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ABSTRACT

We propose a method for bounding the demand elasticity in growing, homogeneous-product markets that requires only minimal data—market price and quantity over a time span as short as two periods. Reminiscent of revealed-preference arguments using choices over time to bound the shape of indifference curves, we use shifts in the equilibrium over time to bound the shape of the demand curve under the assumption that growing demand curves do not cross. We apply the method to assess the effectiveness of the antitrust remedy in the 1952 Du Pont decision, ordering the incumbent manufacturers to license their patents for commercial plastics. Commentators have suggested that the incumbents may have preserved the monopoly outcome by gaming the licensing contracts. The upper bounds on demand elasticities that we compute are significantly less than 1 in many post-remedy years. Such inelastic demand is inconsistent with monopoly, suggesting the remedy may have been effective.

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1. Introduction

Whether structural remedies for antitrust violations are effective in fostering competition has been an ongoing concern for scholars and policymakers. An early advocate of structural remedies, Adams (1951) documented a series of earlier antitrust cases that the agencies won but imposed remedies too weak to restore competition, the so-called pyrrhic victories of antitrust.

To understand why promising remedies might fail to achieve their goals, consider the *Du Pont* decision of 1952, which compelled the incumbent manufacturers in the commercial plastics market to license their manufacturing patents to all applicants. As Areeda, Kaplow, and Edlin (2013) note, remedies do not typically set the licensing terms, leaving them subject to commercial negotiation. This raises the opportunity for incumbent manufacturers to preserve something close to the monopoly outcome by specifying exorbitant fees. Even if incumbents set fees that are deemed "reasonable," they can withhold the tacit knowledge necessary for entrants to compete effectively against them.

In this paper, we study the effectiveness of the antitrust remedy ordered in the aforementioned *Du Pont* decision. The case involved two incumbent chemical manufacturers: the U.S. firm Du Pont and the U.K. firm Imperial Chemical Industries (ICI). The two firms signed a Patents and Processes agreement in 1929, granting each company exclusive licenses for the patents and secret processes controlled by the other and dividing the global market into exclusive territories between them. The U.S. government brought suit under the Sherman Act, alleging an illegal market division. The judge ruled in favor of the government, ordering the defendants to cancel their exclusive-territory arrangements, requiring them to license the patents behind several of their products, most significantly, two types of plastic widely used for commercial purposes: low-density polyethylene (used to make Tupperware food-storage containers) and high-density polyethylene (used to make Hula Hoops).

The remedy ostensibly had the desired procompetitive effect: eleven manufacturers entered by the end of the decade; prices steadily declined and output rose (Backman 1964, p. 71). However, the same price declines and output increases may have arisen in a monopoly market experiencing substantial cost declines, plausibly true for plastics in the 1950s and 60s. The entrants may merely have produced their share of the monopoly quantity, returning most of the rents to the incumbents.

Our study will provide formal evidence for the effectiveness of the remedy that cuts through these criticisms. Formal study is hindered by a paucity of historical data, just yearly aggregate price and quantity data for polyethylene, only available for post-remedy years. We offer a new method that allows solid conclusions to be drawn from these fairly minimal data. The method bounds the demand elasticity for a homogeneous product in a given year based on the assumption that demand is growing, sandwiching the demand curve in the given year between the demand curves in earlier and later periods.

Figure 1 provides some intuition for how the method works. In the example in the figure, the researcher has price and quantity data for two years. The equilibrium point in the first year is e_1 and in the second is, say, e'_2. The researcher wants to bound the slope the period-1 inverse demand curve through e_1. With so little data, there may be little hope to say anything more than inverse demand lies somewhere between the horizontal dotted line, corresponding to an infinitely elastic demand curve, and the dotted vertical line, corresponding to an infinitely inelastic one. But in fact we can say more. Positing a functional form for demand, say linear, and assuming demand is nondecreasing over time, a curve like D can be ruled out because that would put the later equilibrium point e'_2 on a lower demand curve. The demand curve through e_1 must be at least as steep as the line connecting e_1 and e'_2—the line labeled D'—to preserve nondecreasing demand. The comparison of the two equilibrium points leads to a lower bound on the steepness of inverse demand, which translates into an upper bound on the elasticity of demand through e_1.

The tightness of the bound is data-driven. Suppose that the observed equilibrium point in period 2 is e''_2 rather than e'_2. This leads to a tighter bound on the elasticity. The inverse demand through e_1 must be at least as steep as D'' to keep e''_2 from lying on a lower demand curve. The demand curve is funneled from the entire shaded region down into just the dark-shaded part, only leaving room for a relatively inelastic demand. A researcher who finds that demand is as inelastic as D'' may be able to rule out monopoly. Intuitively, the drop from e_1 to a point like e''_2 may be so steep that a monopolist would never have dropped price this much for such a small increase in quantity, even if marginal costs had dropped to zero. The more likely conclusion may be that competitive pressure drove the firms onto the inelastic region of demand.

The example in Figure 1 implicitly assumes that demand is linear, at least locally. We develop variants of the methodology that can narrow the bounds on the demand elasticity considerably if the researcher is willing to assume that demand is globally linear, from the vertical to the horizontal

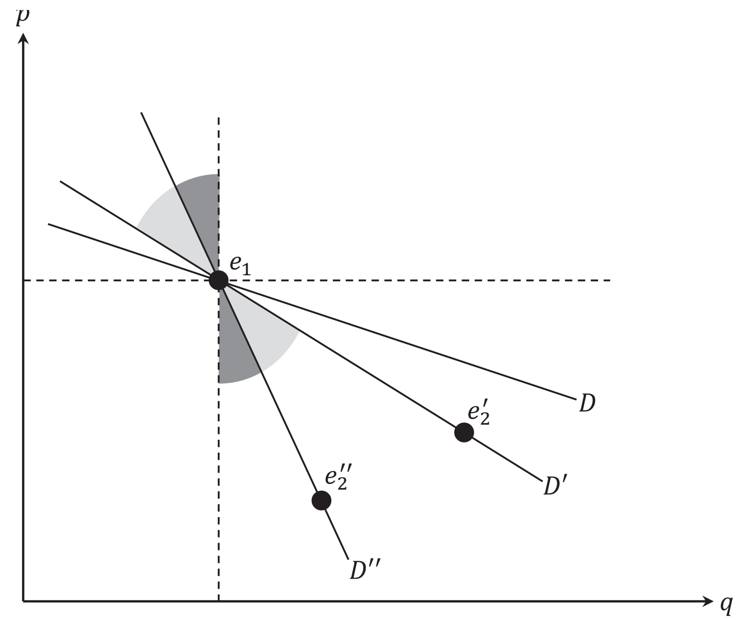

Figure 1: Intuition for the Method Bounding the Demand Elasticity

intercept. The variants incorporate information from the intercepts of other year's demands to iteratively narrow the bounds on the elasticity in a given year. Our methods generalize beyond the linear case. We redo the analysis for another widely assumed functional form, logit demand, and show in the application that the elasticity bounds are often virtually identical to those assuming linear demand. The methodology section provides general formulas allowing any functional form that the researcher chooses to be imposed.

Bounding methodologies that seem initially promising may turn out to produce such wide bounds as to be uninformative. In our application to the polyethylene market, however, the upper bound on the demand elasticity is substantially and statistically significantly below 1 in many sample years. The bounds are robust to alternative products, functional forms, and method variants. Such inelastic demand is inconsistent with monopoly, suggesting the remedy may have been effective. Thus our formal evidence lends support to the use of structural remedies in monopolization cases.

Our work is related to several literatures. One is the literature assessing the effectiveness of structural remedies for antitrust. In addition to the aforementioned, pioneering study by Adams (1951), studies by the U.S Federal Trade Commission (FTC) staff allso make important contributions to the literature. These studies include U.S. Federal Trade Commission (1999), examining the efficacy of divestiture remedies accompanying merger approvals from 1990–94; Farrell, Pautler, and Vita (2009), describing a similar FTC exercise focusing on a sample of hospital mergers; and U.S. Federal Trade Commission (2017), studying all 89 FTC merger orders from 2006–12. This last study judged remedies to be at least a qualified success in 83% of cases.

We also contribute to the industrial-organization literature following Manski (1995) that seeks to bound rather than point-estimate elasticities and other parameters. Haile and Tamer (2003) derive bounds for bids and reserve prices in their incomplete model of English auctions. Ciliberto and Tamer (2009) identify parameter sets in the presence of multiple equilibria, applying their method to an entry game among U.S. airlines. Pakes (2010) and Pakes, Porter, Ho, and Ishii (2015) develop moment inequalities in single and multi-agent settings based on the revealed-preference argument that an observed choice is better in expectation than alternative choices in the feasible set. Our paper applies to a different setting and adopts a different approach—dealing with aggregate price and quantity data in a homogeneous-product market rather than microdata from a differentiated-product market. However, we share the basic idea that equilibrium observations can identify weak inequalities among structural parameters. Rather than "revealed preference," we label the idea "revealed growth" in our title—the idea that, in a growing market, equilibrium points must lie on higher demand curves over time, which can be used to bound demand elasticities.

Our paper is also related to the literature on patent licensing. The most germane papers in this vast literature are those regarding patent pools and those using historical evidence to shed light on contemporary issues in intellectual property. Stocking and Watkins (1946) were the first to formally identify the Patents and Processes agreement between Du Pont and ICI as a patent pool.[1] Lerner, Strojwas and Tirole (2007) identify distinguishing features of pro-competitive patent pools. For example, independent licensing is more likely to be allowed when complementary patents are pooled, and pooling complementary patents is more likely to be procompetitive than substitute patents. Lampe and Moser (2013) study the first U.S. patent pool, formed in 1856 involving sewing machines. Perhaps the closest paper in this literature is Watzinger, Fackler, Nagler and Schnitzer (2017), who examine induced innovation in the context of the 1956 Bell System consent

[1]The formation of a patent pool was hardly exceptional during that period, as Tirole (2014) notes in his Nobel lecture: "A little known fact is that, prior to 1945, most high-tech industries of the time were run by patent pools. But the worry about cartelization through joint marketing led to a hostile decision of the U.S. Supreme Court in 1945 and the disappearance of pools until the recent revival of interest."

decree. The United States filed a monopolization suit against Bell in 1949, alleging that it had foreclosed entry into various related markets such as telecommunications equipment. The consent decree allowed Bell to remain vertically integrated into telecommunications but forced it to license its existing patents royalty-free. The authors find that, while follow-on innovation did not expand in telecommunications equipment, it did in other sectors.

Finally, our paper is related to economic commentaries specifically on the remedy ordered the *Du Pont* case. Some of the commentary is quite critical. In his classic, two-volume history, Reader (1975, p. 417) argues that the prosecution arose out of political pressure at the Department of Justice. Hounshell and Smith (1988, p. 206) criticize the court's verdict for a different reason. "Judge Ryan ... refused to accept the argument that ICI's and Du Pont's agreement resulted in the genuine exchange of scientific and technical information. But the overwhelming historical evidence demonstrates this was indeed the case." By contrast, we provide supporting evidence that the licensing remedy was effective in ending the monopoly and spurring competition in plastics.

The paper is structured as follows. The first part of the paper covers methodology. Section 2 models the situation to which the methodology will be applied. Section 3 presents a method for bounding demand elasticities incorporating local information from the pairwise comparison of equilibria. A series of subsections treats different demand functions, from linear, to logit, to general forms. Section 4 shows how the bounds can be tightened by iteratively incorporating limiting information. Again, a series of subsections treats different demand functions, from linear, to logit, to general forms. The remainder of the paper covers the empirical application to the *Du Pont* decision. Section 5 provides institutional background, Section 6 describes the data, Section 7 presents results, and Section 8 discusses robustness issues. Section 9 concludes. Appendix A provides proofs not included in the text. Online Appendix B provides supplementary figures.

2. Model

This section lays out a general model of a growing market for a homogeneous product used in the analysis. Each period t, the interaction between producers and consumers on the market leads to an equilibrium $e_t \equiv (q_t, p_t)$, where $q_t \geq 0$ is market quantity and $p_t \geq 0$ is market price. The researcher observes the market equilibrium over some time span $t \in \{1, \ldots, T\}$. Let $E \equiv \{e_t \,|\, t = 1, \ldots, T\}$ denote the set of time-series observations of equilibrium.

The analysis can be streamlined without much loss of generality if ties between price observations or quantity observations are ruled out, accomplished by the following definition of distinctness.

Definition. *E is* distinct *if and only if, for all $e_t, e_{t'} \in E$ such that $t \neq t'$, we have $q_{t'} \neq q_{t''}$ and $p_{t'} \neq p_{t''}$.*

Assuming E is distinct entails little loss of generality if one supposes that two observations are never exactly equal if measured to arbitrary precision. We also assume that the market is nontrivial in the sense of involving positive prices and quantities in each period.

Definition. *E is* nontrivial *if and only if, for all $e_t \in E$, we have $q_t, p_t > 0$.*

We will characterize each side of the market in turn starting with producers. Since the goal of our application will be to determine whether the antitrust remedy was effective in changing producer conduct, it is natural to consider producer conduct as an unknown to be determined. Thus, we will be fairly agnostic about producer behavior in the model. Producers may engage in perfect competition, in which case it may be possible to characterize their behavior with a supply curve; or they may engage in some form of imperfect competition, perhaps monopoly, whose behavior is characterized by a supply relation derived from a first-order condition.

Next, consider the consumer side of the market. Consumers are price takers whose behavior is captured by the demand curve $q = D_t(p)$, the functional form of which is posited by the researcher. Assume the functional form obeys the law of demand, i.e., that the demand curve at a given point in time, $D_t(p)$, is nonincreasing in p.

Assumption 1 (Law of Demand). *For all $t \in \{1, \ldots, T\}$, $D_t(p') \geq D_t(p'')$ for all $p'' > p' \geq 0$.*

The assumption of a growing market entails growing demand, formally that $D_t(p)$ is nondecreasing in t.

Assumption 2 (Growing Demand). *For all $t', t'' \in \{1, \ldots, T\}$ such that $t' < t''$, $D_{t'}(p) \leq D_{t''}(p)$ for all $p \geq 0$.*

Not all equilibrium configurations E are consistent with growing demand. To aid the discussion of which inconsistent configurations are ruled out, we introduce notation for subsets determined

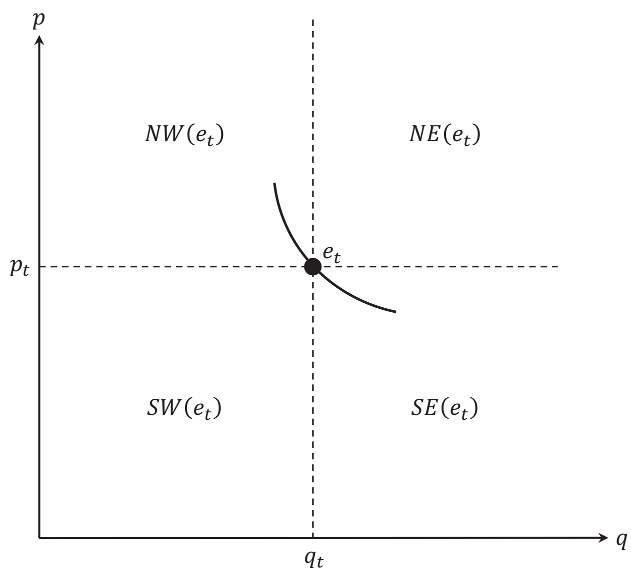

Figure 2: Sets Determined by Compass Position

by compass directions relative to a reference equilibrium point, e_t:

$$
\begin{aligned}
NW(e_t) &\equiv \{(q_{t'}, p_{t'}) \in E \mid q_{t'} < q_t,\ p_{t'} > p_t\} \\
NE(e_t) &\equiv \{(q_{t'}, p_{t'}) \in E \mid q_{t'} > q_t,\ p_{t'} > p_t\} \\
SE(e_t) &\equiv \{(q_{t'}, p_{t'}) \in E \mid q_{t'} > q_t,\ p_{t'} < p_t\} \\
SW(e_t) &\equiv \{(q_{t'}, p_{t'}) \in E \mid q_{t'} < q_t,\ p_{t'} < p_t\}.
\end{aligned}
\tag{1}
$$

Each equilibrium point divides the nonnegative quadrant into four regions corresponding to four relative compass directions; our notation provides a label for these subregions. Figure 2 depicts these compass sets. As the figure indicates, the axes are taken to be oriented in the usual way when graphing inverse demand, i.e., with quantity on the horizontal axis and price on the vertical axis. The fact that the definitions in (1) involve strict inequalities leaves points on the dotted lines through e_t in the figure unclassified, but this is without loss of generality for distinct E, which precludes equilibrium points from sharing coordinates.

Building on the definitions in equation (1), we define subsets of the compass sets depending on the time at which the equilibria appear. For example, define

$$
\begin{aligned}
NW^{-}(e_t) &\equiv \{e_{t'} \in NW(e_t) \mid t' < t\} \\
NW^{+}(e_t) &\equiv \{e_{t'} \in NW(e_t) \mid t' > t\}.
\end{aligned}
\tag{2}
$$

In words, $NW^{-}(e_t)$ is the subset of equilibrium points in $NW(e_t)$ that occur before e_t and $NW^{+}(e_t)$ the subset that occur after e_t. Four compass directions and two relative times entail eight possible combinations and thus eight possible subsets. The other six subsets—$NE^{-}(e_t)$, $NE^{+}(e_t)$, $SE^{-}(e_t)$, $SE^{+}(e_t)$, $SW^{-}(e_t)$, and $SW^{+}(e_t)$—are defined by analogy to equation (2). We will call these *time-augmented compass sets*.

