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INSTRUMENT-BASED VS. TARGET-BASED RULES

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ABSTRACT

We develop a simple model to study rules based on instruments vs. targets. A principal faces a better informed but biased agent and relies on joint punishments as incentives. Instrument-based rules condition incentives on the agent's observable action; target-based rules condition incentives on outcomes that depend on the agent's action and private information. In each class, an optimal rule takes a threshold form and imposes the worst punishment upon violation. Target-based rules dominate instrument-based rules if and only if the agent's information is sufficiently precise. An optimal unconstrained rule relaxes the instrument threshold whenever the target threshold is satisfied.

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1 Introduction

The question of whether to base incentives on agents' actions or the outcomes of these actions arises in various contexts. Perhaps most prominently, scholars and policymakers have long debated on the merits of using instruments vs. targets for monetary policy. The US House of Representatives and several notable economists support the use of a Taylor (1993) rule that guides the interest rate choice of the central bank,[1] whereas numerous central banks such as the Bank of England and the Bank of Canada rely on inflation targeting rules that are based on outcomes.[2] Fiscal policy rules also vary in this dimension, with some US states constraining instruments like tax rates and spending and others using rules contingent on targets like deficits.[3] More recently, these considerations have received attention in the design of environmental policies. Environmental regulation may focus on technology mandates—requirements on firms' production processes, such as the choice of equipment—or on performance standards—requirements on output, such as maximum emission rates.[4,5]

In this paper, we develop a stylized model to study and compare *instrument-based* and *target-based* rules. Using mechanism design, we present a simple theory that elucidates the benefits of each class of rule and shows which class will be preferred as a function of the environment. Additionally, we characterize the optimal unconstrained or *hybrid* rule and examine how combining instruments and targets can improve welfare.

Our model builds on a canonical delegation framework. A principal delegates decision-making to an agent who is biased towards higher actions. The agent's action is observable, but the agent has private information about its value, with a higher agent *type* corresponding to a higher expected marginal benefit of the action for both the principal and the agent. We extend this delegation setting by introducing an observable noisy outcome that is a function of the agent's action and his private information. For example, the agent may be a policymaker who is biased towards expansionary monetary policy relative to society, and the outcome is inflation which depends on the choice of policy and the realization of

[1]The Financial Choice Act of 2017 passed by the US House of Representatives requires the Federal Reserve to report on a rule to Congress. For discussions for and against Taylor rules, see Svensson (2003), Bernanke (2004, 2015), Appelbaum (2014), Blinder (2014), Taylor (2014), Kocherlakota (2016), and the Statement on Policy Rules Legislation signed by a number of economists, available at http://web.stanford.edu/~johntayl/2016_pdfs/Statement_on_Policy_Rules_Legislation_2-29-2016.pdf.

[2]See Bernanke and Mishkin (1997), Bernanke et al. (1999), Mishkin (1999, 2017), and Svensson (2003) for discussions of inflation targeting regimes.

[3]See National Conference of State Legislatures (1999).

[4]See for example US Congress, Office of Technology Assessment (1995) and Goulder and Parry (2008).

[5]These issues are also relevant to organizations, where non-monetary incentives like promotions or firing may be based on both workers' decisions and their performance.

economic shocks about which the agent has ex-ante private information. Due to his bias, the agent's preferred outcome exceeds that of the principal.[6]

As is standard in delegation settings, transfers between the parties are infeasible, but the principal can make use of joint punishments as incentives. That is, the principal can engage in "money burning" by taking measures that mutually harm the principal and the agent, like imposing sanctions or firing the agent. We distinguish between different classes of rules depending on how punishments are structured: we say that the principal's rule is instrument-based if punishments depend only on the agent's action, and the rule is target-based if punishments depend only on the realized outcome. In the context of monetary policy, an instrument-based rule conditions punishments on the choice of policy, whereas a target-based rule conditions punishments on realized inflation.

Under either class of rule, the principal can tailor punishments appropriately so as to incentivize the agent to choose the principal's preferred action, thus eliminating any distortions in decision-making arising from the agent's bias. However, because punishments are costly to the principal, those incentives would not be optimal. An optimal rule for the principal minimizes distortions due to the agent's bias along with the cost of joint punishments. Moreover, as a consequence, this rule induces an action by the agent and a value of welfare for the principal which depend on the class of the rule.[7]

Our analysis begins by showing that, within each class, an optimal rule takes a threshold form, with violation of the threshold leading to the worst punishment. In the case of an optimal instrument-based rule, the principal allows the agent to choose any action up to a threshold and maximally punishes him for exceeding it. The logic is analogous to that in other delegation models (reviewed below); since the agent prefers higher actions than the principal, this punishment structure is optimal to deter the agent from taking actions that are excessively high. In the case of an optimal target-based rule, the principal specifies a threshold for the outcome, maximally punishing the agent if the realized outcome is above it. This punishment structure also incentivizes the agent to not choose excessively high actions, since these actions result in higher outcomes in expectation. High-powered incentives of this form arise in moral hazard settings with hidden action.[8]

Our main result uses this characterization of the optimal rules for each class to compare their performance. We show that target-based rules dominate instrument-based rules if and only if the agent's private information is sufficiently precise. To illustrate, suppose that

[6]Our analysis is unchanged if the agent instead prefers lower outcomes than the principal.

[7]Instead, if punishments were not joint but costly only to the agent, then both rule classes would yield the same solution, as distortions due to the agent's bias could be eliminated at no cost.

[8]See for example Innes (1990) and Levin (2003). Here these incentives arise because punishments cannot depend directly on the agent's action under a target-based rule.

the agent's information is perfect. Then the principal guarantees her preferred action by providing steep incentives under a target-based rule, where punishments do not occur *on path* because the perfectly informed agent chooses the action that delivers the target outcome. This target-based rule strictly dominates any instrument-based rule, as the latter cannot incentivize the agent while giving him enough flexibility to respond to his information. At the other extreme, suppose that the agent has no private information. Then the principal guarantees her ex-ante preferred action with an instrument-based rule that ties the hands of the agent, namely that punishes the agent if any higher action is chosen. This instrument-based rule strictly dominates any target-based rule, as the latter gives the agent unnecessary discretion and requires on-path punishments to provide incentives. Our main result shows that this logic applies more generally as we locally vary the precision of the agent's private information, away from the extremes of perfect and no information.

We additionally show that the benefit of using a target-based rule over an instrument-based rule is decreasing in the bias of the agent and increasing in the severity of punishment. Intuitively, the less biased is the agent, the less costly is incentive provision under a target-based rule, as the principal can deter the agent from choosing high actions with less frequent punishments. Similarly, the harsher is the punishment imposed on the agent for exceeding the target threshold, the less often the principal needs to exercise punishment on path to implement a target outcome. These two forces therefore make target-based rules more appealing than instrument-based rules on the margin.

A natural question is how the principal can combine instruments and targets to improve upon the above rules that rely exclusively on one of these tools. We study the optimal hybrid rule, that is the optimal unconstrained rule in which punishments can depend freely on the agent's action and the realized outcome.[9] We show that this rule admits a simple implementation: the principal sets an instrument threshold which is relaxed whenever a target threshold is satisfied. The optimal hybrid rule dominates instrument-based rules by allowing the agent more flexibility to choose high actions under a target-based criterion, and it dominates target-based rules by more efficiently limiting the agent's discretion with direct punishments. An example of an optimal hybrid rule in the context of monetary policy would be a Taylor rule which, whenever violated, switches to an inflation target. Notably, some policymakers and economists advocated such an approach in the US in the aftermath of the Global Financial Crisis, when the Federal Reserve's policy deviated

[9]This rule yields the principal the highest welfare that she can achieve given the agent's private information.

significantly from the Taylor rule but realized inflation remained near the target.[10]

This paper is related to several literatures. First, the paper fits into the mechanism design literature that studies the tradeoff between commitment and flexibility in policymaking, including Athey, Atkeson, and Kehoe (2005) and Kocherlakota (2016) in the context of monetary policy, and Amador, Werning, and Angeletos (2006) and Halac and Yared (2014, 2018a,b) in the context of fiscal policy. Second, the paper contributes to an extensive literature on delegation in principal-agent settings, which builds on the seminal work of Holmström (1977, 1984).[11] We extend the theoretical frameworks in both of these literatures by introducing an observable outcome that partially reflects the agent's information. Our contribution is to compare incentives based on the agent's action (as in these literatures) with incentives based on the observable outcome, and to characterize optimal incentives which can condition on both variables.

Third, this paper relates to other theoretical literatures on optimal policy design, including in the context of monetary policy where instruments and targets have been analyzed; see, e.g., Barro and Gordon (1983), Rogoff (1985), McCallum and Nelson (2005), Svensson (2005, 2010), and Giannoni and Woodford (2017). We contribute to this work by characterizing rules as optimal mechanisms in a private information setting and by contrasting incentive provision under each class of rule. It is worth noting that models of monetary policy are concerned with additional questions, most importantly the role of inflationary expectations.[12] We address this issue in an extension in Subsection 5.3. We show that the magnitude of the agent's bias, and hence the optimal class of rule, depends on the cost of inflation in a Barro and Gordon (1983) setting.

Finally, there are other papers that relate to our work in that they consider the use of joint punishments as incentives like we do, albeit in different environments. See for example Acemoglu and Wolitzky (2011) and Padró i Miquel and Yared (2012).[13]

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[10] See for example Yellen (2015, 2017) and the discussion in Walsh (2015).

[11] See Alonso and Matouschek (2008), Amador and Bagwell (2013), and Ambrus and Egorov (2017), among others.

[12] Frankel and Kartik (2017) examine how inflationary expectations and thus equilibrium outcomes depend on not only the amount but also the kind of information that a central bank has.

[13] Although the settings and analyses differ in many aspects, our paper also relates to some models of career concerns for experts which study how the information a principal has affects reputational incentives. In particular, Prat (2005) distinguishes between information on actions and on outcomes.

2 Model

We consider a stylized model with a principal and an agent. The agent observes a signal $s \in \{s^L, s^H\}$, which is the agent's private information or *type*, and chooses an action $\mu \in \mathbb{R}$. Given this action choice, an outcome $\pi = \mu - \theta$ is realized, where $\theta \in \mathbb{R}$ is a shock. A possible interpretation is that the action μ is a policy instrument, such as the level of monetary policy expansion; the shock θ is a stochastic macroeconomic fundamental, such as the level of economic slack; and the outcome π is a payoff-relevant outcome, such as the level of inflation.

The agent's signal is informative about the shock. Specifically, we assume that the conditional distribution of the shock is normal with mean equal to the signal, i.e., $\theta|_{s^i} \sim \mathcal{N}(s^i, \sigma^2)$ for $i = L, H$. The precision of the agent's information is given by $\sigma^{-1} > 0$. We take $s^L = -\Delta$ and $s^H = \Delta$ for some $\Delta > 0$ and assume that each signal occurs with equal probability. The shock's unconditional distribution is thus a mixture of two normal distributions and has mean and variance given by

$$\mathbb{E}(\theta) = 0 \ \text{ and } \ Var(\theta) = \sigma^2 + \Delta^2.$$

The principal observes the agent's action μ and the realized shock θ (or, equivalently, the agent's action μ and the realized outcome π). She cannot however deduce the agent's private information s^i from these observations, as the distribution of θ has full support over the entire real line for each signal s^i.

As is standard in settings of delegation, transfers between the principal and the agent are not feasible. Instead, as a function of the action μ and the shock θ, the principal can commit to a continuation value for the principal and the agent, given by $V(\mu, \theta) \in [\underline{V}, \overline{V}]$ for some finite $\underline{V}$ and $\overline{V}$. This continuation value represents rewards and punishments, including formal penalties due to sanction regimes or replacement of the agent, as well as informal penalties in the form of suboptimal continuation play in a dynamic environment (see Halac and Yared, 2018b).[14]

Denote by $\phi(z|\overline{z}, \sigma_z^2)$ the normal density of a variable z with mean $\overline{z}$ and variance σ_z^2, and by $\Phi(z|\overline{z}, \sigma_z^2)$ the corresponding normal cumulative distribution function. The agent's

[14]Our assumption that the signal s is privately observed by the agent can equivalently be interpreted as a restriction on the rewards and punishments represented by $V(\mu, \theta)$, which cannot explicitly condition on the signal.

expected welfare conditional on information s^i and action μ^i, for $i = L, H$, is

$$\int_{-\infty}^{\infty} \left[-\frac{(\mu^i - \theta - \alpha)^2}{2} + V(\mu^i, \theta) \right] \phi(\theta | s^i, \sigma^2) d\theta, \tag{1}$$

where $\alpha > 0$. The principal's expected welfare is

$$\sum_{i=L,H} \frac{1}{2} \int_{-\infty}^{\infty} \left[-\frac{(\mu^i - \theta)^2}{2} + V(\mu^i, \theta) \right] \phi(\theta | s^i, \sigma^2) d\theta. \tag{2}$$

The principal and the agent receive utility as a function of the observable outcome $\pi = \mu - \theta$ which is concave and single-peaked. For both parties, the preferred action μ that maximizes this utility is increasing in the shock θ. However, as captured by the parameter α, the agent is biased relative to the principal. For each signal s^i, the agent's preferred action, or *flexible* action, is equal to $s^i + \alpha$ (this follows from (1) since $\mathbb{E}(\theta | s^i, \sigma^2) = s^i$). This level exceeds the principal's preferred action, or *first-best* action, which is equal to s^i. Therefore, conditional on the signal, the agent always prefers a higher action than the principal. We assume:

Assumption 1. $\alpha \geq 2\Delta$.

Assumption 1 is analogous to an assumption in Halac and Yared (2014), and its role is to take agent types which are relatively "close" to each other, i.e. with Δ relatively small.[15] The implication of this assumption is that the agent's flexible action $s^i + \alpha$ exceeds the first-best action under each signal:

$$s^L < s^H \leq s^L + \alpha < s^H + \alpha. \tag{3}$$

As shown in (1)-(2), the principal and the agent can be jointly rewarded with a high continuation value $V(\mu, \theta) \in \left[\underline{V}, \overline{V}\right]$ or jointly punished with a low such value. This continuation value being common to the two parties captures, in a stark way, the fact that incentivizing the agent is costly for the principal.[16] We will give the principal a sufficient breadth of incentives to use in her relationship with the agent; specifically, our analysis in the next sections assumes:

Assumption 2. $\overline{V} - \underline{V} \geq \dfrac{\alpha^2}{2\phi(1|0,1)}$.

[15] As we show in Subsection 5.1, our results therefore do not rely on a discrete distance between the types.

[16] In Subsection 5.2, we consider an extension in which punishments harm the agent and the principal asymmetrically.

