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PRICING LONG-LIVED SECURITIES IN DYNAMIC ENDOWMENT ECONOMIES

Jerry Tsai
Jessica A. Wachter

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ABSTRACT

We solve for asset prices in a general affine representative-agent economy with isoelastic recursive utility and rare events. Our novel solution method is exact in two special cases: no preference for early resolution of uncertainty and elasticity of intertemporal substitution equal to one. Our results clarify model properties governed by the elasticity of intertemporal substitution, by risk aversion, and by the preference for early resolution of uncertainty. Finally, we show in a general setting that the linear relation between normal-times covariances and expected returns need not hold in a model with rare events.

Jerry Tsai
Department of Economics
University of Oxford
Manor Road Building
Oxford, OX1 3UQ
U.K.
jerry.tsai@economics.ox.ac.uk

Jessica A. Wachter
Department of Finance
2300 SH-DH
The Wharton School
University of Pennsylvania
3620 Locust Walk
Philadelphia, PA 19104
and NBER
jwachter@wharton.upenn.edu

1 Introduction

The framework of representative-agent asset pricing, in which complete markets allows for the diversification of idiosyncratic risks, has for many years delivered benchmark models of the cross-section and time-series of stock prices and returns.¹ These models are at the same time simple and rich in the types of economic intuition they capture. In this paper, we focus on two nested sub-classes of the dynamic representative-agent framework with the goal of clarifying important implications for risk premia and asset prices.

In the first part of the paper, we extend classic cross-sectional results of Merton (1973) and Breeden (1979) to a dynamic setting with recursive utility (Epstein and Zin, 1989) and rare events. This section assumes only isoelastic recursive utility and a Markov structure. Thus this paper unifies and extends existing results such as those of Bansal and Yaron (2004), Barro (2006), Benzoni, Collin-Dufresne, and Goldstein (2011), Drechsler and Yaron (2011), Farhi, Fraiberger, Gabaix, Ranciere, and Verdelhan (2017) Gabaix (2012), Hansen, Heaton, Lee, and Roussanov (2007), Longstaff and Piazzesi (2004) and Wachter (2013). Widening the class of models beyond the traditional diffusion framework of Merton (1973) has dramatic implications for the cross-section. The intertemporal capital asset pricing model (ICAPM) of Merton is a standard justification for the near-universal use of covariance-based factor models in finance. However, the ICAPM relies on conditional log-normality. Without the assumption of conditional log-normality, a factor structure may not hold.

In this section, we also show that the dynamics of the wealth-consumption ratio are principally governed by the elasticity of intertemporal substitution (EIS) and the risk premia in the cross-section relative to a consumption-based model are governed by the preference for early resolution of uncertainty. However, relative to a wealth-based model, risk premia in the cross-section are governed by risk aversion. When risk aversion is equal to one, a rare-event wealth CAPM holds regardless of the EIS. In contrast, a rare-event consumption

¹Textbook treatments include Campbell (2018) and Cochrane (2001).

CAPM holds if there is no preference for early resolution of uncertainty, regardless of risk aversion. Some of these comparative statics have appeared elsewhere in the literature, but the advantage of our framework is that we can show them in a more general setting.

In the second part of the paper, we derive approximate analytical solutions for the pricing of long-lived assets. Our solution method takes as its starting point the widely-used method of Campbell and Shiller (1988) which involves a first-order approximation of the price-dividend ratio by a log-linear function. Previous studies use this method to compute the wealth-consumption ratio (which is necessary for computing other asset prices under recursive utility), and then to compute prices on other assets. While we use the method to compute the wealth-consumption ratio, we then, given the approximation, compute prices on other assets exactly. As a consequence, our method, unlike others, is exact both when the elasticity of intertemporal substitution is equal to one, and when utility is time-additive.² We illustrate our results by extending the model of Wachter (2013) to a case of non-unitary EIS. We show that our technique is notably closer to the solution when the exact problem is solved numerically, than under standard approximations. Allowing for an EIS greater than one improves the fit of the model to the second moments of stock returns and to the riskfree rate.

By deriving approximate analytic solutions we reveal economic intuition that well-known approximations conceal. Our results also reveal the role of risk aversion, the elasticity of intertemporal substitution and the preference for early resolution of uncertainty, in a fully non-parametric setting. This demonstrates that what drives the role of these parameters is not the specific assumptions in current models, but more general principles.

Finally, our understanding of the cross-section of returns, though well-developed empirically, is, theoretically, more marked by ignorance than knowledge. Of the vast range of cross-sectional anomalies, the value premium has received most of the attention, and even here there is at present no consensus on the reason. This paper shows a potential answer as

²In concurrent and independent work, Pohl, Schmedders, and Wilms (2017) derive related results in a discrete-time conditional lognormal setting.

to why this may be the case: that by seeking to explain expected returns via normal-times covariances, the literature may be looking in the wrong place. What matters for expected returns is the covariance during extreme events.

The remainder of the paper proceeds as follows. Section 2 describes our general set-up and derives results for the cross-section. Section 3 describes the affine set-up with analytical solutions. Section 4 quantitatively evaluates the solution method under an example economy.

2 General Model

2.1 The endowment process

Let B_{ct} be a unidimensional Brownian motion and B_{Xt} an n -dimensional Brownian motion, such that B_{ct} and B_{Xt} are independent. For $j = 1, \dots, m$, let N_{jt} be independent Poisson processes. Consider functions $\mu_c : \mathbf{R}^n \rightarrow \mathbf{R}$, $\sigma_c : \mathbf{R}^n \rightarrow \mathbf{R}$, $\mu_X : \mathbf{R}^n \rightarrow \mathbf{R}^n$, $\sigma_X : \mathbf{R}^n \rightarrow \mathbf{R}^{n \times n}$, and $\lambda : \mathbf{R}^n \rightarrow \mathbf{R}^m$. Assume the endowment follows the process

$$\frac{dC_t}{C_{t-}} = \mu_c(X_{t-}) dt + \sigma_c(X_{t-}) dB_{ct} + \sum_{j=1}^m (e^{Z_{cjt}} - 1) dN_{jt}, \quad (1)$$

where X_t is a vector of state variables following the process

$$dX_t = \mu_X(X_{t-}) dt + \sigma_X(X_{t-}) dB_{Xt} + \sum_{j=1}^m Z_{Xjt} dN_{jt}, \quad (2)$$

and where, for all $j = 1, \dots, m$ and t , Z_{cjt} are scalar random variables and Z_{Xjt} an $n \times 1$ vector of random variables. We assume the joint distribution of Z_{cjt} and Z_{Xjt} is time-invariant, and thus suppress the t subscript when not essential for clarity. The intensity for Poisson process N_{jt} is time-varying and given by $\lambda_j(x)$, the j -th element of $\lambda(x)$. We adopt the convention that B_{Xt} , and therefore X_t , are column vectors and that σ_c is a row vector. We use the notation e_j to denote the conforming column vector that has 1 in the

j th position and 0 elsewhere. The literature focuses on the case of disasters ($Z_{cjt} < 0$). However, Tsai and Wachter (2016) show that booms ($Z_{cjt} > 0$) have rich implications for the cross-section of returns.

Consider a stochastic process for dividends:

$$\frac{dD_t}{D_{t-}} = \mu_d(X_{t-}) + \sigma_d(X_{t-})dB_{ct} + \sum_{j=1}^m (e^{Z_{dj}t} - 1) dN_{jt}. \quad (3)$$

We will price the claim to (3). In equilibrium, this asset is in zero net supply. There may be many such assets, but because we will assume complete markets, it suffices to consider each asset in isolation. Similarly to the rare-event outcomes for consumption and for X_t , $Z_{dj}t$ is a random variable with time-invariant distribution for all j .³

Additional notation will be helpful. Let $B_t \equiv [B_{ct}, B_{Xt}^\top]^\top$. For a function $F(c, d, x)$, define

$$\mathcal{J}_F(x, Z_c, Z_d, Z_X) \equiv \frac{F(e^{Z_c}, e^{Z_d}, x + Z_X)}{F(1, 1, x)} - 1. \quad (4)$$

and

$$\begin{aligned} \bar{\mathcal{J}}_F(x) &\equiv E_\nu [\mathcal{J}_F(x, Z_{c1}, Z_{d1}, Z_{X1}), \dots, \mathcal{J}_F(x, Z_{cm}, Z_{dm}, Z_{Xm})]^\top \\ &\equiv [E_{\nu_1} \mathcal{J}_F(x, Z_{c1}, Z_{d1}, Z_{X1}), \dots, E_{\nu_m} \mathcal{J}_F(x, Z_{cm}, Z_{dm}, Z_{Xm})]^\top \end{aligned}$$

where E_{ν_j} denotes expectations taken with respect to the time-invariant joint distribution of Z_{cj}, Z_{dj}, Z_{Xj} . We will use sometimes use the shorthand $\bar{\mathcal{J}}_{Ft} = \bar{\mathcal{J}}_F(X_t)$, and $\mathcal{J}_{Ft}(Z_c, Z_d, Z_X) \equiv \mathcal{J}_F(X_t, Z_c, Z_d, Z_X)$. We will also typically omit either the Z_c or Z_d argument if the function does not depend on consumption or dividends, respectively. Finally, we follow the convention that partial derivatives with respect to a vector are row vectors; for example, $\partial I / \partial X = [\partial I / \partial X_1, \dots, \partial I / \partial X_n]$.

³Dividends are assumed to be perfectly correlated with consumption during normal times. Normal-times dividend risk that is uncorrelated with consumption and the state variables will have no impact on risk premia or on asset prices themselves. Our analysis can easily be extended to allow for dividend risk that is correlated with the state variables.

2.2 Utility

Assume a representative agent with recursively-defined utility

$$V_t = E_t \left[\int_t^\infty f(C_s, V_s) ds \right], \quad (5)$$

with

$$f(C, V) = \frac{\beta}{1 - \frac{1}{\psi}} ((1 - \gamma)V) \left(\left(C ((1 - \gamma)V)^{-\frac{1}{1-\gamma}} \right)^{1 - \frac{1}{\psi}} - 1 \right). \quad (6)$$

where ψ is the elasticity of intertemporal substitution and γ is risk aversion (Duffie and Epstein, 1992b). When $\gamma = 1/\psi$, the recursion in (6) is linear, and (5) reduces to the time-additive case.

We are interested in two limiting cases of (6). When $\psi = 1$, (6) reduces to

$$f(C, V) = \beta ((1 - \gamma)V) \left(\log C - \frac{1}{1 - \gamma} \log ((1 - \gamma)V) \right). \quad (7)$$

(Duffie and Epstein, 1992a). When $\gamma = 1$, (6) reduces to

$$f(C, V) = \frac{\beta}{1 - \frac{1}{\psi}} \left(e^{(1 - \frac{1}{\psi})(\log C - V)} - 1 \right). \quad (8)$$

When $\gamma = \psi = 1$, V_t equivalent to time-additive log utility. In what follows, let $\theta = (1 - \gamma)/(1 - \frac{1}{\psi})$. Then $\theta = 1$ corresponds to time-additive utility, and $\frac{1}{\theta} = 0$ will, in a formal sense, correspond to $\psi = 1$.

Though this framework is quite general, it specifically assumes that the distribution of dividends and consumption, as well as utility, are invariant to scale. This invariance is empirically justified, useful computationally (because it eliminates state variables, and important conceptually (the notation in Equation 4, for example, requires this scale invariance). The economic import of scale invariance is that the level of consumption, while very important for the level of utility, is not important for decision-making. Likewise, the level of dividends, while important for prices, does not affect returns.

2.3 General characterization of the solution

We first characterize the value function, the wealth-consumption ratio, and the state-price density in terms of the state variables.⁴ Here and in the remainder of the paper, Proofs not given in the main text are contained in Appendix A.

Proposition 1 (Value function). *Suppose the representative agent's preference is defined by (5)–(7), where the consumption growth process follows (1) and the state variable process follows (2). In equilibrium, $J(C_t, X_t) = V_t$, where*

$$J(C, x) = \frac{C^{1-\gamma} I(x)^{1-\gamma}}{1-\gamma} \quad (9)$$

for $\gamma \neq 1$ and

$$J(C, x) = \log C + \log I(x)$$

for $\gamma = 1$. The function $I(\cdot)$ satisfies the partial differential equation

$$\begin{aligned} & \frac{\beta}{1-\frac{1}{\psi}} \left[I(x)^{\frac{1}{\psi}-1} - 1 \right] + \mu_c(x) + \frac{1}{2} \text{tr} \left[\left(\frac{1}{I} \frac{\partial^2 I}{\partial x^2} - \frac{\gamma}{I^2} \left(\frac{\partial I}{\partial x} \right)^\top \left(\frac{\partial I}{\partial x} \right) \right) \sigma(x) \sigma(x)^\top \right] \\ & + \frac{1}{I} \frac{\partial I}{\partial x} \mu_X(x) - \frac{1}{2} \gamma \sigma_c(x)^2 + \frac{1}{1-\gamma} \sum_{j=1}^m \lambda_j(x) E_{\nu_j} \left[e^{(1-\gamma)Z_{cj}} \left(\frac{I(x+Z_{Xj})}{I(x)} \right)^{1-\gamma} - 1 \right] = 0 \end{aligned} \quad (10)$$

for $\psi \neq 1$ and

$$\begin{aligned} & -\beta \log I + \mu_c(x) + \frac{1}{2} \text{tr} \left[\left(\frac{1}{I} \frac{\partial^2 I}{\partial x^2} - \frac{\gamma}{I^2} \left(\frac{\partial I}{\partial x} \right)^\top \left(\frac{\partial I}{\partial x} \right) \right) \sigma(x) \sigma(x)^\top \right] \\ & + \frac{1}{I} \frac{\partial I}{\partial x} \mu_X(x) - \frac{1}{2} \gamma \sigma_c(x)^2 + \frac{1}{1-\gamma} \sum_{j=1}^m \lambda_j(x) E_{\nu_j} \left[e^{(1-\gamma)Z_{cj}} \left(\frac{I(x+Z_{Xj})}{I(x)} \right)^{1-\gamma} - 1 \right] = 0. \end{aligned} \quad (11)$$

⁴We assume parameter values are such that solutions exist. We discuss existence of solutions further in Section 3.

for $\psi = 1$.