Consider positioning a new equilibrium point $e_{t'}$ that occurs after e_t in one of the compass sets in the figure. A downward-sloping inverse demand curve through e_t (such as the solid curve drawn in the figure) slices through regions $NW(e_t)$ and $SE(e_t)$. This leaves room to add the new equilibrium point so that it lies on a higher demand curve—thus respecting Assumption 2 that demand is growing—yet still falls in $NW^{+}(e_t)$ or $SE^{+}(e_t)$. The new equilibrium could also easily be added in $NE^{+}(e_t)$ as well, placed on a higher inverse demand in that region. However, there is no way to add the new equilibrium point to $SW^{+}(e_t)$ without having it lie on a lower demand curve. Thus, in a market with growing demand, $SW^{+}(e_t)$ must be empty for all $e_t \in E$.

When E has the property that $SW^{+}(e_t)$ is empty for all $e_t \in E$, we will say that E exhibits *rectangular expansion* because equilibrium points occurring after the reference one e_t lie outside the rectangle determined by e_t and the origin at opposite corners.

Definition. *E exhibits* rectangular expansion *if and only if $SW^{+}(e_t)$ is empty for all $e_t \in E$.*

With this definition in hand, we can encapsulate the analysis from the previous paragraph leads with the following proposition.

Proposition 1. *Assumption 2 that demand is growing implies that E exhibits rectangular expansion.*

Switching perspectives from demand growing as we advance into the future to demand shrinking as we delve into the past, the fact that $SW^{+}(e_t)$ is empty for all $e_t \in E$ is equivalent to $NE^{-}(e_t)$ being empty for all $e_t \in E$. Hence an equivalent condition for E exhibiting rectangular expansion is that $NE^{-}(e_t)$ is empty for all $e_t \in E$. Yet another equivalent condition for E exhibiting rectangular expansion is that for all $e_{t'}, e_{t''} \in E$ such that $t' < t''$, either $q_{t'} \leq q_{t''}$ or $p_{t'} \leq p_{t''}$. That is, E exhibits rectangular expansion if price, quantity, or both grows each period.

3. Method Incorporating Local Information

We begin with our simplest method for bounding demand elasticities. We will refer to this method as *incorporating local information* because it uses the location of other equilibrium points $e_{t'}$ to constrain the position of the demand curve through a given equilibrium point e_t. This stands in contrast with a method introduced later that uses limits of demand curves through other equilibrium points to constrain the position of the demand curve through e_t. We will refer to this later method as *incorporating limiting information*.

Whether incorporating local or limiting information, all of our methods require the researcher to impose a functional-form assumption on demand. The first subsection analyzes the simple and widely used assumption of linear demand. The next subsection analyzes logit demand. The last subsection extends the methods to general demand functions.

3.1. Linear Demand

Suppose the research imposes the assumption that the sequence of demand curves over time is linear: $D_t(p) = a_t - b_t p$ for $t = 1, \ldots, T$. Note that the parameters a_t and b_t are subscripted by t and thus allowed to vary over time, in turn allowing the demand curve to shift over time.

This specification of demand involves two independent parameters, but further requiring the line to pass through the equilibrium point e_t pins it down to a single-parameter family. Focus for now on the slope, b_t, as the key parameter. Letting $\tilde{D}(p, b_t, e_t)$ be the linear demand with slope b_t through equilibrium point $e_t = (p_t, q_t)$, we have

$$D_t(p) = \tilde{D}(p, b_t, e_t) = q_t + b_t(p_t - p). \tag{3}$$

The law of demand (Assumption 1) holds if and only if $b_t \geq 0$.

With knowledge of the slope b_t and equilibrium point e_t, one can solve for the demand intercept,

$$a_t = q_t + b_t p_t, \tag{4}$$

or for the absolute value of the demand elasticity,

$$\epsilon_t \equiv -\tilde{D}_p(p_t, b_t, e_t)\frac{p_t}{q_t} = \frac{b_t p_t}{q_t}, \tag{5}$$

where $\tilde{D}_p$ denotes the partial derivative of the demand function with respect to its first argument. Note that $b_t \geq 0$ implies $\epsilon_t \geq 0$. For brevity, we will drop the "absolute value" modifier and simply call ϵ_t the demand elasticity.

We will derive bounds on demand slope b_t via pairwise comparisons of that period's equilibrium point e_t with the location of the other equilibrium points. Using equation (5), bounds on b_t can then be translated into the desired bounds on ϵ_t.

Before proceeding with the formal analysis, we provide some intuition on how the method works using Figure 3. Each panel compares a reference equilibrium point e_t to another equilibrium point $e_{t'}$. Without further information, all we know is that the linear inverse demand through e_t must be somewhere between the perfectly horizontal and perfectly vertical ones delineated by the dotted lines. What further information the comparison to $e_{t'}$ can provide depends on which of the eight time-augmented compass subsets $e_{t'}$ falls into relative to e_t. By Proposition 1, two of these sets are empty when demand is growing, leaving the six non-empty sets into which $e_{t'}$ can fall, corresponding to the panels in the figure: $SE^+(e_t)$, $NW^+(e_t)$, $NE^+(e_t)$, $NW^-(e_t)$, $SE^-(e_t)$, and $SW^-(e_t)$.

In Panel A, $e_{t'} \in SE^+(e_t)$. Drawing a line connecting e_t and $e_{t'}$, we know that the inverse demand through e_t cannot be less steep than that; otherwise $e_{t'}$ would be forced to lie on a lower demand curve, violating Assumption 2 that demand is growing over time since $t' > t$ as $e_{t'} \in SW^+(e_t)$. The line connecting e_t and $e_{t'}$ thus provides a lower bound on the steepness of the inverse demand through e_t, allowing us to to narrow the location of the inverse demand curve from somewhere between the dotted lines down into the shaded funnel. Recalling that b_t measures the steepness of the direct (i.e., non-inverse) demand, the lower bound on the steepness of the inverse demand translates into an upper bound on b_t.

In Panel B, $e_{t'} \in NW^+(e_t)$. With this compass orientation for $e_{t'}$, the line connecting e_t and $e_{t'}$ provides an upper bound on the steepness of the inverse demand through e_t; any steeper and $e_{t'}$ would be forced to lie on a lower demand curve, violating Assumption 2. The upper bound on

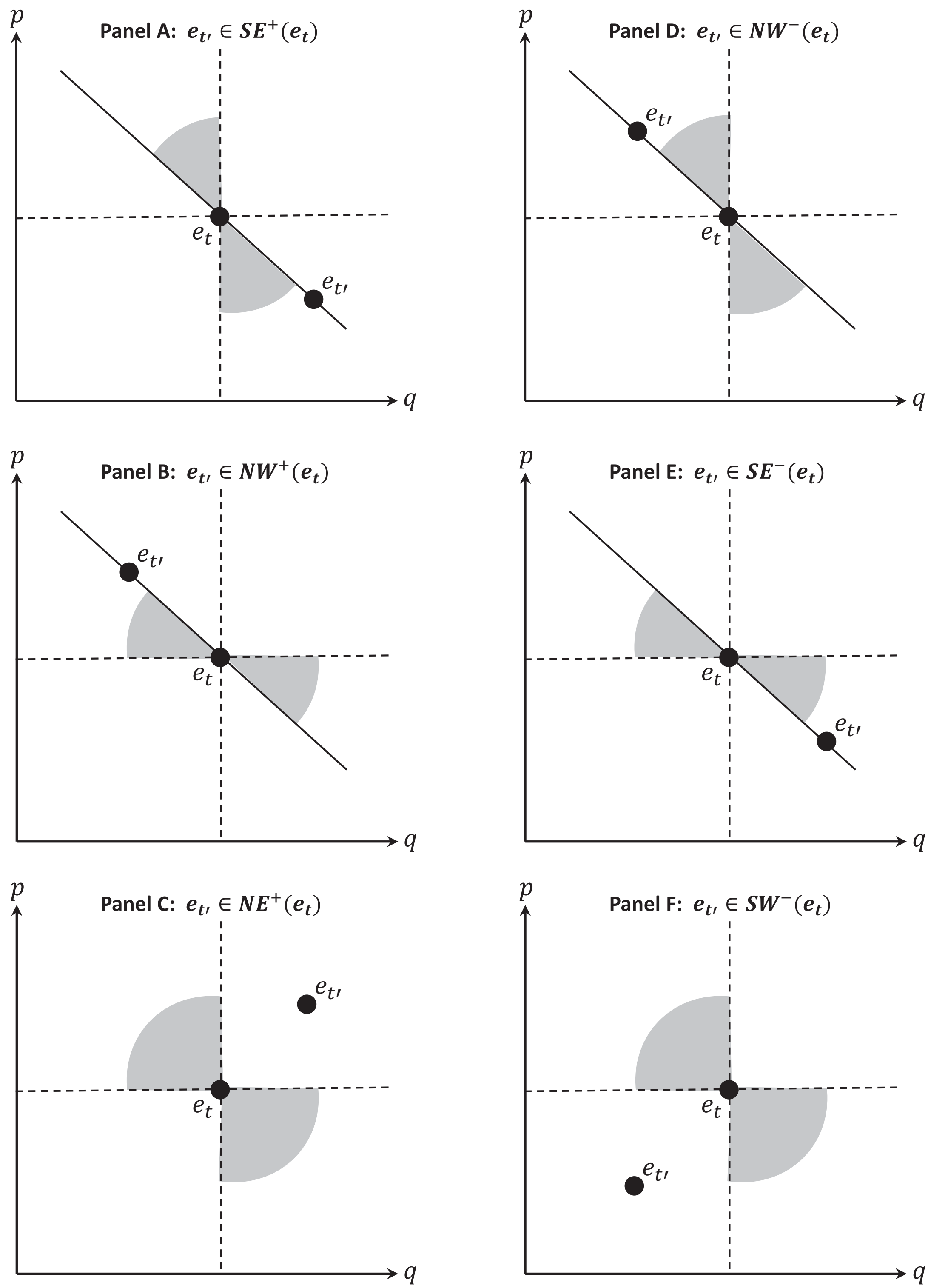

Figure 3: Incorporating Local Information from Pairwise Comparisons to Derive Upper and Lower Bounds

steepness of inverse demand translates into a lower bound on b_t. In Panel C, $e_{t'} \in NE^+(e_t)$. In this case, regardless of which inverse demand between the dotted lines we draw through e_t, it is always possible to place $e_{t'}$ on a higher demand curve. So in this case the pairwise comparison provides no further information to narrow down the location of the inverse demand through e_t.

Panels C–F illustrate cases in which $e_{t'}$ occurs before e_t, so $e_{t'}$ falls into the time-augmented compass sets with minus rather than plus superscripts. Examining each panel, the reader can verify that the bounds are the same as in Panels A–C except that they apply to the opposite compass direction. Thus, $e_{t'} \in NW^-(e_t)$ generates an upper bound on b_t, $e_{t'} \in SE^-(e_t)$ generates a lower bound, and $e_{t'} \in SW^-(e_t)$ provides no information.

To proceed with the formal analysis, first suppose $t' > t$. Then

$$\tilde{D}(p, b_t, e_t) = D_t(p) \leq D_{t'}(p) = \tilde{D}(p, b_{t'}, e_{t'}), \tag{6}$$

where the equalities follow from (3) and the inequality from Assumption 2 of growing demand. Since (6) holds for all $p \geq 0$, it must hold in particular for $p_{t'}$. Substituting $p_{t'}$ into (6) yields $q_t + b_t(p_t - p_{t'}) = \tilde{D}(p_{t'}, b_t, e_t) \leq \tilde{D}(p_{t'}, b_{t'}, e_{t'}) = q_{t'}$, or rearranging,

$$b_t(p_t - p_{t'}) \leq q_{t'} - q_t. \tag{7}$$

The bounds on b_t that can be derived from equation (7) depend on the compass position of $e_{t'}$ relative to e_t. There are three subcases to consider depending on whether $e_{t'}$ is in $NW^+(e_t)$, $NE^+(e_t)$, or $SE^+(e_t)$; the subcase in which $e_{t'} \in SW^+(e_t)$ can be ignored when demand is growing since $SW^+(e_t)$ is empty by Proposition 1. To streamline the analysis, let $B(e_t, e_{t'})$ denote the absolute value of the slope of the linear demand through points e_t and $e_{t'}$:

$$B(e_t, e_{t'}) \equiv \left| \frac{q_{t'} - q_t}{p_{t'} - p_t} \right|. \tag{8}$$

Suppose $e_{t'} \in NW^+(e_t)$. Cross multiplying (7) by $p_t - p_{t'}$ yields

$$b_t \geq \frac{q_{t'} - q_t}{p_t - p_{t'}} = B(e_t, e_{t'}). \tag{9}$$

The first step follows from $e_{t'} \in NW^+(e_t)$, which implies $p_t < p_{t'}$ by (1). The second step follows because the numerator and denominator of the middle fraction are both negative for $e_{t'} \in NW^+(e_t)$. Condition (9) provides a lower bound on the demand slope.

Next, suppose $e_{t'} \in NE^+(e_t)$. Cross multiplying (7) by $p_t - p_{t'}$ yields

$$b_t \geq \frac{q_{t'} - q_t}{p_t - p_{t'}}. \tag{10}$$

The right-hand is negative since the numerator is positive and the denominator is negative for $e_{t'} \in NE^+(e_t)$. Hence, (10) is a weaker condition than the maintained assumption $b_t \geq 0$. Thus this case contributes no useful information to bound b_t.

Next, suppose $e_{t'} \in SE^+(e_t)$. Cross multiplying (7) by $p_t - p_{t'}$ yields

$$b_t \leq \frac{q_{t'} - q_t}{p_t - p_{t'}} = B(e_t, e_{t'}). \tag{11}$$

Cross multiplying did not change the direction of the inequality because $p_t > p_{t'}$ for $e_{t'} \in SE^+(e_t)$. The second step follows because both numerator and denominator of the middle fraction are positive for $e_{t'} \in SE^+(e_t)$. Condition (11) provides an upper bound on the demand slope.

The pairwise comparison of e_t and $e_{t'}$ can be repeated for $t' < t$. Sparing the details, using analysis similar to that above, one can show $b_t \leq B(e_t, e_{t'})$ when $e_{t'} \in NW^-(e_t)$, $b_t \geq B(e_t, e_{t'})$ when $e_{t'} \in SE^-(e_t)$, and no useful information is contributed when $e_{t'} \in SW^-(e_t)$. We have proved the following proposition.

Proposition 2. *Suppose the sequence of linear demands $\tilde{D}(p, b_t, e_t)$ defined in* (3) *satisfies Assumptions 1 and 2. Then $b_t \in [\underline{b}_t, \bar{b}_t]$ for all $t \in \{1, \ldots, T\}$, where*

$$\underline{b}_t \equiv 0 \vee \sup_{e_{t'} \in SE^-(e_t) \cup NW^+(e_t)} B(e_t, e_{t'}) \tag{12}$$

$$\bar{b}_t \equiv \inf_{e_{t'} \in NW^-(e_t) \cup SE^+(e_t)} B(e_t, e_{t'}). \tag{13}$$

A few technical notes about Proposition 2 are in order. The $\vee$ operator denotes the join; i.e., $x \vee y$ denotes the maximum of x and y. The use of this operator in (12) indicates the imposition of a floor of 0 on top of the supremum. We will use this operator repeatedly later as well as $\wedge$ for the meet; i.e., $x \wedge y$ denotes the minimum of x and y. The supremum is taken in (12) rather than the maximum and the infimum in (13) rather than the minimum even though the sets involved are

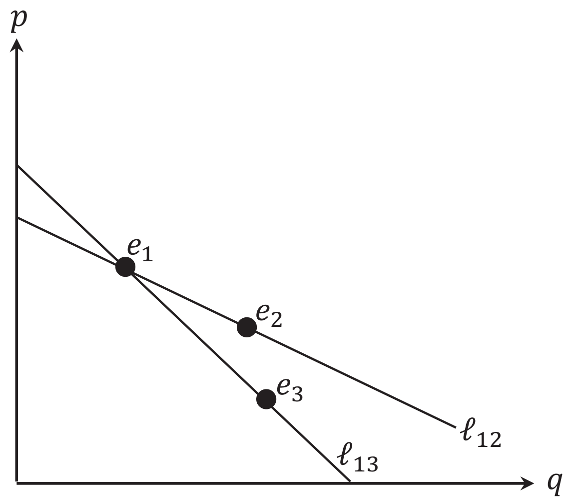

Figure 4: Illustrating Method Incorporating Local Information

discrete so that the possibility that one of these sets is empty can be accommodated. An empty set does not have a maximum or minimum but does have a supremum and infimum; we use the conventional definitions $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$. If the set $SW^-(e_t) \cup NW^+(e_t)$ over which the supremum is taken in (12) is nonempty, then imposing a 0 floor on the supremum is superfluous because $B(e_t, e_{t'})$ is defined to be an absolute value. However, if the set over which the supremum is taken happens to be empty, then $\underline{b}_t = -\infty$ without the imposition of the 0 floor.

Proposition 2 prescribes the following simple algorithm for bounding b_t.[2] The lower bound $\underline{b}_t$ is found by pairing e_t with each of the equilibrium points in $SW^-(e_t)$ and $NW^+(e_t)$, drawing linear demands connecting them, and identifying which is steepest (note, we are here referring to the steepness of the direct, not inverse, demand). The steepness of the identified line gives $\underline{b}_t$. If both $SW^-(e_t)$ and $NW^+(e_t)$ are empty, we set $\underline{b}_t$ to respect the non-negativity constraint on b_t; i.e., $\underline{b}_t = 0$. The upper bound $\bar{b}_t$ is found by pairing e_t with each of the equilibrium points in $NW^-(e_t)$ and $SE^+(e_t)$, drawing linear demands connecting them, and identifying which is the least steep. The steepness of the identified line gives $\bar{b}_t$. If both $NW^-(e_t)$ and $SE^+(e_t)$ are empty, by (13) we have $\bar{b}_t = \inf \emptyset = \infty$. The bounds on b_t can be translated into bounds on ϵ_t using (5): i.e., $\epsilon_t \in [\underline{\epsilon}_t, \bar{\epsilon}_t]$, where $\underline{\epsilon}_t \equiv \underline{b}_t p_t / q_t$ and $\bar{\epsilon}_t \equiv \bar{b}_t p_t / q_t$.