We distinguish between different classes of rules according to how the principal structures incentives. We say that a rule is instrument-based if the principal commits to a continuation value $V(\mu,\theta)$ which depends only on the action μ. A rule instead is target-based if the continuation value $V(\mu,\theta)$ depends only on the realized outcome $\pi = \mu - \theta$. Finally, if the continuation value $V(\mu,\theta)$ depends freely on μ and θ (and therefore freely on μ and π), we say that the rule is hybrid.

We are interested in comparing the performance of these different classes of rules as the environment changes. Our analysis will consider varying the precision of the agent's private information while holding fixed the mean and variance of the shock θ. At one extreme, we can take $\sigma \to \sqrt{Var(\theta)}$ and $\Delta \to 0$, so the agent is uninformed with signal $s^L = s^H = 0$. At the other extreme, we can take $\sigma \to 0$ and $\Delta \to \sqrt{Var(\theta)}$, so the agent is perfectly informed with signal $s^i = \theta$.[17] Note that since Assumption 1 holds for all feasible $\sigma > 0$ and $\Delta > 0$ given $Var(\theta)$ fixed, the assumption implies $\alpha \geq 2\sqrt{Var(\theta)}$.

3 Instrument-Based and Target-Based Rules

We examine rules based on instruments versus targets. Subsection 3.1 and Subsection 3.2 solve for the optimal rule within each class. Subsection 3.3 offers a comparison and shows that the optimal class of rule for the principal depends on the precision of the agent's private information.

3.1 Optimal Instrument-Based Rule

An instrument-based rule specifies an action μ^i for each agent type $i = L, H$ and a continuation value $V(\mu,\theta)$ as a function of the action μ only. Let $V^i \equiv V(\mu^i)$ for $i = L, H$. The allocation $\{\mu^L, \mu^H, V^L, V^H\}$ must satisfy private information, enforcement, and feasibility constraints, as we describe next.

The private information constraint captures the fact that the agent can misrepresent his type. The rule must be such that, for $i = L, H$, an agent of type i has no incentive to deviate privately to action μ^{-i}:

$$\int_{-\infty}^{\infty}\left[-\frac{(\mu^i-\theta-\alpha)^2}{2}+V^i\right]\phi(\theta|s^i,\sigma^2)d\theta \geq \int_{-\infty}^{\infty}\left[-\frac{(\mu^{-i}-\theta-\alpha)^2}{2}+V^{-i}\right]\phi(\theta|s^i,\sigma^2)d\theta. \tag{4}$$

[17] Note that in our setting, welfare under either rule class depends only on the mean and variance of θ. This avoids additional complications stemming from the fact that higher moments of the distribution of θ vary with σ and Δ.

The enforcement constraint captures the fact that the agent can freely choose any action $\mu \in \mathbb{R}$, including actions not assigned to either type. The rule must be such that, for $i = L, H$, an agent of type i has no incentive to deviate publicly to any action $\mu \notin \{\mu^L, \mu^H\}$:

$$\int_{-\infty}^{\infty} \left[-\frac{(\mu^i - \theta - \alpha)^2}{2} + V^i \right] \phi(\theta | s^i, \sigma^2) d\theta \geq \int_{-\infty}^{\infty} \left[-\frac{(\mu - \theta - \alpha)^2}{2} + V(\mu) \right] \phi(\theta | s^i, \sigma^2) d\theta.$$

Note that since the continuation value satisfies $V(\mu) \geq \underline{V}$ for all $\mu \in \mathbb{R}$, the above inequality must hold under maximal punishment, i.e. when $V(\mu) = \underline{V}$. Moreover, since the inequality must then hold for all $\mu \in \mathbb{R}$, it must necessarily hold when μ corresponds to type i's flexible action $s^i + \alpha$. A necessary condition for the enforcement constraint to be satisfied is thus

$$\int_{-\infty}^{\infty} \left[-\frac{(\mu^i - \theta - \alpha)^2}{2} + V^i \right] \phi(\theta | s^i, \sigma^2) d\theta \geq \int_{-\infty}^{\infty} \left[-\frac{(s^i - \theta)^2}{2} + \underline{V} \right] \phi(\theta | s^i, \sigma^2) d\theta \qquad (5)$$

for $i = L, H$, where note that the right-hand side is the agent's minmax payoff.

Constraints (4) and (5) are clearly necessary for an instrument-based rule prescribing $\{\mu^L, \mu^H, V^L, V^H\}$ to be incentive compatible. Furthermore, if these constraints are satisfied, then this allocation can be supported by specifying the worst continuation value $\underline{V}$ following any choice $\mu \notin \{\mu^L, \mu^H\}$. Since such a choice is *off path*, it is without loss to assume that it is maximally punished.

Lastly, feasibility requires that for $i = L, H$,

$$V^i \in [\underline{V}, \overline{V}]. \qquad (6)$$

An optimal instrument-based rule maximizes the principal's expected welfare subject to the private information, enforcement, and feasibility constraints:

$$\max_{\mu^L, \mu^H, V^L, V^H} \sum_{i=L,H} \frac{1}{2} \int_{-\infty}^{\infty} \left[-\frac{(\mu^i - \theta)^2}{2} + V^i \right] \phi(\theta | s^i, \sigma^2) d\theta \qquad (7)$$

subject to, for $i = L, H$, (4), (5), and (6).

Define a *maximally-enforced instrument threshold* μ^* as a rule that prescribes the maximal reward $\overline{V}$ if the agent's action is weakly below a threshold μ^* and the maximal punishment $\underline{V}$ if the action exceeds this threshold. We find:

Proposition 1. *The optimal instrument-based rule specifies $\mu^L = \mu^H = 0$ and $V^L = V^H = \overline{V}$. This rule can be implemented with a maximally-enforced instrument threshold $\mu^* = 0$.*

The optimal instrument-based rule assigns both agent types the action that maximizes the principal's ex-ante welfare. The agent is given no discretion, and punishments occur only off path, if the agent were to publicly deviate to a different action.[18] Note that a rule that induces the first-best action, $\mu^i = s^i$ for $i = L, H$, is available to the principal: the low type can be dissuaded from choosing s^H by specifying an on-path punishment $V^H < \overline{V}$ (and both types can be dissuaded from choosing any action $\mu \notin \{s^L, s^H\}$ by specifying the worst punishment off path). However, because punishment is costly to the principal, Proposition 1 shows that such a rule is strictly dominated by a maximally-enforced instrument threshold $\mu^* = 0$.

To prove Proposition 1, we solve a relaxed version of the program in (7) which ignores the private information constraint (4) for the high type and the enforcement constraints (5) for both types. We show that under Assumption 1, the solution to this relaxed problem entails no discretion, and it thus satisfies (4) for both types. Moreover, Assumption 2 guarantees that (5) is also satisfied for both types.

Proposition 1 is in line with the findings of an extensive literature on delegation, which provides conditions under which threshold delegation with no money burning is optimal. A general treatment can be found in Amador and Bagwell (2013). The analysis in Halac and Yared (2018b) is also related in that it considers enforcement constraints like those in (5) and shows the optimality of maximally-enforced thresholds, where on- and off-path violations lead to the worst punishment. In the current setting with binary signals, enforcement constraints are non-binding by Assumption 2, so punishments occur only off path. Halac and Yared (2018b) study the issues that arise when the analog of this assumption is relaxed in their context. We also address these issues in Subsection 5.1, where we extend our analysis to a continuum of agent types.

3.2 Optimal Target-Based Rule

A target-based rule specifies an action μ^i for each agent type $i = L, H$ and a continuation value $V(\mu, \theta)$ as a function of the outcome $\pi = \mu - \theta$ only. We denote such a continuation value by $V(\pi)$, where note that $V(\pi)$ is defined for $\pi \in \mathbb{R}$ since θ is normally distributed. The allocation $\{\mu^L, \mu^H, \{V(\pi)\}_{\pi\in\mathbb{R}}\}$ must satisfy incentive compatibility and feasibility constraints, as we describe next.

[18] In our extension to a continuum of agent types in Subsection 5.1, the optimal instrument-based rule does provide some discretion to the agent, and it also involves punishments on path.

Incentive compatibility requires that the prescribed action for each agent type solve this type's welfare-maximization problem. Given his private information and the continuation value function specified by the principal, the agent takes into account how his action affects the distribution of outcomes and, thus, continuation values. For $i = L, H$, μ^i must satisfy:

$$\mu^i \in \arg\max_{\mu} \left\{ \int_{-\infty}^{\infty} \left[-\frac{(\mu - \theta - \alpha)^2}{2} + V(\mu - \theta) \right] \phi\left(\theta | s^i, \sigma^2\right) d\theta \right\}. \tag{8}$$

Additionally, feasibility requires

$$V(\pi) \in \left[\underline{V}, \overline{V}\right] \text{ for all } \pi \in \mathbb{R}. \tag{9}$$

An optimal target-based rule maximizes the principal's expected welfare subject to the incentive compatibility and feasibility constraints:

$$\max_{\mu^L, \mu^H, \{V(\pi)\}_{\pi \in \mathbb{R}}} \sum_{i=L,H} \frac{1}{2} \int_{-\infty}^{\infty} \left[-\frac{(\mu^i - \theta)^2}{2} + V(\mu^i - \theta) \right] \phi\left(\theta | s^i, \sigma^2\right) d\theta \tag{10}$$

subject to, for $i = L, H$, (8) and (9).

Note that integration by substitution yields

$$\int_{-\infty}^{\infty} V(\mu - \theta) \phi\left(\theta | s^i, \sigma^2\right) d\theta = \int_{-\infty}^{\infty} V(\pi) \phi\left(\mu - s^i - \pi | 0, \sigma^2\right) d\pi, \tag{11}$$

where we have used the fact that $\phi(\theta | s, \sigma^2) = \phi(\theta - s | 0, \sigma^2)$ since $\phi(\cdot)$ is the density of a normal distribution. Using (11) to substitute in (8), the first-order condition of the agent's problem is

$$\alpha - \left(\mu^i - s^i\right) + \int_{-\infty}^{\infty} V(\pi) \phi'\left(\mu^i - s^i - \pi | 0, \sigma^2\right) d\pi = 0 \text{ for } i = L, H. \tag{12}$$

Condition (12) is necessary for the rule to be incentive compatible. Its solution is $\mu^i = s^i + \kappa$ for $i = L, H$ and some $\kappa \gtreqless 0$, where κ is independent of the agent's type i. The latter observation allows us to simplify the principal's problem as her welfare then also becomes independent of the agent's type i.

Define a *maximally-enforced target threshold* π^* as a rule that prescribes the maximal reward $\overline{V}$ if the outcome is weakly below a threshold π^* and the maximal punishment $\underline{V}$ if the outcome exceeds this threshold. We find:

Proposition 2. *The optimal target-based rule specifies* $\mu^i = s^i + \kappa$, $V(\pi) = \overline{V}$ *if* $\pi \leq \pi^*$, *and* $V(\pi) = \underline{V}$ *if* $\pi > \pi^*$, *for* $i = L, H$, *some* $\kappa \in (0, \alpha)$, *and* $\pi^* = \kappa + \frac{\sigma^2}{\kappa}$. *This rule can be implemented with a maximally-enforced target threshold* π^*.

The optimal target-based rule provides incentives with a maximally-enforced target threshold π^*. Since a higher action μ results in a higher outcome π in expectation, an agent of type i responds to this threshold by choosing an action $s^i + \kappa$ which is below his flexible action $s^i + \alpha$. In contrast to the optimal instrument-based rule, here punishment occurs along the equilibrium path whenever $\pi > \pi^*$, so as to appropriately incentivize the agent. Note that since punishment is costly, the principal limits its frequency by keeping the agent's action above the first-best level. That is, while a rule that induces the principal's preferred action with $\kappa = 0$ is available to the principal, Proposition 2 shows that this rule is strictly dominated by one that allows distortions with $\kappa > 0$. The proposition also shows that the induced expected outcome is below the threshold, i.e. $\mathbb{E}(\pi) = \kappa < \pi^*$. A rule that yields $\mathbb{E}(\pi) = \kappa = \pi^*$ would be suboptimal, as it would entail punishing the agent half of the time (the frequency with which π would exceed π^*). In the optimal rule, the realized outcome π exceeds π^* less than half of the time so that punishment occurs less often.

To prove Proposition 2, we follow a first-order approach and solve a relaxed version of the program in (10) that replaces the incentive compatibility constraint (8) with the first-order condition (12) of the agent's problem. Specifically, we consider a doubly-relaxed problem that takes (12) as a weak inequality constraint (cf. Rogerson, 1985) in order to establish the sign of the Lagrange multiplier on (12) and characterize the solution. We prove that the solution to the relaxed problem takes the threshold form described above, and we show that Assumption 1 and Assumption 2 are sufficient to guarantee the validity of this first-order approach.

High-powered incentives of the form described in Proposition 2 arise in moral hazard settings where, as in our model, rewards and punishments are bounded and enter welfare linearly; see for example Innes (1990) and Levin (2003). These incentives arise here because rewards and punishments cannot directly depend on the agent's action under a target-based rule. As we discuss in Subsection 5.1, Proposition 2 remains valid under a continuum of agent types: since the agent's first-order condition implies that the principal's welfare is independent of the agent's type i, this welfare is also independent of the number of types.

3.3 Optimal Class of Rule

Our main result uses the characterizations in Proposition 1 and Proposition 2 to compare the performance of instrument-based and target-based rules. We find that which class of rule is optimal for the principal depends on the precision of the agent's private information:

Proposition 3. *Take instrument-based and target-based rules and consider changing σ while keeping $Var(\theta)$ unchanged. There exists $\sigma^* > 0$ such that a target-based rule is strictly optimal if $\sigma < \sigma^*$ and an instrument-based rule is strictly optimal if $\sigma > \sigma^*$. The cutoff σ^* is decreasing in the agent's bias α and the worst continuation value $\underline{V}$.*

To see the logic, consider how the principal's welfare under each class of rule changes as we vary the precision of the agent's information σ^{-1}, while keeping the shock variance $Var(\theta)$ unchanged. Since the optimal instrument-based rule gives no flexibility to the agent to use his private information, the principal's welfare under this rule is invariant to σ. Specifically, by Proposition 1 and $Var(\theta) = \mathbb{E}(\theta^2)$ (since $\mathbb{E}(\theta) = 0$), the principal's welfare under the optimal instrument-based rule is given by

$$-\frac{Var(\theta)}{2} + \overline{V},$$

independent of σ. In contrast, using Proposition 2, we can verify that the principal's welfare under the optimal target-based rule is decreasing in σ, that is increasing in the precision of the agent's information. Intuitively, a better informed agent can more closely tailor his action to the shock, and is less likely to trigger punishment by overshooting the threshold specified by the principal. As a result, a higher precision reduces the outcome volatility and the principal's cost of providing high-powered incentives under a target-based rule.