Equation 11 is a special case of (10) as can be seen by taking the limit as $\psi \rightarrow 1$.

Given the value function (9), we can now express the wealth-consumption ratio and the state price density in terms of $I(X_t)$. This ratio will play an important role in our solution method for the affine case.

Corollary 2 (Wealth-consumption ratio). *Let W_t denote the wealth of the representative agent at time t . Then the equilibrium wealth-to-consumption ratio $G^c(X_t) \equiv W_t/C_t$ is a function of X_t and is given by*

$$G^c(x) = \begin{cases} \beta^{-1} I(x)^{1-\frac{1}{\psi}} & \psi \neq 1 \\ \beta^{-1} & \psi = 1. \end{cases} \quad (12)$$

Proof of Corollary 2 Conjecture that the equilibrium wealth-consumption ratio is a function of X_t , namely $G^c(X_t) \equiv W_t/C_t$. Optimality requires that the derivative of f with respect to C_t equals the derivative of J with respect to W_t (Duffie, 1996, Chapter 9). By the chain rule, $\partial J/\partial W = (\partial J/\partial C)G^c(X_t)^{-1}$, so that

$$\frac{\partial f}{\partial C} = \frac{\partial J}{\partial C} \frac{1}{G^c(X_t)}. \quad (13)$$

Furthermore, $V_t = J(C_t, X_t)$. Taking the derivative of (6) with respect to C , and substituting (9) in for V implies

$$\frac{\partial f}{\partial C} = \beta C_t^{-\gamma} I(X_t)^{\frac{1}{\psi}-\gamma}. \quad (14)$$

Combining (14) with (13), and applying (9) to calculate the right hand side of (13), verifies the conjecture and implies (12). \square

Corollary 3 (State-price density). *The state-price density is given by:*

$$\pi_t = \begin{cases} \exp \left\{ -\beta \int_0^t \left((1 - \theta) I(X_s)^{\frac{1}{\psi} - 1} + \theta \right) ds \right\} \beta C_t^{-\gamma} I(X_t)^{\frac{1}{\psi} - \gamma} & \psi \neq 1 \\ \exp \left\{ -\beta \int_0^t \left((1 - \gamma) \log I(X_s) + 1 \right) ds \right\} \beta C_t^{-\gamma} I(X_t)^{1 - \gamma} & \psi = 1 \end{cases} \quad (15)$$

$$\pi_t = \begin{cases} \exp \left\{ -\beta \int_0^t \left((1 - \theta) I(X_s)^{\frac{1}{\psi} - 1} + \theta \right) ds \right\} \beta C_t^{-\gamma} I(X_t)^{\frac{1}{\psi} - \gamma} & \psi \neq 1 \\ \exp \left\{ -\beta \int_0^t \left((1 - \gamma) \log I(X_s) + 1 \right) ds \right\} \beta C_t^{-\gamma} I(X_t)^{1 - \gamma} & \psi = 1 \end{cases} \quad (16)$$

Proof of Corollary 3 Duffie and Skiadas (1994) characterize the state-price density as

$$\pi_t = \exp \left\{ \int_0^t \frac{\partial f}{\partial V}(C_s, V_s) ds \right\} \frac{\partial f}{\partial C} \Big|_{C_t, V_t}$$

The results follow from substituting $V_t = J(C_t, X_t)$ using (9) into (6) and (7) and taking partial derivatives. \square

2.4 Risk premia

Risk premia arise from the co-movement of asset prices with the marginal utility process.

Using Corollary 3, we can write the evolution of π_t in terms of the underlying shocks:

$$\frac{d\pi_t}{\pi_t} = \mu_{\pi_t} dt + \sigma_{\pi_t} dB_t + \sum_{j=1}^m \mathcal{J}_\pi(X_{t-}, Z_{cjt}, Z_{Xjt}) dN_t \quad (17)$$

where expressions for $\mu_{\pi_t} = \mu_\pi(X_t)$, $\sigma_{\pi_t} = \sigma_\pi(X_t)$ and \mathcal{J}_π follow from Ito's Lemma and the homotheticity in the non-locally deterministic terms of (15) and (16).⁵ Here and in what follows, we drop the arguments Z_c or Z_d when these are unnecessary.

We consider the valuation of an asset paying (3). Then, the absence of arbitrage implies that this asset value is $S_t = S(D_t, X_t)$, where

$$S(D_t, X_t) = E_t \int_t^\infty \pi_s D_s ds \quad (18)$$

$$= D_t G(X_t) \quad (19)$$

⁵ The state-price density π_t is not a function of C_t and X_t , and thus \mathcal{J}_π is not strictly speaking defined. To be precise, define $\hat{\pi}_t = C_t^{-\gamma} I(X_t)^{\frac{1}{\psi} - \gamma}$, and replace \mathcal{J}_π in (17) with $\mathcal{J}_{\hat{\pi}_t} \frac{\pi_t}{\hat{\pi}_t}$.

The Markov assumption implies that the right hand side of (18) is a function of D_t and X_t , while the form of (3) implies (19). Note that $G(X_t)$ is the price-dividend ratio. We can then apply Ito's Lemma to conclude that prices evolve according to

$$\frac{dS_t}{S_{t-}} = \mu_{S_{t-}} dt + \sigma_{S_{t-}} dB_t + \sum_{j=1}^m \mathcal{J}_S(X_{t-}, Z_{dj}, Z_{Xjt}) dN_{jt}, \quad (20)$$

for $\mu_{S_t} = \mu_S(X_t)$, $\sigma_{S_t} = \sigma_S(X_t)$, and

$$\mathcal{J}_S(x, Z_d, Z_X) = \frac{S(e^{Z_d}, x + Z_X)}{S(1, x + Z_x)} - 1 = \frac{S(De^{Z_d}, x + Z_X)}{S(D, x + Z_X)} - 1.$$

Given the state-price density, risk premia and prices follow from no-arbitrage pricing. This is equivalent to solving for an equilibrium with a representative agent, provided the state prices are as in Corollary 3.

Lemma 4 (No arbitrage). *Assume state prices given by (17). Suppose an asset has price process given by (20). Then*

$$\mu_\pi(x) + \mu_S(x) + G(x) + \sigma_\pi(x)\sigma_S(x)^\top + \lambda(x)^\top \bar{\mathcal{J}}_{\pi S}(x) = 0. \quad (21)$$

With the no arbitrage condition given in Lemma 4, we can calculate the risk premium of the asset S_t . Note that the expected return on this asset is given by:

$$r^S(x) = \mu_S(x) + G(x) + \lambda(x)^\top \bar{\mathcal{J}}_S(x) \quad (22)$$

Furthermore,

$$\mu_\pi(x) = -r(x) - \lambda(x)^\top \bar{\mathcal{J}}_\pi(x) \quad (23)$$

We now state a result that holds under the general form for the state-price density (17) and for an asset price (20). The proof is straightforward given Lemma 4, (22) and (23).

We suppress the x argument when not essential for clarity.

Theorem 5. *Let $r_t = r(X_t)$ denote the continuously compounded risk-free rate. The continuous-time limit of the risk premium for the asset with price process (20) is*

$$r_t^S - r_t = -\sigma_{\pi t} \sigma_{S t}^\top - \lambda_t^\top (\bar{\mathcal{J}}_{\pi S, t} - \bar{\mathcal{J}}_{\pi, t} - \bar{\mathcal{J}}_{S, t}) \quad (24)$$

$$= -\sigma_{\pi t} \sigma_{S t}^\top - \sum_{j=1}^m \lambda_{jt} E_{\nu_j} [\mathcal{J}_{\pi t}(Z_{cj}, Z_{Xj}) \mathcal{J}_{S t}(Z_{dj}, Z_{Xj})] \quad (25)$$

When there are no Poisson shocks, (25) is a standard pricing result that inspires tests of factor models of the cross section. Elements of $-\sigma_{\pi t}$ are risk prices, while elements of $\sigma_{S t}$ are referred to as risk quantities.⁶ As we will show, elements of $\sigma_{\pi t}$ are determined based on consumption growth, the state variables and the primitive parameters of the utility function. The elements of $\sigma_{\pi t}$ can then be uncovered, up to scaling factors, through OLS regression of stock returns on consumption growth and on the state variables. The Poisson terms represent the expected comovement of the state-price density and the price of the asset during rare events. If a rare disaster coincides with a rise in the stock price, that stock will command a risk premium.

A large empirical literature in asset pricing tests (25), under various specifications for the underlying processes, without the Poisson terms, sometimes with the model restrictions discussed below and sometimes without. Theorem 5 suggests that such tests are misspecified. If risk is not purely Brownian, risk premia need not be linear functions of normal-times covariances. In a recent paper (Tsai and Wachter, 2016), we calibrate a model of rare events to show that this result can account for the value premium. Note also that the convenient separation between prices and quantities of risk, which holds for diffusion processes, fails in the case Poisson risk.⁷

⁶ Below, we will argue that it may make sense to refer to $(\sigma_{\pi t})_k$ rather than $-(\sigma_{\pi t})_k$ as the risk price for for Brownian k depending on whether or not an increase in B_{kt} improves utility.

⁷The above discussion associates risk prices with Brownian shocks. Alternatively, one may associate risk prices with state variables and with consumption. Going between the two is straightforward. To find the prices of risk associated with the state variables and consumption, project σ_S and σ_π on the $(n+1) \times (n+1)$ matrix $\sigma = [\sigma_c e_1, \sigma_X^\top]^\top$. Thus the $(n+1) \times 1$ vector of risk prices is $\sigma_\pi \sigma^\top (\sigma \sigma^\top)^{-1}$ and the $1 \times (n+1)$ vector of risk quantities is $\sigma_S \sigma^\top (\sigma \sigma^\top)^{-1}$. While we continue to use risk prices associated with shocks for convenience, are results hold for this alternative definition.

In what follows, we will emphasize the importance of the compensation for the Poisson risk in (25). The literature has, recently, made progress on identifying this risk. Bai, Hou, Kung, and Zhang (2018) show that anomalies become easier to explain if one includes data over disaster periods. Chabi-Yo, Ruenzi, and Weigert (2017) identify the lower tail dependence of assets on the market portfolio, and show that stocks with high dependence have significantly greater returns than stocks with low dependence. Cremers, Halling, and Weinbaum (2015) and Lu and Murray (2017) identify variation in the probability of disasters using option prices. They show that assets that covary with this probability command higher risk premia. These results support the implications of Corollaries 6 and 8 in what follows, namely that rare-event risk determines expected returns even after controlling for traditional measures such as market or consumption betas.

For the remainder of this section, we assume the endowment process given in (1) and (2) with utility given by (5–8).⁸

2.4.1 Risk premia in the consumption-based model

A natural benchmark is the Consumption CAPM of Breeden (1979). In what follows, we show our results generalize the results in that paper.

Corollary 6 (Rare-event Consumption CAPM). *Risk premia are given by (25) with*

$$\sigma_{\pi t} = \left[-\gamma \sigma_{ct}, \left(\frac{1}{\psi} - \gamma \right) \frac{1}{I} \frac{\partial I}{\partial x} \sigma_{Xt} \right] \quad (26)$$

and

$$\mathcal{J}_{\pi}(x, Z_c, Z_X) = \left(\frac{I(x + Z_X)}{I(x)} \right)^{\frac{1}{\psi} - \gamma} e^{-\gamma Z_c} - 1.$$

Consider the case of no preference for early resolution of uncertainty $\frac{1}{\psi} = \gamma$ and no rare events $\lambda_j = 0$. The model reduces to that of Breeden (1979). If we allow for rare events but continue to assume no preference for early resolution of uncertainty, the model reduces

⁸We assume that $I(X_t) > 0$ for all realizations of X_t . This is a natural assumption given the form of the value function in (9), and it is true in the affine jump-diffusion case explored in the following section.

to a rare-event consumption CAPM, in which what matters is not only covariance with consumption, but covariance with the marginal utility during disaster periods. Because these shocks are not small, they cannot be reduced to the covariance with consumption multiplied by the coefficient of relative risk aversion. This insight is what allows disaster-risk models to explain the equity premium: CRRA implies that large shocks are felt to a greater extent than what we would find a pure quadratic setting.

We consider a shock an improvement in the distribution of future consumption growth if it makes an agent better off. In what follows, we refer to a shock to the k th component of the vector Brownian motion B_{Xt} as Brownian shock k .

Definition 1. *A shock to Brownian motion k , $(dB_{Xt})_k > 0$ is an improvement in the distribution of future consumption growth if and only if $(\frac{\partial J}{\partial X}\sigma_X)_k > 0$, where $(\frac{\partial J}{\partial X}\sigma_X)_k$ is the k th component of the row vector $\frac{\partial J}{\partial X}\sigma_X$.*

Definition 1 allows for a shock to affect multiple state variables through the matrix of loadings σ_X . In the special case where state variables are uniquely identified with a Brownian motion, then σ_X is diagonal, and a shock to a variable is an improvement if and only if it increases the value function.

Definition 2. *If $(dB_t)_k > 0$ constitutes an improvement in the distribution of future consumption growth, then $-(\sigma_{\pi t})_k$ is the price of risk for Brownian shock k . If $(dB_t)_k > 0$ constitutes a deterioration, then, $(\sigma_{\pi t})_k$ is the price of risk.*

While the definition doesn't formally cover the case of shocks to B_{ct} , clearly positive Brownian shocks to consumption increase utility, and hence have a positive price of risk.

Consider, for example, the case where there are shifts to average future consumption growth through μ_c . These lead to higher consumption growth on average, and hence are an improvement. The price of risk for this shock would be the negative of the marginal utility response. Now consider the probability of a rare disaster. Shocks to this variable lead future consumption to be lower and more risky. The price of risk for this shock would

be the marginal utility response itself. The benefit of our definition is that, given a fixed set of preference parameters, both of these shocks would have the same sign.

Given these definitions, there is a general result linking the utility function parameters to the risk prices.

