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ABSTRACT

We develop a theory of optimal government debt in which publicly-issued and privately-issued safe assets are substitutes. While government bonds are backed by future tax revenues, privately-issued safe assets are backed by the future repayment of pools of defaultable private loans. We find that a higher supply of public debt crowds out privately-issued safe assets less than one for one and reduces the interest spread between borrowing and deposit rates. Our main result is that the optimal level of public debt does not fully crowd out private lending and maintains a positive interest spread. Moreover, the optimal level of public debt is higher the more severe are financial frictions.

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1 Introduction

There is a conventional wisdom that public debt can serve as a substitute for privately-issued safe assets when private borrowing is limited. This view is not only the basis for research which focuses on the public provision of liquidity, but it is also the basis for policy proposals promoting the expansion of public liquidity.[1]

In this paper, we study the optimal long-run supply of public and private liquidity. In contrast to previous normative work, we allow for private liquidity with credit frictions and consider the reaction of financial intermediaries to the level of public liquidity. We are motivated by the empirical evidence that public debt increases are associated with reductions in the issuance of other safe assets such as asset-backed securities and money market securities.[2] Our starting premise is that government bonds are backed by future tax revenues, whereas privately-issued safe assets are backed by the future repayment of pools of defaultable private loans. Given this difference, we ask the following questions: How does the supply of public debt interact with the supply of privately-issued safe assets? How does public debt impact interest rates in the economy? And what drives the optimal level of public and private liquidity?

To answer these questions, we introduce privately-issued safe assets to a model of public liquidity. We consider a heterogeneous agent economy with no aggregate risk in which safe assets facilitate households' smoothing of income shocks over time. Safe assets come in two forms: government bonds backed by future tax revenues, and privately-issued safe assets backed by pools of loans to other households. Savers are indifferent between these assets since they provide the same safe deposit rate. Financial intermediaries, however, require a premium to lend to potentially defaulting households versus to the government which is committed to repaying its debt. A household defaults whenever the exogenous cost of doing so is below the cost of repayment, and defaults are idiosyncratic across households. Given the absence of aggregate shocks, there is no aggregate default risk in the economy. This market structure implies the existence of a risk-free deposit rate, equal to the interest rate faced by the government, and a borrowing rate at which households can borrow anonymously. The spread between the borrowing and deposit rates equals the aggregate default rate in the economy.

A natural implication of this construction is that a high enough supply of public debt fully crowds out privately-issued safe assets. In this case, the deposit rate and the borrow-

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[1]See Greenwood, Hanson, and Stein (2016) as an example of a policy proposal for expanding public liquidity.

[2]See for example evidence in Gorton, et al. (2012), Krishnamurthy and Vissing-Jorgensen (2015), and Carlsson et al. (2016)

ing rate—which equal each other—are sufficiently high that no household borrows, there are no defaults, and all households hold positive levels of public debt. A version of Ricardian Equivalence holds since local changes in public debt have no effect on consumption allocations or interest rates. If public debt rises today, households experience lower taxes today and anticipate higher taxes in the future, and they respond by increasing their public debt holdings without changing consumption. The opposite occurs in response to a decrease in public debt.

The interesting role for public debt arises if public debt is sufficiently low. In this case, privately-issued safe assets are no longer zero since borrowing rates are low enough that some households borrow, there are defaults, and a positive interest spread exists between the borrowing rate and the deposit rate. Moreover, Ricardian Equivalence ceases to hold. A local increase in public debt crowds out privately-issued safe assets less than one for one, increases the deposit rate, and reduces the spread between the borrowing and deposit rate.

To understand the logic for this channel, suppose that households were committed to debt repayment. Since default risk would be zero in this case, the borrowing rate would equal the deposit rate. In response to a public debt increase, a saving household's deposits would increase one for one by the same logic as in the case of full crowd out described previously. By analogous reasoning, a borrowing household's debt would decrease one for one with public debt increases.

In contrast to this hypothetical case, households in our environment cannot commit to debt repayment. There is a spread between the borrowing rate and the deposit rate which reflects default risk. Because the government borrows at a lower interest rate than households, a public debt increase cannot mechanically have a neutral impact on a borrower's consumption. Instead, a public debt increase causes a slackening of financial constraints for borrowing households, and these households reduce their borrowing less than one for one with government borrowing. The resultant net increase in total borrowing (household plus government) in the economy puts upward pressure on the deposit rate. Moreover, because each borrowing household borrows less, the probability of default declines, leading to a reduction in the interest spread.

The main result of our paper is that the optimal level of public debt does not fully crowd out privately-issued safe assets. Full crowd out slackens financial constraints, reduces the interest spread to zero, and removes the prevalence of costly defaults. However, by increasing the total supply of safe assets, full crowd out also increases the deposit rate which increases inequality since wealthier households reap higher returns on their savings. The optimal level of public debt trades off more efficient financial markets with

less inequality. These dual considerations imply an optimal policy which admits some financial market inefficiency, which is why optimal policy does not induce full crowd out.

We prove these results analytically in a simple two-period example. Moreover, we verify the robustness of these analytical results numerically in the stationary distribution of the balanced growth path of an infinite horizon economy. This numerical exercise considers the model of public debt of Aiyagari and McGrattan (1998) extended to allow for private financial intermediation with default. We use this exercise to assess the quantitative implications of our theoretical mechanism, and to examine how credit market frictions impact the optimal level of public debt.

Our main quantitative result is that the optimal level of public debt is well below the full crowd out threshold. This level of public debt is between two benchmarks. First, it is below the optimum in the absence of private liquidity. This is because, in our framework, government debt does not relax liquidity constraints as significantly as in the absence of any private financial intermediation. Second, optimal public debt is above the optimum in the absence of default risk (but with private borrowing limits). The presence of default risk increases the liquidity benefit of public debt for two reasons. First, relative to a default-free environment, the fraction of households which directly benefit from more public liquidity increases, since any borrowing household is subject to an interest rate premium relative to the government. Second, the cost of higher public debt through higher borrowing rates is mitigated, since higher public debt also reduces default risk and the interest premium faced by borrowing households. In sum, our work shows that introducing private credit market frictions plays an important role for the determination of optimal public liquidity; the larger are these frictions, the higher is the optimal level of public debt.

Our paper builds on several literatures. First, we contribute to the literature on the optimal supply of public debt in economies with limited private credit.[3] Relative to previous work, we focus on the optimal long-run level of public debt when public debt competes with privately-issued safe assets backed by defaultable loans; this allows us to consider the impact of government debt on interest spreads.[4,5] Second, we contribute

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[3]In addition to the work already mentioned, see Woodford (1990), Holmstrom and Tirole (1998), Azzimonti, de Francisco, and Quadrini (2014), and Angeletos, Collard, and Dellas (2016), among others. Our result that the optimal level of public debt does not fully crowd out the private lending market is in line with results in Yared (2013) and Azzimonti and Yared (2017).

[4]Our work is complementary to the work of Carapella and Williamson (2015) who also study the relationship between public debt and private debt. In their work, public debt is distinguished from private debt because of its role as collateral. In our work, public debt is distinguished from private debt because it is backed by tax revenue as opposed to defaultable private loans.

[5]For related work on the effect of public debt on interest rates and asset prices, see Plosser (1982),

to the literature on optimal public debt management going back to the work of Barro (1979) and Lucas and Stokey (1983).[6] In contrast to this literature, we allow for lump transfers, which removes the role of public debt for smoothing taxes and allows us to focus on public debt's role in providing liquidity. Finally, our paper contributes to the literature on default and credit spreads in heterogeneous agent economies by considering the implications of public debt on credit market conditions.[7]

The paper proceeds as follows. Section 2 illustrates our qualitative results in a two-period example. Section 3 describes the equilibrium of an infinite horizon economy. Section 4 summarizes the results from the quantitative analysis of the infinite horizon economy. Section 5 concludes and the Appendix includes additional results not in the main text.

2 Two-Period Example

We present a simple two-period example which describes the main results of our paper. We show that an increase in the supply of government bonds crowds out privately-issued safe assets less than one for one and reduces the spread between the borrowing rate and the deposit rate. Moreover, we characterize the optimal policy, and we show that the government faces a tradeoff between reducing financial frictions and reducing inequality. In resolving this tradeoff, the optimal supply of government debt does not fully crowd out privately-issued safe assets.

2.1 Environment

2.1.1 Households

There is a continuum of two types of households indexed by $i = \{P, R\}$, each of size $1/2$. Each household's welfare is

$$\mathbb{E} \sum_{t=0,1} \log c_t^i \tag{1}$$

where c_t^i represents the consumption of household of type i at date t. Households have an endowment y_t^i, where we let $y_0^P = 1 - \Delta$ and $y_0^R = 1 + \Delta$ for $\Delta \in (0, 1)$, and we let

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Laubach (2003), Engen and Hubbard (2004), and Gomes, Michaelides Polkovnichenko (2013), among others.

[6]See also Aiyagari, Marcet, Sargent, and Seppala (2002), Werning (2007), and Bhandari, Evans, Golosov, and Sargent (2016), among others.

[7]See for example Athreya (2002), Livshits, MacGee, and Tertilt (2007), and Chatterjee, Corbae, Nakajima, and Ríos-Rull (2007), among others.

$y_1^P = y_1^R = 1$. We therefore refer to R-type households as "rich" households and P-type households as "poor" households.

The resource constraint of the economy is

$$c_t^P + c_t^R \leq y_t^P + y_t^R = 1 \tag{2}$$

for $t = 0, 1$ so that the aggregate endowment in the economy is constant.

The government levies a lump sum tax $T_t \gtrless 0$ uniformly across the population at both dates. At date 0, a household i chooses a quantity of safe deposits $a^i \geq 0$ to buy at price q^a, and these assets pay off a^i with certainty at date 1. In addition, at date 0 a household can sell defaultable bonds $l^i \geq 0$ at price q^l. At date 1, a household receives an idiosyncratic cost of default shock $\kappa \geq 0$ and can decide whether or not to default on these bonds by choosing $d^i = \{0, 1\}$. If $d^i = 0$, the household repays l^i, and if $d^i = 1$, the household does not repay debt l^i, loses assets a^i, and suffers an additive cost to its endowment of size κ. We let κ be determined after decisions are taken at date 0 and before decisions are taken at date 1. The distribution of the shock κ is idiosyncratic across the population of households and is determined according to an exponential probability distribution $f(\kappa) = \exp^{-\kappa}$ with an associated c.d.f. $F(\kappa)$.[8]

Each household faces the following budget constraints at $t = 0$ and $t = 1$, respectively:

$$c_0^i = y_0^i - T_0 - q^a a^i + q^l l^i, \text{ and} \tag{3}$$
$$c_1^i = y_1^i - T_1 + \left(1 - d^i\right)\left(a^i - l^i\right) - d^i \kappa, \tag{4}$$

where κ is heterogeneous across the population and stochastically determined.

Clearly, given a default cost κ, the household at date 1 defaults if $\kappa < l^i - a^i$ and it repays its debt l^i if $\kappa > l^i - a^i$. This implies that

$$c_1^i = \begin{cases} y_1^i - T_1 - \kappa & \text{if } \kappa \leq l^i - a^i \\ y_1^i - T_1 + a^i - l^i & \text{if } \kappa \geq l^i - a^i \end{cases}. \tag{5}$$

The household's problem is to choose c_0^i, c_1^i, a^i, and l^i which maximize (1) given (3) and (5). After substitution of (3) and (5) into the household's welfare (1), this means that the household's maximization problem can be written as:

$$\max_{a^i \geq 0, l^i \geq 0} \left\{ \log\left(y_0^i - T_0 - q^a a^i + q^l l^i\right) + \mathbb{E}_\kappa\left[\log\left(y_1^i - T_1 - \min\left\{\kappa, l^i - a^i\right\}\right)\right] \right\}. \tag{6}$$

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[8]This distributional function assumption is for simplicity. It guarantees that expected total repayment (taking into account default risk) rises with the size of a loan and that the household's problem is concave.