These comparative statics imply that to prove the first part of Proposition 3, it suffices to show that a target-based rule is optimal for high enough precision of the agent's information whereas an instrument-based rule is optimal otherwise. Consider the extreme in which the agent is perfectly informed, that is, $\sigma \to 0$ and $\Delta \to \sqrt{Var(\theta)}$. In this case, the optimal target-based rule sets a threshold $\pi^* = 0$, providing steep incentives and inducing the first-best action. Note that this rule involves no punishments along the equilibrium path, as a perfectly informed agent of type $i = L, H$ chooses $\mu^i = s^i$ to avoid punishment. Consequently, in this limit case, the optimal target-based rule yields welfare

$$\overline{V} > -\frac{Var(\theta)}{2} + \overline{V},$$

and thus it dominates the optimal instrument-based rule.

Consider next the extreme in which the agent is uninformed, that is, $\sigma \to \sqrt{Var(\theta)}$ and $\Delta \to 0$. In this case, the optimal instrument-based rule guarantees the principal her preferred outcome given no information by tying the hands of the agent. Instead, the principal cannot implement her ex-ante optimum with a target-based rule, which gives the agent unnecessary discretion and requires punishments to provide incentives. The optimal target-based rule in this limit case sets a threshold $\pi^* > 0$, inducing an agent of type $i = L, H$ to choose $\mu^i = s^i + \kappa$ for $\kappa > 0$ and yielding welfare

$$-\frac{Var(\theta)}{2} + \overline{V} - \frac{\kappa^2}{2} - \Phi\left(\kappa - \pi^* | 0, \sigma^2\right)\left(\overline{V} - \underline{V}\right) < -\frac{Var(\theta)}{2} + \overline{V}.$$

Thus, this rule is dominated by the optimal instrument-based rule.

The second part of Proposition 3 shows that the benefit of using a target-based rule over an instrument-based rule is decreasing in the bias of the agent and increasing in the severity of punishment. The less biased is the agent, the less costly is incentive provision under a target-based rule, as relatively infrequent punishments become sufficient to deter high actions. Similarly, the harsher is the punishment experienced by the agent for missing the target threshold, the less often punishment needs to be used on the equilibrium path to provide incentives under a target-based rule. In contrast, the optimal instrument-based rule is independent of the agent's bias and the severity of punishment. As such, target-based rules dominate instrument-based rules for a larger range of parameters when the agent's bias is relatively low or punishment is relatively severe.[19]

In addition to the results in Proposition 3, the comparison of instrument-based and target-based rules reveals differences with regards to volatility. Specifically, the two rule classes differ in how they trade off different kinds of volatility: the optimal instrument-based rule minimizes the variance of the agent's action, whereas the optimal target-based rule achieves a lower variance of the outcome. We will return to this distinction in Subsection 5.3, where we extend our model to consider the role of expectations.

4 Hybrid Rules

A hybrid rule combines features of instrument-based and target-based rules, with a continuation value $V(\mu, \theta)$ that depends freely on μ and θ. For $i = L, H$, denote by $V^i(\theta)$ the continuation value assigned to agent type i as a function of the shock θ. Analogous to the

[19] One may also wonder how the comparison of the two rule classes changes if the unconditional variance of the shock increases. The answer is ambiguous in that it depends on the extent to which the increase in $Var(\theta)$ is due to an increase in σ or Δ: if only σ increases, instrument-based rules become more beneficial over target-based rules (as implied by Proposition 3), whereas if only Δ increases, the opposite is true.

program in (7), an optimal hybrid rule solves:

$$\max_{\mu^L,\mu^H,\{V^L(\theta),V^H(\theta)\}_{\theta\in\mathbb{R}}} \sum_{i=L,H} \frac{1}{2}\int_{-\infty}^{\infty}\left[-\frac{(\mu^i-\theta)^2}{2}+V^i(\theta)\right]\phi(\theta|s^i,\sigma^2)d\theta \tag{13}$$

subject to, for $i = L, H$,

$$\int_{-\infty}^{\infty}\left[-\frac{(\mu^i-\theta-\alpha)^2}{2}+V^i(\theta)\right]\phi(\theta|s^i,\sigma^2)d\theta \geq \int_{-\infty}^{\infty}\left[-\frac{(\mu^{-i}-\theta-\alpha)^2}{2}+V^{-i}(\theta)\right]\phi(\theta|s^i,\sigma^2)d\theta, \tag{14}$$

$$\int_{-\infty}^{\infty}\left[-\frac{(\mu^i-\theta-\alpha)^2}{2}+V^i(\theta)\right]\phi(\theta|s^i,\sigma^2)d\theta \geq \int_{-\infty}^{\infty}\left[-\frac{(s^i-\theta)^2}{2}+\underline{V}\right]\phi(\theta|s^i,\sigma^2)d\theta, \tag{15}$$

$$V^i(\theta)\in[\underline{V},\overline{V}] \text{ for all } \theta\in\mathbb{R}. \tag{16}$$

The solution to this program gives the principal the highest welfare that she can achieve given the private information of the agent. Note that constraints (14)-(15) are analogous to (4)-(5) in the program that solves for the optimal instrument-based rule, but they now allow the continuation value to depend on the shock θ in addition to the agent's action μ^i. Note also that by analogous arguments as those used to solve for the optimal target-based rule, the continuation value $V^i(\theta)$ can be equivalently written as a function of the outcome, $V^i(\pi)$. We use this formulation in what follows to ease the interpretation.

Define a *maximally-enforced hybrid threshold* $\{\mu^*,\mu^{**},\{\pi^*(\mu)\}_{\mu\in\mathbb{R}}\}$ as a rule that specifies the maximal reward $\overline{V}$ if the outcome is weakly below a threshold $\pi^*(\mu)$ and the maximal punishment $\underline{V}$ if the outcome exceeds $\pi^*(\mu)$, where this threshold depends on the action μ and satisfies

$$\pi^*(\mu)=\begin{cases}\infty & \text{if } \mu\leq\mu^* \\ h(\mu) & \text{if } \mu\in(\mu^*,\mu^{**}] \\ -\infty & \text{if } \mu>\mu^{**}\end{cases} \tag{17}$$

for cutoffs $\mu^*<\mu^{**}$ and some continuous function $h(\mu)\in(-\infty,\infty)$ with $\lim_{\mu\downarrow\mu^*} h(\mu)=\infty$. The cutoff μ^* is a soft instrument threshold, where any action $\mu\leq\mu^*$ is rewarded independently of the outcome with the maximal continuation value $\overline{V}$. The cutoff $\mu^{**}>\mu^*$ is a hard instrument threshold, where any action $\mu>\mu^{**}$ is punished independently of the outcome with the worst continuation value $\underline{V}$. Intermediate actions $\mu\in(\mu^*,\mu^{**}]$ are maximally rewarded if the outcome satisfies $\pi\leq\pi^*(\mu)$ and maximally punished if the outcome satisfies $\pi>\pi^*(\mu)$. Therefore, an interior target threshold only applies to intermediate actions.

We find:

Proposition 4. *The optimal hybrid rule specifies* $\mu^L < \mu^H$, $V^L(\pi) = \overline{V}$ *for all* π, $V^H(\pi) = \overline{V}$ *if* $\pi \leq \pi^*(\mu^H)$, *and* $V^H(\pi) = \underline{V}$ *if* $\pi > \pi^*(\mu^H)$, *for some* $\pi^*(\mu^H) \in (-\infty, \infty)$. *This rule can be implemented with a maximally-enforced hybrid threshold* $\{\mu^*, \mu^{**}, \{\pi^*(\mu)\}_{\mu \in \mathbb{R}}\}$, *where* $\mu^* = \mu^L$ *and* $\mu^{**} = \mu^H$.

The optimal hybrid rule assigns a low action and the maximal reward to the low type, while specifying a higher action and a target threshold for the high type. To prove this result, we solve a relaxed version of (13)-(16) which ignores the private information constraint (14) for the high type and the enforcement constraints (16) for both types. We establish that the solution to this relaxed problem takes the form described in Proposition 4 and satisfies these constraints.

As implied by Proposition 4, the optimal hybrid rule strictly improves upon the rules studied in Section 3. Intuitively, this rule dominates instrument-based rules by giving the agent more flexibility to respond to his private information while preserving incentives. The reason is that, under a hybrid rule, the principal can allow the agent to choose actions $\mu > \mu^*$ and still deter excessively high actions by using a target-based criterion. Analogously, the optimal hybrid rule dominates target-based rules by more efficiently limiting the agent's discretion to choose actions that are excessively high. The reason is that, under a hybrid rule, the principal can avoid punishments under actions $\mu \leq \mu^*$ and directly punish the agent for actions $\mu > \mu^{**}$, thus reducing the frequency of punishment on path.

In principle, combining instruments and targets could yield rules with complicated forms. Proposition 4 however shows that the optimal hybrid rule admits an intuitive implementation. This rule essentially consists of an instrument threshold μ^* which is relaxed to μ^{**} whenever the target threshold is satisfied. As noted in the Introduction, rules of this form have been advocated in practice in the context of monetary policy.

5 Extensions

We consider three extensions of our model. The first two are important robustness checks: we extend our setting to allow for a continuum of signals and for punishments that harm the agent and the principal asymmetrically. The third extension is of particular interest for applications to monetary and fiscal policy: we consider a welfare function that depends on the deviation of the outcome from its expectation. In all three cases, we show that our main results from Section 3 continue to hold.

5.1 Continuum of Types

Suppose that instead of observing a binary signal $s \in \{-\Delta, \Delta\}$, the agent observes a signal as rich as the shock: $s \in \mathbb{R}$ with $s \sim \mathcal{N}(0, \Delta^2)$. The shock's conditional distribution obeys $\theta|_s \sim \mathcal{N}(s, \sigma^2)$, and thus the unconditional distribution has $\mathbb{E}(\theta) = 0$ and $Var(\theta) = \sigma^2 + \Delta^2$ as in our baseline model. Defining the optimal instrument-based and target-based rules analogously as in Section 3, with formal representations provided in the Online Appendix, we obtain:[20]

Proposition 5. *Consider an extended environment with a continuum of agent types. The optimal instrument-based rule can be implemented with a maximally-enforced instrument threshold $\mu^* \in (-\infty, \infty)$. The optimal target-based rule can be implemented with a maximally-enforced target threshold π^* which coincides with that in Proposition 2. Moreover, take these two classes of rules and consider changing σ while keeping $Var(\theta)$ unchanged. There exists $\sigma^* > 0$ such that a target-based rule is strictly optimal if $\sigma < \sigma^*$ and an instrument-based rule is strictly optimal if $\sigma > \sigma^*$.*

The optimal instrument-based and target-based rules under a continuum of types take the same implementations as under binary types. There are differences, however, with regards to the agent's behavior under the optimal instrument-based rule. In the case of binary types, Proposition 1 shows that both types are bunched at the threshold μ^* and receive the maximal reward. Instead, under a continuum of types, this is only true for types $s \in [s^*, s^{**}]$, where $s^* = \mu^* - \alpha < s^* + \sqrt{2(\overline{V} - \underline{V})} = s^{**}$. Types $s < s^*$ satisfy the instrument threshold strictly by choosing their flexible action $s + \alpha < \mu^*$ and receive the maximal reward, whereas types $s > s^{**}$ break the instrument threshold by choosing their flexible action $s + \alpha > \mu^*$ and receive the maximal punishment. The cutoff s^{**} corresponds to the type that is indifferent between abiding to the instrument threshold and receiving a continuation value $\overline{V}$ versus breaking the threshold and receiving a continuation value $\underline{V}$:

$$\int_{-\infty}^{\infty} \left[-\frac{(\mu^* - \alpha - \theta)^2}{2} + \overline{V} \right] \phi(\theta | s^{**}, \sigma^2) d\theta = \int_{-\infty}^{\infty} \left[-\frac{(s^{**} - \theta)^2}{2} + \underline{V} \right] \phi(\theta | s^{**}, \sigma^2) d\theta.$$

Compared to the analysis in Subsection 3.1, characterizing this optimal instrument-based rule requires more involved arguments. These arguments are related to those developed in Halac and Yared (2018b), and we present them in the Online Appendix in three

[20] We assume that an optimal instrument-based rule is piecewise continuously differentiable. Also, if the program solving for this rule admits multiple solutions that differ only on a countable set of types, we select the solution that maximizes the principal's welfare for those types.

main steps.[21] First, we show that the linearity of the principal and agent's welfare in the continuation value, together with the richness of the information structure, imply that only the extreme values $\{\underline{V}, \overline{V}\}$ are prescribed in an optimal rule; that is, the optimal continuation values are *bang-bang*. Second, appealing to the log concavity of the normal density function, we establish that the optimal continuation values are also monotonic, with types below some cutoff s^{**} being maximally rewarded with value $\overline{V}$ and types above s^{**} being maximally punished with value $\underline{V}$. As an implication, types $s > s^{**}$ must choose their flexible action $s + \alpha$. Finally, we show that for types $s \leq s^{**}$, the optimal rule prescribes an action that is continuous in the agent's type and takes the form of a threshold.[22]

The characterization of the optimal target-based rule under a continuum of types follows the same steps as in Subsection 3.2. In fact, the optimal target threshold π^* is quantitatively identical to that in the case of binary types, so the statement in Proposition 2 applies directly to this continuum-of-types case. As mentioned in Subsection 3.2, the reason is that the agent's first-order condition in (12) implies a choice $\kappa = \mu - s$ by the agent which is independent of his type. This means that the optimal target-based rule is independent of the distribution of the agent's signal s, and therefore the characterization in Proposition 2 holds independently of the number of types.

The second part of Proposition 5 extends Proposition 3 to a continuum of types: we show that target-based rules dominate instrument-based rules if and only if the agent's information is sufficiently precise. The result follows from comparing how an increase in the precision of the agent's signal affects the principal's welfare under each rule class. Under the optimal target-based rule, the effect takes the same form as in Subsection 3.3. Specifically, if the agent's information is more precise, then all types can better tailor their actions to the shock, which lowers the outcome volatility and thus also the need to utilize costly punishments on path. Under the optimal instrument-based rule, things are different. Given the cutoffs s^* and s^{**} defined above, a higher precision allows types $s < s^*$ and $s > s^{**}$ to better tailor their actions to the shock; however, there are no target-based punishments whose frequency is reduced as a result, and moreover nothing changes for types $s \in [s^*, s^{**}]$ who have no discretion. Consequently, we show that the welfare effect of an increase in precision is smaller under the optimal instrument-based rule than under the optimal target-based rule. This comparison allows us to obtain the result in Proposition 5

[21] The arguments in Halac and Yared (2018b) cannot be applied directly to the current setting as the agent's bias in that paper is multiplicative instead of additive.

