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DEBT-MATURITY MANAGEMENT WITH LIQUIDITY COSTS

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ABSTRACT

We characterize the optimal debt-maturity management problem in the presence of liquidity costs. A government issues an arbitrary number of finite-maturity bonds and faces income and interest-rate risk, which can tempt it to default. Optimal issuances are spread out across maturities and are given by the ratio of a value gap over a liquidity coefficient that measures the price impact. The value gap is the proportional difference between the bond prices obtained by discounting with the international interest rates and with the domestic discount factor. This characterization allows us to quantify the contribution of different economic forces—impatience, yield-curve riding, expenditure smoothing, self-insurance, credit risk, and default incentives—in shaping the optimal debt maturity distribution. In an application, we estimate the liquidity coefficients using Spanish debt auction data and exploit the framework to evaluate Spanish debt management practices.

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1 Introduction

Any government faces a large-stakes problem: to design a strategy for the quantity and maturity of its debt. This paper presents a new framework to think about and evaluate that design. We study the optimal debt-maturity management problem of a government subject to income and interest-rate risk, risks that can lead to a default. The framework makes two innovations. First, it puts forth liquidity costs as a central consideration, the notion that the greater a bond auction is, the lower the auction price. Second, we develop an approach to characterize the optimal debt program allowing for an arbitrary number of finite-life bonds. With this approach, we can analytically and quantitatively decompose the different economic forces that influence an optimal debt-management strategy.

Liquidity costs are a common concern to practitioners, but to a large extent, their role has been absent in normative analysis. Here, we embrace the view that liquidity costs are central to the debt management problem. We build on Duffie, Garleanu and Pedersen (2005) and feature a government that auctions bonds to primary dealers who take time to resell bonds to their ultimate holders, investors. The greater the issuance, the longer the resell time. As in Vayanos and Vila (2021), we feature bond markets segmented by maturity and dealers who face higher capital costs than investors. As a result, the time to redeploy bonds is costly and produces a markup between the issuer government's auction price and secondary-bond market prices. Since the resell time increases with the auction size, the auction price falls with the auction size. The sensitivities of auction prices to the auctioned amounts at given maturities are liquidity coefficients that we can measure with auction data.

Before studying the optimal debt management problem, we estimate the liquidity coefficients in a medium-sized economy, Spain. Over the last 20 years, Spain has issued recurrently in only ten maturities. We estimate the *liquidity coefficients* at each of those maturities with data covering more than 2000 bond auctions. Depending on the maturity we consider, a one pp increase in monthly issuances (over annual GDP) is associated with a reduction in auction prices between 8 and 56bps.

We then study the optimal debt management problem of a government that internalizes liquidity costs. In the presence of liquidity costs, the number and types of bonds that the government can issue are not innocuous. For that reason, we allow the government to issue an arbitrary number of finite-life bonds, an exercise hitherto not carried out. Due to the curse of dimensionality, qualitative analysis is often relegated to highly stylized models, and quantitative models only allow for a small number of decaying perpetuities (as in Leland and Toft, 1996).[1] In practice, governments simultaneously issue in many maturities, and perpetuities are a rarity. The approach here makes restrictive assumptions on shocks, but allows for an analytic characterization and a quantitative evaluation of optimal debt management with a realistic debt structure, thus complementing earlier work.

[1] This limitation is easily understood. If we want to construct a yearly model where the government only issues 30-year bonds, we need at least 30 state variables because a 30-year bond becomes a 29-year bond the following year, a 28-year bond the year after, and so on. By contrast, a bond that matures by 5-per cent every year is still a bond that matures by 5-per cent the year after its issuance.

Our analysis uncovers a general principle. The government's problem can be studied as if multiple artificial traders were in charge of issuing debt of a corresponding maturity. Each trader must apply a simple rule:

$$\frac{\text{issuance at maturity } \tau}{\text{GDP}} = \frac{1}{\text{liquidity coefficient at maturity } \tau} \times \text{value gap at maturity } \tau \,.$$

This rule follows from a condition that equates marginal auction revenues to an internal debt valuation. The rule states that optimal issuances of a bond of maturity τ equal the ratio of a value gap over the liquidity coefficient of that maturity. The value gap is the proportional difference between the secondary-market price and a domestic valuation. With liquidity costs, the government's discount rate does not equal the international short-term rate. Thus, the domestic valuation is a counterfactual bond price, computed using the government's discount rate instead of the short-term rate. A positive value gap indicates the desire to arbitrage the difference between market prices and domestic valuations by the artificial traders. The liquidity coefficient modulates the willingness to arbitrage in a given maturity. As a result of this limited arbitrage, it is optimal to issue in all maturities at all times, as commonly done by Treasuries. Since the domestic discount factor rate must be internally consistent at an optimum, a single equilibrium variable summarizes a problem with infinite control variables.[2]

The simple issuance rule holds, regardless of whether the government faces income or interest-rate risk, or has the option to default. These factors impact the bond valuations and prices through their effects on the interest rate and the domestic discount rates. This representation is convenient because it makes it possible to dissect the forces that shape the optimal debt-maturity profile through their impact on these rates. We characterize how these forces affect the level and weighted average maturity (WAM) of debt issuance through the elasticities of domestic valuations and bond prices with respect to the parameters associated with each force.

To flesh out how different forces contribute to the optimal maturity choice, we study the government's problem in steps. In the first step, we abstract from risk. In this case, the domestic rate only depends on the government's impatience relative to international investors, which implies an analytic expression for the optimal steady-state debt profile. We show that, when liquidity costs are homogeneous, the government should issue in all maturities at the steady state, but with a pattern that increases with maturity.[3] The government's greater impatience compared to the market is key to deliver this result. Extending the maturity by one period delays the principal by one period. If the government were as patient as international investors, this delay would not bring any benefit. However, since the government is more impatient, the delay brings it a benefit. Thus, the government always prefers to issue at long maturities but spreads out its issuances to mitigate liquidity costs. We introduce a proof, based on the analysis of elasticities, showing

[2]The domestic discount rate is the solution of a fixed-point problem: A conjectured expenditure path maps to a domestic discount rate. This discount rate generates an issuance path via the optimal rule. Ultimately, the issuance path must be consistent with the expenditure path's debt services. The conjectured expenditure path must coincide with the actual expenditure path obtained applying the issuance rule at the optimum.

[3]Although issuances increase with maturity, the outstanding debt stock decreases with maturity, as in the data. This decreasing maturity profile is an artifact of bonds having a finite life.

that as relative impatience increases issuances increase as well at all maturities, but the average maturity is reduced.

Understanding the role of impatience is key to understanding many of the forces articulated in the economics literature. Throughout a deterministic transition, two forces shape the dynamics of the value gap: *expenditure smoothing*, and *yield-curve riding*. Expenditure smoothing is activated if the government has a positive intertemporal elasticity of substitution. Its desire to smooth expenditures induces higher domestic discounts when expenditures fall. Hence, smoothing acts as a temporary increase in impatience and induces greater overall borrowing and a shortening of debt maturity.[4] Yield-curve riding is activated when there are predictable changes to the yield curve, even if the government is indifferent about the smoothness of its expenditures. Without liquidity costs, the path of interest rates is immaterial for the maturity choice, but not with liquidity costs. For example, if the government faces a temporary increase in short-term rates the value gap narrows, but especially for short-term debt. In this case, the optimal amount of debt falls, but the maturity of issuances increases.[5] A temporary increase in short-term rates is associated with a downward sloping yield curve. Thus, yield-riding is the strategy of altering the debt maturity against the direction of changes in the yield curve, something we observe in practice that has not been rationalized by models without liquidity costs.

After presenting the deterministic environment, we incorporate income and interest-rate risk and consider the possibility of strategic default. Because the state variable is the entire maturity distribution, the characterization of an equilibrium confronts the same computational challenges as heterogeneous-agent models with aggregate shocks—see Krusell and Smith (1998). Here, we introduce a risky-steady state (RSS) approach that allows us to make analytic progress. In the RSS approach, we study the asymptotic limit of an economy as the government waits for an absorbing shock that has not yet occurred. This approach captures, for example, the expectation of possible disaster events and has the advantage of offering an analytic expression that shows how risk modifies bond prices and domestic valuations.

Armed with the RSS approach, we characterize analytically the effects of risk and the option to default. Risk introduces additional forces that shape the issuance profile: *hedging* and *self-insurance*. Absent liquidity costs, the government would hedge risks by building a debt structure that generates capital gains upon an adverse interest-rate shock. Because of liquidity costs, perfect hedging is impossible and, thus, expenditures can jump. The jump in expenditures shows up as a term that reduces domestic discount rates, akin to making the government more patient. This effect has the hallmark of self-insurance behavior where issuances decrease and maturity lengthens, in the expectation of negative shocks.

We then analyze the effects of default. We assume that the government can commit to a debt program

[4]For example, consider an economic recovery where the path of revenues is known to be momentarily low. With liquidity costs, the government cannot smooth expenditures perfectly. As a result, domestic discounts will be high throughout the recovery. High discounts increase the value gap, but more so longer bonds due to compounding. A similar logic applies when the government expects a sizeable due principal.

[5]Of course, if the government also cares about smoothing expenditures, changes in the yield curve carry effects through both *yield-curve riding* and *expenditure smoothing*.

prior to a large shock, but can default strategically when a large shock arrives, in the spirit of Eaton and Gersovitz (1981) or Arellano (2008). Default modifies prices and valuations through two channels: *credit risk* and *default incentives*. First, credit risk is the reflection of the default probability, so bond prices and domestic valuations adjust to compensate for this probability. From the risk-averse government's point of view, credit risk is a form of insurance, and thus it has effects in the opposite direction of self-insurance. Second, domestic discounts also capture incentives, because the government cannot commit ex-ante to default contingent on the realization of certain shocks. Thus, domestic valuations incorporate an additional term which captures the fact that a marginal increase in a given issuance will increase the default probability on every date before the expiration of that bond. The fall in bond prices due to the increase in the endogenous default probability spills over to the revenues of auctions in other issuances. We dub the term that captures this effect, the *revenue-echo*. This term does not appear in the bond market price. The revenue echo unambiguously calls for less debt and an increase in its average maturity. Building on these insights, we move to an application.

An Application. To illustrate one of our framework's many possible applications, we apply it to evaluate Spain's debt management strategy. We calibrate the model through the following steps:

1. We calibrate the income and interest rate processes and default costs to match Spain's risk-free yield curve, income revenues, and default premium.

2. We calibrate the government's impatience to match the average level of issuances.

With this calibration in hand, we construct the implied domestic valuations and model-implied debt maturity distribution. Because there are many maturity profiles consistent with the same average issuance level to GDP, we can contrast the actual debt issuance profile with Spain's historical averages. Spain's debt management strategy is surprisingly close to the optimal policy predicted by the model: the historical average of Spain's issuance profile and that predicted by the model are hump-shaped with a peak at 10-year debt. However, the model recommends that Spain should have issued more in the long end of the yield curve, 15 and 30-year bonds, and less in the shorter end, 3, 5, and 10-year bonds.

We take the optimal debt issuance profile predicted by the model and decompose it into the contribution of each force, to understand why the model recommends an extension of Spain's maturity profile. According to the model, 15-year and 30-year bonds are relatively cheap for the government, given its estimated impatience and the fact that the yield curve is flat at these horizons. This more than compensates for the higher liquidity costs at those maturities. For 3 and 5-year debt, liquidity costs are lower, but the combination of self-insurance and the revenue echo limits the issuances at those levels. Thus, according to the model, Spain's debt strategy either puts excessive weight on the low liquidity of long-term debt or too little weight on the forces that operate at the short end.

We end the paper with a study of a transition to a low-interest environment—–where short-term rates fall by 1 pp—–and study the effects of introducing 50-year debt into that environment. We find that, in the scenario of lower rates, Spain should run fiscal deficits until its debt increases from 106% of GDP to 114% and extends its average maturity by half a year. The lower interest burden would provide Spain some fiscal slack to increase expenditures in the long run. If 50-year debt is available, Spain should issue almost equal amounts of 50-year bonds as it does of 30-year ones. With this strategy, Spain would experience a long-lasting increase in expenditures until debt to GDP reaches 186% and eventually expenditure falls 1 pp from the original situation. We conclude the paper by offering a menu of possible extensions and applications. Our applications showcase the usefulness of an all-encompassing approach that accounts for different economic forces and the pivotal role of liquidity.

Related Literature. There is substantial evidence of liquidity costs in asset markets, as surveyed by Vayanos and Wang (2013) or Duffie (2010).[6] As we noted above, we build on Vayanos and Vila (2021) and Duffie et al. (2005) and motivate liquidity costs through a combination of bond-market segmentation and over-the-counter frictions. These frictions have received a lot of recent attention, especially in light of recent disruptions in the US Treasury markets—see for example Duffie (2020) or Kargar, Lester, Lindsay, Liu, Weill and Zúñiga (2020). Relative to this ample literature, to the best of our knowledge, we are the first to study how the frictions introduced in those papers impact the optimal management of public debt.

From a normative standpoint, two fields, international and public finance, provide guidelines for optimal debt management practices. Typically, in international finance settings income is treated exogenous, and the government chooses a debt profile to maximize a net-present value of domestic consumption. In public finance settings, expenditures are exogenous and the government chooses its debt profile to minimize the net-present value of tax distortions. Aside from this difference, both areas base their prescriptions on common economic forces such as smoothing, insurance, and default incentives. Our starting point is a model in the international finance spirit which we show below also applies in a public finance formulation. Hence, we contribute to both areas by investigating, analytically and quantitatively, how liquidity costs interact with these forces to shape the optimal debt structure.

In public finance, Barro (1979) showed that, absent risk, a government should design its debt profile to smooth tax distortions, akin to consumption smoothing in canonical current account models of international finance. In both instances, smoothing has implications for the stock of debt, but not about maturity—because bond prices are arbitrage-free and the government's discount rate coincides with the short-term rate. Liquidity costs break that relation and open the value gap described above. We show that, with liquidity costs, it is ideal to shorten maturity when the government desires to smooth a temporary decline in revenues. We also underscore the yield-riding force, a force that is not present when the government's discount rate coincides with short-term rates. Yield riding dictates that debt maturity should move in the opposite direction to the slope of the yield curve.

Risk opens the door to insurance. In a stochastic extension of Barro (1979), Lucas and Stokey (1983) show

[6] See also Cammack (1991), Spindt and Stolz (1992), Duffee (1996), Fleming (2002), Green (2004), Fleming and Rosenberg (2008), Pasquariello and Vega (2009), Krishnamurthy and Vissing-Jorgensen (2012), Pelizzon, Subrahmanyam, Tomio and Uno (2016) or Breedon (2018).

that a government that accesses state-contingent debt should smooth taxes in a stochastic sense. Angeletos (2002) demonstrates that a government can implement the complete-markets optimal taxation in Lucas and Stokey (1983) by hedging risks with a menu of fixed-income bonds and Buera and Nicolini (2004) noted that, with the observed volatility of bond prices in the data, governments would have to hold substantial debt positions to implement the Lucas and Stokey (1983) tax sequence. A particular case is studied in Barro (2003), where income is deterministic but the discount rate is stochastic. That paper shows that governments should issue perpetuities to avoid costly debt roll-overs. A similar prescription would also emerge in international finance where roll-over risk of short-term debt may prompt defaults, as in Cole and Kehoe (2000). Lustig, Sleet and Yeltekin (2008) analyze how, even in the presence of short-selling constraints, long-term debt provides a hedge against fiscal shocks. The liquidity costs here inhibit perfect hedging, so that force is quantitatively muted.

Departing from complete markets, Aiyagari, Marcet, Sargent and Seppälä (2002) studied the Lucas and Stokey (1983) problem when the government can only issue a single maturity. Aiyagari et al. (2002) showed that, in that case, self-insurance is a force toward lower debt levels. Since liquidity costs make perfect hedging impossible, self-insurance is also an important force in our framework.

The closest paper to ours is Bhandari, Evans, Golosov and Sargent (2017), who show how self-insurance affects optimal debt maturity in an open economy with distortionary taxes. That paper characterizes, under quasi-linear preferences, the asymptotic moments of debt and maturity in a two-bond case. In that model, the maturity and levels of debt have a positive correlation. In contrast to Bhandari et al. (2017), here, the presence of liquidity costs prescribes issuances of debt at all maturities. In our model, the correlation between debt levels and maturity depends on the nature of the shock. The expectation of a positive spike in interest rates shortens the debt maturity because the yield curve slopes upward. The expectation of a negative expenditure shock makes the government de facto more patient, and thus induces a reduction in debt levels together with the lengthening of maturity. When these shocks materialize, the pattern reverses.

Another close paper is Faraglia, Marcet, Oikonomou and Scott (2018), which, to the best of our knowledge, is the only other paper to study optimal maturity management with finite life bonds. That paper stresses the role of frictions that prevent the quick adjustment of the debt profile. While Faraglia et al. (2018) work with recurrent shocks, allowing for issuances in only a few maturities, we impose no restrictions on the maturities that the government can issue. In our analytic characterization, we only consider small perturbations around the RSS. Both papers share the message that liquidity frictions are essential to reconcile debt management practices with model predictions. Relative to that work, our contribution is the analytic and quantitative decomposition of the economic forces that shape the optimal issuances at different maturities.

Finally, we make contact with the vast literature on sovereign default (see Eaton and Gersovitz, 1981; Arellano, 2008; Aguiar and Amador, 2013). Default is a form of insurance (Zame, 1993) as it helps to smooth consumption in adverse states of nature. In terms of maturity choice, Bulow and Rogoff (1988) identified that long-term debt is prone to debt dilution, the idea that once the long-term debt is issued, the price of a new issuance does not internalize the increased default premia on past debt. Our paper abstracts away from debt dilution because, before a default event, the government can commit to a debt program. Thus, default affects the debt profile only through an insurance effect (when coupled with self-insurance), which shortens maturity, and through the revenue echo, which lengthens it.