In this economy, we refer to $1/q^a$ as the deposit rate and $1/q^l$ as the borrowing rate and the interest spread as corresponding to $1 - q^l/q^a$. Moreover, the total supply of safe assets can be represented by

$$A = \sum_{i=P,R} \frac{1}{2} a^i, \qquad (7)$$

and the total supply of private borrowing can be represented by

$$L = \sum_{i=P,R} \frac{1}{2} l^i. \qquad (8)$$

2.1.2 Government

The government chooses taxes T_0 and T_1 and government debt $B \geq 0$ to satisfy its dynamic budget constraint at dates 0 and 1:

$$0 = T_0 + q^g B \text{ and} \qquad (9)$$
$$0 = T_1 - B \qquad (10)$$

where q^g is the price of government bonds. Note that in contrast to households, the government is committed to always repaying its debt.

2.1.3 Financial Intermediation

There is a set of perfectly competitive financial intermediaries in the economy who trade with households and the government anonymously. These financial intermediaries sell safe assets to households at price q^a, and they buy non-defaultable bonds from the government at price q^g and defaultable bonds from the private sector at price q^l. The bonds purchased from the private sector are all pooled together, independently of how much each individual household borrows, which is why all households sell their private bonds at the same price q^l.[9] Competition in financial intermediation thus requires that

$$q^g = q^a \qquad (11)$$

since the interest rate for riskless lending from intermediaries to the government must equal the interest rate for riskless lending from households to financial intermediaries.

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[9]We have considered an economy where households buy government bonds through financial intermediaries. All of our results are unchanged if households instead buy these bonds directly from the government.

Furthermore, no arbitrage requires that

$$q^l = q^a (1 - D) \tag{12}$$
$$\text{where } D = \frac{\sum_{i=P,R} F(l^i - a^i) l^i}{\sum_{i=P,R} l^i}.$$

$1 - D$ represents an aggregate recovery rate. This equation states that an intermediary can achieve the same expected return—taking default risk into account—by buying non-defaultable government bonds versus buying defaultable private bonds.[10] Note that the probability of being repaid by a household with net borrowing $l^i - a^i$ is $1 - F(l^i - a^i)$, since default occurs if the default cost κ is below $l^i - a^i$, where we define $F(l^i - a^i) = 0$ if $l^i - a^i < 0$. Equation (12) implies that $q^l < q^a$ if $l^i - a^i > 0$ for some i and $q^l \to q^a$ as $l^i \to 0$ for $i = P, R$.[11] As such, if $l^P = l^R = 0$, we write that $q^l = q^a$.

Total lending in the economy must equal total borrowing. Therefore, it follows that A defined in (7) satisfies

$$A = B + \frac{q^l}{q^a} L.^{12} \tag{13}$$

In other words, safe assets are backed by government bonds and pooled risky loans to households. We refer to the sum of pooled risky loans to households as privately-issued safe assets.

2.2 Competitive Equilibrium

2.2.1 Definition of Competitive Equilibrium

Given a government policy $\{B, T_0, T_1\}$, a competitive equilibrium corresponds to a set of prices $\{q^a, q^l\}$ and a level of deposits and borrowing $\{a^i, l^i\}_{i=P,R}$ which satisfy the following conditions:

1. $\{a^i, l^i\}_{i=P,R}$ maximize (6) for $i = P, R$ given T_0, T_1, q^a, and q^l,

2. $\{B, T_0, T_1\}$ satisfy the government budget constraints $(9) - (10)$ given (11),

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[10]All households face the same bond price since they borrow and lend anonymously. An alternative formulation would allow for non-anonymous non-linear pricing for deposits and loans such as in Livshits, MacGee, and Tertilt (2007). Given our focus on interest spreads in the economy, we focus on this simpler formulation for our analysis.

[11]Formally, if $l^P = \epsilon$ and $l^R = \varrho\epsilon$ for some $\epsilon, \varrho > 0$, one can consider the value of (12) as ϵ approaches 0. From L'Hopital's rule and our assumption of an exponential distribution for $F(\cdot)$, we find that the right hand side of (12) approaches q^a as ϵ approaches 0.

[12]This equation can be derived directly by combining the resource constraint (2)—which binds at date 0—with the household's budget constraint (3) and the government's budget constraint (9).

3. $\{a^i, l^i\}_{i=P,R}$ and $\{q^a, q^l\}$ satisfy the no arbitrage condition (12), and

4. $\{a^i, l^i\}_{i=P,R}$ and $\{q^a, q^l\}$ satisfy the market clearing condition (13).

2.2.2 Characterization of Competitive Equilibrium

In considering the household's problem in (6), note that if $q^l < q^a$ it is suboptimal for any household to choose $a^i > 0$ and $l^i > 0$ simultaneously. Any such choice is dominated by choosing instead $a^i - \epsilon$ and $l^i - \epsilon$ for $\epsilon > 0$ arbitrarily small which strictly increases date 0 consumption while keeping date 1 consumption fixed. It follows then that $l^i > 0$ only if $a^i = 0$, which implies that default only occurs if $l^i > \kappa$.[13]

In considering the household's optimal decision, there are two cases to consider. In the first case, $a^i \geq 0$ and $l^i = 0$. Maximization of (6) implies:

$$q^a \frac{1}{y_0^i - q^a (a^i - B)} \geq \frac{1}{y_1^i + (a^i - B)} \tag{14}$$

which holds with equality if $a_i > 0$. Note that we have substituted in for T_0 and T_1 in (14) using $(9)-(11)$. The government finances a tax break in the initial period by borrowing at the same deposit rate at which households save. This means that the ensuing allocation is equivalent to one in which each individual household faces taxes equal to zero and decides to deposit $a^i - B$. In other words, by choosing B, the government is borrowing on behalf of all households at the deposit rate. As we will see, this observation is useful for understanding how public debt affects households' decisions.

In the second case, $a^i = 0$ and $l^i \geq 0$, and default at date 1 occurs if $\kappa < l^i$. Maximization of (6)—where again we substitute in for T_0 and T_1 using $(9)-(11)$—implies

$$q^l \frac{1}{y_0^i + q^a B + q^l l^i} \leq \left(1 - F\left(l^i\right)\right) \frac{1}{y_1^i - B - l^i} \tag{15}$$

which holds with equality if $l^i > 0$. Equation (15) shows that the allocation in this case is equivalent to one in which each individual household faces taxes equal to zero, borrows an amount B at a bond price q^a, and borrows an additional amount l^i at a bond price q^l. This amount l^i is repaid with probability $1 - F\left(l^i\right)$ at date 1.

It is clear in this economy that the rich save more than the poor because of a consumption smoothing incentive. Moreover, because the economy is closed, it is not possible for

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[13]If instead $q^a = q^l$, one can show that the equilibrium consumption and the values of $a^i - l^i$ for each household are uniquely determined. However, the values of a^i and l^i are not uniqely determined. In this case, we select $a^i > 0$ and $l^i = 0$ so as to be consistent with the $q^a > q^l$ case, and this is without loss of generality for our main results.

both the rich and the poor to be borrowing. Therefore, only the poor potentially borrow, and it follows that (12) reduces to

$$q^l = q^a \left(1 - F\left(l^P\right)\right). \tag{16}$$

The below lemma states this formally.

Lemma 1 *In a competitive equilibrium, $a^R > 0$ and $l^R = 0$. Moreover, if $l^P > 0$ then $a^P = 0$.*

There are two cases to consider when describing a competitive equilibrium conditional on government policy B.

High Debt Case We first consider the equilibrium when public debt is very high and completely crowds out privately-issued safe assets.

Lemma 2 (High Public Debt) *Suppose that $B \geq B^* = \Delta/2$. Then $A = B$ (full crowd out), $q^a = q^l = 1$ (zero interest spread), and consumption is unresponsive to local changes in B for $B > B^*$ (local Ricardian equivalence).*

If government debt is sufficiently high, then all deposits are backed by government bonds, and privately-issued safe assets are zero. Both rich and poor households hold positive deposits and do not borrow because the borrowing rate is too high. Given that there is no borrowing, there are no defaults, and the interest spread between the borrowing and deposit rate is zero. Furthermore, Ricardian Equivalence holds locally. An increase in government debt B by $\epsilon > 0$ causes a reduction in taxes in the initial date by $q^a \epsilon$ and an increase in future taxes by ϵ. In response to such a debt increase, both rich and poor households increase deposits by $q^a \epsilon$ without altering consumption. An analogous reasoning holds with respect to a reduction in B starting from $B > B^*$.

Low Debt Case Local Ricardian Equivalence does not hold once government debt becomes sufficiently low. If $B = B^*$, it is the case $a^P = 0$, meaning poor households hold zero deposits, and they cannot reduce deposits in response to a reduction in B. The below proposition describes the equilibrium for $B < B^*$.

Proposition 1 (Low Public Debt) *Suppose that $B < B^*$. Then $A > B$ (no full crowd out), $q^a > q^l$ (positive interest spread), and consumption responds to local changes in B. A local increase in B results in:*

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1. *A reduction in q^a and q^l (higher deposit and borrowing rates),*

2. *An increase in q^l/q^a (lower interest spread), and*

3. *A decrease in $q^l l^P$ by less than the increase in $q^a B$ (partial crowd out).*

Proposition 1 provides the first main result of this paper. For low levels of public debt, government debt does not fully crowd out private debt. The borrowing rate is low enough that there is a positive level of private borrowing, there are defaults in equilibrium, and there is a positive interest spread between the borrowing and deposit rate. Moreover, consumption changes in response to changes in fiscal policy. More specifically, an increase in government debt causes an increase in the deposit and borrowing rates (reduction in q^a and q^l), a decrease in the interest spread (increase in q^l/q^a), and partial crowd out of private borrowing by the poor.

The intuition is that an increase in government debt increases the demand for overall borrowing in the economy (by the government and households), which puts upward pressure on the deposit rate. A rise in the deposit rate puts upward pressure on the borrowing rate which reduces private borrowing and reduces defaults in the overall economy, which is reflected in a lower interest spread.

The mechanism behind this result comes from the fact that the rise in borrowing by the government *partially* crowds out private borrowing. To understand the logic for this channel, suppose that households were committed to debt repayment. Since default risk would be zero in this case, the borrowing rate would equal the deposit rate. What happens if public debt rises by ϵ in this case? A rich household's deposits increase by ϵ by the same logic as in the case of full crowd out described previously. Analogously, a borrowing household's debt decreases by ϵ. All households anticipate higher taxes in the future to finance the public debt increase and would therefore utilize the tax reduction today to either increase deposits or reduce borrowing without impacting consumption.

In contrast to this hypothetical case, households in our environment cannot commit to debt repayment. There is a spread between the borrowing rate and the deposit rate which reflects default risk. Because the government borrows at a lower interest rate than households, a public debt increase cannot mechanically have a neutral impact on a borrower's consumption. If private borrowing were reduced one for one with public borrowing, a borrowing household would maintain the same consumption today but increase its consumption tomorrow. Formally, prior to the change in B, a poor borrowing household's

Euler equation (15) (which binds) after using (16) to substitute in for q^l becomes

$$q^a \frac{1}{y_0^P + q^a B + q^l l^P} = \frac{1}{y_1^P - B - l^P}.$$

Now suppose that government debt increases by ϵ and suppose that crowding out of private borrowing were one for one so that $q^l l^P$ declines by $q^a \epsilon$. If that were the case, the poor household's Euler equation would become

$$q^a \frac{1}{y_0^P + q^a B + q^l l^P} > \frac{1}{y_1^P - B - l^P + \epsilon \left(q^a / q^l - 1\right)}$$

which could never hold as an equality. Therefore, a public debt increase causes a slackening of financial constraints for borrowing households, and these households reduce their borrowing less than one for one with government borrowing. The resultant increase in total borrowing (household plus government) in the economy puts upward pressure on the deposit rate. Moreover, because each borrowing household borrows less, the probability of default declines, leading to a reduction in the interest spread.