[22] We note that our proof in the Online Appendix, and thus the characterization of the optimal instrument-based rule in Proposition 5, apply more broadly to any distribution of types with a log concave density. This includes the exponential, gamma, log-normal, Pareto, and uniform distributions among others; see Bagnoli and Bergstrom (2005).

and hence to establish the robustness of our findings to a continuum of agent types.

5.2 Asymmetric Punishments

Suppose that varying the continuation value has a larger effect on the agent's welfare compared to that of the principal. Specifically, let us modify the agent's expected welfare in (1) so that it is now given by

$$\int_{-\infty}^{\infty} \left[-\frac{(\mu^i - \theta - \alpha)^2}{2} + cV(\mu^i, \theta) \right] \phi(\theta | s^i, \sigma^2) d\theta \tag{18}$$

for $i = L, H$ and some $c > 1$ which we also require to satisfy $c < \alpha/2\Delta$. Note that our baseline model corresponds to setting $c = 1$. Under $c > 1$, any reduction in the continuation value $V(\mu^i, \theta)$ implies a harsher punishment on the agent than on the principal. Defining the optimal instrument-based and target-based rules analogously as in Section 3, with formal representations provided in the Online Appendix, we obtain:

Proposition 6. *Consider an extended environment with asymmetric punishments. The optimal instrument-based rule can be implemented with a maximally-enforced instrument threshold* μ^* *which coincides with that in Proposition 1. The optimal target-based rule can be implemented with a maximally-enforced target threshold* $\pi^* \in (-\infty, \infty)$. *Moreover, take these two classes of rules and consider changing* σ *while keeping* $Var(\theta)$ *unchanged. There exists* $\sigma^* > 0$ *such that a target-based rule is strictly optimal if* $\sigma < \sigma^*$ *and an instrument-based rule is strictly optimal if* $\sigma > \sigma^*$.

The optimal instrument-based and target-based rules under asymmetric punishments take the same implementations as under symmetric punishments. The optimal instrument-based rule is in fact quantitatively identical to that in Subsection 3.1, with $\mu^L = \mu^H = 0$ and $V^L = V^H = \overline{V}$. Intuitively, given $c < \alpha/2\Delta$, it is still the case that providing incentives to separate the two agent types is too costly for the principal, so the optimal instrument-based rule continues to bunch both types. The optimal target-based rule takes the same form as in Subsection 3.2; since the agent's welfare remains linear in the continuation value under (18), high-powered incentives remain optimal. Nonetheless, this rule is not quantitatively identical to that under symmetric punishments. The reason is intuitive: since imposing any given punishment on the agent, and therefore providing incentives, is now less costly for the principal, the optimal target-based rule induces an action closer to first best (i.e., with a lower value of κ) in the case of asymmetric punishments.

The second part of Proposition 6 extends Proposition 3 to asymmetric punishments: we show that target-based rules dominate instrument-based rules if and only if the agent's information is sufficiently precise. The argument is the same as in the case of symmetric punishments. In particular, note that since the optimal instrument-based rule provides no discretion to the agent, the principal's welfare under this rule is invariant to the precision of the agent's information, as in Subsection 3.3. Moreover, since the optimal target-based rule takes the same form as under symmetric punishments, the principal's welfare under this rule increases with the precision of the agent's information, also as in Subsection 3.3. These comparative statics allow us to obtain the result in Proposition 6 and hence to establish the robustness of our findings to asymmetric punishments.

5.3 Role of Expectations

In some applications of our model, the welfare of the principal and the agent may depend not only on the payoff-relevant outcome $\pi = \mu - \theta$, but also on the deviation of this outcome from its expectation, $\pi - \mathbb{E}(\pi)$. For example, in an application to monetary policy, welfare depends not only on inflation, but also on realized output, which is increasing in the difference between actual and expected inflation; see, e.g., Barro and Gordon (1983). In an open economy capital taxation application, welfare depends on tax revenue, which not only depends on the level of taxes but is also increasing in the difference between actual and expected taxes; see, e.g., Aguiar, Amador, and Gopinath (2009). To incorporate these considerations, let

$$\pi^e \equiv \mathbb{E}(\pi) = \mathbb{E}(\mu - \theta) \tag{19}$$

be the expected outcome, computed after the principal commits to a continuation value function $V(\mu^i, \theta)$. That is, the expectation in (19) takes into account the distributions of the shock θ and signal s, as well as the equilibrium behavior of the agent given $V(\mu^i, \theta)$. We modify the agent's expected welfare in (1) so that it is now given by

$$\int_{-\infty}^{\infty} \left[-\frac{(\mu^i - \theta - \beta)^2}{2} + \gamma_a[\mu^i - \theta - \pi^e] + V(\mu^i, \theta) \right] \phi(\theta | s^i, \sigma^2) d\theta \tag{20}$$

for $i = L, H$ and some $\gamma_a \geq 0$ and $\beta \geq 0$. The principal's expected welfare in (2) becomes

$$\sum_{i=L,H} \frac{1}{2} \int_{-\infty}^{\infty} \left[-\frac{(\mu^i - \theta)^2}{2} + \gamma_p[\mu^i - \theta - \pi^e] + V(\mu^i, \theta) \right] \phi(\theta | s^i, \sigma^2) d\theta \tag{21}$$

for some $\gamma_p \geq 0$. In this formulation, γ_a and γ_p are the weights that the agent and the principal place on the deviation of the outcome π from its expectation π^e, and β is the agent's action bias. We note that if $\gamma_p = \gamma_a > 0$ and $\beta = 0$, then this environment corresponds to that in Barro and Gordon (1983).

To solve for the optimal instrument-based and target-based rules, note first that the principal's welfare in (21) is identical to that in (2) in our baseline model, since the expected value of the second term in (21) equals zero by (19). The reason is that the principal internalizes the impact of the agent's action μ^i on the expected outcome π^e when choosing the continuation value function $V(\mu^i, \theta)$. In contrast, given $V(\mu^i, \theta)$ and his private signal s^i, the agent optimally chooses μ^i taking π^e as given, and so he does not internalize how his behavior affects the expected outcome. Nevertheless, it turns out that the agent's welfare in (20) is also equivalent to that in our baseline model. Specifically, note that (20) can be rewritten as

$$\int_{-\infty}^{\infty} \left[-\frac{(\mu^i - \theta - \alpha)^2}{2} + V(\mu^i, \theta) \right] \phi(\theta | s^i, \sigma^2) d\theta - \left[\frac{\gamma_a^2}{2} - \gamma_a(\beta - \pi^e) \right] \tag{22}$$

for $\alpha \equiv \beta + \gamma_a$. Since the agent takes π^e as given, the second term in (22) is constant and independent of the agent's action. Therefore, the agent's welfare is equivalent to that in (1) with a bias α arising from the combination of the direct action bias β and the weight γ_a that the agent places on the deviation of the outcome from its expectation.

These observations imply that this extended environment is mathematically equivalent to our baseline environment. Consequently, all of our results in Section 3 continue to apply:

Proposition 7. *Consider an extended environment in which welfare depends on the difference between the actual outcome and the expected outcome. The optimal instrument-based and target-based rules are as described in Proposition 1 and Proposition 2 respectively. Moreover, take these two classes of rules and consider changing σ while keeping $Var(\theta)$ unchanged. There exists $\sigma^* > 0$ such that a target-based rule is strictly optimal if $\sigma < \sigma^*$ and an instrument-based rule is strictly optimal if $\sigma > \sigma^*$. The cutoff σ^* is decreasing in the agent's bias α and the worst continuation value $\underline{V}$.*

The result that σ^* is decreasing in α is of particular interest in the application to monetary policy. This result says that instrument-based rules are more appealing on the margin relative to target-based rules if the agent is more biased. In the context of Barro and Gordon (1983), with $\gamma_p = \gamma_a \equiv \gamma > 0$ and $\beta = 0$, this implies that instrument-based rules are preferred if the weight γ placed on the *outcome surprise* is sufficiently large. The magnitude of γ in this context would depend on the social benefit of output expansion relative to the cost of high inflation: if the benefit is large enough relative to the cost, the agent's temptation to engage in surprise inflation is significant and instrument-based rules dominate target-based rules.

Relatedly, recall from Subsection 3.3 that the optimal instrument-based rule yields a higher variance of the outcome, but a lower variance of the agent's action, compared to the optimal target-based rule. Applied to monetary policy, this means that the optimal instrument-based rule yields more volatile output and inflation, but lower average inflation, than the optimal target-based rule. This lower average inflation makes the instrument-based rule particularly beneficial when the agent's temptation to inflate, and thus expected inflation, are high.

6 Concluding Remarks

Using mechanism design, we have characterized optimal instrument-based and target-based rules, studied the conditions under which each class is optimal, and characterized the optimal hybrid rule that combines instruments and targets. As discussed in the Introduction, our results may shed light on a number of applications. For example, in the context of monetary policy, our analysis implies that inflation targeting should be adopted if the central bank has significantly superior information relative to the public; otherwise a Taylor rule would perform better. We found that inflation targeting has a larger advantage over Taylor rules if the central bank's inflationary bias is relatively small or the sanctions that can be imposed for missing the inflation target are relatively large. Furthermore, we showed that an optimal hybrid rule would guide the choice of the interest rate as in a Taylor rule but relax these requirements when realized inflation satisfies some specified target, similar to the measures proposed by policymakers in the aftermath of the Global Financial Crisis.

We presented a stylized model with binary signals and symmetric punishments, and we showed in extensions that our main results apply also to settings with a continuum of signals or asymmetric punishments. Our analysis could be further extended in various directions. First, different specifications of asymmetric punishments could be studied by allowing for non-linearities. For example, while the principal may suffer a lower cost than the agent from any given punishment, as in our setting of Subsection 5.2, her cost may also increase at a faster rate with the severity of punishment, becoming closer to that of the agent at higher levels. Note that different specifications could give rise to different *penal codes*. In particular, unlike the extreme punishments that we found to be optimal in our model, the solution could feature punishment that "fits the crime."

A second extension could consider an agent bias that is unknown to the principal and may take different signs. In the application to monetary policy, central bankers may be biased in favor of or against inflation relative to society, and their preferences may not be public. Under binary signals and assumptions analogous to our Assumption 1 and Assumption 2, an instrument-based rule that bunches the agent types regardless of their bias would be optimal, so our characterization in Proposition 1 would remain valid. A characterization of the optimal target-based rule, on the other hand, would have to deal with the problem that assigned actions may now depend on the agent's bias.

Finally, it could be interesting to consider the quantitative representation of the optimal hybrid rule. Using quantitative methods, one could solve for this unconstrained optimal rule in a richer model than the one we have studied, and the analysis could show how predictions may depend on different features of the environment.

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A Appendix

In this Appendix, we provide proofs for the results in Section 3 and Section 4. Proofs for the results in Section 5 can be found in the Online Appendix.

A.1 Proof of Proposition 1

We proceed in three steps.

Step 1. We solve a relaxed version of the program in (7) which ignores (4) for $i = H$ and (5) for $i = L, H$. Step 2 verifies that the solution to this relaxed problem satisfies these constraints.

Step 1a. We show that the solution to this relaxed problem satisfies (4) for $i = L$ as an equality. If this were not the case, then the principal would optimally set $\mu^i = s^i$ and $V^i = \overline{V}$ for $i = L, H$. However, (4) for $i = L$ would then become

$$\int_{-\infty}^{\infty} \left[-\frac{\left(\left(s^L - \theta\right) - \alpha\right)^2}{2} \right] \phi\left(\theta | s^L, \sigma^2\right) d\theta \geq \int_{-\infty}^{\infty} \left[-\frac{\left(\left(s^L - \theta\right) + \left(s^H - s^L - \alpha\right)\right)^2}{2} \right] \phi\left(\theta | s^L, \sigma^2\right) d\theta,$$

which after some algebra yields

$$\left(s^H - s^L\right)\left(s^H - s^L - 2\alpha\right) \geq 0.$$

This inequality contradicts Assumption 1. Thus, (4) for $i = L$ must bind.

Step 1b. We show that the solution to this relaxed problem satisfies $\mu^H \geq \mu^L$. Suppose by contradiction that $\mu^H < \mu^L$. Consider two perturbations, one assigning μ^L and $\overline{V}$ to both types, and another assigning μ^H and $\overline{V}$ to both types. Since these perturbations are feasible and incentive compatible, the contradiction assumption requires that neither of them strictly increase welfare, which requires:

$$s^H \mu^H - \frac{(\mu^H)^2}{2} \geq s^H \mu^L - \frac{(\mu^L)^2}{2} \quad \text{and} \quad s^L \mu^L - \frac{(\mu^L)^2}{2} \geq s^L \mu^H - \frac{(\mu^H)^2}{2}.$$

This is equivalent to

$$\left(\mu^H - \mu^L\right) \left[s^H - \frac{\left(\mu^H + \mu^L\right)}{2} \right] \geq 0 \quad \text{and} \quad \left(\mu^H - \mu^L\right) \left[s^L - \frac{\left(\mu^H + \mu^L\right)}{2} \right] \leq 0.$$

For $\mu^H < \mu^L$, these inequalities require

$$s^H - \frac{\left(\mu^H + \mu^L\right)}{2} \leq 0 \leq s^L - \frac{\left(\mu^H + \mu^L\right)}{2},$$

which cannot hold given $s^H > s^L$. Therefore, $\mu^H \geq \mu^L$.

Step 1c. We show that the solution to this relaxed problem satisfies $V^L = V^H = \overline{V}$. Note first that if $V^L < \overline{V}$, then an increase in V^L is feasible, relaxes constraint (4) for $i = L$, and strictly increases the objective. Hence, $V^L = \overline{V}$, and therefore (4) for $i = L$ (which binds by Step 1a) can be rewritten as:

$$\left(s^L + \alpha\right)\mu^L - \frac{(\mu^L)^2}{2} + \overline{V} = \left(s^L + \alpha\right)\mu^H - \frac{(\mu^H)^2}{2} + V^H. \tag{23}$$

This equation implies that, up to an additive constant independent of the allocation, the high type's welfare satisfies

$$\left(s^H + \alpha\right)\mu^H - \frac{(\mu^H)^2}{2} + V^H = \left(s^L + \alpha\right)\mu^L - \frac{(\mu^L)^2}{2} + \overline{V} + \left(s^H - s^L\right)\mu^H. \tag{24}$$

Now suppose by contradiction that $V^H < \overline{V}$. Then it follows from (23) and Step 1b that $\mu^H > \mu^L$. Substituting (24) into the objective in (7), the principal's welfare up to an additive constant independent of the allocation is equal to

$$\left(s^L + \frac{1}{2}\alpha\right)\mu^L - \frac{(\mu^L)^2}{2} - \frac{1}{2}\left(\alpha - \left(s^H - s^L\right)\right)\mu^H + \overline{V}. \tag{25}$$

Consider a perturbation that reduces μ^H to μ^L and increases V^H to $\overline{V}$. This perturbation is feasible, satisfies (23), and strictly increases the principal's welfare given the representation in (25) and Assumption 1. It follows that $V^H < \overline{V}$ cannot hold, and thus $V^H = \overline{V}$ in the solution.