2 An OTC model of Liquidity Costs

This section presents a wholesale-retail model of the bond market, with over-the-counter (OTC) search frictions, following Duffie et al. (2005) which produce liquidity costs. We consider an *auction* that issues a quantity $\iota$of bonds on a given date t. For now, we focus on obtaining an expression for the liquidity cost as a function of the auction size. In the next section, we estimate these costs from Spanish data and, after that, embed them into the problem of a government that dynamically optimizes its debt.

Time is continuous. The auction participants are a continuum of risk-neutral and deep-pocketed primary dealers. Dealers purchase bonds in the auction, which we refer to as the primary market. Dealers then resell bonds to international investors in a secondary market. Investors face an (exogenous) international risk-free interest rate, r_t^*. For now, we treat r_t^* as a known function of time. Following Duffie et al. (2005), dealers have a higher capital cost than investors. In particular, the dealers' discount rate is $r_t^* + \eta$, where $\eta > 0$ is an exogenous spread. After buying the bonds, investors contact dealers at a certain rate. The Poisson contact flow is $\mu \cdot y_{ss}$ per instant, where μ is a customer flow and y_{ss} a measure of economic activity. Each contact results in a bond purchase by investors. Thus, in an interval Δt, the bonds sold by the dealer are $\mu \cdot y_{ss} \cdot \Delta t$. We assume that dealers extract all the surplus from investors.

The key friction that translates into a liquidity cost as a function of the issuance size is that it takes time for dealers to liquidate their bond portfolios. This friction, together with the fact that dealers have an inventory holding cost, implies that the larger the auction, the longer the waiting time to resell a bond and, thus, the lower the price the dealer is willing to offer in an auction.[7]

Secondary market prices. We consider a single bond that matures τ periods into the future. We normalize the principal payment to one. Prior to maturity, the bond pays an instantaneous coupon δ. The secondary-market price of a bond of maturity ψ is *arbitrage-free* and therefore equals:

$$\psi_t(\tau) = e^{-\int_t^{t+\tau} r_u^* du} + \delta \int_t^{t+\tau} e^{-\int_t^s r_u^* du} ds. \tag{1}$$

The present value formula for $\psi(\tau)$ satisfies the following PDE representation:

$$r_t^* \psi_t(\tau) = \delta + \frac{\partial \psi}{\partial t} - \frac{\partial \psi}{\partial \tau}, \tag{2}$$

with boundary condition $\psi_t(0) = 1$. The first term captures the coupon flow, the second term the capital gains, and the last term captures how the bond's maturity reduces with time. The government repays the

[7] Previous papers have also explored inventory management of financial intermediaries as a source of liquidity costs. See, for instance, Ho and Stoll (1983), Huang and Stoll (1997), Grossman and Miller (1988), or Weill (2007).

principal once the bond matures, $\tau = 0$. This PDE representation will re-emerge throughout the paper.

Bond inventories and primary-market valuations. Now assume that, at time t, a dealer wins an auction and buys the stock ι with maturity τ. Since all contacts with investors result in trade, the bond inventory that remains with the investor at time s after the auction is $\max(\iota(\tau) - \mu y_{ss} \cdot s, 0)$. Hence, the bond inventory is exhausted by $\bar{s}(\iota) = \iota/\mu y_{ss}$ into the future. The intensity of bond sales at the instant s therefore is:

$$\gamma(s;\iota) = \frac{1}{\bar{s}(\iota) - s} \text{ for } s \in [0, \min\{\tau, \bar{s}(\iota)\}) .$$

We can think of $\gamma(s)$ as the Poisson intensity at which dealers sell bonds.

The valuation of a bond from the primary dealer's perspective is $q_{t+s}(\tau)$. Recall that the dealer has a higher cost of capital than investors. Hence, their valuation satisfies:

$$(r^*_{t+s} + \eta) q_{t+s}(\tau - s) = \delta + \frac{\partial q}{\partial t} - \frac{\partial q}{\partial \tau} + \gamma(s) \left(\psi_{t+s}(\tau - s) - q_{t+s}(\tau - s) \right) . \tag{3}$$

The expression is similar to (2). However, the expression also captures that, upon a match, the value of the bond jumps from q to ψ because the dealer extracts all the surplus. The jump arrives with endogenous intensity $\gamma(s)$.

At the date of the auction, $s = 0$, the dealer can enter the auction freely and pay $q_t(\tau)$ for the bond. This price is a function of ι, through the effect on γ. In Appendix C.1, we solve the exact price formula and obtain a first-order linear approximation around small issuances. The solution yields an expression for the auction price as a function of ι:

$$q_t(\tau, \iota) = \underbrace{\psi_t(\tau)}_{\text{market price}} \underbrace{- \frac{1}{2} \bar{\lambda} \psi_t(\tau) \cdot \iota}_{\text{liquidity costs}}, \tag{4}$$

where $\bar{\lambda} = \frac{\eta}{\mu \cdot y_{ss}}$. Intuitively, dealers take a longer time to liquidate their bond inventories as the auction size increases. The longer the waiting time to liquidate a portfolio, the greater the cost. As a result, the greater the auction, the lower the price.

In Equation 4, we interpret $\bar{\lambda} = \frac{\eta}{\mu y_{ss}}$ as liquidity costs. These costs increase linearly with the holding cost (the spread η) and inversely with the contact flow μ (which reduces the holding time). In the rest of the paper, we allow the contact flow μ to depend on τ, interpreting these as segmented bond markets as in Vayanos and Vila (2021). This extension simply leads to a liquidity cost function that depends on τ, $\bar{\lambda}(\tau)$, which we estimate in the next section using bond auction data.

The key difference relative to Vayanos and Vila (2021) is that the market price ψ_t depends on the stock of the outstanding amount of bonds in that paper. We can interpret the price impact in Vayanos and Vila (2021) as *stock* liquidity costs, whereas the ones here as *flow* liquidity costs. If we consider market segmentation by maturity, but not by issuers, then a small issuer will have a minor impact on the price at a given maturity.

Thus, flow liquidity costs are a more suitable assumption for small economies. Later, we describe the effects of flow and stock liquidity differences for the optimal planner problem.

3 Evidence of Liquidity Costs

We now estimate the liquidity costs from Spanish microdata. Spain is an ideal country to measure these costs. First, Spain has an active secondary market from which we can observe market prices. Second, Spain represents a small share of the European sovereign debt market, so its issuances have little impact on Euro area yields. We use detailed data on primary and secondary prices to estimate liquidity costs.

Structure of issuances. The Spanish Treasury issues debt on behalf of the Central Government. Bond issues are grouped into vintages identified by an International Securities Identification Number (ISIN). Each vintage is associated with the same expiration date and the same monthly coupon. After a new vintage is issued, the Treasury often auctions bonds corresponding to that vintage at later dates. Thanks to our conversations with practitioners, we understand that grouping issuances into common vintages simplifies the legal and regulatory transaction costs through standardized legal parameters.

The Treasury catalogs bonds into three categories: "*Letras del Tesoro*" (henceforth, bills) are zero-coupon bonds with approximate original maturities of about 3, 6, 9, 12 or 18 months, "*Bonos*" and "*Obligaciones del Estado*" are constant coupon bonds with approximate maturities of about 3, 5 years (labeled as *Bonos*) or about 10, 15, 30, 50 years (labeled as *Obligaciones del Estado*).[8] We refer to the latter categories simply as bonds. Each vintage falls into a maturity category, although bonds of the same vintage are auctioned at different dates, and thus differ in their maturity.[9]

Re-issuances typically occur within a year window, so issuances within the same vintage have approximately the same maturity rounding up to a year. Thus, we classify bonds into 11 maturity categories that roughly correspond to the Spanish Treasury categories.[10] We define the collection of categories as $\mathcal{M}$ and use it in the rest of the paper.

Auctions. The Spanish Treasury issues debt at competitive auctions. Auction participants include primary dealers and institutional investors. Auctions occur weekly, on two Tuesdays for bills and two Thursdays for bonds. The Treasury publishes tentative auction schedules several months in advance and announces the auction size, i.e., each bond's amount to be issued.

Auctions are sequential: the first round (the competitive round) is open to all market participants, whereas

[8] As commented above, maturities may vary slightly at a monthly frequency: for example, the Treasury may issue five-year bonds with expiration dates anywhere from 54 to 60 months, all issued in Euros. The Treasury also issues inflation-indexed bonds, which we do not analyze as their volume is small.

[9] To give an example, consider an initial issue of five-year maturity bonds. The Treasury, may re-issue bonds of the same ISIN in subsequent months after the first issuance. Economically, this re-issuance increases the stock of bonds of a five-year vintage, but effectively this re-issuance represents an issuance of a four-year bond.

[10] We use the following monthly categories: 3 (between 0 and 4.5 months), 6 (between 4.5 and 7.5 months), 9 (between 7.5 and 10.5 months), 12 (between 10.5 and 14.5 months), 18 (between 14.5 and 20.5 months), 36 (between 20.5 and 44.5 months), 60 (between 44.5 and 74.5 months), 120 (between 74.5 and 144.5 months), 180 (between 144.5 and 210.5 months), 360 (between 210.5 and 410.5 months) and 600 (more than 410.5).

the second round (the non-competitive round) is reserved for frequently-registered participants. We focus only on the first round, which accounts for the bulk of issuances. Auctions are modified Dutch auctions: competitive bidders submit sealed bids of price-quantity pairs. Non-competitive bids only specify a quantity. The allotment works as follows: All non-competitive bids are accepted by default. Competitive bids are sorted by price. The Treasury's offered amount in an auction, net of non-competitive bids, determines the stop-out marginal price. This marginal price is such that all competitive bids at or above the marginal price are accepted. Therefore, the sum of non-competitive and competitive accepted bids equals the supplied amount. The allocation price is calculated in two tiers: bids falling between the marginal and rounded-up weighted average price pay their bid price. Bids above the weighted average price and non-competitive bids pay the weighted average price.[11] Securities are issued three working days after the auction: the corresponding Friday for bills and the corresponding Tuesday for bonds.

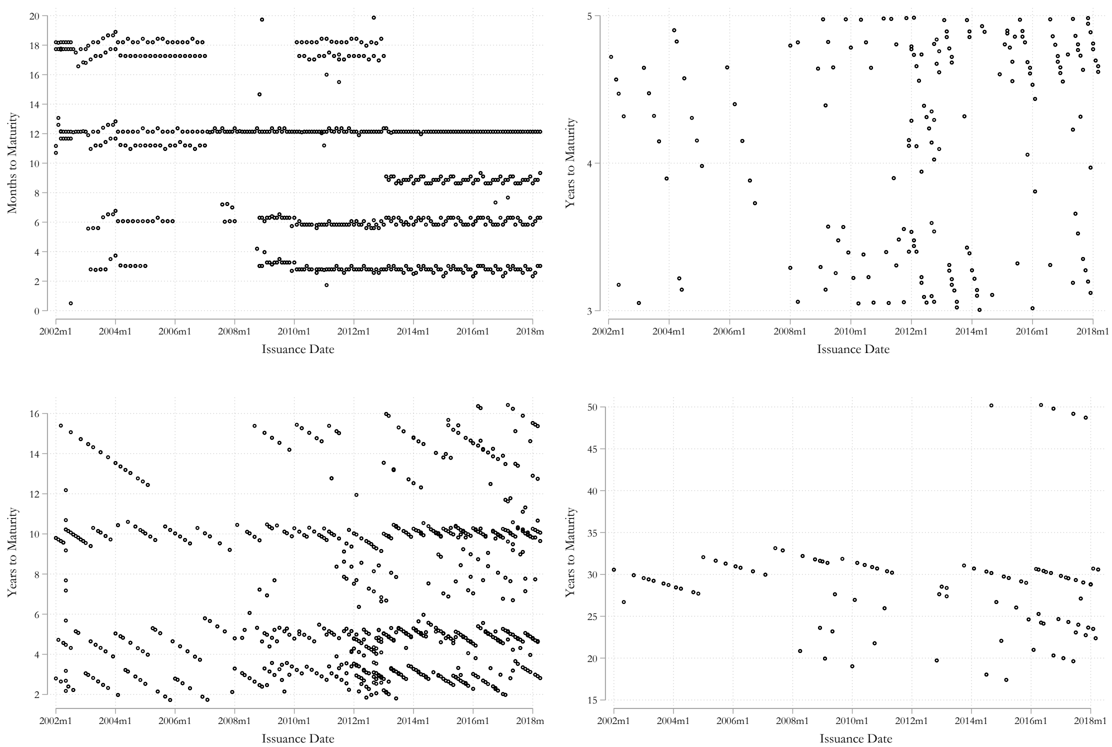


Figure 1: Issuance Pattern.

Notes: Each point in each figure corresponds to an auction. The vertical axis denotes the issued security's maturity at a given date.

Data. The Spanish Treasury provided us with the universe of auctions from January 4, 2002, to April 20, 2018, excluding inflation-indexed instruments, which are small. Adding across auctions, we construct a time series for the Spanish debt stock, accurately approximating the national accounts' stock. The data covers almost the entire universe of debt issuances. For each of the 2,579 auctions, the panel data set includes

[11] A cap on the marginal price is placed to protect some investors from overpaying. The segmentation of bids into competitive and non-competitive is done to protect the auction from collusion, by placing large and frequent investors in the non-competitive segment.

the ISIN code, the coupon, the issuance date, the inception date of the vintage, the expiration date, the total tender amounts received in both the first (competitive) round and second (non-competitive), the total tender amounts allocated, the marginal and the average prices.

Figure 1 displays scatter plots where each dot represents an auction with the date in the x-axis and maturity in the y-axis. The different panels group auctions by their category. We can observe that the Treasury issues in each maturity almost every month. We only observe a few gaps in the 9- and 18-month bills and the 15-year bonds—50-year debt was introduced later in the sample. We also observe a saw-like pattern in all categories, reflecting that issuances in a given vintage have approximately the same maturity.[12] These observations guide our theory. First, the observations call for a theory where economic forces induce continuous issuances in all maturities instead of issuances concentrated in a single maturity. Second, the pattern justifies bundling bonds into specific maturity categories—from the Treasury's perspective, 40-month and 36-month bonds are approximately the same bonds.

Using the ISIN code, we merge this data with secondary bond market prices obtained from Bloomberg. For each ISIN traded on a given day, the data includes the corresponding first and last bid and ask prices. The vintage must have already been open to match auction prices with secondary prices. Since the matched prices are by ISIN, we match identical maturity securities, isolating any other potential legal or regulatory characteristics that could pollute the analysis. As most auctions are re-issuances (85 percent), we match about 80 percent of the total auctions of pre-existing vintages (2,077) with a secondary-market price. We normalize the issuance amount by the month's nominal GDP to account for both changes in the economy's size and the price level.[13] We denote the issuance over GDP at a given date by, $I_t(\tau)$ expressed as a percentage.

We measure the normalized difference between the auction price and the market price with this data. We call this difference the *auction markup* (see, for instance, Song and Zhu, 2018). Consider a maturity τ at date t, for each issuance, we construct the markup using the formula

$$\text{Markup}_t(\tau) \equiv \frac{\psi_t(\tau) - q_t(\tau, \iota)}{\psi_t(\tau)},$$

where $q_t(\tau, \iota)$ is the auction price and $\psi_t(\tau)$ represents the secondary-market price of a bond with the same ISIN. The market price is computed as the average between the bid and ask at the end of the auction's date. A positive ratio indicates that the average participant earned a markup. Otherwise, the participant overpaid. There are multiple prices in each auction corresponding to different bids: the marginal price, the

[12]Take, for example, the 3-year category. Typically, the first auction of a 3-year vintage has a maturity of slightly above 36 months. After the first issuance, bonds of the same vintage are re-issued every following month for about a semester. Hence, by the time the vintage reaches its last auction, the issued bond has a maturity of 30 months. Past the last auction, a new vintage is re-opened, starting again from a maturity above 36 months. The same is true of other categories. Thus, although categories do not exactly correspond to a specific maturity, we observe an almost seasonal pattern centered around the category's maturity—the pattern is not exactly seasonal because the length of the vintages is not uniform.

[13]We also match each auction with the nominal GDP of the corresponding month. We obtain nominal monthly GDP from a dynamic factor model that computes short-term forecasts of the Spanish GDP growth in real-time based on Camacho and Perez-Quiros (2009).

average prices, and the weighted average-above-the-marginal (WAAM) price. We can construct markups for each of these prices. In the rest of the paper, we express markups in basis points.

Table 2 in the Appendix reports the summary statistics for markups and issuances over GDP. The average markup on the marginal price ranges from 1.4 bp to 5.7 bp for bills, and from 7.6 to 30.7 bp for bonds. We also report markups constructed with the WAAM and the average price, which are smaller across all categories because they are constructed from higher bids.[14] In the analysis, we use the WAAM markup.[15] The substantial markups in primary auctions, sometimes positive and sometimes negative, suggest the presence of frictions.

Evidence of liquidity costs in bond issuances. We estimate the liquidity costs articulated in the previous section.[16] We estimate $\bar{\lambda}$ in equation (4) of the previous section through a linear regression. In particular, we estimate $\Lambda(\tau)$ from the following equation:

$$\frac{\psi_t(\tau) - q_t(\tau)}{\psi_t(\tau)} = \alpha(\tau) + \beta_t + \Lambda(\tau) \cdot I_t(\tau) + \epsilon_t(\tau), \tag{5}$$

where t and $\tau \in \mathcal{M}$ are the date and maturity category of the auction. The term $\alpha(\tau)$ is a maturity fixed effect, β_t is a month fixed effect, $I_t(\tau)$ are monthly issuances over yearly GDP, and $\epsilon_t(\tau)$ the error. As in (4), $\Lambda(\tau)$ measures the sensitivity of markups to issuances. $\Lambda(\tau)$ is estimated for each maturity group except for the 50-year bonds, which we discard due to lack of observations. In the quantitative analysis of the model in Section 6, we map the estimates of $\Lambda(\tau)$ to calibrate $\bar{\lambda}$ in model.