2.2.3 Optimal Policy

Let us now consider the problem of a utilitarian government optimally choosing government debt, taking into account its effects on taxes, the deposit rate, and the borrowing rate. Without loss of generality, we can focus our attention on levels of debt $B \leq B^*$, since the choice of debt $B \geq B^*$ entails full crowd out and a consumption allocation which is invariant to debt by Lemma 2.

Using Lemma 1, Proposition 1, and (16), we can write social welfare (up to a multiplicative constant) as:

$$\log \left(y_0^R - q^a \left(a^R - B\right)\right) + \log \left(y_1^R + \left(a^R - B\right)\right) \tag{17}$$
$$+ \log \left(y_0^P + q^a [B + \left(1 - F\left(l^P\right)\right) l^P]\right) + \mathbb{E}_\kappa \left[\log \left(y_1^P - B - \min \left\{\kappa, l^P\right\}\right)\right]$$

where the first line corresponds to the welfare of the rich and the second line corresponds to the welfare of the poor. Note that q^a, a^R, and l^P are all implicit functions of B, where these necessarily satisfy the market clearing condition (13) which can be rewritten as

$$\frac{1}{2} a^R = B + \left(1 - F\left(l^P\right)\right) \frac{1}{2} l^P, \tag{18}$$

and the Euler equations of the rich (14) and the poor (15) which bind with q^a and q^l satis-

fying (16). The optimal choice of B to maximize (17) taking into account market clearing and the Euler equations yields the following first order conditions to the government's program:

$$-\frac{\partial q^a}{\partial B}\left(a^R - B\right)\frac{1}{c_0^R} + \left(\frac{\partial q^a}{\partial B}\left(a^R - B\right) - \frac{\partial l^P}{\partial B}q^a f\left(l^P\right)l^P\right)\frac{1}{c_0^P} + \left(q^a\frac{1}{c_0^P} - \mathbb{E}_\kappa\frac{1}{c_1^P}\right) \geq 0 \quad (19)$$

which holds with equality unless $B = B^*$.

The three terms in this first order condition provide some insight regarding the forces determining the government's optimal choice of debt. The first term captures the fact that a higher level of debt increases the deposit rate, and this increases the welfare of savers. More specifically, since the rich lend $(a^R - B > 0)$, an increase in government debt increases the deposit rate $(\partial q^a/\partial B < 0)$ which benefits the them. The second term captures the fact that a higher level of debt harms borrowers by increasing the deposit and borrowing rates. More specifically, the increase in deposit rate—holding the interest spread fixed—reduces the welfare of borrowers by $\frac{\partial q^a}{\partial B}\left(a^R - B\right)\frac{1}{c_0^P} < 0$. This effect however is mitigated by the reduction in the premium paid by borrowers due to a lower interest spread, and this is captured by $-\frac{\partial l^P}{\partial B}q^a f\left(l^P\right)l^P\frac{1}{c_0^P} \geq 0$, which is strictly positive whenever $l^P > 0$. The last term captures the fact that higher government debt directly relaxes the borrowing constraint of the poor. The poor would like to borrow at the same low rate as the government but are unable to. If the government increases public debt, it is equivalent to the government borrowing more on the behalf of the poor, and this relaxation of borrowing constraints increases social welfare. As such, the third term in (19) which captures the marginal gain from relaxing the poor's borrowing constraint is positive.

The optimal policy thus equalizes the marginal benefit of additional debt to its marginal cost.

Proposition 2 (optimal policy) *The optimal policy which maximizes (17) admits $B < B^*$ (partial crowd out).*

Proposition 2 states that the optimal policy involves an interior level of public debt with partial crowd out of privately-issued safe assets. To understand this result, note that if $B = B^*$, the marginal benefit of a lower interest premium for borrowers in the second term in (19) is zero. This is because borrowers are not engaging in any private borrowing. Moreover, the third term in (19) is zero; the marginal benefit of additional liquidity for borrowers is zero since they can borrow at the same interest rate as the government. As

such, (19) reduces to

$$-\frac{\partial q^a}{\partial B}\left(a^R - B\right)\left(\frac{1}{c_0^R} - \frac{1}{c_0^P}\right)$$

which is negative since $c_0^R > c_0^P$. Because borrowers have a higher marginal utility than savers, higher government debt negatively affects social welfare on the margin. Therefore, the optimal policy involves partial crowd out of privately-issued safe assets, with some private borrowing, some defaults, and a spread between the borrowing and deposit rate.

3 Infinite Horizon Model

We now build on the insights of the two-period model by analyzing the stationary distribution along the balanced growth path of an infinite horizon economy. In this section, we introduce the infinite horizon environment, and in the next section, we describe our numerical strategy for evaluating model and our quantitative results. This numerical exercise allows us to check the robustness of the theoretical results from the two-period model. It also allows us to determine the quantitative implications of our theoretical mechanism, and to examine how credit market frictions impact the optimal level of public debt.

3.1 Environment

Our environment builds on the economy of Aiyagari and McGrattan (1998) by introducing private financial intermediation with household default. We first describe the environment and then focus our analysis on a balanced growth path of this economy in which there are fluctuations in an individual household's consumption, income, and wealth, but per household variables are growing at a constant rate, with a cross-sectional distribution which is constant over time (relative to aggregate income).

3.1.1 Technology

There is a continuum of infinitely lived households of mass 1 who receive idiosyncratic shocks to their labor productivities and supply labor inelastically. Let e_t denote an household's labor productivity, and suppose that this productivity is i.i.d. across households and follows a Markov process over time. There are no aggregate shocks.

Output at date t is produced with a Cobb-Douglas technology, $Y_t = K_t^\alpha \left(z_t N_t\right)^{1-\alpha}$, where K_t is the capital stock, N_t is the aggregate labor input, and z_t is a measure of an aggregate labor-augmenting, exogenous technical efficiency in period t. We assume that $z_t = (1+\phi)^t$ where $\phi > 0$ is the rate of technical progress. Capital depreciates at the

constant rate δ. There are competitive product and factor markets with wage rate w_t and rental rate of capital r_t given by

$$w_t = (1 - \alpha) \frac{Y_t}{N_t} \text{ and } r_t = \alpha \frac{Y_t}{K_t}. \tag{20}$$

3.1.2 Households

A household has preferences

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t U(c_t), \ \beta \in (0, 1) \tag{21}$$

where $U(c_t) = \frac{c_t^{1-\gamma} - 1}{1 - \gamma}$ for $\gamma > 0$. A household with labor income $w_t e_t$ at date t faces lump sum taxes T_t and a linear labor income tax τ_t^n; holds deposits a_t and owes debt l_t; and chooses future deposits a_{t+1} and future debt l_{t+1}. If a household defaults, it pays a default cost $\kappa Y_t > 0$, where κ is i.i.d. across the population distributed with c.d.f. $F(\kappa)$. This formalization captures the fact that aggregate default costs are constant as a share of GDP along the balanced growth path.

As in the two-period economy, given a default cost κY_t, a household defaults ($d_t = 1$) if $l_t - a_t > \kappa Y_t$, and it does not default ($d_t = 0$) if $l_t - a_t < \kappa Y_t$. The household's budget constraint at date t is:

$$c_t = w_t e_t (1 - \tau_t^n) - T_t - \min\{\kappa Y_t, l_t - a_t)\} - q_t^a a_{t+1} + q_t^l l_{t+1}, \tag{22}$$

where q_t^a represents the price of a deposit and q_t^l represents the price of a bond issued by the household at date t. Note that q_t^a and q_t^l are deterministic from the perspective of the household since there are no aggregate shocks.

3.1.3 Government

The government finances expenditures G_t and outstanding debt B_t by borrowing B_{t+1} and by levying lump sum taxes T_t, labor income taxes τ_t^n, and capital income taxes τ_t^k (levied on the return to capital net of depreciation). The government budget constraint at date t satisfies

$$G_t + B_t = T_t + \tau_t^n w_t N_t + \tau_t^k (r_t - \delta) K_t + q_t^g B_{t+1}, \tag{23}$$

where q_t^g is the price of a government bond. We assume that spending G_t equals a fixed fraction of per-capita GDP, with $G = G_t/Y_t$.

3.1.4 Financial Intermediation

As in the two-period model, there is a set of perfectly competitive financial intermediaries who trade with households and the government anonymously. Moreover, they own the capital stock K_t, rent it out to firms at a rental rate r_t, and pay capital income taxes $\tau_t^k (r_t - \delta) K_t$.

As in the two-period model, competition in financial markets requires that the deposit rate equals the interest rate for government borrowing so that intermediaries make zero profit:

$$q_t^g = q_t^a. \tag{24}$$

The deposit rate must equal the after tax return on capital:

$$\frac{1}{q_t^a} = 1 + (r_{t+1} - \delta) \left(1 - \tau_{t+1}^k\right), \tag{25}$$

and this follows from the fact that intermediaries must be indifferent between holding capital and holding government bonds.

Moreover,

$$q_t^l = q_t^a \left(1 - D_{t+1}\right), \tag{26}$$

where $1 - D_{t+1}$ is the aggregate recovery rate at date $t+1$ from private lending in the economy given the distribution of private borrowing l_{t+1} and future default decisions d_{t+1}. This condition guarantees that intermediaries are indifferent between lending to the government and lending to households.

All of the newly issued assets at date t must be backed by newly issued government bonds, newly installed capital, and new loans to households. Formally,

$$A_{t+1} = B_{t+1} + \frac{1}{q_t^a} K_{t+1} + \frac{q_t^l}{q_t^a} L_{t+1}, \tag{27}$$

where A_{t+1} corresponds to the sum of a_{t+1} across the population and L_{t+1} corresponds to the sum of l_{t+1} across the population. Moreover, all of the outstanding assets at date t are backed by government bonds, the non-depreciated capital stock and the after-tax income which it generates, and the recovered private loans:

$$A_t = B_t + \left[1 + (r_t - \delta) \left(1 - \tau_t^k\right)\right] K_t + (1 - D_t) L_t. \tag{28}$$

3.2 Equilibrium along the Balanced Growth Path

Along the balanced growth path, w_t, Y_t, and K_t all grow at the rate ϕ, whereas the interest rate is constant with $r_t = r \; \forall t$. Since capital grows at the same rate of output, we let $K = K_t/Y_t$. Aggregate labor is constant and equal to $N_t = \mathbb{E}(e_t)$ since it is supplied inelastically and there is no aggregate uncertainty.