Figure 4 provides a simple example of the use of the algorithm. To bound the elasticity ϵ_1 of the demand curve through point e_1 in Figure 4, the lines ℓ_{12} and ℓ_{13} connecting e_1 to the other

[2] Stata code for this and subsequent procedures is made available upon request.

equilibrium points are drawn. The steeper of the two, ℓ_{13}, provides the a lower bound on the steepness of the inverse demand curve through e_1, which translates into an upper bound $\bar{b}_1$ on the steepness demand given in Proposition 2 and an upper bound $\bar{\epsilon}_1$ on the demand elasticity. Because $SE^-(e_1)$ and $NW^+(e_1)$ are empty, pairwise comparisons do not yield a lower bound $\underline{\epsilon}_1$ on the elasticity in this example (apart from 0 derived from non-negativity).

3.2. Logit Demand

Instead of imposing linear demand, the researcher may choose to impose logit demand, micro-founded by McFadden (1973), widely used in structural estimation of differentiated-product demand following Berry, Levinsohn, Pakes (1995). In the context of a homogeneous product market under study, logit demand can be specified as

$$D_t(p) = \frac{n_t \exp(-\alpha_t p)}{1+\exp(-\alpha_t p)} = \frac{n_t}{1+\exp(\alpha_t p)}, \tag{14}$$

where n_t is interpreted as a market-size parameter and α_t as a price-sensitivity parameter. For demand to be nonnegative, $n_t \geq 0$; for the law of demand (Assumption 1) to hold, $\alpha_t \geq 0$.

This specification of demand involves two independent parameters, but further requiring the curve to pass through the equilibrium point e_t pins it down to a single-parameter family. Focus for now on price sensitivity, α_t, as this key parameter. Given α_t and equilibrium point (q_t, p_t), equation (14) can be solved for the market-size parameter:

$$n_t = q_t[1+\exp(\alpha_t p_t)]. \tag{15}$$

Substituting for n_t from equation (15) into (14) yields an expression for logit demand in terms of the single unknown parameter α_t and known equilibrium point $e_t = (q_t, p_t)$:

$$\tilde{D}(p, \alpha_t, e_t) = \frac{q_t[1+\exp(\alpha_t p_t)]}{1+\exp(\alpha_t p)}. \tag{16}$$

The elasticity of logit demand is

$$\epsilon_t = -\tilde{D}_p(p_t, \alpha_t, e_t)\frac{p_t}{q_t} = \frac{\alpha_t p_t}{1+\exp(-\alpha_t p_t)}. \tag{17}$$

As in the linear-demand case, here in the logit-demand case here we will derive bounds on the price-sensitivity parameter (α_t) from the pairwise comparison of reference equilibrium point e_t to other equilibrium points $e_{t'}$. Bounds on α_t can then be translated into bounds on ϵ_t via equation (17).

Suppose $t < t'$. To respect Assumption 2 of growing demand, for all $p \geq 0$ we must have $\tilde{D}(p, \alpha_t, e_t) = D_t(p) \leq D_{t'}(p) = \tilde{D}(p, \alpha_{t'}, e_{t'})$. The preceding inequality must hold in particular for $p = p_{t'}$, implying

$$\frac{q_t[1+\exp(\alpha_t p_t)]}{1+\exp(\alpha_t p_{t'})} = \tilde{D}(p_{t'}, \alpha_t, e_t) \leq \tilde{D}(p_{t'}, \alpha_{t'}, e_{t'}) = q_{t'}, \tag{18}$$

or rearranging,

$$q_t[1+\exp(\alpha_t p_t)] \leq q_{t'}[1+\exp(\alpha_t p_{t'})]. \tag{19}$$

Let $A(e_t, e_{t'})$ be the solution to the previous expression treated as an equality, i.e., the value of $\alpha \geq 0$ solving

$$q_t[1+\exp(\alpha p_t)] = q_{t'}[1+\exp(\alpha p_{t'})]. \tag{20}$$

One can show—as done in the proof of the next proposition—that if $e_{t'} \in NW^+(e_t)$, then $A(e_t, e_{t'})$ exists, is unique, and provides a lower bound on the α_t satisfying (19). One can further show that if $e_{t'} \in SE^+(e_t)$, $A(e_t, e_{t'})$ exists, is unique, and provides an upper bound on the α_t satisfying (19). If $e_{t'} \in NE^+(e_t)$, then (20) has no solution over $\alpha \geq 0$ since (19) is satisfied as a strict inequality for all $\alpha_t \geq 0$. In this case, pairwise comparison of e_t and $e_{t'}$ yields no bounding information. The case in which $e_{t'} \in SW^+(e_t)$ can be ignored because the set is empty under Assumption 2 of growing demand.

The pairwise comparison of e_t and $e_{t'}$ can be repeated for $t' < t$. Growing demand then implies the same inequality as (18) with the direction reversed, leading to the same bound but for the opposite compass direction. For details, see the proof of the following proposition, provided in the appendix.

Proposition 3. *Suppose the sequence of logit demands $\tilde{D}(p, \alpha_t, e_t)$ defined in* (16) *satisfies Assumptions 1 and 2. Then $\alpha_t \in [\underline{\alpha}_t, \bar{\alpha}_t]$ for all $t \in \{1, \ldots, T\}$, where*

$$\underline{\alpha}_t \equiv 0 \vee \sup_{e_{t'} \in SE^-(e_t) \cup NW^+(e_t)} A(e_t, e_{t'}) \tag{21}$$

$$\bar{\alpha}_t \equiv \inf_{e_{t'} \in NW^-(e_t) \cup SE^+(e_t)} A(e_t, e_{t'}). \tag{22}$$

Proposition 3 prescribes the following algorithm for bounding α_t. To obtain the lower bound $\underline{\alpha}_t$, pair e_t with each of the equilibrium points in $SW^-(e_t)$ and $NW^+(e_t)$, compute $A(e_t, e_{t'})$ for each pair by solving the nonlinear equation (20). Equation (20) is well-behaved and can be rapidly solved by a grid search, Newton-Raphson, or other standard methods. The largest of these solutions $A(e_t, e_{t'})$ is taken as $\underline{\alpha}_t$. If both $SW^-(e_t)$ and $NW^+(e_t)$ are empty, we set $\underline{\alpha}_t$ to respect the non-negativity constraint on α_t; i.e., $\underline{\alpha}_t = 0$. To obtain the upper bound $\bar{\alpha}_t$, pair e_t with each of the equilibrium points in $NW^-(e_t)$ and $SE^+(e_t)$, compute $\alpha(e_t, e_{t'})$ for each pair by solving the nonlinear equation (20). The smallest of these solutions $A(e_t, e_{t'})$ becomes $\bar{\alpha}_t$. If both $NW^-(e_t)$ and $SE^+(e_t)$ are empty, the formula (22) yields $\bar{\alpha}_t = \inf \emptyset = \infty$, correctly implying we obtain no upper bound in this case. Using equation (17), the bounds on α_t can be translated into bounds on ϵ_t; namely, $\epsilon_t \in [\underline{\epsilon}_t, \bar{\epsilon}_t]$, where

$$\underline{\epsilon}_t \equiv \frac{\underline{\alpha}_t p_t}{1+\exp(-\underline{\alpha}_t p_t)}, \quad \bar{\epsilon}_t \equiv \frac{\bar{\alpha}_t p_t}{1+\exp(-\bar{\alpha}_t p_t)}. \tag{23}$$

3.3. General Demand

The method for incorporating local information from pairwise equilibrium comparisons is readily generalizable to a broad class of functional forms. Suppose the researcher specifies the form for demand $q = D_t(p)$. This demand curve may start out as a multiple-parameter family; but assume that once it is required to pass through e_t, this pins it down to a single-parameter family indexed by θ:

$$D_t(p) = \tilde{D}(p, \theta, e_t). \tag{24}$$

Assume $\tilde{D}(p, \theta, e_t)$ is continuously differentiable of all orders in all arguments. Assume increases in $\theta \in [0, \infty)$ cause demand to rotate. As an accounting convention, assume that the rotation is such that demand becomes steeper (and inverse demand less steep) when θ increases, consistent with the effect of the price-sensitivity parameter α in the logit case. Thus

$$\begin{aligned} \tilde{D}_\theta(p, \theta, e_t) &< 0 \quad \text{if } p > p_t \\ \tilde{D}_\theta(p, \theta, e_t) &= 0 \quad \text{if } p = p_t \\ \tilde{D}_\theta(p, \theta, e_t) &> 0 \quad \text{if } p < p_t, \end{aligned} \tag{25}$$

where $\tilde{D}_\theta$ denotes the partial derivative of demand with respect to its second argument. We also impose the following Inada conditions:

$$\lim_{\theta \to 0} \tilde{D}(p, \theta, e_t) = q_t \tag{26}$$

$$\lim_{\theta \to \infty} \tilde{D}(p, \theta, e_t) = \begin{cases} 0 & p > p_t \\ \infty & p < p_t. \end{cases} \tag{27}$$

Equation (26) implies that $\tilde{D}(p, \theta, e_t)$ becomes infinity inelastic for arbitrarily small θ. Given $\tilde{D}(p, \theta, e_t)$ must pass through e_t, as it becomes infinitely inelastic, its inverse approaches a vertical line at quantity q_t for all $p \geq 0$. Equation (27) implies that $\tilde{D}(p, \theta, e_t)$ becomes infinitely elastic as θ becomes arbitrarily large. Together, (26) and (27) in effect say that the domain of θ_t is rich enough to allow changes in θ to trace out all possible demand slopes from infinitely elastic to infinitely inelastic. We verify in the appendix that conditions (25)–(27) hold when $\tilde{D}(p, \theta, e_t)$ is taken to be the logit demand defined in (16) with α playing the role of θ. Hence the general specification nests logit demand as a special case.

As in the linear and logit cases, we write the price-sensitivity parameter with a subscript, θ_t, to emphasize that it may vary over time. Bounds on θ_t will be derived from the pairwise comparison of e_t to other equilibrium points $e_{t'}$.

Start by supposing $t < t'$. To respect Assumption 2 of growing demand, for all $p \geq 0$ we must have $\tilde{D}(p, \theta_t, e_t) = D_t(p) \leq D_{t'}(p) = \tilde{D}(p, \theta_{t'}, e_{t'})$. The inequality must hold in particular for $p = p_{t'}$, implying $\tilde{D}(p_{t'}, \theta_t, e_t) \leq \tilde{D}(p_{t'}, \theta_{t'}, e_{t'}) = q_{t'}$. Let $\Theta(e_t, e_{t'})$ be the solution to the equation formed by treating the preceding condition as an equality, i.e., the value of $\theta \geq 0$ solving

$$\tilde{D}(p_{t'}, \theta, e_t) = q_{t'}. \tag{28}$$

Sparing the details filled in by the proof of the next proposition, $\Theta(e_t, e_{t'})$ is the key component of the bounds on θ_t for general demands, serving the same role as $A(e_t, e_{t'})$ for logit demands.

Proposition 4. *Suppose the sequence of general demands $\tilde{D}(p, \theta_t, e_t)$ defined in* (24) *satisfies As-*

sumptions 1 and 2. Then $\theta_t \in [\underline{\theta}_t, \bar{\theta}_t]$ *for all* $t \in \{1, \dots, T\}$*, where*

$$\underline{\theta}_t \equiv 0 \vee \sup_{e_{t'} \in SE^-(e_t) \cup NW^+(e_t)} \Theta(e_t, e_{t'}) \tag{29}$$

$$\bar{\theta}_t \equiv \inf_{e_{t'} \in NW^-(e_t) \cup SE^+(e_t)} \Theta(e_t, e_{t'}). \tag{30}$$

The proof is provided in the appendix.

Proposition 4 prescribes an algorithm for bounding θ_t in the case of general demands. This algorithm is analogous to that prescribed by Proposition 3 for bounding α_t in the case of logit demand. The only difference is that instead of solving equation (20) for $A(e_t, e_{t'})$, the more general equation (28) is solved for $\Theta(e_t, e_{t'})$. The algorithms are otherwise the same, including that they involve the same time-augmented compass sets for the upper and lower bounds.

Equation (28) is a nonlinear equation in the single unknown variable θ. This equation is well-behaved for $e_{t'}$ in the relevant subsets contributing to the bounds in Proposition 4: the proof of the proposition shows that for $e_{t'} \in NW(e_t) \cup SE(e_t)$, the left-hand side of (28) is monotonic in θ. Thus standard methods, including a straightforward grid search, can be used to solve (28) and derive $\Theta(e_t, e_{t'})$.

Translating the bounds on θ_t into bounds on the elasticity, defined in the general case as

$$\epsilon_t = -\tilde{D}_p(p_t, \theta_t, e_t)\frac{p_t}{q_t}, \tag{31}$$

requires additional work in the general case because it needs to be shown that condition (25) implies that ϵ_t is nondecreasing in θ_t. This is done in the proof of the following proposition, proved in the appendix.

Proposition 5. *Suppose conditions* (25)–(27) *hold. Then, defining*

$$\underline{\epsilon}_t \equiv -\tilde{D}_p(p_t, \underline{\theta}_t, e_t)\frac{p_t}{q_t}, \quad \bar{\epsilon}_t \equiv -\tilde{D}_p(p_t, \bar{\theta}_t, e_t)\frac{p_t}{q_t}, \tag{32}$$

we have $\epsilon_t \in [\underline{\epsilon}_t, \bar{\epsilon}_t]$.

4. Method Incorporating Limiting Information

If the researcher is willing to assume that the posited functional form applies throughout the whole domain of the demand curve, the elasticity bounds can be tightened by exploiting information

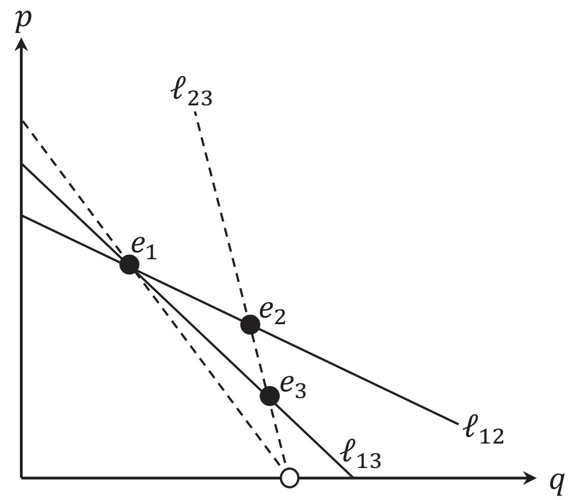

Figure 5: Illustrating Method Incorporating Limiting Information

about the relative position of demand curves for limiting values of prices, $p \to 0$ and $p \to \infty$, rather than observed prices. The analysis proceeds as in the previous subsection by analyzing a sequence of functional forms for demand, from linear, to logit, to general.

Before delving into the technical details, we provide some intuition on how limiting information can help tighten the elasticity bounds using Figure 5. The figure revisits the simple example introduced in Figure 4, carrying over the features from that previous figure as the solid lines. Recall the goal of that simple example was to bound the elasticity ϵ_1 of demand through equilibrium point e_1. The method of incorporating local information from pairwise comparisons allowed us to conclude that the inverse demand through e_1 must be at least as steep as line ℓ_{13}, providing a lower bound of the steepness of inverse demand, translating into an upper bound $\bar{b}_1$ on the steepness of demand and an upper bound $\bar{\epsilon}_1$.

The new features of Figure 5, drawn as dotted lines and open circles, show how to tighten the bound by incorporating information on the position of the limits of demand curves through other equilibrium points. The pairwise comparison of points e_2 and e_3 leads us to conclude that the inverse demand through e_3 must be as steep as line ℓ_{23} for e_2 not to lie on a higher demand curve. But notice that ℓ_{23} intersects ℓ_{13}. Unless the inverse demand curve through e_1 is steeper than ℓ_{23}, parts of the curve will lie above the curve through e_3, violating Assumption 2 that demand is growing. For the whole inverse demand through e_1 to be lower than the whole inverse demand through e_3, the demand curves cannot cross even for prices approaching 0, implying in the case of linear inverse demands that their horizontal intercepts cannot cross. To ensure their horizontal intercepts do not cross, the inverse demand through e_1 must be at least as steep as the dotted line that connects e_1 with the horizontal intercept of ℓ_{23}, drawn as the open circle. This new line through e_1 is even steeper than ℓ_{13}, tightening the lower bound on the steepness of inverse demand, and thus tightening the upper bound on the demand elasticity.

This is just one simple example involving three equilibrium points and linear demands. The general method below will apply to an arbitrary number of equilibrium points and general demand specifications. The general method will preserve the flavor of the example in that the first step will be to compute the bounds incorporating local information, and then the second step will be to compare the limiting positions of the bounding demand curves from the first step.

