

NBER WORKING PAPER SERIES

CONDITIONAL DYNAMICS AND THE MULTI-HORIZON RISK-RETURN TRADE-OFF

Mikhail Chernov
Lars A. Lochstoer
Stig R. H. Lundeby

Working Paper 25361
<http://www.nber.org/papers/w25361>

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
December 2018, Revised January 2020

We thank Hank Bessembinder, Ian Dew-Becker, Greg Duffee, Valentin Haddad, Francis Longstaff, Tyler Muir, Christopher Polk, Seth Pruitt, Tommy Stamland, and Stan Zin for comments on earlier drafts, as well as participants in the seminars and conferences sponsored by the ASU Winter Finance Conference, BI, David Backus Memorial Conference at Ojai, LAEF, NBER LTAM conference, NBIM, Norwegian School of Economics, the SFS Finance Cavalcade, Stanford, the UBC Winter Finance Conference, and UCLA. We appreciate any comments including inadvertently omitted citations. The latest version is available at https://sites.google.com/site/mbchernov/CLL_MHR_latest.pdf. The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research.

NBER working papers are circulated for discussion and comment purposes. They have not been peer-reviewed or been subject to the review by the NBER Board of Directors that accompanies official NBER publications.

© 2018 by Mikhail Chernov, Lars A. Lochstoer, and Stig R. H. Lundeby. All rights reserved. Short sections of text, not to exceed two paragraphs, may be quoted without explicit permission provided that full credit, including © notice, is given to the source.

Conditional Dynamics and the Multi-Horizon Risk-Return Trade-Off
Mikhail Chernov, Lars A. Lochstoer, and Stig R. H. Lundeby
NBER Working Paper No. 25361
December 2018, Revised January 2020
JEL No. C51,G12

ABSTRACT

We propose testing asset-pricing models using multi-horizon returns (MHR). A correctly specified stochastic discount factor prices the cross-section of returns at all horizons. We show that MHR are informative about the model's conditional implications. Different from typical conditioning variables, MHR-implied conditioning variables are endogenous to the model. Further, MHR are economically important as they explicitly test the model's ability to take present values of streams of risky cash flows that accrue at different horizons, a core concept in financial economics. We apply MHR-based testing to prominent linear factor models and show that these models typically do a poor job of pricing longer-horizon returns. We find that strong and, surprisingly, pro-cyclical time-variation in the pricing kernel's factor loadings and implied prices of risk is needed to jointly price returns at multiple horizons in these models.

Mikhail Chernov
Anderson School of Management
University of California, Los Angeles
110 Westwood Plaza, Suite C-417
Los Angeles, CA 90095
and CEPR
and also NBER
mikhail.chernov@anderson.ucla.edu

Stig R. H. Lundeby
Norwegian School of Economics (NHH)
Helleveien 30
Bergen
Norway
stig.lundeby@nhh.no

Lars A. Lochstoer
University of California at Los Angeles
lars.lochstoer@anderson.ucla.edu

1 Introduction

One of the main objectives of empirical asset pricing is to understand how markets value assets with risky cash flows. For instance, there is a large literature developing models to account for the cross-section of expected returns. This literature typically focuses the analysis on a particular return frequency (e.g., monthly or quarterly). Thus, the problem is condensed to taking present values of single-horizon risky cash flows, even though many real-life present-value problems involve risky cash flows at multiple horizons.

In this paper we argue for using returns realized over multiple horizons jointly in tests of asset pricing models. No-arbitrage implies that the h -period stochastic discount factor (SDF) equals the product of the h corresponding single-period SDFs. It is therefore straightforward to derive a model's implication for returns at any horizon. Thus, with multi-horizon returns (MHR) we are testing overidentifying restrictions of the model.

Specifically, we make two key contributions. In the first, theoretical, contribution we derive a set of MHR-based moment conditions in the context of GMM estimation. Further, we show that these moments amount to testing conditional implications of a model. This insight allows us to reduce the MHR-based moments to the familiar single-horizon setting with the addition of managed portfolios. The portfolio weights are represented by the returns over different horizons multiplied by matching SDFs.

There are important differences between the MHR-based conditioning information and typical conditioning variables used in the literature. For one, in contrast to the more popular conditioning variables, such as the dividend-price ratio or the term spread, the MHR-based conditioning variables are endogenous to the model being tested. Thus, the conditioning variables are not selected based on, say, their ex post ability to predict returns, which could raise concerns about data mining. Also, we differ from a well-established literature that uses past returns over a single horizon as conditioning variables. Our approach relies on multiple horizons jointly for the tests to have power. Further, these are not just returns, but returns multiplied by the SDF.

In the second, empirical, contribution we show that misspecification of the temporal dynamics in state-of-the-art models of the SDF, as uncovered by MHR, indeed are quantitatively large. In particular, we consider linear factor models that arguably are the workhorse models for empirical risk-return modeling. We test the minimal requirement that a model prices its own factors at multiple return horizons. We consider eight models: the CAPM, a two-factor model related to [Black, Jensen, and Scholes \(1972\)](#) (the market factor plus a betting-against-beta factor), the [Fama and French \(1993\)](#) three-factor model, the [Carhart \(1997\)](#) four-factor model (the three Fama-French factors plus momentum), the [Fama and French \(2015\)](#) five-factor model, the [Daniel, Mota, Rottke, and Santos \(2019\)](#) five-factor model, the [Stambaugh and Yu \(2017\)](#) four-factor model, and the [Hou, Xue, and Zhang \(2015\)](#) four-factor model.

As an example of the test results, consider the market factor in the Fama-French model. The h -period gross return to this factor is simply the product of the one-period gross returns from t to $t + h$. The model trivially prices the one-period return to this factor, but quickly generates a pricing error when we consider the model's implications for longer-period returns. At the four-year horizon, the model's annualized pricing error for the market factor is 7% – about the same as the market risk premium itself.

This example is not unique. The average annualized pricing error across all factors and models is 4.5% when tested jointly on horizons of 1, 3, 6, 12, 24, and 48 months. This is about the same magnitude as the average annualized factor risk premiums the models were designed to explain in the first place. With the exception of the CAPM, all models are rejected at the 5% level.

The reason the models are rejected is that the model-implied SDFs do not price the test assets conditionally. If the SDF factor loadings are allowed to vary over time there trivially exists loadings that allows each factor model to conditionally price its own factors (there are then K free parameters at each time t to price K factors). However, the presence of such non-constant loadings implies that the factors do not span the unconditional mean-variance efficient frontier, which is the null hypothesis of the factor models.

The evidence prompts us to investigate the reasons behind the conditional misspecification. As a baseline, we note that if factor returns are i.i.d., a model that prices

single-horizon returns will also price MHR. In the data, the factors in many cases turn out to be surprisingly far from i.i.d. As a simple metric, we compute variance ratios of the log gross return to the mean-variance efficient (MVE) combination of the factors in each model. Variance ratios measure the cumulative autocorrelations of log returns, where an i.i.d. process has a variance ratio of one at all horizons. For the market portfolio, there is a well-known slight increase above one and then a subsequent decline after the 15-month horizon. Many of the other models, however, have much stronger patterns. For instance, the [Daniel, Mota, Rottke, and Santos \(2019\)](#) model has a variance ratio that increases to about three at the four-year horizon. Thus, there are strong persistent components in the returns to these factor portfolios.

In fact, i.i.d. returns is not the only scenario that delivers unconditional spanning by the candidate factors. For ease of exposition, consider the SDF of a one-factor model, $M_{t,t+1} = 1 - b(F_{t,t+1} - E(F_{t,t+1}))$, where F is a traded portfolio excess return. Under the null of this model, F is unconditionally mean-variance efficient. This implies that the conditional expected return to F is proportional to its conditional second moment. Thus, while it is not true that the variance ratio of an unconditionally MVE portfolio needs to be one at all horizons, the degree of persistence in returns that we document is much higher than that typically estimated for return second moments.

To assess the economic implications of the factor dynamics, we first show that for many models the annualized Sharpe ratio from investing in the model-implied MVE portfolio is strongly decreasing in the investment-horizon. This occurs as the variances increase faster than at the rate of the horizon due to the persistent components, which makes these trading strategies less attractive as long-run investments than the high single-horizon Sharpe ratios would imply. In fact, we find that buy-and-hold investors with investment horizons of two to four years would pay between 5% and 20% of current wealth to avoid the time-dependence in these returns. In other words, the returns to simple trading strategies that do not engage in factor timing are still exposed to the factor dynamics as the investment horizon increases.

Next, we analyze the nature of the time variation in the factor loadings needed to jointly price factor returns at different frequencies. Given our test assets, F does span the *conditional* mean-variance efficient combination of the test assets for each

model. Thus, there exists a time-varying SDF loading $b_t = Var_t^{-1}(F_{t,t+1})E_t(F_{t,t+1})$ that prices the factor MHR. The evidence discussed above indicates that b_t must exhibit large and persistent time-variation to match the MHR data.

To assess this conclusion, we estimate the conditional means and covariance matrix of the factors for each model to get estimates of b_t . We find that b_t 's indeed are strongly time-varying. In particular, the average of the estimated b_t across factors and models at each time t is about ten, whereas the average standard deviation across models is about four. For all but the Stambaugh-Yu model, the conditional factor models implied by the estimated b_t price the MHR in the sense that pricing errors are smaller and the models are no longer rejected.

The implied prices of risk (maximal conditional Sharpe ratios, $b_t Var_t^{1/2}(F_{t,t+1})$) of each model exhibit substantially different dynamics than those implied by the corresponding constant b models. The average price of risk across all time-varying b_t models is significantly positively correlated with the dividend-price ratio and, surprisingly, negatively correlated with conditional market variance and a recession variable (the negative of industrial production growth). These dynamics, coming from returns to factors that otherwise do well in accounting for the cross-section of single-horizon stock returns, are hard to reconcile with the counter-cyclical price of risk implied by standard asset pricing models.

Related literature. There are many papers that test conditional versions of factor models. For instance, [Boguth, Carlson, Fisher, and Simutin \(2011\)](#), [Ferson and Harvey \(1999\)](#), [Farnsworth, Ferson, Jackson, and Todd \(2002\)](#), [Jagannathan and Wang \(1996\)](#), [Lettau and Ludvigson \(2001\)](#), [Lewellen and Nagel \(2006\)](#), and [Moreira and Muir \(2017\)](#). Our contribution relative to this literature is to show that MHR in asset pricing tests effectively serve as conditioning variables endogenous to the model and that, empirically, multi-horizon factor returns indeed are informative in terms of uncovering novel conditional dynamics of prominent factor models. In contemporaneous and independent work [Haddad, Kozak, and Santosh \(2019\)](#) and [Linnainmaa and Ehsani \(2019\)](#) use different methods to study factor dynamics with a focus on single-horizon returns.

Our paper makes a connection with a literature that seeks to characterize multi-

horizon properties of “zero-coupon” assets, such as bonds, dividends strips, variance swaps, and currencies. Such work includes [Belo, Collin-Dufresne, and Goldstein \(2015\)](#), [van Binsbergen, Brandt, and Kojen \(2012\)](#), [Dahlquist and Hasseltoft \(2013\)](#), [Dew-Becker, Giglio, Le, and Rodriguez \(2015\)](#), [Hansen, Heaton, and Li \(2008\)](#), [Kojen, Lustig, and Nieuwerburgh \(2017\)](#), [Lustig, Stathopoulos, and Verdelhan \(2013\)](#), and [Zviadadze \(2017\)](#). A related strand of the literature considers multiple frequencies of observations when testing models (e.g., [Brennan and Zhang, 2018](#), [Daniel and Marshall, 1997](#), [Jagannathan and Wang, 2007](#), [Kamara, Korajczyk, Lou, and Sadka, 2016](#), [Parker and Julliard, 2005](#)), though none of these consider the implications of a joint test across horizons.

Notation. We use E for expectations and V for variances (a covariance matrix if applied to a vector). A t -subscript on these denotes an expectation or variance conditional on information available at time t , whereas no subscript denotes an unconditional expectation or variance. We use double subscripts for time-series variables, like returns, to explicitly denote the relevant horizon. Thus, a gross return on an investment from time t to time $t + h$ is denoted $R_{t,t+h}$.

2 Linear factor models and multi-horizon returns

In this section we use linear factor models as a motivation for developing a general asset pricing test based on MHR. We offer a reminder that a set of factors that is valid at a single horizon does not necessarily extend to multiple horizons. Thus, we introduce the SDF representation of linear factor models to facilitate derivation of a model’s implications for different return horizons. We then give a simple example of how MHR can reveal conditional model misspecification. We conclude by outlining informally our MHR-based testing approach.