Bond prices are also constant, meaning $q_t^a = q^a$ (and therefore $q_t^g = q^a$) and $q_t^l = q^l \; \forall t$. Government debt and lump sum taxes grow at the same rate as GDP with $B_t/Y_t = B$ and $T_t/Y_t = T$. Labor and capital income tax rates are constant with $\tau_t^n = \tau^n$ and $\tau_t^k = \tau^k$. Given (20), this means that the government budget constraint (23) can be represented in terms of quantities normalized by aggregate income:

$$G + B = T + \tau^n (1 - \alpha) + \tau^k (\alpha - \delta K) + q^a (1 + \phi) B, \tag{29}$$

For a given household, we define normalized consumption $\widetilde{c}_t = c_t/Y_t$. Since Y_t grows at a constant rate ϕ, household preferences (21) can be represented as

$$\left(K^{\alpha/(1-\alpha)} \mathbb{E}(e_t)\right)^{1-\gamma} \mathbb{E} \sum_{t=0}^{\infty} \widetilde{\beta}^t \frac{\widetilde{c}_t^{1-\gamma}}{1-\gamma}, \text{ where } \widetilde{\beta} = \beta (1 + \phi)^{1-\gamma}, \tag{30}$$

and we have taken into account that $Y_0 = \left(\frac{K_0}{Y_0}\right)^{\alpha/(1-\alpha)} \mathbb{E}(e_t)$ in deriving household welfare (30). Define normalized assets and debt $\widetilde{a}_t = a_t/Y_t$ and $\widetilde{l}_t = l_t/Y_t$. Substituting (20) and (29) into (22), the household's budget constraint can be rewritten in terms of normalized quantities:

$$\widetilde{c}_t = y(e_t) - \min\left\{\kappa + B, \widetilde{l}_t - (\widetilde{a}_t - B)\right\} - (1 + \phi)\left(q^a (\widetilde{a}_{t+1} - B) - q^l \widetilde{l}_{t+1}\right) \tag{31}$$

for
$$y(e_t) = (1 - \alpha)(1 - \tau^n) e_t/\mathbb{E}(e_t) + \tau^n (1 - \alpha) + \tau^k (\alpha - \delta K) - G. \tag{32}$$

Note that for the purposes of our later discussion, we have incorporated the level of government debt B directly into the household's budget constraint in its relation to household assets $\widetilde{a}_t$.

If we define the highest realized default cost κ as $\overline{\kappa}$, it follows that a household's choice of normalized private debt $\widetilde{l}_t$ is bounded from above by some $\overline{l} = \overline{\kappa} > 0$, since any debt in excess of this amount would never be repaid. As such, we can write a household's problem at any date t recursively in terms of normalized quantities:

$$V\left(\widetilde{a}, \widetilde{l}, e, \kappa\right) = \max_{\widetilde{c}, \widetilde{a}' \geq 0, \widetilde{l}' \in [0, \widetilde{l}]} \left\{ \frac{\widetilde{c}^{1-\gamma}}{1-\gamma} + \widetilde{\beta}\mathbb{E}\left[V\left(\widetilde{a}', \widetilde{l}', e', \kappa', \right) | e\right] \right\} \tag{33}$$

s.t.

$$\widetilde{c} = y\left(e\right) - \min\left\{\kappa + B, \widetilde{l} - (\widetilde{a} - B)\right\} - (1 + \phi)\left(q^a\left(\widetilde{a}' - B\right) - q^l\widetilde{l}'\right) \tag{34}$$

Note that since $q^a > q^l$, a household never chooses $\widetilde{a}' > 0$ and $\widetilde{l}' > 0$ by analogous logic as in the two-period economy. Therefore, default tomorrow occurs whenever $\kappa < \widetilde{l}'$, which occurs with probability $F\left(\widetilde{l}'\right)$, otherwise a household repays $\widetilde{l}'$.

Let $\omega = \left\{\widetilde{a}, \widetilde{l}, e, \kappa\right\} \in \Omega$ represent the state for a given household and let $\widetilde{a}\left(\omega\right)$ and $\widetilde{l}\left(\omega\right)$ denote the values of $\widetilde{a}$ and $\widetilde{l}$ associated with ω. Letting $\widetilde{a}^*\left(\omega\right)$ and $\widetilde{l}^*\left(\omega\right)$ denote the optimal choices of $\widetilde{a}'$ and $\widetilde{l}'$ which solve $(33) - (34)$ given ω, we can define a probability density function $\Gamma\left(\omega\right)$ in the population under the stationary equilibrium associated with policies $\widetilde{a}^*\left(\omega\right)$ and $\widetilde{l}^*\left(\omega\right)$. Given Γ, we can define the aggregate (normalized) assets $\widetilde{A}$, the aggregate (normalized) private loans $\widetilde{L}$, and the aggregate recovery rate $1 - D$ with

$$\widetilde{A} = \int_{\omega \in \Omega} \widetilde{a}\left(\omega\right)\Gamma\left(\omega\right)d\omega, \tag{35}$$

$$\widetilde{L} = \int_{\omega \in \Omega} \widetilde{l}\left(\omega\right)\Gamma\left(\omega\right)d\omega, \text{ and} \tag{36}$$

$$D = \frac{\int_{\omega \in \Omega} \widetilde{l}\left(\omega\right)\mathbb{I}_{\kappa < \widetilde{l}}\Gamma\left(\omega\right)d\omega}{\int_{\omega \in \Omega} \widetilde{l}\left(\omega\right)\Gamma\left(\omega\right)d\omega}, \tag{37}$$

where $\mathbb{I}_{\kappa < \widetilde{l}}$ is an indicator function which equals one when the default shock is smaller than outstanding debt, and otherwise $\mathbb{I}_{\kappa < \widetilde{l}}$ equals zero. Given these aggregates, the no arbitrage conditions in $(24) - (26)$ can be written as independent of time along the balanced growth path. Moreover, by analogous reasoning, market clearing conditions (27) and (28) can be normalized by output and also written as independent of time.

A competitive equilibrium along the balanced growth path given policies $\left\{B, G, \tau^n, \tau^k, T\right\}$ corresponds to a set of prices $\left\{q^a, q^l, r\right\}$, aggregate quantities $\left\{K, \widetilde{A}, \widetilde{L}\right\}$, an aggregate default rate D, and a joint stationary distribution Γ which satisfy the following conditions:

1. $\widetilde{a}^*\left(\omega\right)$ and $\widetilde{l}^*\left(\omega\right)$ solve the household's optimization program in $(33) - (34)$,

2. Firms maximize profits given wages and rental rates so that (20) holds,

3. The government budget constraint (29) is satisfied,

4. There is no arbitrage in financial markets so that $(24) - (26)$ hold, and

5. The market clears so that (normalized) $(27) - (28)$ hold.

4 Quantitative Exercise

In this section, we quantitatively assess the properties of the stationary distribution along the balanced growth path. We begin by describing our parameter choices for this exercise. We then move to assess the impact of government debt on privately-issued safe assets, interest rates, and inequality. We conclude by characterizing the optimal level of public debt which maximizes social welfare along the balanced growth path.

4.1 Parameters and Computation

We now describe our choice of parameters.

4.1.1 Technology

We choose standard parameters following Aiyagari and McGrattan (1998), with $\alpha = 0.3$ for the capital share of income, $\delta = 0.075$ for the capital depreciation rate, and $\phi = 0.0185$ for the growth rate of labor-augmenting technical change.

We model the process of idiosyncratic labor productivity following Guvenen, Ozkan, and Song (2014):

$$
\begin{aligned}
\ln e_t &= \ln \nu_t + \epsilon_t, \ \epsilon_t \sim N\left(0, \sigma_\epsilon^2\right) \\
\ln \nu_t &= \rho \ln \nu_{t-1} + \eta_t, \ \text{where} \\
\eta_t &= \begin{cases} \eta_{1t} \sim N\left(\mu_1, \sigma_1^2\right) & \text{with probability } 1 - p \\ \eta_{2t} \sim N\left(\mu_2, \sigma_2^2\right) & \text{with probability } p \end{cases}.
\end{aligned}
$$

We let $\sigma_\epsilon^2 = 0.186$, $\rho = 0.979$, $\mu_1 = 0.119$, $\mu_2 = -0.114$, $\sigma_1^2 = 0.325$, $\sigma_2^2 = 0.001$, and $p = 0.49$.[14] The process for the persistent component of earnings, ν_t, is generated using a four-point discretization as in Civale, Diez-Catalan, and Fazilet (2017). The idiosyncratic component of earnings ϵ_t is estimated with two equal probability shocks around 0.

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[14]This process comes from Model 1 in Table 1 for normal times in Guvenen, Ozkan, and Song (2014). Given our discretization of the earnings process, μ_2 is chosen to ensure that the mean of $\log e_t$ is zero in the stationary distribution.

4.1.2 Households

Following Aiyagari and McGrattan (1998), we set the CRRA parameter to $\gamma = 1.5$ and the discount rate to $\beta = 0.971$ so that each time period corresponds to a year.[15]

4.1.3 Government

Following Aiyagari and McGrattan (1998), we choose the value of public debt to GDP $B = 0.67$ and government spending to GDP, $G = 0.20$. Following Domeij and Heathcoate (2004), we set the tax rate on labor income $\tau^n = 0.27$ and the tax rate on capital income to $\tau^k = 0.40$. The value of lump sum taxes T is chosen so as to satisfy the government budget constraint.

4.1.4 Financial Intermediation

We are left to choose the stochastic process for the default cost κ. We let κ take on two possible values, $\{\underline{\kappa}, \overline{\kappa}\}$, where $\Pr\{\kappa = \underline{\kappa}\} = \chi$. We choose $\underline{\kappa}$, $\overline{\kappa}$, and χ jointly so as to target three empirical moments in the U.S. economy: the aggregate recovery rate of private loans, $1 - D$, which is 97.2 percent;[16] the percentage of households with negative net worth with $a_t = 0$, which is 17.6 percent;[17] and the proportion of households in default which is 0.20 percent.[18]

4.1.5 Balanced Growth Path

Table 1 summarizes the calibrated parameters together with targeted moments. Table 2 summarizes the parameters which are set exogenously (that is, outside the model).

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[15]This corresponds to the parameterization in which Aiyagari and McGrattan (1998) focus on the liquidity role of public debt by adjusting changes in public debt with changes in lump sum taxes (see p. 463).

[16]This represents the average charge-off rate on consumer loans from 1995 to 2017. See CORCACBS in FRED, https://fred.stlouisfed.org/series/CORCACBS.

[17]This is the fraction of households with negative net worth in 2013 according to the U.S. Census.

[18]The fraction of households filing for bankruptcy is calculated for 2017 using the same method as Chatterjee, et al (2007). See http://www.uscourts.gov/statistics-reports/caseload-statistics-data-tables.

Table 1. Calibrated Parameters

Parameter	Value	Target	Model	Data
Low default cost, $\underline{\kappa}$	2.000	Recovery rate on private loans	0.972	0.972
High default cost, $\overline{\kappa}$	2.280	Fraction indebted households	0.179	0.176
$\Pr\{\kappa = \underline{\kappa}\}, \chi$	0.116	Percentage households in default	0.20%	0.20%

Table 2. Exogenous Parameters

Parameter	Value
Technology	
Capital share, α	0.30
Depreciation rate, δ	0.075
Growth rate of technology, ϕ	0.0185
Earnings process:	
Variance of temporary shock, σ_ϵ^2	0.186
Normal mixture, $\{\mu_1, \sigma_1^2, \mu_2, \sigma_1^2, p\}$	$\{0.119, 0.325, -0.114, 0.001, 0.49\}$
Autocorrelation, ρ	0.979
Preferences	
CRRA coefficient, γ	1.5
Discount factor, β	0.971
Policy	
Public debt / GDP, B	0.67
Government spending / GDP, G	0.20
Labor income tax rate, τ^n	0.27
Capital income tax rate, τ^k	0.40

We discuss the numerical method used to compute the stationary distribution in detail in the Appendix. The computational algorithm—based on the discretization of the state space—is standard, with the exception that relative to the baseline Aiyagari and McGrattan (1998) model, there are two assets $\widetilde{a}$ and $\widetilde{l}$ and two prices q^a and q^l, instead

of one asset and one price. To simplify the problem, we appeal to the fact that a household never chooses to simultaneously borrow and lend. This allows us to reduce the two state variables $\{\widetilde{a}, \widetilde{l}\}$ in (33) into a single state variable. Our procedure is based on first guessing q^a and q^l, solving for the household's problem, and then updating our guesses of q^a and q^l based on how well the implied equilibrium satisfies the no arbitrage condition (26) and the market clearing condition (27).

Table 3 summarizes the moments from the balanced growth path.