The estimates of $\Lambda(\tau)$ are presented in Figure 2 and further details can be found in Table 3 in the Appendix. The x-axis of the figure corresponds to a maturity category and the y-axis the scale coefficients expressed in bps of markup per percentage of monthly issuances over annual GDP. Each diamond in the figure represents the point estimate and the bars the 1%, 5%, and 10% confidence intervals. We can observe that among *bills*, the maturities placed to the left of the dashed line, we cannot find evidence of liquidity costs. However, for all *bond* categories, those to the right of the dashed line, the liquidity coefficient $\Lambda(\tau)$ is positive and significant at the 1% confidence level. From conversations with practitioners, we understand that the lack of evidence among bills can occur because dealers buy bills to manage liquidity, thus expecting to hold bills to maturity and do not cater to retail investors. We will adapt our quantitative model to account for this.

The estimates of liquidity costs among bonds are not negligible. The coefficient values range from a maximum of 56 for the 30-year bonds to 8 for 3-year bonds. These values are not negligible. Considering that Spain issues, on average, about 17% of GDP per year, adequately managing its bond issuances is important for fiscal outcomes. To get a sense of the numbers, think of the following thought experiment: Suppose that

[14] Using the WAAM, for instance, one-year bonds have the highest markup (5 bp) among bills. Among bonds, markups are typically negative. For example, markups are -33 bp, -42 bp, and -36 bp for the 10, 15, and 30-year bonds, respectively.

[15] The regression estimates for marginal-price markups are insignificant. We interpret this finding as suggesting that the distribution of bids, which we do not observe, is concentrated around the marginal price. If this is the case, the marginal price is not sensitive to the auction size. Hence, auction-size effects are captured by the WAAM markup, which motivates its use.

[16] A large empirical literature has documented the presence of price impact in bond markets: For example, see Madhavan and Smidt (1991), Madhavan and Sofianos (1998), Hendershott and Seasholes (2007) or Naik and Yadav (2003a,b).

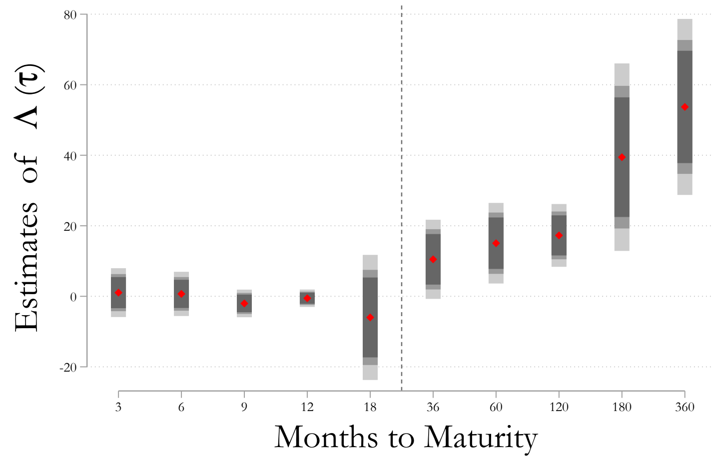


Figure 2: Regression coefficients $\Lambda(\tau)$.
Note: each bar corresponds to an estimate of $\Lambda(\tau)$: diamonds are the point estimates, and the bars correspond to the 1%, 5% and 10% confidence intervals.

Spain has to roll over its debt but can choose to spread issuances in 12 months or re-issue all in a single auction. Applying a linear pricing formula, the relative loss on revenues from concentrating the issuances are $(\Lambda(\tau) - 12 \cdot \Lambda(\tau))$. With an estimate of the liquidity cost $\Lambda(\tau)$ of 10 bp, the loss in revenue is about 1.1% of the total issuances. If Spain issues 17 percent of its GDP annually, it would save up to 0.2 percent of its GDP, around 2.5 billion dollars per year, by designing an optimal strategy.

The results support the notion that liquidity costs matter for debt issuance. Table 4 reports the regression estimates of $\Lambda(\tau)$ using a parametric form that is linear in maturity. Both tables 3 and 4 present robustness estimates breaking the sample into sub-periods . The magnitudes of the estimates are similar and stable across specifications and subsamples, so we are confident that outlier auctions do not drive our results.

4 Debt Management with Liquidity Costs

In this section, we integrate the liquidity costs into the problem of a government that manages its stock of debt at different maturities. The government confronts two processes, $\{y_t, r_t^*\}_{t\geq 0}$, for the variables that represent income revenues and a short-term interest rate processes. For now, these processes are deterministic, an assumption that we abandon below. As $t \to \infty$, these objects converge to their steady states. We use the ss subscript to denote the variable's steady-state value.

4.1 Debt-management problem

The setting is in continuous time which is convenient to produce comparative statics. Other than that, there is nothing particular about continuous time. There is a single, freely-traded good. The government has preferences over expenditure paths, $\{c_t\}_{t\geq 0}$, given by

$$\mathcal{V}_0 = \int_0^\infty e^{-\rho t} U(c_t)\, dt,$$

where $\rho \in (0,1)$ is a discount rate and $U(\cdot)$ is increasing and concave. We assume that $r^*_{ss} < \rho$. Below, we show how these preferences can be modified to incorporate a tax smoothing motive, following the public finance literature.

The government issues bonds that differ by their time to maturity, τ. For now, we let the government chose issuance in a continuum of maturities, $\tau \in [0,T]$, where T is a maximum exogenous maturity. In the quantitative evaluation, we use a discrete number of predetermined maturities. For ease of exposition, we assume that $\delta = r^*_{ss}$, so that bonds trade at par in steady state. The government auctions each bond in the auctions introduced earlier. Thus we treat each maturity as a separate market, in the spirit of Vayanos and Vila (2021).

The outstanding stock of bonds with maturity τ at date t is $f_t(\tau)$, which we refer to as the *debt profile*. The debt profile evolves endogenously according to a partial differential equation (PDE):

$$\frac{\partial f}{\partial t} = \iota_t(\tau) + \frac{\partial f}{\partial \tau}, \tag{6}$$

with boundary condition $f_t(T) = 0$ and, $f_0(\tau)$, given. This law of motion captures that given τ and t, the change in the quantity of bonds of maturity τ, $\partial f/\partial t$, equals the issuances at that maturity, $\iota_t(\tau)$, plus the net flow of bonds, $\partial f/\partial \tau$. The latter term captures the maturing of bonds.[17] When negative, issuances represent purchases. The government's budget constraint is:

$$c_t = y_t - f_t(0) + \int_0^T [q_t(\tau,\iota)\iota_t(\tau) - \delta f_t(\tau)]\, d\tau, \tag{7}$$

where y_t is the income, $-f_t(0)$ principal repayments, $-\delta\int_0^T f d\tau$ the coupon payments, and $\int_0^T q\iota d\tau$ the revenues from all issuances. Finally, $q_t(\tau,\iota)$ is the auction price of a τ-bond at date t, introduced in Section 2. The government solves:

$$V[f_0(\cdot)] = \max_{\{\iota_t(\cdot), f_t(\cdot), c_t\}_{t\geq 0, \tau\in[0,T]}} \int_0^\infty e^{-\rho t} U(c_t)\, dt, \tag{8}$$

subject to (6), (7), the initial condition f_0. The solution V is *a value functional*, as it maps a debt density $f_0(\cdot)$ into a real number.

[17] Issuances, $\iota_t(\tau)$, are chosen from a space of functions $\mathcal{I} : [0,T] \times (0,\infty) \to \mathbb{R}$ that meets some technical conditions. In particular, $\mathcal{I}$ is the space of functions $g_t(\tau)$ on $[0,T] \times [0,\infty)$ such that $e^{-\rho t} g$ is square Lebesgue-integrable.

4.2 Optimal debt management

We employ infinite-dimensional optimization to solve (8). We formulate the Lagrangian:

$$\begin{aligned}\mathcal{L}[\iota, f] &= \int_0^\infty e^{-\rho t} U\left(y_t - f_t(0) + \int_0^T \left[q(t,\tau,\iota)\,\iota_t(\tau) - \delta f_t(\tau)\right] d\tau\right) dt \\ &\quad + \int_0^\infty \int_0^T e^{-\rho t} j_t(\tau) \left(-\frac{\partial f}{\partial t} + \iota_t(\tau) + \frac{\partial f}{\partial \tau}\right) d\tau dt,\end{aligned}$$

where we substituted expenditures out from the budget constraint, (7). The second line attaches a Lagrange multiplier $j_t(\tau)$ corresponding to the constraints imposed by (6). A classic variational argument is that no variations around the optimal issuance, ι, can improve the Lagrangian. This first-order condition implies that:

$$U'(c_t)\left(q(t,\tau,\iota) + \frac{\partial q}{\partial \iota}\iota_t(\tau)\right) = -j_t(\tau). \tag{9}$$

A second condition is that no infinitesimal variation around f can improve the Lagrangian. This requires j to satisfy:

$$\rho j_t(\tau) = -U'(c_t)\,\delta + \frac{\partial j}{\partial t} - \frac{\partial j}{\partial \tau},\ \ \tau \in (0, T], \tag{10}$$

with a boundary condition: $j_t(0) = -U'(c_t)$ and a transversality condition (TVC), $\lim_{t\to\infty} e^{-\rho t} j_t(\tau) = 0$. Relative to a standard optimal control problem where Lagrangians are connected through an ordinary-differential equation (ODE), here Lagrangians are connected through the time and maturity dimensions giving rise to a PDE.

To better interpret the equilibrium conditions, it is convenient to translate j from utils into good units through $v_t(\tau) \equiv -j_t(\tau)/U'(c_t)$. We refer to v as the *domestic valuation* of a (τ, t)-bond. Aided with this definition, we re-express (9) and (10) as

$$\underbrace{\frac{\partial q}{\partial \iota}\iota_t(\tau) + q(t,\tau,\iota)}_{\text{marginal revenue}} = \underbrace{v_t(\tau)}_{\text{domestic valuation}}, \tag{11}$$

and,

$$r_t v_t(\tau) = \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau},\ \tau \in (0, T] \text{ and } v_t(0) = 1, \tag{12}$$

where we define r_t as

$$r_t \equiv \rho - \frac{U''(c_t)\,c_t}{U'(c_t)}\frac{\dot{c}_t}{c_t}. \tag{13}$$

The transformed necessary condition, (11), states that the optimal issuance of a (τ, t)-bond must equate the marginal auction revenue to a marginal cost. The marginal revenue is the price per issuance, q, plus the price impact, $\frac{\partial q}{\partial \iota}\iota$. The marginal cost is encoded in the forward-looking valuation, v_t. The valuation satisfies (12) and thus, shares a remarkable connection with the market-price equation (2): Both the domestic valuation and the price are net present values of the payments flows of each bond. The only difference is the discount that is applied to compute each net-present value. Whereas the market price discounts time with the exogenous short-term rate r_t^*, the domestic valuation does so with the endogenous *domestic-discount rate,* r_t. Since v satisfies the same PDE as ψ in (1), its solutions must be identical after we replace r_t^* by r_t:

$$v_t(\tau) = e^{-\int_t^{t+\tau} r_u du} + \delta \int_t^{t+\tau} e^{-\int_t^s r_u du} ds. \tag{14}$$

We make use of this representation below. The following proposition summarizes the characterization of the solution:[18]

Proposition 1. *[**Optimal issuances**] If a solution $\{c_t, \iota_t(\cdot), f_t(\cdot)\}_{t\geq 0}$ to (8) exists, then domestic valuations $v_t(\tau)$ satisfy the PDE (11), optimal issuances $\iota_t(\tau)$ satisfy the issuance rule (11), and the evolution of the debt profile can be recovered from the law of motion for debt, (6), given the initial condition f_0. Finally, c_t and r_t must be consistent with the budget constraint (7) and (13).*

There is a noteworthy property of the solution, *decentralization*. We can interpret the solution as if the government designates a continuum of traders to be in charge of a single maturity. Each trader must compute a domestic valuation of their corresponding bonds, given an assigned domestic discount r. Each trader then issues bonds of their corresponding maturities according to the rule given by (11), without regard for what other traders are doing. Of course, the assigned discount factor must be internally consistent with the issuances of all traders.

Consider the approximate liquidity cost function (4). Under this approximation, the optimal-issuance condition (11) simplifies further to:

$$\iota_t(\tau) = \frac{1}{\bar{\lambda}} \cdot \underbrace{\frac{\psi_t(\tau) - v_t(\tau)}{\psi_t(\tau)}}_{\text{value gap}} \tag{15}$$

This rule states that the optimal issuance of a (τ, t)-bond is the product of the value gap and the inverse of the liquidity coefficient. When the value gap is positive, a trader should issue debt because the market price exceeds his cost assessment summarized by his valuation. The force that limits that desire to arbitrage is the liquidity cost, which captures a larger issuance's price impact. This force is encoded in $1/\bar{\lambda}$.

Generically, valuations and prices differ, $\psi_t(\tau) \neq v_t(\tau)$, leading to issuances in all maturities, $\iota_t(\tau) \neq 0$. Intuitively, if the government had to issue in a single maturity, it would issue in maturity with the largest value gap. With liquidity costs, the issuance rule dictates that the marginal auction revenue, be equal to valuations. Since marginal revenues decrease with issuance size, the government spread them along many maturities. Issuances in all maturities is a feature documented above, in Section 3, and is a virtue of this model.

To provide statements about how different forces shape the optimal debt maturity, we first define the weighted average maturity (WAM) of issuances:

[18] The solution can be found in appendix C.2.

$$\mu_t \equiv \int_0^\tau s\iota_t(s)\,ds \Big/ \int_0^T \iota_t(z)\,dz.$$

We are interested in the comparative statics of the WAM with respect to parameters that we associate with the different economic forces. Let θ be a parameter of interest. The elasticity of the WAM, $\epsilon^{\mu}_{t,\theta} \equiv \frac{\partial \mu_t}{\partial \theta} \cdot \frac{\theta}{\mu_t}$, is related to the elasticity of issuances at a given maturity, $\epsilon^{\tau}_{t,\theta} \equiv \frac{\partial \iota_t(\tau)}{\partial \theta} \cdot \frac{\theta}{\iota_t(\tau)}$, as established in the following Lemma.

Lemma 1. *[**Monotone Comparative Statics**] Assume that issuances are positive at all maturities, $\iota_t(\tau) > 0$. The elasticity of the WAM with respect to θ, $\epsilon^{\mu}_{t,\theta}$, is positive (negative) as long as the elasticity of issuances at a given maturity, $\epsilon^{\tau}_{t,\theta}$, is increasing (decreasing) in τ, $\frac{d\epsilon^{\tau}_{t,\theta}}{d\tau} > 0$, for all $\tau \in [0, T]$.*

This lemma provides a sufficient condition to obtain monotone comparative static for the weighted average maturity.[19] The lemma is convenient because it provides a simple condition that summarizes the effect of parameter changes on the WAM. Since issuance elasticities only depend on valuations and prices, to understand the effects on maturities all we need is to test whether the elasticities of valuations and prices are monotone in τ. We exploit Lemma 1 when we explain how different forces impact the WAM.

Discussion: Dual problem and debt-management office mandates. Debt-management offices typically have the mandate of minimizing financial expenditures, subject to a path of financing needs (see Greenwood, Hanson, Rudolph and Summers, 2015). A virtue of the model in this paper is that the *dual* representation of the government's problem is consistent with that mandate. Take a given expenditure path $\{c_t\}_{t \geq 0}$, the dual problem is

$$\min_{\{\iota_t(\cdot)\}} \int_0^\infty e^{-\int_0^t r(s)ds} \underbrace{\left(f_t(0) + \int_0^T \delta f_t(\tau)\,d\tau - \int_0^T q(\tau, t, \iota)\iota_t(\tau)\,d\tau \right)}_{\text{Net flow of financial receipts}} dt \tag{16}$$

subject to (13), (6), and (4). In other words, the dual and the original problem yield the same solution issuances ι if the expenditure path in the dual and the primal problems coincide.[20]

4.3 Analysis

Proposition 1 holds for any arbitrary U. To sharpen the results, for the rest of the paper, we assume that $U(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$ so that $r_t = \rho + \sigma \dot{c}_t / c_t$. Before we proceed, it is convenient to explain what happens in the case without liquidity costs, $\bar{\lambda} = 0$. In this case, the solution of the government problem coincides with the solution to a standard consumption-savings problem with a single instantaneous bond, B_t, with the budget constraint $\dot{B}_t = {r^*}_t B_t - y_t + c_t$. Then, any debt profile $f_t(\tau)$ is a solution to the original problem provided that $B_t \equiv \int_0^T \psi_t(\tau) f_t(\tau)\,d\tau$ at all t. Equation (11) in the frictionless case holds trivially as r^*_t and r_t must

[19] The proof can be found in Appendix C.3.
[20] The proof is in Appendix C.4.

be equal.[21] Hence, the optimal maturity structure is not determined as noted early by Barro (1979). Things change with liquidity costs.

The long-run pattern of optimal debt profile. The asymptotic behavior of the solution to (8) can be characterized analytically (Appendix C.6). If $\bar{\lambda}$ is below a threshold, the solution exhibits a pattern of convergence toward zero expenditures as in the case without liquidity costs.[22] However, the maturity distribution is still determined. If $\bar{\lambda} > \bar{\lambda}_o$ a steady state with positive expenditures exists.

When $\bar{\lambda} \geq \bar{\lambda}_o$, the domestic discount factor converges to ρ and the issuances are solved analytically:

$$\iota_{ss}(\tau) = \frac{1}{\bar{\lambda}}\left[1 - \frac{e^{-\rho\tau} + (\delta/\rho)\left(1 - e^{-\rho\tau}\right)}{e^{-r^*_{ss}\tau} + (\delta/r^*_{ss})\left(1 - e^{-r^*_{ss}\tau}\right)}\right], \tag{17}$$

which is positive given that $r^*_{ss} < \rho$. We investigate this expression to describe how issuances vary with maturity.