2.1 Implications of linear factor models for multiple horizons

Consider first the standard factor models for expected returns:

$$E(R_{t,t+1}^i - R_{t,t+1}^f) = \beta_i^\top E(F_{t,t+1}), \quad (1)$$

where $R_{t,t+1}^i$ is the gross one-period return to a portfolio or individual asset i , $R_{t,t+1}^f$ is the corresponding gross risk-free rate, and $F_{t,t+1}$ is a vector of factors representing excess returns on some trading strategies. This model holds for all assets if the factors span the unconditional MVE portfolio.

The model is commonly tested via a regression

$$R_{t,t+1}^i - R_{t,t+1}^f = \alpha_i + \beta_i^\top F_{t,t+1} + \varepsilon_{t+1}^i,$$

where, if Equation (1) holds, we have $\alpha_i = 0$. The question we ask here is whether the model can also account for longer-horizon returns to the test assets.

Consider a model of a stochastic discount factor (SDF), $M_{t,t+1}$, that prices all excess returns, including F itself, via

$$E(M_{t,t+1}(R_{t,t+1}^i - R_{t,t+1}^f)) = 0.$$

An SDF of the form

$$M_{t,t+1} = 1 - b^\top (F_{t,t+1} - E(F_{t,t+1})), \quad (2)$$

$$b = V^{-1}(F_{t,t+1})E(F_{t,t+1}). \quad (3)$$

implies Equation (1). As an implication, $b^\top F_{t,t+1}$ is the unconditional MVE portfolio. See [Cochrane \(2004\)](#) for an extensive discussion of the relationship between linear factor models and the SDF framework.

As is known from extant literature, see e.g., [Grossman, Melino, and Shiller \(1987\)](#), [Levhari and Levy \(1977\)](#), and [Longstaff \(1989\)](#), a factor model does not apply across

all horizons. To see this, consider the two-period SDF implied by Equation (2):

$$\begin{aligned} M_{t,t+2} &= M_{t,t+1}M_{t+1,t+2} = (a - b^\top F_{t,t+1})(a - b^\top F_{t+1,t+2}) \\ &= a^2 - ab^\top F_{t,t+1} - ab^\top F_{t+1,t+2} + b^\top F_{t,t+1}F_{t+1,t+2}^\top b, \end{aligned}$$

where $a = 1 + b^\top E(F_{t,t+1})$. This implies that the corresponding regression for the two-period return $R_{t,t+2}^i$ will essentially feature a new set of factors even if the original single-horizon model is correctly specified.

2.2 Extending to multiple horizons via SDFs

The SDF-based approach is a natural way to translate the model in Equation (1) into its counterpart at any longer horizon h . Indeed, the multi-horizon SDF is simply a product of single-horizon ones. Thus, we cast analysis in this paper in terms of SDFs. Switching over to the SDF language means that the focus on the magnitude of α changes to the focus on whether

$$E(M_{t,t+h}R_{t,t+h}^i) = 1. \quad (4)$$

Simply put, in a correctly specified model the present value of any \$1 investment is indeed \$1.

Returning to the original question of multi-horizon properties of R , we are now in a position to explain the economic insight arising from considering Equation (4). Consider the implication of this Equation for two-period gross returns:

$$\begin{aligned} 1 &= E(M_{t,t+2} \cdot R_{t,t+2}^i) \\ &= E(M_{t,t+1}M_{t+1,t+2} \cdot R_{t,t+1}^i R_{t+1,t+2}^i) \\ &= E(M_{t,t+1}R_{t,t+1}^i) \cdot E(M_{t+1,t+2}R_{t+1,t+2}^i) \\ &\quad + Cov(M_{t,t+1}R_{t,t+1}^i, M_{t+1,t+2}R_{t+1,t+2}^i). \end{aligned}$$

If (4) holds at both one- and two-period horizons, then

$$Cov(M_{t,t+1}R_{t,t+1}^i, M_{t+1,t+2}R_{t+1,t+2}^i) = 0. \quad (5)$$

Thus, by considering MHR we introduce additional information about the dynamic properties of the strategy returns and of the benchmark model. In particular, Equation (5) ensures that past “Euler equation errors” $M_{t,t+1}R_{t,t+1}^i$ do not predict future “Euler equation errors” $M_{t+1,t+2}R_{t+1,t+2}^i$.

2.3 An example

We illustrate this point in a simple example. Assume that a single factor follows:

$$F_{t,t+1} = \mu + \varepsilon_{t+1},$$

where ε_{t+1} is a mean-zero i.i.d. error term, and that the gross risk-free rate is constant and equal to one. Equations (2) and (3) imply the SDF:

$$M_{t,t+1} = 1 - [V(\varepsilon_{t+1})]^{-1} \mu \cdot \varepsilon_{t+1}.$$

Assume that the test asset’s return dynamics are:

$$R_{t,t+1} = 1 + \mu_t + \beta\varepsilon_{t+1} + u_{t+1},$$

where u_{t+1} is independent of ε_{t+1} and takes values $\delta > 0$ and $-\delta$ with equal probabilities. The conditional mean of the strategy, μ_t , takes values μ_H and μ_L depending on whether u_t is positive or negative, respectively. We also assume $E(\mu_t) = \beta\mu$ so that the SDF prices R unconditionally, $E(M_{t,t+1}R_{t,t+1}) = 1$.

Now, consider valuation of the two-period return

$$\begin{aligned} E(M_{t,t+2}R_{t,t+2}) &= E(M_{t,t+1}R_{t,t+1})E(M_{t+1,t+2}R_{t+1,t+2}) \\ &+ Cov(M_{t,t+1}R_{t,t+1}, M_{t+1,t+2}R_{t+1,t+2}) \\ &= 1 + Cov(u_{t+1}, \mu_{t+1}) \\ &= 1 + (\mu_H - \mu_L)\delta/2, \end{aligned}$$

where the second equality follows from the independence between μ_{t+1} and ε_{t+1} , and the dynamics of μ_{t+1} . See Appendix A.1. As a result, multi-horizon returns on the

strategy are mispriced unconditionally even if the single-horizon returns are priced correctly unconditionally.

2.4 Conditional implications

The unconditional test of the model using the 2-period return can be viewed as adding a test asset that is a managed position in the original asset in a one-period return test. Let $z_t \equiv M_{t-1,t}R_{t-1,t}$ and re-write the additional moment condition as $E(M_{t,t+1}z_tR_{t,t+1}) = 1$. This takes us to a test of the conditional implications of the model for one-period returns.

Continuing with the example of the previous section, the two-period mispricing, $(\mu_H - \mu_L)\delta/2$, is connected to the degree of predictability in R that the candidate SDF does not price conditionally. Indeed, $E_t(M_{t,t+1}R_{t,t+1}) = 1 + \mu_t - E(\mu_t)$, which equals 1 only if $\mu_H = \mu_L$.

A test that uses z_t has a number of important differences from the traditional conditional tests of asset pricing models. For one, the managed portfolio weight z_t is endogenous to the model, in contrast to typical conditioning variables used in the literature (e.g., the dividend-price ratio). As such they do not require an auxiliary search for variables that predict returns, which raises concerns about data mining. Also, the model implies that $E(z_t) = 1$ and that z_t is a function of the parameters in the SDF.

If we had access to the full information set of the marginal investor, we could have tested the SDF model conditionally using single-horizon returns (SHR). Our proposal to focus on MHR arises from the lack of access to that full information set. Obviously, MHR are not the only information subset that one could explore. The literature on managed portfolios is addressing the same issue.

We find using MHR attractive because it represents a direct test of the model's ability to correctly take present values of streams of risky cash flows that accrue at different horizons. Such present value problems are fundamental to many economic applications. Examples include consumption-savings and capital budgeting problems, as well

as venture capital and private equity valuations. Further, correct long-horizon risk-adjustment is important to long-horizon investors. As the example in this section shows, a strategy that has an alpha of zero at the one-period horizon may in fact deliver nonzero risk-adjusted returns when evaluated at a longer horizon. A reasonable requirement for a benchmark model is then that it can account for returns to well-known trading strategies at longer horizons as well.

The next section develops our testing methodology which is applicable to any model that respects the Law of One Price (LOOP) and to any set of test assets.

3 Testing asset pricing models using MHR

In this section, we develop a GMM-based test using MHR that is applicable to any asset pricing model that satisfies LOOP. The latter implies

$$E_t(M_{t,t+1}R_{t,t+1}^i) = 1,$$

for any asset i .

The SDF framework offers a natural way to propagate the model implications across multiple horizons. The multi-horizon SDF and returns are simple products of their single-horizon counterparts:

$$\begin{aligned} M_{t,t+h} &= \prod_{j=1}^h M_{t+j-1,t+j}, \\ R_{t,t+h}^i &= \prod_{j=1}^h R_{t+j-1,t+j}^i. \end{aligned}$$

LOOP still holds, see Equation (4), or, for excess returns,

$$E_t(M_{t,t+h}(R_{t,t+h}^i - R_{t,t+h}^f)) = 0, \tag{6}$$

where $R_{t,t+h}^f$ is the h -period return to a reference asset, typically a “risk-free” asset like a U.S. Treasury bill. See Appendix A.2. The unconditional version of this condition can be easily tested jointly for multiple horizons h in a GMM framework.

We have discussed the informational content of adding MHR in Equation (5) for the two-period case. A generalization to $(h + 1)$ -period returns, under covariance-stationarity, is

$$E(M_{t-h,t+1}R_{t-1,t+1}^i) = 1 + \sum_{j=0}^{h-1} Cov(M_{t-h+j,t}R_{t-h+j,t}^i, M_{t,t+1}R_{t,t+1}^i). \quad (7)$$

If the model of the SDF is correctly specified, Equation (4) holds for any h . Therefore, the sum of covariances in Equation (7) should equal zero. In other words, discounted returns $(M_{t,t+1}R_{t,t+1}^i)$ should not be predictable by lagged discounted returns of any horizon.

The virtue of this representation is that it allows us to construct a test on the basis of moment conditions whose residuals are not serially correlated. This improves the small-sample performance of the test. See Hodrick (1992) for a similar argument. Specifically, the moment conditions for test assets $i = 1, \dots, I$ that we consider are of the form:

$$f_{t+1}^i = \begin{pmatrix} M_{t,t+1}R_{t,t+1}^i - 1 \\ z_{i,t}^{(h_2)}(M_{t,t+1}R_{t,t+1}^i - 1) \\ \vdots \\ z_{i,t}^{(h_n)}(M_{t,t+1}R_{t,t+1}^i - 1) \end{pmatrix}, \quad (8)$$

where the conditional variable is

$$z_{i,t}^{(h)} = \sum_{j=0}^{h-1} M_{t-h+j,t}R_{t-h+j,t}^i, \quad (9)$$

and n is the number of horizons used in the test and $\{h_j\}_{j=2}^n$ are the set of horizons used in addition to the single-period horizon. In subsequent tests we use the horizons 3, 6, 12, 24, and 48 months in addition to the one-period (monthly) horizon. See Appendix A.3. The null hypothesis is $E(f_{t+1}^i) = 0$ for all i , and the test is thus an

unconditional test of the conditional properties of the asset pricing model.

A common approach in the literature is to introduce conditioning information via instrumental variables, such as dividend-price ratios (e.g., [Hansen and Singleton, 1982](#), [Hodrick and Zhang, 2001](#)). Denoting such variables by \tilde{z}_t , this strategy implies that $\tilde{z}_t(R_{t,t+1}^i - R_{t,t+1}^f)$ is just an excess return on another asset. Thus it could be incorporated as a new test asset using the moments outlined in Equation (8). This logic highlights the conceptual difference between the existing and our approaches. While the former relies on exogenously selected conditioning variables, the latter is using those dictated by a given model and set of test assets.

The test falls into the standard GMM framework, where:

$$g_T(\theta) = \frac{1}{T} \sum_{t=1}^T \begin{pmatrix} f_t^1(\theta) \\ f_t^2(\theta) \\ \vdots \\ f_t^I(\theta) \end{pmatrix},$$

where θ are the parameters in the SDF to be estimated. The objective function is as usual:

$$\operatorname{argmin}_{\theta} g_T(\theta)^\top W g_T(\theta),$$

where W is an $(I \times n) \times (I \times n)$ positive definite weighting matrix (e.g., [Hansen and Singleton, 1982](#)). Relevant test statistics and parameter standard errors can be found using the usual GMM toolkit.

An important feature of the test is that the moment conditions, f_t are uncorrelated across time under the null of the model. Thus, when estimating the covariance matrix of the moment conditions (the spectral density matrix, S) one does not need to account for leads or lags, which implies that $S(\theta) = V(f_t(\theta))$.