Table 3. Moments from Balanced Growth Path

Variable	Value
Capital / GDP, K	2.01
Deposit rate, $1/q^a - 1$	4.4%
Borrowing rate, $1/q^l - 1$	7.2%
Privately-issued safe assets / GDP, $\widetilde{L}$	0.161

The distribution of after-tax income, $w_t e_t (1 - \tau_t^n) - T_t$, along the balanced growth path and its comparison to the U.S. data is summarized in Table 4.[19] Our cross-sectional distribution of income is very close to that in the data.

Table 4. Income Distribution in Model vs Data

Variable	Data (PSID)	Model
Q1 share	4.5	5.7
Q2 share	9.9	12.0
Q3 share	15.3	13.2
Q4 share	22.8	26.7
Q5 share	47.5	42.4
Gini	0.42	0.37

The distribution of net worth $\widetilde{a} - \widetilde{l}$, along the balanced growth path and its comparison to the U.S. data is summarized in Table 5.[20] Our cross-sectional distribution of wealth is close to the difference between the bottom 40 percent and the top 60 percent of the wealth distribution. We note that as is common in heterogeneous agent economies, our model misses the concentration of wealth in the top quintile (see De Nardi, Fella, and Pardo, 2016). In the Appendix, we extend the environment to allow for shocks to household net

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[19]The distribution from PSID is from Kruger, Mitman, and Perri (2016) Table 1.

[20]The distribution from SCF is from Krueger, Mitman, and Perri (2016) Table 1.

worth following Hubmer, Krusell, and Smith (2017). We show that such an extension allows us to better match the empirical concentration at the top and does not change the qualitative predictions of our numerical model.

Table 5. Wealth Distribution in Model vs Data

Variable	Data (SCF)	Model
Q1 share	-0.2	-5.6
Q2 share	1.2	6.1
Q3 share	4.6	16.6
Q4 share	11.9	29.3
Q5 share	82.5	53.7
Gini	0.78	0.60

4.2 Effect of Changing Public Debt

We now explore the effect of changing the level of public debt B. Starting from our computed stationary distribution, we change B, keeping all other exogenous parameters fixed and accommodating changes in B with changes in lump sum taxes T, as in the two-period model of Section 2. In principle, given the government budget constraint (29), higher public debt can be associated with increases in any combination of the labor income tax rate, capital income tax rate, or lump sum tax. Our formulation which focuses on adjusting the lump sum tax T allows us to consider the role of public debt as a substitute for private liquidity without confusing this role with that of redistributive taxation which affects the income process $y(e_t)$.

More formally, the household's problem in $(33) - (34)$ is equivalent to letting the household issue some non-defaultable debt $B - \widetilde{a}' \leq B$ at price q^a and defaultable debt $\widetilde{l}' \leq \bar{l}$ at price $q^l < q^a$ given an income process $y(e_t)$. Since households clearly prefer to issue non-defaultable debt, an increase in government debt B is equivalent to relaxing financing constraints on households by allowing them to issue more non-defaultable debt. In this sense, public liquidity here serves as a substitute for private liquidity, and an increase in government debt relaxes household's borrowing constraints without directly affecting the distribution of income.

4.2.1 Crowd Out of Privately-Issued Safe Assets

Figure 1 illustrates the impact of changing public debt B on total assets $\widetilde{A}$ and on the capital stock plus privately-issued safe assets, $\frac{1}{q^a}K + \frac{q^l}{q^a}\widetilde{L}$. As in the two-period economy of Section 2, an increase in public debt increases total assets in the economy while simultaneously crowding out privately-issued safe assets. In this dynamic environment with capital, higher public debt also crowds out capital, an effect not captured in our two-period model. This crowding out effect reduces the level of output along the balanced growth path.

More specifically, an increase in government debt by 1 percent of GDP reduces privately-issued safe assets by 0.051 percent of GDP and the capital stock by 0.076 percent of GDP.[21]

Figure 1: Public Debt and Assets

As in the two-period economy, the crowd-out of privately-issued assets is partial. For example, in the neighborhood of a public debt to GDP ratio of 67 percent, the average borrowing household reduces its borrowing by 0.29 percent of GDP in response to an increase in public debt to GDP by 1 percent. Recall from our discussion of the two-

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[21]The crowding out of privately-issued safe assets in the model is qualitatively consistent with the empirical evidence of crowd out (e.g., Gorton, et al, 2012 and Krishnamurthy and Vissing-Jorgensen, 2015). The magnitude of the coefficient in the model cannot be directly compared to the coefficient in the data, however, since many privately-issued safe assets in the data are also backed by capital projects.

period model that, under Ricardian Equivalence, an increase in public debt reduces private borrowing one for one for every borrowing household. However, in the presence of financial market frictions, this is no longer the case since the government is able to borrow at a lower interest rate than households, so that an increase in public debt slackens borrowing households' financial constraints.[22]

4.2.2   Interest Rates and Default

Since an increase in public debt raises overall borrowing demand, this puts upward pressure on the deposit rate. Figure 2 displays the positive relationship between public debt and the deposit rate $(1/q^a - 1)$.
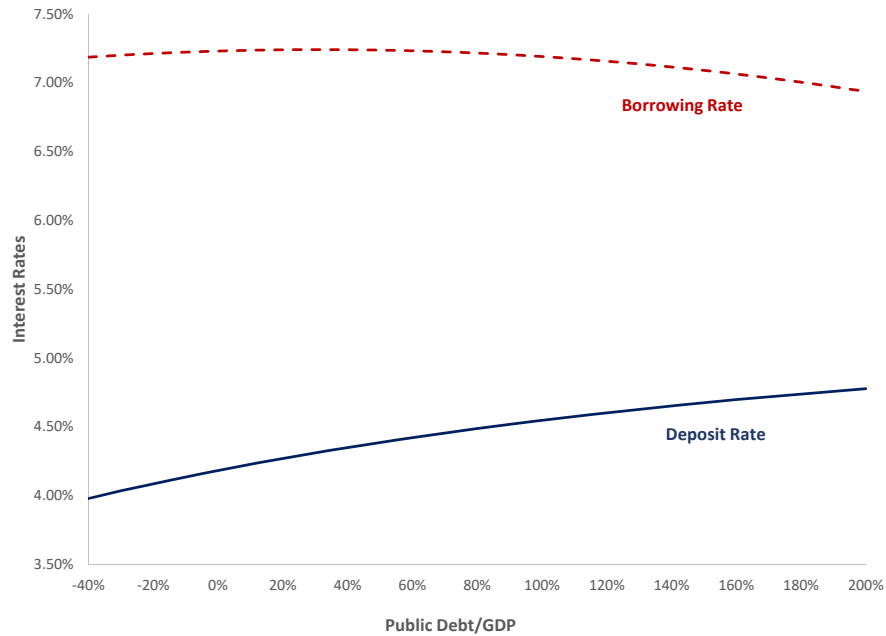

Figure 2: Public Debt and Interest Rates

Our analysis suggests that an increase in government debt by 1 percent of GDP increases the deposit rate by 0.003 percent. This effect is consistent with numerous empirical studies which find that higher government debt increases interest rates.[23]

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[22]Note that in the presence of Ricardian Equivalence, measured crowd out of privately-issued safe assets in the overall economy (not per borrower) is mechanically less that one for one and equal to the fraction of households that are privately borrowing and are reducing their individuals loans one for one. Barro, Fernández-Villaverde, Levintal, and Mollerus (2017) make a similar point in their model of safe assets where Ricardian Equivalence holds.

[23]The quantitative magnitude of this estimate is in the lower the range of the effect documented in the literature. See Haugh, Ollivaud, and Turner (2009) Table 1 for a survey of these estimates.

Note that the impact of higher public debt on the borrowing rate $(1/q^l - 1)$ is non-monotonic; it is positive for low debt and negative for high debt. The positive effect is due to the higher deposit rate. The negative effect is due to the reduction in default risk in the economy given the crowd out of private borrowing. Figure 3 shows that the fraction of households in default declines as public debt rises.
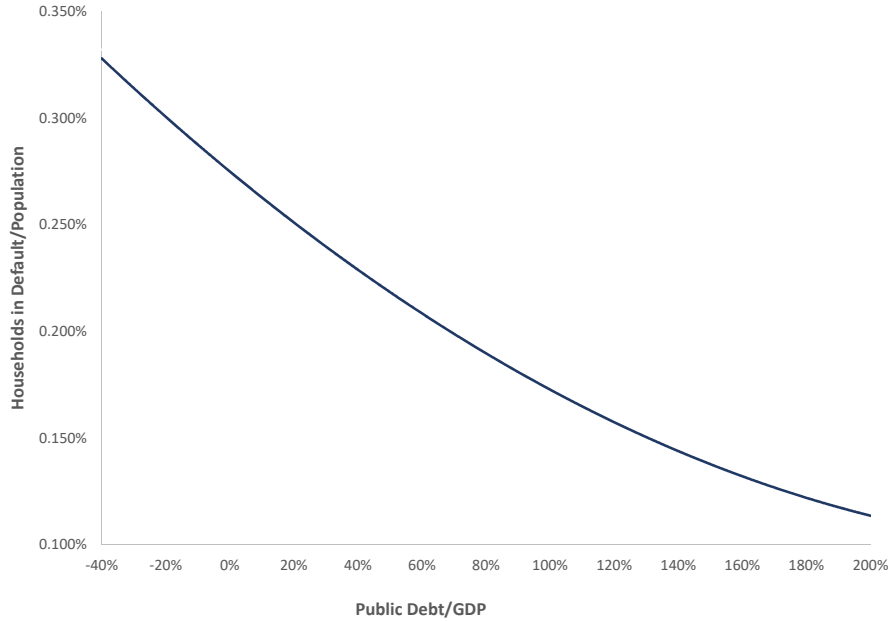

Figure 3: Public Debt and Defaults

Figure 4 displays the interest rate implications of this decline in defaults. As in the two-period model, as public debt rises, the interest spread—measured here as $1/q^l - 1/q^a$—falls. Since this spread is positively correlated with the number of aggregate defaults, it falls as defaults fall. More specifically, an increase in government debt by 1 percent of GDP reduces the interest spread by 0.0043 percent. The effect of public debt on the interest spread is qualitatively consistent with the empirical evidence. For example, Cortes (2003) and Krishnamurthy and Vissing-Jorgensen (2012) document a negative relationship between public debt and various measures of interest spreads, such as the spread between the return on corporate bonds and the return on government debt.[24]

In sum, Figures 2-4 depict how higher public debt reduces financial market inefficiencies by reducing interest spreads and reducing default. A government is able to reduce these inefficiencies since it can commit to repaying debt and can consequently borrow at a cheaper interest rate than the private sector.

