl}(\alpha) < 0 \text{ and } \frac{dw_2^{pl}(\alpha)}{d\alpha} > 0 \text{ for each } \alpha \in (0,1). \tag{35}$$

As expected, the gap value in the recession is strictly negative. Moreover, a greater wealth-share for optimists increases the gap value. Optimists' wealth share improves welfare because it increases asset prices and aggregate demand, which brings the economy closer to efficient output. Importantly, this effect is an *aggregate demand externality*. Since optimists' wealth share is an *aggregate* state variable, individual investors do not take into account the effect of their decisions on the gap value (regardless of the planner's belief). This neglect motivates the prudential policy interventions in the boom that we analyzed earlier.

**Equivalence of prudential policies.** We next consider the gap value function in the boom. We establish our main result in this section: for small changes, PMP can replicate the welfare benefits of macroprudential policy.

**Proposition 6.** *Consider the setup of Proposition 5. For small policy changes, the welfare effects of PMP are the same as the welfare effects of tightening the leverage limit directly:*

$$\frac{dw_1^{pl}\left(\alpha, l, Q_1\left(\cdot, \tilde{l}\right)\right)}{d\tilde{l}}|_{\tilde{l}=l} = \frac{dw_1^{pl}\left(\alpha, \tilde{l}\right)}{d\tilde{l}}|_{\tilde{l}=l}. \tag{36}$$

To show this result, note that the policies $\tilde{l}$ and $Q_1\left(\cdot, \tilde{l}\right)$ lead to identical equilibrium allocations except for the asset price in the boom state. Using this observation and the definition of the gap value in (34), the welfare difference between the two policies can be written as

$$w_1^{pl}\left(\alpha, l, Q_1\left(\cdot, \tilde{l}\right)\right) - w_1^{pl}\left(\alpha, \tilde{l}\right) = \int_0^\infty e^{-\left(\rho+\lambda_1^{pl}\right)t}\left(W\left(Q_1\left(\alpha_{t,1}, \tilde{l}\right)\right) - W\left(Q^*\right)\right)dt. \tag{37}$$

Here, $\alpha_{t,1}$ denotes optimists' wealth share when the economy starts with $\alpha_{0,1} = \alpha$, follows policy $\tilde{l}$, and reaches time $t$ without transitioning into recession. Since $W(Q_{t,1}) < W(Q^*)$ for $Q_{t,1} < Q^*$, this expression implies that PMP always yields lower welfare than the equivalent tightening of the leverage limit. However, since $W(Q_{t,1})$ is maximized at $Q_{t,1} = Q^*$, these welfare differences are second order when the prudential policy is used in small doses (so that

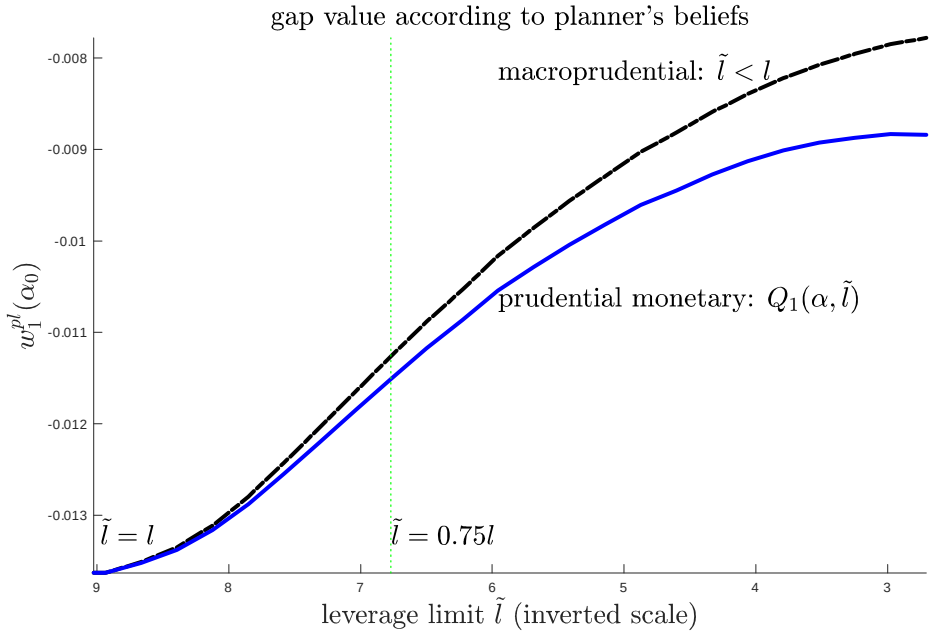


Figure 6: The planner's gap value as a function of the effective leverage ratio starting with $a_0 = 0.85$ and $s = 1$ for a direct tightening (dashed line) and an equivalent tightening via PMP (solid line). The vertical dotted line illustrates the leverage tightening studied earlier.

$Q_{t,1}$ remains close to $Q^*$). Therefore, as formalized by Eq. (36), the two policies have identical first-order effects on welfare.

Figure 6 illustrates this result for the numerical example in Section 4.2.2. We take the planner's beliefs to be the average of optimists' and pessimists' beliefs, $\lambda_s^{pl} = (\lambda_s^o + \lambda_s^p)/2$. The black dash-dotted line in Figure 6 illustrates that, if feasible, a direct tightening of the leverage limit would improve the gap value. The solid blue line illustrates that an indirect tightening via PMP also increases the gap value. In fact, for small policy changes, PMP has the same welfare impact as a direct tightening.

Intuitively, PMP and macroprudential policy both increase optimists' wealth share if there is a transition to recession. This mitigates output gaps and improves welfare. Relative to macroprudential policy, PMP slows down the economy in the boom. Starting from an efficient output, these costs are second order, whereas the benefits of improving output in the recession are first order. Therefore, PMP achieves similar welfare benefits as macroprudential policy.

## 6. Optimal prudential monetary policy

So far, we have established that monetary policy can have prudential benefits by effectively tightening an existing leverage limit. In this section, we analyze the determinants of optimal PMP in our setting. We first characterize the optimal prudential policy as the solution to a

recursive optimization problem. We then solve the problem numerically and investigate the comparative statics of optimal policy.

For each $\alpha$, suppose the planner sets an arbitrary price level $Q_1 \leq Q^*$ subject to the restriction that the price level weakly declines after the transition. Given $Q_1$, optimists' wealth share after transition is determined by the function $\alpha_2(\alpha, \overline{\omega}_1, Q_1) \in [0,1]$. This is a continuous and piecewise differentiable function that is equal to $\alpha_2(\alpha, \infty)$ if optimists' leverage limit does not bind (that is, if $\omega_1^o(\alpha, \infty, Q_1) < \overline{\omega}_1(\alpha)$) and is equal to the solution to (31) if the limit binds. Using this notation, we can recursively formulate the planner's optimization problem in the boom state $s = 1$ as

$$\begin{aligned}
\left(\rho + \lambda_1^{pl}\right) w_1^{pl}(\alpha) &= \max_{Q_1} W(Q_1) - W(Q^*) + \frac{dw_1^{pl}(\alpha)}{d\alpha}\dot{\alpha} + \lambda_1^{pl} w_2^{pl}(\alpha_2) \qquad (38)\\
\text{where } \dot{\alpha} &= \frac{\alpha(1-\alpha)\lambda_1^p}{1-\alpha_2}\left(1 - \frac{\alpha_2}{\alpha}\right)\\
\alpha_2 &= \alpha_2(\alpha, \overline{\omega}_1, Q_1)\\
\text{and } Q_1 &\in [Q_2(\alpha_2(\alpha, \overline{\omega}_1, Q_1)), Q^*].
\end{aligned}$$

The second line uses Eq. (23) to describe the evolution of optimists' wealth share absent a transition, $\dot{\alpha} = \frac{d\alpha_{t,1}}{dt}$, as a function of their induced wealth share after transition, $\alpha_2 = \alpha_{t,2}$ (as well as their current wealth share, $\alpha = \alpha_{t,1}$).[10]

The analytical solution to problem (38) is complicated. However, a numerical solution is straightforward. Moreover, we can glean some intuition by considering the local optimality conditions. Specifically, for an interior solution $Q_1 \in (Q_2, Q^*)$, the optimality condition (for decreasing $Q_1$ further) can be written as

$$\begin{aligned}
\frac{dW(Q_1)}{dQ_1} &= \frac{d\alpha_2}{d(-Q_1)}\left[\lambda_1^{pl}\frac{dw_2^{pl}(\alpha_2)}{d\alpha_2} + \frac{d\dot{\alpha}}{d\alpha_2}\frac{dw_1^{pl}(\alpha)}{d\alpha}\right] \qquad (39)\\
\text{where } \frac{d\dot{\alpha}}{d\alpha_2} &= -\lambda_1^p \frac{(1-\alpha)^2}{(1-\alpha_2)^2}.
\end{aligned}$$

The left-hand side of Eq. (39) captures the costs of the policy via its impact on the output gap in period 1. This term is positive since $Q_1 < Q^*$: decreasing the asset price in the boom below the efficient level worsens the output gap. The right-hand side captures the welfare effects of the policy via its impact on optimists' wealth share. We have $\frac{d\alpha_2}{d(-Q_1)} > 0$: lowering the asset price increases optimists' wealth share after transition. We also have $\frac{dw_2^{pl}(\alpha_2)}{d\alpha_2} > 0$: increasing optimists' wealth share after transition mitigates the output gaps in the recession. Hence, the first term inside the brackets is positive and captures the benefits of PMP.

On the other hand, we also have $\frac{d\dot{\alpha}}{d\alpha_2} < 0$: if there is no transition, the policy slows down

[10] In problem (38), we ignore the zero lower bound constraint on the interest rate. In numerical solutions (described subsequently), we check and verify that this constraint doesn't bind at the optimal solution.

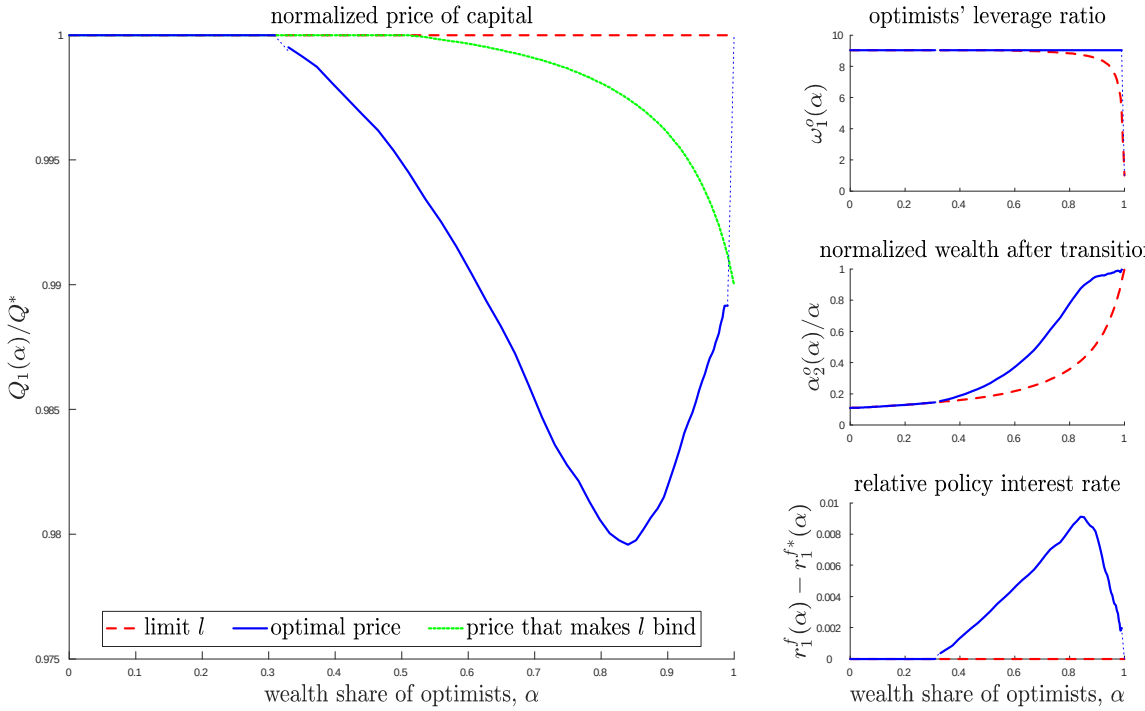


Figure 7: Equilibrium with optimal PMP (blue solid line) and without PMP (red dashed line) given the leverage limit $l$. The green dotted line in the left panel illustrates the minimum price decline necessary to make optimists' leverage limit bind.

the accumulation of optimists' wealth share. This illustrates potential *dynamic* costs of PMP. In a dynamic setting, optimists' wealth share can also be useful in future recessions and thus prudential policies always involve a trade-off—even when they do not affect the output gap in the boom.[11]

**Numerical illustration.** Figure 7 illustrates the optimal monetary policy corresponding to the numerical example in Section 4.2.2. As a benchmark, the red dashed lines illustrate the equilibrium without PMP but with the simple leverage limit $\overline{\omega}_1(\alpha, l) = l = 9.03$. Recall that this leverage limit is chosen so that (absent PMP) it binds for optimists when $\alpha < 0.5$ but not when $\alpha \geq 0.5$. The green dotted line in the left panel illustrates the minimum price decline necessary to make the leverage limit bind for optimists—price reductions smaller than this level have no prudential benefits since they are undone by endogenous risk adjustments by optimists.

The blue solid line in the left panel of Figure 7 illustrates the optimal price that solves problem (38). With this parameterization, the planner does not use monetary policy for prudential purposes when $\alpha < 0.33$. In this range, the leverage limit is already tight, and tightening it

[11] In Caballero and Simsek (forthcoming), we investigate this trade-off in the context of macroprudential policy. We show that the benefits from an immediate transition to recession are often larger than the costs of deepening future recessions (in view of discounting). In particular, under regularity conditions and starting from a no-policy benchmark, adopting some macroprudential policy improves welfare.

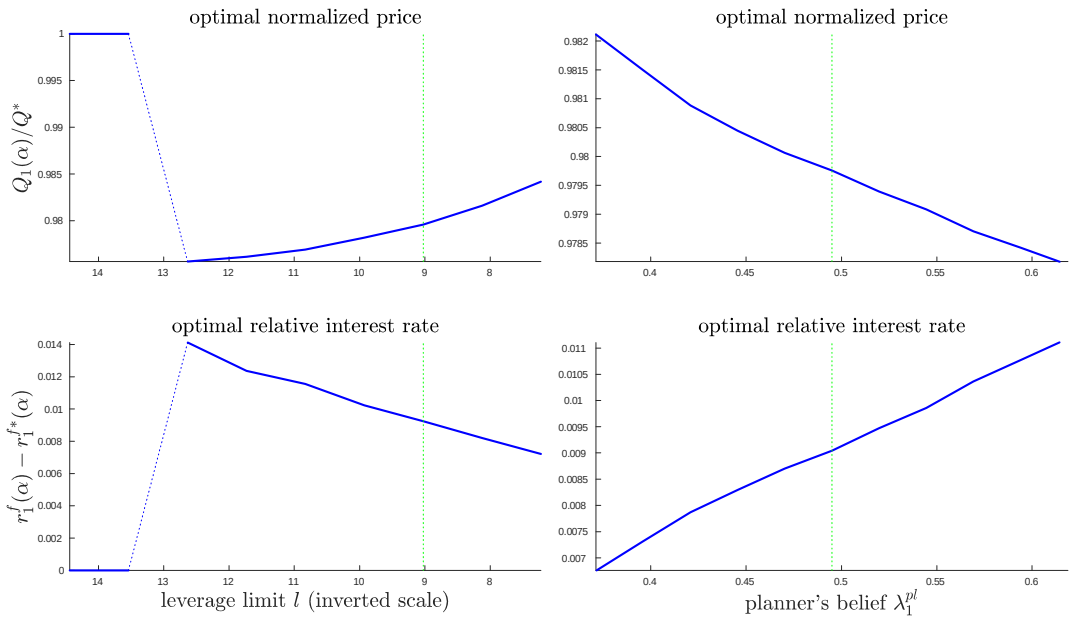


Figure 8: Comparative statics of the optimal PMP price level (top panels) and the interest rate (bottom panels) for $\alpha = 0.85$ with respect to changing the parameter on the x-axis. The vertical dotted lines illustrate the benchmark parameters (used in earlier figures).

further via PMP does not create large enough benefits to compensate for the costs imposed by slowing down the accumulation of optimists' wealth share [cf. Eq. (39)]. In contrast, the planner uses PMP over the range $\alpha \in [0.33, 0.99]$. Moreover, the degree of tightening relative to the conventional policy benchmark is non-monotonic in optimists' wealth share. In particular, the planner tightens the policy more as optimists' wealth share increases toward $\alpha = 0.85$ and tightens it less beyond this level. Hence the policy is most useful when optimists' wealth share lies in an intermediate range. Two forces make the policy relatively less attractive for large $\alpha$. First, since optimal private leverage drops as $\alpha$ rises, the policy becomes costlier as the planner needs to reduce the price even further to make optimists' leverage limit bind and gain traction (as illustrated by the green dotted line). Second, the policy is less useful because there is less speculation. In fact, for $\alpha \simeq 0.99$, these countervailing forces are strong enough that the planner stops using the policy altogether (as illustrated by the jump in the blue solid line).