Corollary 7. *A shock to the distribution of future consumption growth has a positive price of risk if and only if $\gamma > 1/\psi$.*

Proof. The result follows from the prices of risk (26), and from the fact that, given (9), the sign of $(\frac{\partial J}{\partial X}\sigma_X)_k$ is equal to the sign of $\frac{1}{I_t}(\frac{\partial I}{\partial X}\sigma_X)_k$. \square

Namely, if the agent has a preference for an early resolution of uncertainty, then shocks to the distribution of consumption are associated with a positive risk premium. If there is no preference for the timing of the resolution of uncertainty then these shocks do not have a risk premium.

Why is it that pricing of the state variables depends on the preference for early resolution of uncertainty? To understand the appearance of these additional terms with recursive utility, one must first understand the *disappearance* of the terms when utility is time-additive, the central insight of Breeden (1979). Because of complete markets, we first think of the investor as choosing an optimal consumption process. All that matters, when weighing the desire to purchase contingent claims, is setting the marginal utility of this consumption process against the marginal cost, namely the price, of the claim. Because of the symmetry of risk and time, marginal utility depends only on consumption itself. If we view the dynamic process of consumption as representing a compound lottery only outcomes of this compound lottery are what needs to be hedged, not the manner in which uncertainty is resolved.

Under recursive utility, the desire to smooth consumption across states and time is of course not the same – the agent has a dislike for uncertainty itself. This manifests itself as an additional term in marginal utility. If the risk of negative outcomes increases, the agent enjoys current consumption less, and marginal utility is higher. The agent considers

this when purchasing contingent claims. Claims are more expensive if they fail to pay off in states of the world when uncertainty is high.⁹

2.4.2 Risk premia in the wealth-based model

The theorems in Section 2.4.1 show additional terms relative to the consumption CAPM. These additional terms depend on the preference for the early resolution of uncertainty. However, many studies use the market portfolio rather than consumption in cross-sectional regressions because asset returns are less noisy. Indeed, the original ICAPM of Merton (1973) is written purely in terms of asset returns. Campbell (1993) derives an ICAPM in a discrete-time homoskedastic setting with recursive utility; this is extended by Campbell, Giglio, Polk, and Turley (2018) to allow for heteroskedasticity. An ICAPM also holds in our general setting. The sign of prices of risk on the state variables no longer depends on a preference for an early resolution of uncertainty, but rather on whether risk aversion exceeds one.

Using (9) and (12) we rewrite the value function as a function of wealth and of X_t :

$$J(C_t, X_t) = J\left(W_t \frac{1}{G^c(X_t)}, X_t\right) = \beta^{1-\gamma} \frac{W_t^{1-\gamma}}{1-\gamma} I(X_t)^{\frac{1}{\psi}(1-\gamma)}. \quad (27)$$

Likewise, from (15) it follows that

$$\begin{aligned} \pi_t &= \exp\left\{-\beta \int_0^t \left((1-\theta)I(X_s)^{\frac{1}{\psi}-1} + \theta\right) ds\right\} \beta^{1-\gamma} W_t^{-\gamma} G^c(X_t)^\gamma I(X_t)^{\frac{1}{\psi}-\gamma} \\ &= \exp\left\{-\beta \int_0^t \left((1-\theta)I(X_s)^{\frac{1}{\psi}-1} + \theta\right) ds\right\} \beta^{1-\gamma} W_t^{-\gamma} I(X_t)^{\frac{1}{\psi}(1-\gamma)} \end{aligned} \quad (28)$$

⁹Another interpretation (Hansen, 2012) relates the concept of uncertainty preference to the idea of consumption as a durable good. That is, an agent with a preference for early resolution of uncertainty behaves similar to one who consumes at a lower frequency. Intuitively, if one is more willing to substitute across time than over states, then exactly when this consumption takes place is less important. Under this interpretation, the extra term in (15) reflects marginal utility over long-term consumption rather than immediate consumption because of the information it contains about the consumption distribution. What then determines risk premia are risks to consumption over multiple periods.

Moreover, from $W_t = C_t G^c(X_t)$, it follows that wealth evolves according to

$$\frac{dW_t}{W_{t-}} = \mu_{wt-} dt + \sigma_{wt-} dB_t + \sum_{j=1}^m (e^{Z_{wjt}} - 1) dN_{jt},$$

where

$$\sigma_{wt} = \left[\sigma_{ct}, \left(1 - \frac{1}{\psi}\right) \frac{1}{I(X_t)} \frac{\partial I}{\partial X} \sigma_{Xt} \right] \quad (29)$$

and

$$Z_{wjt} = Z_{cjt} + \left(1 - \frac{1}{\psi}\right) \log \left(\frac{I(X_{t-} + Z_{Xjt})}{I(X_{t-})} \right). \quad (30)$$

In the approximate affine model of the following section, Z_{wjt} will not depend on X_{t-} . A rare-event ICAPM then follows from Theorem 5:

Corollary 8 (Rare-event ICAPM). *Risk premia satisfy the following:*

$$\begin{aligned} r_t^S - r_t &= \gamma \sigma_{wt} \sigma_{St}^\top \\ &+ (1 - \gamma) \left[0, -\frac{1}{\psi} \frac{1}{I} \frac{\partial I}{\partial X} \sigma_{Xt} \right] \sigma_{St}^\top - \sum_{j=1}^m \lambda_{jt} E_{\nu_j} [\mathcal{J}_{\pi t}(Z_{wj}, Z_{Xj}) \mathcal{J}_{St}(Z_{dj}, Z_{Xj})] \end{aligned} \quad (31)$$

where (for a given X_t , Z_{cj} , and Z_{Xj}) Z_{wjt} is defined as (30) and \mathcal{J}_π can be expressed as

$$\mathcal{J}_\pi(x, Z_w, Z_X) = \left(\frac{I(x + Z_X)}{I(x)} \right)^{\frac{1}{\psi}(1-\gamma)} e^{-\gamma Z_w} - 1. \quad (32)$$

Note the parallels between the role of wealth and of consumption in this ICAPM versus the consumption CAPM. Also note the parallels between the compensation for Poisson risk and for Brownian risk in (31) and (32).

Consider first the special case with only Brownian risk. Then Corollary 8 is precisely the ICAPM of Merton (1973), derived under the more general condition of recursive utility. Risk premia are a weighted average of the covariance of returns with aggregate wealth and the covariance of returns with investment opportunities. If an asset rises in price when investment opportunities increase, the investor with $\gamma > 1$ demands less of the asset than

he would otherwise. In equilibrium, the risk premium rises to compensate. The investor with $\gamma < 1$ seeks to hold more than otherwise, and the risk premium falls to compensate.

Now consider the special case when $\gamma = 1$. Then a rare-event CAPM holds. Risk premia depend only on the covariance with wealth during normal times and during rare events. This holds regardless of the value of the EIS. When $\gamma \neq 1$ and there is Poisson risk, a rare-event ICAPM holds, in which the covariance of investment opportunities and asset returns during disasters also have the potential to determine risk premia.

We derive a wealth-based analogue of Corollary 7. The analogue to the distribution of future consumption growth is the investment opportunity set. We can define whether a shock is an improvement to the investment opportunity set using the identical value function criterion in Definition 1. We can also define the price of risk using the analogous criterion in Definition 2.¹⁰

Corollary 9. *Define prices of risk as in Definitions 1 and 2. Then a shock to the investment opportunity set has a positive price of risk in the wealth-based model if and only if risk aversion is greater than 1.*

Why the difference between the comparative statics in Corollary 7 and Corollary 9? It is because wealth, or, depending on one's point of view, consumption, already contains an endogenous response of the agent to changes in investment opportunities. This change is mediated through the elasticity of intertemporal substitution. The following corollary follows directly from (12) and from (9).

Corollary 10. *When $\psi > 1$, a shock representing an improvement in investment opportunities decreases the wealth-consumption ratio and a shock representing a deterioration increases the ratio. The opposite holds when $\psi < 1$*

¹⁰Specifically, if $(dB_t)_k > 0$ constitutes an improvement in the investment opportunity set, then $\frac{1}{\psi}(1 - \gamma) \frac{1}{I(X_t)} \frac{\partial I}{\partial X} \sigma_{X_t}$ is the price of risk. If $(dB_t)_k < 0$ constitutes an improvement, then the negative of this quantity is the price of risk.

3 Affine Model

We now make assumptions on the dynamics that imply exact solutions for the cases $\psi = 1$ and $\psi = \frac{1}{\gamma}$, and approximate solutions otherwise. We assume that drift and volatility in the C_t and X_t processes, as well as the jump probability, are affine functions of the state variables X_t . That is, for a column vector x of length n , define

$$\mu_c(x) = k_0 + k_1x \quad (33a)$$

$$\sigma_c^2(x) = u_0 + u_1x \quad (33b)$$

$$\mu_X(x) = K_0 + K_1x \quad (33c)$$

$$(\sigma_X(x)\sigma_X(x)^\top)_{ij} = (U_0)_{ij} + (U_1)_{ij}x \quad (33d)$$

$$\lambda(x) = l_0 + l_1x, \quad (33e)$$

where k_0 and u_0 are scalars, k_1 and u_1 are $1 \times n$, l_0 is $m \times 1$, l_1 is $m \times n$, K_0 is $n \times 1$, K_1 and U_0 are $n \times n$ matrices, and U_1 can be thought of as an $n \times n \times n$ matrix in a sense that will be made more precise below.

Finally, l_0 is a column vector of length m . This is similar to the affine structure defined by Duffie, Pan, and Singleton (2000), except in that case it is a specification of the endowment process rather than the discount rate. This structure can accommodate time-varying rare disasters as in Wachter (2013), as well as time-variation in the mean and standard deviation of consumption growth, as in Bansal and Yaron (2004). The model can accommodate rare events that affect the mean and standard deviation of the consumption growth process (Benzoni, Collin-Dufresne, and Goldstein, 2011; Drechsler and Yaron, 2011; Tsai and Wachter, 2016), and self-exciting jumps (Nowotny, 2011). The model can also accommodate a stationary dividend-consumption ratio, while still allowing dividends to temporarily respond more to disasters (Longstaff and Piazzesi, 2004). Like Eraker and Shaliastovich (2008), we solve for prices in a general continuous-time affine economy; we depart from their approach in that we do not use the standard approximation to the price-dividend

ratio.

Assumption (33) place requirements on the functional form of conditional means, variances, and covariances for the processes C_t and X_t . Specifically, consider (33d). From (2), it follows that $\sigma_X(x)\sigma_X(x)^\top$ is the normal-times conditional variance of X_t . That is, it is the instantaneous variance over an interval without rare events. Given two linear combinations of X_t , $a^\top X_t$ and $b^\top X_t$, for column vectors a, b , the normal-times conditional covariance of $a^\top X_t$ with $b^\top X_t$ is $a^\top \sigma_X(x)\sigma_X(x)^\top b$. Assumption (33d) implies that this conditional covariance is linear in x . To see this, note that from (33d) it follows that

$$a^\top \sigma(x)\sigma(x)^\top b = \sum_{i,j} a_i(\sigma(x)\sigma(x)^\top)_{i,j} b_j \quad (34)$$

$$= \sum_{i,j} a_i(U_0)_{ij} b_j + \sum_{i,j} a_i((U_1)_{ij}x) b_j. \quad (35)$$

For a fixed $i, j = 1, \dots, N$, $(U_1)_{ij}$ is a row vector. Let $(U_1)_{ij,k}$ be the k th element of this row vector so that

$$a_i((U_1)_{ij}x) b_j = \sum_k a_i(U_1)_{ij,k} x_k b_j = \sum_k a_i(U_1)_{ij,k} b_j x_k$$

From (35), it then follows that

$$a^\top \sigma(x)\sigma(x)^\top b = a^\top U_0 b + a^\top U_1 b x,$$

where $a^\top U_1 b$ is formally defined to be the column vector with k th element $a_i(U_1)_{ij,k} b_j$.¹¹

To complete the specification, we consider dividend-paying assets with cash flows of the

¹¹In (33d), we assume the existence of a solution $\sigma_X(X_t)$ for all X_t ; otherwise the variance matrix for X_t is not well-defined. This restriction allows for a rich set of cases. For example, X_t could follow a multivariate Cox, Ingersoll, and Ross (1985) process with uncorrelated shocks, but with interactions of the variables introduced through K_1 . Then $U_0 = 0$ and the column vector $(U_1)_{ij} \propto e_j^\top$ if $i = j$ and zero otherwise. With additional restrictions X_t will be non-negative and there will exist a $\sigma_X(X_t)$ satisfying (33d). Other examples include processes that are a mix of homoskedastic multivariate Gaussian terms and square root terms. Examples include models in Huang and Kilic (2018) and Seo and Wachter (2016).

form (3), where

$$\mu_d(x) = k_0^d + k_1^d x, \quad (36)$$

$$\sigma_c(x)\sigma_d(x) = u_0^{cd} + u_1^{cd} x, \quad (37)$$

with scalar k_0^d and u_0^{cd} , with k_1^d and u_1^{cd} row vectors in \mathbf{R}^n .

In what follows, define a $m \times 1$ vector $Z_c = [Z_{c1}, \dots, Z_{cm}]^\top$ and an $m \times n$ matrix $Z_X = [Z_{X1}, \dots, Z_{Xm}]^\top$. Given a vector x , we use e^x to denote the exponential of each element in x . To evaluate expressions at $\gamma = 1$, apply $\lim_{\gamma \rightarrow 1} \frac{1}{1-\gamma} e^{(1-\gamma)y} - 1 = y$.

3.1 Value function

We first solve for the value function. Theorem 11 shows that the PDE (10) has an exact solution in the cases $\psi = 1$ and $\psi = 1/\gamma$. Outside of these cases, there is no known exact solution.

We can, however, find an approximate analytical solution. We approximate the wealth-consumption ratio by a log-linear function of the state variables, following Campbell and Shiller (1988) and Campbell and Viceira (1999). In a continuous-time setting, the analogue of the discrete-time Euler equation is the PDE for the value function. Chacko and Viceira (2005) show that, by log-linearizing the expression for the wealth-consumption ratio within this PDE, one obtains a PDE that admits an analytical solution. In this section, we generalize their results.