4 Testing linear factor models using MHR

In this section we apply the general methodology of the previous section to models featuring linear SDFs. These models represent the mainstream approach towards understanding the pricing of risk in the cross-section of returns.

4.1 Adopting the general test to linear models

We slightly re-write a K -factor model in Equation (2) as

$$M_{t,t+1} = 1 - b^\top (F_{t,t+1} - \mu),$$

to emphasize the need to estimate $\mu = E(F_{t,t+1})$. Under the null hypothesis of these models, $b^\top F_{t,t+1}$ is an unconditionally mean-variance efficient portfolio, which implies that it prices excess returns to all assets both conditionally and unconditionally (Hansen and Richard, 1987).

Guaranteeing that this SDF prices the risk-free rate conditionally requires adding auxiliary assumptions that are not explicit in the settings that are traditionally used for testing linear factor models. Because our goal is to assess the original models' performance, we make a slight adjustment to the moment conditions to ensure we do not reject the models based on mispricing of the multi-period risk-free rates, something that they were not designed to match.

Specifically, we note that predicting discounted gross returns, MR^i , as in the covariance condition in Equation (7), is equivalent to predicting discounted excess returns, $M(R^i - R^f)$, if the model prices the risk-free asset. We therefore use the MHR moment conditions:

$$f_{t+1}^i = \begin{pmatrix} M_{t,t+1}(R_{t,t+1}^i - R_{t,t+1}^f) \\ z_{i,t}^{(h_2)} M_{t,t+1}(R_{t,t+1}^i - R_{t,t+1}^f) \\ \vdots \\ z_{i,t}^{(h_n)} M_{t,t+1}(R_{t,t+1}^i - R_{t,t+1}^f) \end{pmatrix}.$$

The resulting $K + I \times n$ GMM moments are:

$$g_T(b, \mu) = \frac{1}{T} \sum_{t=1}^T \begin{pmatrix} F_{t,t+1} - \mu \\ f_t^1(b, \mu) \\ f_t^2(b, \mu) \\ \vdots \\ f_t^I(b, \mu) \end{pmatrix}.$$

Note that the managed portfolio weights $z_{i,t}^{(h)}$ in each f^i are exactly the same as in Equation (9), that is, they still depend on gross returns rather than excess ones.

We consider the factors themselves as the set of test assets. Because the factors in the literature are designed as zero-investment long-short portfolios, we construct $R^i = R^f + F^i$ for each factor i . The reason for this choice of test assets is three-fold.

First, in this case it is clear that the model can price these single-horizon excess returns unconditionally. We will in fact estimate b such that the single-horizon returns to the factors themselves are priced without error. That is in line with the standard [Black, Jensen, and Scholes \(1972\)](#) regressions in Equation (2), as the regression imposes the sample mean of the factors in the estimation of α_i . Thus, any rejection must be due to the joint test of the models' pricing of longer-horizon returns. To achieve this, we use the weighting matrix:

$$W_{(K+I \times n) \times (K+I \times n)} = \begin{bmatrix} I_K & 0_{K \times n} & 0_n & \cdots & 0_n \\ 0_{n \times K} & Q & 0_n & \cdots & 0_n \\ 0_n & 0_n & Q & \cdots & 0_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_n & 0_n & 0_n & \cdots & Q \end{bmatrix},$$

where I_K is the $K \times K$ identity matrix, 0_n is an $n \times n$ matrix of zeros, and

$$Q_{(n \times n)} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Second, the factors in these models are created from mechanical trading strategies in order to price a documented empirical spread in the cross-section of expected returns. Thus, a natural requirement for a well-specified model is that the model can price these strategies at any horizon. As an example, in a factor model that uses the Fama and French (1993) HML factor, the present value of a \$1 investment in the risk-free rate and a position in the HML (value) factor should be \$1 regardless of the holding period.

Third, this choice of test assets implies that there exists an SDF with time-varying loadings b_t instead of a constant b from Equation (3), that does price the returns at any horizon. We discuss this alternative hypothesis in more detail in a later section.

4.2 Data

We select our models based on their historical importance, recent advancements, and data availability. Specifically, we include the CAPM, CAPM combined with the BAB factor (Frazzini and Pedersen, 2014, Black, Jensen, and Scholes, 1972, Novy-Marx and Velikov, 2016), Fama and French 3- and 5-factor models, FF3 and FF5, respectively (Fama and French, 1993, Fama and French, 2015), a version of the FF5 models with hedged unpriced risks (Daniel, Mota, Rottke, and Santos, 2019), FF3 and momentum (Carhart, 1997) and the four-factor models of Hou, Xue, and Zhang (2015) and Stambaugh and Yu (2017).

The Fama-French 5-factor model includes the market factor (MKT), the value factor (HML), the size factor (SMB), the profitability factor (RMW; see also Novy-Marx, 2013), and the investment factor (CMA; see also Cooper, Gulen, and Schill, 2008). These data and the momentum factor MOM (Jegadeesh and Titman, 1993) are provided on Kenneth French’s webpage. The returns are monthly and the sample is from July 1963 to June 2017.

The hedged versions of these factors studied by Daniel, Mota, Rottke, and Santos (2019) (DMRS) are available on Kent Daniel’s webpage. The sample period is July 1963 to June 2017. The factors studied by Hou, Xue, and Zhang (2015) (HXZ)

are MKT, SMB, I/A (investment-to-assets) and ROE (return on equity), and are available on Lu Zhang’s website. The sample is from January 1967 to December 2017. [Stambaugh and Yu \(2017\)](#) propose two factors intended to capture stock mispricing, in addition to the existing MKT and SMB factors: PERF and MGMT. We denote this four-factor model as SY. These data are available on Robert Stambaugh’s webpage. The sample period for these factors starts January 1963 and ends December 2016.

Given the recent critique by [Novy-Marx and Velikov \(2016\)](#), we depart from the BAB factor construction of [Frazzini and Pedersen \(2014\)](#). We use the value-weighted beta- and size-sorted portfolios on Kenneth French’s webpage as the building blocks for constructing this factor, following [Fama and French \(2015\)](#) and [Novy-Marx and Velikov \(2016\)](#). Specifically, we construct four value-weighted portfolios: (1) small size, low beta, (2) small size, high beta, (3) big size, low beta, and (4) big size, high beta. The size cutoffs are the 40th and 60th NYSE percentiles. For betas, we use the 20th and 80th NYSE percentiles. Denote these returns as $R_{sl}, R_{sh}, R_{bl}, R_{bh}$, respectively, where s denotes small size, ℓ denotes low beta, b denotes big size, and h denotes high beta. We also compute the prior beta for each of the four portfolios and shrink towards 1 with a value of 0.5 on the historical estimate. We denote these as $\beta_{sl,t}, \beta_{bl,t}, \beta_{sh,t}$, and $\beta_{bh,t}$. We construct these portfolios using the 25 size and market beta sorted portfolio returns, as well as the corresponding market values and 60-month historical betas, given on Kenneth French’s webpage.

The factor return is then constructed as follows:

$$BAB_{t,t+1} = \frac{1}{\beta_{\ell,t}} \left(\frac{1}{2} R_{sl,t,t+1} + \frac{1}{2} R_{bl,t,t+1} - R_{f,t,t+1} \right) - \frac{1}{\beta_{h,t}} \left(\frac{1}{2} R_{sh,t,t+1} + \frac{1}{2} R_{bh,t,t+1} - R_{f,t,t+1} \right),$$

where $\beta_{\ell,t} = \frac{1}{2}\beta_{sl,t} + \frac{1}{2}\beta_{bl,t}$, and $\beta_{h,t} = \frac{1}{2}\beta_{sh,t} + \frac{1}{2}\beta_{bh,t}$. As a result, the conditional market beta of BAB should be close to zero, as in [Frazzini and Pedersen \(2014\)](#).

Finally, we get the monthly risk-free rate from CRSP and create the real risk-free rate by subtracting realized monthly inflation from the nominal rate. The inflation data are from CRSP as well.

4.3 MHR pricing errors and model tests

We start by computing annualized pricing errors for each factor in each model across horizons. The pricing errors should be understood as the net present value of an h -period \$1 buy-and-hold investment in the gross factor return. Since the models are estimated to match one-period returns unconditionally, non-zero net present values are due to mispricing of the conditional factor return. To facilitate comparison we annualize errors so that each reported number reflects the same period irrespective of the horizon. Thus, the pricing error for a factor F^i at horizon h is $12/h \times E_T(z_{i,t}^{(h)} M_{t,t+1} F_{t,t+1}^i)$. The horizons for reported errors range from 1 to 48 months.

Figure 1 displays the pricing errors for the first four factor models. The top left panel shows that the pricing errors of the CAPM are small across horizons, always less than 1% annualized. Thus, for the market model, a constant b coefficient in the SDF works well for pricing market returns at any horizon.

The top right panel shows the MKT+BAB model. In this case, pricing errors are much larger for both factors. For the BAB factor, the annualized pricing error increases with horizon (in absolute value) to almost 10% per year at the 48-month horizon. That is about twice the average annualized monthly returns on this factor.

The bottom left panel shows the corresponding pricing errors for the FF3 model. Here the pricing errors for the MKT and SMB factors are moderate, whereas the pricing errors for the HML factor is about 5% p.a. at the 2- and 4-year horizons. The bottom right plot shows the Carhart model (FF3+MOM), where the pricing errors get very large, exceeding 50% p.a. for the 4-year MOM return.

Panel A of Table 1 gives the p -values of the J -test of each model. With the exception of the CAPM, the models are rejected.¹ We calculate the mean absolute pricing error (MAPE) for each model as the mean of the absolute value of the annualized pricing errors across the factors and horizons. For the CAPM, the MAPE is only 0.7%, for

¹We would like to emphasize the meaning of failure to reject in this context. The MKT factor is capable of pricing itself at multiple horizons. That does not imply, however, that the MKT model is well-specified. As we know, it is easily rejected by a cross-section of equity returns.

the CAPM+BAB it is 3%, for the FF3 it is 1.4%, and for the Carhart model it is 7.6%.

Figure 2 shows the pricing errors for the remaining four models. The top left panel shows pricing errors for the FF5 model. Again pricing errors increase in absolute value with horizon, which is natural as the models price one-period factor returns perfectly. Three of the five factors (MKT, RMW, and CMA) have absolute pricing errors in excess of 5% p.a. at the 4-year horizon.

A similar picture emerges in the top right panel for the FF5_{DMRS} model. Its pricing errors exceed 10% p.a. for two factors (their versions of the MKT and SMB factors) and 5% for their version of the CMA factor. The two bottom plots show the pricing errors for the SY and HXZ models. For these models, pricing errors are even larger, with the largest pricing error exceeding 100% p.a. Longer-run returns are noisy, so one should not overinterpret any one factor’s mispricing.

Overall, Table 1 shows that all the models are strongly rejected. We conclude from this that the current benchmark models for risk-adjustment do a poor job accounting for MHRs. The average MAPE across all eight models is 4.6%, which is about the same as the annualized factor risk premiums that these models were originally designed to match.

Table 1 also displays the annualized maximal Sharpe ratios, $[E(F)^\top V(F)^{-1} E(F)]^{1/2}$, implied by each factor model. A higher Sharpe ratio implies that the factors are closer to spanning the unconditional MVE portfolio. As is well-known, the Sharpe ratio of the MKT factor is much lower than the maximal Sharpe ratios in more recent multi-factor models. For instance, the SY model has an annualized Sharpe ratio of 1.7 compared to 0.4 for the CAPM.

Figure 3 shows each model’s MAPE plotted against the respective maximal Sharpe ratios. Interestingly, there is a positive relation. The higher a model’s Sharpe ratio the closer it should be to spanning the unconditionally mean-variance efficient portfolio and thus the lower the pricing errors should be. The opposite being the case indicates that the search for high Sharpe ratio models has increased the complexity of the conditional dynamics, consistent with the findings in [Haddad, Kozak, and Santosh](#)

(2019). Thus, there is a need for understanding the economic effects and drivers of these dynamics.

4.4 Factor dynamics and long-horizon investment

In order to gain more intuition about the rejection results, we evaluate both statistical and economic metrics that are relevant for long-horizon investors. Because we have implemented formal inference in the previous section, we no longer test the models. We rather highlight their properties that are responsible for the reported rejections.

We focus on the in-sample mean-variance efficient combination of each model’s factors in order to facilitate cross-model comparison and to reduce the overall dimensionality of the presentation. In particular, we calculate for each model the unconditional MVE combination of the factors and scale the positions such that its volatility is the same as the volatility of market returns. We denote (excess) returns on the MVE portfolio by $MVE_{t,t+1}$. The returns $R_{t,t+1}$ are computed by adding the gross-risk free rate.