Proposition 2. ***[Maturity]*** *Let $\delta = r^*_{ss}$. Then steady state issuances increase with maturity, $\frac{\partial}{\partial\tau}\left[\iota_{ss}(\tau)\right] > 0$.*

To explain the increasing pattern, note that $\frac{\partial}{\partial\tau}\left[\iota_{ss}(\tau)\right] = -\frac{1}{\bar{\lambda}}\frac{\partial}{\partial\tau}\upsilon_{ss}(\tau)$ and that, if we set $r_t = \rho$ in (14), we obtain

$$\frac{\partial}{\partial\tau}\left[\upsilon_{ss}(\tau)\right] = -(\rho - \delta)\,e^{-\tau\rho} < 0, \tag{18}$$

a negative derivative since $\rho > \delta = r^*_{ss}$. The logic is simple. An increase in maturity delays the principal repayment by one instant, which is discounted by ρ, but through the delay, it costs an additional coupon δ. This trade-off is evaluated τ periods ahead, hence the discount by $e^{-\tau\rho}$.[23] Thus, to equalize marginal revenues to every valuation, the government must spread out issuances across all maturities. Since the valuation of longer-term debt decreases with maturity, the government accepts a lower marginal revenue on longer maturities. Hence, we obtain a pattern of issuances in all maturities, but one that is increasing in maturity. All in all, impatience leads to an increasing issuance profile that is counterfactual compared to the hump-shaped pattern of issuances described in Section 3. Later, we explain that with another force, risk, the model naturally produces the hump shape that we observe in the data.

In contrast to the issuance profile, the debt profile in (17) is consistent with the data, because it is *decreasing in maturity*. This property follows because long term debt becomes short term debt with time, as in (6): as debt is the integral of issuances, $f_{ss}(\tau) = \int_\tau^T \iota_{ss}(\tau')\,d\tau'$, it is decreasing in maturity—since issuances are positive at all maturities: $\frac{\partial}{\partial\tau}\left[f_{ss}(\tau)\right] = -\iota_{ss}(\tau) < 0$.

[21] Given that the yield curve is arbitrage-free and the discount factor coincides with the interest rate, there is no way for the government to restructure debt to reduce the cost of servicing debt. All bonds are redundant, but the path of consumption is consistent with ${r^*}_t = r_t$ and an intertemporal budget constraint. See Appendix C.5.

[22] Appendix C.6 shows that if $\bar{\lambda} \leq \bar{\lambda}_o$, there is no steady state; expenditures decrease asymptotically at the exponential rate $r^*_{ss} - \rho$ and r_t converges to a limit value $r_\infty\left(\bar{\lambda}\right)$. The asymptotic discount rate $r_\infty\left(\bar{\lambda}\right)$ is increasing and continuous in $\bar{\lambda}$ with bounds $r_\infty\left(\bar{\lambda}_o\right) = \rho$ and $r_\infty(0) = r^*_{ss}$.

[23] If the bond is not issued at par, we must take into account the reduction in the bond price, $\frac{\partial}{\partial\tau}\left[\psi_t\right]$ in the issuance decision, but the logic follows.

At the steady state, a single force, *impatience,* governs the steady-state issuance volume. The following result describes how the relative impatience of the government matters for the amount of borrowing and the average maturity of issuances:

Proposition 3. *[**Relative impatience**] Define the relative impatience of the government as $\Delta \equiv \rho - r^*$. In the steady state, issuances increase with relative impatience, $\frac{\partial}{\partial \Delta}\left[\iota_{ss}(\tau)\right] > 0$, for any $0 < \delta = r^* < \rho$. Furthermore, if $0 = \delta < \Delta$,* then

$$\epsilon^{\tau}_{ss,\Delta} = \frac{\Delta}{\exp(\Delta\tau) - 1} > 0.$$

Furthermore, $\epsilon^{\tau}_{ss,\Delta}$ is decreasing in τ and, hence, the WAM falls with the government's relative impatience.

The proposition states that, as we increase impatience ρ, the government issues more debt at the steady state, but with a lower average maturity.[24] The increase in debt is intuitive and follows because a more impatient government is willing to take a greater price impact, thus, issuing more at all maturities. The intuition for the reduction in the WAM follows because issuance elasticities are higher for bonds of short maturity. The intuition is also clarified by equation 18. That equation tells us what is the benefit for the government of delaying a principal payment in time. That benefit is proportional to the spread $(\rho - \delta)$, which increases with impatience, discounted by time. Thus, an increase in impatience is discounted less at shorter maturities. However, as we increase impatience, shorter maturities respond more and this has the effect of increasing the elasticity of issuances. The effect of an increase in the steady-state interest rate operates in the opposite direction, reducing the overall issuances but increasing the WAM.

Dynamics. Next, we explain the forces that shape the maturity profile along a transition. The dynamics of the debt profile are governed by two forces encoded in the value gap: *expenditure smoothing* and *yield-curve riding*. Expenditure smoothing is captured through the dynamics of r_t, whereas yield riding goes through the dynamics of r^*_t.

First, we provide a formal characterization of the effects of smoothing and yield-riding on the debt profile:[25]

Proposition 4. [***Dynamic Forces***] *Assume zero-coupon bonds, $\delta = 0$, and let income and interest rates revert to their steady-state values at an exponential rate α,*

$$x_t = x_{ss} + (x_t - x_{ss}) \exp(-\alpha\tau) \text{ for } x \in \{y, r\}.$$

Then, we have the following effects:

*1. [**Expenditure smoothing**] The issuance semi-elasticities with respect to a small negative decline in income, $\varepsilon = y_{ss} - y_0 \gtrsim 0$, when $\bar{\lambda} \to \infty$, are positive*

$$\lim_{\bar{\lambda}\to\infty} \epsilon^{\tau}_{0,\varepsilon} = \frac{\sigma}{y_{ss}} \frac{1 - \exp(-\alpha\tau)}{\exp((\rho - r^*)\tau) - 1} > 0,$$

[24]The proof can be found in Appendix C.7.
[25]The proof can be found in Appendix C.8.

and increasing with maturity. Hence, issuances increase and the WAM decreases, i.e., $\lim_{\bar{\lambda}\to\infty} \epsilon^{\mu}_{0,\varepsilon} > 0$.

*2. [**Yield-riding**] Let* $\sigma = 0$ *(risk neutral government). The issuance elasticities with respect to a small positive increase in interest rates,* $r^*_0 = r^*_{ss} + \varepsilon$*, are negative*

$$\epsilon^{\tau}_{0,\varepsilon} = -\frac{1}{\alpha}\frac{1-\exp(-\alpha\cdot\tau)}{\exp((\rho - r^*_{ss})\,\tau) - 1} < 0.$$

and decreasing with maturity. Hence, the WAM increases, i.e., $\epsilon^{\mu}_{0,r^*_{ss}} > 0$.

Proposition 4 provides a characterization of the effects produced by smoothing and yield riding.[26] To sharpen the prediction, we simplify the calculations by considering zero-coupon bonds, $\delta = 0$, and work with the limit as $\bar{\lambda} \to \infty$, such that the consumption path can be obtained analytically.

The first item of the Proposition characterizes how the smoothing force impacts optimal debt management. To isolate the expenditure smoothing force, we fix the short-term rate at its steady state. Expenditure smoothing refers to the desire to smooth expenditures throughout a transition. In the proposition, we study a temporary drop in income which translates into an identical drop in expenditures that reverts to its steady-state value. Because consumption is expected to grow, domestic discount rates are temporarily high, and then mean revert as long as $\sigma > 0$. The elasticity $\epsilon^{\tau}_{0,\varepsilon}$ is positive, reflecting the desire to issue debt to smooth expenditures as long as $\sigma > 0$. In contrast with the classic study of Barro (1979), with liquidity costs, expenditure smoothing affects the maturity distribution just as impatience affects the optimal maturity at steady state. In fact, an increasing path of expenditures produces a temporary increase in r_t. Since the average discount that applies to short-term valuations is higher, the elasticity of short-term issuances is more sensitive to the perturbation and this has the effect of reducing the average maturity.

Yield-curve riding influences the optimal debt profile through bond prices; in particular, through the slope of the yield curve. To isolate this force, we set σ to zero so that $v(\tau)$ does not change with the perturbation. Just as a temporary increase in r_t reduces valuations, a temporary increase in $r^*_t(\tau)$ reduces market prices. From the perspective of the government, this decreases the marginal auction revenues, and more so for longer bonds. As a result, when short-term rates are temporarily high (a downward-slopping yield curve) the stock of debt decreases. In turn, the WAM increases because issuances of long term debt are more elastic to the rate. Hence, the optimal policy stipulates that maturity should move in the opposite of the slope of the yield curve, hence the term yield-curve riding.

With a finite inter-temporal elasticity of substitution, $\sigma > 0$, and liquidity costs, $\bar{\lambda} > 0$, changes in the short-term rate r^*_t carry effects through both yield-curve riding and expenditure smoothing. In particular, the path of rates affects the expenditure path through the financial cost of debt. Unlike smoothing, yield-curve riding is a force germane to liquidity costs.[27]

[26] Notice that we employ semi-elasticities instead of elasticities as the size of the shock is very small. Lemma1 holds in both cases.

[27] As discussed above, without liquidity costs, the domestic discount coincides with the short-term rate, $r_t = r^*_t$, so the effects on valuations and rates are identical. Thus, the margin of adjustment is the growth rate of expenditures (and total debt), without a prediction regarding maturity.

Vanishing costs. We showed that without liquidity costs, the distribution is undetermined. However, in the limit as $\bar{\lambda} \to 0$, the asymptotic distribution is determinate and given by

$$\lim_{\bar{\lambda} \to 0} \iota_\infty(\tau) = \frac{1 - e^{-r^*_{ss}\tau}}{1 - e^{-r^*_{ss}T}} \Xi,$$

where Ξ is a constant that guarantees zero expenditures.[28] Thus, in this limit, the debt profile shares the qualitative feature that the debt issuances are tilted toward long-term bonds. The takeaway from the discussion is that even if liquidity costs are small, the optimal maturity distribution is determinate.

Discussion: irrelevance of roll-over frequency. It is important to clarify that the frequency of roll-over has nothing to do with the preference for long-term debt. The roll-over frequency would be a consideration in an environment where, at an initial date, the government has to chose to issue permanently in a given maturity, τ. In the problem studied here, the envelope theorem guarantees that, when considering the issuances at different maturities at any point in time, the government only compares the payment flows associated with each bond, disregarding of the maturity of the bonds it will use to refinance those cash-flows. As a result, although the future rollover date enters in the domestic valuations, 12, the rollover frequency does not enter the optimal issuance rule, 15.

4.4 Public finance redux

Here we show that the framework above can be adapted to settings with tax distortions, for example, as in Bhandari et al. (2017). To do so, we assume that households supply labor according to GHH preferences—with an inverse Frisch elasticity of ν. As is usual in the literature, let the government decide on savings on behalf of households. We modify the government's problem and assume that government expenditures, g_t, are random and follow a Poisson process. Production is linear in hours, h_t, so the real wage is set to one. The government sets a linear tax, η_t, on hours worked so that labor tax receipts, w_t, are given by $w_t = \eta_t \cdot h_t$. The government's budget constraint is given by:

$$w_t + \int_0^T q_t(\tau)\iota_t(\tau)\, d\tau = g_t + \left[f_t(0) + \delta \int_0^T f_t(\tau)\, d\tau \right]. \tag{19}$$

The objective of the government is to maximize the household's welfare:

$$\max_{\{\iota_t(\cdot), f_t(\cdot), c_t, h_t, \eta_t\}} \int_0^\infty e^{-\rho t} U\left(c_t - h_t^{1+\nu}/(1+\nu)\right) dt \tag{20}$$

subject to the KFE (6), the modified budget constraint (19), the initial condition, f_0, and the optimal labor supply, $\eta_t = h_t^\nu$. Next, we show how the debt-management problem can be reformulated isomorphically to the version we encountered earlier.

Proposition 5. [***Issuances Public Finance***] *Let the domestic valuations $v_t(\tau)$ satisfy the PDE (11), optimal*

[28] See the proof in appendix C.9.

issuances $\iota_t(\tau)$ satisfy the issuance rule (11), and suppose that the evolution of the debt distribution can be recovered from the law of motion for debt, (6), given the initial condition f_0. Also, let the domestic rate r_t be given by:

$$r_t = \rho - (U''/U' - W''/W')\,\dot{x}_t$$

where

$$x_t \equiv y_t + \int_0^T q_t(\iota,\tau)\iota(\tau,t)\,d\tau - \left[f_t(0) + \delta\int_0^T f_t(\tau)\,d\tau\right],$$

for $y_t = -g_t$, and where $W(x) \equiv \left\{c | c - \chi^{-\frac{1}{1+\nu}} \cdot c^{\frac{1}{1+\nu}} = x\right\}$ is an indirect utility. Finally, suppose that

$$c_t = W(x_t)\,,\ \ h_t = W(x_t)^{\frac{1}{1+\nu}} \ \ \text{and } \eta_t = 1 - W(x_t)^{\frac{\nu}{1+\nu}}\,.$$

Then the path $\{\iota_t(\cdot), f_t(\cdot), c_t, h_t, \eta_t\}_{t\geq 0}$ induced by x_t is a solution to (20) if $x_t \geq -\nu(1+\nu)^{-\frac{1+\nu}{\nu}}$.

The proposition shows that the debt-management problem is reformulated as a problem with modified preferences, which for the purpose of the optimal debt profile only modifies the domestic discount rate. The discount rate itself depends on the variable x_t, which represents a private current account deficit. This problem now admits negative income, y_t, If we now solve for the path of x_t, we obtain the equilibrium consumption from W. The problem does feature a constraint, a lower bound on x_t, with the interpretation that it is the point of maximal extraction of resources from the private sector, associated with the peak of the Laffer curve. We can observe that the domestic discount rate captures elements of both expenditure smoothing but also smoothing of tax distortions. We do not pursue an application of optimal taxation in this paper, but we note one can compute a solution using the algorithm discussed in this paper and obtain $\{c_t, \eta_t, h_t\}$ from Proposition 5.

5 Risk and Default

This section incorporates risk and endogenous default into the environment. These two features allow our model to encompass most of the previous literature analyzing optimal debt management. In this section we discuss the theoretical implications whereas in Section 6 we calibrate it and take it to the data.

5.1 Extending the model with risk and default

Aggregate risk. We model risk as a single random jump event. We now let the exogenous state $X_t = [y_t, r_t^*]$, follow a compound Poisson process with intensity ϕ. Denote by t^o the date of a jump event. We introduce notation to distinguish pre- from post-jump variables: for any variable at t we denote its value prior to the jump by a hat, i.e., $\hat{x}_t$, and use x_t to express its value after the jump. Upon a jump, a new value of X is drawn from a distribution $G(\cdot|\hat{X}_{t^o})$, where G conditions on $\hat{X}_{t^o}$.[29] We let X_{t^o} take values in the

[29] Formally, X_t is right-continuous. If the jump occurs at t^o, the left-limit $\hat{X}_{t^o} \equiv \lim_{s\uparrow t^o}\hat{X}_s$ jumps to some new $X_{t^o} \sim G(\cdot;\hat{X}_{t^o})$.

compact space $[y_l, y_h] \times [r_l^*, r_h^*]$. We denote the time t conditional expectation over $G(\cdot; X_{t^o})$ by $\mathbb{E}_t^X[\cdot]$. After the jump, x_t follows a mean-reverting process:

$$\dot{x}_t = -\alpha^x(x_t - x_{ss})$$

where α^x is the mean-reversion rate. There are no further jumps thereafter. Finally, we define by $J_t^X[x_t] \equiv x_t/\hat{x}_t$ the jump in x, conditional on the realization of X_t.

The default option. The government can default only upon the arrival of a jump. If the government defaults, it does so on all bonds and is barred from international markets after that. The government's autarky value is

$$V_{t^o}^D \equiv V^D(X_{t^o}) \equiv \int_{t^o}^{\infty} e^{-\rho(s-t^o)} U\left((1-\kappa)y(s)\right) ds + \varepsilon. \tag{21}$$

The autarky value is the discounted value of only spending revenues, assuming that a fraction κ of revenues is lost, plus a zero-mean random variable, ε. The variable ε captures the randomness around the decision to default. Denote by $\Theta(\cdot)$ and $\theta(\cdot)$ the CDF and PDF of $V_{t^o}^D$, respectively.[30] The government defaults if the autarky value is greater than the value of continuing to service debts, $V_{t^o} \equiv V\left[\hat{f}_{t^o}(\cdot), X_{t^o}\right]$, the perfect-foresight value of starting with an initial debt profile, $\hat{f}_{t^o}(\cdot)$. Thus, the default probability is $\mathbb{P}\left\{V_t^D > V_t\right\} = 1 - \Theta(V_t)$. To ease notation, we use $\Theta_t \equiv \Theta(V_t)$ and $\theta_t \equiv \theta(V_t)$.

Bond prices. Bond prices now include interest rate risk and default. It turns out that the PDE (2) still holds, with only a modification to the discount rate:

$$\hat{\xi}_t^*(\tau) \cdot \hat{\psi}_t(\tau) = \delta + \frac{\partial \hat{\psi}}{\partial t} - \frac{\partial \hat{\psi}}{\partial \tau}, \; ; \hat{\psi}_t(0) = 1. \tag{22}$$

where $\hat{\xi}_t^*(\tau)$ is a risk-adjusted rate:

$$\hat{\xi}_t^*(\tau) \equiv \underbrace{\hat{r}_t^*}_{\text{short-term rate}} + \phi \cdot \left(1 - \mathbb{E}_t^X\left[\underbrace{\Theta_t}_{\text{credit risk}} \cdot \underbrace{J^X[\psi_t(\tau)]}_{\text{interest risk}}\right]\right). \tag{23}$$

From (23), we observe that $\hat{\xi}_t^*(\tau)$ equals the short-term rate plus a τ-specific risk premium. This risk premium is affected by Θ_t, which captures *credit risk,* and by $J^X[\psi_t(\tau)]$ which captures the *interest-rate risk*. To understand these terms, suppose that default is certain when a jump occurs. In that case, the bond is discounted at the rate $\hat{r}_t^* + \phi$, because, by waiting one more period, the investor should discount time at $\hat{r}_t^*$ and the hazard of default at ϕ.