Theorem 11. *The value function takes the form (9), with*

$$I(x) \simeq \exp\{a + b^\top x\}, \quad (38)$$

where a is a scalar and b is $n \times 1$. When $\psi \neq 1$,

$$a = \frac{1}{i_1} \left(\frac{1}{1 - 1/\psi} (i_1 \log \beta + i_0 - \beta) + k_0 - \frac{1}{2} \gamma u_0 + b^\top K_0 + \frac{1}{2} (1 - \gamma) b^\top U_0 b + \frac{1}{1 - \gamma} \left(E_\nu [e^{(1-\gamma)(Z_c + Z_X b)} - 1] \right)^\top l_0 \right), \quad (39)$$

and

$$\frac{1}{2} (1 - \gamma) b^\top U_1 b - i_1 b^\top + b^\top K_1 + k_1 - \frac{1}{2} \gamma u_1 + \frac{1}{1 - \gamma} \left(E_\nu [e^{(1-\gamma)(Z_c + Z_X b)} - 1] \right)^\top l_1 = 0, \quad (40)$$

with $i_1 = e^{E[\log(\beta I(X_t)^{\frac{1}{\psi}-1})]}$ and $i_0 = i_1(1 - \log i_1)$. For $\psi = 1$, (38) is exact, and (39) and (40) hold with $\lim_{\psi \rightarrow 1} i_1 = \beta$ and $\lim_{\psi \rightarrow 1} \frac{1}{1 - \frac{1}{\psi}} (i_1 \log \beta + i_0 - \beta) = 0$. Given (38), the wealth-consumption ratio follows from Corollary 2.

It follows immediately from (40) that the vector b is nonzero if and only if at least one state variable affects the consumption distribution directly, either through the mean (k_1), the variance u_1 , or the jump probability l_1 .

3.2 State price density

We next characterize the state-price density and the riskfree rate.

Theorem 12. *The state-price density is given by*

$$\frac{d\pi_t}{\pi_{t-}} = \mu_{\pi t-} dt + \sigma_{\pi t-} dB_t + \sum_{j=1}^m (e^{Z_{\pi j}} - 1) dN_{jt}, \quad (41)$$

where

$$\sigma_{\pi t} \simeq \left[-\gamma \sigma_{ct}, \left(\frac{1}{\psi} - \gamma \right) b^\top \sigma_{Xt} \right]^\top \quad (42)$$

and

$$Z_{\pi j} \simeq -\gamma Z_{cj} + \left(\frac{1}{\psi} - \gamma \right) b^\top Z_{Xj}. \quad (43)$$

where b is given in Theorem 11. Furthermore,

$$\mu_{\pi t} = -r_t - \sum_{j=1}^m \lambda_{jt} E_{\nu_j} [e^{Z_{\pi j}} - 1]$$

where r_t , the riskfree rate, is given by

$$\begin{aligned} r_t \simeq & \beta + \frac{1}{\psi} (k_0 + k_1 X_t) - \frac{1}{2} \gamma \left(1 + \frac{1}{\psi} \right) (u_0 + u_1 X_t) - \frac{1}{2} \left(\gamma - \frac{1}{\psi} \right) \left(1 - \frac{1}{\psi} \right) (b^\top U_0 b + (b^\top U_1 b) X_t) \\ & + \left(E_\nu \left[\left(1 - \frac{1}{\theta} \right) (e^{(1-\gamma)(Z_c + Z_X b)} - 1) - \left(e^{-\gamma Z_c + (\frac{1}{\psi} - \gamma) Z_X b} - 1 \right) \right] \right)^\top (l_0 + l_1 X_t). \end{aligned} \quad (44)$$

The approximations are exact in the case of $\psi = 1$ and $\gamma = \frac{1}{\psi}$.

This theorem shows that derivatives with respect to the value function in Corollaries 6 and 8 can be replaced with the simpler constant vector b . Moreover, the rare events' impact on marginal utility π_t can be replaced by the simpler expression (43).

3.3 Equity prices in the affine model

The value of equity is an integral of expected future dividends paid increasingly far in the future. That equity takes the form (18) is powerful information that we can exploit to derive an analytical solutions. Rather than solving for (18) all at once, we can solve for individual components. These components have an economic interpretation: they are prices of future dividends paid at specific points in time, otherwise known as equity strips (Lettau and Wachter, 2007). Once we have these components, the solution for the overall price is a one-dimensional integral over the horizon, where the integrand is a well-behaved function.

It is here that we depart most significantly from the previous literature in our solution method. The previous literature uses log-linearization to compute the price-dividend ratio (see Appendix B). However, by applying the economic insight that the market is an integral of expected future dividends, we can reduce the problem from two approximations to only

one, and give exact analytical solutions in the special cases of time-additive utility and $\psi = 1$.

The following theorem gives the value of the equity strip with maturity τ .

Theorem 13. *Let $H(D, x, \tau)$ denote the price of an asset that pays dividend D , τ years in the future. Then*

$$H(D, x, \tau) \simeq D \exp \{ a_\phi(\tau) + b_\phi(\tau)^\top x \}, \quad (45)$$

where functions $a_\phi(\tau) : [0, \infty) \rightarrow \mathbf{R}$ and $b_\phi(\tau) : [0, \infty) \rightarrow \mathbf{R}^n$ solve

$$\begin{aligned} \frac{\partial a_\phi(\tau)}{\partial \tau} &= k_0^d - \frac{1}{\psi} k_0 - \beta + \frac{1}{2} \gamma \left(1 + \frac{1}{\psi} \right) u_0 - \gamma u_0^{cd} + b_\phi(\tau)^\top K_0 \\ &+ \frac{1}{2} \left(1 - \frac{1}{\psi} \right) \left(\gamma - \frac{1}{\psi} \right) b^\top U_0 b + \frac{1}{2} b_\phi(\tau)^\top U_0 b_\phi(\tau) + \left(\frac{1}{\psi} - \gamma \right) b_\phi(\tau)^\top U_0 b \\ &+ \left(E_\nu \left[\left(\frac{1}{\theta} - 1 \right) \left(e^{(1-\gamma)(Z_c + Z_X b)} - 1 \right) + \left(e^{-\gamma Z_c + Z_a + Z_X (b_\phi(\tau) + (1/\psi - \gamma)b)} - 1 \right) \right] \right)^\top l_0, \end{aligned} \quad (46)$$

and

$$\begin{aligned} \left(\frac{\partial b_\phi(\tau)}{\partial \tau} \right)^\top &= k_1^d - \frac{1}{\psi} k_1 + \frac{1}{2} \gamma \left(1 + \frac{1}{\psi} \right) u_1 - \gamma u_1^{cd} + b_\phi(\tau)^\top K_1 \\ &+ \frac{1}{2} \left(1 - \frac{1}{\psi} \right) \left(\gamma - \frac{1}{\psi} \right) b^\top U_1 b + \frac{1}{2} b_\phi(\tau)^\top U_1 b_\phi(\tau) + \left(\frac{1}{\psi} - \gamma \right) b_\phi(\tau)^\top U_1 b \\ &+ \left(E_\nu \left[\left(\frac{1}{\theta} - 1 \right) \left(e^{(1-\gamma)(Z_c + Z_X b)} - 1 \right) + \left(e^{-\gamma Z_c + Z_a + Z_X (b_\phi(\tau) + (1/\psi - \gamma)b)} - 1 \right) \right] \right)^\top l_1. \end{aligned} \quad (47)$$

The approximations are exact if utility is time-additive or if $\psi = 1$.

Given the above result, the price of the claim to all future dividends follows immediately.

Corollary 14. *Let $S(D_t, X_t)$ denote the time t price of an asset that pays the stream of dividends given by (3), then*

$$S(D_t, X_t) = \int_0^\infty H(D_t, X_t, \tau) d\tau = D_t G(X_t), \quad (48)$$

where H is defined in (45) and G , the price-dividend ratio, satisfies

$$G(X_t) \simeq \int_0^\infty e^{a_\phi(\tau) + b_\phi(\tau)^\top X_t} d\tau. \quad (49)$$

The approximation is exact if utility is time-additive or if $\psi = 1$.

While we have written Theorems 12 and 13, as well as Corollary 14 in terms of approximations, both results are exact given the state-price density in Theorem 12. Furthermore, these results are all exact in the case of $\psi = 1$ and time-additive utility.¹²

Corollary 14 applies to the asset that pays aggregate consumption as dividend, that is, $\mu_d = \mu_c$, $\sigma_d = \sigma_c$ and $Z_{dj} = Z_{cj}$ for all $j = 1, \dots, m$. It follows that this theorem provides an alternative way to solve for the wealth-consumption ratio $G^c(X_t)$. Indeed, when we set $\psi = 1$ in the equations in this theorem, we find $a_\phi(\tau) = -\beta\tau$ and $b_\phi(\tau) = 0$, verifying that $G^c(X_t) = \beta^{-1}$. However, when $\psi \neq 1$, the wealth consumption ratio calculated using (49), is not the same as (12). Which one is a better approximation? We return to this question in Section 4.

We now turn to risk premia. Despite the potential complexity of this model, risk premia for equity strips always take a simple form. First note that the expected return on zero-coupon equity is

$$r^{H(\tau)}(x) = \mu_{H(\tau)}(x) + \lambda(x)^\top \bar{J}_{H(\tau)}(x)$$

Corollary 15. 1. Consider the claim to the dividend τ years in the future, where dividends follow the process (3). The risk premium on this claim equals

$$\begin{aligned} r_t^{H(\tau)} - r_t &\simeq \gamma \sigma_{ct} \sigma_{dt} - \left(\frac{1}{\psi} - \gamma \right) b^\top \sigma_{Xt} \sigma_{Xt}^\top b_\phi(\tau) \\ &\quad - \sum_j \lambda_{jt} E_{\nu_j} \left[\left(e^{-\gamma Z_{cj} + \left(\frac{1}{\psi} - \gamma \right) b^\top Z_{\pi j}} - 1 \right) \left(e^{Z_{dj} + b_\phi(\tau)^\top Z_{Xj}} - 1 \right) \right] \end{aligned} \quad (50)$$

¹²Note, however, that Theorem 11 is approximate in the case of time-additive utility. The distinction is that it is not necessary to obtain the value function to price securities when utility is time-additive. The value function does have an exact solution in this case, but we do not give it here.

2. Consider the claim to the stream of dividends. The risk premium on this claim equals

$$r_t^S - r_t \simeq \gamma \sigma_{ct} \sigma_{dt} - \left(\frac{1}{\psi} - \gamma \right) b^\top \sigma_{Xt} \sigma_{Xt}^\top \frac{1}{S} \left(\frac{\partial S}{\partial X} \right)^\top - \sum_j \lambda_{jt} E_{\nu_j} \left[\left(e^{-\gamma Z_{cj} + \left(\frac{1}{\psi} - \gamma \right) b^\top Z_{\pi j}} - 1 \right) \mathcal{J}_{St}(Z_{dj}, Z_{Xj}) \right], \quad (51)$$

where

$$\frac{1}{S} \frac{\partial S}{\partial x} = \frac{1}{G} \frac{\partial G}{\partial x} = \int_0^\infty \frac{e^{a_\phi(\tau) + b_\phi(\tau)^\top x}}{\int_0^\infty e^{a_\phi(u) + b_\phi(u)^\top x} du} b_\phi(\tau) d\tau \quad (52)$$

and

$$\mathcal{J}_S(x, Z_d, Z_X) = \int_0^\infty \frac{e^{a_\phi(\tau) + b_\phi(\tau)^\top x}}{\int_0^\infty e^{a_\phi(u) + b_\phi(u)^\top x} du} (e^{Z_d + b_\phi(\tau)^\top Z_X} - 1) d\tau \quad (53)$$

These expressions are exact if utility is time-additive or $\psi = 1$.

Proof The first result follows from the general expression for risk premia in Theorem 5, the expression for the state-price density (Theorem 12), and the price of the asset (Theorem 13) in the affine case. The second result follows from similar reasoning, except that we apply Corollary 14 in place of Theorem 13. \square

The complex asset S_t is a portfolio of zero-coupon claims. The weighted-average forms of (52) and (53) reflect this.

3.4 A guide to using these results

The previous results suggest the following step-by-step guide for tackling a solving multi-dimensional dynamic asset pricing economy.

1. Model the underlying stochastic structure as in (33).
2. Compute the stochastic process for the state-price density in Theorem 12, namely $\sigma_{\pi t}$ and $Z_{\pi j}$. This requires computation of the coefficient b , which is given explicitly in Theorem 11.

3. Compute the riskfree rate (also given in Theorem 12).
4. Calculate the price of assets of interest (Theorem 13 and Corollary 14).
5. Using the price, Ito's Lemma gives the stochastic process, and risk premia are given, in closed form, by Theorem 5).
6. For unconditional moments, one can readily simulate from the process in (33) (approximating the continuous-time dynamics with a discrete-time process as necessary), calculate a time series of price-dividend ratios from (49), dividend growth from (3), and from there, a series of returns.¹³

A strength of the method is how it lends itself, not only to closed-form solutions, but also to simulation. Given a simulated series of consumption growth, dividend growth, returns, and scaled prices, one can compute any unconditional moments. Because prices are available in closed form (up to the solution of ODEs), this method is very fast and does not suffer from the curse of dimensionality.

3.5 Existence of solutions

Thus far we have assumed existence of the solution to the agent's maximization problem.¹⁴ Pohl, Schmedders, and Wilms (2015) give examples in which such solutions fail to exist, but where one can derive an apparent approximate solution (which would have no valid economic interpretation). They also provide a simple diagnostic that helps in determining existence. They prove that, if the wealth-consumption ratio exists for time-additive utility at a given EIS ψ , then it will also exist for recursive utility with that ψ and $\gamma \geq 1/\psi$. Namely, at a given level of curvature, a preference for early resolution of uncertainty makes the problem more well-behaved. They show their result in a discrete-time economy in which state variables are conditionally normally distributed. It would be of interest to

¹³For details, see Tsai and Wachter (2016), Appendix C.