Step 1d. We show that the solution to this relaxed problem satisfies $\mu^L = \mu^H = 0$. By Step 1b, if $\mu^L \neq \mu^H$, then $\mu^H > \mu^L$. However, a perturbation that reduces μ^H to μ^L is then feasible, satisfies (23), and strictly increases the principal's welfare given the representation in (25) and Assumption 1, yielding a contradiction. It follows that $\mu^L = \mu^H$, and since $\mathbb{E}(\theta) = 0$, the principal's welfare in (7) is then maximized at $\mu^L = \mu^H = 0$.

Step 2. We verify that the solution to the relaxed problem in Step 1 satisfies the constraints of the original problem. Since $\mu^L = \mu^H$ and $V^L = V^H$, constraint (4) for $i = H$ is satisfied. As for the constraints in (5), given $\mu^L = \mu^H = 0$, these require, for $i = L, H$,

$$\overline{V} - \underline{V} \geq \frac{\left(s^i + \alpha\right)^2}{2}.$$

By Assumption 1, this inequality holds for $i = L, H$ if

$$\overline{V} - \underline{V} \geq \frac{\alpha^2/4 + \alpha^2 + \alpha^2}{2},$$

which is satisfied by Assumption 2 and the fact that $2\phi(1|0,1) < 0.5$.

Step 3. We verify that a maximally-enforced instrument threshold $\mu^* = 0$ implements the solution. Given (1) and (3), conditional on choosing an action $\mu \leq \mu^*$ and receiving continuation value $\overline{V}$, the agent's optimal action choice is $\mu = \mu^*$ regardless of his type. Moreover, conditional on choosing an action $\mu > \mu^*$ and receiving continuation value $\underline{V}$, the agent's optimal choice is $s^i + \alpha$ for each $i = L, H$. The enforcement constraints in (5) guarantee that the agent has no incentive to deviate to $\mu > \mu^*$.

A.2 Proof of Proposition 2

We proceed in two steps.

Step 1. We follow a first-order approach by solving a relaxed version of the program in (10) that replaces (8) with the agent's first-order condition (12). Step 2 verifies the validity of this approach.

As noted in the text, the solution to (12) is $\mu^i = s^i + \kappa$ for $i = L, H$ and some $\kappa \gtreqless 0$. Hence, the relaxed problem, up to an additive constant independent of the allocation, can be written as:

$$\max_{\kappa, \{V(\pi)\}_{\pi \in \mathbb{R}}} \left\{ -\frac{\kappa^2}{2} + \int_{-\infty}^{\infty} V(\pi)\, \phi(\kappa - \pi|0, \sigma^2) d\pi \right\} \tag{26}$$

subject to

$$\alpha - \kappa + \int_{-\infty}^{\infty} V(\pi)\, \phi'(\kappa - \pi|0, \sigma^2) d\pi = 0, \tag{27}$$

$$V(\pi) \in \left[\underline{V}, \overline{V}\right] \text{ for all } \pi \in \mathbb{R}. \tag{28}$$

Step 1a. Denote by λ the Lagrange multiplier on (27). We show that $\lambda < 0$. To do this, we consider a doubly-relaxed problem in which constraint (27) is replaced with an inequality constraint (cf. Rogerson, 1985):

$$\alpha - \kappa + \int_{-\infty}^{\infty} V(\pi)\phi'(\kappa - \pi|0,\sigma^2)d\pi \leq 0. \tag{29}$$

Since this is an inequality constraint, the multiplier satisfies $\lambda \leq 0$. We show that (29) holds as an equality in the solution to the doubly-relaxed problem, and thus this problem is equivalent to (26)-(28) with $\lambda < 0$. Suppose by contradiction that (29) holds as a strict inequality. Then to maximize (26) the principal chooses $\kappa = 0$ and $V(\pi) = \overline{V}$ for all π. However, substituting back into the left-hand side of (29), using the fact that $\phi'(\kappa - \pi|0,\sigma^2) = \frac{\pi-\kappa}{\sigma^2}\phi(\kappa - \pi|0,\sigma^2)$, yields

$$\alpha + \overline{V}\int_{-\infty}^{\infty} \frac{\pi}{\sigma^2}\phi(-\pi|0,\sigma^2)d\pi = \alpha \leq 0,$$

which is a contradiction since $\alpha > 0$. Therefore, (29) holds as an equality in the doubly-relaxed problem and $\lambda < 0$.

<u>Step 1b.</u> We show that the solution to (26)-(28) satisfies $V(\pi) = \overline{V}$ if $\pi \leq \pi^*$ and $V(\pi) = \underline{V}$ if $\pi > \pi^*$, for some $\pi^* \in (-\infty, \infty)$. Denote by $\overline{\psi}(\pi)$ and $\underline{\psi}(\pi)$ the Lagrange multipliers on the upper bounds and the lower bounds on $V(\pi)$. The first-order condition with respect to $V(\pi)$ is

$$\phi(\kappa - \pi|0,\sigma^2) + \lambda\phi'(\kappa - \pi|0,\sigma^2) + \underline{\psi}(\pi) - \overline{\psi}(\pi) = 0. \tag{30}$$

Suppose that $V(\pi)$ is strictly interior with $\underline{\psi}(\pi) = \overline{\psi}(\pi) = 0$. Then (30) yields

$$-\frac{1}{\lambda} = \frac{\phi'(\kappa - \pi|0,\sigma^2)}{\phi(\kappa - \pi|0,\sigma^2)} = \frac{\pi - \kappa}{\sigma^2}. \tag{31}$$

Since the right-hand side of (31) is strictly increasing in π whereas the left-hand side is constant, it follows that (31) holds for only one value of $\pi \in (-\infty, \infty)$, which we label π^*. By (30) and (31), the solution has $V(\pi) = \overline{V}$ if $\pi \leq \pi^*$ and $V(\pi) = \underline{V}$ if $\pi > \pi^*$.

<u>Step 1c.</u> We show that $\pi^* > \kappa$ and $\kappa \in (0, \alpha)$. To show the first inequality, recall from Step 1a that $\lambda < 0$; hence, (31) yields $\pi^* > \kappa$. To show $\kappa < \alpha$, note that by Step 1b, (27) can be rewritten as

$$\alpha - \kappa - \phi\left(\kappa - \pi^*|0,\sigma^2\right)\left(\overline{V} - \underline{V}\right) = 0. \tag{32}$$

Since $\phi\left(\kappa - \pi^*|0,\sigma^2\right)\left(\overline{V} - \underline{V}\right) > 0$, (32) requires $\kappa < \alpha$.

We are left to show that $\kappa > 0$. By Step 1b, we can write the Lagrangian of the principal solving for the optimal level of κ and π^* as

$$-\frac{\kappa^2}{2} + \left(1 - \Phi(\kappa - \pi^*|0, \sigma^2)\right)\overline{V} + \Phi(\kappa - \pi^*|0, \sigma^2)\underline{V} \\ + \lambda \left[\alpha - \kappa - \phi(\kappa - \pi^*|0, \sigma^2) \left(\overline{V} - \underline{V}\right)\right]. \tag{33}$$

The first-order condition with respect to κ is

$$-\kappa - \phi(\kappa - \pi^*|0, \sigma^2) \left(\overline{V} - \underline{V}\right) - \lambda \left[1 + \phi'(\kappa - \pi^*|0, \sigma^2) \left(\overline{V} - \underline{V}\right)\right] = 0, \tag{34}$$

and the first-order condition with respect to π^* is

$$\phi(\kappa - \pi^*|0, \sigma^2) \left(\overline{V} - \underline{V}\right) + \lambda\phi'(\kappa - \pi^*|0, \sigma^2) \left(\overline{V} - \underline{V}\right) = 0. \tag{35}$$

Substituting (35) into (34) yields

$$-\lambda = \kappa. \tag{36}$$

Since $\lambda < 0$ by Step 1a, (36) implies $\kappa > 0$.

Step 2. We verify the validity of the first-order approach: we establish that the choice of κ in the relaxed problem satisfies (8) and therefore corresponds to the agent's global optimum.

Step 2a. We begin by showing that the agent has no incentive to choose some $\kappa' \neq \kappa$, $\kappa' \leq \pi^*$. Differentiating the first-order condition (32) with respect to κ yields

$$-1 - \phi'(\kappa - \pi^*|0, \sigma^2) \left(\overline{V} - \underline{V}\right). \tag{37}$$

Note that (37) is strictly negative for all $\kappa \leq \pi^*$, and thus the agent's welfare is strictly concave over this range. Since by Step 1c the solution to the relaxed problem sets $\kappa < \pi^*$, we conclude that this κ is a maximum and dominates any other $\kappa' \leq \pi^*$.

Step 2b. We next show that the agent has no incentive to choose some $\kappa' \neq \kappa$, $\kappa' > \pi^*$. To prove this, we first establish that in the solution to the relaxed problem, given π^*, κ satisfies $\kappa - \pi^* \leq -\sigma$. Suppose by contradiction that $\kappa - \pi^* > -\sigma$. Note that by (31) and (36), $\kappa - \pi^* = -\frac{\sigma^2}{\kappa}$. Hence, the contradiction assumption implies $\kappa > \sigma$. Substituting $\kappa - \pi^* = -\frac{\sigma^2}{\kappa}$ into (32) yields

$$\alpha - \kappa - \phi\left(-\frac{\sigma^2}{\kappa}|0, \sigma^2\right) \left(\overline{V} - \underline{V}\right) = 0. \tag{38}$$

Since the left-hand side of (38) is decreasing in κ and (by the contradiction assumption) $\kappa > \sigma$, (38) requires

$$\alpha - \sigma - \phi\left(-\sigma|0,\sigma^2\right)\left(\overline{V} - \underline{V}\right) > 0.$$

Multiply both sides of this equation by $\sigma > 0$ to obtain:

$$\sigma(\alpha - \sigma) - \sigma\phi\left(-\sigma|0,\sigma^2\right)\left(\overline{V} - \underline{V}\right) > 0. \tag{39}$$

Note that since $0 < \sigma < \sqrt{Var(\theta)}$ and, by Assumption 1, $\sqrt{Var(\theta)} \leq \alpha/2$, we have $\sigma\left(\alpha - \sigma\right) < \alpha^2/2$. Hence, (39) yields

$$\frac{\alpha^2}{2\sigma\phi\left(-\sigma|0,\sigma^2\right)} > \overline{V} - \underline{V}.$$

However, this inequality violates Assumption 2 since $\sigma\phi\left(-\sigma|0,\sigma^2\right) = \phi\left(1|0,1\right)$. Thus, given π^*, κ satisfies $\kappa - \pi^* \leq -\sigma$.

We can now establish that the agent has no incentive to deviate to $\kappa' \neq \kappa$, $\kappa' > \pi^*$. Consider some $\kappa' > \pi^*$ that is a local maximum for the agent. The difference in welfare for the agent from choosing the value of κ given by the solution to the relaxed problem versus κ' is equal to

$$\alpha\kappa - \frac{\kappa^2}{2} - \left(\alpha\kappa' - \frac{(\kappa')^2}{2}\right) + \left(\Phi\left(\kappa' - \pi^*|0,\sigma^2\right) - \Phi\left(\kappa - \pi^*|0,\sigma^2\right)\right)\left(\overline{V} - \underline{V}\right). \tag{40}$$

Note that by the arguments in Step 1c and κ and κ' satisfying the agent's first-order condition, it follows that both κ and κ' are between 0 and α. Thus, (40) is bounded from below by

$$-\frac{\alpha^2}{2} + \left(\Phi\left(\kappa' - \pi^*|0,\sigma^2\right) - \Phi\left(\kappa - \pi^*|0,\sigma^2\right)\right)\left(\overline{V} - \underline{V}\right). \tag{41}$$

Since (32) is satisfied for both κ and κ' and $\kappa' > \pi^* > \kappa$, we must have $\phi\left(\kappa - \pi^*|0,\sigma^2\right) > \phi\left(\kappa' - \pi^*|0,\sigma^2\right)$. Moreover, by the symmetry of the normal distribution, $\phi\left(\kappa - \pi^*|0,\sigma^2\right) = \phi\left(-\left(\kappa - \pi^*\right)|0,\sigma^2\right)$ and thus $\Phi\left(-\left(\kappa - \pi^*\right)|0,\sigma^2\right) < \Phi\left(\kappa' - \pi^*|0,\sigma^2\right)$. Therefore, (41) is bounded from below by

$$-\frac{\alpha^2}{2} + \left(\Phi\left(-\left(\kappa - \pi^*\right)|0,\sigma^2\right) - \Phi\left(\kappa - \pi^*|0,\sigma^2\right)\right)\left(\overline{V} - \underline{V}\right). \tag{42}$$

Since, as shown above, $\kappa - \pi^* \leq -\sigma$, we obtain that (42) is itself bounded from below by

$$-\frac{\alpha^2}{2} + \left(\Phi\left(\sigma|0,\sigma^2\right) - \Phi\left(-\sigma|0,\sigma^2\right)\right)\left(\overline{V} - \underline{V}\right) = -\frac{\alpha^2}{2} + \left(\Phi\left(1|0,1\right) - \Phi\left(-1|0,1\right)\right)\left(\overline{V} - \underline{V}\right) > 0,$$

where the last inequality follows from Assumption 2 and the fact that $\phi(1|0,1) < \Phi(1|0,1) - \Phi(-1|0,1)$. Therefore, the agent strictly prefers κ over κ'.

A.3 Proof of Proposition 3

We begin by proving the following lemma:

Lemma 1. *Consider changing σ while keeping $Var(\theta)$ unchanged. The principal's welfare is independent of σ under the optimal instrument-based rule and it is strictly decreasing in σ under the optimal target-based rule.*

Proof. By Proposition 1, an optimal instrument-based rule sets $\mu^i = 0$ and $V^i = \overline{V}$ for $i = L, H$. Since $Var(\theta) = \mathbb{E}(\theta^2) - (\mathbb{E}(\theta))^2 = \mathbb{E}(\theta^2)$ (by $\mathbb{E}(\theta) = 0$), the principal's welfare under this rule is equal to $-\frac{Var(\theta)}{2} + \overline{V}$, which is independent of σ.