To help clarify the iterative nature of the method, some new notation is in order. Re-express the bounds incorporating local information presented in the previous section by adding a single star to them to indicate that these bounds constitute the first step of an iterative procedure, thus writing $\underline{b}_t^* \equiv \underline{b}_t$, $\bar{b}_t^* \equiv \bar{b}_t$, $\underline{\alpha}_t^* \equiv \underline{\alpha}_t$, $\bar{\alpha}^* \equiv \bar{\alpha}_t$, $\underline{\theta}_t^* \equiv \underline{\theta}_t$, $\bar{\theta}^* \equiv \bar{\theta}_t$, $\underline{\epsilon}_t^* \equiv \underline{\epsilon}_t$, and $\bar{\epsilon}_t^* \equiv \bar{\epsilon}_t$.

4.1. Linear Demand

This section provides a formal derivation of the new bounds incorporating limiting information in the case of linear demands. We will analyze new bounds emerging from the limits $p \to 0$ and $p \to \infty$ in turn. Start by considering the limit $p \to 0$ and comparing the equilibrium points e_t and $e_{t'}$ for $t' > t$. Assumption 2 of growing demand implies $\tilde{D}(p, b_t, e_t) \leq \tilde{D}(p, b_{t'}, e_{t'})$ for all $p \geq 0$, implying in particular that $\tilde{D}(0, b_t, e_t) \leq \tilde{D}(0, b_{t'}, e_{t'})$. Substituting from for $\tilde{D}$ from (3) into this inequality, and further substituting $p = 0$, yields $q_t + b_t p_t \leq q_{t'} + b_{t''} p_{t'}$. Notice this is a condition on the relative position of the inverse demand curves' horizontal intercepts. Rearranging,

$$b_t \leq \frac{1}{p_t}(q_{t'} - q_t + b_{t'} p_{t'}) \leq \frac{1}{p_t}(q_{t'} - q_t + \bar{b}_{t'}^* p_{t'}), \tag{33}$$

where the second inequality follows from Proposition 2. Condition (33) provides an additional upper bound on b_t incorporating information on a different equilibrium point's upper bound $\bar{b}_{t'}^*$ from the prior iteration.

Supposing instead that $t' < t$, we can apply the analysis from the previous paragraph, just reversing the inequalities, to obtain

$$b_t \geq \frac{1}{p_t}(q_{t'} - q_t + b_{t'} p_{t'}) \geq \frac{1}{p_t}(q_{t'} - q_t + \underline{b}^*_{t'} p_{t'}). \tag{34}$$

This additional lower bound incorporates information on a different equilibrium point's lower bound $\underline{b}^*_{t'}$ from a prior iteration.

Next, consider the limit $p \to \infty$. Start by supposing $t' > t$. The fact that $\tilde{D}(p, b_t, e_t) \leq \tilde{D}(p, b_{t'}, e_{t'})$ for all $p \geq 0$ implies $\lim_{p\to\infty} \tilde{D}(p, b_t, e_t) \leq \lim_{p\to\infty} \tilde{D}(p, b_{t'}, e_{t'})$. These limits are just the vertical intercepts of the respective inverse demands. Hence, the preceding inequality implies $p_t + q_t/b_t \leq p_{t'} + q_{t'}/b_{t'}$, or after rearranging,

$$b_t \left(p_{t'} - p_t + \frac{q_{t'}}{b_{t'}} \right) \geq q_t. \tag{35}$$

One can show that the factor in parentheses is always positive when $t' > t$ and demand is growing.[3] Cross multiplying by the positive factor in parentheses yields

$$b_t \geq \frac{q_t}{p_{t'} - p_t + q_{t'}/b_{t'}} \geq \frac{q_t}{p_{t'} - p_t + q_{t'}/\underline{b}^*_{t'}}. \tag{36}$$

Again, we have an additional lower bound incorporating information on a different equilibrium point's lower bound $\underline{b}^*_{t'}$ from the prior iteration.

Supposing instead that $t' < t$, we can apply the preceding analysis, just reversing inequalities, to obtain

$$b_t \left(p_{t'} - p_t + \frac{q_{t'}}{b_{t'}} \right) \leq q_t. \tag{37}$$

An important difference here is that the factor in parentheses cannot be unambiguously signed when $t' < t$. When the factor in parentheses is non-positive, (37) holds for all b_t. In that case, the condition provides no useful bounding information. When the factor in parentheses is positive,

[3] It is immediate that the factor in parentheses in equation (35) is positive if $p_{t'} > p_t$ since $q_{t'} > 0$ and $b_{t'} \geq 0$. Suppose instead that $p_{t'} < p_t$. Then $e_{t'} \in SW^+(e_t) \cup SE^+(e_t)$. Proposition 1 rules out $e_{t'} \in SW^+(e_t)$ under Assumption 2, leaving $e_{t'} \in SE^+(e_t)$. Then

$$b_{t'} \leq \bar{b}_{t'} \leq B(e_t, e_{t'}) = \frac{q_{t'} - q_t}{p_t - p_{t'}} < \frac{q_{t'}}{p_t - p_{t'}}.$$

The first and second steps follow from Proposition 2. The next step follows from the definition of $B(e_t, e_{t'})$ from (8) and from $q_{t'} > q_t$ and $p_t > p_{t'}$ for $e_{t'} \in SE^+(e_t)$. The last step follows from $q_t > 0$ for nontrivial equilibrium set E. Cross multiplying by $p_t - p_{t'}$, which is positive, and rearranging proves that the term in parentheses is positive.

cross multiplying by it preserves the direction of the inequality, yielding

$$b_t \leq \frac{q_t}{p_{t'} - p_t + q_{t'}/b_{t'}} \leq \frac{q_t}{p_{t'} - p_t + q_{t'}/\bar{b}^*_{t'}}. \tag{38}$$

To emphasize, condition (38) only provides a potentially useful upper bound if the denominator of the last fraction is positive, i.e.,

$$p_{t'} + \frac{p_{t'}}{\bar{b}^*_{t'}} > p_t. \tag{39}$$

If (39) does not hold, we must ignore (38) lest we conclude that b_t is bounded above by the negative number on the right-hand side of (38), violating the law of demand (Assumption 1), which implies $b_t \geq 0$.

The bounding exercise that generated conditions (33)–(38) can be repeated comparing e_t to all the other equilibrium points $e_{t'}$. If the tightest of the resulting bounds incorporating limiting information is tighter than the corresponding bound incorporating local information, we take the former as the new bound, indicated with two stars in the superscript. Otherwise, the new bound is just set to the old bound. We have the following proposition.

Proposition 6. *Suppose the sequence of linear demands $\tilde{D}(p, b_t, e_t)$ defined in* (3) *satisfies Assumptions 1 and 2. Then $b_t \in [\underline{b}^{**}_t, \bar{b}^{**}_t]$ for all $t \in \{1, \ldots, T\}$, where*

$$\underline{b}^{**}_t \equiv \underline{b}^*_t \vee \sup_{t'<t} \left\{ \frac{1}{p_t}(q_{t'} - q_t + \underline{b}^*_{t'} p_{t'}) \right\} \vee \sup_{t'>t} \left\{ \frac{q_t}{p_{t'} - p_t + q_{t'}/\underline{b}^*_{t'}} \right\} \tag{40}$$

$$\bar{b}^{**}_t \equiv \bar{b}^*_t \wedge \inf_{t'<t} \left\{ \frac{q_t}{p_{t'} - p_t + q_{t'}/\bar{b}^*_{t'}} \;\middle|\; p_{t'} + \frac{q_{t'}}{\bar{b}^*_{t'}} > p_t \right\} \wedge \inf_{t'>t} \left\{ \frac{1}{p_t}(q_{t'} - q_t + \bar{b}^*_{t'} p_{t'}) \right\}. \tag{41}$$

Some technical remarks about the proposition are in order. Note that, as do equations (12)–(13), (40)–(41) involve suprema and infima rather than maximums and minimums to accommodate possibly empty sets, as will be the case when computing bounds for the first and last equilibrium points, i.e., e_1 or e_T. Note also that it is conceivable that continuing the iterations could result in yet tighter bounds, for example computing $\bar{b}^{***}_t$ by replacing $\bar{b}^*$ with $\bar{b}^{**}_t$ on the right-hand side of (41), and so on. We conjecture that convergence is achieved by the stage shown in (40) and (41), so there is no gain from further iterating.[4]

[4] Our Stata code that implements the bounds in Proposition 6 allows continued iteration. In the empirical application to the *Du Pont* decision and all Monte Carlos we have tried, however, iterations stop at $\underline{b}^{**}_t$ and $\bar{b}^{**}_t$. We have managed a general proof that iteration stops at $\underline{b}^{**}_t$ and $\bar{b}^{**}_t$ for many cases, but a handful of cases remain open questions.

The bounds $\underline{b}_t^{**}$ and $\bar{b}_t^{**}$ incorporating limiting information are at least as tight as $\underline{b}_t^*$ and $\bar{b}_t^*$. This is true by definition because the latter bounds appear on the right-hand side of (40) and (41). If the limiting information does not tighten the bounds, by default they return to the first stage that incorporates just local information. The formulas in (40) and (41) are sufficiently complicated that it is hard to tell whether $\underline{b}_t^{**}$ and $\bar{b}_t^{**}$ can ever be strictly tighter than $\underline{b}_t^*$ and $\bar{b}_t^*$. Figure 5, which we now have the language to say derives $\bar{b}_1^{**}$ in the simple example with those three equilibrium points, suggests that $\underline{b}_t^{**}$ and $\bar{b}_t^{**}$ can be strictly tighter than $\underline{b}_t^*$ and $\bar{b}_t^*$ in some cases. We will indeed see that substantial tightening is possible when we turn to the empirical application to the *Du Pont* decision.

Substituting the new bounds into the elasticity formula (5), we have $\epsilon_t \in [\underline{\epsilon}_t^{**}, \bar{\epsilon}_t^{**}]$, where $\underline{\epsilon}_t^{**} \equiv \underline{b}_t^{**} p_t/q_t$ and $\bar{\epsilon}_t^{**} \equiv \bar{b}_t^{**} p_t/q_t$.

4.2. Logit Demand

The method for incorporating limiting information assuming logit demand is similar to that assuming linear demand. Start by considering the limit $p \to 0$ and comparing equilibrium points e_t and $e_{t'}$ for $t' > t$. Assumption 2 implies $\tilde{D}(p, \alpha_t, e_t) \le \tilde{D}(p, \alpha_{t'}, e_{t'})$ for all $p \ge 0$. In particular, this inequality must hold for $p = 0$: i.e., $\tilde{D}(0, \alpha_t, e_t) \le \tilde{D}(0, \alpha_{t'}, e_{t'})$. Substituting for $\tilde{D}$ from (3) in the previous inequality yields $q_t[1+\exp(\alpha_t p_t)] \le q_{t'}[1+\exp(\alpha_{t'} p_{t'})]$, or upon rearranging,

$$\alpha_t \le \frac{1}{p_t} \ln\left(\frac{q_{t'}}{q_t}[1+\exp(\alpha_{t'} p_{t'})]-1\right) \le \frac{1}{p_t} \ln\left(\frac{q_{t'}}{q_t}[1+\exp(\bar{\alpha}_{t'}^* p_{t'})]-1\right). \tag{42}$$

For $t' < t$, we obtain the reverse inequality:

$$\alpha_t \ge \frac{1}{p_t} \ln\left(\frac{q_{t'}}{q_t}[1+\exp(\alpha_{t'} p_{t'})]-1\right) \ge \frac{1}{p_t} \ln\left(\frac{q_{t'}}{q_t}[1+\exp(\underline{\alpha}_{t'}^* p_{t'})]-1\right). \tag{43}$$

To explore the other limit, $p \to \infty$, suppose $t' > t$. Expressing its implication in terms of a ratio rather than a difference, Assumption 2 implies

$$\lim_{p\to\infty} \left[\frac{\tilde{D}(p, \alpha_t, e_t)}{\tilde{D}(p, \alpha_{t'}, e_{t'})}\right] \le 1. \tag{44}$$

Evaluation of this limit is somewhat involved, so the details are deferred to the appendix proof. The result, however, is simple. We show that a necessary condition for (44) is $\alpha_t \geq \alpha_{t'}$. In words, under logit demand, for demand to grow over time requires the price-sensitivity parameter to shrink over time. Using the result from Proposition 3 that $\alpha_{t'} \geq \underline{\alpha}^*_{t'}$, this leads to the lower bound $\alpha_t \geq \underline{\alpha}^*_{t'}$ for all $t' > t$. For $t' < t$, the reverse inequality is required: $\alpha_t \leq \alpha_{t'} \leq \bar{\alpha}^*_{t'}$.

The following proposition summarizes the preceding analysis. The appendix proof fills in the omitted detail.

Proposition 7. *Suppose the sequence of logit demands $\tilde{D}(p, b_t, e_t)$ defined in* (16) *satisfies Assumptions 1 and 2. Then $\alpha_t \in [\underline{\alpha}^{**}_t, \bar{\alpha}^{**}_t]$ for all $t \in \{1, \ldots, T\}$, where*

$$\underline{\alpha}^{**}_t \equiv \underline{\alpha}^*_t \vee \sup_{t'<t} \left\{ \frac{1}{p_t} \ln \left(\frac{q_{t'}}{q_t} [1 + \exp(\underline{\alpha}^*_{t'} p_{t'})] - 1 \right) \right\} \vee \sup_{t'>t} \underline{\alpha}^*_{t'}. \tag{45}$$

$$\bar{\alpha}^{**}_t \equiv \bar{\alpha}^*_t \wedge \inf_{t'<t} \bar{\alpha}^*_{t'} \wedge \inf_{t'>t} \left\{ \frac{1}{p_t} \ln \left(\frac{q_{t'}}{q_t} [1 + \exp(\bar{\alpha}^*_{t'} p_{t'})] - 1 \right) \right\}. \tag{46}$$

Equations (45)–(46) could be streamlined by consolidating the bounds obtained the first stage, $\underline{\alpha}^*_t$ and $\bar{\alpha}^*_t$, into one of the other sets, but we have kept them separate to emphasize the point that $\underline{\alpha}^{**}_t$ and $\bar{\alpha}^{**}_t$ are weakly tighter than $\underline{\alpha}^*_t$ and $\bar{\alpha}^*_t$.

As done in equation (23), the bounds on α_t can be translated into bounds on ϵ_t; namely, $\epsilon_t \in [\underline{\epsilon}^{**}_t, \bar{\epsilon}^{**}_t]$, where

$$\underline{\epsilon}^{**}_t \equiv \frac{\underline{\alpha}^{**}_t p_t}{1 + \exp(-\underline{\alpha}^{**}_t p_t)}, \quad \bar{\epsilon}^{**}_t \equiv \frac{\bar{\alpha}^{**}_t p_t}{1 + \exp(-\bar{\alpha}^{**}_t p_t)}. \tag{47}$$

The complex formulas in Proposition 7 for logit demand look very different from those in Proposition 6 for linear demand. We will see in the empirical application to the *Du Pont* decision, however, that they generate nearly identical bounds on the elasticities. Reassuringly, the bounds incorporating limiting information are more robust to functional-form assumptions than the formulas might suggest.

4.3. General Demand

The logic used to derive bounds incorporating limiting information for logit demand carries over to general demand. We have the following proposition, proved in the appendix.

Proposition 8. *Suppose the sequence of general demands $\tilde{D}(p, \theta_t, e_t)$ defined in* (24) *satisfies As-*

sumptions 1 and 2 as well as conditions (25)–(27)*. Then* $\theta_t \in [\underline{\theta}_t^{**}, \bar{\theta}_t^{**}]$ *for all* $t \in \{1, \ldots, T\}$*, where*

$$\underline{\theta}_t^{**} \equiv \underline{\theta}_t^{*} \vee \sup_{t'<t}\left\{\theta \;\middle|\; \lim_{p\to 0}\left[\frac{\tilde{D}(p,\theta,e_t)}{\tilde{D}(p,\underline{\theta}_{t'}^{*},e_{t'})}\right] = 1\right\} \vee \sup_{t'>t}\left\{\theta \;\middle|\; \lim_{p\to\infty}\left[\frac{\tilde{D}(p,\theta,e_t)}{\tilde{D}(p,\underline{\theta}_{t'}^{*},e_{t'})}\right] = 1\right\} \tag{48}$$

$$\bar{\theta}_t^{**} \equiv \bar{\theta}_t^{*} \wedge \inf_{t'<t}\left\{\theta \;\middle|\; \lim_{p\to \infty}\left[\frac{\tilde{D}(p,\theta,e_t)}{\tilde{D}(p,\bar{\theta}_{t'}^{*},e_{t'})}\right] = 1\right\} \wedge \inf_{t'>t}\left\{\theta \;\middle|\; \lim_{p\to 0}\left[\frac{\tilde{D}(p,\theta,e_t)}{\tilde{D}(p,\bar{\theta}_{t'}^{*},e_{t'})}\right] = 1\right\}. \tag{49}$$

The algorithm for computing bounds incorporating limiting information prescribed in the proposition is an iterative procedure. For example, the lower bounds $\underline{\theta}_{t'}^{*}$ incorporating local information are computed in a first iteration; the $\underline{\theta}_{t'}^{*}$ for all equilibrium points $e_{t'} \in E$ are then used to compute the new lower bound $\underline{\theta}_t^{**}$ for the single equilibrium point e_t in the second iteration. Lower bounds from the first iteration are used in the computation of lower bounds in the second iteration; upper bounds from the first iteration are used to compute upper bounds in the second iteration.