¹⁴See Schroder and Skiadas (1999) for existence theorems in a portfolio choice setting.

extend their results to the present setting, which allows for non-Gaussian shocks, and is of course in continuous time. Such a result would be very useful because, as Theorem 11 states, the time-additive utility case can be solved in closed-form, and one need not worry about phantom solutions created by log-linearization. Moreover, the overwhelming majority of empirical applications assume early resolution of uncertainty.¹⁵

Even if a solution to (10) exists, it is possible that prices of dividends do not exist at all maturities. This can occur if one is pricing long-term bonds (so the dividends in question are interest payments that are fixed) in a model with time-varying disaster risk. The problem is that long-term real default-free bonds are very safe, having negative risk premia, and infinite prices (Wachter, 2013). The problem does not occur when the “dividends” are actual dividends, namely risky cash flows for which a premium is required. Finally, even if a solution for the zero-coupon claims exists, a third difficulty is that the indefinite integral in (48) may fail to converge because the price of these claims grows exponentially as the horizon increases. In a parametric example, Tsai and Wachter (2016) show that it suffices to guarantee that $b_\phi(\tau)$ converges to a fixed value as $\tau \rightarrow \infty$ and that $\lim_{\tau \rightarrow \infty} a(\tau)/\tau < 0$. The latter condition implies that $H(X_t, \tau)$ decays geometrically, and hence that the indefinite integral (essentially an infinite sum) converges.

4 Numerical accuracy in a calibrated economy

As an example of these techniques, consider a model with time-varying risk of disaster. Assume $n = m = 1$ and $X_t = \lambda_t$, the disaster probability. Furthermore, let $\mu_X(x) = \kappa_\lambda(\bar{\lambda} - x)$ and $\sigma_X(x) = \sigma_\lambda\sqrt{x}$. We set $Z_{Xt} = 0$, and $Z_{dt} = \phi Z_{ct}$. We assume $Z_{ct} < 0$.

¹⁵It is likely that their results do extend, as they rely simply on the characterization of the wealth-consumption ratio as the solution to a fixed-point problem, and the Cauchy-Schwartz inequality.

Equations 39 and 40 have closed-form solutions with

$$a = \frac{1}{i_1} \left(\left(1 - \frac{1}{\psi}\right)^{-1} (i_1 \log \beta + i_0 - \beta) + \mu - \frac{1}{2} \gamma \sigma^2 + b \kappa \bar{\lambda} \right), \quad (54)$$

$$b = \frac{(\kappa + i_1) - \sqrt{(\kappa + i_1)^2 - 2\sigma_\lambda^2 E_\nu [e^{(1-\gamma)Z_c} - 1]}}{(1-\gamma)\sigma_\lambda^2}. \quad (55)$$

When $\psi = 1$, these equations reduce to those in Wachter (2013).¹⁶ It follows from (55) that $b < 0$ regardless of the preference parameters. Therefore, an increase in the probability of a rare disaster always decreases the investor's utility. Applying the definitions from Section 2.4, the price of risk for λ_t (relative to the CCAPM) is equal to $(\sigma_{\pi t})_2 = \left(\frac{1}{\psi} - \gamma\right) b \sigma_\lambda \sqrt{\lambda_t}$, and thus is positive if and only if $\gamma > \frac{1}{\psi}$.

Applying Corollary 2, we find the following approximation for the wealth-consumption ratio:

$$G^c(\lambda_t) \simeq \beta^{-1} \exp \left\{ \left(1 - \frac{1}{\psi}\right) (a + b \lambda_t) \right\}. \quad (56)$$

It follows from Theorem 5 that the premium for bearing λ_t -risk is positive if $\psi > 1$ and $\gamma > 1/\psi$ or if $\psi < 1$ and $\gamma < 1/\psi$. In the former case, the wealth-consumption ratio decreases in λ_t , and the agent prefers an early resolution of uncertainty (so the price of λ_t -risk is positive). In the latter case, the wealth-consumption ratio increases in λ_t and the agent prefers a late resolution of uncertainty (so the price of λ_t -risk is negative).

It follows from Corollary 14 that the price-dividend ratio satisfies

$$G(\lambda_t) \simeq \int_0^\infty e^{a_\phi(\tau) + b_\phi(\tau)^\top \lambda_t} d\tau. \quad (57)$$

¹⁶ a and b in this paper are equal to a and b in the previous paper divided by $1 - \gamma$.

where

$$a'_\phi(\tau) = \mu_D - \frac{1}{\psi}\mu - \beta + \left(\frac{1}{2}\left(1 + \frac{1}{\psi}\right) - \phi\right) \gamma\sigma^2 + \kappa\bar{\lambda}b_\phi(\tau) \quad (58)$$

$$b'_\phi(\tau) = \frac{1}{2}\sigma_\lambda^2 b_\phi(\tau)^2 + \left(\left(1 - \frac{1}{\theta}\right)b\sigma_\lambda^2 - \kappa\right) b_\phi(\tau) + \frac{1}{2\theta} \left(\frac{1}{\theta} - 1\right) b^2\sigma_\lambda^2 + E_\nu \left[\left(\frac{1}{\theta} - 1\right) (e^{(1-\gamma)Z_t} - 1) + e^{(\phi-\gamma)Z_t} - 1 \right], \quad (59)$$

and $a_\phi(0) = b_\phi(0) = 0$.

We calibrate the model to the first two moments of aggregate market (modeled as the dividend claim) and Treasury bill returns, assuming $\psi = 2$. Data on the distribution for disasters come from Barro and Ursúa (2008). We use their benchmark cutoff of 10% to determine whether a consumption decline constitutes a disaster. To evaluate numerical accuracy under more extreme conditions, we also consider a cutoff of 15%. We assume $\bar{\lambda} = 0.0286$, the unconditional probability of a disaster in OECD countries as reported by Barro and Ursúa. See Tsai and Wachter (2015) for further discussion on the parameters.¹⁷ The model reconciles the high equity premium and return volatility with a low volatility of consumption growth and a risk aversion γ of 3.

Panel A of Figure 1 shows the wealth-consumption as a function of the disaster probability. We compare (56) with an exact solution computed using Chebyshev polynomials.¹⁸ The approximation is highly accurate for the 10% minimum disaster size calibration. It remains accurate even for the 15% cutoff.

Panel B of Figure 1 shows the price-dividend ratio as a function of the disaster probability. We compare (57) to the exact solution computed using Chebyshev polynomials. We also compare it to the standard log-linear approximation of the price-dividend ratio (see Appendix B). Note that while our method does require a log-linear approxima-

¹⁷Other parameters are as follows: discount rate $\beta = 0.01$, normal-times consumption growth $\mu = 0.0195$, consumption growth volatility $\sigma = 0.0125$, dividend growth $\mu_D = 0.04$, $\phi = 3$, mean-reversion $\kappa_\lambda = 0.12$, and volatility $\sigma_\lambda = 0.081$.

¹⁸We solve (A.18) using the Chebyshev polynomial method, and then convert the solution to the wealth-consumption ratio using (12).

tion to the wealth-consumption ratio to find the coefficients b , it does not require log-linearization thereafter. While (57) is nearly indistinguishable from the numerical solution, log-linearization is notably less accurate. That is, log-linearization appears to be accurate when applied to the wealth-consumption ratio, but much less so when applied to the price-dividend ratio. This makes sense, because the approximations are based on proximity to the long-run mean, and the price-dividend ratio exhibits much more variation than the wealth-consumption ratio. Fortunately, as we show, it is unnecessary to approximate the price-dividend ratio.

The inaccuracies in log-linearization become more apparent when one considers the equity premium. Using Corollary 15, we find the following expression for the risk premium on the dividend claim:

$$r_t^S - r_t = \underbrace{\gamma\phi\sigma_c^2 - \left(\frac{1}{\psi} - \gamma\right)\frac{1}{G}\frac{\partial G}{\partial\lambda}b\sigma_\lambda^2\lambda_t}_{\lambda\text{-premium}} - \lambda_t E_\nu \left[\left(e^{-\gamma Z_c} - 1 \right) \left(e^{\phi Z_c} - 1 \right) \right],$$

where

$$\frac{1}{G}\frac{\partial G}{\partial\lambda} = \int_0^\infty \frac{e^{a_\phi(\tau)+b_\phi(\tau)\lambda}}{\int_0^\infty e^{a_\phi(u)+b_\phi(u)\lambda} du} b_\phi(\tau) d\tau, \quad (60)$$

We consider two cases, one where $\psi = 1$ (in which case our method is exact) and the other where $\psi = 2$. The first panel shows the price-dividend ratios under the analytical method and the log-linear approximation method described in Appendix B. The second panel shows the λ -premium, which on average accounts for about half of the total equity premium.¹⁹

Approximating $G(\lambda_t)$ by a linear function implies that (60) is replaced with a constant coefficient. However, (60) is a concave function as Figure 2 shows. This concavity reflects an important economic effect that log-linearization obscures. The price-dividend ratio is a sum of claims of varying duration. When the disaster probability increases, claims to dividends in the long-term fall in price by more than claims to dividends in the short-term because, as in simple models of bond-pricing, higher duration implies greater sensitivity to

¹⁹The other components of the equity premium are constant in λ_t and so we leave them out.

state variables. These long-term claims have greater risk premia, again, because of their higher duration. As the disaster probability increases, risk premia on all claims increase, but claims on the long-term assets increase by more. At the same time, these assets have a lower weight in the overall market. Figure 2 shows that, as a result, not only does log-linearization imply an inaccurate relation between the state variable and the equity premium, it also overstates the equity premium. Finally, because the discount rate is overstated, it understates the level of the price-dividend ratio, as shown in Panel A.

Finally, we return to an issue raised the previous section. Our method suggests two ways of approximating the wealth-consumption ratio. The first method is to use (56). We apply this method in Panel A of Figure 1 and show it already to be already quite accurate. The second method is to treat the wealth-consumption ratio as a special case of the price-dividend ratio, using the log-linear approximation to compute the coefficients b , and then using (57).

We show the results in Figure 3, and compare them to the exact numerical solution. We refer to the first method as log-linearization and the second as analytical, to be consistent with the terminology in the previous figures. We consider $\gamma = 3$, and look at the case of $\psi = 3$ and $\psi = 1/3$. In the latter case, utility is time-additive and the second method is exact. The figure clearly shows that the second method, using (57), is a closer approximation to the true wealth-consumption ratio.

It may seem surprising that (57) achieves greater accuracy, given that it uses a log-linearization of the wealth-consumption ratio as an input. However, note that (57) does not require the entire solution for the wealth-consumption ratio. It only requires the coefficient b , because this is what enters into the state-price density. Indeed, in the special case of time-additive utility (in the right panel), the analytical method is exact because these responses are zero, as Theorem 12 shows. The time-additive case is an extreme manifestation of a more general result: the prices of risk are calculated more precisely than the level of the value function. It is the prices of risk that that enter the computation for asset prices more generally.

5 Conclusion

In this paper, we have extended classic results on the cross-section to the setting of rare events. When there are no rare events, and utility is time-additive, our results reduce to the consumption CAPM of Breeden (1979). When there are no rare events and risk aversion is equal to one, our results reduce to the wealth CAPM of Sharpe (1964). In the rare-event versions of these models, risk premia are not necessarily determined by covariances with consumption in the first case, nor in the second case are risk premia necessarily determined by covariances with wealth. Moving beyond these knife-edge cases, the sign of risk premia relative to the consumption CAPM is determined by the agent's preference for early resolution of uncertainty, while the sign of risk premia relative to the wealth CAPM is determined by whether risk aversion is below or above one. While versions of these models without rare events lead to the usual factor structure, when rare events can occur, there is again no reason to assume the general factor structure holds. This is perhaps surprising given that that the factor structure has dominated empirical asset pricing for many years.

In the second part of the paper we specialized to an affine structure and solve explicitly for the prices of long-lived assets. These assets are integrals of prices of equity strips: claims to dividends at specific points in time. Our solution relies on an approximation for the wealth-consumption ratio. It is fully exact in two special cases: EIS of one and time-additive utility. In all other cases, asset prices are exact given the approximate solution of the wealth-consumption ratio. Despite the richness of the problem, our formulas for prices and risk premia are quite simple. Besides being highly accurate, our approach preserves the important intuition that long-lived assets are sums (or integrals) of individual claims. As we show, log-linearization obscures this insight, leading to an overstatement of risk premia and an understatement of prices. The ease and accuracy with which the equilibrium can be computed in this framework allows for many potential applications beyond what the literature has considered, such as jumps in the state variables, learning, and multiple sources of risk across different asset classes.