Figure 8 illustrates the comparative statics of the optimal policy. To facilitate exposition, we describe the effects for a particular level of optimists' wealth share, $\alpha = 0.85$ (the same wealth share we considered in the previous section). The top panels display the change in the optimal price level as we vary a single parameter. The bottom panels display the change in the optimal interest rate relative to the conventional policy benchmark with output stabilization.

The left panels show the effect of changing the leverage limit, $l$. When the leverage limit is very loose, the planner does not use prudential policy because it is easily undone by optimists, illustrating Proposition 3. There is a threshold leverage limit below which the planner uses

monetary policy. Once the leverage limit is below this threshold, tightening it further makes the planner use PMP less. Hence, the leverage limit and PMP are *complements* in the high-$l$ range but they become *substitutes* in the low-$l$ range.

The right panels illustrate the effect of changing the planner's belief about the probability of transition into recession, $\lambda_1^{pl}$. As expected, when the planner believes the recession is more likely, she utilizes PMP more and reduces the asset price by a greater amount. Recall that in our model the recession is driven by a decline in asset valuations. Hence, the planner should calibrate PMP by monitoring the *tail risk in asset prices*—the chance of a sufficiently large decline in asset valuations to trigger a demand recession.

## 7. Prudential policies with "shadow banks"

In practice, a major concern with macroprudential policy is that there are lightly regulated institutions—typically referred to as shadow banks—that can circumvent regulatory constraints. Stein (2013) noted that in these environments PMP might have an advantage over macroprudential policy "because it gets in all of the cracks." We next evaluate the performance of macroprudential policy and PMP in our model when some of the high-valuation agents can avoid regulatory leverage limits. We find that whether PMP is more effective than macroprudential policy depends on the nature of the leverage limits faced by shadow banks.

First consider the case in which shadow banks face a binding leverage limit, e.g., due to financial frictions or self-imposed limits, even though they are not constrained by the regulatory limit (see Remark 2). In this case, shadow banks and regular banks are both constrained (although perhaps for different reasons), so our earlier analysis for PMP applies. In particular, PMP can replicate the prudential effects of tightening the leverage limit *for all banks*, whereas macroprudential policy is weaker because a direct regulatory tightening of the leverage limit applies only to regular banks. This broader impact illustrates that PMP can indeed be more effective than macroprudential policy.

Next consider the other extreme case in which shadow banks do not face a binding leverage limit (whereas regular banks face a binding limit). We next analyze this case and find that, while PMP remains useful, it is weakened by the same general equilibrium forces that weaken macroprudential policy.

Formally, suppose a subset of optimists are not subject to the leverage constraint, $\omega_{1,t} \leq \overline{\omega}_{1,t}$. We refer to these agents as *unconstrained optimists* ("shadow banks"), and refer to the remaining fraction of optimists as *constrained optimists* ("regular banks"). We let $\beta \in (0,1)$ denote the relative fraction of optimists' wealth that is held by unconstrained optimists. Hence, the wealth share of unconstrained and constrained optimists is given by, respectively, $\alpha\beta$ and $\alpha(1-\beta)$. As before, the total wealth share of optimists (including both types) and pessimists is given by, respectively, $\alpha$ and $1-\alpha$. The rest of the model is unchanged.

In Appendix A.5, we characterize the equilibrium and generalize our earlier results, Propo-

sitions 2 and 4, to this extended setting (see Propositions 7 and 8). Here, we illustrate our findings with a numerical example.

Specifically, consider the same example we analyzed in Section 4.2.2. In particular, the current leverage limit barely binds when optimists have half of the wealth share. The planner would like to tighten the existing limit by a quarter, $\tilde{l} = 0.75l$. The difference is that a fraction $\beta$ of optimists are unconstrained. Figure 9 plots the equilibrium functions for three different policy specifications over the range $\alpha \in [0.4, 0.9]$ and $\beta \in [0, 1]$. The lines corresponding to $\beta = 0$ match the earlier equilibria without unconstrained optimists (also plotted in Figure 4). The rest of the surfaces illustrate the effect of unconstrained optimists.

In this setting, macroprudential policy that tightens the leverage limit has smaller prudential benefits compared to the earlier analysis (cf. Proposition 2 in the appendix). To see this, compare the benchmark with the current limit (illustrated with red lines) with a direct tightening of the limit (illustrated with black lines). The top two left panels show constrained and unconstrained optimists' leverage ratios, respectively. In the benchmark, constrained and unconstrained optimists have similar leverage ratios (since the leverage limit barely binds). The proposed tightening of the leverage limit reduces constrained optimists' leverage ratio while *raising* unconstrained optimists' leverage ratio. The top right panel illustrates optimists' wealth share after a transition to recession. Macroprudential policy improves optimists' wealth share in the recession but less so than in the case without unconstrained optimists ($\beta = 0$).

Intuitively, tightening the leverage limit reduces financial stability risk, since it increases asset prices after transition to recession. Unconstrained optimists respond by taking greater risks. This reduces (but does not fully eliminate) the effectiveness of macroprudential policy. Consequently, macroprudential policy improves asset prices in the recession but less so than in the case without unconstrained optimists.

In this setting, while PMP can still replicate the prudential benefits of macroprudential policy (cf. Proposition 8 in the appendix), it is weakened by the reaction of unconstrained optimists. To see this, consider the equilibrium with PMP that replicates the prudential effects of a direct tightening of the leverage limit (illustrated with blue lines). The two panels on the right show the prudential benefits of PMP: it achieves the same wealth share for optimists after transition to recession, and the same asset prices in recession as a direct tightening of the leverage limit. The two panels in the bottom left show that, as before, PMP achieves these outcomes by increasing the interest rate and lowering asset prices during the boom. The two panels in the top left show that PMP increases the leverage ratio of constrained optimists (since it pushes them against the leverage limit) *and* the leverage ratio of unconstrained optimists.[12]

These results illustrate that, when some high-valuation agents are not subject to any (regulatory or non-regulatory) leverage limit, PMP is subject to similar limitations as macroprudential

[12] In fact, unconstrained optimists increase their leverage ratio even more than when the planner directly tightens the leverage limit. These agents obtain the same wealth share after transition as in direct tightening. However, they now achieve this outcome by taking on greater leverage since the price drop after transition is smaller.

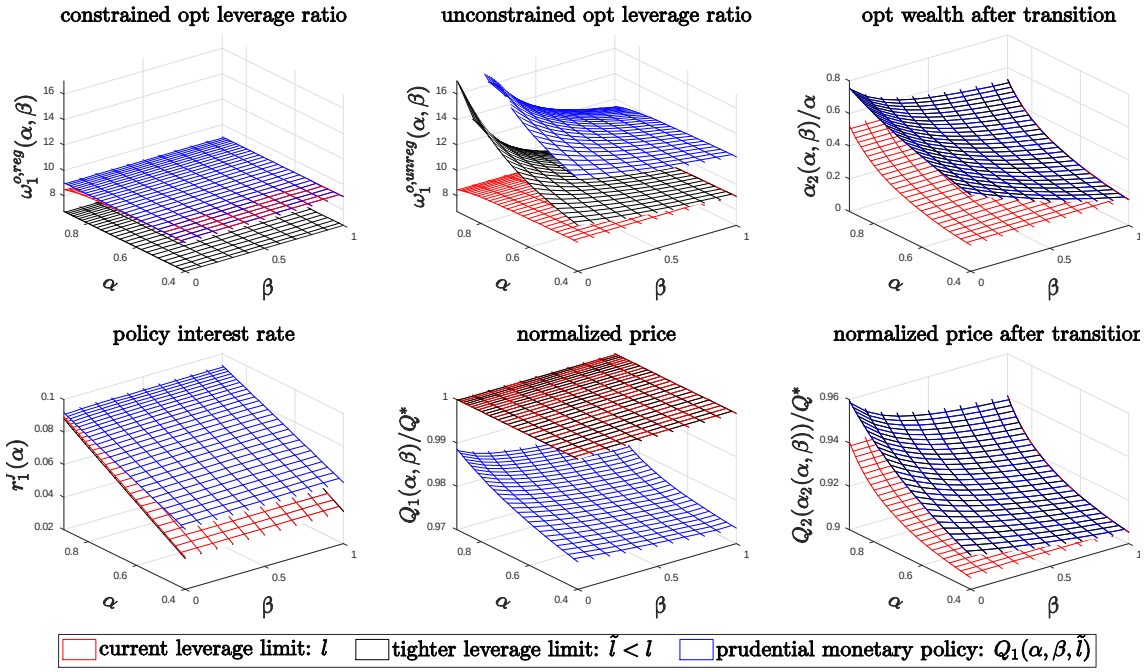


Figure 9: Equilibrium functions in the boom state $s = 1$ with unconstrained optimists for different specifications of the leverage limit and PMP. $\beta$ is the fraction of optimists' wealth held by unconstrained optimists.

policy: unconstrained agents respond to *either* policy by increasing their leverage and risk taking. This finding is consistent with recent empirical evidence showing that a contractionary monetary policy shock *increases* lending by shadow banks (see Elliott et al. (2019); Drechsler et al. (2019)).

## 8. Final remarks

We propose a model of asset price booms with speculation that, under some conditions, justifies using PMP to reduce the severity of future recessions. PMP aims to reduce the social cost of concentrating risk in leveraged, high-valuation agents ("optimists" or "banks"). PMP achieves this goal by lowering the asset price during the boom, which reduces the asset price decline after a transition to recession. This reduction supports highly-levered agents' balance sheets in the recession, which in turn raises asset prices (and hence further reduces the price drop) and softens the recession.

In our model, PMP sets the interest rate in the boom higher than "rstar." In contrast, a recent literature on forward guidance suggests that, in a recession with constrained interest rates, the central bank could stimulate the economy by promising to keep the interest rates in the boom lower than "rstar." These policies can be reconciled by observing that forward guidance largely concerns the "recovery" phase of the boom *immediately after the recession*. The central

bank can fulfill its forward guidance promises during the recovery and switch to PMP later in the boom if levered speculation becomes prevalent. Our comparative statics results suggest this strategy is more valuable if the probability of a risk-centric recession rises as the boom persists.

We have simplified the analysis by assuming prices are fully sticky. If prices are somewhat flexible, then a natural concern is that PMP could also exert downward pressure on inflation in the boom—in addition to creating negative output gaps. Two considerations mitigate this concern. First, while PMP creates negative output gaps in the boom, it also *mitigates the negative output gaps in the recession*. If price-setting agents are forward looking, as in the standard New Keynesian model, then the anticipation of smaller output gaps in the recession would reduce their incentive to cut prices in response to PMP. Second, the central bank could in principle apply PMP in the boom in time-varying fashion to mitigate its side-effects on inflation. In particular, combining PMP with forward guidance (as we discussed earlier) can be especially valuable as the two policies are likely to have opposing effects on inflation.

In our setup, the economy transitions from boom to recession with an exogenous probability. In practice, policymakers are concerned that raising interest rates in the boom can *endogenously* trigger a recession that could otherwise be avoided. These concerns suggest amplification mechanisms induced by a hike in interest rates. One possibility is that the interest rate hike comes as a surprise and damages the balance sheets of high-valuation investors (as in Kekre and Lenel (2019)). This mechanism suggests PMP is still useful but should be pre-announced or be part of a policy rule (as in our model). Alternatively, investors can have extrapolative beliefs, in which case an interest rate hike can lower asset prices and exacerbate pessimism. This mechanism suggests PMP can backfire when asset prices have been steady or declining, but it can be *even more powerful* when asset prices have been increasing. In the latter case, PMP would mitigate the excessive optimism in the boom driven by extrapolative beliefs, in addition to reducing the price crash once the optimism subsides.

In our model, there is a single risky asset, which we interpret as a weighted average of all risky assets according to their weights in banks' portfolios (see Remark 1). This bank-centric perspective suggests PMP should *primarily monitor bond prices,* and to some extent also house prices, but not necessarily stock prices. Specifically, the central bank can stabilize bond yields by raising the policy rate when the (term or credit) spreads are low and cutting it when spreads spike. In fact, if markets are somewhat segmented, then the central bank might be able to stabilize bond yields *even without changing the policy interest rate*: by selling bonds when the spreads are compressed and purchasing them when the spreads rise. We leave a formal analysis of *targeted PMP* along these lines for future work.

## References

Acharya, V., Naqvi, H., 2018. On reaching for yield and the coexistence of bubbles and negative bubbles. Journal of Financial Intermediation.

Acharya, V. V., Plantin, G., 2019. Monetary easing, leveraged payouts and lack of investment. NBER Working Paper No. 26471.

Adrian, T., de Fontnouvelle, P., Yang, E., Zlate, A., 2017. Macroprudential policy: a case study from a tabletop exercise. Economic Policy Review 23 (1), 1âĂŞ30.

Adrian, T., Duarte, F., 2016. Financial vulnerability and monetary policy.

Adrian, T., Liang, N., 2018. Monetary policy, financial conditions, and financial stability. International Journal of Central Banking.

Adrian, T., Shin, H. S., 2010. Financial intermediaries and monetary economics. In: Handbook of monetary economics. Vol. 3. Elsevier, pp. 601–650.

Allen, F., Barlevy, G., Gale, D., 2019. Asset price booms and macroeconomic policy: a risk-shifting approach. In: 2019 Meeting Papers. No. 587. Society for Economic Dynamics.

Bailey, M., Dávila, E., Kuchler, T., Stroebel, J., 2019. House price beliefs and mortgage leverage choice. The Review of Economic Studies 86 (6), 2403–2452.

Berger, D. W., Milbradt, K., Tourre, F., Vavra, J., 2018. Mortgage prepayment and path-dependent effects of monetary policy. NBER working paper No. 25157.

Bernanke, B. S., Kuttner, K. N., 2005. What explains the stock market's reaction to federal reserve policy? The Journal of finance 60 (3), 1221–1257.

Bigio, S., Zilberman, E., 2019. Speculation-driven business cycles. working paper.

Borio, C., Zhu, H., 2012. Capital regulation, risk-taking and monetary policy: a missing link in the transmission mechanism? Journal of Financial stability 8 (4), 236–251.

Borio, C. E., 2014. Monetary policy and financial stability: what role in prevention and recovery?

Borio, C. E., Drehmann, M., 2009. Assessing the risk of banking crises–revisited. BIS Quarterly Review, March.

Borovička, J., 2020. Survival and long-run dynamics with heterogeneous beliefs under recursive preferences. Journal of Political Economy 128 (1), 000–000.

Brunnermeier, M. K., Simsek, A., Xiong, W., 2014. A welfare criterion for models with distorted beliefs. The Quarterly Journal of Economics 129 (4), 1753–1797.