More specifically, to compute the lower bound $\underline{\theta}_t^{**}$, one solves the equation for θ,

$$\lim_{p\to 0}\left[\frac{\tilde{D}(p,\theta,e_t)}{\tilde{D}(p,\underline{\theta}_{t'}^{*},e_{t'})}\right] = 1 \tag{50}$$

for each $t' < t$ and takes the largest solution. One then solves a similar equation, just involving a different limit, for θ,

$$\lim_{p\to \infty}\left[\frac{\tilde{D}(p,\theta,e_t)}{\tilde{D}(p,\underline{\theta}_{t'}^{*},e_{t'})}\right] = 1 \tag{51}$$

for each $t' > t$ and takes the largest solution. The new bound $\underline{\theta}_t^{**}$ is set to whichever is largest of (a) the bound from the method incorporating local information, $\underline{\theta}_t^{*}$, (b) the largest solution over $t' < t$ to (50), and (c) the largest solution over $t' > t$ to (51). The upper bound $\bar{\theta}_t^{**}$ is computed analogously.

To translate the bounds on θ_t into bounds on the elasticity ϵ_t, the arguments behind Proposition 5, which we will not repeat here, continue to apply. Defining

$$\underline{\epsilon}_t^{**} \equiv -\tilde{D}_p(p_t, \underline{\theta}_t^{**}, e_t)\frac{p_t}{q_t}, \quad \bar{\epsilon}_t^{**} \equiv -\tilde{D}_p(p_t, \bar{\theta}_t^{**}, e_t)\frac{p_t}{q_t}, \tag{52}$$

we have $\epsilon_t \in [\underline{\epsilon}_t^{**}, \bar{\epsilon}_t^{**}]$.

5. Institutional Background

That completes the discussion of general methods. We next turn to the empirical application, beginning with some institutional background on the plastics market and the *Du Pont* decision.

The use of plastic in consumer goods—in the products themselves and their packaging—is so widespread today that it is hard to envision the world in the 1950s and 1960s when plastic was initially developed and diffused commercially. Polyethylene was among the first commercially developed plastics. We focus on two types, low- and high-density. Low-density polyethylene was famously used to make Tupperware food-storage containers (Clarke 1999, p. 2) and high-density polyethylene to make Hula Hoops (Fenichell 1996, p. 264). Polyethylene remains the highest-volume commercial plastic in terms of global sales.

In the late 1930s, the U.S. Department of Justice sought to prosecute Du Pont, a U.S. manufacturer of polyethylene, and its U.K. co-conspirator Imperial Chemical Industries (ICI), for their Patents and Processes agreement. Signed in 1929, the agreement granted each company exclusive licenses for the patents and secret processes controlled by the other and divided the global market into exclusive territories between them. Views differ on whether the main motive for the agreement was the sharing of complementary technologies or the monopolization achieved by market division (Hounshell and Smith 1999, p. 190). The Department of Justice viewed the Patents and Processes agreement as an illegal market division, in violation of Section 1 of the Sherman Act. The military significance of the chemical industry led Franklin Roosevelt's administration to lobby to suspend legal action during the Second World War (Reader 1975, p. 432). After the war, the case proceeded to trial. The prosecution won a liability verdict in 1951, and Judge Sylvester Ryan ordered a remedy in 1952.

Judge Ryan's order cancelled the exclusive-territory arrangements between Du Pont and ICI and required them to license patents and secret processes involved in the manufacture of several of their products to all applicants at reasonable royalty rates. The incumbents' most significant patents covered three products: polyethylene, nylon, and neoprene. Judge Ryan did not order the compulsory licensing of neoprene. Of the remaining products, commentators have contended that only the polyethylene patents garnered substantial commercial interest (Whitney 1958, p. 217). Thus, we focus on the structural remedy's effect on the polyethylene market.

Whether compulsory licensing solves the monopolization problem is the subject of debate in the antitrust literature. In their seminal legal casebook, Areeda, Kaplow, and Edlin (2013) note that the terms of such licensing typically remain subject to commercial negotiations of the parties. This leaves open the possibility that the licensor preserves the monopoly outcome by specifying exorbitant rates.[5] At one point, ICI was asking for a fixed fee of $500,000 and royalties amounting to eight percent of sales revenue (Fortune, 1954). Since this royalty applied to revenue not profit, it would have added to markups, whether as much as the monopoly level may be hard for industry outsiders to know. Expanding the incumbents' scope to manipulate the licensing outcome in their favor, patents often do not relate all the information necessary for entrants to replicate a production process. While Judge Ryan ordered incumbents to provide accompanying manuals and training, the question remains whether these materials were sufficient to allow the entrants to operate on an even footing with incumbents.

The remedy ostensibly had a dramatic effect on the polyethylene market (Backman 1964, p. 71). Seven manufacturers entered in 1953–56 and four more in 1956–59. Prices steadily declined and output rose. On the face of these facts, one might be tempted to conclude that Judge Ryan's remedy achieved its purpose of turning a monopoly into a more competitive market. However, the same price declines and output increases may have arisen in a monopoly market experiencing substantial cost declines, plausibly realistic for plastics in the 1950s and 60s. The fact of entry seems to disprove monopoly unless it is thought that the entrants are merely producing their share of the monopoly quantity, returning most of the rents to the licensor. Thus our study will try to produce evidence for effective remedy that cuts through these criticisms.

To the extent the remedy was successful, a contributing factor may have been that the market was monopolized not by a single firm but by two independent firms, which used the Patents and Processes agreement to facilitate collusion. With the agreement cancelled by Judge Ryan's remedy, the independent operators may have been thrown into competition together to license their technologies. This may have disciplined exorbitant licensing fees and thwarted an attempt to preserve the monopoly outcome. That said, the question remains whether incumbents with a history of collusion needed the explicit legal agreement to maintain a cooperative relationship.

[5] Judge Ryan himself foresaw the potential difficulty in defining a reasonable royalty rate: "While it is true that as to [royalty rates] question might well be raised as to whether they were arrived at after arm's length commercial negotiations, they do nevertheless furnish guide posts for future determination of the amount at which such royalties should be set. But, in any event, as to these products and all others, there is also available for judicial finding the sum a prudent licensee would pay under all existing circumstances." (US vs. ICI, 1952, p. 227–228.)

Table 1: Descriptive Statistics

		Low-density polyethylene		High-density polyethylene	
Variable	Units	Mean	Std. dev.	Mean	Std. dev.
Revenue	Billion $ per year	0.37	0.10	0.13	0.06
Price	$ per pound	0.19	0.07	0.22	0.09
Sales quantity	Billion pounds per year	2.37	1.27	0.78	0.61
Production quantity	Billion pounds per year	2.57	1.41	0.92	0.69

Source: U.S. Tariff Commission (various years).

6. Data

Our dataset consists of annual price and quantity data, aggregated across all firms the U.S. market, for two plastic products, low- and high-density polyethylene, in the period following the imposition of the remedy in the *Du Pont* (1952) case. The foundation of our dataset is provided by Table 1 of Lieberman (1984), which presents the data he obtained from the annual reports, *Synthetic Organic Chemicals: United States Production and Sales*, issued by the U.S. Tariff Commission. The earliest date that price and quantity are reported for polyethylene is 1958, so Lieberman's data begin then. Lieberman ends the series in 1972 because in subsequent years the OPEC crisis created an oil shock which, together with the subsequent recession, disrupted supply and demand in the plastics market.

Rather than taking Lieberman's data directly, we returned to the original U.S. Tariff Commission source for two reasons. First, we noticed some typographical errors. Second, he recorded production for his quantity variable. Since we are interested in estimates of demand, we went back and collected sales for our quantity variable.[6] Our final dataset consists of prices (measured in nominal dollars per pound) and sales quantities (measured in billion pounds) for low- and high-density polyethylene from 1958–72. The price series is an average wholesale price, computed by dividing total annual industry revenue by industry quantity sold.

Table 1 provides descriptive statistics. Prices are roughly the same across the products. Quan-

[6] Production equals the sum of sales, change in inventories, contract production using the customer's raw materials, and production consumed directly by the manufacturer. While we believe sales best captures quantity demanded, we repeat the empirical analysis using production as our quantity measure. The results, presented in Figure B2 in Appendix B, are qualitatively close to those using sales for quantity.

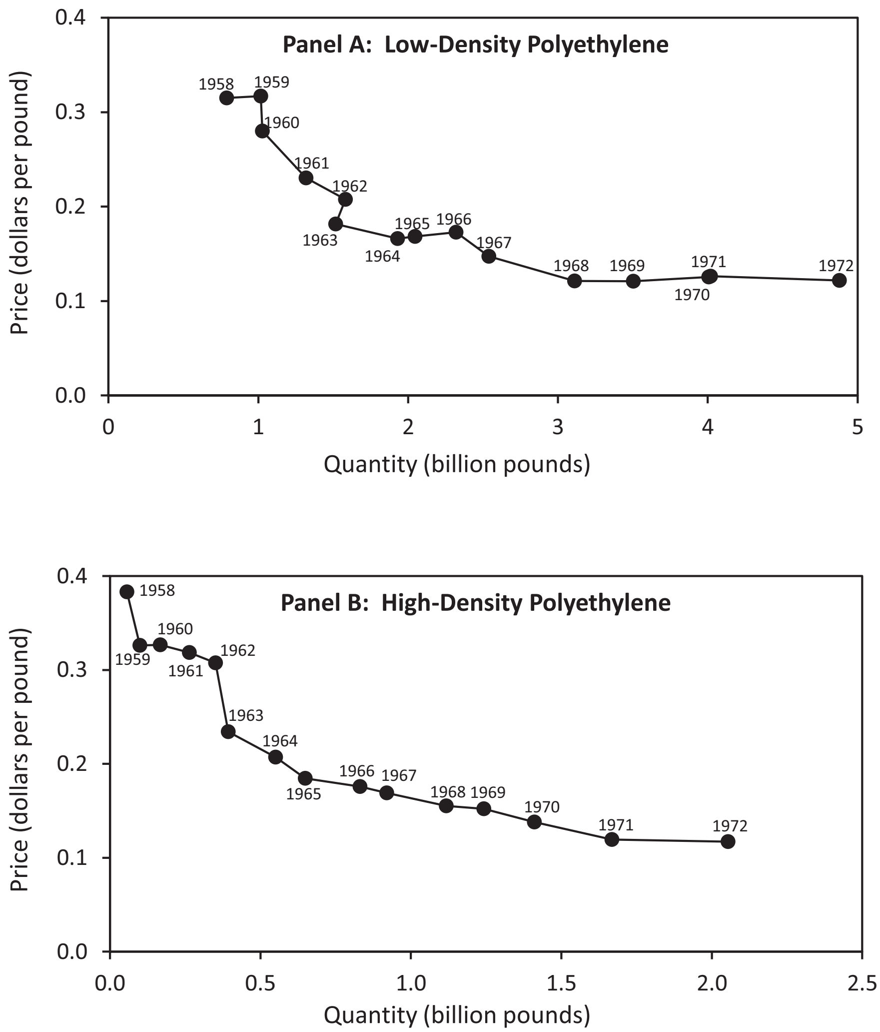

Figure 6: Evolution of Equilibrium in the Plastics Market. *Source:* U.S. Tariff Commission (various years).

tity sold in the market for low-density polyethylene was about twice that in the market for high-density polyethylene.

Figure 6 displays the evolution of equilibrium over time for the two products in the two panels. The curves are of course not demand; each dot is an equilibrium point resulting from the interaction of demand and a supply relation in the given year. The figure shows how these equilibrium points shift over time. The predominant pattern is for equilibrium to shift to the southeast each year. With one exception, the equilibrium never shifts southwest over time, thus exhibiting rectangular expansion.[7]

The lone exception is 1963 for low-density polyethylene. Our analysis excludes that product-

[7]The equilibrium set E in this application satisfies the other maintained assumptions from the model section, that E is distinct and nontrivial.

year. As a robustness check, we redo the analysis preserving rectangular expansion by dropping 1962 rather than 1963. The results, reported in Figure B1 in Appendix B, are quantitatively quite similar. We think the most likely cause of this violation of rectangular expansion is that the market may have been growing but not fast enough to offset a small perturbation in price and/or quantity or a small observation error that made it look like demand shrunk that year. Section 8 discusses several approaches to robust estimation that allows for such perturbations.

7. Empirical Results

This section presents empirical results using our methods for bounding elasticity of demand ϵ_t. The nature of our dataset happens to allow only meaningful upper, not lower, bounds to be obtained. We are not able to obtain meaningful lower bounds because $SE^-(e_t)$ and $NW^+(e_t)$ happen to be empty for all $e_t \in E$ in our dataset. A lower bound of 0 is obtained by default—as a consequence of the law of demand (Assumption 1). Thus ϵ_t will always be bounded in an interval extending from 0 to the upper bound we derive. We perform all our analyses assuming both linear and logit demands to gauge robustness to functional form and for two products, low- and high-density polyethylene.

The results are presented graphically in Figure 7, low-density polyethylene in Panels A–B and high-density polyethylene in Panels C–D. For each product, the top panels present results using the method incorporating local information from pairwise equilibrium comparisons, generating the upper bound $\bar{\epsilon}_t^*$. The bottom panels present results using the method incorporating limiting information, generating the upper bound $\bar{\epsilon}_t^{**}$. For reference, the precise numerical elasticities are provided in Table 2.

The figure displays bootstrapped confidence intervals around upper bounds $\bar{\epsilon}_t^*$ or $\bar{\epsilon}_t^{**}$ as whiskers at the top of the bars. Because our upper bounds involve an extreme order statistic (in particular the minimum of results from pairwise comparisons), standard bootstrapping methods are invalid. We instead use the bootstrapping method for extreme order statistics proposed by Zelterman (1993). We defer a detailed discussion of the Zelterman (1993) bootstrap to Section 8.1. We chose to display a two-tailed 90% confidence interval because this allows easy visualization of the one-tailed test of whether the elasticity bound is less than 1 at the standard (5%) significance level. The starred significance levels in Table 2 are also based on the Zelterman (1993) bootstrap.

Focus first on the results in Panel A for low-density polyethylene using the method incorpo-

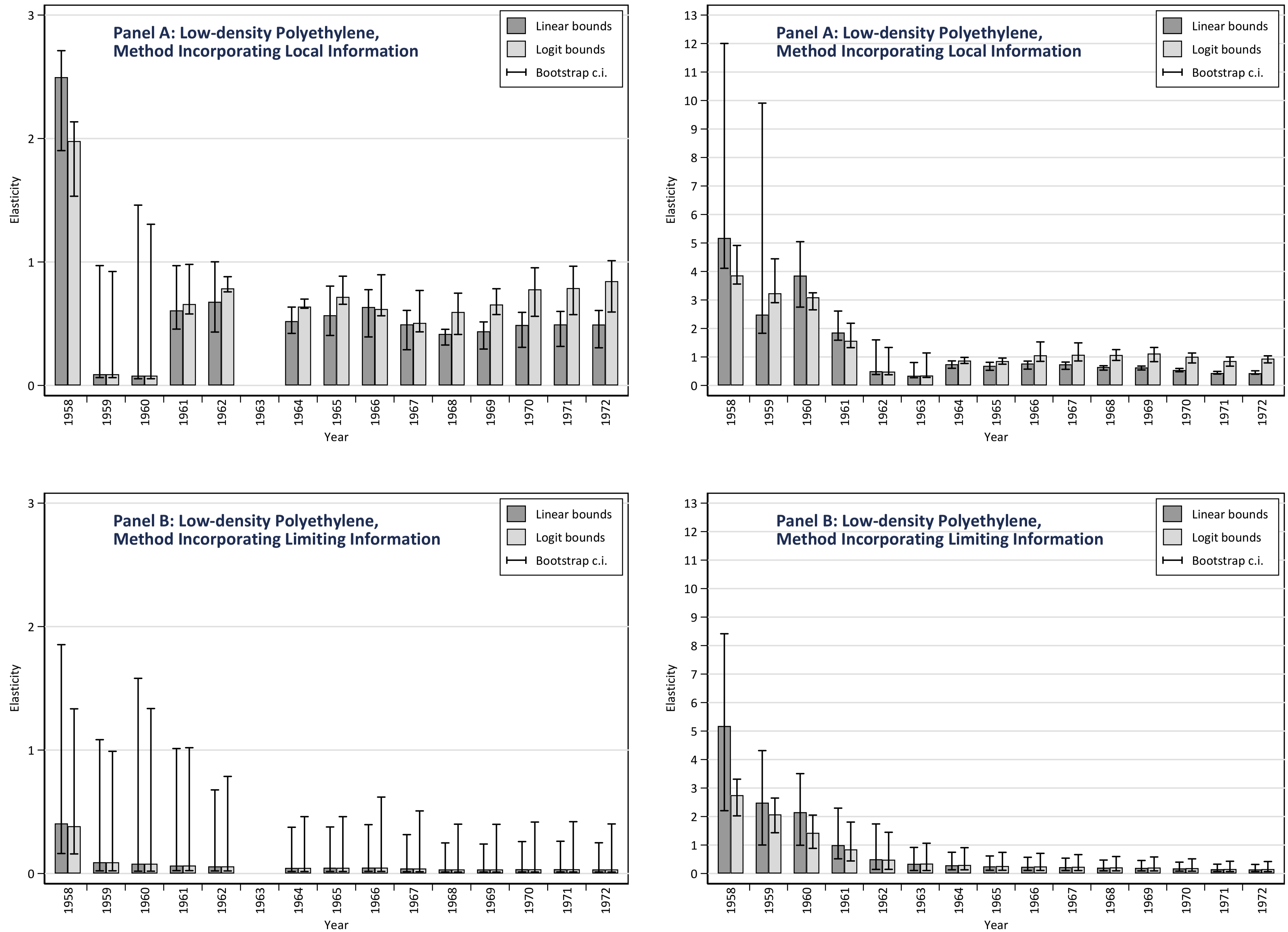

Figure 7: Elasticity Bounds. *Notes:* Shaded bars are elasticity bounds assuming linear demand (dark bars) or logit demand (light bars). Whiskers at top of bars are bootstrapped 90% confidence intervals on the upper elasticity bound using Zelterman's (1993) technique.