In turn, assume that repayment is certain, $\Theta_t = 1$. In this case, the risk premium is

$$\phi\left(1 - \mathbb{E}_t^X\left[J^X[\psi_t(\tau)]\right]\right) = \phi \mathbb{E}_t^X\left[\left(\hat{\psi}_t(\tau) - \psi_t(\tau, X)\right)/\hat{\psi}_t(\tau)\right],$$

[30] Note that the government can hold foreign bonds ($f_t(\tau) < 0$) but in that case, foreign investors expropriate its assets if the country defaults.

which captures the capital losses from the change in the price of τ-bond given a jump in the short-term rate r_t^*. Naturally, $\hat{\xi}_t^*(\tau)$ depends on τ because capital losses differ by maturity. When $\Theta_t < 1$, then the capital loss is treated as 100 percent in default states.

Government problem. With the default option, the government's problem is represented recursively:

$$V_0 \equiv V[f_0(\cdot), X_0] = \max_{\{\iota_t, f_t, c_t\}} \mathbb{E}_0 \left[\int_0^{t^o} e^{-\rho t} U(\hat{c}_t)\, dt + e^{-\rho t^o} \mathbb{E}_{t^o}^X \left[\max\left\{ V_{t^o}, V_{t^o}^D \right\} \right] \right] \tag{24}$$

subject to the law of motion of debt, (6), the budget constraint, (7), and the pricing equation (22). In this problem, the government commits to a debt program at time zero, before the shock. However, the government cannot commit to repay after the shock. Commitment to a debt program is a natural benchmark because the government has nothing to gain from defaulting prior to the shock. This assumption seems reasonable if the debt management office operates independently from political parties or if governments can commit to a fiscal rule. Without commitment, the solution involves solving a game between the government and the investors as in Eaton and Gersovitz (1981).

5.2 Default-adjusted valuations

Here we adapt the solution to the government problem to allow for risk and default. The approach is the same as above, except that we now consider the jumps in prices and revenues, and the default option.[31]

Proposition 6. *(Necessary conditions) If a solution $\left\{ \hat{c}_t, \hat{\iota}_t(\cdot), \hat{f}_t(\cdot), \hat{\psi}_t(\cdot) \right\}_{t \geq 0}$ to (24) exists, it satisfies the same conditions of the perfect-foresight solution (Proposition 1) with valuations satisfying the following PDE,*

$$\hat{\xi}_t(\tau)\, \hat{v}_t(\tau) \quad = \quad \delta + \frac{\partial \hat{v}}{\partial t} - \frac{\partial \hat{v}}{\partial \tau};\ \hat{v}_t(0) = 1, \tag{25}$$

where $\hat{\xi}_t(\tau)$ is a risk-adjusted discount rate:

$$\hat{\xi}_t(\tau) = \underbrace{\rho + \sigma \frac{\dot{\hat{c}}_t}{\hat{c}_t}}_{\hat{r}_t} + \phi \cdot \left(1 - \mathbb{E}_t^X \left[\left(\underbrace{\Theta_t}_{\text{credit risk}} + \underbrace{\Omega_t}_{\text{revenue echo}} \right) \cdot \underbrace{J^X[v_t(\tau)]}_{\text{hedging}} \cdot \underbrace{J^X[U_t']}_{\text{self-insurance}} \right] \right), \tag{26}$$

and the revenue echo, Ω_t, is given by:

$$\Omega_t \equiv \theta_t \cdot U_t' \cdot \int_0^T \int_{\max\{t+m-T,0\}}^t e^{-\int_z^t (r_u^* - \hat{r}_u) du} \cdot \frac{\psi_t(m)}{2\bar{\lambda}} \cdot \left(1 - \left(\frac{\hat{v}_z(m+t-z)}{\hat{\psi}_z(m+t-z)} \right)^2 \right) dz dm.$$

Additional Forces. Proposition 6 showcases that the decentralization in Proposition 1 carries over to the problem with risk and default. The PDE corresponding to the valuations, (25), is similar to the correspond-

[31] The proof can be found in Appendix C.11.

ing one without risk, with only a modification in the maturity-specific discount, $\xi_t(\tau)$. In this general case, in addition to impatience and smoothing, this discount rate also captures additional forces: *hedging, self-insurance, credit risk,* and *default incentives* (revenue-echo), as we explain next.

Credit risk is captured by the repayment probability, Θ_t, which is also present in bond prices. Just like bond prices, we also encounter a jump in the valuations $J^X[v_t(\tau)]$, which acts like a "capital loss" in valuations produced by risk. The jump in valuations, $J^X[v_t(\tau)]$, captures the desire to *hedge* risk. Relative to the bond price, we encounter two new terms: a jump in marginal utilities $J^X[U_t']$ and a term we dub as the *revenue echo* Ω_t. The jump term, $J^X[U_t']$, is a ratio of marginal utilities that give us an "exchange rate" between states, the relative value of goods before *vis-à-vis* after the shock. For example, if a shock produces a drop in expenditures, the ratio of marginal utilities associated with that state is greater than one. We will see that this jump accounts for *self-insurance.* In fact, if the government could perfectly insure against shocks with a bond portfolio, its consumption would not jump.

The revenue echo is a subtle force. With an endogenous default decision, when the government issues a τ-bond at time t it must factor in that the bond will impact the probability of default until its expiration date. The increased likelihood of default will impact the auction revenues in all maturities issued on the dates prior to the expiration of the issued bond. With commitment, because markets are forward-looking, the government must also take into account how the prices of bonds issued in the past are affected by the change in the default likelihood. The revenue-echo effect Ω_t captures these calculations. We interpret the formula behind the revenue echo with the aid of figure 3. The x-axis represents time and the y-axis maturity. Consider a marginal increase in the stock of τ_0 bond at date t_0. In the time and maturity dimension, the marginal issuance has a marginal increase in the profile $f_t(\tau)$ in every $\{t, \tau\}$ point along the $(1, -1)$ direction starting from $\{t_0, \tau_0\}$. Fix an arbitrary date in the future, denoted by $t > t_0$ in the figure. The marginal increase in the stock of debt impacts the default probability at t, depending on the valuation of the maturity of the bond at that date; hence the term $\theta_t(V_t) \cdot U'(c_t) \cdot v_t(\tau)$. The impact on the probability of default is encoded into information that travels backward in time, along the rays $(-1, 1)$, affecting all the prices. The double integral in Ω_t adds the decline in revenues on all issuances prior to t that have not expired by t, for any t prior to the expiration of the bond in consideration. A further walk through the formula is presented in Appendix C.10.

Risky steady state. Although Proposition 6 characterizes the solution with risk and default and helps characterize effects, the computation of f is infeasible. Although we consider a single shock instead of a recurrent Poisson process, the problem remains intractable without an approximation. The reason is that any date t before the shock is associated with an expenditure jump which is a function of f. The potential jump affects issuance choices through the valuations. Hence, a solution to the transition is a fixed point, not over a single distribution as in the perfect foresight case, but over a set of debt distributions. An analogous challenge appears in incomplete market models with aggregate shocks. Although there are several numerical approaches to deal with this issue, e.g., Krusell and Smith (1998), Reiter (2009), Ahn, Kaplan, Moll, Winberry and Wolf (2017), or Boppart, Krusell and Mitman (2018), we do not apply those

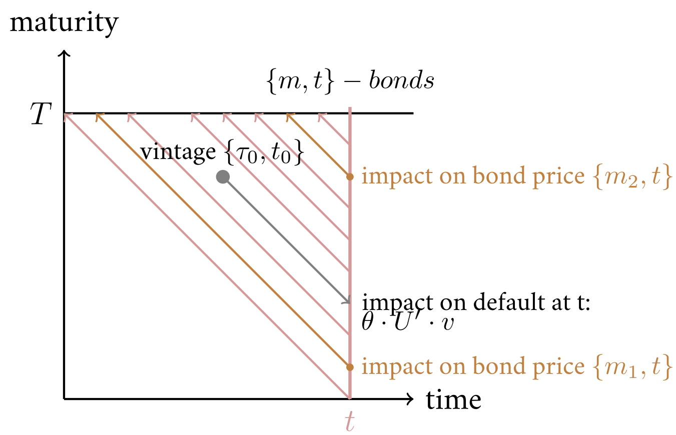


Figure 3: Illustration of the revenue-echo effect (axes inverted).

methods because we are interested in analytical predictions.

To circumvent this challenge, we develop a risky steady-state (RSS) approach that does allow an exact solution to a debt profile with risk and default, a solution that we compute and contrast with the data in the next section. In our context, the RSS is the asymptotic limit before the jump realization. In other words, the RSS is the state of convergence of the solution, when the shock has been expected to arrive forever but has not yet materialized. Since jumps happen once, the solution after the jump is the perfect foresight solution. The assumption of a single shock is reasonable because the aggregate shocks we consider correspond to "disaster shocks" discussed, for instance, by Barro and Ursua (2010). In the rest of the paper, we denote a variable x_t in the RSS by x_{rss}.

We obtain an explicit solution to the valuations in the RSS:

$$\hat{v}_{rss}(\tau) = \delta \cdot \int_0^{\tau} e^{-\hat{\xi}_{rss}\cdot(\tau-s)} ds + e^{-\hat{\xi}_{rss}\cdot\tau} \text{ and } \hat{\psi}_{rss}(\tau) = \delta \cdot \int_0^{\tau} e^{-\hat{\xi}^*_{rss}\cdot(\tau-s)} ds + e^{-\hat{\xi}^*_{rss}\cdot\tau},$$

where

$$\hat{\xi}_{rss}(\tau) = \rho + \phi\left(1 - \mathbb{E}^X\left[(\Theta_0 + \Omega_0) J^X\left[U_0' v_0(\tau)\right]\right]\right) \text{ and } \hat{\xi}^*_{rss}(\tau) = \hat{r}^*_{ss} + \phi\left(1 - \mathbb{E}^X\left[\Theta_0 J^X\left[\psi_0(\tau)\right]\right]\right), \tag{27}$$

and $\Theta_0 = \Theta(V_0)$ and where $v_0(\tau)$ and $\psi_0(\tau)$ are the valuations of the perfect-foresight problem with initial conditions $\hat{f}_{rss}(\tau)$ and X_{rss}. The RSS can be computed as a fixed point in the space of expenditures, as we explain in the Appendix E.

We exploit the RSS to provide a characterization of how these risk-related forces shape the level and WAM of issuances.

Proposition 7. *[**Stochastic Forces**] Consider a RSS with $0 = \delta < r^* < \rho$. We have the following:*

1. [Self-Insurance] Assume default is not possible, $\Theta_0 = 1$, $\Omega_0 = 0$ and consider an expected small negative shock to income $\varepsilon = y_{ss} - y_0$, which reverts back immediately to its steady state value, i.e., $\alpha^y \to \infty$. Then, the *issuance semi-elasticities are negative:*

$$\lim_{\bar{\lambda}\to\infty}\lim_{\alpha^y\to\infty}\epsilon^{\tau}_{rss,\varepsilon} = -\sigma\frac{1-\exp\left(-\phi\tau\right)}{\exp\left(\left(\rho-r^*_{ss}\right)\tau\right)+\exp\left(-\phi\tau\right)-1} < 0.$$

If the government cannot issue at maturities below $\tau^* \equiv \frac{1}{\phi}\log\left(\frac{\phi+(\rho-r^*_{ss})}{(\rho-r^*_{ss})}\right)$ *then* $\lim_{\bar{\lambda}\to\infty}\lim_{\alpha^y\to\infty}\epsilon^{\mu}_{rss,\varepsilon} > 0$ *and the WAM increases.*

2. [Credit Risk] Let the government be risk-neutral, $\sigma = 0$*. Consider a small increase in the default probability,* $\hat{\Theta}_0 = \Theta_0 + \varepsilon$ *. Then issuance semi-elasticities are zero,*

$$\lim_{\bar{\lambda}\to\infty}\epsilon^{\tau}_{rss,\varepsilon} = 0,$$

and issuances do not change with the shock.

3. [Revenue Echo] Let $\sigma = 0$*. Consider small increase in the revenue echo,* $\hat{\Omega}_0 = \Omega_0 + \varepsilon$*. Then the issuance semi-elasticities are negative:*

$$\lim_{\bar{\lambda}\to\infty}\epsilon^{\tau}_{rss,\varepsilon} = -\frac{\exp\left(-\left(\rho-r^*_{ss}\right)\tau\right)\left(\exp\left(\phi\tau\right)-1\right)}{\left[1-\exp\left(-\left(\rho-r^*_{ss}\right)\tau\right)\right]\left(1+\Theta_0\left(\exp\left(\phi\tau\right)-1\right)\right)} < 0.$$

If the default probability is very low, $\Theta_0 \approx 1$*, then* $\lim_{\bar{\lambda}\to\infty}\epsilon^{\mu}_{rss,\varepsilon} > 0$*, and the WAM increases.*

Proposition 7 describes how the different forces introduced here impact the RSS issuances and their WAM.[32] As before, we consider the limit $\bar{\lambda}\to\infty$ of the model where issuances vanish and analyze a drop in income that approximates a one-time jump, $\alpha^y \to \infty$, to isolate the effect of self-insurance from the effects of smoothing. The first item of the proposition characterizes self-insurance in isolation because no item in equations (27) responds to the perturbation except for the jump in marginal utilities, $J^X\left[U'_0\right]$. Recall that $J^X\left[U'_0\right]$ enters the valuation equation (27) in the same way as the impatience coefficient ρ, but with the opposite sign. Thus, the overall effect of the expected perturbation is similar to reducing the domestic discount rate, which reduces issuances and lengthens the WAM.[33] These are hallmarks of self-insurance behavior: taking less debt to reduce payments and extending the maturity to avoid large principal payments when income is low. Notice how the elasticity of issuances scales with σ, which captures the degree of risk aversion, and how the elasticities vanish as risk vanishes, $\phi \to 0$.

The second item in the proposition describes the effect of credit risk. Because we set risk aversion to zero, $\sigma = 0$, and in the limit as $\bar{\lambda}\to\infty$ revenue echo is negligible, this shock is akin to introducing a lottery that erases debts. In this case, risk enters both in the valuation and price equations, (27) and these terms cancel each other in the issuance rule. This is a natural outcome since both the government and investors are risk-neutral. Things are different in the risk-averse case, $\sigma > 0$. If the government is risk-averse,

[32] The proof can be found in Appendix C.12.

[33] Notwithstanding, the effect of self-insurance on the WAM can be ambiguous in certain cases, as the issuance elasticities can be negative for maturities close to expiration.

default is a form of insurance (an idea found, for example, in Zame, 1993); thus, default reduces the impact of self-insurance. We can see this in the valuations in (27) because greater credit risk reduces the effect of the jump in marginal utilities. Thus, we can establish that credit risk per se has no effects on issuance, but coupled with risk aversion, it induces greater issuances at shorter maturities as it mitigates the need for self-insurance.

The final item in the proposition establishes how the revenue echo effect influences issuances and maturity. An increase in the revenue echo captures the idea that issuances have a marginal impact on the default probability. Even though the government is risk neutral, it recognizes that by increasing the default probability it reduces prices at all maturities. Thus, the government considers the effect on marginal revenues into the future and back in time. The revenue echo is positive, and thus, it acts like making the government more patient. Thus, this term reduces issuances and increases the average maturity. With debt dilution, there would be a potentially offsetting effect.

Above, we allude to the idea that the jump in valuations $J^X\left[v_t\left(\tau\right)\right]$, accounts for hedging. To understand why, recall our discussion of maturity-management models under complete markets, Duffie and Huang, 1985; Angeletos, 2002; Buera and Nicolini, 2004. The insight in these studies is that, in the presence of income and interest rate risk, the government should design its debt profile to insulate itself from jumps in expenditures. To do so, the debt profile should produce capital gains that offset the change in the net present value of revenues or debt service produced by a shock. With liquidity costs, that perfect insulation is impossible.[34] Now, consider what happens to issuances before and after a shock. If valuations jump after a shock, the issuance rule tells the government to issue the debt with the value gap that jumps by more, i.e., akin to issuing more in the maturities that become cheaper and less in maturities that become more expensive, a strategy akin to generating capital gains.

The focus of Proposition 7 is on a single shock to income, but we can use it to anticipate more general effects. For example, abstract from default and consider a small shock that either increases or decreases interest rates. Again, we can consider a reversion of the shock. Because of risk-aversion, we can anticipate that this shock will provoke a jump in expected utilities, $\mathbb{E}^X\left[J^X\left[U_0'\right]\right]$, even though the impact on interest rates is symmetric and of equal probability. Thus, this shock will produce a self-insurance behavior: less debt and an extension of maturities.

In the next section, we build on these results to discuss a quantitative application. Before, we discuss some modeling decisions.

Discussion: on stock liquidity costs and limited commitment. As we noted above, the distinction between liquidity costs here and those in Vayanos and Vila (2021) is that, in that model, the price of debt falls with the stock of debt f at a given maturity whereas here it falls with the flow, ι. With stock liquidity

[34] Liquidity costs prevent reallocating a portfolio quickly so realizing capital gains is not immediate and is costly. Even without liquidity costs, income shocks may not be correlated with the international interest rates or the number of maturities may not be enough to span the space of shocks. We discuss a generalization of the conditions that guarantee market completion discussed in Duffie and Huang (1985), Angeletos (2002), or Buera and Nicolini (2004) in Appendix D.

costs, the optimal solution to the debt issuance problem must also consider that expectations of future issuances impact current prices, not only that current issuances impact prices, as happens here. Suppose the government can commit to an issuance policy. In addition to the liquidity costs, valuations will be affected by a revenue echo effect similar to the one provoked by default. That altered revenue echo effect would capture the impact of past and future prices of bonds of the same maturity due to the stock liquidity costs.

The solution and computation of a version of our model with default and without commitment to a debt program would be more complex. The solution's complexity follows because, without commitment, the government treats its future self as a different agent. Thus, it considers how its current decision will affect its future self's incentives to dilute debt, thus impacting prices today in its issuance choice. This insight is known from Aguiar, Amador, Hopenhayn and Werning (see Forthcoming), which studies a two-bond deterministic environment and shows that throughout a transition, the government would control its stock of short-term debt. Many quantitative papers in international finance introduce risk and numerically solve models with two perpetuities that feature debt dilution and insurance.[35] The solution to this problem should display a revenue echo effect similar to the one here but which no longer considers the impact on past prices.