To evaluate the principal's welfare under an optimal target-based rule, consider the Lagrangian taking into account the conditional variance term (which is exogenous and thus excluded from (33)):

$$-\frac{\sigma^2}{2} - \frac{\kappa^2}{2} + \left(1 - \Phi(\kappa - \pi^*|0,\sigma^2)\right)\overline{V} + \Phi(\kappa - \pi^*|0,\sigma^2)\underline{V}$$
$$+\lambda\left[\alpha - \kappa - \phi(\kappa - \pi^*|0,\sigma^2)\left(\overline{V} - \underline{V}\right)\right].$$

The derivative with respect to σ is:

$$-\sigma + \left(\overline{V} - \underline{V}\right)\left[\int_{\kappa-\pi^*}^{\infty}\left(-\frac{\sigma^2 - z^2}{\sigma^3}\right)\phi(z|0,\sigma^2)dz + \lambda\frac{\sigma^2 - (\kappa - \pi^*)^2}{\sigma^3}\phi(\kappa - \pi^*|0,\sigma^2)\right].$$

The first term is strictly negative. Using (31) and (36) to substitute in for λ and π^*, the sign of the second term is the same as the sign of

$$-\int_{-\frac{\sigma^2}{\kappa}}^{\infty}\left(\sigma^2 - z^2\right)\phi(z|0,\sigma^2)dz - \kappa\left[\sigma^2 - \left(\frac{\sigma^2}{\kappa}\right)^2\right]\phi\left(-\frac{\sigma^2}{\kappa}|0,\sigma^2\right). \tag{43}$$

We next show that this expression is strictly negative, which proves the claim. To show this, consider the derivative of (43) with respect to κ:

$$\left[\left(\sigma^2 - \left(\frac{\sigma^2}{\kappa}\right)^2\right)\frac{\sigma^2}{\kappa^2} - \left(\sigma^2 - \left(\frac{\sigma^2}{\kappa}\right)^2\right) - 2\left(\frac{\sigma^2}{\kappa}\right)^2 - \left(\sigma^2 - \left(\frac{\sigma^2}{\kappa}\right)^2\right)\frac{\sigma^2}{\kappa^2}\right]\phi\left(-\frac{\sigma^2}{\kappa}|0,\sigma^2\right).$$

This derivative takes the same sign as

$$-\left(\sigma^2 - \left(\frac{\sigma^2}{\kappa}\right)^2\right) - 2\left(\frac{\sigma^2}{\kappa}\right)^2,$$

which is strictly negative. Hence, since $\kappa > 0$, it suffices to show that the sign of (43) is weakly negative for $\kappa \to 0$. By the definition of variance, the first term in (43) goes to zero as $\kappa \to 0$. The second term in (43) can be rewritten as:

$$-\sigma^2 \kappa \phi\left(-\frac{\sigma^2}{\kappa}|0, \sigma^2\right) + \frac{\sigma^4}{\kappa}\phi\left(-\frac{\sigma^2}{\kappa}|0, \sigma^2\right). \tag{44}$$

As $\kappa \to 0$, the first term in (44) goes to zero. Moreover, applying L'Hopital's Rule on

$$\frac{1/\kappa}{\phi\left(-\frac{\sigma^2}{\kappa}|0, \sigma^2\right)^{-1}}$$

shows that the second term also goes to zero. □

We now proceed with the proof of Proposition 3. By Lemma 1, welfare under the optimal instrument-based rule is invariant to σ, whereas welfare under the optimal target-based rule is decreasing in σ. To prove the first part of the proposition, it thus suffices to show that a target-based rule is optimal at one extreme, for $\sigma \to 0$, and an instrument-based rule is optimal at the other extreme, for $\sigma \to \sqrt{Var(\theta)}$. This is what we prove next.

Consider first the case of $\sigma \to 0$. By the arguments in Step 1c and Step 2b of the proof of Proposition 2, $0 < \kappa \leq \sigma$. Hence, $\kappa \to 0$ as $\sigma \to 0$. Moreover, as $\sigma \to 0$, $\phi(z|0, \sigma^2)$ corresponds to a Dirac's delta function, with cumulative distribution function $\Phi(z|0, \sigma^2) = 0$ if $z < 0$ and $\Phi(z|0, \sigma^2) = 1$ if $z \geq 0$. Therefore, since $\kappa - \pi^* < 0$ in the optimal target-based rule, the limit of the principal's welfare under this rule, as $\sigma \to 0$, is given by

$$\lim_{\sigma \to 0}\left\{-\frac{\kappa^2}{2} + \left(1 - \Phi\left(\kappa - \pi^*|0, \sigma^2\right)\right)\overline{V} + \Phi\left(\kappa - \pi^*|0, \sigma^2\right)\underline{V}\right\} = \overline{V}.$$

Since the principal's welfare under the optimal instrument-based rule is $-\frac{Var(\theta)}{2} + \overline{V}$, it follows that the optimal target-based rule dominates the optimal instrument-based rule.

Consider next the case of $\sigma \to \sqrt{Var(\theta)}$ and thus $\Delta \to 0$. Since κ in the optimal target-based rule satisfies equation (38), the solution in this case admits $\kappa > 0$. The principal's welfare under the optimal target-based rule is then equal to

$$-\frac{Var(\theta)}{2} + \overline{V} - \frac{\kappa^2}{2} - \Phi\left(\kappa - \pi^*|0, \sigma^2\right)\left(\overline{V} - \underline{V}\right).$$

Since this is strictly lower than $-\frac{Var(\theta)}{2} + \overline{V}$, it follows that the optimal instrument-based rule dominates the optimal target-based rule.

Finally, to prove the second part of the proposition, note that the principal's welfare under the optimal instrument-based rule is independent of the agent's bias α and the punishment $\underline{V}$. Thus, it suffices to show that the principal's welfare under the optimal target-based rule is decreasing in α and $\underline{V}$. The former follows from the fact that the derivative of the Lagrangian in (33) with respect to α is equal to λ, which is strictly negative by Step 1a in the proof of Proposition 2. To evaluate how welfare changes with $\underline{V}$, consider the representation of the program in (10). A reduction in $\underline{V}$ relaxes constraint (9). Since this constraint is binding in the solution (by Step 1b of the proof of Proposition 2), it follows that a reduction in $\underline{V}$ strictly increases the principal's welfare under the optimal target-based rule.

A.4 Proof of Proposition 4

We proceed in three steps.

Step 1. We solve a relaxed version of (13)-(16) which ignores (14) for $i = H$ and (15) for $i = L, H$. Step 2 verifies that the solution to this relaxed problem satisfies these constraints.

Step 1a. We show that the solution satisfies (14) for $i = L$ as an equality. The proof of this claim is analogous to that in Step 1a of the proof of Proposition 1 and thus omitted.

Step 1b. We show that the solution satisfies $\mu^H \geq \mu^L$. The proof of this claim is analogous to that in Step 1b of the proof of Proposition 1 and thus omitted.

Step 1c. We show that the solution satisfies $V^L(\theta) = \overline{V}$ for all θ. If $V^L(\theta) < \overline{V}$ for some θ, then an increase in $V^L(\theta)$ is feasible, relaxes constraint (14) for $i = L$, and strictly increases the objective. The claim follows.

Step 1d. We show that the solution satisfies $V^H(\theta) = \underline{V}$ if $\theta < \theta^*$ and $V^H(\theta) = \overline{V}$ if $\theta \geq \theta^*$, for some $\theta^* \in (-\infty, \infty)$. Let $\frac{1}{2}\lambda$ be the Lagrange multiplier on (14) and denote by $\overline{\psi}(\theta)$ and $\underline{\psi}(\theta)$ the Lagrange multipliers on the upper bounds and the lower bounds on

$V^H(\theta)$. The first-order condition with respect to $V^H(\theta)$ yields

$$\frac{1}{2}\phi(\theta|s^H,\sigma^2) - \frac{1}{2}\lambda\phi(\theta|s^L,\sigma^2) + \underline{\psi}(\theta) - \overline{\psi}(\theta) = 0. \tag{45}$$

Suppose that $V^H(\theta)$ is strictly interior with $\underline{\psi}(\theta) = \overline{\psi}(\theta) = 0$. Then (45) implies

$$\lambda = \frac{\phi\left(\theta - s^H|0,\sigma^2\right)}{\phi\left(\theta - s^L|0,\sigma^2\right)}. \tag{46}$$

Since the right-hand side of (46) is strictly increasing in θ whereas the left-hand side is constant, it follows that (46) holds only for one value of $\theta \in (-\infty,\infty)$, which we label θ^*. By (45) and (46), the solution has $V(\theta) = \underline{V}$ if $\theta < \theta^*$ and $V(\theta) = \overline{V}$ if $\theta \geq \theta^*$.

Step 1e. We show that the solution satisfies $\mu^L \in \left[s^L, s^H\right]$ and $\mu^H \in \left[s^L, s^H\right]$ with $\mu^H > \mu^L$. Since $\lambda > 0$, it follows that θ^* satisfying (46) is strictly interior. Hence, given Step 1b, the binding constraint (14) for $i = L$ implies $\mu^H > \mu^L$. The principal's first-order condition with respect to μ^L yields

$$\mu^L = s^L + \frac{\lambda}{1+\lambda}\alpha,$$

which implies that μ^L, and thus μ^H, exceed s^L. The first-order condition with respect to μ^H yields

$$\mu^H = s^H - \frac{\lambda}{1-\lambda}\left(\alpha - 2\Delta\right),$$

and the second-order condition yields $\lambda < 1$. Using Assumption 1, it follows that μ^H, and thus μ^L, are weakly below s^H.

We end this step by observing that since $\lambda < 1$ and $s^H = -s^L = \Delta$, (46) implies $\theta^* < 0$.

Step 2. We verify that the solution to the relaxed problem satisfies the constraints of the original problem. The binding constraint (14) for $i = L$ implies

$$\overline{V} - \left(1 - \Phi\left(\theta^*|s^L,\sigma^2\right)\right)\overline{V} - \Phi\left(\theta^*|s^L,\sigma^2\right)\underline{V} = \left(s^L+\alpha\right)\mu^H - \frac{(\mu^H)^2}{2} - \left(s^L+\alpha\right)\mu^L + \frac{(\mu^L)^2}{2}. \tag{47}$$

Since $s^H > s^L$ and $\mu^H > \mu^L$, the right-hand side of (47) is strictly smaller than

$$\left(s^H+\alpha\right)\mu^H - \frac{(\mu^H)^2}{2} - \left(s^H+\alpha\right)\mu^L + \frac{(\mu^L)^2}{2}. \tag{48}$$

Moreover, the left-hand side of (47) is strictly larger than

$$\overline{V}-\left(1-\Phi\left(\theta^{*}|s^{H},\sigma^{2}\right)\right)\overline{V}-\Phi\left(\theta^{*}|s^{H},\sigma^{2}\right)\underline{V}. \tag{49}$$

Therefore, (48) is strictly larger than (49), implying that constraint (14) for $i=H$ is satisfied.

To verify that constraint (15) for $i=L$ is satisfied, recall from Step 1e that $\mu^{L}\in[s^{L},s^{H}]$. Given this range, the low type's welfare in the optimal rule is no less than that under $\mu^{L}=s^{L}$, and thus (15) for $i=L$ is guaranteed to hold if

$$\overline{V}-\underline{V}\geq\frac{\alpha^{2}}{2},$$

which is satisfied by Assumption 2.

Finally, we verify that constraint (15) for $i=H$ is also satisfied. Note that by constraint (14) for $i=H$ being satisfied, the high type's welfare in the optimal rule is no less than that achieved from mimicking the low type under $\mu^{L}=s^{L}$. Thus, (15) for $i=H$ is guaranteed to hold if

$$\overline{V}-\underline{V}\geq\frac{(2\Delta+\alpha)^{2}}{2},$$

which is satisfied by Assumption 1 and Assumption 2.

Step 3. We verify that a maximally-enforced hybrid threshold $\{\mu^{*},\mu^{**},\{\pi^{*}(\mu)\}_{\mu\in\mathbb{R}}\}$ implements the solution. Let $\mu^{*}=\mu^{L}$ and $\mu^{**}=\mu^{H}$. Construct the function $\pi^{*}(\mu)$ as described in (17), with $h(\mu)$ solving

$$\begin{aligned}\overline{V}-\left(1-\Phi\left(\mu-h\left(\mu\right)|s^{L},\sigma^{2}\right)\right)\overline{V}-\Phi\left(\mu-h\left(\mu\right)|s^{L},\sigma^{2}\right)\underline{V}\\ =\left(s^{L}+\alpha\right)\mu-\frac{\mu^{2}}{2}-\left(s^{L}+\alpha\right)\mu^{L}+\frac{(\mu^{L})^{2}}{2}.\end{aligned} \tag{50}$$

The left-hand side of (50) is increasing in $\mu-h\left(\mu\right)$ and the right-hand side is increasing in μ. Note that $\lim_{\mu\downarrow\mu^{*}}h\left(\mu\right)=\infty$ and, by (47), $h\left(\mu^{**}\right)=\mu^{**}-\theta^{*}$. If follows that a solution for $h\left(\mu\right)$ exists and $\mu-h\left(\mu\right)$ is increasing in μ.

We verify that both agent types choose their prescribed actions, $\mu^{L}=\mu^{*}$ and $\mu^{H}=\mu^{**}$, under this maximally-enforced hybrid threshold. By Step 2, neither type has incentives to deviate to $\mu>\mu^{**}$, as the best such deviation entails choosing $\mu=s^{i}+\alpha$ for $i=L,H$ which is suboptimal by (15). The low type has no incentive to deviate to $\mu=\mu^{**}$ by (14) for $i=L$, and this type has no incentive to deviate to $\mu<\mu^{*}$ either as he is better off by instead choosing $\mu^* < s^L + \alpha$ and receiving the same continuation value. The high type has no incentive to deviate to $\mu \leq \mu^*$, as the best such deviation entails choosing $\mu^* < s^H + \alpha$ which is suboptimal by (14) for $i = H$. Therefore, it only remains to be shown that neither type has incentives to deviate to $\mu \in (\mu^*, \mu^{**})$. This follows immediately from (50) for the low type, as this equation ensures that the low type is indifferent between choosing μ^* and choosing any $\mu \in (\mu^*, \mu^{**})$. To show that the high type has no incentive to deviate, combine (47) and (50) to obtain:

$$\left(\Phi\left(\theta^*|s^L, \sigma^2\right) - \Phi\left(\mu - h\left(\mu\right)|s^L, \sigma^2\right)\right)\left(\overline{V} - \underline{V}\right) = \left(s^L + \alpha\right)\mu^H - \frac{(\mu^H)^2}{2} - \left(s^L + \alpha\right)\mu + \frac{\mu^2}{2}. \tag{51}$$

Since $s^H > s^L$ and $\mu^H > \mu$, the right-hand side of (51) is strictly smaller than

$$\left(s^H + \alpha\right)\mu^H - \frac{(\mu^H)^2}{2} - \left(s^H + \alpha\right)\mu + \frac{\mu^2}{2}. \tag{52}$$

Moreover, note that $\mu - h(\mu) < \theta^*$ for $\mu \in (\mu^*, \mu^{**})$, and $\theta^* < 0$ by Step 1e. Hence, the left-hand side of (51) is strictly larger than

$$\left(\Phi\left(\theta^*|s^H, \sigma^2\right) - \Phi\left(\mu - h\left(\mu\right)|s^H, \sigma^2\right)\right)\left(\overline{V} - \underline{V}\right). \tag{53}$$

Therefore, (52) is strictly larger than (53), implying that the high type has no incentive to deviate to $\mu \in (\mu^*, \mu^{**})$.