Statistical assessment

The null hypothesis that the SDF in Equation (2) is correctly specified tells us something about dynamics of factors. Specifically, it implies that conditional mean of factor returns is proportional to their conditional second moment. See Appendix A.4. That our test rejects the models implies that this requirement does not hold in the data.

One way to illustrate dynamics of returns is to compare variance ratios of MVE returns across multiple horizons. We take logs of returns ($r = \log R$). The h -horizon variance ratio is then calculated as:

$$VR(h) = \frac{V(r_{t,t+1} + r_{t+1,t+2} + \dots + r_{t+h-1,t+h})}{h \times V(r_{t,t+1})}.$$

For further intuition, note that the variance ratio can be written as a weighted average of autocorrelations of log returns at lags up to horizon h . One benchmark for return dynamics is that of i.i.d. In this case, the variance ratio is equal to 1 at any horizon.

Figure 4 displays the variance ratios for each model. The market factor in the CAPM displays familiar dynamics where the variance ratio increases slightly from 1 to almost 1.2 at the annual horizon and subsequently decreases towards 1 at the 4-year horizon. The latter decrease is consistent with a long-run mean-reverting component in market returns.

All the other models display markedly stronger stronger departures from the i.i.d. baseline. To start with the most extreme cases, the $FF5_{DMRS}$ and the MKT+BAB models have strongly increasing variance ratios exceeding 2 at 12 to 24 months. For $FF5_{DMRS}$, the 4-year variance ratio is about 3. That is, a 4-year investor holding this portfolio is subject to, per unit of time, triple the variance of an investor with a monthly holding period.

The variance ratio of the FF5 model peaks at about 1.8 at the 2-year horizon, while the SY model has a variance ratio in excess of 2 at the 4-year horizon. The Carhart and HXZ models have slowly increasing variance ratios that end up at about 1.6 at the 4-year horizon. Only the CAPM and FF3 models have variance ratios that revert to around 1 at the longer horizons.

The variance ratios indicate not only departures from i.i.d. but also very strong persistent components in the returns of the model-implied MVE portfolios. This observation is something that was not emphasized in the literature, to the best of our knowledge.

Economic assessment

We offer another perspective on this conclusion by assessing the economic impact of these dynamics for long-horizon investors. Specifically, we compute Sharpe ratios and certainty equivalents for holding periods with different horizons. First, we calculate the annualized Sharpe ratio by horizon for each of the MVE portfolios. Excess returns at horizon h is the average h -period gross return minus the h -period gross risk-free rate, where both of these are calculated by multiplying together h one-period gross returns. The Sharpe ratio is then the mean excess return divided by the standard deviation of excess returns. We annualize by multiplying with $\sqrt{12/h}$.

To obtain a benchmark Sharpe ratio that corresponds to the null hypothesis, we draw from the original MVE and risk-free returns with replacement 10,000 artificial histories of the same length as our sample. This procedure imposes i.i.d. dynamics on the bootstrapped returns. The procedure retains the same unconditional distribution of returns and does not impose normality.

Figures 5 and 6 display the Sharpe ratios for each model and their benchmarks under the null. Across all models Sharpe ratios are declining in the horizon. As the benchmark Sharpe ratios illustrate, the pattern in of itself is not surprising. What is different for some models is much steeper decline than in the benchmark. The larger steepness coincides with instances of strong increases in the variance ratio.

The persistent returns lead to higher long-run variance which depresses long-run Sharpe ratios. The economic magnitude is particularly large for the four models in Figure 6. For instance, in the case of the SY model the 4-year Sharpe ratio is half of that in the benchmark.

These observations suggest a large economic loss to long-term investors who do not account for the factor dynamics in their allocations. As a next step of our analysis, we quantify the welfare costs of the conditional factor dynamics. Specifically, we ask how much a CRRA agent with over end-of-period wealth would pay to get the i.i.d. version of the MVE returns as opposed to the actual MVE returns. In particular, the amount this agent would be willing to pay, for a given investment horizon h , out of an initial wealth of \$1 is found as:

$$wc(h) = 1 - \left(\frac{ER_{t,t+h}^{1-\gamma}}{[ER_{t,t+1}^{1-\gamma}]^h} \right)^{\frac{1}{1-\gamma}},$$

where γ is risk aversion. See Appendix A.5. Figure 7 gives these amounts for each model as a function of the horizon when $\gamma = 5$. The peak value ranges between 5% and 20% for the different models.

Overall the plots in Figures 5 - 7 indicate that investors who do not engage in factor timing are nevertheless exposed to important dynamics in the SDF factor loadings that strongly affect their utility of the investment. Next, we ask which conditional dynamics enable the models to jointly account for the MHRs.

5 Pricing factor MHR

The rejection of the models is a consequence of factor dynamics unaccounted for in the linear SDF specification. In this section, we consider these dynamics. Full accounting for the uncovered role of dynamics and proposing a convincing alternative to each model is beyond the scope of this paper. We have a more modest objective of providing an illustration of what accounting for these dynamics might entail and to suggest a path for future research.

5.1 Estimating time-varying SDF loadings

Given that the test asset are the SDF factors themselves, there exists an SDF with time-varying coefficients that prices MHR to these test assets:

$$M_{t,t+1} = 1 - b_t^\top (F_{t,t+1} - \mu_t), \quad (10)$$

where $\mu_t = E_t(F_{t,t+1})$. Here, b_t and μ_t are $K \times 1$ vectors.

Indeed, Equations (6) and (10) imply, when applied to the factors themselves, that:

$$b_t = V_t(F_{t,t+1})^{-1} \mu_t.$$

Given this value of b_t , the Euler equation errors $M_{t,t+1}R_{t,t+1}$ are not forecastable. Thus, the SDF (10) prices MHR correctly per Equation (7).

Thus, in order to obtain b_t , we explicitly estimate μ_t and $V_t(F_{t,t+1})$ for each model. We emphasize that, because of the illustrative nature of our exercise, the estimation is in-sample. We estimate the conditional monthly variance-covariance matrix of the factor returns using the multivariate CCC-GARCH method of [Bollerslev \(1990\)](#). We estimate conditional mean of each element k of the vector of factors F using a simple regression model that is motivated by the uncovered strong dependencies in factor returns:

$$F_{t,t+1}^i = \beta_{i,0} + \sum_{j=1}^n \beta_{i,h_j} x_{i,t}^{(h_j)} + \varepsilon_{t+1}^i, \quad (11)$$

where $x_{i,t}^{(h)} = \sum_{j=1}^h F_{t-j,t-j+1}^i$. Note that the predictive variables $x_{i,t}^{(h)}$ are different from the conditioning variables $z_{i,t}^{(h)}$ that we use in our GMM tests.

We use the post LASSO approach of [Belloni and Chernozhukov \(2013\)](#) to estimate the regression (11) for each factor. That is, we use the LASSO to select strong predictive variables and, because the LASSO yields biased return estimates, we next use OLS with these selected regressors to get the conditional factor risk premium estimate, μ_t^i .

We select h_j in Equation (11) to be 1, 3, 12, and 48 months for a total of four predictive variables. The predictive variable set is chosen to account for persistent components in returns with possibly multiple frequencies. We consider slightly fewer horizons than in our GMM test to reduce the correlation between return predictors in the LASSO.

5.2 Estimation results

Table 2 presents test results along with pricing errors and Sharpe ratios. The results suggest that, overall, our first cut at estimating b_t delivers reasonable results. Nevertheless, it can be improved, at least for certain models.

The p -values of our test now fail to reject at the 10% level with the exception of the SY (p -value = 0.2%) and the HXZ (p -value = 9.1%) models. We note that our inference does not account for estimation errors in μ_t and V_t thereby making the reported p -values the lower bounds on the correct numbers. Thus, the only model that is likely to be rejected is SY.²

Economically, we see improvements across the board. The pricing errors are smaller as compared to the constant b case reported in Table 1. Their magnitude ranges between 14% (FF3+MOM) to 82% (FF5) as a fraction of the original errors.

We next consider the Sharpe ratios of the model-implied unconditional MVE portfolios. See Appendix A.6 for the derivation. Table 2 shows that the Sharpe ratios for

²To clarify, the rejection of SY in this setting suggests that the basic factor dynamics that we've posited in order to estimate b_t is not appropriate for that model. Still, there exists a b_t that prices the SY factors correctly. Given the illustrative nature of our exercise we do not perform extensive modeling to find such a b_t .

the models with time-varying b_t is generally higher than those from the constant b versions of the models. For instance, the MKT+BAB and FF3 models have Sharpe ratios that are about 1.3 times higher than in the constant b cases. Thus, the estimated time-variation in b_t indeed leads to an improved pricing kernel also when viewed from the standard single-horizon perspective.

The CAPM and the SY models are exceptions, with Sharpe ratios of the model-implied unconditional MVE portfolios at about the same level as the case for the constant b models. That is consistent with the lack of rejection in the constant b case for the CAPM and rejection of the SY model with time-varying b_t .

5.3 Sources of variation in b_t

The estimation procedure delivers an estimate of b_t for each factor in each model. We have two related objectives regarding these estimates. First, we would like to characterize general variation in b_t that is required to price MHR successfully. Second, we would like to check if b_t is related to standard conditioning variables, such as the dividend-price ratio.

To streamline the presentation we discuss variation in the average b_t across factors for each model. Table 3 presents main results. The mean of this average b_t differs across models due to the volatility of the pricing factors and their Sharpe ratios. The cross-model average b_t is about 10. Importantly, the time-variation in the average b_t 's are large, with a cross-model average of 4.3. The CAPM has the least time variation in the estimated b_t , consistent with the constant- b version of this model not being rejected. The other models exhibit stronger time-variation with FF5_{DMRS} having the largest standard deviation of b_t of 6.9.

The first column of Table 4 compresses the evidence further by averaging b_t across the models, thus focusing the common variation in b_t across both factors and models. Nevertheless, there is still substantial time-variation in that b_t , with a standard deviation of 3.6. The annualized serial correlation of 0.4 translates into 0.93 at the monthly frequency of the data. Thus, there is a highly variable and persistent common component in the b_t 's across factors and models.

Panel (A) of Figure 8, which displays the time series of that average b_t , supports this intuition showing a range from about 1 to 20. As seen from the shaded NBER recession bars in the Figure, the common variation across factors and models in the SDF loadings tends to be *negatively* associated with recessions.

There is a long tradition in the literature to model b_t as a linear function of variables that are related to aggregate discount rates. See, e.g., [Ferson and Harvey \(1999\)](#) for an early example, and [Moreira and Muir \(2017\)](#) for a recent example. Motivated by this work, we check if our estimate of b_t is related to the market dividend-price ratio, the term spread, market variance, and a recession variable (the negative of annual industrial production growth, overlapping monthly observations).

We implement the analysis for b_t . Also, in order to gain more transparent economic interpretation of the results, we consider the associated prices of risk $\lambda_t = b_t V_t^{1/2}$ for each factor, a.k.a. maximal conditional Sharpe ratios. The results of the grand average λ_t across factors and models are displayed in the second column of Table 4 and panel (B) of Figure 8.³

The price of risk is significantly negatively related to the recession variable and market variance. This is striking and unexpected. We would venture to say most market participants view recessions and spikes in volatility as bad times, but this is not reflected in the prices of risk of these factor models. Either the factor models are missing an important component of risk or investor expectations are not rational and the price of risk dynamics we estimate in the data reflects investor expectational errors in these bad times. For instance, investors could be persistently surprised by the severity of recessions and/or volatility spikes, leading to lower expected returns in bad times as prices initially under-react.

More reassuringly, the dividend-price ratio and the term spread are positively associated with fluctuations λ_t , as theory and intuition suggests should be the case, albeit with weaker statistical significance. The overall adjusted R^2 is 25%. Thus, the lion's share of the estimated common variation in the prices of risk in these models is unexplained.

³Results for factor- and model-specific λ_t are available upon request.

6 Conclusion

One of the principal goals of empirical asset pricing is to provide a stochastic discount factor that appropriately takes the present-value of any risky cash flow accruing at any future horizon. The main focus of the literature, however, has been on developing models of single-horizon expected returns. That disparity motivates us to develop a GMM-based test that uses multi-horizon returns (MHR) to evaluate the ability of any asset pricing model to price risky cash flows that accrue at different horizons. We argue that MHR are appealing also because the conditioning variables they imply are endogenous to the model being tested.

The test rejects a set of prominent linear factor models, and we find that the average pricing errors are similar to the average factor risk premiums the models are designed to explain in the first place. The reason the models do a poor job pricing longer-horizon returns is that the implied conditional properties of risk pricing are strongly at odds with dynamic properties of the factors associated with these models. Because long-run investment entails exposure to conditional return dynamics even in the absence of factor timing, these dynamics show up as large mispricing in longer-run returns.