The solution to the optimal debt management problem without commitment and with stock liquidity costs, as in Vayanos and Vila (2021), would inherit this complexity. This point relates to the issue of lack of commitment in public finance models. Lucas and Stokey (1983), anticipated that a government that lacked commitment would ex-post attempt to modify bond prices (or the public's stochastic-discount factor) to make capital gains. Along those lines, Debortoli, Nunes and Yared (2018) show that a government can avoid these commitment costs by issuing long-term debt, the opposite of what occurs in Aguiar et al. (see Forthcoming). Lack of commitment lies outside the scope of our analysis.

6 Quantitative Application

In this section, we calibrate the model and conduct a normative evaluation of Spain's debt management to showcase the applicability of the theory. We begin by describing Spain's debt management strategy and then calibrate the model to evaluate this strategy. We then compare the RSS debt profile generated by the model with the historical data and contrast the model predictions for the responses of revenue and the interest-rate shocks with the data.

Spain's debt management strategy. Before we begin with our evaluation of Spain's debt management policy, we first run regressions on the auction data presented in Section 3. The dependent variables are the total issuances and the weighted average maturity (WAM) of auctioned amounts in each quarter. The independent variables are the primary deficit, the principal amortizations over GDP due, and the level and

[35] For example, Arellano and Ramanarayanan (2012), Bianchi, Hatchondo and Martinez (Forthcoming), and Hatchondo, Martinez and Sosa-Padilla (2016) study debt management with two-assets. In Sánchez, Sapriza and Yurdagul (2018) and Bocola and Dovis (2018) the government refinances every period by choosing the debt amount and a single maturity. Hatchondo, Martinez and Roch (2018) compares optimal holdings of long-term debt under full commitment in a version of Eaton and Gersovitz (1981) to a version without commitment.

slope of the yield curve, of the same quarter. To obtain a time series for the level and slope factors of the Spanish yield curve, we use estimates of a dynamic Nelson-Siegel four-factor model, which are used in the construction of the Spanish yield curve—see Diebold and Rudebusch (2012).

We summarize data and the regression coefficients in the four panels of Figure 6—the values are found in Table 5 in the Appendix. Panel (a) depicts the issuances against the deficit and the principal due. We observe a clear pattern: Spain consistently issued debt at about 10% of GDP, with a slight downward trend during the 2000s. By 2007, the primary deficit began a dramatic increase, surging up to 20 percent of GDP. Principal amortizations increased at a slower pace, as the debt increase would take time to mature. Whereas the deficit has fallen continually since 2009, principal amortizations continued to pile up during the sovereign debt crisis. The coefficients, which we obtain from a simple regression, are presented graphically in Panel (c). Diamonds represent the point estimates and bars the confidence intervals. The takeaway is that issuances correlate positively with deficit and principals—approximately 0.75. Issuances are negatively correlated, although not significant, to the level of the yield curve, which corresponds to the short-term rate.

In terms of the drivers of the WAM, Panel (b) reports the yearly WAM against the level and slope of the yield curve. We observe significant changes in the average issuance maturity, which correlate with the yield-curve factors. Panel (d) corroborates this and presents the regression coefficients. The level of short-term rates negatively correlates with the WAM, whereas the slope of the yield curve, correlates positively—deficits and principal payments affect the maturity modestly. Below we investigate whether these correlations are also prescriptions obtained from the model.

Calibration. We calibrate the model to fit the Spanish data over the period January 2002 to April 2018. We externally calibrate the parameters that are exogenous to Spain's debt management office. In particular, we feed the model with the estimated liquidity costs of Section 2. We back out the rest of the parameters to replicate moments. In line with the evidence presented in Section 3, we modify the government's problem so that it issues in the discrete maturities of the set $\mathcal{M}$.[36] The government's budget constraint (7) is modified to:

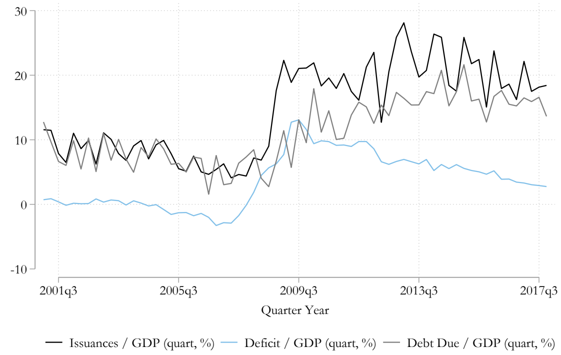
$$c_t = y_t - f_t(0) + \sum_{\tau \in \mathcal{M}} \left[q_t(\tau, \iota)\iota_t(\tau) - \delta f_t(\tau) \right]. \tag{28}$$

We solve the model using a finite-difference method described in Appendix E, interpreting the frequency as monthly, so that, consistent with the data, bonds are issued every period. The computation of a risky steady state takes less than 30 seconds on a personal desktop and a full transition only a few minutes.

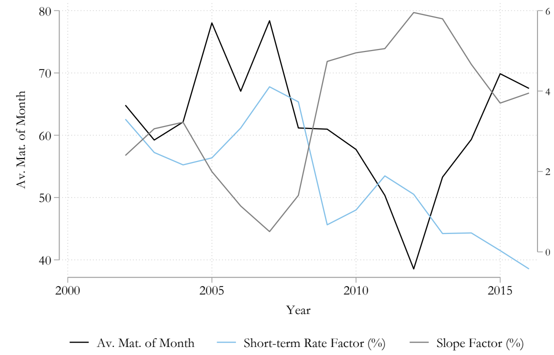
As we explained, our model is suited for bond issuances but not bill issuances, because we cannot estimate the auction price impact at those very short maturities. Because bills are short-lived, they represent only a minor fraction of the debt stock, and thus, only a small fraction of the debt-to-GDP ratio. Still, we want to account for bill issuances in the model, and to do so, we treat the issuances of bills as exogenous.

We also modify the liquidity costs, $\bar{\lambda}(\tau)$, so that they now vary with maturity, as in the data. The optimal issuance rule is modified by replacing $\bar{\lambda}$ by $\bar{\lambda}(\tau)$ of the corresponding maturity. To map the theoretical to

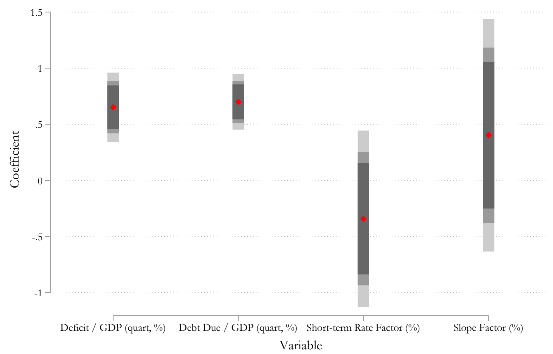
[36]Namely, 3, 6, 9, 12, and 18 months and 3, 5, 10, 15, and 30 years. We exclude 50-year bonds.



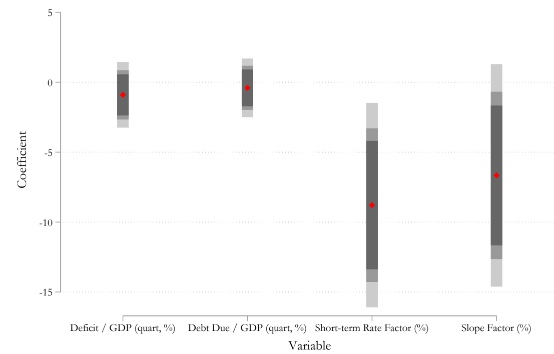
(a) Issuances, deficit and amortizations



(b) Maturity, level and slope factors



(c) Regression coefficients of issuances



(d) Regression coefficients of maturity

Figure 4: Issuance Pattern

Panel (a) depicts the quarterly debt issuances, principal amortizations, and the primary deficit as a fraction of quarterly GDP from 2001 Q1 and 2017 Q3. Panel (b) depicts the yearly series for the average maturity in the year (weighted by issuances size) against the level and slope factors of the yield curve. Panels (c) and (d) report the regression coefficients of quarterly issuances and quarterly maturity against the quarterly principal amortizations, the quarterly primary deficit, and the quarterly average of the daily level and slope factors.

the estimated values in equations (4) and (5), we use a monthly aggregation:[37]

$$\frac{q_t(\tau)}{\psi_t(\tau)} - 1 \approx -\underbrace{\frac{1}{2}\bar{\lambda}\,(\tau)}_{\Lambda(\tau)}\underbrace{\frac{1}{\Delta t}\int_t^{t+\Delta t} \iota_t(\tau)dt}_{I_t(\tau)},$$

where $\Delta t = 1/12$. Hence, $\bar{\lambda}\,(\tau \in \{36, 60, 120, 180, 360\}) = 2\Lambda\,(\tau) = 2 \cdot 10^{-2} \cdot \{8, 12, 17, 40, 56\}$.[38] The rest of the parameters and their values are presented in Table 6.

We set the (real) coupon rate δ to 2.4 percent, based on the average nominal coupon rate of 4.2 percent in the period, which we compute from the microdata, and the average 1-year ahead inflation expectation of 1.8 percent, derived from inflation-linked swaps (ILS).

[37] As output is normalized to one in the steady state, issuances are expressed as a share of average GDP.

[38] We ignore the maturity-specific average markup $\alpha\,(\tau)$ of the empirical specification (5), as its impact of the optimal issuance strategy is quantitatively meaningless.

To calibrate $r^*_{rss}(\tau)$, we employ estimates of the Spanish and French nominal yield curve estimates.[39] For every date, we have estimates of the two yield curves from which we construct sample averages. We construct a zero-coupon real yield curve $r^*(\tau)$, as the difference between the nominal zero-coupon yield curve $i(\tau)$ on Spanish debt minus the expected euro inflation curve $\pi(\tau)$ and the same for French yields. We interpret the French real curve as the *risk-free curve* and treat the difference as the default premium of Spain. We display the average real yield curve for Spain in the dashed red line in Panel (a) of figure 6 which we use to calibrate a process for r^*_t.

Following Barro (2006), we consider the possibility of a large crisis in which revenues collapse and the short-term rate spikes. With this disaster, we capture an event with a meaningful decline in fiscal revenues, like those observed during the 2008 financial crisis or the Covid-19 crisis together with a flight-to-quality episode. We calibrate the realization intensity ϕ, using the slope of the yield curve.

We calibrate the real short-term rate process to produce the RSS risk-free yield curve, which anticipates the disaster event. We set the steady-state real rate r^*_{ss} to -0.4 percent, such that the 1-year yield on risk-free bonds in the model is 1.6%, coinciding with the real 1-year yield on our target for the risk-free yield curve. The rest of the curve is generated with a limited number of parameters. We calibrate the arrival rate of the shock ϕ to 2 to replicate the slope of the yield curve at short-end (0-18 months), and the persistence α^r to 0.0004, to replicate the almost-flat slope at the long-end (15-30 years). We set the after-shock value $r^*_{t_0}$ to 3.3 percent, such that the 30-year real rate is 2.9 percent (5.2 nominal rate - 2.3 expected inflation). This process generates a yield curve that approximates well the yield curve we construct from the data—the solid blue line in Panel (a) in Figure 6 corresponds to the yield curve generated by this process.

We set the average income y_{ss} to one as a normalization. We treat a disaster as a 5 percent fall in revenues corresponding to a large recession in the data in Barro and Ursua (2010). The average frequency is set to 10 years, roughly a major shock per decade. We set $\alpha^y \doteq 0.2$ to match estimates of the persistence.[40]

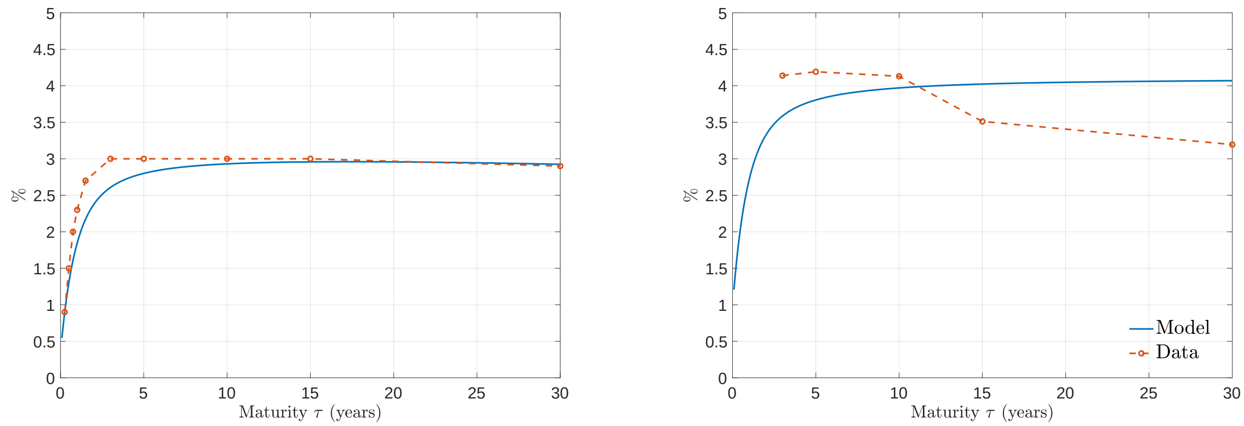
Regarding the parameters that govern credit risk, the output loss after default, κ, is set to 2 percent, following Aguiar and Gopinath (2006). Following the discrete-choice literature, the distribution of the shift to the autarky value, ε, is a logistic with coefficient ς. We set ς to 2.3, to reproduce a 3-month bond credit risk premium of 60 bps. The unconditional default probability is 0.56 percent—roughly one default every 190 years, close to the reported frequencies in Reinhart and Rogoff (2009).

We set the inverse of the intertemporal elasticity of substitution, σ, to 2, a standard value. Finally, we calibrate the subjective discount factor, ρ, to replicate the average of total annual issuances over GDP (17 percent). We obtain the best fit when we set ρ to 4.1 percent.

Panel (b) of Figure 6 compares the domestic valuations in the model (solid blue line) with their empirical counterparts (dashed red line), constructed by inverting the optimal issuance rule, $v(\tau) = \psi(\tau) -$

[39] We employ a dynamic 3-factor model provided to us by Jens Christensen.

[40] The detrending uses a Hodrick-Prescott filter with a parameter of 1600. We estimate an AR(1) process $\log y_t = \rho_y \log y_{t-1} + \sigma_y \epsilon^y_{t-1}$ for the detrended seasonally adjusted output. The estimated persistence of quarterly income (ρ_y) is 0.95. This corresponds to a value of $\alpha^y = (1-\rho_y)/(3 \times \Delta t) = 0.2$.



(a) Bonds

(b) Domestic valuations

Figure 5: Zero-coupon yield curves of bonds and domestic valuations.

Notes: Panel a: Zero-coupon real yield curve for Spain. The empirical counterpart is constructed as the difference between the nominal zero-coupon yield curve on Spanish debt minus the expected euro inflation curve, derived from ILS plus 32 bps to all maturities to compensate for the constant in the auction markup estimates. Panel b: Zero-coupon domestic valuations yield curve for Spain. The empirical counterpart is constructed using the optimal issuance formula with bond prices, issuances and liquidity costs taken from the data.

$\bar{\lambda}(\tau)\,\iota(\tau)$, where $\psi(\tau)$ is the market price of a zero-coupon bond in the data, $\bar{\lambda}(\tau)$ are the empirical liquidity costs and $\iota(\tau)$ the average issuances per maturity over the period.

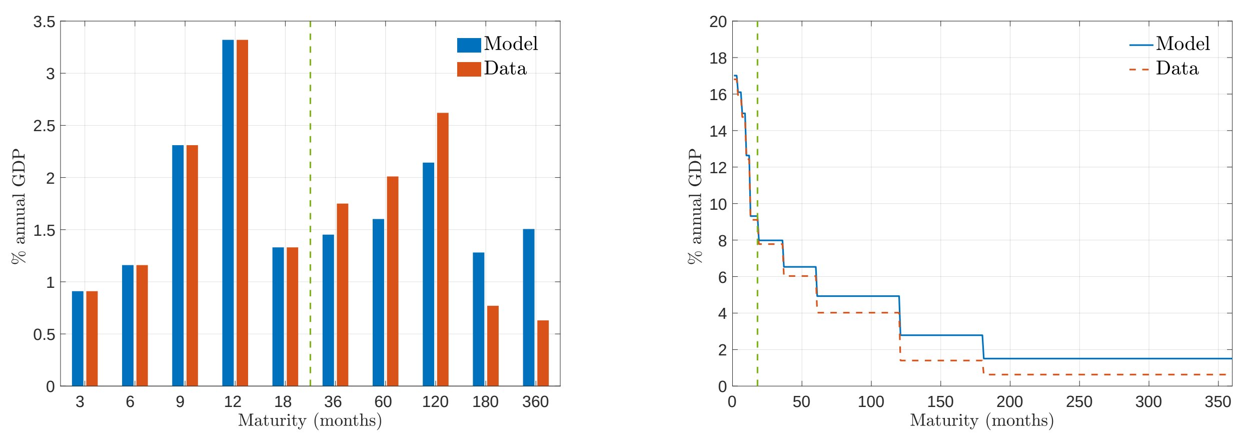
	Description	Value	Target / source
Bonds			
T	Maximum maturity (years)	30	Maximum maturity of Spanish bonds
r^*_{ss}	Steady-state real rate (p.a)	-0.4%	1-year risk-free real rate of 1.6%
ϕ	Shock arrival rate	2	Slope of the yield curve (0-18 months)
$r^*_{t_0}$	After-shock rate (p.a)	3.3%	30-year real rate 2.9%
α^r	Rate shock persistence	0.0004	Slope of the yield curve (15-30 years)
δ	Coupon (p.a)	2.4%	Average real coupon 2.4%
Income			
y_{ss}	Steady state income	1	
y_{t_0}	After-shock income	0.95	Large recession in Barro and Ursua (2010)
α^y	Income shock persistence	0.2	GDP shocks persistence
Default			
κ	Post-default output loss	2%	Literature
ς	Parameter preference shock	2.3	3-month default premium 60 bp
Preferences			
σ	1/IES	2	Literature
ρ	Discount rate (p.a.)	4.1%	Annual issuance over GDP 17 percent

Table 1: List of calibrated parameter values.