[24]We find similar patterns when analyzing net chargeoff rates on consumer bank loans, a proxy for aggregate default. Details available upon request.
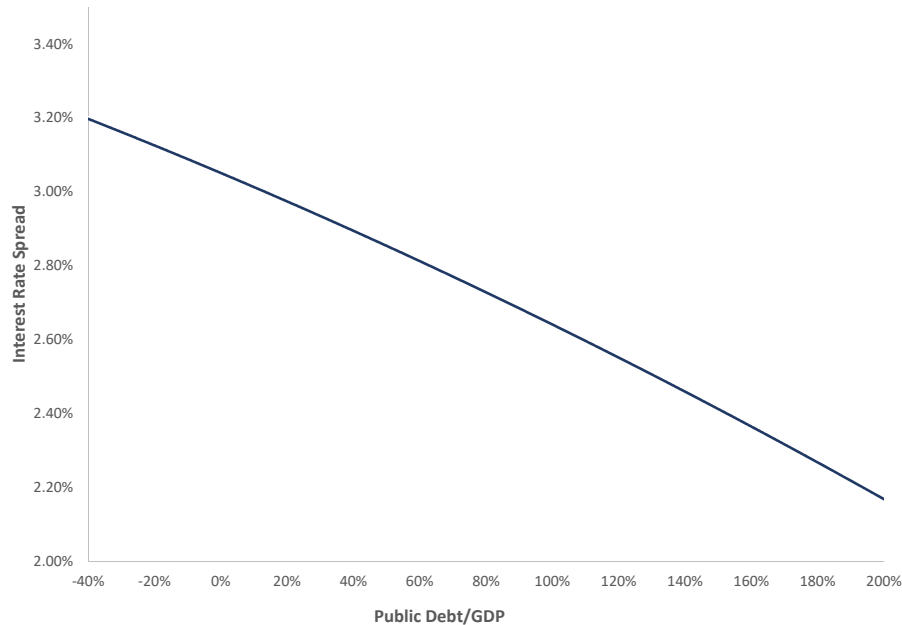
Figure 4: Public Debt and the Interest Spread

4.2.3   Inequality

Figure 5 illustrates the distributional consequences of increasing public debt. The y-axis in Figure 5 represents the average welfare for each quintile of the welfare distribution. We have demeaned welfare for each group by the welfare under a government debt to GDP ratio -40 percent, which is the minimum of the range in the figure. We represent the change in welfare relative to this baseline in consumption equivalent terms.

Changes in public debt have very different consequences for different segments of the population. For illustration, an increase in public debt from -40 to 100 percent of GDP reduces the welfare of the bottom quintile by 0.72 percent in consumption equivalent terms and increases the welfare of top quintile by 1.17 percent in consumption equivalent terms.

These observation are consistent with our results in the two-period example, and the intuition is similar. As public debt rises, interest rates rise, and this benefits the wealthy who save and harms the poor who borrow. Therefore, the rich in the top quintiles are made strictly better off as public debt rises, while the poor in the bottom quintile are made strictly worse off. Therefore, even though an increase in public debt makes financial markets more efficient by relaxing financial constraints, it also increases inequality.
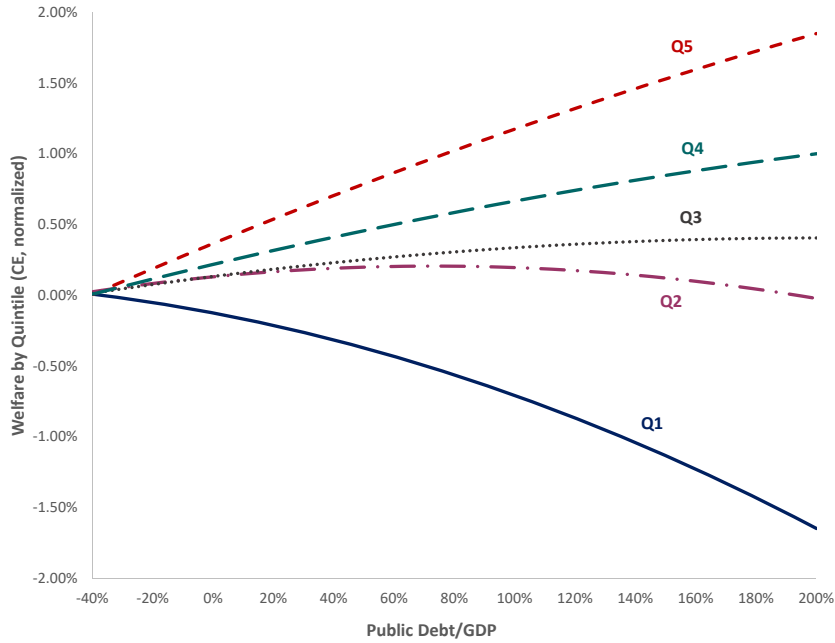
Figure 5: Public Debt and Welfare Inequality

4.3   Optimal Public Debt

We now consider the optimal level of public debt which maximizes social welfare along the balanced growth path.[25] We then describe how this value depends on financial market frictions.

4.3.1   Optimal Public Debt in Benchmark Model

Figure 6 depicts aggregate welfare as a function of public debt (in consumption equivalent terms and demeaned by the welfare under a government debt to GDP ratio -40 percent). For low values of public debt, welfare rises in public debt since the benefit of reducing financial market inefficiencies outweighs the cost of rising inequality and a lower capital stock. For high values of public debt, the marginal benefit of reducing financial frictions declines by more than the marginal cost of rising inequality and a lower capital stock. For this reason overall welfare declines.

We find that the optimal value of public debt is 145 percent of GDP. Relative to our starting point with a value of debt equal to 67 percent of GDP, the interest spread now is 2.4 percent versus 2.8 percent and the fraction of indebted households is 14.4 percent versus 17.9 percent. The optimal value of public debt does not fully crowd out

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[25]Social welfare along the balanced growth path corresponds to the expectation of (30) across the population.
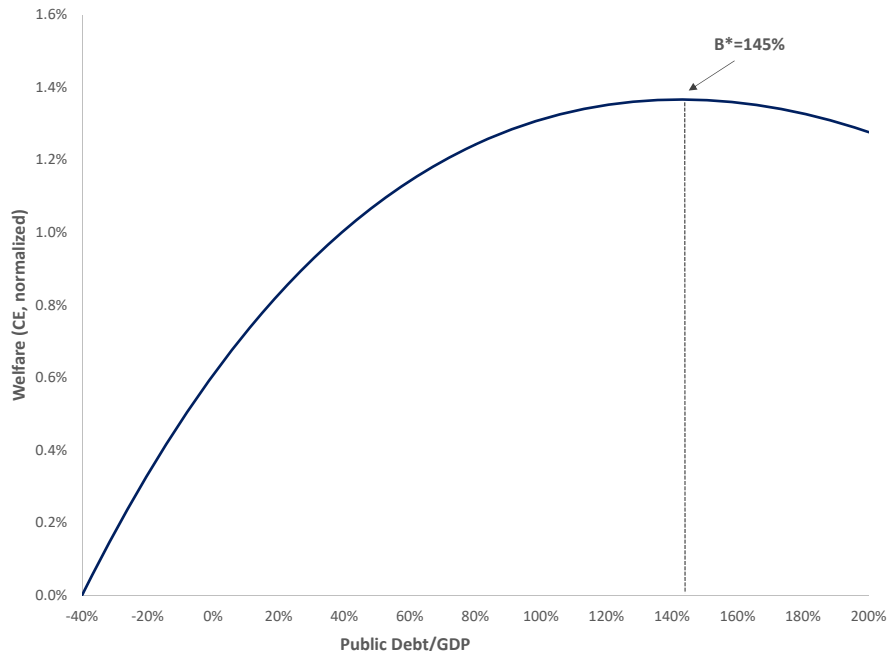
27

Figure 6: Public Debt and Welfare

privately-issued safe assets which are equal to 13 percent of GDP. A partial equilibrium calculation suggests that the value of public debt required for such full crowd out would be significantly larger at 393 percent of GDP. Our quantitative result is thus consistent with the qualitative results in the two-period model in Proposition 2.[26,27]

4.3.2   Role of Financial Frictions

Table 6 considers the effect of different levels of financial frictions on the economy. Column 1 describes the benchmark environment under the optimal public debt policy. Column 2 considers the optimal policy in an economy without private borrowing with $\bar{l} = 0$, in which case the optimal value of public debt is 198 percent of GDP.[28] Column 3 considers the optimal policy in an economy with private borrowing $\bar{l} > 0$, but without default with

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[26]The analog of Proposition 2 holds in our two-period environment if we incorporate a capital stock since the marginal cost of reducing the capital stock starting from full crowd out is zero.

[27]Given the difficulty in calculating the stationary distribution in the absence of borrowing constraints (see Krebs, 2004), we cannot explicitly calculate the value of government debt associated with full crowd out. Our partial equilibrium calculation holds the deposit rate fixed and the distribution of income $y(e_t)$ fixed and determines the value of debt needed to relax the borrowing limit of all households.

[28]For comparison, the optimal value of public debt in the model of Aiyagari and McGrattan (1998) when changes in public debt are accommodated with lump sum taxes is 140 percent (p. 463). We achieve a different result here since we consider a different process for income, have different levels of capital and labor income taxes, and have an exogenous labor supply.

$\Pr\{\kappa = \overline{\kappa}\} = 1$ (so that default is too costly for everyone). In this case, the optimal value of public debt is -30 percent of GDP. By construction, since there are no defaults in either environments, the interest spread is zero, and the main role played by government debt is in its ability to relax the borrowing constraint imposed by the private borrowing limit $\overline{l}$.

Table 6. Role of Financial Frictions

Variable	Benchmark	No Borrowing	No Default
Public debt / GDP	1.45	1.98	-0.30
Capital / GDP	1.96	1.99	1.99
Deposit rate	4.7%	4.6%	4.6%
Borrowing rate	7.1%	4.6%	4.6%
Privately-issued safe assets / GDP	0.13	0	0.43
Fraction indebted households	0.144	0	0.346

Our optimal value of public debt to GDP of 145 percent is between these two scenarios. It is below the optimal value of debt in the absence of private borrowing since government debt is not as important for relaxing liquidity constraints in our setup. It is also above the optimal value of debt in the absence of default.[29] This suggests that credit market frictions play an important role in determining the optimal level of public debt. The presence of default risk in our environment increases the liquidity benefit of public debt for two reasons.

First, relative to a default-free environment with borrowing, the fraction of households which directly benefit from public liquidity increases in our framework, since any borrowing household is subject to an interest rate premium relative to the government. To see this intuitively, turn to the optimality condition (19) for public debt in the two-period economy. Note that the third term in (19) applies to any borrowing household in our environment. In contrast, in an environment without default, the analog of the third term in (19) only applies to households at the borrowing limit $\overline{l}$, since these are the only credit constrained households.

To illustrate this points, Figure 7 compares the welfare effect of higher public debt in our benchmark environment relative to a default-free environment. The welfare of

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[29]Note that by construction, the no borrowing and the no default scenarios are equivalent in terms of steady state allocations. However, the optimal value of public debt differs in the two scenarios because of the presence of a private borrowing limit $\overline{l}$ which normalizes this optimal value.

the bottom quintile and the welfare of the top quintile are represented in consumption equivalent terms, normalized at their value at a public debt to GDP ratio of -40 percent. The figure shows that as public debt rises, the relative decline in welfare for the lowest quintile is significantly mitigated in our environment relative to an environment without default. As a comparison, this difference is not as apparent for the highest quintile, for which the change in welfare is closer across the two environments.
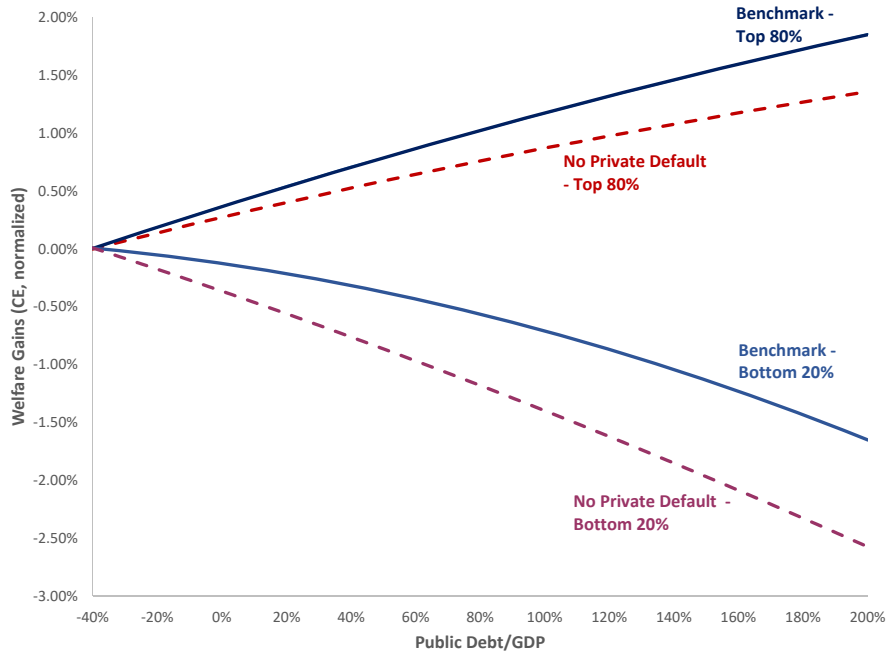