Appendix

A Proof of Theorems

Proof of Proposition 1 For convenience, let $J_t = J(C_t, X_t)$. The Hamilton-Jacobi-Bellman (HJB) equation is given by:

$$\mathcal{D}J + f(C, J) = 0. \quad (\text{A.1})$$

Substituting (9) into (6)–(7) yields:

$$f(C_t, J_t) = \begin{cases} J_t \beta \theta \left[I(X_t)^{\frac{1}{\psi}-1} - 1 \right] & \psi \neq 1, \\ -J_t \beta (1 - \gamma) \log I(X_t) & \psi = 1. \end{cases} \quad (\text{A.2})$$

$$+ \sum_{j=1}^m \lambda_j(x) E_{\nu_j} [J(C e^{Z_{cj}}, x + Z_{Xj}) - J(C, x)] \quad (\text{A.3})$$

By Ito's Lemma:

$$\begin{aligned} \frac{\mathcal{D}J}{J} = \frac{1}{J} & \left(\frac{\partial J}{\partial C} C \mu_c(x) + \frac{\partial J}{\partial X} \mu_X(x) + \frac{1}{2} \frac{\partial^2 J}{\partial C^2} C^2 \sigma_c^2(x) + \frac{1}{2} \text{tr} \left[\frac{\partial^2 J}{\partial X^2} \sigma(x) \sigma(x)^\top \right] \right. \\ & \left. + \sum_{j=1}^m \lambda_j(x) E_{\nu_j} [J(C e^{Z_{cj}}, x + Z_{Xj}) - J(C, x)] \right), \quad (\text{A.4}) \end{aligned}$$

Equation 9 implies:

$$\frac{1}{J} \frac{\partial J}{\partial C} = \frac{1 - \gamma}{C}, \quad \frac{1}{J} \frac{\partial^2 J}{\partial C^2} = \frac{-\gamma(1 - \gamma)}{C^2}, \quad \frac{1}{J} \frac{\partial J}{\partial X} = \frac{1 - \gamma}{I} \frac{\partial I}{\partial X}, \quad (\text{A.5})$$

$$\frac{1}{J} \frac{\partial^2 J}{\partial X^2} = (1 - \gamma) \left(\frac{1}{I} \left(\frac{\partial^2 I}{\partial X^2} \right)^2 - \frac{\gamma}{I^2} \left(\frac{\partial I}{\partial X} \right)^\top \left(\frac{\partial I}{\partial X} \right) \right), \quad (\text{A.6})$$

and

$$\frac{J(C e^{Z_{cj}}, x + Z_{Xj})}{J} = e^{(1-\gamma)Z_{cj}} \left(\frac{I(x + Z_{Xj})}{I(x)} \right)^{1-\gamma}. \quad (\text{A.7})$$

Substituting (A.5–A.7) into (A.4) yields:

$$\begin{aligned} \frac{\mathcal{D}J}{J} = & (1 - \gamma) \mu_c(x) + \frac{1 - \gamma}{I} \frac{\partial I}{\partial X} \mu_X(x) - \frac{1}{2} \gamma (1 - \gamma) \sigma_c(x)^2 \\ & + \frac{1 - \gamma}{2} \text{tr} \left[\left(\frac{1}{I} \left(\frac{\partial^2 I}{\partial X^2} \right)^2 - \frac{\gamma}{I^2} \left(\frac{\partial I}{\partial X} \right)^\top \left(\frac{\partial I}{\partial X} \right) \right) \sigma(x) \sigma(x)^\top \right] \\ & + \sum_{j=1}^m \lambda_j E_{\nu_j} \left[e^{(1-\gamma)Z_{cj}} \left(\frac{I(x + Z_{Xj})}{I(x)} \right)^{1-\gamma} - 1 \right]. \quad (\text{A.8}) \end{aligned}$$

Finally, substituting (A.2) and (A.8) into (A.1) yields (10) and verifies the form (9) for $\psi \neq 1$. Analogously, substituting (A.3) and (A.8) into (A.1) yields (11), and verifies (9) for $\psi = 1$. \square

Proof of Lemma 4 Let S_t be the price of the asset that pays a continuous dividend stream D_t . Then by no arbitrage,

$$S_t = E_t \left[\int_t^\infty \frac{\pi_s}{\pi_t} D_s ds \right]. \quad (\text{A.9})$$

Multiplying each side of (A.9) by π_t implies

$$\pi_t S_t = E_t \left[\int_t^\infty \pi_u D_u du \right]. \quad (\text{A.10})$$

The same equation must hold at any time $s > t$:

$$\pi_s S_s = E_t \left[\int_s^\infty \pi_u D_u du \right]. \quad (\text{A.11})$$

Combining (A.10) and (A.11) implies

$$\pi_t S_t = E_t \left[\pi_s S_s + \int_t^s \pi_u D_u du \right]. \quad (\text{A.12})$$

Adding $\int_0^t \pi_u D_u du$ to both sides of (A.12) implies

$$\pi_t S_t + \int_0^t \pi_u D_u du = E_t \left[\pi_s S_s + \int_0^s \pi_u D_u du \right]. \quad (\text{A.13})$$

Therefore, $\pi_t S_t + \int_0^t \pi_u D_u du$ is a martingale. By Ito's Lemma:

$$\begin{aligned} \pi_t S_t + \int_0^t \pi_u D_u du &= \pi_0 S_0 + \int_0^t \pi_u S_u \left(\mu_{\pi,u} + \mu_{S,u} + \frac{D_u}{S_u} + (\sigma_{\pi,u})(\sigma_{S,u})^\top \right) du \\ &\quad + \int_0^t \pi_u S_u (\sigma_{S,u} + \sigma_{\pi,u}) dB_u + \sum_j \sum_{0 < u_{ij} \leq t} \left(\pi_{u_{ij}} S_{u_{ij}} - \pi_{u_{ij}^-} S_{u_{ij}^-} \right), \end{aligned} \quad (\text{A.14})$$

where $u_{ij} = \inf\{u : N_{ju} = i\}$ (namely, the time that the i th type j jump occurs). Adding and subtracting the jump compensation term from (A.14) yields:

$$\begin{aligned} \pi_t S_t + \int_0^t \pi_u D_u du &= \pi_0 S_0 + \int_0^t \pi_u S_u \left(\mu_{\pi,u} + \mu_{S,u} + \frac{D_u}{S_u} + \sigma_{\pi,u} \sigma_{S,u}^\top + \lambda_u^\top \bar{\mathcal{J}}_{\pi S}(X_u) \right) du \\ &\quad + \int_0^t \pi_u S_u (\sigma_{S,u} + \sigma_{\pi,u}) dB_u \\ &\quad + \sum_j \sum_{0 < u_{ij} \leq t} \left(\pi_{u_{ij}} S_{u_{ij}} - \pi_{u_{ij}^-} S_{u_{ij}^-} \right) - \int_0^t \lambda_u^\top \pi_u S_u \bar{\mathcal{J}}_{\pi S}(X_u) du, \end{aligned} \quad (\text{A.15})$$

where we use the definition of an intensity and the fact that

$$\int_0^t \lambda_u^\top \pi_u S_u \bar{\mathcal{J}}_{\pi S}(X_u) du = \int_0^t \lambda_{u^-}^\top \pi_{u^-} S_{u^-} \bar{\mathcal{J}}_{\pi S}(X_{u^-}) du$$

because the integrals differ on a set of measure zero. The second and third terms on the right-hand side of (A.15) have zero expectation. Therefore the first term in (A.15) must also have zero expectation, and it follows that the integrand of this term must equal zero. \square

Proof of Corollary 8 It follows from Theorem 5 and Corollary 6 that

$$r(x)^S - r(x_0) = \gamma \sigma_c(x) e_1^\top \sigma_S(x) - \left(\frac{1}{\psi} - \gamma \right) \left[0, \frac{1}{I(X_t)} \frac{\partial I}{\partial X} \sigma_X(x) \right] \sigma_S(x)^\top - \sum_{j=1}^m \lambda_j(x) E_{\nu_j} \left[\left(\left(\frac{I(x + Z_{Xj})}{I(x)} \right)^{\frac{1}{\psi} - \gamma} e^{-\gamma Z_{cj}} - 1 \right) \bar{\mathcal{J}}_S(x) \right].$$

The result follows from adding and subtracting the expression $\gamma \left(1 - \frac{1}{\psi} \right) \left[0, \frac{1}{I} \frac{\partial I}{\partial X} \sigma_{Xt} \right] \sigma_S^\top$ and substituting in $\sigma_w(x)$ and Z_{wjt} using (29) and (30). \square

Proof of Theorem 11 Conjecture that $I(x)$ is approximately exponential affine. Then

$$\frac{1}{I} \frac{\partial I}{\partial x} \simeq [b_1, \dots, b_n] = b^\top, \quad (\text{A.16})$$

$$\frac{1}{I} \frac{\partial^2 I}{\partial x^2} \simeq \begin{bmatrix} b_1^2 & \dots & b_1 b_n \\ \vdots & \dots & \vdots \\ b_n b_2 & \dots & b_n^2 \end{bmatrix} = b b^\top. \quad (\text{A.17})$$

For $\psi \neq 1$, substitute (33), (A.16), and (A.17) into (10) of Proposition 1 to find:

$$\begin{aligned} \beta I(x)^{\frac{1}{\psi} - 1} &= \beta - \left(1 - \frac{1}{\psi} \right) (k_0 + k_1 x) + \frac{1}{2} \gamma \left(1 - \frac{1}{\psi} \right) (u_0 + u_1 x) - \left(1 - \frac{1}{\psi} \right) (b^\top K_0 + b^\top K_1 x) \\ &\quad - \frac{1}{2} \left(1 - \frac{1}{\psi} \right) (1 - \gamma) \left(b^\top U_0 b + (b^\top U_1 b) x \right) - \frac{1}{\theta} \left(E_\nu [e^{(1-\gamma)(Z_c + Z_X)b} - 1] \right)^\top (l_0 + l_1 x). \end{aligned} \quad (\text{A.18})$$

We use the log-linear approximation

$$\beta I(X_t)^{\frac{1}{\psi} - 1} \simeq i_0 - i_1 \log \left(\beta^{-1} I(X_t)^{1 - \frac{1}{\psi}} \right), \quad (\text{A.19})$$

where $i_1 = e^{E[\log(\beta I(X_t)^{1/\psi - 1})]}$, and $i_0 = i_1(1 - \log i_1)$. Substituting (A.19) into (A.18) and matching coefficients yields (40) and (39), verifying the conjecture.

For $\psi = 1$ we follow a similar derivation and note that $\log I(x) = a + b^\top x$. The HJB

(11) can be rewritten as:

$$\begin{aligned} \beta (a + b^\top x) &= (k_0 + k_1 x) - \frac{1}{2} \gamma (u_0 + u_1 x) + (b^\top K_0 + b^\top K_1 x) \\ &+ \frac{1}{2} (1 - \gamma) (b^\top U_0 b + (b^\top U_1 b) x) + \frac{1}{1 - \gamma} (E_\nu [e^{(1-\gamma)(Z_c + Z_X b)} - 1])^\top (l_0 + l_1 x). \end{aligned} \quad (\text{A.20})$$

We then match coefficients as above. To show that the limits work out as stated, see the Lemma below. \square

Lemma A.1. *Let $y = (k_0, k_1, u_0, u_1, K_0, K_1, U_0, U_1, l_0, l_1, \nu, \gamma)$. Let $I(X, \psi; y) = \exp \{a(\psi) + b(\psi)^\top X\}$ denote the value function as a function of ψ with $\psi \neq 1$. Suppose $I(X, \psi; y)$ is well-defined at (y, ψ) for $\psi \in (1 - \epsilon, 1 + \epsilon) \setminus \{1\}$ with solutions $b(\psi)$ and $a(\psi)$ given by (40) and (39). Let $\tilde{I}(X; y)$ denote the value function with $\psi = 1$, $\tilde{I}(X; y)$ is well defined at y , with solutions \tilde{b} and \tilde{a} as described in Theorem 11. Furthermore, assume $\lim_{\psi \rightarrow 1} \frac{\partial I(X, \psi; y)}{\partial \psi} < \infty$ exists. Then, $\lim_{\psi \rightarrow 1} a(\psi) = \tilde{a}$ and $\lim_{\psi \rightarrow 1} b(\psi) = \tilde{b}$.*

Proof of Lemma A.1 Note that

$$i_1 = \exp \left(E \left[\log \left(\beta I(X_t, \psi; y)^{\frac{1}{\psi} - 1} \right) \right] \right) = \beta \exp \left(\left(\frac{1}{\psi} - 1 \right) E[\log I(X_t, \psi; y)] \right).$$

Since $\lim_{\psi \rightarrow 1} 1/\psi - 1 = 0$, the above expression converges to β . Next, we look at the limit of $(1 - 1/\psi)^{-1} (i_1 \log \beta + i_0 - \beta)$. For convenience, we denote $I(X, \psi; y)$ by I_y

$$\frac{1}{1 - 1/\psi} (i_1 \log \beta + i_0 - \beta) = \beta \left(E[\log I_y] e^{(\frac{1}{\psi} - 1) E[\log I_y]} + \frac{1}{1 - 1/\psi} \left(e^{(\frac{1}{\psi} - 1) E[\log I_y]} - 1 \right) \right)$$

When $\psi \rightarrow 1$, $1/\psi - 1 \rightarrow 0$ and the first term in the bracket converges to $E[\log I_y]$. Apply l'Hopital's rule to the second term:

$$\begin{aligned} \lim_{\psi \rightarrow 1} \frac{1}{1 - 1/\psi} \left(e^{(\frac{1}{\psi} - 1) E[\log I_y]} - 1 \right) &= \lim_{\psi \rightarrow 1} - \frac{\exp(E[\log I_y])^{\frac{1}{\psi} - 1} - 1}{\frac{1}{\psi} - 1} \\ &= -E[\log I_y]. \end{aligned}$$

Therefore, as $\psi \rightarrow 1$, $\theta(i_1 \log \beta + i_0 - \beta) = 0$, that is, $\lim_{\psi \rightarrow 1} a(\psi) = \tilde{a}$. \square

Proof of Theorem 12 Equation 41 follows from Ito's Lemma applied to Corollary 3. Equations 42 and 43 follow from Corollary 6, substituting for $I(x)$ from Theorem 11.