Evaluation and force decomposition. Armed with the calibration, we parse out whether, given the parameters that rationalize the average debt to GDP ratio of Spain during the period, the average issuance and debt profiles in Spain are close to optimal vis-à-vis the profile generated by the model. Recall that we have deduced only one preference parameter, but the issuance profile contains information on average issuances for five exogenous maturities (3, 5, 10, 15, and 30 years), leaving us with four untargeted moments.

Panel (a) in Figure 6 contrasts the average annual *issuances* over GDP by maturity in the data against the corresponding values in the RSS of the model. Panel (b) compares the average *outstanding* debt profile in the data and the model.[41] Maturities below 18 months are matched by construction. Hence, we base the evaluation on the fit to maturities above 18 months. The key message is that the model replicates relatively well Spain's debt-management strategy, though it is slightly biased towards long maturities. The WAM of issuances produced by the model, 6.2 years, is higher than the one observed in the data, 4.7 years. As a result, the model has a distribution of *outstanding* debt with more debt at the curve's long end.

As we can observe, the model reproduces the hump-shape issuances that peak at ten-year maturities. The reason behind the hump shape is the liquidity costs. The fit is surprisingly good and indicates that Spain has conducted close to optimal policies. The room for improvement is that Spain should issue more at the long end of the curve, 50% more in 15 years and twice in 30-year debt, and about 20% less in 3, 5, and 10-year debt. Next, we present a force decomposition to shed light on what explains this recommendation.



(a) Issuances

(b) Debt stock

Figure 6: Issuance and debt stock by maturities.

Notes: Panel a: Average yearly issuances as a percentage of annual GDP (data) versus RSS issuances (model). Panel b: Long-run debt stock, computed using perpetual inventory from issuances.

The force decomposition builds on Proposition 6, and consists of turning on and off the terms associated with the different forces in the different terms in the interest and discount rates. To do this, we compute

[41] For consistency, we construct the data counterpart as if the observed average issuances are maintained indefinitely.

counterfactual values for v and ψ by suppressing different terms, and use the optimal issuance rule to calculate counterfactual issuances that isolate particular forces.

We present the decomposition outcomes through the waterfall chart in Figure 6. For ease of exposition, we perform the decomposition for only three maturities, namely the 3-, 10-, and 30-year issuances. 3-year and 30-year bonds are displayed in panels (a) and (b), respectively, whereas 10-year bonds are shown in Appendix F. The decomposition corresponding to 5- and 15-year bonds are close to those for 3 and 30-year bonds, respectively. Each bar in the panels represents the change in the issuance produced by the force. The first column represents the deterministic steady-state issuances with common liquidity coefficients, in which the only force is impatience.[42] The second column introduces the impact on issuances due to the upward-sloping risk-free yield curve. The third column captures the effect of replacing the common liquidity coefficient by the estimated heterogeneous liquidity costs.[43] The subsequent bars correspond to the contribution of the self-insurance (including hedging), credit risk, and revenue echo terms. The final column is the sum of all the forces, and it reproduces the baseline.[44]

Let us now discuss why the model calls for less issuances of 3-, 5-, and 10-year bonds and more of 15- and 30-year bonds. It is clear from all panels that, with only impatience, all issuances would increase, and more so at the longest horizons. Thus, with only impatience, the debt-to-GDP ratio would be substantially higher. Impatience is offset at all maturities after we introduce the actual bond prices: this is because $\xi^*_{rss}(\tau) > r^*_{ss}$ due to the presence of upward risk in the short term. This effect is more important for long maturities, as the yield curve is upward-sloping.[45] From the third force, heterogeneous liquidity, we can observe that the hump shape in the optimal issuances is provoked by the greater liquidity of 3- and 10-year bonds, which contributes positively to issuance. By contrast, 30-year issuances drop by almost 50% after considering their higher liquidity costs.

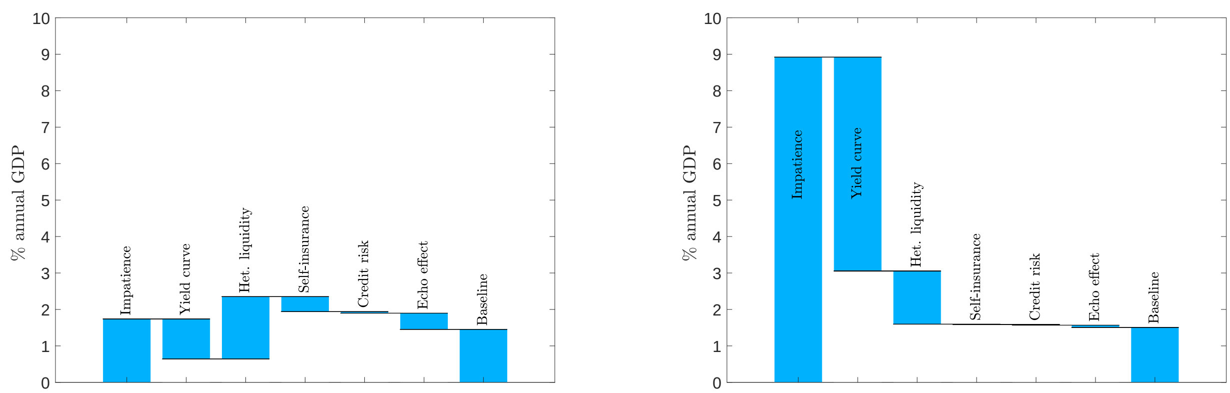
We can observe that the additional forces, self-insurance, default risk, and the revenue echo, are not significantly important for the 30-year bond issuances. The reason is similar to the intuition discussed in Section 4 regarding why impatience matters for middle maturities, but much less for long ones. As we explained, this is because long-term issuances are not very elastic to small changes in domestic discount rates. This is not the case with 3- and 10-year bonds. For example, for 3-year bonds, the combined effect of self-insurance and hedging produces a reduction of almost 0.5 pp in issuances, and the revenue echo accounts for another 0.5 pp reduction. The impact of credit risk is small as its effect on valuations is approximately compensated by that on bond prices. According to Proposition 7, the theoretical effect is exactly zero when the government is risk-neutral.

[42]We start by assuming no issuances, and then compute the issuances we would obtain if $\xi_{rrs} = \rho$ and $\xi^*_{rss} = r^*_{ss}$, as in the steady-state with homogeneous liquidity costs $\bar{\lambda}$. We calibrate the common $\bar{\lambda}$ such that the total volume of issuances coincides with the earlier calibration.

[43]In this case, we obtain issuances by replacing $\bar{\lambda}$ with the heterogeneous liquidity costs $\bar{\lambda}(\tau)$.

[44]Thus, we compute ξ_t with the terms $\phi\mathbb{E}^X\left[1 - J^X[v_0(\tau)] \cdot J[U_0]\right]$, $\phi\mathbb{E}^X\left[1 - \Theta_{rss} \cdot J^X[v_0(\tau)] \cdot J[U_0]\right]$, and $\phi\mathbb{E}^X\left[1 - (\Theta_{rss} + \Omega_{rss}) \cdot J^X[v_0(\tau)] \cdot J[U_0]\right]$, respectively.

[45]Greenwood et al. (2015) highlight the trade-off between its desire to issue "cheap" short-term securities and its desire to manage fiscal risk. This trade-off is also captured in this paper.



(a) 3-year bond (b) 30-year bond

Figure 7: Force decomposition.
Notes: Panel a: Decomposition of the RSS issuance of 3-year bonds. Panel b: Decomposition of the RSS issuance of 30-year bonds.

All in all, our methodology suggests that Spain either overweighted the liquidity costs of its long-term debt, issuing too little in the long end of the yield curve, or underweighted self-insurance and the revenue echo effect at the short end.

Dynamic responses Up to this point, the discussion has centered on the forces that affect the optimal maturity profile in the RSS. Next, we turn to the forces that impact the dynamics: yield-curve riding and smoothing. As we explained above, we cannot exactly solve the optimal responses with large recurrent shocks. Instead, we consider "small" perturbations around the RSS, such that the impulse responses of the model are approximately linear.[46] The difference here is that we approximate the stochastic dynamics around the RSS and not the DSS. In order to do so, we compute valuations and prices using modified version of the short-term and discount rates (23-26) where we freeze the default probabilities, the revenue-echo term and the jumps to their RSS values:

$$\xi_t(\tau) = \rho + \sigma \frac{\dot{\hat{c}}_t}{\hat{c}_t} + \phi \cdot \left(1 - \mathbb{E}_t^X \left[(\Theta_0 + \Omega_0) \cdot J^X \left[v_0(\tau)\right] \cdot J^X \left[U_0'\right]\right]\right).$$

and

$$\hat{\xi}_t^*(\tau) = \hat{r}_t^* + \phi \cdot \left(1 - \mathbb{E}_t^X \left[\Theta_0 \cdot J^X \left[\psi_0(\tau)\right]\right]\right).$$

The motivation is that small shocks do not alter the RSS objects significantly. Naturally, if the deviation is zero, the approximation yields the RSS interest and discount rates.

[46] This approach is standard in the heterogeneous-agent literature, either by computing the first-order perturbation around the deterministic steady state (DSS) numerically, as in Reiter (2009) or Ahn et al. (2017), or computing MIT shocks from the DSS, as in Boppart et al. (2018).

Figure 6 displays the dynamics that follow income and interest-rate shocks. We treat the income shock as a temporary 1 percent decline in income. We consider a temporary 0.1 percent increase in the short-term rate for the interest rate shock. In both cases, we set the persistence to 0.2, which produces the same rate of mean reversion as a discrete-time AR(1) process with a quarterly persistence of 0.95.

The same forces that operate in the deterministic model of Section 4, namely smoothing and yield-riding, drive the dynamic responses to these shocks. The solid blue line depicts the responses to the income shock. In the case of income shocks, market prices are constant. Hence, the desire to smooth expenditures is the only factor shaping the debt dynamics. Panels (c) and (d) show how the fall in revenues produces a decline in expenditures on impact, followed by a recovery. The initial fall in expenditure growth leads to an increase in the domestic discount, reverting to the steady state (panel b). As the discount increases, valuations decrease, which acts as a temporary increase in impatience. The optimal issuance rule (15) dictates an increase in the issuances at all maturities (panel e) and a decrease in the WAM (panel f), as we should expect from Proposition 4.[47] The same pattern emerges in the simple correlations obtained from the data, which reveals that Spain's debt program is consistent with the qualitative prescriptions from the model.[48]

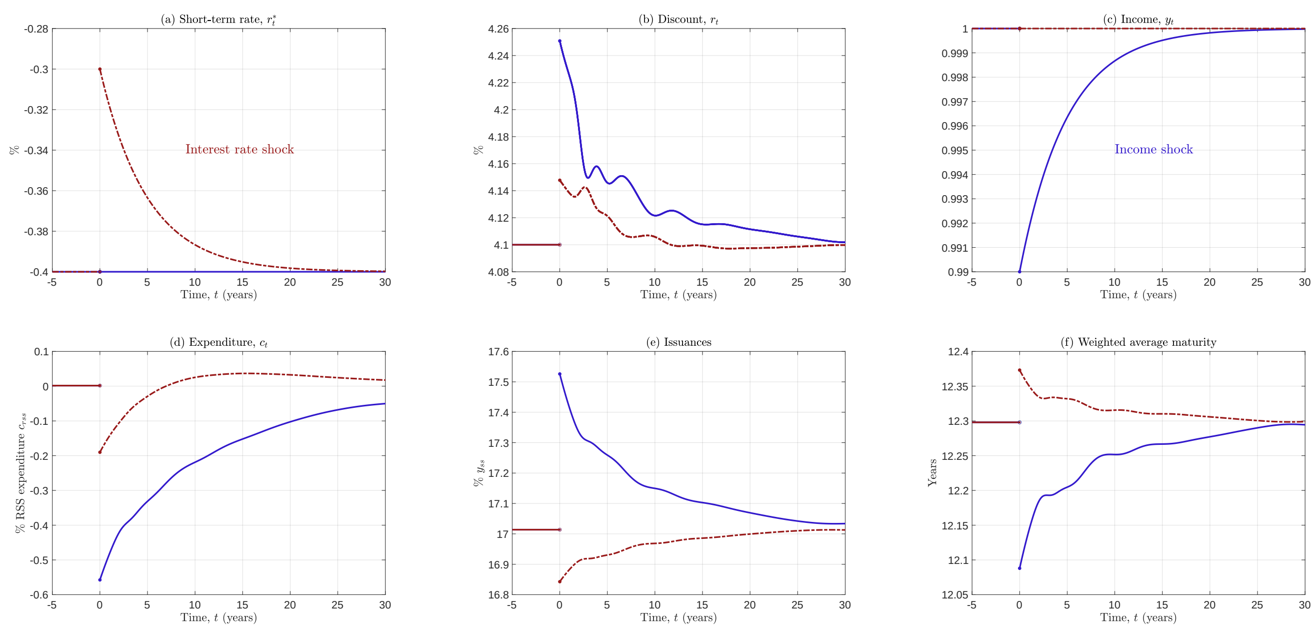


Figure 8: Impulse response functions to income (solid blue) and interest rate (dashed red) shocks.

The shock to the interest rates, depicted in red in Panel (a), also produces a pattern consistent with the theoretical predictions of Proposition 4. On impact, expenditure falls, and the domestic discount jumps,

[47] Notice that this is the WAM of issuances at or above 3 years, as issuances below that maturity remain constant in our algorithm.

[48] In terms of smoothing, we also study the optimal maturity profile when the government expects a large principal payment in the future. In particular, in Appendix F we consider that, at time 0, the government faces the RSS distribution that we perturb with an additional stock of 10-year zero-coupon bonds that amount to 10% of GDP. The responses are consistent with the empirical pattern.

tracking the rate's path, as shown in panels (b) and (c). This impact narrows the value gap across all maturities. The gap then widens again as expenditures recover. The initial effect of a narrower value gap is a decrease in all issuances, as shown in Panel (e). This effect captures the notion that, upon an interest rate increase, the government sacrifices present expenditure to mitigate a higher debt burden. A noticeable feature is that the interest-rate shock produces an increase in the WAM which occurs exactly when the yield curve slopes downward.

Application: Lower long-term rates and a 50-year bond. The fiscal stress provoked by the Covid-19 crisis has brought new considerations to debates on optimal debt-management practices. Two themes stand out: first, the possibility of a transition to low-interest environments, and second, the introduction of bonds of very long maturity, such as fifty-year or hundred-year debt. We now exploit the framework to compute optimal debt strategies in this novel environment.

Through the model, we can interpret a decline in the transition to a low interest-rate environment as a permanent reduction in both the steady-state short-term rate, r^*_{ss} and ρ.[49] Thus, we modify the previous approach to study the transition from an initial RSS to a terminal RSS. To compute a transition, we first compute both the initial and the terminal RSS. Then, starting from the debt profile of the initial RSS, we compute the path of consumption consistent with issuance rules given by the price and valuation equations (22) and (25) where the terminal RSS approximates the jump terms.

Using this approach, we study a transition from the original RSS presented above to a RSS with lower rates. In particular, we study a 1pp reduction in long-term rates—engineered via a reduction the jump in short-term rates from 3.3% to 2.1%—together with a 1pp decline in ρ. We compare the transitional dynamics to a low interest-rate environment with the transitional dynamics to a low interest-rate environment where Spain can also issue 50-year debt. To study the effects of introducing 50-year bonds, we modify $\bar{\lambda}(\tau)$. In particular, we use the parametric estimation of $\bar{\lambda}(\tau)$ to obtain a value for $\bar{\lambda}(600)$. According to our microfoundation, the value of $\bar{\lambda}(600)$ is associated with a customer flow μ. Thus, we correct all $\lambda(\tau)$ to keep a constant total customer flow. Because the linear form of $\bar{\lambda}$ is steep, the customer flow associated with 50-year debt is tiny and has only a minor impact on other liquidity coefficients.

The results are presented in Figure 6. Panel (a) shows the issuance profile for the baseline, the low interest-rates environment, and the low interest-rate environment with 50-year debt. As we can observe, the low-interest environment induces greater issuances at all maturities: the issuance flow increases from 17.0% to 17.7% percent of GDP, and the WAM increases marginally from 12.3 to 12.4 years, which leads to an increase in the RSS stock of debt from 106% to 114% of GDP. Using our force decomposition, we find that the flattening of the yield curve is the primary source of this difference, more than compensates for the lower government impatience. More striking differences emerge when we introduce 50-year debt. First, although the liquidity costs associated with 50-year debt are almost 50 percent higher than that of 30-year debt, the greater relative impatience of the government also produces a higher value gap, which leads to issuances in 50-year debt of a similar amount to 30-year debt. As a result, issuances over GDP increase

[49] This occurs if we introduce growth into the model and assume that low interest-rates are linked to low growth.