Figure 7: Welfare Gain in Benchmark vs No Private Default

The second reason why the presence of default risk increases the liquidity benefit of public debt is that the cost of higher public debt through higher borrowing rates is mitigated in our framework, since higher public debt also reduces default risk and the interest premium faced by borrowing households. To see this intuitively, turn again to the optimality condition (19) for public debt in the two-period economy. The second term in (19) which captures the cost of higher borrowing costs for borrowing households includes the mitigating effect of a lower interest spreads resulting from higher public debt. The analog of this term in an environment without default does not include such an effect. Figure 8 illustrates this point by showing that in our environment, the borrowing rate is not very responsive to increasing public debt since the increase in the deposit rate is mitigated by the reduction in the interest spread. In contrast, in an environment without default, there is a much larger increase in the borrowing rate which hurts borrowers.
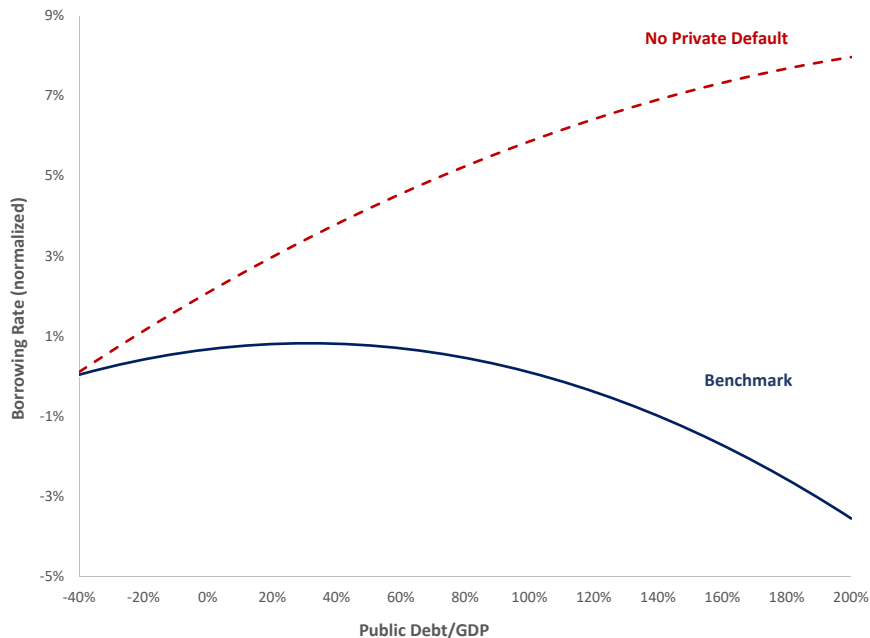
Figure 8: Borrowing Rate Effect of Public Debt in Benchmark vs No Private Default

5 Conclusion

We have presented a theory of the optimal long-run supply of public and private liquidity in an environment in which government debt competes with privately-issued safe assets. According to our model, an increase in public debt crowds out privately-issued safe assets less than one to one, decreases interest spreads, and increases deposit rates. These results are qualitatively consistent with the empirical evidence. Our main result is that while an increase in public liquidity does reduce financial market frictions, it also comes at a cost of higher inequality. As such, the goal of optimal policy should not be to induce full crowd out of privately-issued safe assets since this would increase inequality. We find that the optimal level of public debt is rising in the level of financial market frictions.

Our analysis leaves several important avenues for further study. First, we have abstracted away from privately-issued safe assets backed by capital projects, and a natural question for future analysis concerns the extent to which public debt may serve to crowd-in capital by facilitating borrowing for the purpose of investment. Second, and relatedly, there is typically a maturity mismatch between privately-issued safe assets which are short-term securities and the pools of loans backing them which are long-term. This mismatch may impact the relative efficiency of private versus public liquidity and may also inform the government's choice of optimal government debt maturity. Finally, we

have ignored the interaction between fiscal policy and monetary policy in the provision of public liquidity. A better understanding of how these interact in the provision of liquidity, both theoretically and empirically, is an important area for further research.

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7 Appendix

7.1 Proof of Theoretical Results

Proof of Lemma 1. We first establish that $a^R > 0$ which implies from the discussion in the text that $l^R = 0$. Suppose by contradiction that a $a^R = 0$. Since $B > 0$, from (13) and the fact that optimality requires of $l^P > 0$ only if $a^P = 0$, it follows that $a^P > 0$ so that (14) binds for $i = P$ and $l^P = 0$. There are two cases to consider. In the first case, consider if $l^R = 0$. (14) for $i = R$ requires

$$q^a \frac{1}{1 + \Delta + q^a B} \geq \frac{1}{1 - B}. \tag{38}$$

but this contradicts (14) which binds for $i = P$. In the second case, consider if instead $l^R > 0$, then (15) binds for $i = R$, and and substitution of (12) into (15) implies

$$q^a \frac{1}{1 + \Delta + q^a B + q^l l^R} = \frac{1}{1 - B - l^R}. \tag{39}$$

Equations (13), (14) for $i = P$ (which binds), and (39) all imply that

$$\frac{1}{1 - B - l^R} < \frac{1}{1 - B + a^P},$$

which is a contradiction since $a^P > -l^R$. Therefore, $a^R > 0$ and $l^R = 0$. The discussion in the text implies the second part of the lemma.■

    Proof of Lemma 2. Suppose that $B \geq B^*$. Let us assume and later establish that the constraint that $a^i \geq 0$ does not bind. This means that $l^P = l^R = 0$. Therefore, $q^a = q^l$ from (16) and there is no default so that (2) binds. (14) becomes

$$\frac{1}{1 + \Delta - q^a (a^R - B)} = \frac{1}{q^a} \frac{1}{1 + (a^R - B)} \quad \text{and} \tag{40}$$

$$\frac{1}{1 - \Delta - q^a (a^P - B)} = \frac{1}{q^a} \frac{1}{1 + (a^P - B)}, \tag{41}$$

and the summation of these two first order conditions taking into account (13) implies that $q^a = 1$. Substitution of $q^a = 1$ into (40) and (41) implies that $a^R = B + \Delta/2$ and $a^P = B - \Delta/2$. Therefore, $c_0^P = c_1^P = 1 - \Delta/2$ and $c_0^R = c_1^R = 1 + \Delta/2$ so that consumption is unresponsive to local changes in B.

    We are left to establish that the constraint that $a^P \geq 0$ does not bind. Suppose that

it did bind. From (13), this implies that $a^R \geq 2B \geq \Delta$ and (40)—which must continue to hold—implies that $q^a \leq 1$. There are two cases to consider. Suppose first that $l^P = 0$. In this case, $a^P = 0$ and $a^R = 2B$ from (13). The fact that $a^P \geq 0$ is a binding constraint implies from (14) that

$$\frac{1}{1 - \Delta + q^a B} > \frac{1}{q^a}\frac{1}{1 - B},$$

but this implies that $q^a > 1$, which is a contradiction. If instead $l^P > 0$, then (15) taking into account (16) implies

$$\frac{1}{1 - \Delta + q^a B + q^l l^P} = \frac{1}{q^a}\frac{1}{1 - B - l^P},$$

but since $l^P > 0$, this also implies that $q^a > 1$, which is a contradiction. This establishes that the constraint that $a^P \geq 0$ cannot be binding.∎

Proof of Proposition 1. Suppose that $B < B^*$. From the proof of Lemma 2, it cannot be the case that the constraint that $a^P \geq 0$ is not binding since in that situation, $q^a = 1$, $a^R = B + \Delta/2$ and $a^P = B - \Delta/2$, but this violates the constraint that $a^P \geq 0$. Therefore, $a^P = 0$. As such the economy is characterized by (40) and (15) for $i = P$. We can rewrite (15) for $i = P$ as

$$\frac{1}{1 - \Delta + q^a \left(B + (1 - F(l^P)) l^P\right)} \leq \frac{1}{q^a}\frac{1}{1 - B - l^P}, \tag{42}$$

where we have taken into account (16).

Equation (42) binds whenever $l^P > 0$. We first establish that (42) must bind. If it does not bind, then $l^P = 0$ which means that (42) becomes

$$\frac{1}{1 - \Delta + q^a B} < \frac{1}{q^a}\frac{1}{1 - B}. \tag{43}$$

Since $B < B^*$, (43) implies $q^a < 1$. Moreover, (13) implies that $a^R = 2B$. Substitution into (40) given $B < B^*$ implies $q^a > 1$, which is a contradiction. Therefore, (42) holds with equality. (13) taking into account (16) can be rewritten as

$$\frac{1}{2}a^R = B + \left(1 - F(l^P)\right)\frac{1}{2}l^P. \tag{44}$$

(40), (42) which binds, and (44) provides a system of three equations and three unknowns $\{q^a, a^R, l^P\}$ for a given B. (40) implies that $a^R - B$ rises as q^a declines. Substitution of

37

(44) into (40) and (42) which binds implies that

$$q^a = \frac{1}{1 - \frac{1}{2}F(l^P)l^P} \tag{45}$$

which means that l^P declines when q^a declines. Now consider (44). Note that $(1 - F(l^P))l^P = \exp^{-l^P}l^P$ is rising in l^P as long as $l^P < 1$, which must hold to guarantee $c_1^P > 0$. Therefore, by our above reasoning, $-(a^R - B) + (1 - F(l^P))l^P$ declines as q^a declines, which from (44) means that B must rise as q^a declines. Finally, consider the value of q^l, given the representation in (45):

$$q^l = \frac{1 - F(l^P)}{1 - \frac{1}{2}F(l^P)l^P}.$$

It follows that the derivative of this term with respect to l^P given the functional form for F takes the same sign as

$$-\left(\frac{1}{2}\exp^{l^P}(1 - l^P) + \frac{1}{2}\right) \tag{46}$$

which is negative since $l^P < 1$. Therefore, q^l declines as l^P declines.

These observations imply that as B rises, q^a declines. Since l^P declines, q^l declines from (46) and q^l/q^a rises given (16).

From (40) it follows that as B rises and q^a falls, c_0^R falls. From (2), this means that c_0^P rises. This last observation implies the last result that $q^l l^P$ declines by less than the increase in $q^a B$.∎

Proof of Proposition 2. By Proposition 1, $l^P \geq 0$ is strictly decreasing in B, and there is no loss maximizing welfare with respect to l^P.[30] Consider the analog of (19) with respect to l^P:

$$-\frac{\partial q^a}{\partial l^P}(a^R - B)\left(\frac{1}{c_0^R} - \frac{1}{c_0^P}\right) - f(l^P)q^a l^P \frac{1}{c_0^P} + \frac{\partial B}{\partial l^P}\left(q^a \frac{1}{c_0^P} - \mathbb{E}_\kappa \frac{1}{c_1^P}\right) \leq 0 \tag{47}$$

which holds with equality whenever $l^P > 0$. Suppose that $B = B^*$ with $l^P = 0$ is optimal and consider (47). Note that $\frac{\partial q^a}{\partial l^P}$ can be derived from (45) and is finite and negative at $l^P = 0$. Furthermore, (42) which holds with equality together with (45) define B as a function of l^P:

$$B = \frac{1}{2}\left(\Delta - l^P\left(2 - F(l^P)(1 + (1 - \Delta)/2)\right)\right).$$

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[30]The same analysis holds by taking the derivative with respect to B; we pursue this route to simplify the steps.