Caballero, R. J., Simsek, A., forthcoming. A risk-centric model of demand recessions and speculation. The Quarterly Journal of Economics.

Campbell, J. Y., 2014. Empirical asset pricing: Eugene fama, lars peter hansen, and robert shiller. The Scandinavian Journal of Economics 116 (3), 593–634.

Cao, D., 2017. Speculation and financial wealth distribution under belief heterogeneity. The Economic Journal.

Cochrane, J. H., 2011. Presidential address: Discount rates. The Journal of finance 66 (4), 1047–1108.

Collard, F., Dellas, H., Diba, B., Loisel, O., 2017. Optimal monetary and prudential policies. American Economic Journal: Macroeconomics 9 (1), 40–87.

Curdia, V., Woodford, M., 2010. Credit spreads and monetary policy. Journal of Money, credit and Banking 42, 3–35.

Dávila, E., Korinek, A., 2017. Pecuniary externalities in economies with financial frictions. The Review of Economic Studies 85 (1), 352–395.

Drechsler, I., Savov, A., Schnabl, P., 2019. How monetary policy shaped the housing boom. NBER working paper no. 25964.

Elliott, D., Meisenzahl, R. R., Peydró, J.-L., 2019. Nonbanks, banks, and monetary policy: Us loan-level evidence since the 1990s.

Farhi, E., Werning, I., 2016. A theory of macroprudential policies in the presence of nominal rigidities. Econometrica 84 (5), 1645–1704.

Fostel, A., Geanakoplos, J., 2008. Leverage cycles and the anxious economy. The American Economic Review 98 (4), 1211–1244.

Geanakoplos, J., 2010. The leverage cycle. NBER macroeconomics annual 24 (1), 1–66.

Gertler, M., Karadi, P., 2015. Monetary policy surprises, credit costs, and economic activity. American Economic Journal: Macroeconomics 7 (1), 44–76.

Gilchrist, S., López-Salido, D., Zakrajšek, E., 2015. Monetary policy and real borrowing costs at the zero lower bound. American Economic Journal: Macroeconomics 7 (1), 77–109.

Gourio, F., Kashyap, A. K., Sim, J. W., 2018. The trade offs in leaning against the wind. IMF Economic Review 66 (1), 70–115.

Hanson, S. G., Stein, J. C., 2015. Monetary policy and long-term real rates. Journal of Financial Economics 115 (3), 429–448.

Heimer, R., Simsek, A., 2019. Should retail investors' leverage be limited? Journal of Financial Economics 132 (3), 1–21.

Iachan, F. S., Nenov, P. T., Simsek, A., 2015. The choice channel of financial innovation. NBER working paper no. 21686.

IMF, 2015. Monetary policy and financial stability. IMF Staff Report.

Jordà, Ò., Schularick, M., Taylor, A. M., 2013. When credit bites back. Journal of Money, Credit and Banking 45 (s2), 3–28.

Kekre, R., Lenel, M., 2019. Monetary policy, redistribution, and risk premia. Princeton University working paper.

Kiyotaki, N., Moore, J., 1997. Credit cycles. Journal of political economy 105 (2), 211–248.

Kocherlakota, N., 2014. Discussion of 2014 usmpf monetary policy report. Tech. rep., Federal Reserve Bank of Minneapolis.

Korinek, A., Simsek, A., 2016. Liquidity trap and excessive leverage. American Economic Review 106 (3), 699–738.

Lian, C., Ma, Y., Wang, C., 2018. Low interest rates and risk taking: Evidence from individual investment decisions. Review of Financial Studies.

Lorenzoni, G., 2008. Inefficient credit booms. The Review of Economic Studies 75 (3), 809–833.

Maddaloni, A., Peydró, J.-L., 2011. Bank risk-taking, securitization, supervision, and low interest rates: Evidence from the euro-area and the us lending standards. the review of financial studies 24 (6), 2121–2165.

Martinez-Miera, D., Repullo, R., 2019. Monetary policy, macroprudential policy, and financial stability.

Mian, A., Straub, L., Sufi, A., 2019. Indebted demand. Harvard University working paper.

Morris, S., Shin, H. S., 2014. Risk-taking channel of monetary policy: A global game approach. Unpublished working paper, Princeton University.

Rajan, R. G., 2006. Has finance made the world riskier? European Financial Management 12 (4), 499–533.

Rognlie, M., Shleifer, A., Simsek, A., 2018. Investment hangover and the great recession. American Economic Journal: Macroeconomics 10 (2), 113–53.

Scheinkman, J. A., Xiong, W., 2003. Overconfidence and speculative bubbles. Journal of political Economy 111 (6), 1183–1220.

Shiller, R. J., 2014. Speculative asset prices. American Economic Review 104 (6), 1486–1517.

Simsek, A., 2013a. Belief disagreements and collateral constraints. Econometrica 81 (1), 1–53.

Simsek, A., 2013b. Speculation and risk sharing with new financial assets. The Quarterly journal of economics 128 (3), 1365–1396.

Smets, F., 2014. Financial stability and monetary policy: How closely interlinked? International Journal of Central Banking 10 (2), 263–300.

Stein, J., 2012. Monetary policy as financial stability regulation. The Quarterly Journal of Economics 127 (1), 57–95.

Stein, J., 2013. Overheating in credit markets: Origins, measurement, and policy responses: speech given to the symposium on restoring household financial stability after the great recession, federal reserve bank of st. louis, st. louis, missouri, february 07, 2013. Tech. rep.

Stein, J., 2014. Incorporating financial stability considerations into a monetary policy framework: a speech at the international research forum on monetary policy, washington, dc, march 21, 2014. Tech. rep.

Svensson, L. E., 2017. Cost-benefit analysis of leaning against the wind. Journal of Monetary Economics 90, 193–213.

Svensson, L. E., 2018. The future of monetary policy and macroprudential policy. In: The Future of Central Banking: An ECB Colloquium Held in Honour of Vitor Constancio.

Weidmann, J., 2018. The relationship between monetary and macroprudential policies - black and white or shades of grey?: dinner speech at the annual meeting of the cebra, frankfurt am main, 20 august 2018. Tech. rep., Central Bank Research Association (CEBRA).

Woodford, M., 2012. Inflation targeting and financial stability. NBER working paper no. 17967.

# A. Appendix: Omitted results and derivations

This appendix presents the analyses omitted from the main text. Sections A.1-A.4 present the derivations and proofs omitted from Sections 2-5, respectively. Section A.5 characterizes the equilibrium with unconstrained shadow banks and presents the results omitted from Section 7. Section A.6 presents the details of the numerical exercise.

## A.1. Omitted derivations in Section 2

We start by deriving the investors' optimality conditions. We then prove Lemmas 1-3.

Recall that we allow for a leverage limit only in the boom state $s = 1$ (cf. Eq. (6)). To unify the notation, we derive the optimality conditions for a slightly more general problem that might feature a leverage limit also in the other states. Our results concern the special case with $\overline{\omega}_{t,s} = \infty$ for $s \neq 1$.

Recall that the investor's portfolio problem is given by (7). The corresponding HJB equation is

$$\begin{aligned}\rho V_{t,s}^i\left(a_{t,s}^i\right) = \max_{c,\omega} \log c + \frac{\partial V_{t,s}^i}{\partial a}\left(a_{t,s}^i\left(r_{t,s}^f + \omega\left(r_{t,s} - r_{t,s}^f\right)\right) - c\right) \\ + \frac{\partial V_{t,s}^i\left(a_{t,s}^i\right)}{\partial t} + \lambda_s^i\left(V_{t,s'}^i\left(a_{t,s}^i\left(1 + \omega\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}}\right)\right) - V_{t,s}^i\left(a_{t,s}^i\right)\right) \\ \text{s.t. } \omega \leq \overline{\omega}_{t,s}.\end{aligned} \tag{A.1}$$

In view of log utility, the solution has the functional form

$$V_{t,s}^i\left(a_{t,s}^i\right) = \frac{\log\left(a_{t,s}^i/Q_{t,s}\right)}{\rho} + v_{t,s}^i. \tag{A.2}$$

The first term in the value function captures the effect of holding a greater capital stock (or greater wealth), which scales the investor's consumption proportionally at all times and in all states. The second term, $v_{t,s}^i$, is the normalized value function when the investor holds one unit of the capital stock (or wealth, $a_{t,s}^i = Q_{t,s}$). This functional form also implies

$$\frac{\partial V_{t,s}^i}{\partial a} = \frac{1}{\rho a_{t,s}^i}.$$

The first order condition for $c$ then implies

$$c_{t,s}^i = \rho a_{t,s}^i. \tag{A.3}$$

The first order condition for $\omega$ implies

$$\frac{\partial V_{t,s}^i}{\partial a} a_{t,s}^i\left(r_{t,s} - r_{t,s}^f\right) + \lambda_s^i \frac{\partial V_{t,s'}^i\left(a_{t,s'}^i\right)}{\partial a} a_{t,s}^i \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}} \geq 0,$$

with equality if $\omega = \overline{\omega}_{t,s}$. After substituting for $\frac{\partial V_{t,s}^i}{\partial a}$ and $\frac{\partial V_{t,s'}^i}{\partial a}$ and rearranging terms, this implies

$$r_{t,s} - r_{t,s}^f + \lambda_s^i \frac{a_{t,s}^i}{a_{t,s'}^i} \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}} \geq 0, \tag{A.4}$$

with equality if $\omega = \overline{\omega}_{t,s}$.

**Proof of Lemma 1.** Combining Eq. $(A.3)$ with Eq. (9) implies that aggregate consumption is a constant fraction of aggregate wealth,

$$c^o_{t,s} + c^p_{t,s} = \rho Q_{t,s} k_{t,s}.$$

Combining this result with the goods market clearing condition (12), we obtain Eq. (14).

Next note that combining Eqs. (11) and (14), the efficient level of utilization solves (16), which we replicate here:

$$\delta'(\eta^*)\,\eta^* = \rho.$$

This equation has a unique and positive solution because the left-hand side is strictly increasing with $\lim_{\eta\to\infty} \delta'(\eta)\,\eta = \infty$ (since $\delta$ is strictly increasing and convex).

Combining Eqs. (14) and (16), we obtain Eq. (17). Finally, Substituting Eq. (14) into Eq. (5), we obtain Eq. (15). □

**Proof of Lemma 2.** Recall that the investors' portfolio optimality condition is given by Eq. $(A.4)$. After substituting $a^i_{t,s} = \alpha^i_{t,s} Q_{t,s} k_{t,s}$ [cf. Eq. (19)], this gives Eq. (21) in the main text.

Eq. (22) holds by the definition of investors' wealth shares. Note also that Eq. (22), together with Eqs. (19) and (20), implies the asset market clearing condition (9). Therefore, the asset market is in equilibrium as long as Eq. (22) holds.

Finally, consider the evolution of investors' wealth shares in (23). After substituting optimal consumption from $(A.3)$ into the budget constraint in (7), type $i$ investors' wealth evolves according to

$$\frac{da^i_{t,s}/dt}{a^i_{t,s}} = r^f_{t,s} + \omega^i_{t,s}\left(r_{t,s} - r^f_{t,s}\right) - \rho.$$

Combining this with Eq. (9), aggregate wealth evolves according to

$$\frac{d\left(Q_{t,s}k_{t,s}\right)/dt}{Q_{t,s}k_{t,s}} = r^f_{t,s} + \left(r_{t,s} - r^f_{t,s}\right) - \rho.$$

Combining these expressions with $\alpha^i_{t,s} = \frac{a^i_{t,s}}{Q_{t,s}k_{t,s}}$ [cf. Eq. (19)], we obtain:

$$\frac{\dot{\alpha}^i_{t,s}}{\alpha^i_{t,s}} = \left(\omega^i_{t,s} - 1\right)\left(r_{t,s} - r^f_{t,s}\right). \tag{A.5}$$

Next recall that the portfolio optimality condition (21) holds with equality for pessimists. Applying this equation, we obtain:

$$r_{t,s} - r^f_{t,s} = -\lambda^p_s \frac{\alpha^p_{t,s}}{\alpha^p_{t,s'}} \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}}. \tag{A.6}$$

Likewise, applying Eq. (20) for type $i$ investors, we obtain:

$$\omega^i_{t,s} - 1 = \left(\frac{\alpha^i_{t,s'}}{\alpha^i_{t,s}} - 1\right) \frac{Q_{t,s'}}{Q_{t,s'} - Q_{t,s}}. \tag{A.7}$$

Substituting Eqs. $(A.6)$ and $(A.7)$ into Eq. $(A.5)$, we obtain Eq. (23). □

**Proof of Lemma 3.** Since Eq. (21) holds as equality for both types of investors, we have $\lambda_s^o \frac{\alpha_{t,s}^o}{\alpha_{t,s'}^o} = \lambda_s^p \frac{\alpha_{t,s}^p}{\alpha_{t,s'}^p}$. Combining this Eq. (9), we obtain (24). Substituting this into Eq. (23), we obtain (25). Finally, substituting it into Eq. (21), we obtain the risk balance condition (26). □

## A.2. Omitted derivations in Section 3

As we describe in the main text, for the rest of the analysis we often simplify the notation by dropping the subscript $o$ from optimists' wealth share:

$$\alpha_{t,s} \equiv \alpha_{t,s}^o.$$

Pessimists' wealth share is the complement of this expression, $\alpha_{t,s}^p = 1 - \alpha_{t,s}$.

We characterize the equilibrium starting in the recession state under the following parametric restrictions:

**Assumption A1.** $\delta(0) - \left(\rho + \lambda_2^i\right) < g_2 < \delta(\eta^*) - \rho < g_3$.

**Proof of Proposition 1.** Under Assumption A1, we conjecture an equilibrium in which the recovery state $s = 3$ features positive interest rates, efficient asset prices, and efficient factor utilization, $r_{t,3}^f > 0, Q_{t,3} = Q^*$ and $\eta_{t,3} = \eta^*$. The recession state $s = 2$ features an interest rate of zero, lower asset prices, and inefficient factor utilization, $r_{t,2}^f = 0, Q_{t,2} < Q^*$ and $\eta_{t,2} < \eta^*$. We also establish Eq. (27): that is, the equilibrium price in the recession state can be represented as a strictly increasing function of optimists' wealth share: $Q_{t,2} = Q_2(\alpha_{t,2})$ where $Q_2(\cdot)$ is a strictly increasing function.

Note that for $s \in \{2, 3\}$ the leverage limit doesn't bind. Therefore, Lemma 3 applies and facilitates the analysis.

First consider the recovery state. There is no speculation since $\lambda_3^i = 0$ for each $i$. Substituting this transition probability into Eq. (26), we obtain, $r_{t,3}^f = r_{t,3}$. After substituting for the market return from Eq. (15), and using $\dot{Q}_{t,3} = 0$ (since $Q_{t,3} = Q^*$ is constant), we obtain:

$$r_{t,3}^f = \rho + g_3 - \delta(\eta^*) > 0. \tag{A.8}$$

The inequality follows from Assumption A1. Hence, in the recovery state, the interest rate is constant and strictly positive and the equilibrium asset price and factor utilization levels are efficient.

Next consider the recession state. There is some speculation since investors have heterogeneous beliefs, $\lambda_2^o > \lambda_2^p$ [cf. Assumption 1]. Substituting Eq. (15) into Eq. (26) and using the conjecture $Q_{t,3} = Q^*$, we obtain Eq. (28) in the main text. Substituting the conjecture $r_{t,2}^f = 0$, we further obtain:

$$\rho + g_2 - \delta\left(\frac{Q_{t,2}}{Q^*}\eta^*\right) + \frac{\dot{Q}_{t,2}}{Q_{t,2}} + \overline{\lambda}_{t,2}\left(1 - \frac{Q_{t,2}}{Q^*}\right) = 0. \tag{A.9}$$

Next consider the extreme cases $\alpha_{t,2} \in \{0, 1\}$. These cases are the same as if there is a single belief type $i \in \{o, p\}$. In particular, since there is no speculation, the price is constant within the state, that is: $Q_{t,2} \equiv Q_2^i$ and thus $\dot{Q}_{t,2} = 0$. Therefore, Eq. $(A.9)$ can be written as

$$\rho + g_2 - \delta\left(\frac{Q_2^i}{Q^*}\eta^*\right) + \lambda_2^i\left(1 - \frac{Q_2^i}{Q^*}\right) = 0.$$

Under Assumption A1, there exists a solution that satisfies $Q_2^i \in (0, Q^*)$. Using $\lambda_2^o > \lambda_2^p$ (Assumption 1), it is easy to check that $Q_2^o > Q_2^p$. In particular, the price is greater under optimists' beliefs than under pessimists' beliefs.