We directly calculate $\mu_{\pi t}$ and then back out the riskfree rate from the no-arbitrage condition (23). First consider $\psi \neq 1$. We apply Ito's lemma to (15) to find

$$\begin{aligned} \mu_{\pi t} = & -\beta \left((1 - \theta) I(X_t)^{\frac{1}{\psi} - 1} + \theta \right) - \gamma \mu_c(X_t) + \frac{1}{2} \gamma (\gamma + 1) \sigma_c^2(X_t) \\ & + \left(\frac{1}{\psi} - \gamma \right) \frac{1}{I} \frac{\partial I}{\partial X} \mu_X(X_t) + \frac{1}{2} \left(\frac{1}{\psi} - \gamma \right)^2 \text{tr} \left(\frac{1}{I} \frac{\partial^2 I}{\partial X^2} \sigma_X(X_t) \sigma_X(X_t)^\top \right). \end{aligned}$$

Substituting in for $I(X_t)$ and its derivatives using (A.17–A.18), together with (33), we find

$$\begin{aligned} \mu_{\pi t} \simeq & -\beta - \frac{1}{\psi} (k_0 + k_1 x) + \frac{1}{2} \gamma \left(1 + \frac{1}{\psi} \right) (u_0 + u_1 x) + \frac{1}{2} \left(1 - \frac{1}{\psi} \right) \left(\gamma - \frac{1}{\psi} \right) \left(b^\top U_0 b + (b^\top U_1 b) x \right) \\ & - \left(1 - \frac{1}{\theta} \right) \left(E_\nu \left[e^{(1-\gamma)(Z_c + Z_X b)} - 1 \right] \right)^\top (l_0 + l_1 x). \quad (\text{A.21}) \end{aligned}$$

For $\psi = 1$, apply the same argument using (16) to find:

$$\begin{aligned} \mu_{\pi t} = & -\beta \left((1 - \gamma) \log I(X_t) + 1 \right) - \gamma \mu_c(X_t) + (1 - \gamma) \frac{1}{I} \frac{\partial I}{\partial X} \mu_X(X_t) + \frac{1}{2} \gamma (\gamma + 1) \sigma_c^2(X_t) \\ & + \frac{1}{2} (1 - \gamma)^2 \text{tr} \left(\frac{1}{I} \frac{\partial^2 I}{\partial X^2} \sigma_X(X_t) \sigma_X(X_t)^\top \right) \\ = & -\beta - (k_0 + k_1 x) + \gamma (u_0 + u_1 x) - \left(E_\nu \left[e^{(1-\gamma)(Z_c + Z_X b)} - 1 \right] \right)^\top (l_0 + l_1 x). \quad (\text{A.22}) \end{aligned}$$

The risk-free rate then follows from the no-arbitrage condition (23). The exact result for time-additive utility follows from the fact that (15) reduces to

$$\pi_t = e^{-\int_0^t \beta ds} \beta C_t^{-\gamma}$$

when $\theta = 0$. This is the standard form of the state-price density under time-additive utility and constant relative risk aversion.

□

We prove a no-arbitrage theorem for equity strips, analogous to the result for long-lived assets (Lemma 4). We omit the proof, which follows along lines similar to that of Lemma 4).

Lemma A.2. *Let $H(D_t, X_t, T - t)$ denote the time- t price of a single future dividend payment at time $T > t$. For fixed T , define $H_t = H(D_t, X_t, T - t)$. Define $\mu_{H,t}$ and $\sigma_{H,t}$ such that*

$$\frac{dH_t}{H_t} = \mu_{H,t} dt + \sigma_{H,t} dB_t + \sum_{j=1}^m \mathcal{J}_H(X_{t-}, Z_{dt}) dN_{jt}. \quad (\text{A.23})$$

Then no-arbitrage implies that

$$\mu_{\pi,t} + \mu_{Ht} + \sigma_{\pi,t} \sigma_{Ht}^\top + \lambda_t^\top \bar{\mathcal{J}}_{Ht} = 0. \quad (\text{A.24})$$

Proof of Theorem 13 Conjecture (45). As in the proof of Lemma A.2, fix T and define $H_t = H(D_t, X_t, T - t)$, which follows (A.23). Let $\tau = T - t$. It follows from Ito's Lemma that

$$\sigma_H(x, \tau) \simeq [\sigma_d, b_\phi(\tau)^\top \sigma_X(x)]^\top, \quad (\text{A.25})$$

and

$$\begin{aligned} \mu_H(x, \tau) \simeq & (k_0^d + k_1^d x) + b_\phi(\tau)^\top (K_0 + K_1 x) - (a'_\phi(\tau) + b'_\phi(\tau)^\top x) \\ & + \frac{1}{2} \left(b_\phi(\tau)^\top U_0 b_\phi(\tau) + (b_\phi(\tau)^\top U_1 b_\phi(\tau)) x \right), \end{aligned} \quad (\text{A.26})$$

where $b'_\phi(\tau) = [b'_{\phi_1}(\tau), \dots, b'_{\phi_n}(\tau)]^\top$ denotes the vector of derivatives with respect to τ .

Also, by Ito's Lemma,

$$\mathcal{J}_H(x, Z_d, Z_X) = e^{Z_{dj} + b_\phi(\tau)^\top Z_{Xj}} - 1. \quad (\text{A.27})$$

Substituting (42–44) and (A.25–A.27) into the no-arbitrage condition (A.24) implies:

$$\begin{aligned}
0 = & -\beta - \frac{1}{\psi}(k_0 + k_1x) + \frac{1}{2}\gamma \left(1 + \frac{1}{\psi}\right) (u_0 + u_1x) + (k_0^d + k_1^d x) + b_\phi(\tau)^\top (K_0 + K_1x) \\
& - \left(\frac{\partial a_\phi(\tau)}{\partial \tau} + \left(\frac{\partial b_\phi(\tau)}{\partial \tau} \right)^\top x \right) + \frac{1}{2} \left(1 - \frac{1}{\psi}\right) \left(\gamma - \frac{1}{\psi} \right) \left(b^\top U_0 b + (b^\top U_1 b) x \right) - \gamma(u_0^{cd} + u_1^{cd}x) \\
& + \frac{1}{2} \left(b_\phi(\tau)^\top U_0 b_\phi(\tau) + (b_\phi(\tau)^\top U_1 b_\phi(\tau)) x \right) + \left(\frac{1}{\psi} - \gamma \right) \left(b_\phi(\tau)^\top U_0 b + (b_\phi(\tau)^\top U_1 b) x \right) \\
& + \left(\left(\frac{1}{\theta} - 1 \right) E_\nu \left[e^{(1-\gamma)(Z_c + Z_X b)} - 1 \right] + E_\nu \left[e^{-\gamma Z_c + Z_d + Z_X (b_\phi(\tau) + (1/\psi - \gamma)b)} - 1 \right] \right)^\top (l_0 + l_1x).
\end{aligned}$$

Matching the constant terms implies (46) and matching the terms multiplying x implies (47), satisfying the conjecture.

For $\psi = 1$ and $\psi = 1/\gamma$, (42–44) hold with equality. The conjecture that (45) holds with equality is therefore satisfied. \square

B Approximating the price-dividend ratio by a log-linear function

An alternative to the approximate solution method we propose is to log-linearize the price-dividend ratio. That is, consider

$$\frac{1}{G(x)} \approx g_0 - g_1 \log(G(x)), \tag{B.1}$$

where $g_1 = e^{E_\nu[-\log G]}$ and $g_0 = g_1(1 - \log g_1)$. Conjecture

$$G(x) \simeq e^{\hat{a}_\phi + \hat{b}_\phi^\top x},$$

where \hat{a}_ϕ is a scalar $\hat{b}_\phi = [\hat{b}_{\phi 1}, \dots, \hat{b}_{\phi n}]$ is a column vector. Ito's Lemma then implies that

$$\sigma_{St} = \left[\sigma_{dt}, \hat{b}_\phi \sigma_{Xt} \right]^\top, \tag{B.2}$$

$$\mu_{St} = (k_0^d + k_1^d x) + \hat{b}_\phi^\top (K_0 + K_1 x) + \frac{1}{2} \left(\left(\hat{b}_\phi^\top U_0 \hat{b}_\phi \right) + \left(\hat{b}_\phi^\top U_1 \hat{b}_\phi \right) x \right), \quad (\text{B.3})$$

and, if $\psi \neq 1$,

$$\mathcal{J}_{\pi S}(x, Z_c, Z_d, Z_X) = e^{-\gamma Z_c + Z_d + Z_X \left(\frac{1}{\psi} - \gamma \right) b + \hat{b}_\phi}. \quad (\text{B.4})$$

Substituting (A.21), (42) along with (B.2), (B.3) and (B.4) into the no-arbitrage condition (21) implies:

$$\begin{aligned} 0 = & -\beta - \frac{1}{\psi}(k_0 + k_1 x) + \frac{1}{2}\gamma \left(1 + \frac{1}{\psi} \right) (u_0 + u_1 x) + (k_0^d + k_1^d x) + \hat{b}_\phi^\top (K_0 + K_1 x) \\ & + \frac{1}{2} \left(1 - \frac{1}{\psi} \right) \left(\gamma - \frac{1}{\psi} \right) (b^\top U_0 b + (b^\top U_1 b) x) + \frac{1}{2} \left(\hat{b}_\phi^\top U_0 \hat{b}_\phi + \left(\hat{b}_\phi^\top U_1 \hat{b}_\phi \right) x \right) + \frac{1}{G_t} \\ & - \gamma(u_0^{cd} + u_1^{cd} x) + \left(\frac{1}{\psi} - \gamma \right) \left(\hat{b}_\phi^\top U_0 b + \left(\hat{b}_\phi^\top U_1 b \right) x \right) \\ & + E_\nu \left[\left(\frac{1}{\theta} - 1 \right) \left(e^{(1-\gamma)(Z_c + Z_X b)} - 1 \right) + \left(e^{-\gamma Z_c + Z_d + Z_X (\hat{b}_\phi + (\frac{1}{\psi} - \gamma) b)} - 1 \right) \right]^\top (l_0 + l_1 x). \end{aligned}$$

Matching coefficients on x :

$$\begin{aligned} 0 = & -g_1 \hat{b}_\phi^\top - \frac{1}{\psi} k_1 + \frac{1}{2}\gamma \left(1 + \frac{1}{\psi} \right) u_1 + k_1^d + \hat{b}_\phi^\top K_1 - \gamma u_1^{cd} + \frac{1}{2} \left(1 - \frac{1}{\psi} \right) \left(\gamma - \frac{1}{\psi} \right) b^\top U_1 b + \left(\frac{1}{\psi} - \gamma \right) \hat{b}_\phi^\top U_1 b \\ & + \frac{1}{2} \hat{b}_\phi^\top U_1 \hat{b}_\phi + \left(E_\nu \left[\left(\frac{1}{\theta} - 1 \right) \left(e^{(1-\gamma)(Z_c + Z_X b)} - 1 \right) + e^{-\gamma Z_c + Z_d + Z_X (\hat{b}_\phi + (\frac{1}{\psi} - \gamma) b)} - 1 \right] \right)^\top l_1, \end{aligned}$$

and \hat{a}_ϕ is given by

$$\begin{aligned} \hat{a}_\phi = & \frac{1}{g_1} \left(g_0 - \beta - \frac{1}{\psi} k_0 + \frac{1}{2}\gamma \left(1 + \frac{1}{\psi} \right) u_0 + k_0^d + \hat{b}_\phi^\top K_0 - \gamma u_0^{cd} + \frac{1}{2} \left(1 - \frac{1}{\psi} \right) \left(\gamma - \frac{1}{\psi} \right) b^\top U_0 b + \frac{1}{2} \hat{b}_\phi^\top U_0 \hat{b}_\phi \right. \\ & \left. + \left(\frac{1}{\psi} - \gamma \right) \hat{b}_\phi^\top U_0 b + E_\nu \left[\left(\frac{1}{\theta} - 1 \right) \left(e^{(1-\gamma)(Z_c + Z_X b)} - 1 \right) + \left(e^{-\gamma Z_c + Z_d + Z_X (\hat{b}_\phi + (\frac{1}{\psi} - \gamma) b)} - 1 \right) \right]^\top l_0 \right). \end{aligned}$$

References

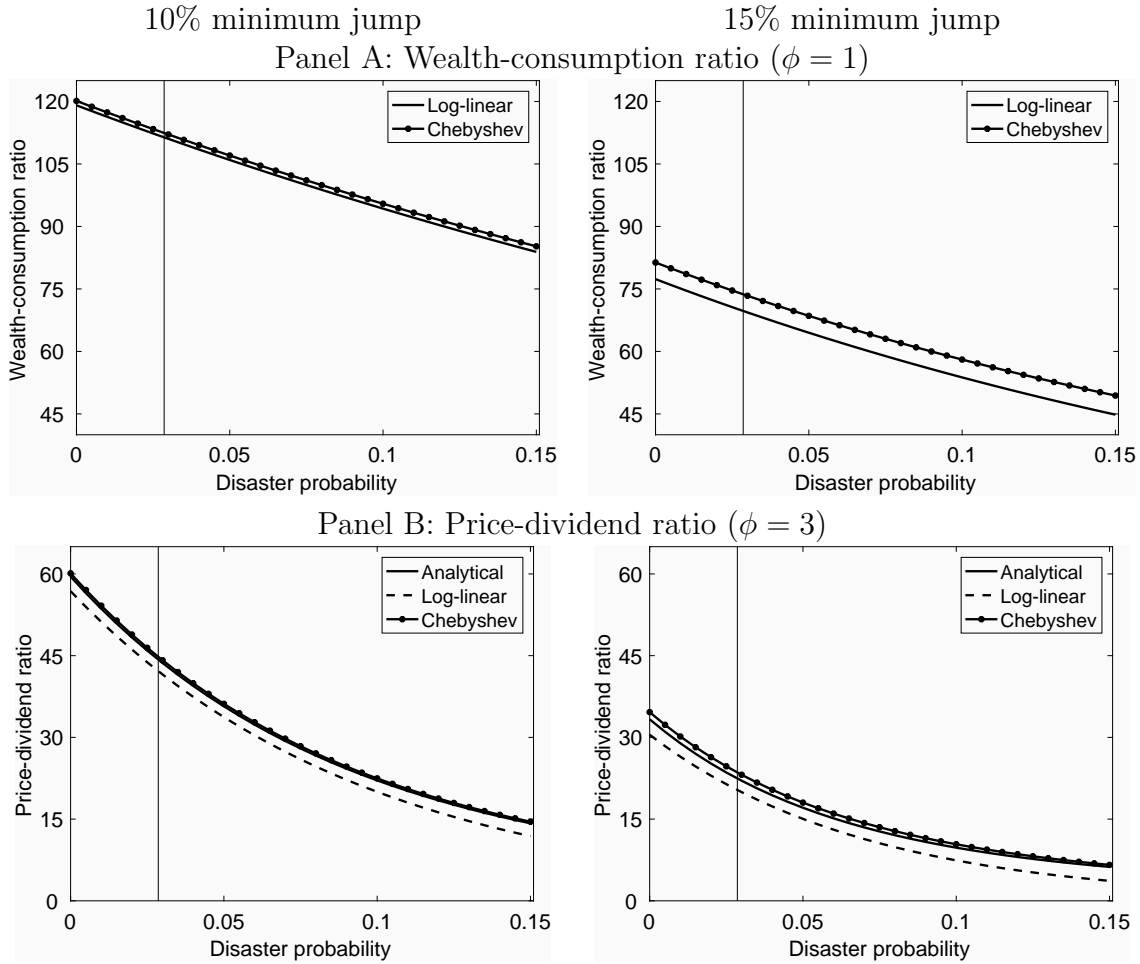
- Bai, Hang, Kewei Hou, Howard Kung, and Lu Zhang, 2018, The CAPM strikes back? An equilibrium model with disasters, forthcoming, *Journal of Financial Economics*.
- Bansal, Ravi, and Amir Yaron, 2004, Risks for the long run: a potential resolution of asset pricing puzzles, *Journal of Finance* 59, 1481–1509.
- Barro, Robert J., 2006, Rare disasters and asset markets in the twentieth century, *Quarterly Journal of Economics* 121, 823–866.
- Barro, Robert J., and José F. Ursúa, 2008, Macroeconomic crises since 1870, *Brookings Papers on Economic Activity* 39, 255–350.
- Benzoni, Luca, Pierre Collin-Dufresne, and Robert S. Goldstein, 2011, Explaining asset pricing puzzles associated with the 1987 market crash, *Journal of Financial Economics* 101, 552 – 573.
- Breeden, Douglas T., 1979, An intertemporal asset pricing model with stochastic consumption and investment opportunities, *Journal of Financial Economics* 7, 265–296.
- Campbell, John Y., 1993, Intertemporal asset pricing without consumption data, *American Economic Review* 83, 487–512.
- Campbell, John Y., 2018, *Financial Decisions and Markets: A Course in Asset Pricing*. (Princeton University Press Princeton, NJ).
- Campbell, John Y., Stefano Giglio, Christopher Polk, and Robert Turley, 2018, An intertemporal CAPM with stochastic volatility, forthcoming, *Journal of Financial Economics*.
- Campbell, John Y., and Robert J. Shiller, 1988, The dividend-price ratio and expectations of future dividends and discount factors, *Review of Financial Studies* 1, 195–228.