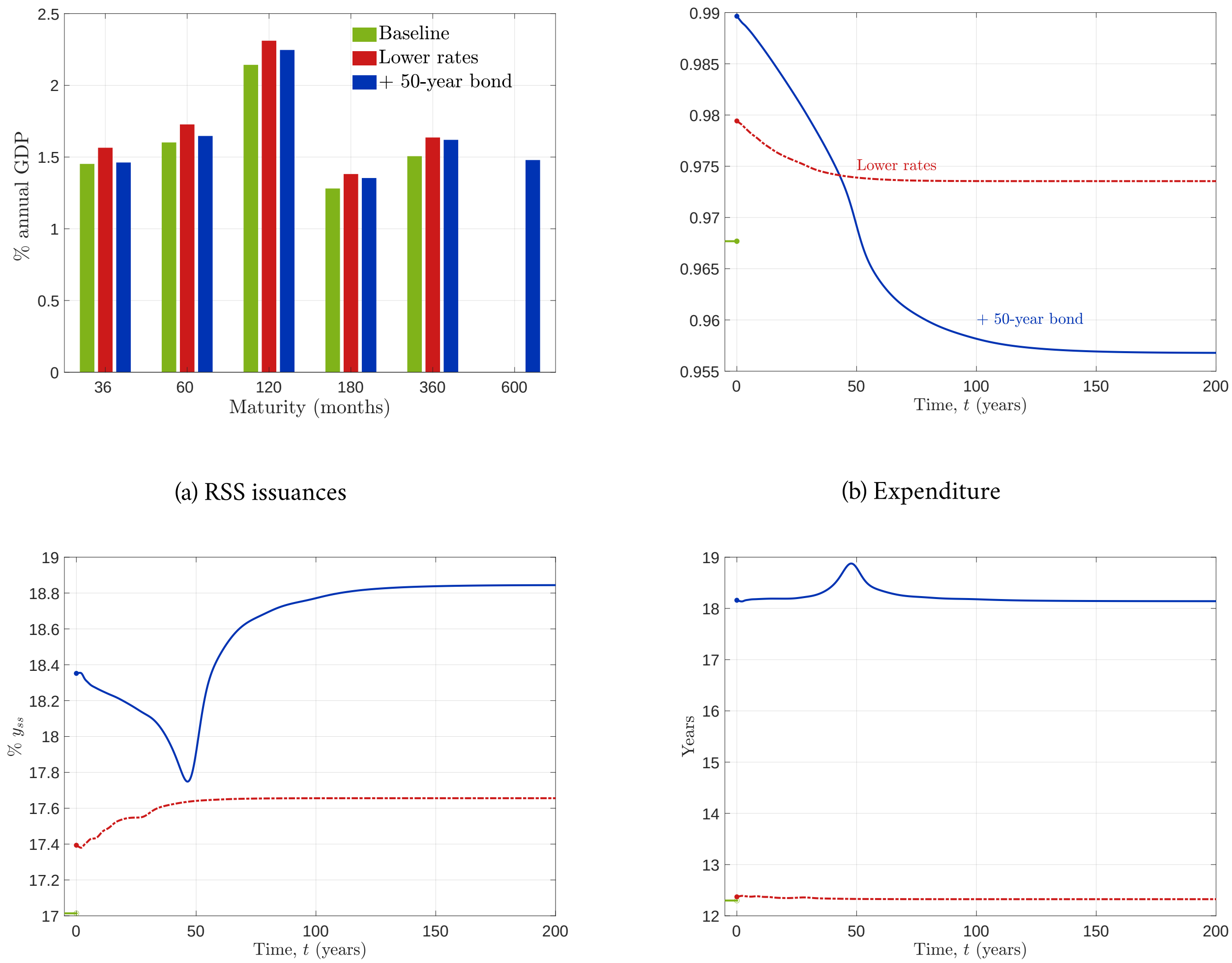
to 18.9% and the WAM to 18.8 years. The increase in the WAM leads to a whopping increase in the RSS debt-to-GDP, to 186% of GDP.

Panels (b), (c), and (d) illustrate the path of expenditures and the level and WAM of issuances, respectively, throughout both transitions to the new regimes. Let us first discuss the transition to a low-interest-rate environment—depicted in the red dashed line. The shock lowers rates but also makes Spain more patient. As we explained above, the shock provokes a higher issuance level and an increase in the WAM; interestingly, on impact, the shock induces an expenditure boom. After the initial boom, expenditures continue to fall toward a higher expenditure level, which is possible with a higher debt stock because of the lower debt-service cost. Consider now the transition that occurs in tandem with the possibility to issue fifty-year debt. The new transition also features an initial boom, but the economy returns to a substantially lower expenditure level. In this case, the boom occurs because of the lower rates and the willingness to issue 50-year debt. As we noted, opening up markets for 50-year debt induces the country to run a much higher level of long-run debt to GDP in the RSS. Interestingly, the initial boom in debt issuances leads to a substantial increase in the debt principal 50 years later. After those fifty years, Spain will have issued continuously in fifty-year debt, so the debt services sequence is smoother after that. The path of expenditure initially increases but eventually reaches a lower point as the 50-year debt begins to accumulate, adding up to more significant financial expenses. Throughout the transition, the country slowly builds up its stock of debt. Thus, the transition toward lower expenditures is prolonged, taking almost a hundred years until the economy reaches its new RSS expenditure level. Naturally, although the RSS level of expenditures is lower the country experiences a welfare gain at the start of the transition. In a policy evaluation, we should contrast this welfare gain against any costs associated with opening up debt markets of specific maturity.

7 Conclusions

This paper aims to present a new approach to study debt management problems with debt instruments that resemble those that governments issue in practice. A central feature of the framework is liquidity costs that limit immediate re-balancing across maturities. The main challenge of these problems is that the state variable is a distribution. The paper showcases a general principle: optimal issuances are proportional to the gap between the market price of debt and its domestic valuation. The presence of risk and default modifies the formulas for prices and valuations, but the optimal issuance rule remains the same. The paper highlights classic forces and uncovers new ones that shape the optimal debt-maturity distribution.

As the first step in this new direction, the framework naturally faces limitations. One limitation is that we only obtain exact solutions when assuming that shocks occur only once. Thus, perturbations around the RSS only capture the expected impulse response up to a first-order. An extension that would admit recurrent shocks would face the same computational challenges as heterogeneous agent models with aggregate shocks. Second, we consider a specific form of liquidity costs. However, as we stress throughout the paper, the framework can be extended to allow for more general liquidity costs that, for instance, vary with maturity or with the amount of outstanding debt. Finally, when we study default, we assume that the



(a) RSS issuances

(b) Expenditure

(c) Issuances

(d) WAM

Figure 9: RSS and transitional Dynamics to low interest-rate environment (dashed red) and additional long-term bond (solid blue).

government can commit to an issuance path. Without commitment, we would have to compute terms that capture how current actions discipline the future self.

As a mode of conclusion, we end the paper with a discussion of our theory's other potential applications.

Other Applications and Extensions. In recent policy debates, Cochrane (2015) proposes that governments should switch to using consols.[50] We can easily adapt the framework to evaluate a switch to consols. We can adapt the applications in the previous section to evaluate the benefits of introducing consols. We can also extend the framework to allow for different values of coupons at different points in time. To do so, we can treat debt that differs in maturity and coupons as different markets. If one wants to force issuances with a specific coupon at a given time, one can set the liquidity costs of bonds with other coupons to infinity.

[50] The idea behind the proposal is that by avoiding the maturity of large principle payments in one date, the country can avoid substantial interest-rate risk—a similar point is made in Debortoli et al. (2018).

We can also let the coupon structure be endogenous by forcing issuances to be in only one coupon at a time. The version here only incorporated income and interest rate risk. We could introduce bonds that differ in their currency denomination to study exchange rate risk. To do so, we would only need to treat each bond category as a different market. Finally, we can extend the model to incorporate shocks to λ itself, interpreted as liquidity risk in the spirit of Calvo (1988) or Cole and Kehoe (2000). These extensions would introduce new forces to the optimal debt strategy.

References

Achdou, Yves, Jiequn Han, Jean-Michel Lasry, Pierre-Louis Lions, and Benjamin Moll, "Income and wealth distribution in macroeconomics: A continuous-time approach," Technical Report, National Bureau of Economic Research 2017.

Aguiar, Mark and Gita Gopinath, "Defaultable debt, interest rates and the current account," *Journal of International Economics*, 2006, *69* (1), 64 – 83.

_ **and Manuel Amador**, "Sovereign Debt: A Review," Technical Report, National Bureau of Economic Research August 2013.

_ **,** _ **, Hugo Hopenhayn, and Iván Werning**, "Take the short route: Equilibrium default and debt maturity," *Econometrica*, Forthcoming.

Ahn, S, Greg Kaplan, Benjamin Moll, Thomas Winberry, and Christian Wolf, "Micro Heterogeneity and Aggregate Consumption Dynamics," *NBER Macroeconomics Annual, forthcoming*, 2017.

Aiyagari, S Rao, Albert Marcet, Thomas J Sargent, and Juha Seppälä, "Optimal taxation without state-contingent debt," *Journal of Political Economy*, 2002, *110* (6), 1220–1254.

Angeletos, George-Marios, "Fiscal policy with noncontingent debt and the optimal maturity structure," *The Quarterly Journal of Economics*, 2002, *117* (3), 1105–1131.

Arellano, Cristina, "Default Risk and Income Fluctuations in Emerging Economies," *The American Economic Review*, 2008, *98* (3), 690–712.

_ **and Ananth Ramanarayanan**, "Default and the Maturity Structure in Sovereign Bonds," *Journal of Political Economy*, 2012, *120* (2), pp. 187–232.

Barro, Robert, "Optimal Management of Indexed and Nominal Debtl," *Annals of Economics and Finance*, 2003, *4*, 1 – 15.

_ **and José Ursua**, "Barro-Ursua Macroeconomic Data," 2010.

Barro, Robert J, "On the determination of the public debt," *Journal of Political Economy*, 1979, *87*, 940–971.

Barro, Robert J., "Rare Disasters and Asset Markets in the Twentieth Century," *The Quarterly Journal of Economics*, 2006, *121* (3), 823–866.

Bhandari, Anmol, David Evans, Mikhail Golosov, and Thomas J. Sargent, "Fiscal Policy and Debt Management with Incomplete Markets*," *The Quarterly Journal of Economics*, 2017, *132* (2), 617–663.

Bianchi, Javier, Juan Carlos Hatchondo, and Leonardo Martinez, "International reserves and rollover risk," *American Economic Review*, Forthcoming.

Bocola, Luigi and Alessandro Dovis, "Self-Fulfilling Debt Crises: A Quantitative Analysis," *Working Paper*, 2018.

Boppart, Timo, Per Krusell, and Kurt Mitman, "Exploiting MIT shocks in heterogeneous-agent economies: the impulse response as a numerical derivative," *Journal of Economic Dynamics and Control*, 2018, *89* (C), 68–92.

Breedon, "On the Transactions Costs of UK Quantitative Easing," *Journal of Banking and Finance*, 2018, *88*, 347–356.

Buera, Francisco and Juan Pablo Nicolini, "Optimal maturity of government debt without state contingent bonds," *Journal of Monetary Economics*, 2004, *51* (3), 531–554.

Bulow, Jeremy I. and Kenneth Rogoff, "Sovereign Debt: Is To Forgive To Forget?," Working Paper 2623, National Bureau of Economic Research June 1988.

Calvo, Guillermo A, "Servicing the public debt: The role of expectations," *The American Economic Review*, 1988, pp. 647–661.

Camacho, Maximo and Gabriel Perez-Quiros, "Spain Short Term Indicator of Growth," Working Papers 0912, Bank of Spain Working Papers Homepage June 2009.

Cammack, Elizabeth B., "Evidence on Bidding Strategies and the Information in Treasury Bill Auctions," *Journal of Political Economy*, 1991, *99* (1), 100–130.

Cochrane, John H, "A new structure for US federal debt," *Hoover Institution Economics Working Papers*, 2015.

Cole, Harold L. and Timothy J. Kehoe, "Self-Fulfilling Debt Crises," *The Review of Economic Studies*, 2000, *67* (1), pp. 91–116.

Debortoli, Davide, Ricardo Nunes, and Pierre Yared, "Optimal Taxation and Debt Management without Commitment," Technical Report, National Bureau of Economic Research 2018.

Diebold, Francis X. and Glenn D. Rudebusch, *Yield Curve Modeling and Forecasting: The Dynamic Nelson-Siegel Approach* number 9895. In 'Economics Books.', Princeton University Press, 2012.

Duffee, Gregory R., "Idiosyncratic Variation of Treasury Bill Yields," *The Journal of Finance*, 1996, *51* (2), 527–551.

Duffie, Darrell, "Presidential address: Asset price dynamics with slow-moving capital," *The Journal of finance*, 2010, *65* (4), 1237–1267.

— , "Still the World's Safe Haven? Redesigning the U.S. Treasury Market After the COVID-19 Crisis," June 2020. Hutchins Center Working Paper 62.

— **and Chi-Fu Huang**, "Implementing Arrow-Debreu Equilibria by Continuous Trading of Few Long-Lived Securities," *Econometrica*, 1985, *53* (6), 1337–1356.

— **, Nicolae Garleanu, and Lasse Heje Pedersen**, "Over-the-Counter Markets," *Econometrica*, 2005, *73* (6), 1815–1847.

Eaton, Jonathan and Mark Gersovitz, "Debt with potential repudiation: Theoretical and empirical analysis," *The Review of Economic Studies*, 1981, *48* (2), 289–309.

Faraglia, Elisa, Albert Marcet, Rigas Oikonomou, and Andrew Scott, "Government Debt Management: The Long and the Short of It," *The Review of Economic Studies*, 10 2018, *86* (6), 2554–2604.

Fleming, Michael J., "Are Larger Treasury Issues More Liquid? Evidence from Bill Reopenings," *Journal of Money, Credit and Banking*, 2002, *34* (3), pp. 707–735.

— **and Joshua V. Rosenberg**, "How do Treasury Dealers Manage their Positions," 2008. Frederal Reserva Bank of New York Staff Reports Staff Report no. 299.

Green, T. Clifton, "Economic News and the Impact of Trading on Bond Prices," *The Journal of Finance*, 2004, *59* (3), 1201–1233.

Greenwood, Robin, Samuel Gregory Hanson, Joshua S. Rudolph, and Lawrence Summers, "The Optimal Maturity of Government Debt," in David Wessel, ed., *The $13 Trillion Question: How America Manages Its Debt*, Brookings Institution Press, 2015, chapter 1, pp. 1–41.

Grossman, Sanford J and Merton H Miller, "Liquidity and market structure," *the Journal of Finance*, 1988, *43* (3), 617–633.

Hatchondo, Juan Carlos, Leonardo Martinez, and Cesar Sosa-Padilla, "Debt dilution and sovereign default risk," *Journal of Political Economy*, 2016, *124* (5), 1383–1422.

— **,** — **, and Francisco Roch**, "Overborrowing," 2018.

Hendershott, Terrence and Mark S Seasholes, "Market maker inventories and stock prices," *American Economic Review*, 2007, *97* (2), 210–214.

Ho, Thomas S.Y. and Hans R. Stoll, "The dynamics of dealer markets under competition," *The Journal of finance*, 1983, *38* (4), 1053–1074.

Huang, Roger D and Hans R Stoll, "The components of the bid-ask spread: A general approach," *The Review of Financial Studies*, 1997, *10* (4), 995–1034.

Kargar, Mahyar, Benjamin T. Lester, David Lindsay, Shuo Liu, Pierre-Oliver Weill, and Diego Zúñiga, "Corporate Bond Liquidity During the COVID-19 Crisis," June 2020.

Krishnamurthy, Arvind and Annette Vissing-Jorgensen, "The aggregate demand for treasury debt," *Journal of Political Economy*, 2012, *120* (2), 233–267.

Krusell, Per and Anthony A Smith Jr, "Income and wealth heterogeneity in the macroeconomy," *Journal of political Economy*, 1998, *106* (5), 867–896.

Leland, Hayne and Klaus Toft, "Optimal Capital Structure, Endogenous Bankruptcy, and the Term Structure of Credit Spreads," *The Journal of Finance*, 1996, *51* (3), 987–1019.

Lucas, Robert E and Nancy L Stokey, "Optimal fiscal and monetary policy in an economy without capital," *Journal of monetary Economics*, 1983, *12* (1), 55–93.

Lustig, Hanno, Christopher Sleet, and Sevin Yeltekin, "Fiscal hedging with nominal assets," *Journal of Monetary Economics*, May 2008, *55* (4), 710–727.

Madhavan, Ananth and George Sofianos, "An empirical analysis of NYSE specialist trading," *Journal of Financial Economics*, 1998, *48* (2), 189–210.

_ **and Seymour Smidt**, "A Bayesian model of intraday specialist pricing," *Journal of Financial Economics*, 1991, *30* (1), 99–134.

Naik, Narayan Y and Pradeep K Yadav, "Do dealer firms manage inventory on a stock-by-stock or a portfolio basis?," *Journal of Financial Economics*, 2003, *69* (2), 325–353.

_ **and** _ , "Risk management with derivatives by dealers and market quality in government bond markets," *The Journal of Finance*, 2003, *58* (5), 1873–1904.

Pasquariello, Paolo and Clara Vega, "The on-the-run liquidity phenomenon," *Journal of Financial Economics*, 2009, *92* (1), 1 – 24.

Pelizzon, Loriana, Marti G Subrahmanyam, Davide Tomio, and Jun Uno, "Sovereign credit risk, liquidity, and European Central Bank intervention: Deus ex machina?," *Journal of Financial Economics*, 2016, *122* (1), 86–115.

Reinhart, Carmen M and Kenneth S Rogoff, *This time is different: Eight centuries of financial folly*, Princeton University Press, 2009.

Reiter, Michael, "Solving heterogeneous-agent models by projection and perturbation," *Journal of Economic Dynamics and Control*, 2009, *33* (3), 649–665.

Sánchez, Juan M, Horacio Sapriza, and Emircan Yurdagul, "Sovereign default and maturity choice," *Journal of Monetary Economics*, 2018, *95*, 72–85.

Song, Zhaogang and Haoxiang Zhu, "Quantitative easing auctions of Treasury bonds," *Journal of Financial Economics*, 2018, *128* (1), 103 – 124.

Spindt, Paul A. and Richard W. Stolz, "Are US treasury bills underpriced in the primary market?," *Journal of Banking & Finance*, 1992, *16* (5), 891 – 908.

Vayanos, Dimitri and Jean-Luc Vila, "A Preferred-Habitat Model of the Term Structure of Interest Rates," *Econometrica*, January 2021, *89* (1), 77–112.

Vayanos, Dimitry and Jiang Wang, "Market Liquidity: Theory and Empirical Evidence," in George Constantinides, Milton Harris, and Rene Stulz, eds., *Handbook of the Economics of Finance*, 2013, chapter 19.

Wang, Chong, Neng Wang, and Jinqiang Yang, "Optimal consumption and savings with stochastic income and recursive utility," *Journal of Economic Theory*, 2016, *165*, 292 – 331.

Weill, Pierre-Olivier, "Leaning against the wind," *The Review of Economic Studies*, 2007, *74* (4), 1329–1354.

Zame, William R., "Efficiency and the Role of Default When Security Markets are Incomplete," *The American Economic Review*, 1993, *83* (5), 1142–1164.