Next consider the intermediate cases, $\alpha_{t,2} \in (0,1)$. In this case we combine Eq. $(A.9)$ with Eq. (25) for state $s = 2$ to obtain a system of differential equations for $(\alpha_{t,2}, Q_{t,2})$:

$$\begin{aligned} &\rho + g_2 - \delta\left(\frac{Q_{t,2}}{Q^*}\eta^*\right) + \frac{\dot{Q}_{t,2}}{Q_{t,2}} + \overline{\lambda}_{t,2}\left(1 - \frac{Q_{t,2}}{Q^*}\right) = 0, \\ &\dot{\alpha}_{t,2} = -\alpha_{t,2}\left(1-\alpha_{t,2}\right)\left(\lambda_2^o - \lambda_2^p\right). \end{aligned} \tag{A.10}$$

This is similar to the differential equation system for the recession state in Caballero and Simsek (forthcoming). Following similar steps, the system is saddle path stable: for any $\alpha_{t,2}$, there exists a unique equilibrium price level $Q_{t,2} \in [Q_2^p, Q_2^o)$ such that the solution satisfies $\lim_{t\to\infty} \alpha_{t,2} = 0$ and $\lim_{t\to\infty} Q_{t,2} = Q_2^p$. Since the system is stationary, the solution can be written as a function of optimists' wealth share, $Q_{t,2} = Q_2(\alpha)$. In Caballero and Simsek (forthcoming), we show that $Q_2(\alpha)$ is strictly increasing in $\alpha$. Since $Q_2^p < Q_2^o < Q^*$, this establishes Eq. (27) in the main text and completes the proof. □

To facilitate a numerical solution, we also convert the differential equation in $(A.10)$ into a differential equation in $\alpha$-domain. In particular, differentiating $Q_{t,2} = Q_2(\alpha_{t,2})$ with respect to time, we obtain:

$$\dot{Q}_{t,2} = Q_2'(\alpha_{t,2})\,\dot{\alpha}_{t,2}.$$

Combining this with Eq. $(A.10)$, we obtain:

$$\frac{Q_2'(\alpha)}{Q_2(\alpha)} = \frac{1}{\alpha(1-\alpha)\left(\lambda_2^o - \lambda_2^p\right)}\left(\rho + g_2 - \delta\left(\frac{Q_2(\alpha)}{Q^*}\eta^*\right) + \overline{\lambda}_2(\alpha)\left(1 - \frac{Q_2(\alpha)}{Q^*}\right)\right).$$

The equilibrium price function is the solution to this system subject to the boundary conditions $Q_2(0) = Q_2^p$ and $Q_2(1) = Q_2^o$. Figure 3 in the main text illustrates the solution for a particular parameterization.

## A.3. Omitted derivations in Section 4

This section presents the proofs omitted from Section 4. Section A.3.1 concerns the equilibrium in the boom state in a benchmark case without PMP (analyzed in Section 4.1). Section A.3.2 concerns the case with PMP (analyzed in 4.2).

### A.3.1. Equilibrium in the boom without PMP

We characterize the equilibrium under the following parametric restriction.

**Assumption A2.** $r_1^{f,p} \equiv \rho + g_1 - \delta(\eta^*) - \lambda_1^p\left(\frac{Q^*}{Q_2^p} - 1\right) > 0.$

Here, $Q_2^p = Q_2(0) < Q^*$ denotes the asset price in the recession state when pessimists dominate the economy. Assumption A2 ensures that the boom features a positive interest rate even if pessimists dominate.

When there is no leverage limit, $\overline{\omega}_1 = \infty$, Assumption A2 is sufficient to characterize the equilibrium. We first characterize the equilibrium in this case and derive Eq. (29). We then consider the case with

a leverage limit. We characterize the equilibrium under a stronger assumption and prove Proposition 2. Throughout, we conjecture that the equilibrium outcomes can be described as a function of optimists' wealth share, $\alpha_{t,1}$ (as well as the leverage limit function, $\overline{\omega}_1(\cdot)$). In particular, we denote the wealth-weighted average belief with

$$\overline{\lambda}_1(\alpha) = \alpha\lambda_1^o + (1-\alpha)\lambda_1^p.$$

We also denote optimists' wealth share after transition with $\alpha_{t,2} = \alpha_2(\alpha_{t,1}, \overline{\omega}_1)$.

**Equilibrium in the boom state $s = 1$ without a leverage limit.** Suppose $\overline{\omega}_1 = \infty$. Under Assumption A2, we conjecture an equilibrium in which the interest rate is positive, $r_{t,1}^f > 0$, and the asset price is at its efficient level, $Q_{t,1} = Q^*$.

First consider the corner cases $\alpha_{t,1} = 0$ and $\alpha_{t,1} = 1$. Equivalently, $\alpha_{t,1}^i = 1$ for some belief type $i$. Using Eq. (21), which applies as equality for type $i$ investors, we obtain:

$$r_1^{f,i} = \rho + g_1 - \delta(\eta^*) - \lambda_1^i\left(\frac{Q^*}{Q_2^i} - 1\right). \tag{A.11}$$

Here, we substituted $r_{t,1}$ from Eq. (15). Under Assumption A2, there exists a solution that satisfies $r_1^{f,i} > 0$ for each $i \in \{o, p\}$. Since $\lambda_1^o < \lambda_1^p$, we also have $r_1^{f,o} > r_1^{f,p}$: the equilibrium interest rate is greater when optimists dominate the economy.

Next consider the intermediate cases, $\alpha_{t,1} \in (0,1)$. Note that Lemma 3 applies (since $\overline{\omega}_1 = \infty$). Using Eq. (24), we obtain Eq. (29) in the main text, which we replicate here:

$$\alpha_2(\alpha, \infty) = \alpha\frac{\lambda_1^o}{\overline{\lambda}_1(\alpha)} < \alpha.$$

Using Eq. (25), we obtain

$$\dot{\alpha}_{t,1}/\alpha_{t,1} = (1-\alpha)(\lambda_1^p - \lambda_1^o) > 0.$$

Using Eq. (26), we also solve for the equilibrium interest rate as

$$r_1^f(\alpha, \infty) = \rho + g_1 - \delta(\eta^*) - \overline{\lambda}_1(\alpha)\left(\frac{Q^*}{Q_2(\alpha_2(\alpha, \infty))} - 1\right). \tag{A.12}$$

Under Assumption A2, $r_1^f(\alpha, \infty) > 0$ for each $\alpha$.

Finally, we combine Eqs. (29) and (20) to solve for the corresponding leverage ratio in closed form:

$$\omega_1^o(\alpha, \infty) = 1 + \frac{1 - \frac{\lambda_1^o}{\overline{\lambda}_1(\alpha)}}{\frac{Q^*}{Q_2\left(\frac{\alpha\lambda_1^o}{\overline{\lambda}_1(\alpha)}\right)} - 1} > 1. \tag{A.13}$$

As expected, optimists have above-average leverage during the boom, which induces a decline in their wealth share after transition to the recession.

**Equilibrium in the boom state $s = 1$ with a leverage limit.** Next consider the case with the leverage limit. In this case, we need additional regularity conditions that ensure Eq. (30) has a unique solution in the relevant range.

**Assumption A3.** $Q_2'(\alpha_2) < \frac{Q^* - Q_2(\alpha_2)}{1-\alpha_2}$ for $\alpha_2 \in (0,1)$; and $Q_2\left(\frac{\alpha\lambda_1^o}{\overline{\lambda}_1(\alpha)}\right) > Q^*\alpha\left(1 - \frac{\lambda_1^o}{\lambda_1^p}\right)$ for $\alpha \in (0,1)$.

These conditions concern the price function in the recession state. They are satisfied by our numerical solutions (they are also sufficient conditions, i.e., they can be relaxed further). The first part says that the slope of the price function is not too large. Since $Q_2(1) = Q_2^o < Q^*$, this condition will always hold if $Q_2(\alpha_2)$ is a linear function. Therefore, it holds as long as $Q_2(\alpha_2)$ does not deviate from linearity too much. The second part requires that either the price decline after transition to the recession is not too large, or the extent of speculation during the boom is not too large. For instance, when $\alpha = 1$, the requirement is $Q_2^o > Q^* \left(1 - \frac{\lambda_1^o}{\lambda_1^p}\right)$. This holds if $Q_2^o$ is close to $Q^*$ or if $\lambda_1^o$ is not substantially smaller than $\lambda_1^p$.

Assumption A3 is useful to establish the following lemma, which we use to prove Proposition 2. We first state and prove the lemma. We then prove Proposition 2.

**Lemma 5.** *Consider the following function:*

$$f(\alpha_2; \alpha, \overline{\omega}_1) = 1 - \frac{\alpha_2}{\alpha} - (\overline{\omega}_1 - 1)\left[\frac{Q^*}{Q_2(\alpha_2)} - 1\right],$$

*where $\alpha, \overline{\omega}_1$ are parameters such that $\alpha \in (0,1), \overline{\omega}_1 \leq \omega_1^o(\alpha, \infty)$. Under Assumption A3, $f(\alpha_2) = 0$ has a unique solution that satisfies $\alpha_2 \in [\alpha_2(\alpha, \infty), \alpha)$.*

**Proof.** We first show that there exists a solution that lies in the desired interval. We have

$$\begin{aligned} f(\alpha_2(\alpha, \infty)) &= 1 - \frac{\alpha_2(\alpha, \infty)}{\alpha} - (\overline{\omega}_1 - 1)\left[\frac{Q^*}{Q_2(\alpha_2(\alpha, \infty))} - 1\right] \\ &\geq 1 - \frac{\alpha_2(\alpha, \infty)}{\alpha} - (\omega_1^o(\alpha, \infty) - 1)\left[\frac{Q^*}{Q_2(\alpha_2(\alpha, \infty))} - 1\right] = 0. \end{aligned}$$

Here, the inequality in the second line follows since $\overline{\omega}_1 \leq \omega_1^o(\alpha, \infty)$ and $Q_2(\alpha_2(\alpha, \infty)) < Q^*$, and the equality follows from the definition of $\omega_1^o(\alpha, \infty)$. We also have

$$f(\alpha) = -(\overline{\omega}_1 - 1)\left[\frac{Q^*}{Q_2(\alpha_2)} - 1\right] < 0.$$

It follows that there exists a solution in $[\alpha_2(\alpha, \infty), \alpha)$.

We next show that the derivative of $f$ is strictly negative at each zero of $f$:

$$f'(\alpha_2) < 0 \text{ for each } \alpha_2 \in [\alpha_2(\alpha, \infty), \alpha) \text{ and } f(\alpha_2) = 0. \tag{A.14}$$

This establishes that $f$ has a unique zero in the desired interval. To establish this claim, we first evaluate the derivative

$$f'(\alpha_2) = -\frac{1}{\alpha} + (\overline{\omega}_1 - 1)\frac{Q^*}{(Q_2(\alpha_2))^2} Q_2'(\alpha_2).$$

Hence, $f'(\alpha_2) < 0$ as long as

$$\alpha(\overline{\omega}_1 - 1)\frac{Q^*}{Q_2(\alpha_2)}\frac{Q_2'(\alpha_2)}{Q_2(\alpha_2)} < 1.$$

Note that we require this to hold when $f(\alpha_2) = 0$. This implies

$$\alpha(\overline{\omega}_1 - 1)\frac{Q^*}{Q_2(\alpha_2)} = (\alpha - \alpha_2)\frac{Q^*}{Q^* - Q_2(\alpha_2)}.$$

Combining the last two displayed equations, we need to show

$$Q_2'(\alpha_2) < \frac{Q^* - Q_2(\alpha_2)}{1-\alpha_2}\frac{1-\alpha_2}{\alpha-\alpha_2}\frac{Q_2(\alpha_2)}{Q^*}. \tag{A.15}$$

Using the first part of Assumption A3, we have

$$Q_2'(\alpha_2) < \frac{Q^* - Q_2(\alpha_2)}{1-\alpha_2}. \tag{A.16}$$

Using the second part of Assumption A3, we also have

$$1 \leq \frac{1-\alpha_2(\alpha,\infty)}{\alpha-\alpha_2(\alpha,\infty)}\frac{Q_2(\alpha_2(\alpha,\infty))}{Q^*} \leq \frac{1-\alpha_2}{\alpha-\alpha_2}\frac{Q_2(\alpha_2)}{Q^*}. \tag{A.17}$$

Here, the first inequality follows from Assumption A3 since $\frac{1-\alpha_2(\alpha,\infty)}{\alpha-\alpha_2(\alpha,\infty)} = \frac{\lambda_1^p}{\alpha\left(\lambda_1^p-\lambda_1^o\right)}$ after substituting $\alpha_2(\alpha,0) = \frac{\alpha\lambda_1^o}{\lambda_1(\alpha)}$ [cf. Eq. (29)]. The second inequality follows since $\alpha_2(\alpha,\infty) \leq \alpha_2$ implies $\frac{1-\alpha_2(\alpha,\infty)}{\alpha-\alpha_2(\alpha,\infty)} \leq \frac{1-\alpha_2}{\alpha-\alpha_2}$ and $Q_2(\alpha_2(\alpha,\infty)) \leq Q_2(\alpha_2)$. Combining Eqs. $(A.16)$ and $(A.17)$ establishes Eq. $(A.15)$. This in turn establishes Eq. $(A.14)$ and shows that there is a unique solution. □

**Proof of Proposition 2.** We characterized the equilibrium without a leverage limit earlier in this section. Suppose there is a leverage limit. For the corner cases, $\alpha_{t,1} = 0$ or $\alpha_{t,1} = 1$, or the intermediate cases in which the leverage limit doesn't bind, the equilibrium is the same as before.

Consider the remaining cases: that is, $\alpha_{t,1} \in (0,1)$ and the leverage limit binds $\overline{\omega}_1(\alpha) \leq \omega_1^o(\alpha,\infty)$. In these cases, optimists' leverage ratio is determined by the limit:

$$\omega_1^o(\alpha,\overline{\omega}_1) = \overline{\omega}_1(\alpha). \tag{A.18}$$

Substituting this into Eq. (20) for the boom state $s = 1$, we obtain Eq. (30) in the main text, which we replicate here:

$$\begin{aligned}\frac{\alpha_2(\alpha,\overline{\omega}_1)}{\alpha} &= 1-(\overline{\omega}_1(\alpha)-1)\left[\frac{Q_1}{Q_2}-1\right],\\ \text{where } Q_1 &= Q^* \text{ and } Q_2 = Q_2(\alpha_2(\alpha,\overline{\omega}_1)).\end{aligned}$$

The solution to this equation corresponds to the zero of the function, $f(\alpha_2;\alpha,\overline{\omega}_1)$, defined in Lemma 5. Therefore, under Assumption A3, there is a unique solution that satisfies $\alpha_2(\alpha,\overline{\omega}_1) \geq \alpha_2(\alpha,\infty)$.