We find that the conditional risk prices in the SDFs that do price longer-horizon factor returns are negatively correlated with those implied by the benchmark models. Specifically, the models with time-varying SDF factor loadings imply that risk prices are pro-cyclical and low when the conditional market return volatility is high. Further, the variation in these factor loadings is substantial and not strongly connected to existing conditional variables, even if ignoring the counterintuitive signs. This sort of variation is puzzling. Future research should focus on understanding the economic forces behind the evidence.

References

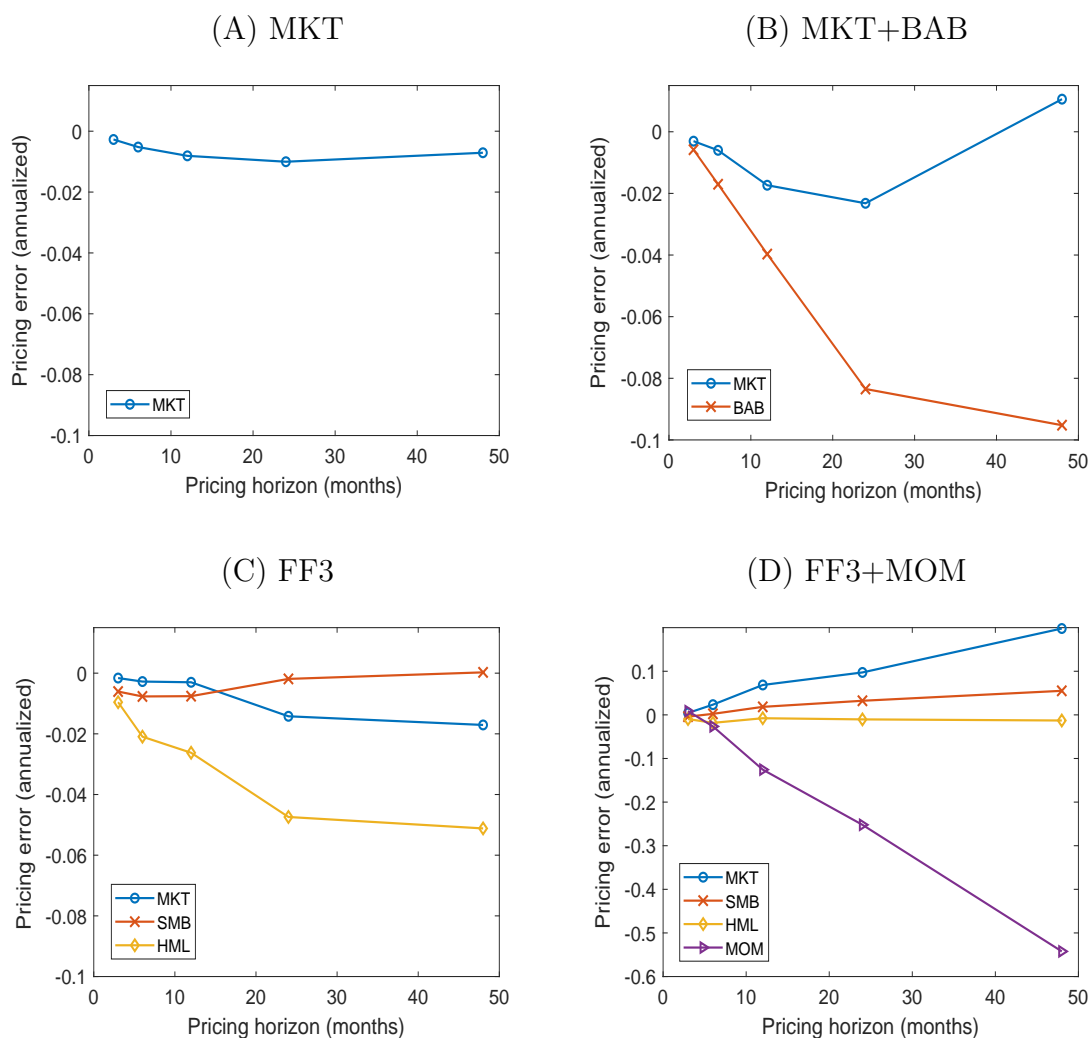
- Belloni, Alexandre, and Victor Chernozhukov, 2013, Least squares after model selection in high-dimensional sparse models, *Bernoulli* 19, 521–547.
- Belo, Frederico, Pierre Collin-Dufresne, and Robert Goldstein, 2015, Dividend dynamics and the term structure of dividend strips, *Journal of Finance* 70, 1115–1160.
- Black, Fischer, Michael Jensen, and Myron Scholes, 1972, The capital asset pricing model: Some empirical tests, in Michael Jensen, ed.: *Studies in the Theory of Capital Markets* (Praeger Publishers Inc.).
- Boguth, Oliver, Murray Carlson, Adlai Fisher, and Mikhail Simutin, 2011, Conditional risk and performance evaluation: Volatility timing, overconditioning, and new estimates of momentum alphas, *Journal of Financial Economics* 102, 363–389.
- Bollerslev, Tim, 1990, Modelling the coherence in short-run nominal exchange rates: a multivariate generalized ARCH model, *Review of Economics and Statistics* 72, 498–505.
- Brennan, Michael, and Yuzhao Zhang, 2018, Capital asset pricing with a stochastic horizon, working paper.
- Carhart, Mark, 1997, On persistence in mutual fund performance, *Journal of Finance* 52, 57–82.
- Cochrane, John, 2004, *Asset Pricing* (Princeton University Press: Princeton, NJ) revised edn.
- Cooper, Michael, Huseyn Gulen, and Michael Schill, 2008, Asset growth and the cross-section of stock returns, *Journal of Finance* 63, 1609–1651.
- Dahlquist, Magnus, and Henrik Hasseltoft, 2013, International bond risk premia, *Journal of International Economics* 90, 17–32.
- Daniel, Kent, and David Marshall, 1997, Equity-premium and risk-free-rate puzzles at long horizons, *Macroeconomic Dynamics*.
- Daniel, Kent, Lira Mota, Simon Rottke, and Tano Santos, 2019, The cross section of risk and return, *Review of Financial Studies*, forthcoming.
- Dew-Becker, Ian, Stefano Giglio, Anh Le, and Marius Rodriguez, 2015, The price of variance risk, working paper.

- Fama, Eugene, and Kenneth French, 1993, Common risk factors in the returns on stocks and bonds, *Journal of Financial Economics* 33, 3–56.
- , 2015, A five-factor asset pricing model, *Journal of Financial Economics* 116, 1–22.
- Farnsworth, Heber, Wayne Ferson, David Jackson, and Steven Todd, 2002, Performance evaluation with stochastic discount factors, *Journal of Business* 75, 473–503.
- Ferson, Wayne E., and Campbell R. Harvey, 1999, Conditioning variables and the cross-section of stock returns, *Journal of Finance* 54, 1325–1360.
- Frazzini, Andrea, and Lasse Pedersen, 2014, Betting against beta, *Journal of Financial Economics* 111, 1–25.
- Grossman, Sanford, Angelo Melino, and Robert Shiller, 1987, Estimating the continuous-time consumption-based asset pricing model, *Journal of Business and Economic Statistics* 5, 315–327.
- Haddad, Valentin, Serhiy Kozak, and Shri Santosh, 2019, Factor timing, *Review of Financial Studies*, forthcoming.
- Hansen, Lars, John Heaton, and Nan Li, 2008, Consumption strikes back? Measuring long-run risk, *Journal of Political Economy* 116, 260 – 302.
- Hansen, Lars, and Scott Richard, 1987, The role of conditioning information in deducing testable restrictions implied by dynamic asset pricing models, *Econometrica* 55, 587–613.
- Hansen, Lars Peter, and Kenneth J. Singleton, 1982, Generalized instrumental variables estimation of nonlinear rational expectations models, *Econometrica* 50, 1269–1286.
- Hodrick, Robert, and Xiaoyan Zhang, 2001, Evaluating the specification errors of asset pricing models, *Journal of Financial Economics* 62, 327–376.
- Hodrick, Robert J., 1992, Dividend yields and expected stock returns: Alternative procedures for inference and measurement, *The Review of Financial Studies* 5, 357–386.
- Hou, Kewei, Chen Xue, and Lu Zhang, 2015, Digesting anomalies: An investment approach, *The Review of Financial Studies* 28, 650–705.
- Jagannathan, Ravi, and Yong Wang, 2007, Lazy investors, discretionary consumption, and the cross-section of stock returns, *Journal of Finance* 62, 1623–1661.

- Jagannathan, Ravi, and Zhenyu Wang, 1996, The conditional capm and the cross-section of expected returns, *Journal of Finance* 51, 3–53.
- Jegadeesh, Narasimhan, and Sheridan Titman, 1993, Returns to buying winners and selling losers: Implications for stock market efficiency, *Journal of Finance* 48, 65–91.
- Kamara, Avraham, Robert A. Korajczyk, Xiaoxia Lou, and Ronnie Sadka, 2016, Horizon pricing, *Journal of Financial and Quantitative Analysis* 51, 1769–1793.
- Koijen, Ralph, Hanno Lustig, and Stijn Van Nieuwerburgh, 2017, The cross-section and time series of stock and bond returns, *Journal of Monetary Economics* 88, 50–69.
- Lettau, Martin, and Sydney Ludvigson, 2001, Resurrecting the (c)capm: A cross-sectional test when risk premia are time-varying, *Journal of Political Economy* 109, 1238–1287.
- Levhari, David, and Haim Levy, 1977, The capital asset pricing model and the investment horizon, *Review of Economic Studies* 59, 92–104.
- Lewellen, Jonathan, and Stefan Nagel, 2006, The conditional capm does not explain asset pricing anomalies, *Journal of Financial Economics* 79, 289–314.
- Linnainmaa, Juhani, and Sina Ehsani, 2019, Factor momentum and the momentum factor, working paper.
- Longstaff, Francis, 1989, Temporal aggregation and the continuous-time capital asset pricing model, *Journal of Finance* 44, 871–887.
- Lustig, Hanno, Andreas Stathopoulos, and Adrien Verdelhan, 2013, The term structure of currency carry trade risk premia, working paper.
- Moreira, Alan, and Tyler Muir, 2017, Volatility-managed portfolios, *Journal of Finance* 72, 1611–1644.
- Novy-Marx, Robert, 2013, The other side of value: The gross profitability premium, *Journal of Financial Economics* 108, 1–28.
- , and Mihail Velikov, 2016, A taxonomy of anomalies and their trading costs, *The Review of Financial Studies* 29, 104–147.
- Parker, Jonathan, and Christian Julliard, 2005, Consumption risk and the cross-section of expected returns, *Journal of Political Economy* 113, 185–222.