Table 2: Upper Bounds on Demand Elasticity

	Low-density polyethylene				High-density polyethylene			
	Incorporating local information		Incorporating limiting information		Incorporating local information		Incorporating limiting information	
	Linear	Logit	Linear	Logit	Linear	Logit	Linear	Logit
1958	2.50	1.98	0.41	0.38	5.18	3.86	5.18	2.75
1959	0.09***	0.09**	0.09	0.09***	2.48	3.24	2.49	2.08
1960	0.08	0.08	0.08*	0.08*	3.85	3.09	2.15	1.43
1961	0.61***	0.66**	0.06*	0.06*	1.86	1.57	2.15	0.85
1962	0.68*	0.79***	0.06***	0.06***	0.50	0.49	0.50	0.49
1963					0.34***	0.35*	0.34***	0.35
1964	0.52***	0.64***	0.05***	0.05***	0.75***	0.87**	0.29***	0.30***
1965	0.57***	0.72***	0.05***	0.05***	0.69***	0.86***	0.25***	0.26***
1966	0.64***	0.62***	0.05***	0.05***	0.77***	1.06	0.23***	0.25***
1967	0.49***	0.51***	0.04***	0.04***	0.74***	1.08	0.22***	0.24***
1968	0.42***	0.60***	0.03***	0.03***	0.65***	1.07	0.20***	0.21***
1969	0.44***	0.66***	0.03***	0.03***	0.63***	1.12	0.20***	0.21***
1970	0.49***	0.78**	0.03***	0.03***	0.54***	1.01	0.18***	0.19***
1971	0.49***	0.79**	0.03***	0.03***	0.44***	0.86**	0.15***	0.16***
1972	0.49***	0.84*	0.03***	0.03***	0.43***	0.94	0.15***	0.16***

Notes: Missing entries correspond to observation dropped to preserve rectangular expansion. Significantly different from 1 in a one-tailed test at the *ten-percent level, **five-percent level, ***one-percent level based on the Zelterman (1993) bootstrap.

rating local information. Within the panel, focus first on the dark-shaded bars, representing the elasticity bounds imposing linear demand. The results are remarkable. Aside from the first year, in which the upper bound is quite high at 2.50, the upper bound on the elasticity is consistently low, never higher than 0.68—which is still quite inelastic.

The upper bound can be much lower than this, in particular, 0.09 and 0.08 in 1959 and 1960. The robustness of the extremely low bounds in those years is called into question, however, by the wide confidence intervals around them, in 1960 even straying above the threshold of 1 that might be consistent with monopoly behavior. The next section will provide a more detailed discussion of robustness. For now, we will note that a consistent finding in our results—holding across specifications, methods, and products—is that the upper bounds in the early years in the sample are either quite high, or have wide confidence intervals, or both.[8]

[8] We suspect the weak results for early years are not a symptom of an idiosyncratically high elasticity in that year; we have no evidence to suggest that polyethylene demand was markedly different in early compared to later years.

Setting aside the admittedly weak results for the early years and focusing on the remaining 1961–72 sample, we see that the upper bound is below 0.5 a majority of the time, significantly less than 1 at the 1% level in a one-tailed test for all of these years except 1962, in which the test is significant at the 10% level. Such low elasticities are inconsistent with a monopoly outcome, which in theory leads to an equilibrium in the elastic region of demand.

The light-shaded bars represent elasticity bounds assuming logit rather than linear demand. The results are consistent across the two functional forms in Panel A. The logit bound can be tighter or looser than that under linear: in the early years, logit demand generates tighter bounds (or at least tighter bootstrap whiskers); in later years, linear demand tends to generate tighter bounds. Overall, the two specifications track each other closely, rising and falling together over the sample, suggesting the results are robust to functional-form assumptions. Setting aside the weak results in the early years (1958–60), in the later years (1961–72), the elasticity bound under logit is as low as 0.51. The bound is significantly less than 1 at the 1% level for a majority of these years at least the 5% level for all these later years except 1972, when it is significant at just the 10% level.

Panel B presents results for low-density polyethylene using the method incorporating limiting information. The estimates for linear and logit demand are almost identical; both are are sharply tighter than in Panel A. When limiting information is incorporated, the upper bounds on the elasticity shrink down to around 0.4 in the first year and extremely close to 0 in all later years. The confidence intervals suggest that the extremeness of these estimates may not be particularly robust: the whisker representing the upper confidence-interval threshold is an order of magnitude higher than the estimated elasticity bound, which shrinks as low as 0.03 in the last five years. The low elasticity bounds obtained in the first stage from the pairwise comparison of the years 1959 and 1960 incorporating local information generates extreme limits that propagate across all the other years in the second stage incorporating that limiting information. Bootstrap subsamples which happen not to include information from this pairwise comparison will be higher, widening the confidence interval. That said, the upper confidence-interval thresholds, although higher than the corresponding bounds, are still only about half the size of those in Panel A, suggesting that the

Two other causes are more likely. First, early years lack prior-year information to bound the elasticities. Second, our methods work by first bounding the demand slope, multiplying by price and dividing by quantity to derive the elasticity. As Figure 6 shows, polyethylene prices were relatively high and quantities low in the early years. For a given slope bound, adjusting by price and quantity will naturally lead to high elasticity bounds.

method incorporating limiting information can lead to a meaningful improvement over the method incorporating only local information. In the last five years of the sample, not only are the elasticities less than 1 at the 1% significance level in a one-tailed test, they are less than 0.42 at the 1% significance level for both linear and logit demands—indeed, less than 0.32 at the 1% significance level for linear demand.

The results incorporating limiting information strengthen the conclusion that demand was very inelastic in some post-remedy years, far from the range consistent with monopoly. The bounds are virtually numerically identical whether linear or logit demand is assumed. The crucial assumption behind this method appears to be not the specific functional form assumed but, whatever that form is, that it is consistent across the domain of prices.

Panels C–D present elasticity bounds for the other product, high-density polyethylene. In this market, the bounds are much wider in the early years compared to low-density polyethylene, necessitating a different vertical-axis scale to display. The bounds narrow considerably by 1963, the upper bound reaching as low as 0.34 for linear demand and 0.35 for logit demand in that year. The bounds remain well within the inelastic region under linear demand. Under logit demand the upper bound rises up above 1 in 1966 and remains around that level for the rest of the sample. The bound is significantly less than 1 at at least the 10% level in only four of the years. It should be emphasized that the logit-demand bounds do not imply that the elasticity was this high but that this method at least cannot rule such high elasticities out. Overall, the upper bound on the elasticity is higher in the high-density market than low-density market, but even in the high-density market, elastic demand and monopoly is precluded in some years in the post-remedy period.

Panel D applies the method incorporating limiting information to the high-density polyethylene market. As in Panel C, the bounds (or the whiskers for the upper confidence-interval threshold) are not particularly tight in the first five years; but by 1964 the upper bound has shrunk below 0.3, and continues to shrink steadily, down to around 0.16 by 1972. After 1964, the bounds are all significantly less than 1 at the 1% level. Regardless of the functional form assumed, incorporating limiting information provides extremely tight bounds on the elasticity of demand in the high-density market. For nearly a decade, demand in each market was well below the level consistent with monopoly according to this method.

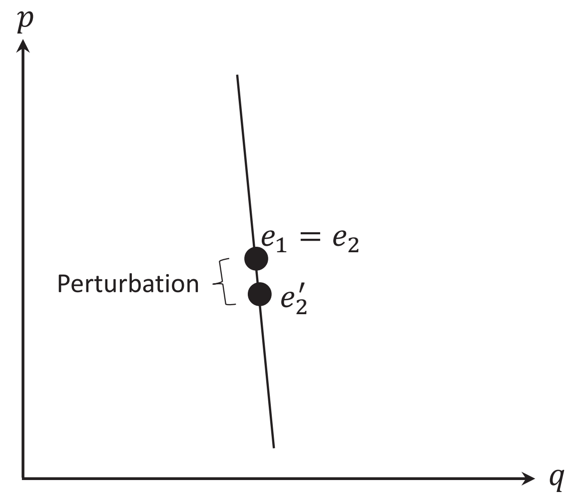

Figure 8: Vulnerability of Method to Observation Errors in Slow-Growing Market

8. Robustness

Our method implicitly assumes price and quantity are perfectly observed. Observation errors can create potential problems for our method. These problems are likely to be relatively inconsequential if the market is growing rapidly over the sample but may be acute if the market experiences periods of only slow growth.

Figure 8 illustrates the potential vulnerability. To take an extreme example, suppose a monopoly serves a market that is unchanging over time. Suppose that the true equilibrium point e_1 continue into period 2 as e_2, but due to an observation error or other small perturbation, e_2' is instead recorded in the data. Using our method, say assuming linear demand, pairwise comparison of e_1 and e_2' implies that the inverse demand curve in both periods would have to be at least as steep as the line drawn. If the perturbation happened to nudge e_2' almost directly south of e_1, we would conclude that demand was almost perfectly inelastic, erroneously rejecting monopoly control of the market. Rapid market growth—driving e_1 and e_2 apart instead of lying on top of each other—would attenuate the effect of a small perturbation from e_2 to e_2' on the estimated elasticity bound.

The concern is not just theoretical. In our application, one might be concerned that the extremely low elasticity bounds of 0.08 and 0.09 in Panel A of Figure 7 were generated by a single perturbation from 1959 to 1960 in the low-density polyethylene market rather than being a robust consequence of many pairwise comparisons. The configuration of the 1959 and 1960 equilibrium points in Panel A of Figure 6 resembles that in Figure 8, with the points lying close enough together that a small southward perturbation could generate an almost vertical line connecting them and a elasticity bound close to zero.

We address the robustness concern in two ways. The next subsection discusses the computation of bootstrapped confidence intervals. If the estimates are unduly influenced by the position of one or two equilibria, we will see higher estimates in bootstrapped subsamples from which they are omitted, widening the confidence intervals. Subsection 8.2 introduces an estimator that leaves out the most influential equilibrium points. We will see that the leave-out estimator appears to stabilize after leaving out the single most influential equilibrium point, the result corresponding closely to the bootstrapped 95% upper confidence-interval threshold.

8.1. Bootstrapping

A natural way to generate error bounds around our estimates is to apply a resampling technique such as boostrapping to the pairwise comparisons between equilibrium points. Standard bootstrap methods are invalid in our context because our upper elasticity bound involves an extreme order statistic, the minimum over pairwise comparisons between a given equilibrium point and others. As Bickel and Freedman (1981) note, it is impossible to draw a pseudosample generating an order statistic more extreme than that generated with the original data because any pseudosample is a subset of the original data. Hence the bootstrapped distribution will be bounded by—rather than centered on—the extreme order statistic estimated from the original data.

Zelterman (1993) provides a technique for circumventing this problem. Instead of sampling the data directly, he proposes sampling the spacings among the highest k observations. Suitably normalized, these spacings asymptotically have an iid exponential distribution. Since the asymptotic result holds for a broad class of distributions of the underlying data, Zelterman classifies the technique as semiparametric, the sole parameter being the k anchoring the order statistic to which the sampled spacings are appended when simulating the extreme order statistic.

Formally, consider applying Zelterman's (1993) technique to bootstrap a confidence interval around $\bar{\epsilon}^*$, the upper bound on the demand elasticity from our first-stage method incorporating local information. Let $\bar{\epsilon}_t^{[1]} \leq \bar{\epsilon}_t^{[2]} \leq \cdots \leq \bar{\epsilon}_t^{[T-1]}$ be elasticity bounds derived from pairwise comparisons between e_t and the other equilibrium points ordered from smallest to largest; superscripts on these terms thus denote the order of these order statistics. Let $d_t^{[i]} \equiv i(\bar{\epsilon}_t^{[i+1]} - \bar{\epsilon}_t^{[i]})$ denote the normalized spacing between two of these ordered bounds. Note that the normalization needed to generate the exponential distribution asymptotically is simply multiplication by i, the degree of the order statistics involved in the spacing. Let $\mathbf{d}_t \equiv \{d_t^{[i]} \,|\, i = 1, \ldots, k\}$ denote the set of the normalized spacings observed in the data between the k lowest pairwise elasticity bounds. One draws a number of pseudosamples $\tilde{\mathbf{d}}_t = \{\tilde{d}_{ti}\}$ of size k with replacement from $\mathbf{d}_t$. The bootstrapped elasticity is computed as

$$\tilde{\epsilon}_t^* \equiv \bar{\epsilon}_t^{[k+1]} - \sum_{i=1}^{k} \tilde{d}_{ti}/i. \tag{53}$$

Intuitively, k draws of normalized spacings are subtracted (after reversing the normalization by dividing by i) from the observed $k+1$st order statistic $\bar{\epsilon}_t^{[k+1]}$, which serves as a sort of anchor for the procedure, to arrive at the simulated minimum $\tilde{\epsilon}_t^*$. Following Zelterman's (1993) recommendation, we take $k = \lfloor T-1 \rfloor /3$.[9,10]

Bootstrapping the bound $\bar{\epsilon}^{**}$, which incorporates limiting information, is more involved because it is a two-stage estimator, and the outcome of the second stage depends in a complicated way on the first stage. We adopt what in our view is the most natural alternative: applying Zelterman's (1993) technique to simulate a bootstrapped estimate from the normalized spacings observed in each separate stage, and then taking the minimum over the result from each stages to generate the final bootstrapped bound $\tilde{\epsilon}_t^{**}$.[11,12]

[9]Zelterman (1993) notes that the asymptotic argument requires $k \to \infty$ as well as $T \to \infty$. Our sample of pairwise comparisons is small since it comes from a relatively short time series. We still proceed to apply the technique and the recommendation for k but note that we may be straining the asymptotic arguments behind the bootstrap. On a positive note, despite the small size of T and k in our application, the bootstrapped confidence intervals appear quite sensible and well behaved.

[10]We take 10,000 bootstrap draws to provide the precision demanded by 99% confidence intervals. The bootstrap procedure is written in Stata code, available on request from the authors. Even for the logit specification, which involves numerical solution of a nonlinear equation for each draw, running the procedure with 10,000 draws only takes a few minutes on a desktop computer.

[11]An alternative would be to bootstrap the demand parameter rather than the elasticity directly. In the case of linear demand, b_t could be bootstrapped. Then the first stage bound ($\bar{b}^*$) could be treated equivalently to the pairwise comparisons between e_t and the other equilibrium points incorporating limiting information in the second stage (the expressions in braces in equation (41)). We tried this alternative, and the results were nearly identical to the alternative we adopted.

[12]While the bootstrapped confidence intervals around $\bar{\epsilon}^{**}$ should typically be tighter than around $\tilde{\epsilon}_t^*$, randomness in the draws can lead the reverse to happen in isolated cases. We see this in for example in Table 2, in the 1961 entries for low-density polyethylene: the one-tailed test is more significant for the first-stage method incorporating local information than the second-stage method incorporating limiting information. In virtually all of the rest of the entries, the second-stage confidence intervals are tighter than the first-stage.

8.2. Leave-Out Estimators

Rather than starting with an estimator that is potentially vulnerable to small perturbations and drawing a confidence interval around it to gauge its vulnerability, an alternative approach is to propose a robust estimator to begin with. This is the approach taken in this subsection in which we propose a series of leave-out estimators.

In particular, $\bar{L}_t^*(1)$ leaves the most influential observation out of the calculation of the upper bound on the demand elasticity; formally, $\bar{L}_t^*(1) \equiv \max_{t' \neq t} \bar{\epsilon}^* \langle e_{t'} \rangle$, where $\bar{\epsilon}^*\langle S \rangle$ is the upper bound on the elasticity restricting the sample to $E \setminus S$ rather than using all of E as does $\bar{\epsilon}^*$. Analogously, the leave-two-out estimator $\bar{L}_t^*(2)$ leaves the two most influential observations out: $\bar{L}_t^*(2) \equiv \max_{t',t'' \neq t} \bar{\epsilon}^* \langle e_{t'}, e_{t''} \rangle$. We can also analogously define the leave-out estimators associated with the method incorporating limiting information: $\bar{L}_t^{**}(1) \equiv \max_{t' \neq t} \bar{\epsilon}^{**} \langle e_{t'} \rangle$ and $\bar{L}_t^{**}(2) \equiv \max_{t',t'' \neq t} \bar{\epsilon}^{**} \langle e_{t'}, e_{t''} \rangle$, where $\bar{\epsilon}^{**}\langle S \rangle$ is defined analogously to $\bar{\epsilon}^*\langle S \rangle$ except that the former incorporates limiting information. Leave-out estimators with orders higher than two can also be defined.[13]

Figure 9 presents the results from the leave-out estimators. The shaded bars repeat the elasticity bounds from Figure 7. The difference are the whiskers, which before represented boostrapped confidence intervals but now represent the extension to the upper bound when the one or two most influential pairwise comparisons are left out. The thick whisker is the extention due to the leave-one-out estimator and the thin whisker to the leave-two-out estimator.

Across the four panels A–D, we see that the leave-out estimators lead to a jump in the upper bound for the early years and across all years for the method incorporating limiting information. These results confirm the claim that the tight bounds for 1959 and 1960 were due to the positioning of 1960 almost vertically below 1959. When one or the other is left out, this causes the upper bound in 1959 and 1960 in Panel A to jump up to a level more consistent with the other years. This extremely tight bound in the first stage led to extreme limits that propogated across the other years in the second stage incorporating limiting information. When one or the other year is left out of the first stage, we see in Panel B that the bounds incorporating limiting information jump from

[13] Although the jackknife is also a leave-out estimator, its resemblance to $\bar{L}_t^*(1)$ and our other leave-out estimators ends there. The jackknife averages over estimates from leave-out samples whereas $\bar{L}_t^*(1)$ takes the worst case—in the case of upper bounds their maximum—over leave-out samples. Further, the jackknife averages over pseudovalues (a suitably weighted difference between the original and leave-out estimates), whereas $\bar{L}_t^*(1)$ is computed directly from the leave-out estimates, and the computation does not involve the original estimate. The jackknife is invalid in our setting for the same reason given by Bickel and Freedman (1981) that the bootstrap is invalid.