This characterizes optimists' wealth share after transition to recession, $\alpha_2(\alpha,\overline{\omega}_1)$. Next consider the comparative statics of the solution with respect to the leverage limit. Implicitly differentiating the equation $f(\alpha_2;\alpha,\overline{\omega}_1) = 0$ with respect to $\overline{\omega}_1$,

$$\frac{d\alpha_2}{d\overline{\omega}_1} = \frac{\frac{Q^*}{Q_2(\alpha_2)}-1}{f'(\alpha_2)} < 0.$$

Here, the inequality follows since $\frac{Q^*}{Q_2(\alpha_2)} - 1 > 0$ and $f'(\alpha_2) < 0$ [cf. Eq. $(A.14)$]. It follows that the solution is strictly decreasing in $\overline{\omega}_1$, that is, $\frac{d\alpha_2(\alpha,\overline{\omega}_1)}{d\overline{\omega}_1(\alpha)} < 0$. In particular, decreasing the leverage limit increases optimists' wealth share after transition.

Next consider optimists' wealth share growth if there is no transition to recession, $\dot{\alpha}_{t,1}/\alpha_{t,1}$. Using

Eq. (23), we obtain:

$$\frac{\dot{\alpha}_{t,1}}{\alpha_{t,1}} = \lambda_1^p \frac{1-\alpha_{t,1}}{1-\alpha_2\left(\alpha_{t,1},\overline{\omega}_1\right)}\left(1-\frac{\alpha_2\left(\alpha_{t,1},\overline{\omega}_1\right)}{\alpha_{t,1}}\right) \leq \left(1-\alpha_{t,1}\right)\left(\lambda_1^p-\lambda_1^o\right). \tag{A.19}$$

Here, the weak inequality is satisfied as equality when the leverage limit doesn't bind (i.e., when $\alpha_2$ is given by Eq. (29)). It is also easy to see that $\dot{\alpha}_{t,1}/\alpha_{t,1}$ is a decreasing function of $\alpha_2$: if optimists obtain a greater wealth share after transition to recession, then their wealth share grows more slowly if there is no transition. Combining this observation with $\frac{d\alpha_2(\alpha,\overline{\omega}_1)}{d\overline{\omega}_1(\alpha)} < 0$, we also find $\frac{d(\dot{\alpha}_{t,1}/\alpha_{t,1})}{d\overline{\omega}_1(\alpha)} < 0$. Hence, decreasing the leverage limit slows down the growth of optimists' wealth share absent transition.

Finally, consider the equilibrium interest rate. Using Eq. (21), which holds as equality for pessimists, we obtain

$$r_1^f\left(\alpha,\overline{\omega}_1\right) = \rho + g_1 - \delta\left(\eta^*\right) - \lambda_1^p \frac{1-\alpha}{1-\alpha_2\left(\alpha,\overline{\omega}_1\right)}\left(\frac{Q^*}{Q_2\left(\alpha_2\left(\alpha,\overline{\omega}_1\right)\right)}-1\right). \tag{A.20}$$

Here, we substituted $r_{t,1}$ from Eq. (15) and $Q_{t,1} = Q^*, Q_{t,2} = Q_2\left(\alpha_{t,2}\right)$. This is the risk balance condition according to pessimists (cf. Eq. (26)). The condition characterizes the output-stabilizing interest rate given investors' wealth shares. Assumption A2 ensures that $r_1^f\left(\alpha,\overline{\omega}_1\right) > 0$ also in this case. This characterizes the equilibrium with a leverage limit and completes the proof of the proposition. □

### A.3.2. Equilibrium in the boom with PMP

**Proof of Proposition 3.** Provided in the main text.

**Proof of Proposition 4.** First consider the effect of the leverage limit, $\tilde{\omega}_1$. Since $\tilde{\omega}_1\left(\alpha\right) < \omega_1\left(\alpha,\infty\right)$, optimists' wealth share, $\alpha_2\left(\alpha,\tilde{\omega}_1\right)$, is characterized as the unique solution to the following equation (see the proof of Proposition 2):

$$\frac{\alpha_2\left(\alpha,\tilde{\omega}_1\right)}{\alpha} = 1-\left(\tilde{\omega}_1\left(\alpha\right)-1\right)\left[\frac{Q^*}{Q_2\left(\alpha_2\left(\alpha,\tilde{\omega}_1\right)\right)}-1\right]. \tag{A.21}$$

We will show (constructively) that there exists a PMP that replicates the wealth share. Let $\tilde{\alpha}_2 = \alpha_2\left(\alpha,\overline{\omega}_1,\tilde{Q}_1\right)$ denote optimists' wealth share after transition with PMP. In the conjectured equilibrium, optimists' leverage limit binds (since $\tilde{\alpha}_2 = \alpha_2\left(\alpha,\tilde{\omega}_1\right) > \alpha_2\left(\alpha,\infty\right)$). Therefore, optimists' wealth share is the solution to

$$\frac{\tilde{\alpha}_2}{\alpha} = 1-\left(\overline{\omega}_1\left(\alpha\right)-1\right)\left[\frac{\tilde{Q}_1\left(\alpha\right)}{Q_2\left(\tilde{\alpha}_2\right)}-1\right]. \tag{A.22}$$

We next claim that, for appropriately chosen $\tilde{Q}_1\left(\alpha\right)$, this equation holds for $\tilde{\alpha}_2 = \alpha_2\left(\alpha,\tilde{\omega}_1\right)$.

To this end, let $\tilde{Q}_1\left(\alpha\right)$ denote the unique solution to

$$\left(\overline{\omega}_1\left(\alpha\right)-1\right)\left[\frac{\tilde{Q}_1\left(\alpha\right)}{Q_2\left(\alpha_2\left(\alpha,\tilde{\omega}_1\right)\right)}-1\right] = \left(\tilde{\omega}_1\left(\alpha\right)-1\right)\left[\frac{Q^*}{Q_2\left(\alpha_2\left(\alpha,\tilde{\omega}_1\right)\right)}-1\right]. \tag{A.23}$$

Hence, $\tilde{Q}_1\left(\alpha\right)$ is the asset price that replicates optimists' wealth decline after accounting for the endogenous price adjustment in the recession. After rearranging this expression, we can solve for $\tilde{Q}_1\left(\alpha\right)$ in closed form:

$$\tilde{Q}_1\left(\alpha\right) = Q_2\left(\alpha_2\left(\alpha,\tilde{\omega}_1\right)\right)\left(1+\frac{\tilde{\omega}_1\left(\alpha\right)-1}{\overline{\omega}_1\left(\alpha\right)-1}\left[\frac{Q^*}{Q_2\left(\alpha_2\left(\alpha,\tilde{\omega}_1\right)\right)}-1\right]\right). \tag{A.24}$$

Since $\tilde{\omega}_1(\alpha) < \overline{\omega}_1(\alpha)$, it is easy to check that $\tilde{Q}_1(\alpha) < Q^*$. Since $\tilde{\omega}_1(\alpha) > 1$, we also have $\tilde{Q}_1(\alpha) > Q_2(\alpha_2(\alpha, \tilde{\omega}_1))$. In particular, there exists a unique $\tilde{Q}_1(\alpha) \in (Q_2(\alpha_2(\alpha, \tilde{\omega}_1)), Q^*)$ that satisfies Eq. $(A.23)$.

We next substitute Eq. $(A.23)$ into Eq. $(A.21)$, which proves our claim that Eq. $(A.22)$ holds with $\tilde{\alpha}_2 = \alpha_2(\alpha, \tilde{\omega}_1)$. We can also check that (under Assumption A3) this equation has a unique solution. This proves $\alpha_2\left(\alpha, \overline{\omega}_1, \tilde{Q}_1\right) = \alpha_2(\alpha, \tilde{\omega}_1)$. Note that Eq. $(A.24)$ implies $\frac{\partial \tilde{Q}_1(\alpha)}{\partial \tilde{\omega}_1(\alpha)} > 0$, which completes the proof of the first part of the proposition .

Next consider the interest rate corresponding to PMP. Since the policy applies only at an infinitesimal instant, it does not affect the price drift, $\dot{Q}_{t,1} = 0$. In particular, the instantaneous return to capital is given by $\tilde{r}_1 = \rho + g_1 - \delta\left(\frac{\tilde{Q}_1(\alpha)}{Q^*}\eta^*\right)$ [cf. Eq. (15)]. Combining this with Eq. (21) for pessimists, we obtain the following analogue of Eq. $(A.20)$:

$$\tilde{r}_1^f = \rho + g_1 - \delta\left(\frac{\tilde{Q}_1(\alpha)}{Q^*}\eta^*\right) - \lambda_1^p \frac{1-\alpha}{1-\alpha_2(\alpha, \tilde{\omega}_1)}\left(\frac{\tilde{Q}_1(\alpha)}{Q_2(\alpha_2(\alpha, \tilde{\omega}_1))} - 1\right).$$

Using Eq. $(A.22)$ to substitute for the price decline, we can rewrite this as

$$\tilde{r}_1^f = \rho + g_1 - \delta\left(\frac{\tilde{Q}_1(\alpha)}{Q^*}\eta^*\right) - \lambda_1^p \frac{1-\alpha}{\alpha}\frac{\alpha - \alpha_2(\alpha, \tilde{\omega}_1)}{1-\alpha_2(\alpha, \tilde{\omega}_1)}\frac{1}{\overline{\omega}_1(\alpha) - 1}. \tag{A.25}$$

Absent prudential policy, the interest rate is characterized by Eq. $(A.20)$. After substituting for the price decline from (20), we can rewrite this expression as

$$r_1^f(\alpha, \overline{\omega}_1) = \rho + g_1 - \delta(\eta^*) - \lambda_1^p \frac{1-\alpha}{\alpha}\frac{\alpha_2(\alpha, \overline{\omega}_1) - \alpha}{1-\alpha_2(\alpha, \overline{\omega}_1)}\frac{1}{\omega_1(\alpha, \overline{\omega}_1) - 1}. \tag{A.26}$$

Here, $\omega_1(\alpha, \overline{\omega}_1)$ denotes the equilibrium leverage ratio.

Next note that $\delta\left(\frac{\tilde{Q}_1(\alpha)}{Q^*}\eta^*\right) < \delta(\eta^*)$ since $\tilde{Q}_1(\alpha) < Q^*$. Note also that $\frac{\alpha - \alpha_2(\alpha, \tilde{\omega}_1)}{1-\alpha_2(\alpha, \tilde{\omega}_1)} < \frac{\alpha - \alpha_2(\alpha, \overline{\omega}_1)}{1-\alpha_2(\alpha, \overline{\omega}_1)}$ since $\alpha_2(\alpha, \tilde{\omega}_1) > \alpha_2(\alpha, \overline{\omega}_1)$. Finally, note that $\frac{1}{\overline{\omega}_1(\alpha)-1} \leq \frac{1}{\omega_1(\alpha, \overline{\omega}_1)-1}$ since $\omega_1(\alpha, \overline{\omega}_1) \leq \overline{\omega}_1(\alpha)$. Combining these observations with Eqs. $(A.25)$ and $(A.26)$ proves that $\tilde{r}_1^f = r_1^f\left(\alpha, \overline{\omega}_1, \tilde{Q}_1\right) > r_1^f(\alpha, \overline{\omega}_1)$: PMP raises the interest rate.

Finally, consider how raising the target leverage limit $\tilde{\omega}_1(\alpha)$ affects the interest rate. Since raising the leverage limit increases $\tilde{Q}_1(\alpha)$, it also increases the effective depreciation rate, $\delta\left(\frac{\tilde{Q}_1(\alpha)}{Q^*}\eta^*\right)$. Since raising the leverage limit reduces $\alpha_2(\alpha, \tilde{\omega}_1)$, it also increases the term $\frac{\alpha - \alpha_2(\alpha, \tilde{\omega}_1)}{1-\alpha_2(\alpha, \tilde{\omega}_1)}$. Combining these observations with $(A.25)$ proves that raising the leverage limit decreases $\tilde{r}_1$, that is: $\frac{\partial r_1^f\left(\alpha, \overline{\omega}_1, \tilde{Q}_1\right)}{\partial \tilde{\omega}_1(\alpha)} < 0$. In particular, targeting a lower effective leverage limit $\tilde{\omega}_1(\alpha)$ requires a higher interest rate, completing the proof. □

**Proof of Proposition 5.** We have the following closed-form solution for the price function:

$$Q_1\left(\alpha, \tilde{l}\right) = \begin{cases} Q^* & \text{if } \omega_1\left(\alpha, \tilde{l}\right) < \overline{\omega}_1\left(\alpha, \tilde{l}\right) \\ Q_2\left(\alpha_2\left(\alpha, \tilde{l}\right)\right)\left(1 + \frac{\overline{\omega}_1(\alpha, \tilde{l}) - 1}{\overline{\omega}_1(\alpha, l) - 1}\left[\frac{Q^*}{Q_2(\alpha_2(\alpha, \tilde{l}))} - 1\right]\right) < Q & \text{if } \omega_1\left(\alpha, \tilde{l}\right) = \overline{\omega}_1\left(\alpha, \tilde{l}\right) \end{cases}. \tag{A.27}$$

Here, the first line corresponds to the case in which the leverage limit does not bind under $\tilde{l}$. In this case, the monetary authority does not use PMP. The second line corresponds to the case in which the leverage limit binds. In this case, the monetary authority uses PMP. Moreover, Eq. $(A.24)$ provides a closed-form

solution for the asset price level.

One difference from Proposition 4 concerns the characterization of the interest rate. Since the policy is applied dynamically, the price drift, $\dot{Q}_{t,1}$, is not necessarily zero, which affects the level of the interest rate. To characterize this effect, note that:

$$\begin{aligned}\dot{Q}_{t,1} &= \frac{\partial Q_1\left(\alpha,\tilde{l}\right)}{\partial_+\alpha}\dot{\alpha}_{t,1} \\ &= \frac{\partial Q_1\left(\alpha,\tilde{l}\right)}{\partial_+\alpha}\lambda_1^p\alpha_{t,1}\frac{1-\alpha_{t,1}}{1-\alpha_{t,2}}\left(\alpha_{t,1}-\alpha_{t,2}\right) \\ &= \frac{\partial Q_1\left(\alpha,\tilde{l}\right)}{\partial_+\alpha}\lambda_1^p\alpha\left(1-\alpha\right)\frac{\alpha-\alpha_2\left(\alpha,\tilde{l}\right)}{1-\alpha_2\left(\alpha,\tilde{l}\right)}. \end{aligned} \tag{A.28}$$

Here, the second line substitutes the evolution of optimists' wealth share from Eq. (23) and the third line substitutes $\alpha_{t,1}=\alpha$ and $\alpha_{t,2}=\alpha_2\left(\alpha,\tilde{l}\right)$. The expression $\frac{\partial Q_1\left(\alpha,\tilde{l}\right)}{\partial_+\alpha}$ corresponds to the right-derivative of the function characterized in $(A.27)$.[13] We can check that the right-derivative, $\frac{\partial Q_1\left(\alpha,\tilde{l}\right)}{\partial_+\alpha}$, is continuous in $\tilde{l}$ and equal to 0 when $\tilde{l}=l$ (because $Q_1\left(\alpha,l\right)=Q^*$ for each $\alpha$). Consequently, when viewed as a function of $\tilde{l}$, the price drift, $\dot{Q}_{t,1}$, is also continuous in $\tilde{l}$ and equal to 0 when $\tilde{l}=l$.

Next note that, following similar steps as in the proof of Proposition 4, the interest rate in this case can be written as

$$\tilde{r}_1^f=\rho+g_1+\dot{Q}_{t,1}-\delta\left(\frac{Q_1\left(\alpha,\tilde{l}\right)}{Q^*}\eta^*\right)-\lambda_1^p\frac{1-\alpha}{1-\alpha_2\left(\alpha,\tilde{l}\right)}\left(\frac{Q_1\left(\alpha,\tilde{l}\right)}{Q_2\left(\alpha_2\left(\alpha,\tilde{l}\right)\right)}-1\right),$$

where $\dot{Q}_{t,1}$ is given by Eq. $(A.28)$. When viewed as a function of $\tilde{l}$, the interest rate $\tilde{r}_1^f$ is continuous in $\tilde{l}$, and it is equal to the benchmark interest rate $r_1^f\left(\alpha,l\right)$ when $\tilde{l}=l$. Recall that the benchmark rate is strictly positive for each $\alpha\in(0,1)$ [cf. Section 4.1]. Therefore, $\tilde{r}_1^f>0$ for each $\alpha\in(0,1)$ as long as $\tilde{l}$ is in a sufficiently small neighborhood of $l$. In particular, PMP doesn't violate the zero lower bound constraint on the interest rate. □

## A.4. Omitted derivations in Section 5

We first present a general characterization of investors' expected values. We then prove Lemma 4 and Proposition 6.