Differentiation of this equation implies that $\frac{\partial B}{\partial l^P}$ is finite at $l^P = 0$. Using this information, substitution into (47) given $l^P = 0$ and $B = B^*$ implies

$$- \left(a^R - B \right) \left(\frac{1}{c_0^R} - \frac{1}{c_0^P} \right) \leq 0.$$

From (44), $a^R - B > 0$, which means that this first order condition can only hold if $c_0^R \leq c_0^P$. However, this contradicts the fact that if $B = B^*$, then $c_0^R = 1 + \Delta/2 > c_0^P = 1 - \Delta/2$.∎

7.2 Algorithm for Quantitative Exercise

To solve the model, we take into account that (33) can be rewritten as a function of a single state variable since an optimal decision for the household requires that $\widetilde{a}' > 0$ only if $\widetilde{l}' = 0$ and $\widetilde{l}' > 0$ only if $\widetilde{a}' = 0$. Define

$$m = -\widetilde{a} + B + \frac{q^l}{q^a}\widetilde{l}.$$

It follows then that (33) can be rewritten as:

$$V\left(m, e, \kappa\right) = \max_{c, m'} \left\{ \frac{\widetilde{c}^{1-\gamma}}{1-\gamma} + \widetilde{\beta}\mathbb{E}\left[V\left(m', e', \kappa'\right)|e\right] \right\}$$

$$\text{s.t.}$$

$$m' \leq B + \frac{q^l}{q^a}\bar{l},$$

$$\widetilde{c} = y\left(e\right) - \widetilde{m}\left(m\right) + \left(1 + \phi\right)q^a m', \text{ and}$$

$$\widetilde{m}\left(m\right) = \begin{cases} m & \text{if } m \leq B \\ B + \frac{q^a}{q^l}\left(m - B\right) & \text{if } m \in \left(B, B + \frac{q^l}{q^a}\kappa\right) \\ B + \kappa & \text{if } m > B + \frac{q^l}{q^a}\kappa \end{cases}.$$

For any $\left\{B, q^a, q^l\right\}$, we compute the solution to this problem using standard discretization methods. In particular, we discretize m using a fine grid and compute policy rules given the state space $\omega = \{m, e, \kappa\}$. These rules are subsequently used to compute the stationary distribution $\Gamma\left(\omega\right)$.[31]

To solve the equilibrium prices given B, we guess $q^a \in \left[\widetilde{\beta}/\left(1 + \phi\right), 1\right]$.

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[31]The capital to labor ratio K in this case which enters additively in $y\left(e\right)$ is pinned down by the choice of q^a given equations (20) and (25).

1. Given any $q^a \in \left[\tilde{\beta}/(1+\phi), 1 \right]$, we solve for a value of q^l such that under the implied distribution $\Gamma(\omega)$, the no arbitrage condition (26) is satisfied. This is done by solving for $\Gamma(\omega)$ given $\{q^a, q^l\}$, then choosing a new guess for q^l which satisfies (26) under $\Gamma(\omega)$, and repeating the process until the new guess equals the old guess.

2. We perform a bisection method iteration to solve for q^a (and the implied q^l from step 1). This bisection method builds on the fact that satisfaction of the market clearing condition in (27) which relates total assets to government debt, capital, and total private borrowing. Therefore, if a guess for $\{q^a, q^l\}$ implies that assets exceed borrowing, the new guess for q^a increases, whereas the oppositve occurs if borrowing exceeds assets. This iteration stops once (27) is satisfied.

7.3   Economy with Net Worth Shocks

In this section, we consider an extension of our model which introduces exogenous shocks to household net worth. This extension allows us to better match the empirical concentration of wealth at the top. We describe the extension and show that our main result—that the optimal level of public debt does not induce full crowd out—is robust.

Following Hubmer, Krusell, and Smith (2017), households are subject to i.i.d. shocks to their net worth in every period. More specifically, a household exiting period $t-1$ with net worth $a_t - l_t$, enters period t with net worth $\xi(a_t - l_t)$, where $\xi \geq 0$ is distributed with c.d.f. $H(\xi)$, where the expected value of ξ is 1. These net worth shocks are a reduced form representation of differences in returns across households due to factors such as imperfect diversification and transaction costs. As has been shown in the literature, incorporating such shocks allows a heterogeneous agent model to better quantitatively match the empirical concentration of wealth at the top. We follow Hubmer, Krusell, and Smith (2017) by abstracting from the microfoundations for these shocks. This means that the distribution of net worth shocks is invariant to the level of public debt.

As in the two-period economy, given a default cost κY_t, a household defaults ($d_t = 1$) if $\xi(l_t - a_t) > \kappa Y_t$, and it does not default ($d_t = 0$) if $\xi(l_t - a_t) < \kappa Y_t$. The household's budget constraint at date t is:

$$c_t = w_t e_t (1 - \tau_t^n) - T_t - \min\{\kappa Y_t, \xi(l_t - a_t)\} - q_t^a a_{t+1} + q_t^l l_{t+1}.$$

Given this structure, it is still the case that a household only chooses $l_t > 0$ if $a_t = 0$ and vice versa, which simplifies the analysis.

Note that the net worth shock ξ is i.i.d. and independent of a_{t+1} and l_{t+1}. Therefore,

the sum of a_{t+1} and l_{t+1} equals the sum of ξa_{t+1} and ξl_{t+1} across the population, respectively. As such, the aggregate market clearing constraints $(27) - (28)$ are not affected by the existence of the net worth shocks. Note further that given the highest realized default cost κ as $\overline{\kappa}$ and the highest realized net worth shock $\overline{\xi}$, it follows that a household's choice of normalized private debt $\widetilde{l}_t$ is bounded from above by some $\overline{l} = \overline{\kappa}/\overline{\xi} > 0$, since any debt in excess of this amount (conditional on $\overline{\xi}$) would never be repaid.

The definition of the equilibrium under this extension is isomorphic to the environment in the main text with the exception that the state ω now incorporates the shock ξ, and (37) is replaced by:

$$D = \frac{\int_{\omega \in \Omega} \xi \widetilde{l}(\omega)\, \mathbb{I}_{\kappa < \xi \widetilde{l}} \Gamma(\omega)\, d\omega}{\int_{\omega \in \Omega} \widetilde{l}(\omega)\, \Gamma(\omega)\, d\omega}.$$

For our computation, we choose the same parameters as in the model without net worth shocks with a few exceptions. We choose the discount rate β such that at our benchmark level of a public debt to GDP ratio of 67 percent, the deposit rate equals 4.4 percent, as in the model without net worth shocks.[32] Such an adjustment allows us to more appropriately contrast the conclusions of an analysis with and without net worth shocks. This adjustment yields a choice of $\beta = 0.982$. In addition, we choose the stochastic process for the default cost κ and the stochastic process for the net worth shock ξ jointly. As in the main text, we let κ take on two possible values, $\{\underline{\kappa}, \overline{\kappa}\}$, where $\Pr\{\kappa = \underline{\kappa}\} = \chi$. We also let ξ take on two possible values $\{\underline{\xi}, \overline{\xi}\}$, where the probability of each shock is chosen to ensure that the expected value of ξ is 1. These parameters are chosen to target the same moments as in the main text plus one additional moment: the share of wealth held by the top 60 percent of households, which is 99 percent.[33,34]

The values of the net worth shocks imply that with a 96 percent probability, household net worth increases by 4 percent in absolute value, and with a 4 percent probability, household net worth decreases by 96 percent in absolute value. This feature allows us to match the concentration of wealth in the top 60 percent of the wealth distribution by inducing more precautionary savings by positive net worth households.

The third column in Table 7 describes the distribution of wealth in our extended environment relative to a model without net worth shocks, and it illustrates the importance of this shock for better matching this concentration of wealth at the top, particularly for the top quintile.

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[32] An analogous adjustment to the discount factor is performed across experiments in the related work of Aiyagari and McGrattan (1998).

[33] The share of wealth held by the top 60 percent of households is from Krueger, Mitman, and Perri (2016), Table 1, using the SCF.

[34] Our exercise yields parameters $\{\underline{\kappa}, \overline{\kappa}, \chi\} = \{0.567, 0.678, 0.109\}$.

Table 7. Wealth Distribution in Model vs Data

Variable	Data (SCF)	Benchmark Model	Model with Net Worth Shocks
Q1 share	−0.2	−5.6	−1.6
Q2 share	1.2	6.1	2.6
Q3 share	4.6	16.6	10.1
Q4 share	11.9	29.3	24.1
Q5 share	82.5	53.7	64.9
Gini	0.78	0.60	0.63

The effect of changes in public debt on the economy is similar in this extended model as under our benchmark and can be described by analogous figures to Figures 1-5. We find that introducing net worth shocks lowers the optimal level of public debt to -15 percent of GDP. Higher public debt is more costly in this extended framework since the ensuing higher interest rates exacerbate the degree of wealth inequality. Moreover, net worth shocks also reduce the liquidity benefit of public debt. This is because a net worth shock can occasionally wipe out a poor household's private debt, but it does not change the poor household's lump sum tax burden which finances the interest on public debt.

Despite this difference in the level of optimal public debt, we find that the role of financial frictions is similar in this extended model. More specifically, the optimal value of public debt is below the optimum in the absence of private liquidity—which is -5 percent of GDP—and above the optimum in the absence of default risk—which is -25 percent of GDP. Table 8, which is analogous to Table 6, summarizes the effect of financial frictions in an environment with net worth shocks. The first column describes the baseline environment with private borrowing and default under the optimal policy. The second column describes the optimal policy without private borrowing and third column describes the optimal policy with private borrowing in the absence of default.

Table 8. Role of Financial Frictions with Net Worth Shocks

Variable	Benchmark	No Borrowing	No Default
Public debt / GDP	-0.15	-0.05	-0.25
Capital / GDP	2.16	2.20	2.12
Deposit rate	3.8%	3.7%	4.0%
Borrowing rate	7.2%	3.7%	4.0%
Privately-issued safe assets / GDP	0.06	0	0.08
Fraction indebted households	0.202	0	0.256

To check robustness, we have also explored another approach which sets the discount factor β to 0.991, as in the benchmark model of Aiyagari and McGrattan (1998).[35] The effect of changes in public debt on the economy is qualitatively similar in this exercise, and we find that the optimal level of public debt is now positive and equal to 12 percent of GDP. The optimal value of public debt is below the optimum in the absence of private liquidity—which is 30 percent of GDP—and above the optimum in the absence of default risk—which is 0 percent of GDP.[36]

In sum, our introduction of net worth shocks—which increases wealth inequality—does not impact our conclusion regarding the importance of credit frictions for increasing the optimal level of public debt. Note that while net worth shocks significantly reduce the optimal level of public debt in our experiment, we conjecture that the introduction of other factors which increase wealth inequality could also *increase* the optimal level of public debt. For example, as is noted in Hubmer, Krusell, and Smith (2017), allowing for heterogeneous discount factors would allow us to better match this concentration of wealth at the top. Such an extension would also cause a utilitarian planner to care more about more patient and wealthier households in determining optimal policy, which would presumably lead to a higher optimal level of public debt. Given the presence of numerous potential factors underlying wealth inequality—which are beyond the scope of this paper—we leave a full exploration of these various factors and their impact on the optimal level of public debt to future work.

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[35]In this exercise, the parameters of the default cost shock and the net worth shock are chosen to target the same empirical moments that we previously described.

[36]As is noted in Aiyagari and McGrattan (1998) the optimal value of public debt in a heterogeneous agent framework is very sensitive to the value of β.