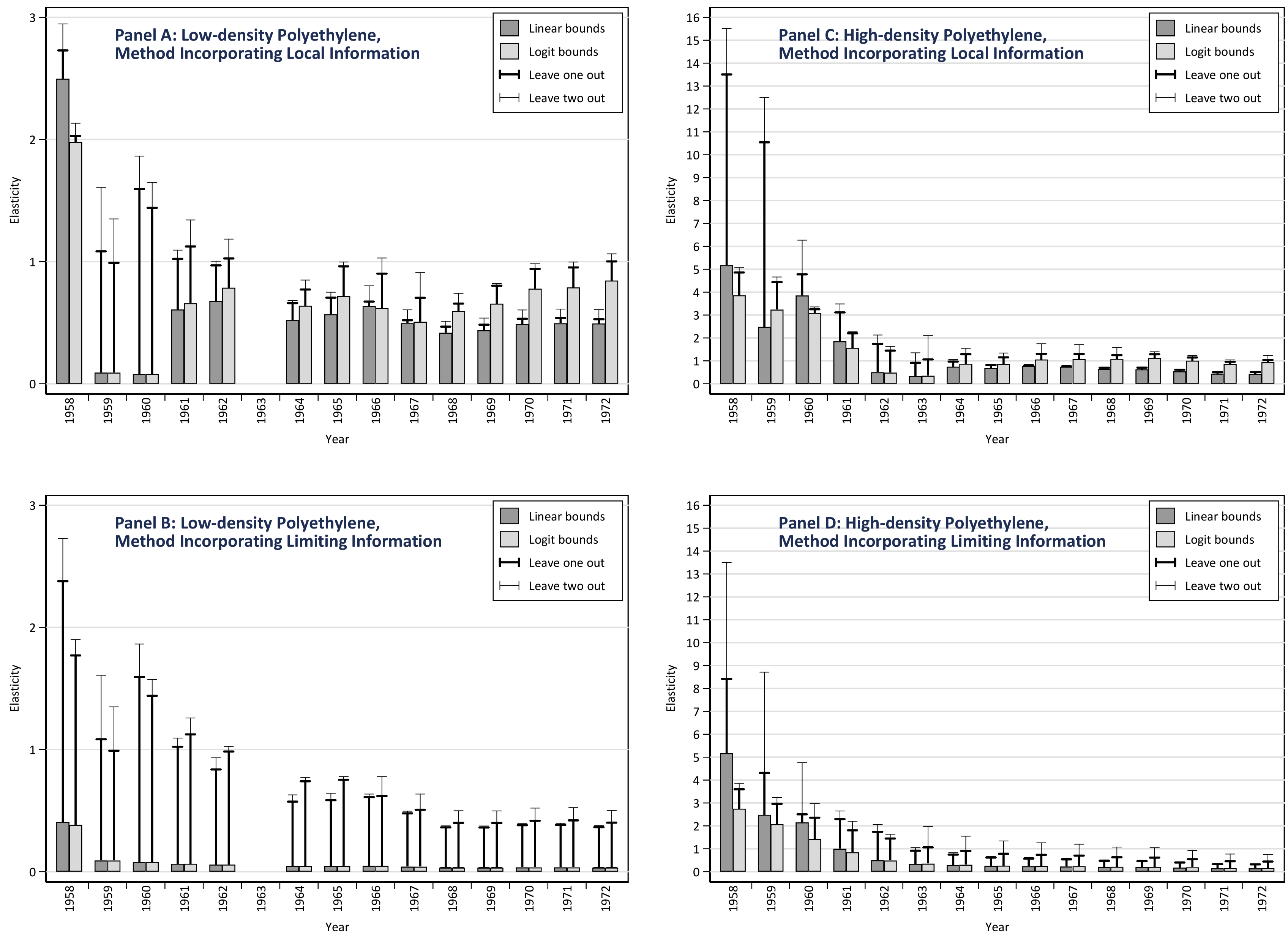

Figure 9: Leave-Out Estimators. *Notes:* Shaded bars are the same elasticity bounds as in Figure 7 for linear demand (dark bars) and logit demand (light bars). Whisker extensions at top represent leave-out estimators: the thick extension represents the leave-tightest-one-out estimator, and the thin extension the leave-tightest-two-out estimator.

near zero to something more moderate.

While the leave-out estimators lead to higher elasticity bounds than the original in Figure 9, we can still take some reassurance from the figure. First, while the bounds jump up in some early years, in later years, the leave-out estimators remain below the threshold of 1 above which monopoly cannot be ruled out. In particular, for linear demand in the market for low-density polyethylene in Panel A, $\bar{L}_t^*(1)$ is below 0.5 in a number of later years. For both linear and logit demand, $\bar{L}_t^{**}(1)$ is below 0.5 for in all of the last six years in Panel B. The leave-out bounds are nearly as tight in the later years for high-density polyethylene in Panels B and D.

Also reassuring is the stability of the leave-out estimators: in years after the first three, the thin whisker does not extend much beyond the thick one, implying that the leave-one-out and leave-two-out estimators are quite close. Leaving one pairwise comparison out appears to be sufficient to protect against the vulnerability to small perturbations illustrated in Figure 8.

Figure 10 plots the two approaches to robustness against each other: the 95% bootstrapped upper confidence threshold from Figure 7 on the horizontal axis against the leave-one-out estimator from Figure 9 on the vertical axis. The two approaches to robustness produce almost identical results, lying almost precisely on the 45-degree line. The small departures in Panel B are mostly above the 45-degree line, indicating that the leave-one-out estimator produces a larger estimate, and thus would provide a more conservative test that the bound is below the threshold (1) used to rule out monopoly behavior. Both results produce many observations inside the dotted box with dimensions less than 1, the threshold below which we can rule out monopoly behavior.

9. Conclusion

This paper provided a methodology for bounding the elasticity of demand that works in growing markets for homogeneous products. The underlying idea is that the demand curve through a given equilibrium point cannot be either so steep or so flat that it passes below earlier equilibria or above later equilibria without violating the assumption that demand is nondecreasing over time. These inequality conditions place bounds on the elasticity of demand in any given year. The method requires minimal information, working with as few as two time-series observations on aggregate prices and quantities.

A potential drawback of any methodology delivering bounds rather than point estimates is that

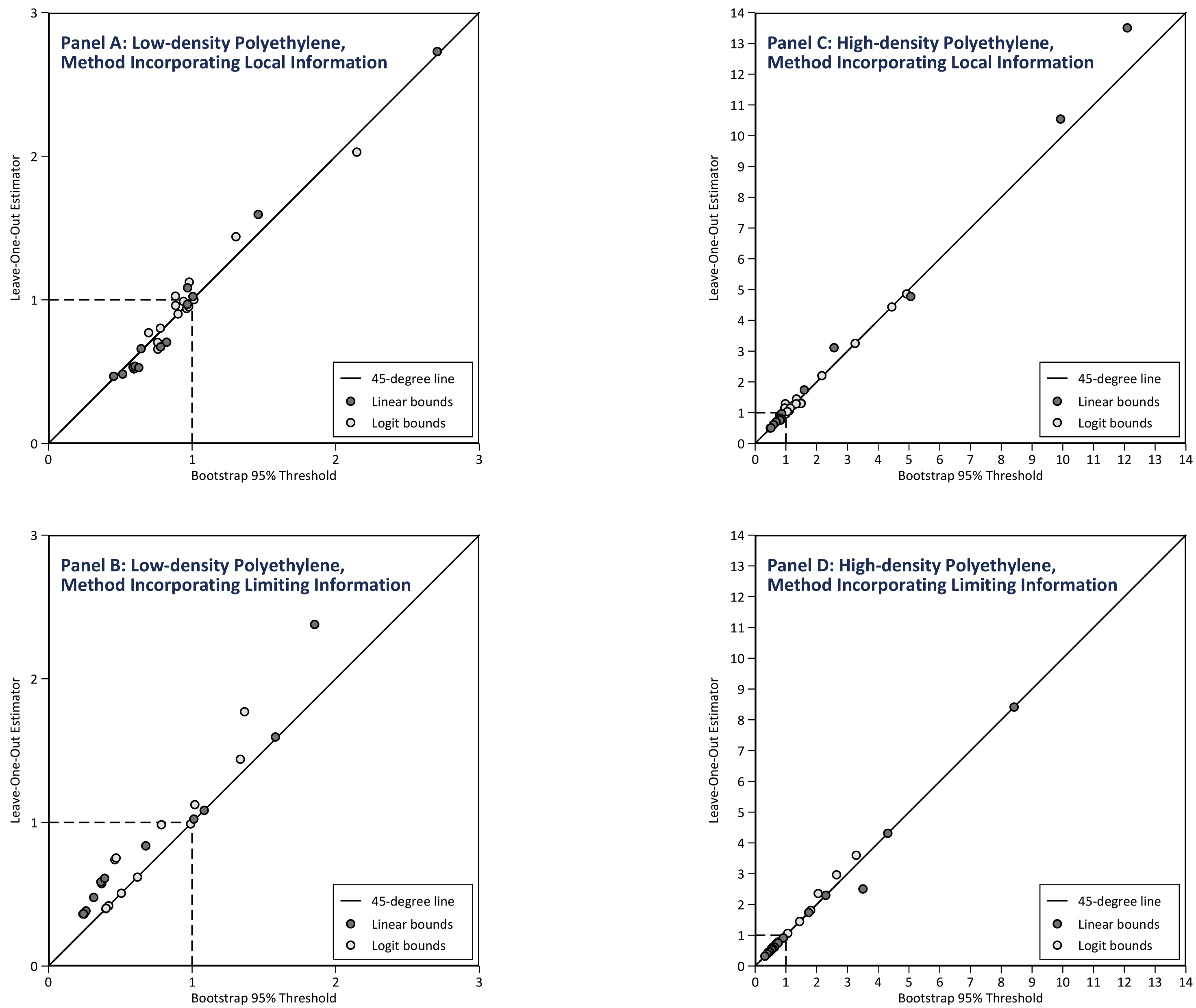

Figure 10: Comparing Bootstrap Theshold to Leave-One-Out Estimator. *Notes:* Plot of bootstrapped 95% upper confidence threshold from Figure 7 versus leave-one-out estimator from Figure 9.

the resulting bounds may be so wide as to be uninformative. In our empirical application to the polyethylene market in the 1958–72 period after a licensing remedy was ordered, the methodology turned out to quite informative. A good snapshot of the results is provided by Table 2. There, with few exceptions, we see a cascade of three-starred entries in every year in 1964–72, indicating an upper bound on the demand elasticity that is significantly less than 1 at the 1% level. This is true across products (low- versus high-density polyethylene), demand specifications (linear versus logit), and methodologies (incorporating local versus limiting information). The main exception is the column for high-density polyethylene assuming logit demand using the method incorporating only local information. While we do not see the same cascade of three-starred results in that column, we still see elasticity bounds significant at the 10% level or better in four of the years.

Based on our finding that the elasticity of demand was bounded in the elastic region, in many years bounded well away at strong significance levels, we reject the contention that monopoly behavior effected by the Patents and Processes agreement between Du Pont and ICI continued after Judge Ryan's remedy in the *Du Pont* case.

The configuration of equilibrium points in 1959 and 1960 in the low-density polyethylene market, with the 1960 lying close to but almost vertically below 1959, raised a question of robustness. Assuming price and quantity are perfectly observed, this configuration provides strong evidence against monopoly. A monopolist would never have dropped its price even by the modest amount experienced unless it would have generated a more substantial gain in quantity, leaving competitive pressure as a plausible explanation of what drove the price drop. These considerations were reflected in the extremely low upper bounds on the demand elasticity for those years, as low as 0.08. This first-stage local comparison propagated into extremely low second-stage bounds incorporating limiting information for all years. This happened because the vertical intercepts for the inverse demands through 1959 and 1960 (or the inverse-demand limits for logit demand) needed to be extremely high to reflect the extremely inelastic demands, forcing later years to also have extremely high vertical intercepts and low elasticities. If the configuration of equilibrium points in 1959–60 were not an actual vertical drop but a small perturbation to fairly stable equilibrium, the extremely tight bounds may have been spurious.

We dealt with the robustness issue in two ways. First, we provided bootstrapped confidence intervals around the estimates using a procedure due to Zelterman (1993) for extreme order statistics.

Second, we proposed estimators leaving out the most influential equilibria for pairwise comparison. The leave-one-out estimator generated almost exactly the same result as the 95% bootstrapped upper confidence threshold across products, functional-forms, and methods. The tight bounds for early years such as 1959 and 1960 using the method incorporating local information were shown not to be robust, nor were the extremely tight bounds across all years using the method incorporating limiting information. However, the elasticity bounds remained significantly less than 1 in many years, a robust finding across products, functional forms, and methods. After one the most influential observation was left out, leaving more out did not appear to have much effect on the more robust findings in the last decade of our sample.

The *Du Pont* application serves as a proof of concept for our bounding methodology, which could be applied to any growing market involving homogeneous products. Based on our investigation into robustness, we strongly suggest using a leave-one-out version of our estimator or displaying bootstrapped confidence intervals. We hope other researchers will find value in our methodology and find it easy to apply using the Stata code available on request.

In theory, one should be able to obtain analogous elasticity bounds in the opposite case of a declining market simply by reversing all the signs and inequalities. This is theoretically true but requires several practical caveats. First, even for markets experiencing secular declines, it may be hard to contend that the demand curve is definitively shifting back each year. Consumers may be losing their taste for a product, but natural population growth may offset this taste change. Second, even if suppliers are not investing much, the spillover of technology from other industries may lower costs in the market, shifting supply out. Equilibrium points may end up shifting to the southwest over time, offering little useful information to bound elasticities.

Appendix A: Proofs

Proof of Proposition 3: The text defines $A(e_t, e_{t'})$ as the solution for α in (20), rearranged here as

$$g(\alpha, p_t, p_{t'}) = \frac{q_{t'}}{q_t}, \tag{A1}$$

where

$$g(\alpha, p_t, p_{t'}) \equiv \frac{1+\exp(\alpha p_t)}{1+\exp(\alpha p_{t'})}. \tag{A2}$$

Before proceeding, we prove several useful facts about g. First,

$$g(0, p_t, p_{t'}) = 1. \tag{A3}$$

Next, differentiating (A2),

$$\frac{\partial g(\alpha, p_t, p_{t'})}{\partial \alpha} = \frac{p_t \exp(\alpha p_t) - p_t \exp(\alpha p_{t'}) + (p_t - p_{t'}) \exp(\alpha p_t) \exp(\alpha p_{t'})}{[1+\exp(\alpha p_{t'})]^2}. \tag{A4}$$

Equation (A4) implies g is monotonic in α, increasing for $p_t > p_{t'}$ and decreasing for $p_t < p_{t'}$. To compute its limit as $\alpha \to \infty$, we first rearrange (A2) as

$$g(\alpha, p_t, p_{t'}) = \frac{1+\exp(-\alpha p_t)}{\exp(\alpha(p_{t'} - p_t)) + \exp(-\alpha p_t)}. \tag{A5}$$

Then it is apparent that

$$\lim_{\alpha \to \infty} g(\alpha, p_t, p_{t'}) = \begin{cases} 0 & p_t < p_{t'} \\ \infty & p_t > p_{t'}. \end{cases} \tag{A6}$$

To apply these facts, first suppose $p_t > p_{t'}$. Then the facts from the previous paragraph imply that g increases from 1 to ∞ as α increases from 0 to ∞. Equation (A1) has a solution if and only if $q_{t'} > q_t$. But $p_t > p_{t'}$ and $q_{t'} > q_t$ imply that $e_{t'} \in SE(e_t)$. Next, suppose $p_{t'} > p_t$. Then the facts from the previous paragraph imply that g decreases from 1 to 0 as α increases from 0 to ∞. Equation (A1) has a solution if and only if $q_{t'} < q_t$. But $p_t > p_{t'}$ and $q_{t'} < q_t$ imply that $e_{t'} \in NW(e_t)$. Therefore, (A1) has a solution if and only if $e_{t'} \in SE(e_t) \cup NW(e_t)$. If (A1) has a solution, the monotonicity of g implies that this solution is unique.

The text analyzes the case in which $t < t'$, showing that (19) is a necessary condition for Assumption 2 not to be violated. One can verify that (19) is satisfied for all $\alpha \geq 0$ if $e_{t'} \in NE^+(e_t)$, implying that this subset does not contribute to bounds on α_t. One can verify that (19) is satisfied for all $\alpha \geq A(e_t, e_{t'})$ if $e_{t'} \in NW^+(e_t)$, implying that this subset contributes lower bounds on α_t. One can verify that (19) is satisfied for all $\alpha \leq A(e_t, e_{t'})$ if $e_{t'} \in SE^+(e_t)$, implying that this subset contributes upper bounds on α_t.