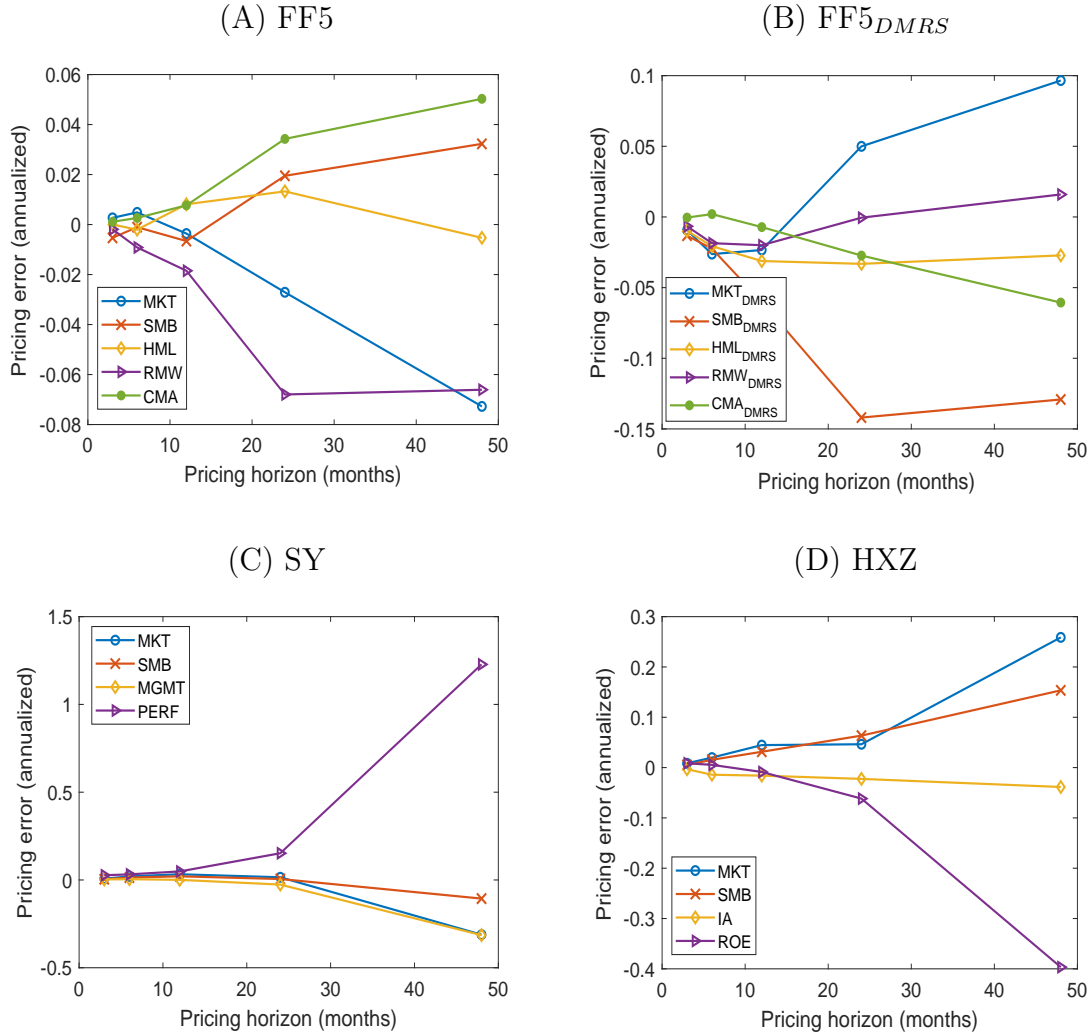
- Stambaugh, Robert, and Yuan Yu, 2017, Mispricing factors, *The Review of Financial Studies* 30, 1270–1315.
- van Binsbergen, Jules, Michael Brandt, and Ralph Koijen, 2012, On the timing and pricing of dividends, *American Economic Review* 102, 1596–1618.
- Zviadadze, Irina, 2017, Term-structure of consumption risk premia in the cross-section of currency returns, *Journal of Finance* 72, 1529–1566.

Figure 1
Term structure of annualized factor pricing errors I



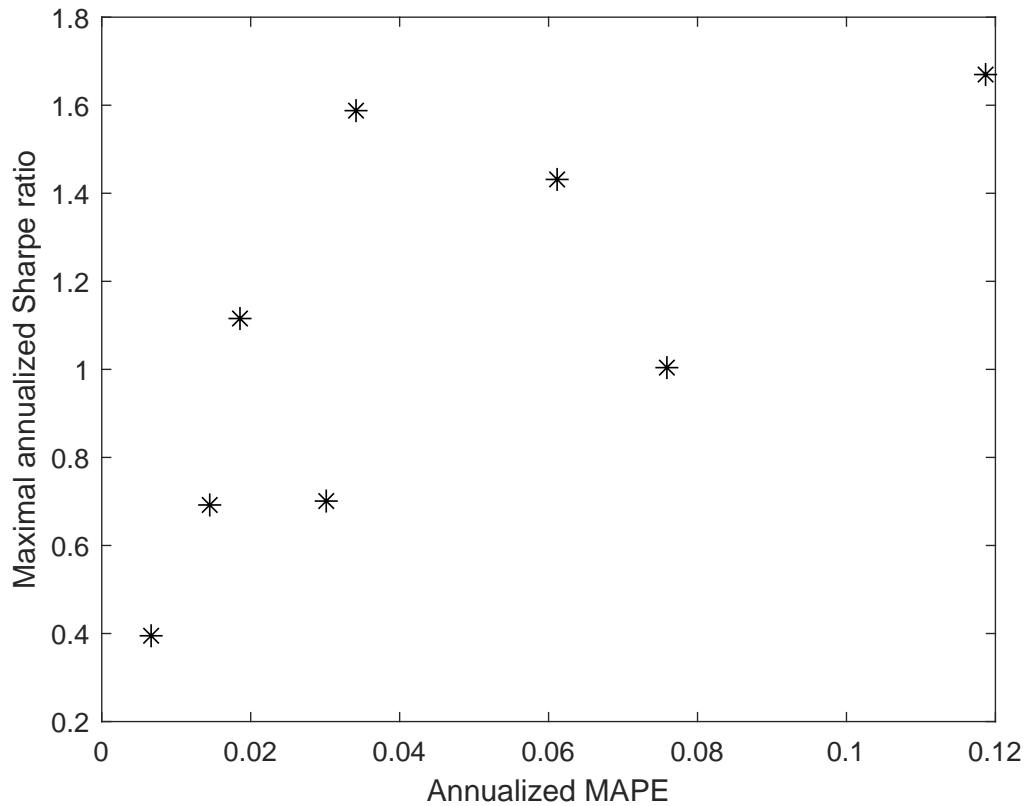
The panels show factor pricing errors for various models at horizons 3, 6, 12, 24, and 48 months. Annualized pricing errors at horizon h are $12/h \times E_T(z_{i,t}^{(h)} M_{t,t+1} F_{t,t+1}^i)$, where E_T denotes the sample average, $z_{i,t}^{(h)}$ is the endogenous conditioning variable for factor i at horizon h described in the main text, and $F_{t,t+1}^i$ is the return to factor i . The population average of a correctly specified model is zero. The sample is monthly, from 1963 to 2017.

Figure 2
Term structure of annualized factor pricing errors II



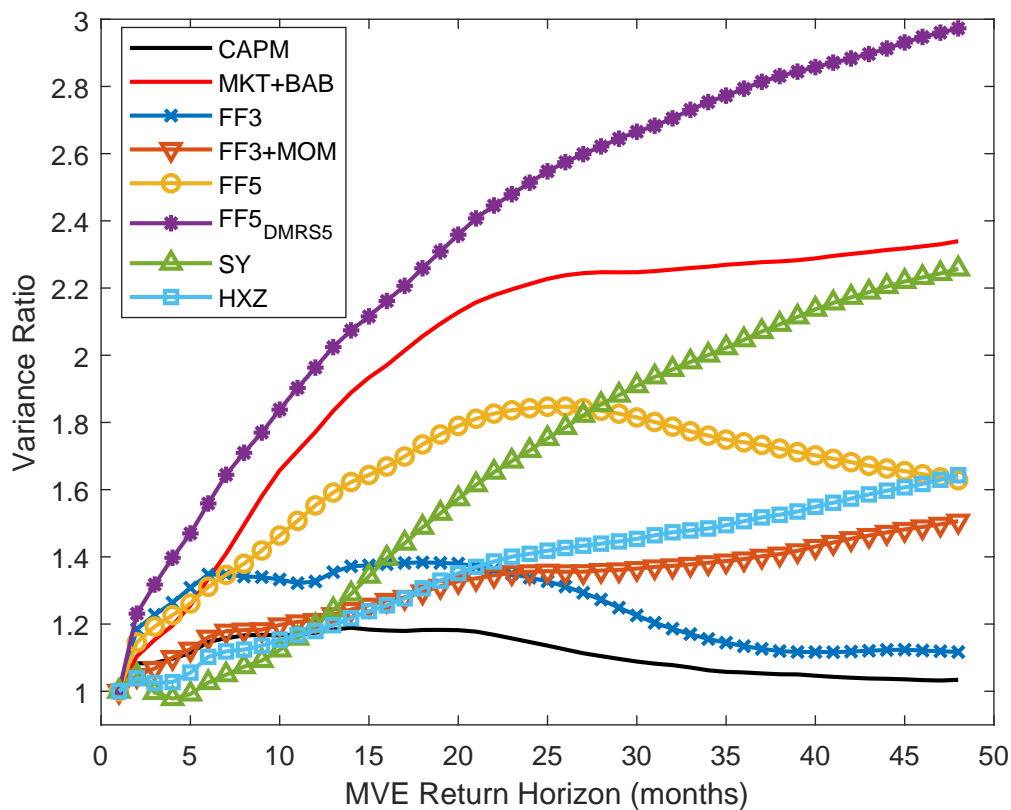
The panels show factor pricing errors for various models at horizons 3, 6, 12, 24, and 48 months. Annualized pricing errors at horizon h are $12/h \times E_T(z_{i,t}^{(h)} M_{t,t+1} F_{t,t+1}^i)$, where E_T denotes the sample average, $z_{i,t}^{(h)}$ is the endogenous conditioning variable for factor i at horizon h described in the main text, and $F_{t,t+1}^i$ is the return to factor i . The population average of a correctly specified model is zero. The sample is monthly, from 1963 to 2017 for $FF5$ and $FF5_{DMRS}$, 1963 to 2016 for SY, and 1967 to 2017 for HXZ.

Figure 3
Max Sharpe ratio of single-horizon factor model vs.
multi-horizon pricing errors



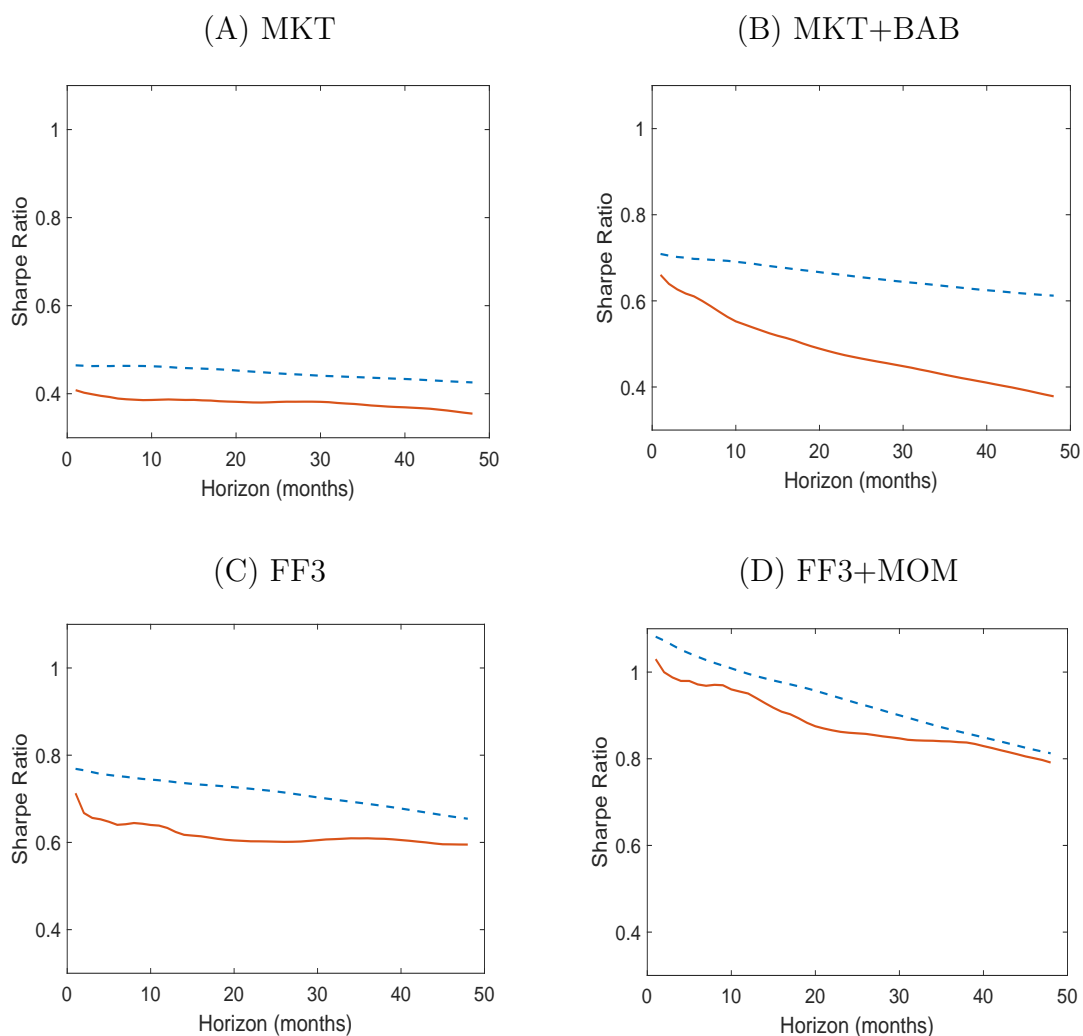
The figure plots the annualized maximal in-sample Sharpe ratio combination of the factors in each model against the annualized mean absolute pricing error (MAPE) of the corresponding model, when the model is estimated using one-period returns and tested on excess factor returns with horizons 1, 3, 6, 12, 24, and 48 months. The sample is monthly, from 1963 to 2017 for all models except SY, which is 1963 to 2016, and HXZ, which is 1967 to 2017.