Let the superscript $b\in\{pl,o,p\}$ denote the belief corresponding to either the planner or the corresponding investor type. Our analysis in the main text requires only the planner's belief, $b=pl$. Here, we characterize investors' value function also with respect to their own belief $b=o$ or $b=p$ (as this would be required for other welfare criteria, e.g., the standard Pareto criterion).

For $i\in\{o,p\}$, let $V_{t,s}^{i,b}\left(a_{t,s}^i\right)$ denote type $i$ investors' expected value when she has wealth $a_{t,s}^i$, evaluated

[13] Note that the function is piecewise differentiable so the right-derivative always exists. The equation depends on the right-derivative (as opposed to left) because $\dot{\alpha}_{t,1}>0$, so $\alpha_{t,1}$ grows over time.

according to belief $b$. In view of log utility, we conjecture the following version of Eq. $(A.2)$:

$$V_{t,s}^{i,b}\left(a_{t,s}^{i}\right)=\frac{\log\left(a_{t,s}^{i}/Q_{t,s}\right)}{\rho}+v_{t,s}^{i,b}. \tag{A.29}$$

Note that this function implies $\frac{\partial V_{t,s}^{i,b}}{\partial a}=\frac{1}{\rho a_{t,s}^{i}}$. Using this expression as well as the investors' consumption choice, $c_{t,s}^{i}=\rho a_{t,s}^{i}$, we obtain the following version of the HJB equation $(A.1)$:

$$\begin{aligned}\rho V_{t,s}^{i,b}\left(a_{t,s}^{i}\right)-\frac{\partial V_{t,s}^{i,b}\left(a_{t,s}^{i}\right)}{\partial t}&=\log\rho a_{t,s}^{i}+\frac{1}{\rho}\left(r_{t,s}^{f}+\omega_{t,s}^{i}\left(r_{t,s}-r_{t,s}^{f}\right)-\rho\right)\\&\quad+\lambda_{s}^{b}\left(V_{t,s'}^{i,b}\left(a_{t,s}^{i}\left(1+\omega\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s}}\right)\right)-V_{t,s}^{i,b}\left(a_{t,s}^{i}\right)\right).\end{aligned} \tag{A.30}$$

Note that this expression evaluates the value function along the equilibrium path and according to transition probability with respect to belief $b$, $\lambda_{s}^{b}$.

Substituting Eq. $(A.29)$ into Eq. $(A.30)$, we obtain a differential equation for the normalized value:

$$\begin{aligned}\rho v_{t,s}^{i,b}-\frac{\partial v_{t,s}^{i,b}}{\partial t}&=\log\rho+\log Q_{t,s}+\frac{1}{\rho}\left(r_{t,s}-\rho-\frac{\dot{Q}_{t,s}}{Q_{t,s}}+\left(\omega_{t,s}^{i}-1\right)\left(r_{t,s}-r_{t,s}^{f}\right)+\lambda_{s}^{b}\log\left(1+\omega\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s'}}\right)\right)\\&\quad+\lambda_{s}^{b}\left(v_{t,s'}^{i,b}-v_{t,s}^{i,b}\right).\end{aligned}$$

To simplify this expression, we substitute $r_{t,s}=\rho+\frac{\dot{Q}_{t,s}}{Q_{t,s}}+g_{s}-\delta\left(\frac{Q_{t,s}}{Q^{*}}\eta^{*}\right)$ using Eq. (15). We also substitute for $\left(\omega_{t,s}^{i}-1\right)\left(r_{t,s}-r_{t,s}^{f}\right)=\frac{\dot{\alpha}_{t,s}^{i}}{\alpha_{t,s}^{i}}$ from Eq. $(A.5)$. Finally, we substitute for $1+\omega\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s'}}=\frac{\alpha_{t,s'}^{i}}{\alpha_{t,s}^{i}}$ using Eq. (20). After these substitutions, we obtain:

$$\begin{aligned}\rho v_{t,s}^{i,b}-\frac{\partial v_{t,s}^{i,b}}{\partial t}&=\log\rho+\log Q_{t,s}+\frac{1}{\rho}\left(g_{s}-\delta\left(\frac{Q_{t,s}}{Q^{*}}\eta^{*}\right)+\frac{\dot{\alpha}_{t,s}^{i}}{\alpha_{t,s}^{i}}+\lambda_{s}^{b}\log\left(\frac{\alpha_{t,s'}^{i}}{\alpha_{t,s}^{i}}\right)\right)\\&\quad+\lambda_{s}^{b}\left(v_{t,s'}^{i,b}-v_{t,s}^{i,b}\right).\end{aligned} \tag{A.31}$$

We have thus characterized the normalized value function, $v_{t,s}^{i,b}$, as a solution to the differential equation in $(A.31)$. This equation applies for any belief $b\in\{o,p,pl\}$, including investors' own belief $b=i$, and it applies regardless of whether the leverage limit binds. The terms that feature $Q_{t,s}$ capture potential welfare losses due to inefficient factor utilization. The term $g_{s}$ captures the welfare effect of expected growth. The term $\frac{\dot{\alpha}_{t,s}^{i}}{\alpha_{t,s}^{i}}+\lambda_{s}^{b}\log\left(\frac{\alpha_{t,s'}^{i}}{\alpha_{t,s}^{i}}\right)$ captures the welfare effect of speculation that reshuffles investors' wealth shares across states.

As we describe in the main text, we decompose the normalized value into two components [cf. (33)]:

$$v_{t,s}^{i,b}=v_{t,s}^{i*,b}+w_{t,s}^{i,b},$$

Here, $v_{t,s}^{i*,b}$ is *the frictionless value function*, which is found by solving Eq. $(A.31)$ with $Q_{t,s}=Q^{*}$ for each $t,s$. This captures all determinants of welfare except for suboptimal factor utilization (including the benefits/costs from speculation). The residual, $w_{t,s}^{b}$, corresponds to *the gap value function.* This captures the welfare losses due to suboptimal factor utilization.

To further characterize the gap value, note that $v_{t,s}^{i,b}$ and $v_{t,s}^{i*,b}$ both solve Eq. $(A.31)$ with $Q_{t,s}$ and $Q_{t,s} = Q^*$, respectively. Taking the difference of these equations, and using $w_{t,s}^{i,b} = v_{t,s}^{i,b} - v_{t,s}^{i*,b}$, we obtain

$$\begin{aligned} \rho w_{t,s}^b - \frac{\partial w_{t,s}^b}{\partial t} &= W\left(Q_{t,s}\right) + \lambda_s^b \left(w_{t,s'}^b - w_{t,s}^b\right), \\ \text{where } W\left(Q_{t,s}\right) &= \log \frac{Q_{t,s}}{Q^*} - \frac{1}{\rho}\left(\delta\left(\frac{Q_{t,s}}{Q^*}\eta^*\right) - \delta\left(\eta^*\right)\right). \end{aligned}$$

Here, we have simplified the notation $w_{t,s}^b \equiv w_{t,s}^{i,b}$ since the gap value depends on the belief type $b$ but not the investor type $i$.

Applying this equation for the planner's belief $b = pl$ gives Eq. (34) in the main text. Integrating Eq. (34) forward, we also obtain:

$$w_{t,s}^{pl} = \int_t^\infty e^{-\left(\rho+\lambda_s^b\right)\left(\tilde{t}-t\right)}\left(W\left(Q_{\tilde{t},s}\right) + \lambda_s^{pl} w_{\tilde{t},s'}^{pl}\right) d\tilde{t}. \tag{A.32}$$

Hence, the gap value captures an appropriately discounted present value of instantaneous welfare gaps. Note that $W\left(Q_{t,s}\right)$ is a strictly concave function maximized at $Q_{t,s} = Q^*$ with maximum value $W\left(Q^*\right) = 0$. Therefore, Eq. $(A.32)$ also implies $w_{t,s}^{pl} \leq 0$ for each $t, s$.

**Proof of Lemma 4.** Next consider the gap value in the steady state. Since the model is stationary, the gap value is a function of optimists' wealth share,

$$w_{t,2}^{pl} = w_2^{pl}\left(\alpha_{t,s}\right),$$

for some function $w_2^{pl}\left(\cdot\right)$. Differentiating this expression, we have:

$$\begin{aligned} \frac{\partial w_{t,s}^{pl}}{\partial t} &= \frac{dw_2^{pl}\left(\alpha_{t,s}\right)}{d\alpha}\dot{\alpha}_{t,s} \\ &= -\frac{dw_2^{pl}\left(\alpha_{t,s}\right)}{d\alpha}\alpha_{t,2}\left(1-\alpha_{t,2}\right)\left(\lambda_2^o - \lambda_2^p\right). \end{aligned}$$

Note that $w_{t,3}^{pl} = 0$ since $Q_{t,3} = Q^*$. Finally, recall that we have $Q_{t,2} = Q_2\left(\alpha\right) < Q^*$, where $Q_2\left(\alpha\right)$ is a strictly increasing function. Substituting these expressions into Eq. (34) for state $s = 2$, we characterize the gap value as the solution to a differential equation in $\alpha$-domain:

$$\left(\rho + \lambda_2^{pl}\right) w_2^{pl}\left(\alpha\right) + \frac{dw_2^{pl}\left(\alpha\right)}{d\alpha}\alpha\left(1-\alpha\right)\left(\lambda_2^o - \lambda_2^p\right) = W\left(Q_2\left(\alpha\right)\right).$$

We analyze the solution to this differential equation in Caballero and Simsek (forthcoming). In particular, since $W\left(Q_2\left(\alpha\right)\right)$ is strictly increasing in $\alpha$ (since $Q_2\left(\alpha\right) < Q^*$), $w_2^{pl}\left(\alpha\right)$ is also strictly increasing in $\alpha$. We also have $w_2^{pl}\left(\alpha\right) < 0$ for each $\alpha$ [cf. $(A.32)$]. This proves Eq. (35). □

**Proof of Proposition 6.** Using Eq. $(A.32)$, we can write the planner's gap value with policy $\tilde{l}$ as

$$w_1^{pl}\left(\alpha_{0,1}, \tilde{l}\right) = \int_0^\infty e^{-\left(\rho+\lambda_1^{pl}\right)t}\left(W\left(Q^*\right) + \lambda_1^{pl} w_{t,2}^{pl}\left(\alpha_{t,2}\right)\right) dt. \tag{A.33}$$

Here, $\alpha_{t,2}$ denotes optimists' wealth share if the economy transitions to recession at time $t$. Likewise, we

write the planner's gap value with policy $Q_1\left(\alpha,\tilde{l}\right)$ as

$$w_1^{pl}\left(\alpha_{0,1},l,Q_1\left(\alpha,\tilde{l}\right)\right)=\int_0^\infty e^{-\left(\rho+\lambda_1^{pl}\right)t}\left(W\left(Q_1\left(\alpha,\tilde{l}\right)\right)+\lambda_1^{pl}w_{t,2}^{pl}\left(\alpha_{t,2}\right)\right)dt. \tag{A.34}$$

Next note that, using Proposition 5, optimists' wealth share after transition, $\alpha_{t,2}$ (conditional on $\alpha_{t,1}$), is the same under policies $\tilde{l}$ and $Q_1\left(\cdot,\tilde{l}\right)$. Combining this observation with Eq. (23), we also find that the evolution of optimists' wealth share absent transition, $\dot{\alpha}_{t,1}/\alpha_{t,1}$, is the same under both policies. Consequently, optimists' wealth share follows an identical path under both policies. In view of this observation, after taking the difference of Eqs. $(A.34)$ and $(A.33)$, we obtain Eq. (37) in the main text.

Next note that Eq. $(A.27)$ implies (for a given $\alpha\in(0,1)$) that the prudential asset price level is differentiable in $\tilde{l}$ with a finite derivative. Note also that $Q_1\left(\alpha,l\right)=Q^*$. Therefore, taking the derivative of Eq. (37) with respect to $\tilde{l}$ and evaluating at $\tilde{l}=l^*$, we obtain:

$$\begin{aligned}\frac{dw_1^{pl}\left(\alpha,l,Q_1\left(\alpha,\tilde{l}\right)\right)}{d\tilde{l}}|_{\tilde{l}=l}-\frac{dw_1^{pl}\left(\alpha,\tilde{l}\right)}{d\tilde{l}}|_{\tilde{l}=l} &= \int_0^\infty e^{-\left(\rho+\lambda_1^{pl}\right)t}\frac{dW\left(Q^*\right)}{dQ_{t,1}}\frac{dQ_1\left(\alpha_{t,1},\tilde{l}\right)}{d\tilde{l}}|_{\tilde{l}=l}dt\\ &= 0.\end{aligned}$$

Here, the first line applies the chain rule and the second line uses the observation that $\frac{dW(Q^*)}{dQ_{t,1}}=0$ [cf. Eq. (34)]. Rearranging this expression establishes Eq. (36) and completes the proof. □

## A.5. Omitted results in Section 7

This section presents the results omitted from Section 7. In particular, we characterize the equilibrium in which a fraction of optimists are unconstrained. We establish Propositions 7 and 8 that generalize Propositions 2 and 4 to this extended setting. We relegate the proofs to the end of the section.

### A.5.1. Prudential policies with shadow banks

Recall that we assume a subset of optimists (*unconstrained optimists*) are not subject to the leverage constraint, $\omega_{1,t}\leq\overline{\omega}_{1,t}$. The remaining optimists (*constrained optimists*) are subject to the leverage constraint as before. We view the unconstrained optimists in this exercise as the model counterpart to "shadow banks" that circumvent the regulatory limit and also do not face non-regulatory limits (whereas the constrained optimists are the model counterpart to "banks"). We let $\beta\in(0,1)$ denote the relative fraction of optimists' wealth that is held by unconstrained optimists. Hence, the wealth share of unconstrained and constrained optimists is given by, respectively, $\alpha\beta$ and $\alpha\left(1-\beta\right)$. As before, the total wealth share of optimists (including both types) and pessimists is given by, respectively, $\alpha$ and $1-\alpha$. The rest of the model is unchanged.

To characterize the equilibrium, consider first the recession state $s=2$. Conditional on the total mass of optimists, $\alpha_2$, the equilibrium is the same as before. This is because we assume optimists face no constraints from state 2 onwards, which implies that constrained and unconstrained optimists are effectively the same from this point forward. In particular, the equilibrium price in the recession can be written as $Q_{t,2}=Q_2\left(\alpha_{t,2}\right)$, where $Q_2\left(\cdot\right)$ is the price function characterized in Proposition 1.

Next consider the equilibrium in the boom state, $s=1$. In this case, there are two state variables: the total mass of optimists, $\alpha\in(0,1)$, and the fraction of unconstrained optimists, $\beta\in(0,1)$. Therefore, we

denote the equilibrium variables as functions of two state variables, in addition to the leverage limit and PMP functions. In particular, $\alpha_2(\alpha,\beta,\overline{\omega}_1(\cdot),Q_1(\cdot))$ and $\beta_2(\alpha,\beta,\overline{\omega}_1(\cdot),Q_1(\cdot))$ denote, respectively, the total mass of optimists and the fraction of unconstrained optimists that obtain if there is an instantaneous transition to recession. To simplify the notation, we suppress the dependence of these functions on some or all of their arguments as long as the appropriate arguments are clear from the context.