Online Appendix for
"Instrument-Based vs. Target-Based Rules"

by Marina Halac and Pierre Yared

In this Online Appendix, we provide proofs for the results in Section 5 of the paper.

B Proofs for Section 5

B.1 Proof of Proposition 5

We first study the optimal instrument-based and target-based rules separately and then compare them.

B.1.1 Optimal Instrument-Based Rule

The program that solves for the optimal instrument-based rule under a continuum of types is analogous to that in (7) in Section 3.1 of the paper:

$$\max_{\{\mu(s),V(s)\}_{s\in\mathbb{R}}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[-\frac{(\mu(s)-\theta)^2}{2}+V(s)\right]\phi(\theta|s,\sigma^2)\phi(s|0,\Delta^2)d\theta ds$$

subject to, for all $s,s'\in\mathbb{R}$,

$$\int_{-\infty}^{\infty}\left[-\frac{(\mu(s)-\theta-\alpha)^2}{2}+V(s)\right]\phi(\theta|s,\sigma^2)d\theta \geq \int_{-\infty}^{\infty}\left[-\frac{(\mu(s')-\theta-\alpha)^2}{2}+V(s')\right]\phi(\theta|s,\sigma^2)d\theta,$$

$$\int_{-\infty}^{\infty}\left[-\frac{(\mu(s)-\theta-\alpha)^2}{2}+V(s)\right]\phi(\theta|s,\sigma^2)d\theta \geq \int_{-\infty}^{\infty}\left[-\frac{(s-\theta)^2}{2}+\underline{V}\right]\phi(\theta|s,\sigma^2)d\theta,$$

$$V(s)\in[\underline{V},\overline{V}].$$

Integrating over θ, this program, up to a constant independent of the allocation, can be rewritten as:

$$\max_{\{\mu(s),V(s)\}_{s\in\mathbb{R}}} \int_{-\infty}^{\infty} \left(s\mu(s) - \frac{\mu(s)^2}{2} + V(s) \right) \phi(s|0,\Delta^2) ds \tag{54}$$

subject to, for all $s, s' \in \mathbb{R}$,

$$(s+\alpha)\mu(s) - \frac{\mu(s)^2}{2} + V(s) \geq (s+\alpha)\mu(s') - \frac{\mu(s')^2}{2} + V(s'), \tag{55}$$

$$(s+\alpha)\mu(s) - \frac{\mu(s)^2}{2} + V(s) \geq \frac{(s+\alpha)^2}{2} + \underline{V}, \tag{56}$$

$$V(s) \in [\underline{V}, \overline{V}]. \tag{57}$$

An instrument-based rule specifying $\{\mu(s), V(s)\}_{s\in\mathbb{R}}$ is incentive compatible if this allocation satisfies (55)-(56), and it is incentive compatible and feasible, or incentive feasible for short, if the allocation satisfies (55)-(57). As mentioned in the paper, we assume that an optimal instrument-based rule is piecewise continuously differentiable. Additionally, if the program above admits multiple solutions that differ only on a countable set of types, we select the solution that maximizes the principal's welfare for those types.

We proceed in four steps. Step 1 establishes some preliminary results that we use in the subsequent steps. Step 2 shows that any optimal instrument-based rule must prescribe bang-bang continuation values. Step 3 shows that in any such rule, either all types receive the maximal continuation value, or there exists an interior cutoff s^{**} such that only types above s^{**} receive the worst continuation value. Step 4 concludes the proof by characterizing the allocation of actions and showing that such an interior cutoff s^{**} indeed exists in any optimal instrument-based rule.

Step 1. *We establish some preliminary results.*

The next lemma follows from standard arguments; see Fudenberg and Tirole (1991):

Lemma 2. *$\{\mu(s), V(s)\}_{s\in\mathbb{R}}$ satisfies the private information constraint (55) if and only if: (i) $\mu(s)$ is nondecreasing, and (ii) the following local private information constraints are satisfied:*

1. *At any point s at which $\mu(\cdot)$, and thus $V(\cdot)$, are differentiable,*

$$\mu'(s)(s+\alpha-\mu(s)) + V'(s) = 0.$$

2. *At any point s at which $\mu(\cdot)$ is not differentiable,*

$$\lim_{s'\uparrow s}\left\{(s+\alpha)\mu(s')-\frac{\mu(s')^2}{2}+V(s')\right\}=\lim_{s'\downarrow s}\left\{(s+\alpha)\mu(s')-\frac{\mu(s')^2}{2}+V(s')\right\}.$$

The private information constraints imply that the derivative of the agent's welfare with respect to s is $\mu(s)$. Hence, in an incentive compatible rule, the welfare of type $s\in\mathbb{R}$ satisfies

$$(s+\alpha)\mu(s)-\frac{\mu(s)^2}{2}+V(s)=\lim_{\underline{s}\to-\infty}\left\{(\underline{s}+\alpha)\mu(\underline{s})-\frac{\mu(\underline{s})^2}{2}+V(\underline{s})+\int_{\underline{s}}^{s}\mu(\widetilde{s})d\widetilde{s}\right\}. \quad (58)$$

Following Amador, Werning, and Angeletos (2006), we can substitute (58) into the principal's objective in (54) to rewrite this objective as

$$\lim_{\underline{s}\to-\infty}\left\{(\underline{s}+\alpha)\mu(\underline{s})-\frac{\mu(\underline{s})^2}{2}+V(\underline{s})+\int_{\underline{s}}^{\infty}\mu(s)Q(s)ds\right\}, \quad (59)$$

where

$$Q\left(s\right)\equiv 1-\Phi(s|0,\Delta^2)-\alpha\phi(s|0,\Delta^2).$$

Note that

$$Q'(s)=-\phi(s|0,\Delta^2)-\alpha\phi'(s|0,\Delta^2),$$

and thus $Q'(s)<0$ if $s<\widehat{s}\equiv\Delta^2/\alpha$ and $Q'(s)>0$ if $s>\widehat{s}$. (Observe that this property on $Q'(s)$ holds for some $\widehat{s}$ for any density function ϕ that is log concave.) We next describe two functions that we will use in our proofs.

Lemma 3. *Given $s^L\leq s^M\leq s^H$, define the functions*

$$\begin{aligned} B^L(s^L,s^M) &= \int_{s^L}^{s^M}\left(s-s^L-\alpha\right)\phi(s|0,\Delta^2)ds+\alpha\phi(s^M|0,\Delta^2)(s^M-s^L),\\ B^H(s^H,s^M) &= \int_{s^M}^{s^H}\left(s-s^H-\alpha\right)\phi(s|0,\Delta^2)ds+\alpha\phi(s^M|0,\Delta^2)(s^H-s^M). \end{aligned}$$

Then $B^L(s^L,s^M)>0$ if $Q'(s)<0$ for all $s\in(s^L,s^M)$, $B^L(s^L,s^M)<0$ if $Q'(s)>0$ for all $s\in(s^L,s^M)$, $B^H(s^H,s^M)>0$ if $Q'(s)>0$ for all $s\in(s^M,s^H)$, and $B^H(s^H,s^M)<0$ if $Q'(s)<0$ for all $s\in(s^M,s^H)$.

Proof. Consider the claims about $B^L(s^L,s^M)$. Note that $B^L(s,s^M)|_{s=s^M}=0$, and hence

$B^L(s^L,s^M) = -\int_{s_L}^{s^M} \frac{dB^L(s,s^M)}{ds} ds$. Moreover,

$$\frac{dB^L(s,s^M)}{ds} = -\int_s^{s^M} \phi(\widetilde{s}|0,\Delta^2)d\widetilde{s} + \alpha\phi(s|0,\Delta^2) - \alpha\phi(s^M|0,\Delta^2),$$

and thus $\frac{dB^L(s,s^M)}{ds}|_{s=s^M} = 0$. Therefore, $B^L(s^L,s^M) = \int_{s_L}^{s^M}\int_s^{s^M} \frac{d^2B^L(\widetilde{s},s^M)}{d\widetilde{s}^2} d\widetilde{s} ds$, where

$$\frac{d^2B^L(s,s^M)}{ds^2} = \phi(s|0,\Delta^2) + \alpha\phi'(s|0,\Delta^2).$$

Note that $\frac{d^2B^L(s,s^M)}{ds^2} > 0$ if $Q'(s) < 0$, $\frac{d^2B^L(s,s^M)}{dB^2} = 0$ if $Q'(s) = 0$, and $\frac{d^2B^L(s,s^M)}{ds^2} < 0$ if $Q'(s) > 0$. The claims about $B^L(s^L,s^M)$ follow.

The proof for the claims about $B^H(s^H,s^M)$ is analogous and thus omitted. □

Step 2. *We show that if $\{\mu(s),V(s)\}_{s\in\mathbb{R}}$ is an optimal instrument-based rule, then $V(s) \in \{\underline{V},\overline{V}\}$ for all $s \in \mathbb{R}$.*

Take any solution to the program in (54)-(57). We proceed in three sub-steps.

Step 2a. *We show $V(s)$ is left-continuous at each $s \in \mathbb{R}$.*

Suppose by contradiction that there exists s at which $V(s)$ is not left-continuous. Denote the left limit by $\{\mu(s^-),V(s^-)\} = \lim_{s'\uparrow s}\{\mu(s'),V(s')\}$. By Lemma 2,

$$(s+\alpha)\mu(s) - \frac{\mu(s)^2}{2} - (s+\alpha)\mu(s^-) + \frac{\mu(s^-)^2}{2} = V(s^-) - V(s).$$

Given $\alpha > 0$ and the fact that $\mu(s^-) < \mu(s)$ by Lemma 2, this implies

$$s\mu(s) - \frac{\mu(s)^2}{2} - s\mu(s^-) - \frac{\mu(s^-)^2}{2} < V(s^-) - V(s).$$

It follows that a perturbation that assigns $\{\mu(s^-),V(s^-)\}$ to type s is incentive feasible, strictly increases the principal's welfare from type s, and does not affect the principal's welfare from types other than s. Hence, $V(s)$ must be left-continuous at each $s \in \mathbb{R}$.

Step 2b. *We show $V(s)$ is a step function over any interval $[s^L,s^H]$ with $V(s) \in (\underline{V},\overline{V})$.*

By the private information constraints, $V(s)$ is piecewise continuously differentiable and nonincreasing. Suppose by contradiction that there is an interval $[s^L,s^H]$ over which $V(s)$ is continuously strictly decreasing in s and satisfies $\underline{V} < V(s) < \overline{V}$. By Lemma 2, $\mu(s)$ must be continuously strictly increasing over the interval, and without loss we can take an interval over which $\mu(s)$ is continuously differentiable. Moreover, by the properties of the normal distribution, we can take either an interval above $\widehat{s}$ with $Q'(s) > 0$ for all $s \in [s^L, s^H]$ or an interval below $\widehat{s}$ with $Q'(s) < 0$ for all $s \in [s^L, s^H]$. We consider each possibility in turn.

Case 1: Suppose $Q'(s) < 0$ for all $s \in [s^L, s^H]$. We show that there exists an incentive feasible perturbation that rotates the increasing schedule $\mu(s)$ clockwise over $[s^L, s^H]$ and strictly increases the principal's welfare. Define

$$\overline{\mu} = \frac{1}{(s^H - s^L)} \int_{s^L}^{s^H} \mu(s) ds.$$

For given $\tau \in [0, 1]$, let $\widetilde{\mu}(s, \tau)$ be the solution to

$$\widetilde{\mu}(s, \tau) = \tau \overline{\mu} + (1 - \tau)\, \mu(s), \tag{60}$$

which clearly exists. Define $\widetilde{V}(s, \tau)$ as the solution to

$$(s+\alpha)\widetilde{\mu}(s,\tau) - \frac{\widetilde{\mu}(s,\tau)^2}{2} + \widetilde{V}(s,\tau) = (s^L+\alpha)\mu(s^L) - \frac{\mu(s^L)^2}{2} + V(s^L) + \int_{s^L}^{s} \widetilde{\mu}(\widetilde{s},\tau) d\widetilde{s}. \tag{61}$$

The original allocation corresponds to $\tau = 0$. We consider a perturbation where we increase τ marginally above zero if and only if $s \in [s^L, s^H]$. Note that differentiating (60) and (61) with respect to τ yields

$$\begin{aligned} \frac{d\widetilde{\mu}\,(s,\tau)}{d\tau} &= \overline{\mu} - \mu(s), & (62)\\ \frac{d\widetilde{\mu}\,(s,\tau)}{d\tau}\,(s + \alpha - \widetilde{\mu}(s,\tau)) + \frac{d\widetilde{V}(s,\tau)}{d\tau} &= \int_{s^L}^{s} \frac{d\widetilde{\mu}(\widetilde{s},\tau)}{d\tau} d\widetilde{s}. & (63) \end{aligned}$$

Substituting (62) in (63) yields that for a type $s \in [s^L, s^H]$, the change in the agent's welfare from a marginal increase in τ, starting from $\tau = 0$, is equal to

$$D\,(s) \equiv \int_{s^L}^{s} (\overline{\mu} - \mu(\widetilde{s}))\, d\widetilde{s}.$$

We begin by showing that the perturbation satisfies constraints (55)-(57). For the private information constraint (55), note that $D(s^L) = D(s^H) = 0$, so the perturbation leaves the welfare of types s^L and s^H (and that of types $s < s^L$ and $s > s^H$) unchanged. Using Lemma 2 and the representation in (58), it then follows from equation (61) and the fact that $\widetilde{\mu}(s, \tau)$ is nondecreasing in s that the perturbation satisfies constraint (55) for all s and any $\tau \in [0, 1]$.