Figure 4
Variance Ratios



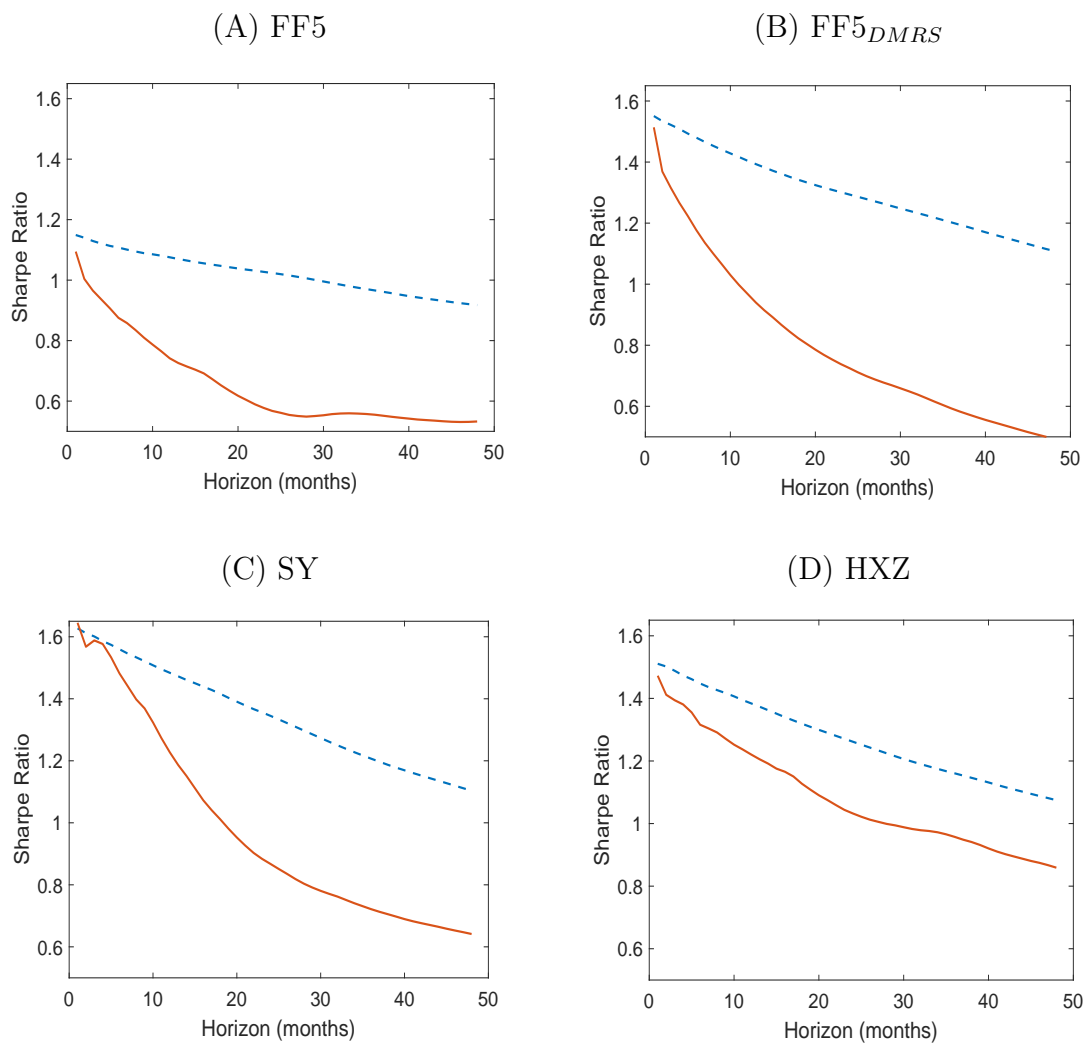
The figure plots variance ratios for the MVE portfolio from each factor model. In particular, for each model we consider the in-sample MVE combination of the factors normalized to have the same return volatility as the market factor. We then add the gross real risk-free rate to this factor return, take logs and compute the variance ratio for each model from horizons 1 to 48 months. If the factor returns are i.i.d., the variance ratio is 1 at all horizons. The variance ratio at horizon h is related to the cumulative autocorrelations of the return series from horizons 1 through h . The sample is monthly, from 1963 to 2017 for all models except SY, which is 1963 to 2016, and HXZ, which is 1967 to 2017.

Figure 5
Term structure of Sharpe ratios I



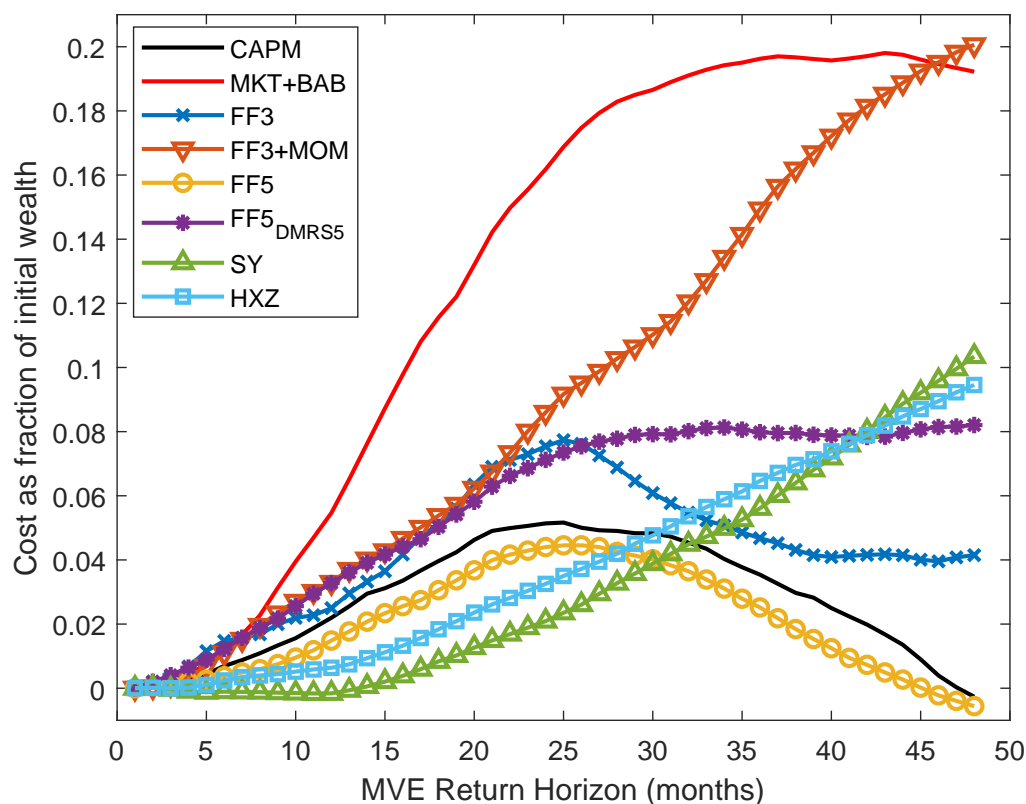
The figure plots the annualized Sharpe ratio of each model's MVE portfolio for different holding periods. For each model we consider the in-sample MVE combination of the factors normalized to have the same return volatility as the market factor. We then add the gross real risk-free rate to this factor return and get h -period returns to this portfolio as $R_{t,t+h} = R_{t,t+1} \times R_{t+1,t+2} \times \dots \times R_{t+h-1,t+h}$. The h -period risk-free rate is found in the same way. We then calculate the h -period annualized Sharpe ratio as $\sqrt{12/h} \times E(R_{t,t+h} - R_{t,t+h}^f) / V^{1/2}(R_{t,t+h} - R_{t,t+h}^f)$ for horizons 1 to 48 months (solid, red line). The dashed, blue line gives the corresponding Sharpe ratios using a bootstrap approach that creates i.i.d. factor returns. The sample is monthly, from 1963 to 2017 for all models.

Figure 6
Term structure of Sharpe ratios II



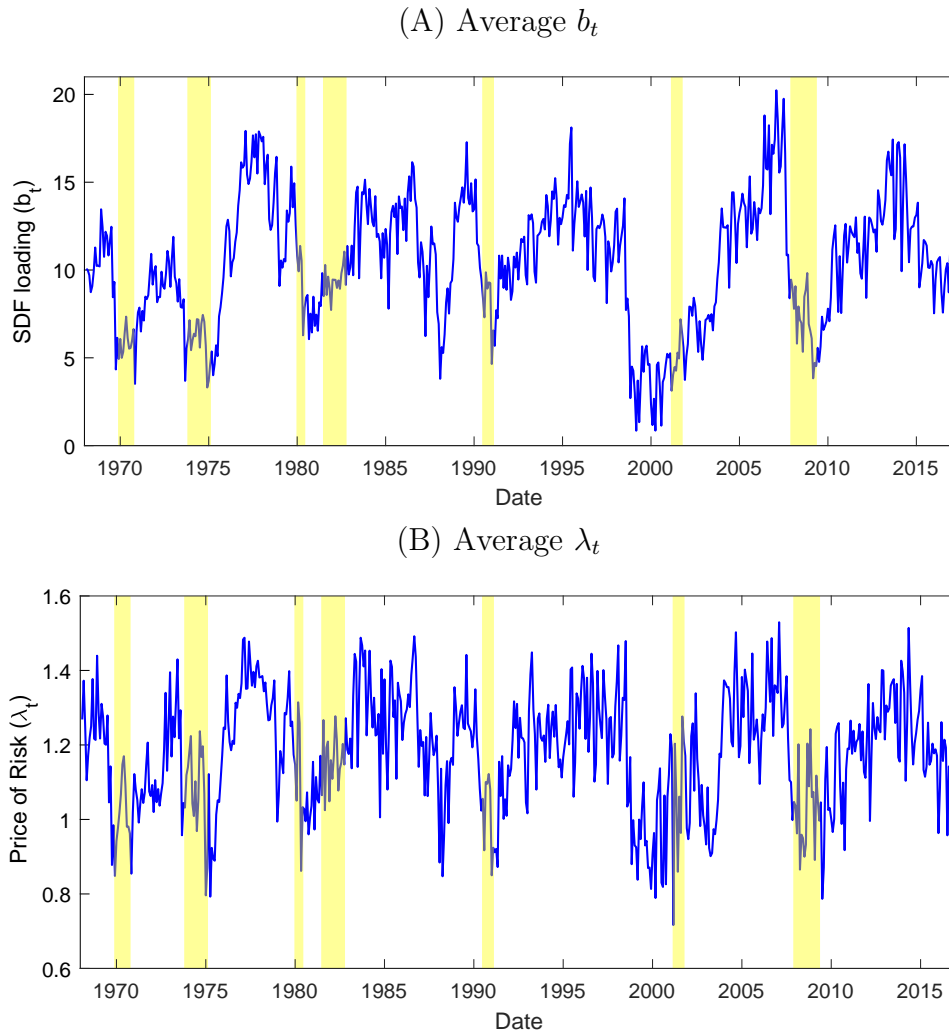
The figure plots the annualized Sharpe ratio of each model's MVE portfolio for different holding periods. For each model we consider the in-sample MVE combination of the factors normalized to have the same return volatility as the market factor. We then add the gross real risk-free rate to this factor return and get h -period returns to this portfolio as $R_{t,t+h} = R_{t,t+1} \times R_{t+1,t+2} \times \dots \times R_{t+h-1,t+h}$. The h -period risk-free rate is found in the same way. We then calculate the h -period annualized Sharpe ratio as $\sqrt{12/h} \times E(R_{t,t+h} - R_{t,t+h}^f) / V^{1/2}(R_{t,t+h} - R_{t,t+h}^f)$ for horizons 1 to 48 months (solid, red line). The dashed, blue line gives the corresponding Sharpe ratios using a bootstrap approach that creates i.i.d. factor returns. The sample is monthly, from 1963 to 2017 for all models except SY, which is 1963 to 2016, and HXZ, which is 1967 to 2017.

Figure 7
A Measure of the Welfare Costs of the Factor Dynamics



The figure plots what an h -period CRRA agent with utility over final period wealth would be willing to pay as a fraction of initial wealth to get i.i.d. returns on each model's MVE factor with the same one-period return distribution as the actual factor. The factor for each model is the in-sample MVE combination of the model's factors normalized to have the same volatility as the market factor. We then add the gross real risk-free rate to this factor return and get h -period returns to this portfolio as $R_{t,t+h} = R_{t,t+1} \times R_{t+1,t+2} \times \dots \times R_{t+h-1,t+h}$. If the MVE factor is i.i.d. or the dynamics are such that they do not impact the investors utility, the amount the agent would be willing to pay is zero. The risk aversion is set to 5. The sample is monthly, from 1963 to 2017 for all models except SY, which is 1963 to 2016, and HXZ, which is 1967 to 2017.