Much of our earlier analysis applies in this setting. In particular, Eq. (23), which characterizes the growth rate of agents' wealth shares absent a state transition, applies for all agents. Applying this equation for constrained and unconstrained optimists (in state $s=1$), we obtain:

$$\begin{aligned}\frac{d(\alpha(1-\beta))/dt}{\alpha(1-\beta)} &= \lambda_1^p\frac{1-\alpha}{1-\alpha_2}\left(1-\frac{\alpha_2(1-\beta_2)}{\alpha(1-\beta)}\right)\\ \frac{d(\alpha\beta)/dt}{\alpha\beta} &= \lambda_1^p\frac{1-\alpha}{1-\alpha_2}\left(1-\frac{\alpha_2\beta_2}{\alpha\beta}\right)\end{aligned} \tag{A.35}$$

Solving these equations for $\dot{\alpha}$ and $\dot{\beta}$, we characterize the wealth dynamics as follows:

$$\begin{aligned}\frac{\dot{\alpha}}{\alpha} &= \lambda_1^p\frac{1-\alpha}{1-\alpha_2}\left(1-\frac{\alpha_2}{\alpha}\right),\\ \frac{\dot{\beta}}{\beta} &= \lambda_1^p\frac{1-\alpha}{1-\alpha_2}\frac{\alpha_2}{\alpha}\left(1-\frac{\beta_2}{\beta}\right).\end{aligned} \tag{A.36}$$

Given $\alpha_2$, optimists' total wealth share follows the same equation as before (cf. Eq. $(A.19)$). Given $\beta_2$ and $\alpha_2$, the relative wealth share of unconstrained optimists follows a similar equation. Below, we will verify that the equilibrium features $\alpha_2<\alpha$ and $\beta_2<\beta$. Combining this observation with $(A.36)$ implies $\dot{\alpha}>0$ and $\dot{\beta}>0$. Optimists' total wealth share (resp. unconstrained optimists' relative wealth share) grows absent transition to recession, because these agents take on greater risk and earn higher returns compared to pessimists (resp. constrained optimists).

It remains to characterize the functions $\alpha_2$ and $\beta_2$. To this end, note that the portfolio optimality condition (21) holds as equality for unconstrained optimists and as a weak inequality for constrained optimists. Combining these observations, we obtain:

$$\lambda_1^o\frac{\alpha(1-\beta)}{\alpha_2(1-\beta_2)}\geq\lambda_1^o\frac{\alpha\beta}{\alpha_2\beta_2}=\lambda_1^p\frac{1-\alpha}{1-\alpha_2}. \tag{A.37}$$

Note also that Eq. (20), which relates agents' wealth shares after transition to their leverage ratio, applies for all agents. Using this condition for constrained and unconstrained optimists, we obtain:

$$\frac{\alpha_2(1-\beta_2)}{\alpha(1-\beta)} = 1-(1-\omega_1^{o,reg})\left(\frac{Q_1}{Q_2(\alpha_2)}-1\right) \tag{A.38}$$

$$\frac{\alpha_2\beta_2}{\alpha\beta} = 1-(1-\omega_1^{o,unreg})\left(\frac{Q_1}{Q_2(\alpha_2)}-1\right). \tag{A.39}$$

Given current $\alpha$, the current price level $Q_1$ and the price function after transition $Q_2(\cdot)$, the equilibrium levels of $\alpha_2,\beta_2$ (as well as those for $\omega_1^{o,reg},\omega_1^{o,unreg}$) can be characterized by solving Eqs. $(A.37-A.39)$.

Consider the case in which constrained optimists' leverage constraint binds (the other case is the same as in the previous section without a leverage limit). In this case, we have $\omega_1^{o,reg}=\overline{\omega}_1$. Substituting

this into Eq. $(A.38)$, we obtain:

$$\frac{\alpha_2\left(1-\beta_2\right)}{\alpha\left(1-\beta\right)}=1-\left(\overline{\omega}_1-1\right)\left[\frac{Q_1}{Q_2\left(\alpha_2\right)}-1\right]. \tag{A.40}$$

As before, this expression describes constrained optimists' relative wealth share as a function of the leverage limit and the price drop after transition. Solving for $\beta_2$ from Eq. $(A.37)$, and substituting into Eq. $(A.40)$, we further obtain:

$$\frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha}-\frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right)=1-\left(\overline{\omega}_1-1\right)\left[\frac{Q_1}{Q_2\left(\alpha_2\right)}-1\right]. \tag{A.41}$$

This equation generalizes Eq. (30) (which we analyzed extensively in previous sections) to cases with $\beta>0$. In particular, the equation characterizes $\alpha_2$ given $Q_1, Q_2\left(\alpha_2\right)$ and $\overline{\omega}_1$.

Note also that the left-hand side of Eq. $(A.41)$ is an increasing function of $\alpha_2$. Hence, as before the equation can be visualized as the intersection of two increasing relations between $\alpha_2$ and $Q_2$.

Our next result summarizes this characterization for the benchmark case without PMP, $Q_1=Q^*$, and establishes the comparative statics with respect to $\beta$. Importantly, the result also establishes the comparative statics with respect to the leverage limit $\overline{\omega}_1$ and generalizes our earlier result about macroprudential policy (Proposition 2) to this setting.

**Proposition 7.** *Suppose Assumptions A1-A3 hold and that a fraction, $\beta\in(0,1)$, of optimists' wealth is held by unconstrained optimists that face no leverage limits. Consider the benchmark equilibrium without PMP, $Q_1(\alpha)=Q^*$. Fix levels $\alpha,\beta\in(0,1)$ that are associated with some binding leverage limit, $\overline{\omega}_1(\alpha,\beta)<\omega_1^{o,reg}(\alpha,\beta,\infty)$. Absent transition to recession, $\alpha$ and $\beta$ follow the dynamics in $(A.36)$. After transition, $\alpha_2$ is characterized as the solution to Eq. $(A.41)$ and $\beta_2$ is characterized as the solution to $(A.37)$. In equilibrium, $\alpha_2<\alpha,\beta_2<\beta$ and $\dot{\alpha}>0,\dot{\beta}>0$: optimists' total wealth share and unconstrained optimists' relative wealth share shrink after transition to recession and grow absent transition. Moreover, $\alpha_2$ satisfies the following comparative statics:*

*(i) Increasing the relative wealth share of unconstrained optimists, $\beta$, decreases optimists' wealth share after transition, $\frac{d\alpha_2(\alpha,\beta,\overline{\omega}_1(\cdot))}{d\beta}<0$. In the limit as $\beta\to1$, optimists' wealth share approaches its level in the equilibrium without leverage limits, $\alpha_2(\alpha,\infty)$.*

*(ii) Macroprudential policy that decreases the leverage limit increases optimists' wealth share after a transition to recession, $\frac{d\alpha_2(\alpha,\beta,\overline{\omega}_1(\cdot))}{d\overline{\omega}_1(\alpha,\beta)}<0$. Increasing the relative wealth share of unconstrained optimists, $\beta$, reduces the effectiveness of macroprudential policy, $\frac{\partial}{\partial\beta}\frac{d\alpha_2(\alpha,\beta,\overline{\omega}_1(\cdot))}{d\overline{\omega}_1(\alpha,\beta)}>0$.*

This result verifies the conventional wisdom that the presence of less constrained agents reduces the strength of macroprudential policy. The first part shows that, as the relative wealth share of unconstrained optimists grows, optimists take on greater risk and their wealth share declines by a greater magnitude after transition to recession. The second part shows that (as long as some optimists are constrained, $\beta<1$) macroprudential policy that tightens leverage limits mitigates the decline in optimists' wealth share but less so than in the earlier setting without unconstrained optimists.

Next consider PMP that lowers the current asset price level, $Q_1(\alpha,\beta)\leq Q^*$. As illustrated by Eq. $(A.40)$, PMP reduces constrained optimists' exposure to recession. As illustrated by Eq. $(A.41)$, this in turn increases the wealth share of optimists after transition to recession, $\alpha_2$. Eq. $(A.41)$ further suggests that, as before, PMP affects the equilibrium in much the same way as a decrease in $\overline{\omega}_1$. Our next

result verifies this intuition. Specifically, we generalize our main result showing that monetary policy can replicate the prudential effects of directly tightening a leverage limit (cf. Proposition 4).

**Proposition 8.** *Suppose Assumptions A1-A3 hold and that a fraction, $\beta \in (0,1)$, of optimists' wealth is held by unconstrained optimists that face no leverage limits. Fix some $\alpha, \beta \in (0,1)$ and consider the setup of Proposition 4. In particular, consider an alternative leverage limit $\tilde{\omega}_1(\cdot)$ that agrees with $\overline{\omega}_1(\cdot)$ everywhere except for $(\alpha, \beta)$ and that satisfies $\tilde{\omega}_1(\alpha,\beta) < \min\left(\overline{\omega}_1(\alpha,\beta), \omega_1^{o,reg}(\alpha,\beta\infty)\right)$. Then:*

*(i) There exists $\tilde{Q}_1(\alpha,\beta) < Q^*$ such that the PMP (with the original leverage limit) generates the same effect on constrained and unconstrained optimists' wealth shares after transition as the alternative leverage limit (without PMP):*

$$\alpha_2\left(\alpha,\beta,\overline{\omega}_1,\tilde{Q}_1\right) = \alpha_2(\alpha,\beta,\tilde{\omega}_1) \text{ and } \beta_2\left(\alpha,\beta,\overline{\omega}_1,\tilde{Q}_1\right) = \beta_2(\alpha,\beta,\tilde{\omega}_1).$$

*Targeting a lower effective limit requires targeting a lower asset price, $\frac{\partial \tilde{Q}_1(\alpha,\beta)}{\partial \tilde{\omega}_1(\alpha,\beta)} > 0$.*

*(ii) PMP requires setting a higher interest rate than the benchmark without policy:*

$$r_1^f\left(\alpha,\beta,\overline{\omega}_1,\tilde{Q}_1\right) > r_1^f(\alpha,\beta,\overline{\omega}_1).$$

*Targeting a lower effective limit requires setting a higher interest rate, $\frac{\partial r_1^f\left(\alpha,\beta,\overline{\omega}_1,\tilde{Q}_1\right)}{\partial \tilde{\omega}_1(\alpha,\beta)} < 0$.*

This result follows from the same steps as in Proposition 4. In particular, the monetary authority can choose $\tilde{Q}_1$ so that optimists' total wealth share and the equilibrium price in the recession settle at the same level as if the regulator had directly tightened the leverage limit. In fact, conditional on optimists' wealth share $\alpha_2$, the replicating $\tilde{Q}_1$ that the planner needs to set is characterized as the solution to the same equation $(A.23)$ as in our earlier analysis.

### A.5.2. Omitted derivations

We first state and prove a generalization of Lemma 5, which implies that Eq. $(A.41)$ has a unique solution (when $Q_1 = Q^*$). We then prove Propositions 7 and 8.

**Lemma 6.** *Consider the following function:*

$$f(\alpha_2;\alpha,\beta,\overline{\omega}_1) = 1 - (\overline{\omega}_1 - 1)\left[\frac{Q^*}{Q_2(\alpha_2)} - 1\right] - \frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha} - \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right),$$

*where $\alpha, \beta, \overline{\omega}_1$ are parameters such that $\alpha, \beta \in (0,1), \overline{\omega}_1 \leq \omega_1^o(\alpha,\beta,\infty)$. Under Assumption A3, $f(\alpha_2) = 0$ has a unique solution that satisfies $\alpha_2 \in (\alpha_2(\alpha,\infty), \alpha)$.*

**Proof.** Following similar steps as in Lemma 5, it is easy to check that $f(\alpha_2(\alpha,\infty)) > 0$ and $f(\alpha) < 0$. This establishes that there exists a solution that lies in the desired interval, $\alpha_2 \in (\alpha_2(\alpha,\infty), \alpha)$.

We next show that the derivative of $f$ is strictly negative at each zero of $f$, that is:

$$f'(\alpha_2) < 0 \text{ for each } \alpha_2 \in (\alpha_2(\alpha,\infty), \alpha) \text{ and } f(\alpha_2) = 0.$$

This establishes that $f$ has a unique zero in the desired interval. To prove the claim, we first evaluate the derivative

$$f'(\alpha_2) = (\overline{\omega}_1 - 1)\frac{Q^*}{(Q_2(\alpha_2))^2}Q_2'(\alpha_2) - \frac{1}{1-\beta}\left(\frac{1}{\alpha} + \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1}{1-\alpha}\right).$$

Hence, $f'(\alpha_2) < 0$ as long as

$$\frac{\overline{\omega}_1 - 1}{Q_2(\alpha_2)} \frac{Q^*}{Q_2(\alpha_2)} Q_2'(\alpha_2) < \frac{1}{1-\beta}\left(\frac{1}{\alpha} + \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1}{1-\alpha}\right).$$

Note that we require this to hold when $f(\alpha_2) = 0$. This implies:

$$\frac{\overline{\omega}_1 - 1}{Q_2(\alpha_2)} = \frac{1}{Q^* - Q_2(\alpha_2)}\left(1 - \frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha} - \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right)\right).$$

Combining the last two displayed equations, we need to show

$$\begin{aligned} Q_2'(\alpha_2) &< \frac{Q^* - Q_2(\alpha_2)}{1-\alpha_2}\frac{Q_2(\alpha_2)}{Q^*} g(\alpha_2, \alpha, \beta) \\ \text{where } g(\alpha_2, \alpha, \beta) &= \frac{(1-\alpha_2)\frac{1}{1-\beta}\left(\frac{1}{\alpha} + \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1}{1-\alpha}\right)}{1 - \frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha} - \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right)}. \end{aligned}$$

Note that, in the proof of Lemma 5, we already established this inequality for $\beta = 0$ (under Assumption A3). Hence, it suffices to show that $g(\alpha_2, \alpha, \beta) \geq g(\alpha_2, \alpha, 0)$. This inequality holds because,

$$g(\alpha_2, \alpha, \beta) > \frac{1-\alpha_2}{1 - \frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha} - \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right)} > \frac{1-\alpha_2}{1-\frac{\alpha_2}{\alpha}} = g(\alpha_2, \alpha, 0).$$

Here, the first inequality follows because $\frac{1}{1-\beta}\left(\frac{1}{\alpha} + \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1}{1-\alpha}\right) < \frac{1}{\alpha}$. The second inequality follows because $\alpha_2 > \alpha_2(\alpha, \infty) = \frac{\alpha\lambda_1^o}{\alpha\lambda_1^o + (1-\alpha)\lambda_1^p}$ implies $\frac{\alpha_2}{1-\alpha_2} > \frac{\lambda_1^o}{\lambda_1^p}\frac{\alpha}{1-\alpha}$, which in turn implies $\frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha} - \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right) > \frac{\alpha_2}{\alpha}$. This establishes the claim and completes the proof of the lemma. □

**Proof of Proposition 7.** First consider the characterization of $\alpha_2$. Earlier, we established that Eq. (*A*.41) characterizes $\alpha_2$. Lemma 6 implies that there exists a unique solution that satisfies $\alpha_2 \in (\alpha_2(\alpha, \infty), \alpha)$. Combining this with Eq. (*A*.36) also implies $\dot{\alpha} > 0$.

Next note that Eq. (*A*.37) characterizes $\beta_2$ conditional on $\alpha_2$. Note also that $\alpha_2 > \alpha_2(\alpha, \infty) = \frac{\alpha\lambda_1^o}{\alpha\lambda_1^o + (1-\alpha)\lambda_1^p}$ implies $\frac{\alpha_2}{1-\alpha_2} > \frac{\lambda_1^o}{\lambda_1^p}\frac{\alpha}{1-\alpha}$. Combining this with Eq. (*A*.37), we obtain $\frac{\beta_2}{\beta} = \frac{\lambda_1^o}{\lambda_1^p}\frac{\alpha/(1-\alpha)}{\alpha_2/(1-\alpha_2)} < 1$. This proves $\beta_2 < \beta$. Combining this with Eq. (*A*.36) also implies $\dot{\beta} > 0$.