We complete the proof by analyzing the case in which $t > t'$. To respect Assumption 2, we must have

$$\tilde{D}(p, \alpha_t, e_t) = D_t(p) \geq D_{t'}(p) = \tilde{D}(p, \alpha_{t'}, e_{t'}). \tag{A7}$$

Substituting $p = p_{t'}$,

$$\frac{q_t[1+\exp(\alpha_t p_t)]}{1+\exp(\alpha_t p_{t'})} = \tilde{D}(p_{t'}, \alpha_t, e_t) \geq \tilde{D}(p_{t'}, \alpha_{t'}, e_{t'}) = q_{t'}, \tag{A8}$$

or rearranging,

$$q_t[1+\exp(\alpha_t p_t)] \geq q_{t'}[1+\exp(\alpha_t p_{t'})]. \tag{A9}$$

One can verify that (A9) is satisfied for all $\alpha \geq 0$ if $e_{t'} \in SW^-(e_t)$, implying that this subset does not contribute to bounds on α_t. One can verify that (A9) is satisfied for all $\alpha \geq A(e_t, e_{t'})$ if $e_{t'} \in SE^-(e_t)$, implying that this subset contributes lower bounds on α_t. One can verify that (A9) is satisfied for all $\alpha \leq A(e_t, e_{t'})$ if $e_{t'} \in NW^-(e_t)$, implying that this subset contributes upper bounds on α_t. *Q.E.D.*

Verifying Logit Satisfies General Demand Conditions: Let $\tilde{D}(p, \theta, e_t)$ be the logit demand defined in (16) after substituting $\theta = \alpha$. We will verify that this $\tilde{D}(p, \theta, e_t)$ satisfies conditions (25)–(27).

We verify (25) by direct differentiation. Differentiating $\tilde{D}(p, \theta, e_t)$ with respect to its first argument and substituting $p = p_t$,

$$\tilde{D}_p(p_t, \theta, e_t) = \frac{-\theta q_t}{1+\exp(-\theta p_t)}. \tag{A10}$$

Differentiating (A10) with respect to θ and substituting $\theta = \theta_t$,

$$\tilde{D}_{p\theta}(p_t, \theta_t, e_t) = \frac{-q_t\,[1+(1+\theta_t p_t)\exp(-\theta_t p_t)]}{[1+\exp(-\theta_t p_t)]^2}, \tag{A11}$$

which is negative, verifying (25).

To verify (26), we can use the notation introduced in the proof of Proposition 3 to write $\tilde{D}(p, \theta, e_t) = q_t g(\theta, p_t, p)$. Equation (A3) then implies $\tilde{D}(p, 0, e_t) = q_t$, verifying (26).

To verify (27),

$$\lim_{\theta\to\infty} \tilde{D}(p, \theta, e_t) = q_t \lim_{\theta\to\infty} g(\theta, p_t, p) = \begin{cases} 0 & p > p_t \\ \infty & p < p_t, \end{cases} \tag{A12}$$

where the last equality follows from (A6). *Q.E.D.*

Proof of Proposition 4: The proof closely follows the logic of the proof of Proposition 3, so we omit most of it for brevity. Here, we fill in the detail that if (25)–(27) hold and $e_{t'} \in NW(e_t) \cup SE(e_t)$, then (28) has a unique solution.

Suppose $e_{t'} \in NW(e_t)$. Condition (26) implies $\lim_{\theta\to 0} \tilde{D}(p_{t'}, \theta, e_t) = q_t < q_{t'}$, where the inequality follows from $e_{t'} \in NW(e_t)$. Condition (27) implies $\lim_{\theta\to\infty} \tilde{D}(p_{t'}, \theta, e_t) = \infty$ since $p_{t'} < p_t$ for $e_{t'} \in NW(e_t)$. Condition 25 implies that $\tilde{D}(p_{t'}, \theta, e_t)$ is increasing in θ since $p_{t'} < p_t$. Together, these results imply that $\tilde{D}(p_{t'}, \theta, e_t)$ is below $q_{t'}$ for low θ and monotonically increases in θ until it exceeds $q_{t'}$ for high θ. Thus, the solution $\Theta(e_t, e_{t'})$ of (28) exists and is unique.

Suppose $e_{t'} \in SE(e_t)$. Condition (26) implies $\lim_{\theta\to 0} \tilde{D}(p_{t'}, \theta, e_t) = q_t > q_{t'}$, where the inequality follows from $e_{t'} \in SE(e_t)$. Condition (27) implies $\lim_{\theta\to\infty} \tilde{D}(p_{t'}, \theta, e_t) = 0$ since $p_{t'} > p_t$ for $e_{t'} \in SE(e_t)$. Condition (25) implies that $\tilde{D}(p_{t'}, \theta, e_t)$ is decreasing in θ since $p_{t'} > p_t$. Together, these results imply that $\tilde{D}(p_{t'}, \theta, e_t)$ is above $q_{t'}$ for low θ and monotonically decreases in θ until it falls below $q_{t'}$ for high θ. Thus, again, the solution $\Theta(e_t, e_{t'})$ of (28) exists and is unique. *Q.E.D.*

Proof of Proposition 5: The first step of the proof is to show that (25) implies $\tilde{D}_{p\theta}(p_t,\theta,e_t) \le 0$. Suppose for the sake of contradiction that $\tilde{D}_{p\theta}(p_t,\theta,e_t) > 0$. Since $\tilde{D}$ is continuously differentiable of all orders, $\tilde{D}_{p\theta}(p_t,\theta,e_t)$ is continuous in its first argument, implying that there exists $\delta > 0$ such that $\tilde{D}_{p\theta}(p,\theta,e_t) > 0$ for all $p \in (p_t-\delta, p_t+\delta)$. Thus

$$0 < \int_{p_t-\delta}^{p_t+\delta} \tilde{D}_{p\theta}(p,\theta,e_t)dp = \tilde{D}_{\theta}(p_t+\delta,\theta,e_t) - \tilde{D}_{\theta}(p_t-\delta,\theta,e_t). \tag{A13}$$

But (25) implies $\tilde{D}(p_t-\delta,\theta,e_t) > 0$ and $\tilde{D}(p_t+\delta,\theta,e_t) < 0$, implying their difference is negative, not positive as in (A13), a contradiction.

The fact that $\tilde{D}_{p\theta}(p_t,\theta,e_t) \le 0$ implies $\frac{\partial}{\partial\theta}[-\tilde{D}_p(p_t,\theta,e_t)]$, implying ϵ_t as defined in (31) for general demand is nondecreasing in θ, implying $\epsilon_t \in [\underline{\epsilon}_t, \bar{\epsilon}_t]$. *Q.E.D.*

Proof of Proposition 7: For concreteness, suppose $t' > t$. (The proof supposing $t' < t$ is similar and omitted.) It remains to show that $\alpha_t \ge \alpha_{t'}$ is a necessary condition for (44). We have

$$\lim_{p\to\infty}\left\{\frac{\tilde{D}(p,\alpha_t,e_t)}{\tilde{D}(p,\alpha_{t'},e_{t'})}\right\} = \lim_{p\to\infty}\left\{\frac{q_t[1+\exp(\alpha_t p_t)]}{1+\exp(\alpha_t p)} \cdot \frac{1+\exp(\alpha_{t'} p)}{q_{t'}[1+\exp(\alpha_{t'} p_{t'})]}\right\} \tag{A14}$$

$$= \frac{q_t[1+\exp(\alpha_t p_t)]}{q_{t'}[1+\exp(\alpha_{t'} p_{t'})]} \lim_{p\to\infty} \exp(p(\alpha_{t'}-\alpha_t)) \tag{A15}$$

$$= \begin{cases} \infty & \alpha_t < \alpha_{t'} \\ \dfrac{q_{t'}[1+\exp(\alpha_{t'} p_{t'})]}{q_t[1+\exp(\alpha_t p_t)]} & \alpha_t = \alpha_{t'} \\ 0 & \alpha_t > \alpha_{t'}. \end{cases} \tag{A16}$$

Equation (A14) follows from substituting for $\tilde{D}$ from (16), (A15) follows from dividing numerator and denominator by $\exp(\alpha_{t'} p)$ and taking limits, and (A16) follows from evaluating the remaining limit. We see that (44) is violated if $\alpha_t < \alpha_{t'}$. *Q.E.D.*

Proof of Proposition 8: Start by analyzing the limit $p \to 0$. For $t' > t$, we have

$$\lim_{p\to 0}\left[\frac{\tilde{D}(p,\theta_t,e_t)}{\tilde{D}(p,\bar{\theta}^*_{t'},e_{t'})}\right] \ge \lim_{p\to 0}\left[\frac{\tilde{D}(p,\theta_t,e_t)}{\tilde{D}(p,\theta_{t'},e_{t'})}\right] \ge 1. \tag{A17}$$

The first inequality follows from Assumption 2. To see the second inequality, note that in the limit as $p \to 0$, eventually $p \le p_{t'}$. But (25) then implies $\tilde{D}_{\theta}(p,\theta,e_{t'}) \ge 0$, in turn implying $\tilde{D}(p,\theta_{t'},e_{t'}) \le \tilde{D}(p,\bar{\theta}^*_{t'},e_{t'})$ in the limit as $p \to 0$ since $\theta_{t'} \le \bar{\theta}^*_{t'}$ by Proposition 4.

Similarly, one can argue that (25) implies that the numerator on the left-hand side of (A17), and thus the entire left-hand side, is increasing in θ_t. Condition (A17) treated as an equality,

$$\lim_{p\to 0}\left[\frac{\tilde{D}(p,\theta,e_t)}{\tilde{D}(p,\bar{\theta}^*_{t'},e_{t'})}\right], \tag{A18}$$

thus provides an equation in θ that can be solved to provide a lower bound on θ_t. The new bound $\underline{\theta}^{**}_t$ in (48) is set to be no lower than the highest of the solutions to (A18) for $t' > t$.

The analyses for $t' < t$ and for the other limit, $p \to \infty$ are similar and omitted for brevity. *Q.E.D.*

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Online Appendix B: Supplementary Figures

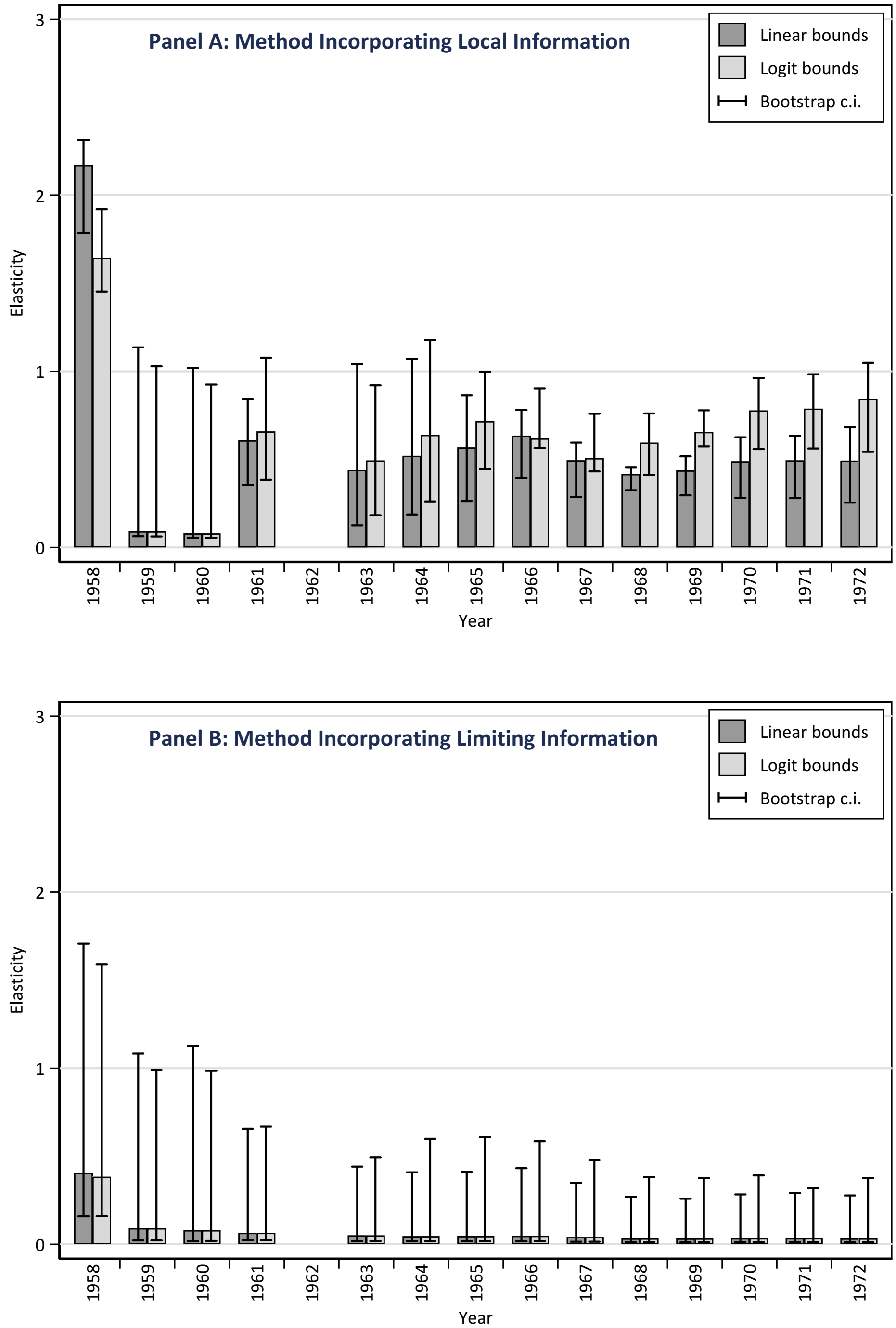

Figure B1: Elasticity Bounds for Low-Density Polyethylene Dropping Alternate Year to Preserve Rectangular Expansion. *Notes:* Rather than dropping 1963 to preserve property of rectangular expansion for equilibrium points, as done in Panels A–B of Figure 7, the same goal is achieved here by dropping 1962. See Figure 7 for additional notes.

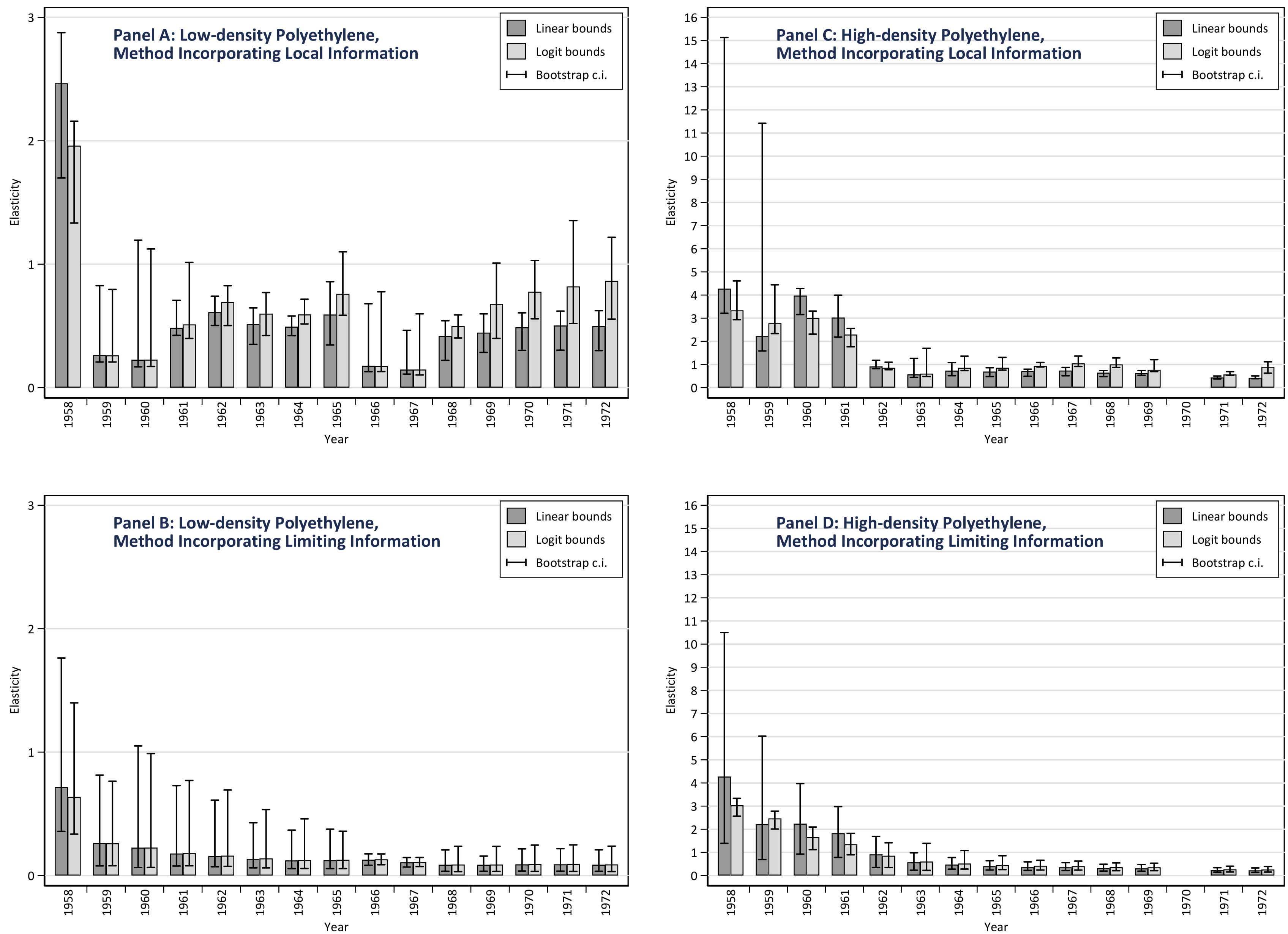

Figure B2: Elasticity Bounds Using Production Data for Quantity. For low-density polyethylene, the whole set of equilibrium points E exhibits rectangular expansion using production rather than sales data for quantity, so 1963 no longer needs to be dropped. For high-density polyethylene, 1970 now has to be dropped to preserve rectangular expansion. See Figure 7 for additional notes.