Figure 8
Time series of average risk premiums



Panel A plots the average b_t across all models and factors, where b_t is first averaged across the factors in a model at each time t and then this quantity is averaged across models for each t . Panel B shows the corresponding average of prices of risk, λ_t , across models. The yellow bars indicate NBER recessions. The sample is monthly, from 1967 to 2016, reflecting the common sample across all models.

Table 1: MHR tests of linear factor models

Panel A:	CAPM	MKT+BAB	FF3	FF3+MOM
p-value	0.186	0.039	0.002	0.026
MAPE	0.007	0.030	0.014	0.076
Sharpe Ratio	0.395	0.701	0.692	1.004

Panel B:	FF5	FF5_{DMRS}	SY	HXZ
p-value	0.004	0.001	0.001	0.011
MAPE	0.019	0.034	0.119	0.061
Sharpe Ratio	1.116	1.588	1.670	1.431

The first row of each panel gives the p -value from the GMM J -test given in Sections 3 and 4, where the linear factor models are estimated on the one-period factor returns and tested on multi-horizon factor returns. The second row displays the annualized mean absolute price error ($MAPE$) across the test assets. The returns horizons used are 1, 3, 6, 12, 24, and 48 months. The table also reports the sample Sharpe ratio of the in-sample MVE combination of each model's factors. The sample is monthly, from 1963 to 2017 for all models except SY, which is 1963 to 2016, and HXZ, which is 1967 to 2017.

Table 2: MHR tests of conditional linear factor models

Panel A:	CAPM	MKT+BAB	FF3	FF3+MOM
p-value	0.810	0.783	0.692	0.715
MAPE(b_t)/MAPE(b)	22.5%	26.9%	68.1%	14.5%
SR(b_t)/SR(b)	100.3%	131.1%	130.5%	121.3%

Panel B:	FF5	FF5_{DMRS}	SY	HXZ
p-value	0.117	0.572	0.002	0.091
MAPE(b_t)/MAPE(b)	82.5%	63.5%	43.3%	55.8%
SR(b_t)/SR(b)	124.0%	114.3%	97.4%	111.8%

This table reports test statistics from the factor models with time-varying SDF loadings b_t , as opposed to the constant b model tests given in Table 1. The first row of each panel gives the p -value from the GMM J -test (see Sections 3 and 5 for details). The returns horizons used in the test are 1, 3, 6, 12, 24, and 48 months. The second row gives the mean absolute pricing errors (MAPE) as a fraction of the MAPE from the constant b version of the model from Table 1. The third row gives the sample annualized Sharpe ratio of the unconditional MVE portfolio as implied by the model with time-varying b_t , reported as a fraction of the Sharpe ratio from the constant b model's unconditional MVE portfolio. The sample is monthly, from 1963 to 2017 for all models except SY, which is 1963 to 2016, and HXZ, which is 1967 to 2017.

Table 3: Properties of time-varying SDF loadings

Panel A:	CAPM	MKT+BAB	FF3	FF3+MOM
Mean(b_t)	3.133	4.513	4.494	6.809
StDev(b_t)	1.343	3.117	3.105	3.954
Auto(annual)(b_t)	0.275	0.151	0.182	0.293
Beta 1 (dp)	-0.291 (-1.328)	1.170 (1.424)	0.918 (1.253)	1.526 (1.493)
Beta 2 (Term)	0.073 (0.968)	0.047 (0.216)	0.166 (0.856)	0.319 (0.974)
Beta 3 (Mkt Var)	-7.216*** (-5.708)	-10.204*** (-6.042)	-6.424*** (-3.816)	-10.122*** (-3.789)
Beta 4 (-IP)	-0.854 (-0.446)	-13.756** (-2.460)	-8.626** (-1.989)	-19.477*** (-2.919)
R_{adj}^2	0.633	0.283	0.112	0.211
Panel B:	FF5	FF5_{DMRS}	SY	HXZ
Mean(b_t)	9.257	21.122	13.259	13.649
StDev(b_t)	4.583	6.890	5.609	5.755
Auto(annual)(b_t)	0.293	0.376	0.336	0.518
Beta 1 (dp)	3.392*** (3.054)	6.581*** (3.370)	3.638*** (2.916)	1.917 (1.019)
Beta 2 (Term)	0.510 (1.465)	1.014** (2.066)	1.355*** (3.596)	-0.354 (-0.571)
Beta 3 (Mkt Var)	-9.831*** (-3.315)	-23.972*** (-7.339)	-19.126*** (-6.506)	-19.674*** (-3.836)
Beta 4 (-IP)	-11.079 (-1.423)	-25.238** (-1.974)	-27.351*** (-2.787)	-17.415* (-1.660)
R_{adj}^2	0.161	0.370	0.388	0.276

The table shows various sample statistics for b_t computed as the average of a given model's SDF factor loadings across factors at each t . The first three rows of each panel gives the mean, standard deviation, and annual autocorrelation. The next rows give the regression coefficients of a regression of b_t onto standard conditioning variables: (1) the market log dividend-price ratio, (2) the difference between the 10-year Treasury bond yield and the 3-month Treasury bill yield (the term spread), (3) the conditional market return variance estimated using EGARCH(1,1) (times 100 for scaling), and (4) the negative of annual log industrial production growth (IP). Heteroskedasticity and autocorrelation adjusted t -statistics are given in parentheses. The last row provides the adjusted R^2 from these regressions. The sample is monthly, from 1963 to 2017 for all models except SY, which is 1963 to 2016, and HXZ, which is 1967 to 2017.

Table 4: Properties of average time-varying SDF loadings and prices of risk

	Average b_t	Average λ_t
Mean	10.443	1.180
StDev	3.603	0.155
Autocorr(annual)	0.444	0.204
Beta 1 (dp)	2.735*** (2.746)	0.024** (2.310)
Beta 2 (Term)	0.437 (1.467)	0.004* (1.673)
Beta 3 (Mkt Var)	-14.193*** (-5.820)	-0.119*** (-4.815)
Beta 4 (-IP)	-17.563*** (-3.040)	-0.256*** (-4.109)
R_{adj}^2	0.433	0.253

The table reports the same statistics as those in Table 3. The first column, however, gives these statistics for the average of b_t across models. The last column gives these statistics for the average annualized price of risk, λ_t , across the models. The sample is monthly, from 1967 to 2016, reflecting the common sample across all models.

A Derivations

A.1 Two-period valuation

The two-period valuation involves solving for the covariance:

$$\begin{aligned}
 Cov(M_{t,t+1}R_{t,t+1}, M_{t,t+1}R_{t,t+1}) &= Cov\left(\begin{matrix} (1 - \sigma_\varepsilon^{-2}\mu\varepsilon_{t+1})(1 + \mu_t + \beta\varepsilon_{t+1} + u_{t+1}), \\ (1 - \sigma_\varepsilon^{-2}\mu\varepsilon_{t+2})(1 + \mu_{t+1} + \beta\varepsilon_{t+2} + u_{t+2}) \end{matrix}\right) \\
 &= Cov\left(\begin{matrix} (1 - \sigma_\varepsilon^{-2}\mu\varepsilon_{t+1})u_{t+1}, \\ (1 - \sigma_\varepsilon^{-2}\mu\varepsilon_{t+2})\mu_{t+1} \end{matrix}\right) \\
 &= Cov(u_{t+1}, \mu_{t+1}),
 \end{aligned}$$

where $\sigma_\varepsilon^2 = V(\varepsilon_{t+1})$. The second and third equalities follow as ε_{t+1} is i.i.d. and uncorrelated with u_{t+1} and, thus, with μ_{t+1} . Because $E_t u_{t+1} = 0$,

$$\begin{aligned}
 Cov(u_{t+1}, \mu_{t+1}) &= E[u_{t+1}\mu_{t+1}] \\
 &= \Pr\{u_{t+1} = \delta\} \delta\mu_H + \Pr\{u_{t+1} = -\delta\} (-\delta)\mu_L \\
 &= \frac{\delta}{2}(\mu_H - \mu_L),
 \end{aligned}$$

since u_{t+1} equals δ with probability 0.5 and $-\delta$ with probability 0.5, and since $\mu_{t+1} = \mu_H$ if $u_{t+1} = \delta$ and $\mu_{t+1} = \mu_L$ if $u_{t+1} = -\delta$.

A.2 Law of one price and MHR

It is immediate from the law of iterated expectations, that an SDF that prices a set of single-horizon returns conditionally, that is,

$$E_t[M_{t,t+1}R_{t,t+1}] = 1,$$

also prices multi-horizon returns to the same set of assets:

$$E[M_{t-h,t+1}R_{t-h,t+1}] = 1 \text{ for any } h \geq 1.$$

Indeed, that can be shown by recursively iterating on the following equation for $h = 1, 2, \dots$:

$$\begin{aligned}
 E[M_{t-h,t+1}R_{t-h,t+1}] &= E[M_{t-h,t}R_{t-h,t}M_{t,t+1}R_{t,t+1}] \\
 &= E[M_{t-h,t}R_{t-h,t}E_t[M_{t,t+1}R_{t,t+1}]] = E[M_{t-h,t}R_{t-h,t}],
 \end{aligned}$$

where the last equality follows if the model prices the single-horizon returns conditionally.

A.3 No serial correlation in residuals

That residuals are not autocorrelated follows from Equation (4). For simplicity, consider one horizon, h . We have that

$$E(f_t^i \cdot f_{t+1}^i) = E(f_t^i \cdot E_t(f_{t+1}^i)) = E(f_t^i \cdot z_{i,t}^{(h)} E_t(M_{t,t+1} R_{t,t+1}^i - 1)) = 0,$$

because, under the null, $E_t(M_{t,t+1} R_{t,t+1}^i - 1) = 0$ for all t .

A.4 Factor dynamics implied by constant SDF loadings

We apply the SDF in Equations (2) and (3) to conditional pricing of factors themselves, $E_t(M_{t,t+1} F_{t,t+1}) = 0$. Denoting $E(F_{t,t+1}) = \mu$ and $V(F_{t,t+1}) = \Sigma$, we have

$$0 = E_t((1 + \mu^\top \Sigma^{-1} \mu - \mu^\top \Sigma^{-1} \cdot F_{t,t+1}) F_{t,t+1}^\top).$$

Therefore,

$$E_t(F_{t,t+1}^\top) = (1 + \mu^\top \Sigma^{-1} \mu)^{-1} \mu^\top \Sigma^{-1} \cdot E_t(F_{t,t+1} F_{t,t+1}^\top).$$

A.5 Welfare cost

The expression is obtained by equalizing utilities under i.i.d. returns and the ones in the data:

$$E[((1 - \text{wc}(h)) R_{t,t+h}^{i.i.d.})^{1-\gamma}] = (1 - \text{wc}(h))^{1-\gamma} [E(R_{t,t+1}^{1-\gamma})]^h = E(R_{t,t+h}^{1-\gamma}).$$

Here we exploit $E[(R_{t,t+h}^{i.i.d.})^{1-\gamma}] = [E((R_{t,t+1}^{i.i.d.})^{1-\gamma})]^h$. We also assume that the unconditional properties of one-period actual and hypothetical i.i.d. returns are the same.

A.6 Unconditional MVE with time-varying SDF factor loadings

Equation (10) implies that the SDF corresponding to the time-varying b_t features a time-varying intercept:

$$M_{t,t+1} = (1 + b_t^\top \mu_t) - b_t^\top F_{t,t+1}.$$

This SDF is equivalent to one with a constant intercept. Consider

$$\begin{aligned}\tilde{M}_{t,t+1} &= 1 - (1 + b_t^\top \mu_t)^{-1} b_t^\top F_{t,t+1} \equiv 1 - \tilde{F}_{t,t+1} \\ &= (1 - E(\tilde{F}_{t,t+1})) - (\tilde{F}_{t,t+1} - E(\tilde{F}_{t,t+1})) \propto 1 - \tilde{b}(\tilde{F}_{t,t+1} - \tilde{\mu}),\end{aligned}$$

where $\tilde{F}_{t,t+1} = (1 + b_t^\top \mu_t)^{-1} b_t^\top F_{t,t+1}$, $\tilde{\mu} = E(\tilde{F}_{t,t+1})$, and $\tilde{b} = (1 - E(\tilde{F}_{t,t+1}))^{-1}$.

This representation of the SDF conditionally prices the same set of excess returns as the original SDF $M_{t,t+1}$. This alternative formulation of a valid excess return SDF, makes it clear that the return to $\tilde{b}F_{t,t+1}$ is perfectly conditionally and unconditionally correlated with the SDF. Given the latter, it is an unconditional MVE return.

Armed with the unconditional MVE returns we can compute their Sharpe ratios using the standard approach.