- Campbell, John Y., and Luis M. Viceira, 1999, Consumption and portfolio decisions when expected returns are time-varying, *Quarterly Journal of Economics* 114, 433–495.
- Chabi-Yo, Fousseni, Stefan Ruenzi, and Florian Weigert, 2017, Crash sensitivity and the cross-section of expected stock returns, Forthcoming, *Journal of Financial and Quantitative Analysis*.
- Chacko, George, and Luis Viceira, 2005, Dynamic consumption and portfolio choice with stochastic volatility in incomplete markets, *Review of Financial Studies* 18, 1369–1402.
- Cochrane, John H., 2001, *Asset Pricing*. (Princeton University Press Princeton, NJ).
- Cox, John C., Jonathan C. Ingersoll, and Stephen A. Ross, 1985, A theory of the term structure of interest rates, *Econometrica* 53, 385–408.
- Cremers, Martijn, Michael Halling, and David Weinbaum, 2015, Aggregate Jump and Volatility Risk in the Cross-Section of Stock Returns, *The Journal of Finance* 70, 577–614.
- Drechsler, Itamar, and Amir Yaron, 2011, What’s vol got to do with it, *Review of Financial Studies* 24, 1–45.
- Duffie, Darrell, 1996, *Dynamic Asset Pricing Theory*. (Princeton University Press Princeton, NJ).
- Duffie, Darrell, and Larry G Epstein, 1992a, Asset pricing with stochastic differential utility, *Review of Financial Studies* 5, 411–436.
- Duffie, Darrell, and Larry G. Epstein, 1992b, Stochastic differential utility, *Econometrica* 60, 353–394.
- Duffie, Darrell, Jun Pan, and Kenneth Singleton, 2000, Transform analysis and asset pricing for affine jump-diffusions, *Econometrica* 68, 1343–1376.

- Duffie, Darrell, and Costis Skiadas, 1994, Continuous-time asset pricing: A utility gradient approach, *Journal of Mathematical Economics* 23, 107–132.
- Epstein, Larry, and Stan Zin, 1989, Substitution, risk aversion and the temporal behavior of consumption and asset returns: A theoretical framework, *Econometrica* 57, 937–969.
- Eraker, Bjorn, and Ivan Shaliastovich, 2008, An equilibrium guide to designing affine pricing models, *Mathematical Finance* 18, 519–543.
- Farhi, Emmanuel, Samuel P. Fraiberger, Xavier Gabaix, Romain Ranciere, and Adrien Verdelhan, 2017, Crash risk in currency markets, forthcoming, *Quarterly Journal of Economics*.
- Gabaix, Xavier, 2012, An exactly solved framework for ten puzzles in macro-finance, *Quarterly Journal of Economics* 127, 645–700.
- Hansen, Lars Peter, 2012, Dynamic valuation decomposition within stochastic economies, *Econometrica* 80, 911–967.
- Hansen, Lars Peter, John Heaton, Junghoon Lee, and Nikolai Roussanov, 2007, *Intertemporal Substitution and Risk Aversion* Elsevier vol. 6 chap. 61, pp. 3967–4056.
- Huang, Darien, and Mete Kilic, 2018, Gold, Platinum, and Expected Stock Returns, Working paper, Cornell University and University of Southern California Rochester, NY.
- Lettau, Martin, and Jessica A. Wachter, 2007, Why is long-horizon equity less risky? A duration-based explanation of the value premium, *Journal of Finance* 62, 55–92.
- Longstaff, Francis A., and Monika Piazzesi, 2004, Corporate earnings and the equity premium, *Journal of Financial Economics* 74, 401–421.
- Lu, Zhongjin, and Scott Murray, 2017, Bear Beta, forthcoming, *Journal of Financial Economics*.

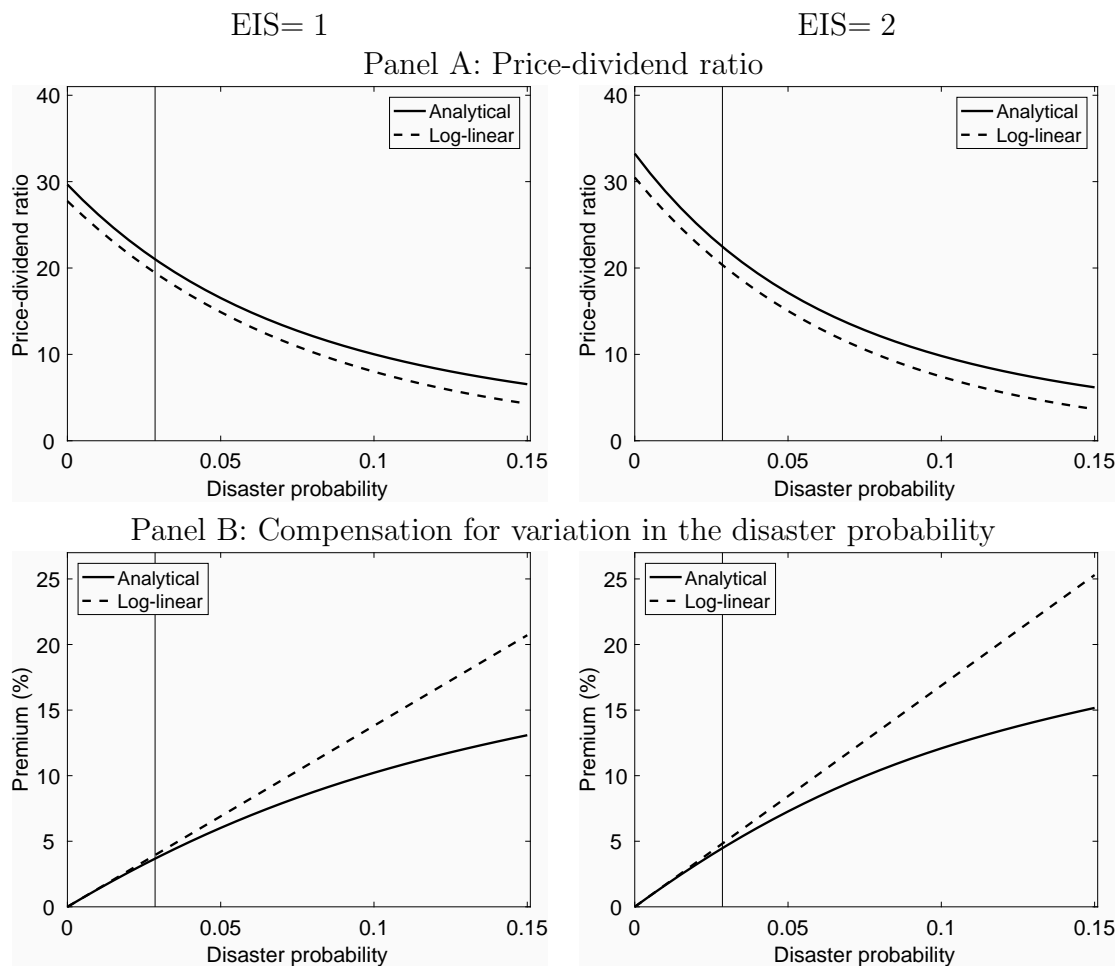
- Merton, Robert C., 1973, An intertemporal capital asset pricing model, *Econometrica* 41, 867–887.
- Nowotny, Michael, 2011, Disaster begets crisis: The role of contagion in financial markets, Working paper, Boston University.
- Pohl, Walter, Karl Schmedders, and Ole Wilms, 2015, Higher-Order Effects in Asset Pricing Models with Long-Run Risks, Working paper, University of Zurich.
- Pohl, Walter, Karl Schmedders, and Ole Wilms, 2017, Higher-Order Effects in Asset Pricing Models with Long-Run Risks, forthcoming, *Journal of Finance*.
- Schroder, Mark, and Costis Skiadas, 1999, Optimal consumption and portfolio selection with stochastic differential utility, *Journal of Economic Theory* 89, 68–126.
- Seo, Sang Byung, and Jessica A. Wachter, 2016, Do rare events explain CDX tranche spreads?, Working paper, University of Houston and University of Pennsylvania.
- Sharpe, W., 1964, Capital asset prices: A theory of market equilibrium under conditions of risk, *Journal of Finance* 19, 425–444.
- Tsai, Jerry, and Jessica A. Wachter, 2015, Disaster Risk and Its Implications for Asset Pricing, *Annual Review of Financial Economics* 7, 219–252.
- Tsai, Jerry, and Jessica A. Wachter, 2016, Rare Booms and Disasters in a Multisector Endowment Economy, *Review of Financial Studies* 29, 1113–1169.
- Wachter, Jessica A., 2013, Can time-varying risk of rare disasters explain aggregate stock market volatility?, *The Journal of Finance* 68, 987–1035.

Figure 1: Accuracy of analytical approximations: wealth-consumption and price-dividend ratios



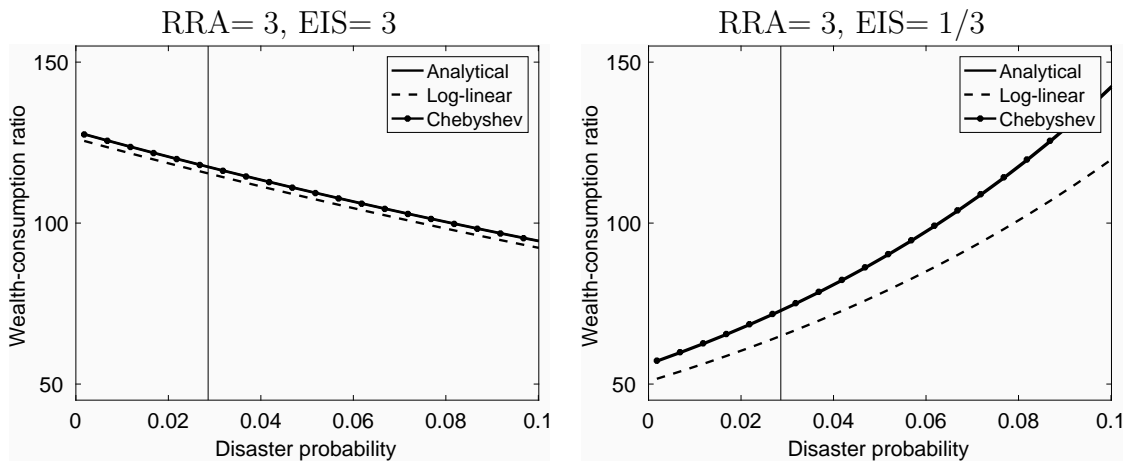
Notes: This figure shows the wealth-consumption ratio (Panel A) and the price-dividend ratio (Panel B) in a model with time-varying risk of rare disaster. We compare our analytical approximation with the exact solution computed using Chebyshev polynomials. For the price-dividend ratio, we also compare our approximation to a log-linear approximation (in the case of the wealth-consumption ratio, the two approximations are the same). Relative risk aversion equals 3 and the EIS equals 2. The solid line denotes the unconditional mean of the disaster probability.

Figure 2: Accuracy of analytical approximations: price-dividend ratios and risk premia



Notes: This figure shows the price-dividend ratio (Panel A) and the annual risk premium for variation in the disaster probability (Panel B) for two values of the EIS, in a model with time-varying risk of rare disaster. We compare our analytical approximation (which is exact for EIS equal to 1) to the log-linear approximation. We assume a minimum disaster size of 15% and relative risk aversion equal to 3. The solid line denotes the unconditional mean of the disaster probability.

Figure 3: Comparing two approximations for the wealth-consumption ratio



Notes: We compute a first-stage solution to the wealth-consumption ratio using log-linearization (“log-linear”). We then use our analytical method to recompute the wealth-consumption ratio as an integral of zero-coupon claims (“Analytical”). For comparison, we show the exact numerical solution. We assume a minimum disaster size of 15% and relative risk aversion equal to 3. When the EIS is equal to 1/3 (right panel), the analytical approximation is exact. The solid line denotes the unconditional mean of the disaster probability.