Next consider the comparative statics of $\alpha_2$ with respect to $\beta$. Recall that $\alpha_2$ is characterized as the unique solution to the equation, $f(\alpha_2; \alpha, \beta, \overline{\omega}_1) = 0$, where $f(\cdot)$ is defined in Lemma 6. Implicitly differentiating the equation with respect to $\beta$, we obtain:

$$\frac{d\alpha_2}{d\beta} = \frac{\partial f(\alpha_2; \alpha, \beta, \overline{\omega}_1)/\partial\beta}{-\partial f(\alpha_2; \alpha, \beta, \overline{\omega}_1)/\partial\alpha_2},$$

where the derivatives are evaluated at the solution. From the proof of Lemma 6, we also know that $f'(\alpha_2; \alpha, \beta, \overline{\omega}_1) < 0$ when $f(\alpha_2) = 0$. Hence, $\frac{d\alpha_2}{d\beta} < 0$ as long as $\partial f(\alpha_2; \alpha, \beta, \overline{\omega}_1)/\partial\beta < 0$. The latter

inequality holds because:

$$\begin{aligned}\frac{\partial f\left(\alpha_2;\alpha,\beta,\overline{\omega}_1\right)}{\partial\beta} &= -\frac{\partial}{\partial\beta}\left(\frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha}-\frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right)\right)\\ &= -\frac{\partial}{\partial\beta}\left(\frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha}-\frac{\lambda_1^o}{\lambda_1^p}\frac{1-\alpha_2}{1-\alpha}\right)+\frac{\lambda_1^o}{\lambda_1^p}\frac{1-\alpha_2}{1-\alpha}\right)\\ &= -\left(\frac{\alpha_2}{\alpha}-\frac{\lambda_1^o}{\lambda_1^p}\frac{1-\alpha_2}{1-\alpha}\right)\left(\frac{\partial}{\partial\beta}\frac{1}{1-\beta}\right)<0.\end{aligned}$$

Here, the last inequality follows since $\frac{\alpha_2}{\alpha}>\frac{\lambda_1^o}{\lambda_1^p}\frac{1-\alpha_2}{1-\alpha}$ (since $\alpha_2>\alpha_2\left(\alpha,\infty\right)$) and $\frac{\partial}{\partial\beta}\frac{1}{1-\beta}>0$. This proves $\frac{d\alpha_2}{d\beta}<0$.

Next consider the limit as $\beta\to 1$. For any $\alpha_2>\alpha_2\left(\alpha,\infty\right)$, we have

$$\begin{aligned}\lim_{\beta\to 1} f\left(\alpha_2;\alpha,\beta,\overline{\omega}_1\right) &= \lim_{\beta\to 1}\left[\begin{array}{c}1-\left(\overline{\omega}_1-1\right)\left[\frac{Q^*}{Q_2(\alpha_2)}-1\right]+\frac{\lambda_1^o}{\lambda_1^p}\frac{1-\alpha_2}{1-\alpha}\\ -\frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha}-\frac{\lambda_1^o}{\lambda_1^p}\frac{1-\alpha_2}{1-\alpha}\right)\end{array}\right]\\ &= -\infty\end{aligned}$$

Here, the last line follows because $\frac{\alpha_2}{\alpha}>\frac{\lambda_1^o}{\lambda_1^p}\frac{1-\alpha_2}{1-\alpha}$ and $\lim_{\beta\to 1}\frac{-1}{1-\beta}=-\infty$. This also implies $\lim_{\beta\to 1}\alpha_2=\alpha_2\left(\alpha,\infty\right)$ because $\alpha_2$ is characterized as the unique solution to the equation, $f\left(\alpha_2;\alpha,\beta,\overline{\omega}_1\right)=0$, over the range $\alpha_2\in\left(\alpha_2\left(\alpha,\infty\right),\alpha\right)$.

Next consider the comparative statics with respect to the leverage limit, $\overline{\omega}_1=\overline{\omega}_1\left(\alpha,\beta\right)$. Following similar steps, we obtain:

$$\begin{aligned}\frac{d\alpha_2}{d\overline{\omega}_1} &= \frac{-\partial f\left(\alpha_2;\alpha,\beta,\overline{\omega}_1\right)/\partial\overline{\omega}_1}{\partial f\left(\alpha_2;\alpha,\beta,\overline{\omega}_1\right)/\partial\alpha_2}\\ &= \frac{\frac{Q^*}{Q_2(\alpha_2)}-1}{\left(\overline{\omega}_1-1\right)\frac{Q^*}{\left(Q_2(\alpha_2)\right)^2}Q_2'\left(\alpha_2\right)-\frac{1}{1-\beta}\left(\frac{1}{\alpha}+\frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1}{1-\alpha}\right)}<0.\end{aligned}$$

Here, the first equality evaluates the partial derivatives and the second inequality uses $\partial f\left(\alpha_2;\alpha,\beta,\overline{\omega}_1\right)/\partial\alpha_2<0$.

Finally, consider the sign of the cross-partial derivative $\frac{\partial}{\partial\beta}\frac{d\alpha_2}{d\overline{\omega}_1}$. We have

$$\begin{aligned}sign\left(\frac{\partial}{\partial\beta}\frac{d\alpha_2}{d\overline{\omega}_1}\right) &= sign\left(\frac{\partial}{\partial\beta}\frac{1}{1-\beta}\left(\frac{1}{\alpha}+\frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1}{1-\alpha}\right)\right)\\ &= sign\left(\frac{\partial}{\partial\beta}\frac{1}{1-\beta}\left(\frac{1}{\alpha}+\frac{\lambda_1^o}{\lambda_1^p}\frac{1}{1-\alpha}\right)-\frac{\lambda_1^o}{\lambda_1^p}\frac{1}{1-\alpha}\right)\\ &= sign\left(\frac{\partial}{\partial\beta}\frac{1}{1-\beta}\left(\frac{1}{\alpha}+\frac{\lambda_1^o}{\lambda_1^p}\frac{1}{1-\alpha}\right)\right)>0.\end{aligned}$$

This proves $\frac{\partial}{\partial\beta}\frac{d\alpha_2}{d\overline{\omega}_1}>0$ and completes the proof. □

**Proof of Proposition 8.** The proof follows similar steps to Proposition 4. Using Eq. *A*.41, $\alpha_2$ corresponding to the alternative leverage limit is characterized as the unique solution to:

$$\frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha}-\frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right)=1-\left(\tilde{\omega}_1-1\right)\left[\frac{Q^*}{Q_2\left(\alpha_2\right)}-1\right]. \tag{A.42}$$

Likewise, $\alpha_2$ corresponding to the PMP (with the current leverage limit) is characterized as the solution to:

$$\frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha}-\frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right)=1-(\overline{\omega}_1-1)\left[\frac{\tilde{Q}_1}{Q_2(\alpha_2)}-1\right]. \tag{A.43}$$

Next note that the proof of Proposition 4 establishes that there is a unique level of $\tilde{Q}_1$ that ensures Eq. $(A.23)$ holds. Let $\tilde{Q}_1$ denote this level, that is:

$$(\overline{\omega}_1-1)\left[\frac{\tilde{Q}_1}{Q_2(\alpha_2)}-1\right]=(\tilde{\omega}_1-1)\left[\frac{Q^*}{Q_2(\alpha_2)}-1\right]. \tag{A.44}$$

Substituting $\tilde{Q}_1$ into Eq. $(A.43)$ ensures that this equation is the same as Eq. $(A.42)$. Therefore, the solutions are the same. Hence, there exists a PMP that replicates $\alpha_2$ that results from the alternative leverage limit. Recall also that $\beta_2$ is characterized by Eq. $(A.37)$ conditional on $\alpha_2$. Thus, the same PMP also replicates $\beta_2$ that results from the alternative leverage limit. Note also that Eq. $(A.44)$ implies $\frac{\partial\tilde{Q}_1}{\partial\tilde{\omega}_1}>0$. This completes the proof of the first part.

Next consider the interest rate corresponding to PMP. Note that Eq. (21) continues to hold as equality for pessimists. This implies that the interest rate is given by:

$$\tilde{r}_1^f=\rho+g_1-\delta\left(\frac{\tilde{Q}_1}{Q^*}\eta^*\right)-\lambda_1^p\frac{1-\alpha}{1-\alpha_2}\left(\frac{\tilde{Q}_1}{Q_2(\alpha_2)}-1\right).$$

Using Eq. $(A.38)$ (that describes the wealth share after transition for constrained optimists) to substitute for the price decline, we obtain:

$$\tilde{r}_1^f=\rho+g_1-\delta\left(\frac{\tilde{Q}_1}{Q^*}\eta^*\right)-\lambda_1^p\frac{1-\alpha}{1-\alpha_2}\left(1-\frac{\alpha_2(1-\beta_2)}{\alpha(1-\beta)}\right)\frac{1}{\overline{\omega}_1-1}. \tag{A.45}$$

For the benchmark without any prudential policy, following similar steps we obtain:

$$r_1^{f,b}=\rho+g_1-\delta(\eta^*)-\lambda_1^p\frac{1-\alpha}{1-\alpha_2^b}\left(1-\frac{\alpha_2^b\left(1-\beta_2^b\right)}{\alpha(1-\beta)}\right)\frac{1}{\omega_1^b-1}. \tag{A.46}$$

Here, $\alpha_2^b,\beta_2^b,\omega_1^b$ denote the equilibrium variables in the benchmark, which are potentially different than the equilibrium with PMP. In particular, recall from Proposition 7 that $\alpha_2>\alpha_2^b$. Combining this with Eq. $(A.37)$, we further obtain $\beta_2<\beta_2^b$. PMP decreases the fraction of optimists' wealth held by unconstrained optimists, because they react to the policy by increasing their risks more than constrained optimists.

Next note that the wealth shares satisfy the following identity:

$$\begin{aligned}\frac{1-\alpha}{1-\alpha_2}\left(1-\frac{\alpha_2(1-\beta_2)}{\alpha(1-\beta)}\right) &= (1-\alpha)\frac{1-\frac{\alpha_2(1-\beta_2)}{\alpha(1-\beta)}}{1-\alpha_2}\\ &= (1-\alpha)\left(1-\frac{\alpha_2}{1-\alpha_2}\left[\frac{1-\beta_2}{\alpha(1-\beta)}-1\right]\right).\end{aligned} \tag{A.47}$$

Here, the term in the brackets is positive because $\beta_2<\beta$. This identity holds for the pair, $(\alpha_2,\beta_2)$, as well as the pair, $\left(\alpha_2^b,\beta_2^b\right)$. Combining the identity with the inequalities, $\alpha_2>\alpha_2^b$ and $\beta_2<\beta_2^b$ (which

implies $1-\beta_2 > 1-\beta_2^b$), we further obtain:

$$\frac{1-\alpha}{1-\alpha_2}\left(1-\frac{\alpha_2\left(1-\beta_2\right)}{\alpha\left(1-\beta\right)}\right) < \frac{1-\alpha}{1-\alpha_2^b}\left(1-\frac{\alpha_2^b\left(1-\beta_2^b\right)}{\alpha\left(1-\beta\right)}\right).$$

Note also that $\frac{1}{\tilde{\omega}_1-1} \leq \frac{1}{\omega_1^b-1}$ since $\omega_1^b \leq \overline{\omega}_1$. Finally, we also have $\delta\left(\frac{\tilde{Q}_1}{Q^*}\eta^*\right) < \delta\left(\eta^*\right)$ since $\tilde{Q}_1 < Q^*$. Combining these inequalities with Eqs. $(A.45)$ and $(A.46)$ proves that $\tilde{r}_1^f > r_1^{f,b}$: that is, PMP sets a higher interest rate than in the benchmark without prudential policies.

Finally, consider how raising the target leverage limit $\tilde{\omega}_1$ affects the interest rate corresponding to PMP. Since raising the leverage limit increases $\tilde{Q}_1$, it also increases the effective depreciation rate, $\delta\left(\frac{\tilde{Q}_1}{Q^*}\eta^*\right)$. Since raising the leverage limit reduces $\alpha_2$, it also increases $\beta_2$ (and reduces $1-\beta_2$). Combining this with the identity in $(A.47)$ implies that raising the leverage limit raises the term, $\frac{1-\alpha}{1-\alpha_2}\left(1-\frac{\alpha_2(1-\beta_2)}{\alpha(1-\beta)}\right)$. Combining these observations with $(A.45)$ proves that raising the target leverage limit decreases the interest rate, that is: $\frac{\partial \tilde{r}_1^f}{\partial \tilde{\omega}_1} < 0$. In particular, targeting a lower effective leverage limit $\tilde{\omega}_1$ requires setting a higher interest rate, completing the proof. □

## A.6. Details of the numerical exercise

For depreciation, we use the functional form

$$\delta\left(\eta\right) = \underline{\delta} + \left(\overline{\delta}-\underline{\delta}\right)\frac{\left(\eta-\underline{\eta}\right)^{1+1/\varepsilon}}{1+1/\varepsilon} \text{ for } \eta \geq \underline{\eta} \tag{A.48}$$

(and $\delta\left(\eta\right) = \underline{\delta}$ for $\eta < \underline{\eta}$) given some constants $\underline{\delta}, (\overline{\delta}-\underline{\delta}), \underline{\eta}, \varepsilon > 0$. This functional form implies that decreasing factor utilization below the efficient level, $\eta^*$, reduces the depreciation rate until $\underline{\eta} < \eta^*$, but it has no effect on depreciation beyond this level. Raising factor utilization above the efficient level increases capital depreciation.

In our numerical examples, we set

$$\underline{\eta} = 0.97,\ \underline{\delta} = 0.04, \overline{\delta} = 0.087, \varepsilon = 20.$$

These choices ensure that the efficient factor utilization and the corresponding depreciation rate are given by [cf. Eq. (16)] with

$$\eta^* = 1 \text{ and } \delta\left(\eta^*\right) = 0.041.$$

In particular, we normalize the efficient factor utilization to one. The choice of $\underline{\eta} = 0.97$ (together with a relatively high elasticity, $\varepsilon = 20$) implies that underutilizing capital by up to 3 percent is not too costly, since it is compensated by a relatively large decline in depreciation. Underutilizing capital beyond this level is much costlier as there is no compensation in terms of reduced depreciation. In our examples, this means that underutilizing capital in the recession is much costlier than underutilizing capital during the boom.

For the discount rate, we set

$$\rho = 0.04.$$

This choice (together with the specification for the depreciation function) ensures that Assumption 2

holds. For the productivity level, we set $A = 1$. This does not play a role as it scales all variables. These choices imply that the efficient asset price level is given by [cf. Eq. (17)]:

$$Q^* = \frac{A\eta^*}{\rho} = 25.$$

For the productivity *growth* rates, we set

$$\begin{aligned} g_3 = g_1 &= 0.1 - (\rho - \delta(\eta^*)) = 0.101 \\ g_2 &= -0.05 - (\rho - \delta(\eta^*)) = -0.049. \end{aligned}$$

These choices satisfy $g_2 < \min(g_1, g_3)$. They also imply that, with no state changes or belief disagreements and if capital were perfectly utilized, then the (risk-adjusted) return to capital would be equal to 10% in the boom and the recovery states and -5% in the recession state [cf. (15)]. In particular, the transition from the boom to the recession represents a 15% shock to asset valuations.

For beliefs, we set

$$\begin{aligned} \lambda_1^o &= 0.09 \text{ and } \lambda_1^p = 0.9 \\ \lambda_2^o &= 4.97 \text{ and } \lambda_2^p = 0.49 \end{aligned}$$

(and $\lambda_3^o = \lambda_3^p = 0$). These beliefs satisfy Assumption 1: compared to pessimists, optimists assign a smaller probability to a transition from boom to recession but a greater probability to a transition from recession to recovery. When combined with the remaining parameters, these values satisfy Assumptions A1-A2, the regularity conditions we need in order to obtain an equilibrium with a positive interest rate in the boom state and a zero interest rate (and suboptimal asset price level) in the recession state.

Recall that we also need Assumption A3 (which is a regularity condition) to ensure that there is a unique equilibrium when optimists' leverage limit binds (cf. Appendix A.3.1). This condition depends on the equilibrium price function in the recession, $Q_2(\alpha)$. Figure 3 plots the price function corresponding to the parameters described above. We verify that, when combined with the remaining parameters, this function satisfies Assumption A3.