To prove that the perturbation satisfies the enforcement constraint (56), we show that the welfare of types $s \in [s^L, s^H]$ weakly rises when τ increases marginally. Since $D(s^L) = D(s^H) = 0$, it is sufficient to show that $D(s)$ is concave over (s^L, s^H) to prove that $D(s) \geq 0$ for all s in this interval. Indeed, we verify:

$$\begin{aligned} D'(s) &= \overline{\mu} - \mu(s), \\ D''(s) &= -\mu'(s) < 0. \end{aligned}$$

Lastly, observe that constraint (57) is satisfied for $\tau > 0$ small enough. This follows from the fact that $V(s) \in (\underline{V}, \overline{V})$ for $s \in [s^L, s^H]$ in the original allocation.

We next show that the perturbation strictly increases the principal's welfare. Using the representation in (59), the change in the principal's welfare from an increase in τ is equal to

$$\int_{s^L}^{s^H} \frac{d\widetilde{\mu}(s,\tau)}{d\tau} Q(s) ds.$$

Substituting with (62) and the expression for $Q(s)$ yields that this is equal to

$$\int_{s^L}^{s^H} (\overline{\mu} - \mu(s)) \left(1 - \Phi(s|0,\Delta^2) - \alpha\phi(s|0,\Delta^2)\right) ds.$$

This is an integral over the product of two terms. The first term is strictly decreasing in s since $\mu(s)$ is strictly increasing over $[s^L, s^H]$. The second term is also strictly decreasing in s; this follows from $Q'(s) < 0$ for all $s \in [s^L, s^H]$. Therefore, these two terms are positively correlated with one another, and thus the change in the principal's welfare is strictly greater than

$$\int_{s^L}^{s^H} (\overline{\mu} - \mu(s)) \, ds \int_{s^L}^{s^H} \left(1 - \Phi(s|0,\Delta^2) - \alpha\phi(s|0,\Delta^2)\right) ds,$$

which is equal to 0. It follows that the change in the principal's welfare from the perturbation is strictly positive. Hence, if $V(s)$ is strictly interior and $Q'(s) < 0$ over a given interval, then $V(s)$ must be a step function over the interval.

Case 2: Suppose $Q'(s) > 0$ for all $s \in [s^L, s^H]$. Recall that $\mu(s)$ is continuously strictly increasing over $[s^L, s^H]$. We begin by showing that the enforcement constraint (56) cannot bind for all $s \in [s^L, s^H]$. Suppose by contradiction that it does. Using the representation of the agent's welfare in (58), this implies

$$\int_{s}^{s^H} (\widetilde{s} + \alpha - \mu(\widetilde{s}))d\widetilde{s} = 0$$

for all $s \in [s^L, s^H]$, which requires $\{\mu(s), V(s)\} = \{s + \alpha, \underline{V}\}$ for all $s \in (s^L, s^H)$. However, this contradicts the assumption that $V(s) \in (\underline{V}, \overline{V})$ for all $s \in [s^L, s^H]$. Hence, the enforcement constraint cannot bind for all types in the interval, and without loss we can take an interval with this constraint being slack for all $s \in [s^L, s^H]$.

We next show that there exists an incentive feasible perturbation that strictly increases the principal's welfare. Specifically, consider drilling a hole around a type s^M within $[s^L, s^H]$ so that we marginally remove the allocation around this type. That is, type s^M can no longer choose $\{\mu(s^M), V(s^M)\}$ and is indifferent between jumping to the lower or upper limit of the hole. With some abuse of notation, denote the limits of the hole by s^L and s^H, where the perturbation marginally increases s^H from s^M. Since the enforcement constraint is slack for all $s \in [s^L, s^H]$, the perturbation is incentive feasible. The change in the principal's welfare from the perturbation is equal to

$$\int_{s^M}^{s^H} \left(\mu'(s^H)(s - \mu(s^H)) + V'(s^H)\right) \phi(s|0, \Delta^2) ds$$
$$+ \frac{ds^M}{ds^H} \left(s^M \mu(s^L) - \frac{\mu(s^L)^2}{2} + V(s^L) - s^M \mu(s^H) + \frac{\mu(s^H)^2}{2} - V(s^H) \right) \phi(s^M|0, \Delta^2).$$

Note that by the private information constraint for type s^H,

$$\mu'(s^H)(s^H + \alpha - \mu(s^H)) + V'(s^H) = 0, \tag{64}$$

and by indifference of type s^M,

$$(s^M + \alpha)\mu(s^L) - \frac{\mu(s^L)^2}{2} + V(s^L) = (s^M + \alpha)\mu(s^H) - \frac{\mu(s^H)^2}{2} + V(s^H). \tag{65}$$

Substituting with these expressions, the change in the principal's welfare is equal to

$$\mu'(s^H) \int_{s^M}^{s^H} \left(s - s^H - \alpha\right) \phi(s|0, \Delta^2) ds + \frac{ds^M}{ds^H} \alpha \left(\mu(s^H) - \mu(s^L)\right) \phi(s^M|0, \Delta^2). \tag{66}$$

Differentiating (65) with respect to s^H and substituting with (64) yields

$$\frac{ds^M}{ds^H} = \mu'(s^H)\frac{(s^H - s^M)}{\mu(s^H) - \mu(s^L)}.$$

Substituting back into (66) and dividing by $\mu'(s^H) > 0$, we find that the change in the principal's welfare takes the same sign as

$$B^H(s^H, s^M) = \int_{s^M}^{s^H} \left(s - s^H - \alpha\right)\phi(s|0,\Delta^2)ds + \alpha\phi(s^M|0,\Delta^2)(s^H - s^M).$$

Since $Q'(s) > 0$ for all $s \in [s^M, s^H]$, Lemma 3 implies $B^H(s^H, s^M) > 0$, and thus the perturbation strictly increases the principal's welfare. Hence, if $V(s)$ is strictly interior and $Q'(s) > 0$ over a given interval, then $V(s)$ must be a step function over the interval.

Step 2c. *We show* $V(s) \in \{\underline{V}, \overline{V}\}$ *for all* $s \in \mathbb{R}$.

Suppose by contradiction that $V(s) \in (\underline{V}, \overline{V})$ for some s. By the previous steps and Lemma 2, s belongs to a stand-alone segment $(s^L, s^H]$, such that $\mu(s) = \mu$ and $V(s) = V$ for all $s \in (s^L, s^H]$, with $V \in (\underline{V}, \overline{V})$ (by assumption), and $\mu(s)$ jumps at s^L and s^H.

We first show that the enforcement constraint must be slack for all $s \in (s^L, s^H)$. Express the enforcement constraint as the difference between the left-hand and right-hand sides of (56), so that this constraint must be weakly positive and it is equal to zero if it binds. By the private information constraints, the derivative of the enforcement constraint with respect to s is equal to $\mu(s) - \alpha - s$. Since $\mu(s)$ is constant over $(s^L, s^H]$, it follows that the enforcement constraint is strictly concave over the interval, and therefore slack for all $s \in (s^L, s^H)$.

We next show that there exists an incentive feasible perturbation that strictly increases the principal's welfare. We consider perturbations that marginally change the constant action μ and continuation value V. As we describe next, the perturbation that we perform depends on the shape of the function $Q(s)$ over $(s^L, s^H]$:

Case 1: Suppose $\int_{s^L}^{s^H} Q(s^L)ds < \int_{s^L}^{s^H} Q(s)ds$. Consider a perturbation that marginally changes the action by $d\mu > 0$ and reduces V in order to keep type s^H equally well off. This means that $\frac{dV}{d\mu}$ is given by

$$s^H + \alpha - \mu + \frac{dV}{d\mu} = 0. \tag{67}$$

Note that for any arbitrarily small $d\mu > 0$, this perturbation makes the lowest types s in $(s^L, s^H]$, arbitrarily close to s^L, jump either to the allocation of type s^L or to their flexible allocation under maximal punishment $\{s + \alpha, \underline{V}\}$, where we let the perturbation introduce the latter. In the limit as $d\mu$ goes to zero, the change in the principal's welfare due to the perturbation is thus equal to[23]

$$\int_{s^L}^{s^H} \left(s - \mu + \frac{dV}{d\mu}\right) \phi(s|0,\Delta^2) ds \tag{68}$$
$$+\frac{ds^L}{d\mu}\left(s^L\mu(s^L) - \frac{\mu(s^L)^2}{2} + V(s^L) - s^L\mu + \frac{\mu^2}{2} - V\right)\phi(s^L|0,\Delta^2),$$

where the following indifference condition holds:

$$(s^L + \alpha)\mu - \frac{\mu^2}{2} + V = (s^L + \alpha)\mu(s^L) - \frac{\mu(s^L)^2}{2} + V(s^L).$$

To verify that the perturbation is incentive feasible, note that the enforcement constraint is slack for all $s \in (s^L, s^H)$, V is strictly interior, and the welfare of types s^L and s^H remains unchanged with the perturbation. Hence, the perturbation is incentive feasible for $d\mu$ arbitrarily close to zero.

To verify that the perturbation strictly increases the principal's welfare, substitute (67) and the indifference condition of type s^L into (68) to obtain:

$$\int_{s^L}^{s^H} \left(s - s^H - \alpha\right) \phi(s|0,\Delta^2) ds + \frac{ds^L}{d\mu}\alpha\left(\mu(s^L) - \mu\right)\phi(s^L|0,\Delta^2). \tag{69}$$

Differentiating the indifference condition of type s^L and substituting with (67) yields

$$\frac{ds^L}{d\mu} = \frac{s^H - s^L}{\mu - \mu(s^L)}.$$

Substituting back into (69), we find that the change in the principal's welfare takes the same sign as

$$B^H(s^H, s^L) = \int_{s^L}^{s^H} \left(s - s^H - \alpha\right) \phi(s|0,\Delta^2) ds + \alpha\phi(s^L|0,\Delta^2)(s^H - s^L),$$

which can be rewritten as

$$B^H(s^H, s^L) = \int_{s^L}^{s^H} \int_{s^L}^{s} Q'(\widetilde{s}) d\widetilde{s} ds = \int_{s^L}^{s^H} (Q(s) - Q(s^L)) ds.$$

[23]The arguments that follow are unchanged if $\{\mu(s^L), V(s^L)\}$ is replaced with $\{s^L + \alpha, \underline{V}\}$ for the cases where the enforcement constraint binds.

By the assumption that $\int_{s^L}^{s^H} Q(s^L)ds < \int_{s^L}^{s^H} Q(s)ds$, the above expression is strictly positive. The perturbation therefore strictly increases the principal's welfare, yielding a contradiction.

Case 2: Suppose $\int_{s^L}^{s^H} Q(s^L)ds \geq \int_{s^L}^{s^H} Q(s)ds$. Since $Q'(s) \neq 0$ almost everywhere, there must exist $s^h \in (s^L, s^H]$ such that $\int_{s^L}^{s^h} Q(s^L)ds > \int_{s^L}^{s^h} Q(s)ds$. Then consider a perturbation where, for $s \in (s^L, s^h]$, we marginally change the action by $d\mu < 0$ and increase V in order to keep type s^h equally well off. This perturbation makes types arbitrarily close to s^L jump up to the allocation of the stand-alone segment. Arguments analogous to those in Case 1 above imply that the perturbation is incentive feasible. Moreover, following analogous steps as in that case yields that the implied change in the principal's welfare takes the same sign as

$$-B^H(s^h, s^L) = -\int_{s^L}^{s^h} \left(s - s^h - \alpha\right) \phi(s|0, \Delta^2)ds - \alpha\phi(s^L|0, \Delta^2)(s^h - s^L),$$

which can be rewritten as

$$-B^H(s^h, s^L) = -\int_{s^L}^{s^h} \int_{s^L}^{s} Q'(\widetilde{s})d\widetilde{s}ds = -\int_{s^L}^{s^h} (Q(s) - Q(s^L))ds.$$

By the assumption that $\int_{s^L}^{s^h} Q(s^L)ds > \int_{s^L}^{s^h} Q(s)ds$, the above expression is strictly positive. The perturbation therefore strictly increases the principal's welfare, yielding a contradiction.

Step 3. *We show that if $\{\mu(s), V(s)\}_{s\in\mathbb{R}}$ is an optimal instrument-based rule, then either $V(s) = \overline{V}$ for all $s \in \mathbb{R}$, or there exists some $s^{**} \in (\widehat{s}, \infty)$ such that $V(s) = \overline{V}$ if $s \leq s^{**}$ and $V(s) = \underline{V}$ is $s > s^{**}$.*

Take any solution to the program in (54)-(57). We proceed in two sub-steps.

Step 3a. *We show that if $V(s^{**}) = \underline{V}$ for some $s^{**} \in \mathbb{R}$, then $s^{**} \geq \widehat{s}$.*

By Step 2a, if $V(s^{**}) = \underline{V}$ for some s^{**}, then $V(s) = \underline{V}$ over an interval $(s^L, s^H]$ that contains s^{**}. Take the largest such interval. We establish that $s^L \geq \widehat{s}$. Suppose by contradiction that $s^L < \widehat{s}$ and take a subinterval $(s^L, s^h]$ below $\widehat{s}$. Note that the enforcement constraint requires $\mu(s) = s + \alpha$ for all $s \in (s^L, s^h]$. Then we can perform a perturbation that rotates the action schedule clockwise over this interval, analogous to the perturbation used in Case 1 in Step 2b. By the arguments in that case, this perturbation is incentive feasible. In particular, note that since the perturbation weakly increases the welfare of all types $s \in (s^L, s^h]$ while simultaneously changing their action away from $s + \alpha$, it follows that the perturbation must necessarily increase $V(s)$ above $\underline{V}$. Moreover, by $Q'(s) < 0$ for all types $s \in (s^L, s^h]$ (by this interval being below $\widehat{s}$), the perturbation strictly increases the principal's welfare, yielding a contradiction.

Step 3b. *We show that if $V(s^{**}) = \underline{V}$ for some $s^{**} \in \mathbb{R}$, then $V(s) = \underline{V}$ for all $s \geq s^{**}$.*

Suppose by contradiction that $V(s^{**}) = \underline{V}$ for $s^{**} \in (-\infty, \infty)$ and $V(s) > \underline{V}$ for some $s > s^{**}$. By Step 3a above, $s^{**} \geq \widehat{s}$. Moreover, by Step 2a, there exist $s^H > s^L \geq s^{**}$ such that $V(s) = \overline{V}$ for all $s \in (s^L, s^H]$.

We begin by establishing that $\mu(s) = \mu$ for all $s \in (s^L, s^H]$ and some μ. Suppose by contradiction that $\mu'(s) > 0$ at some $s' \in (s^L, s^H]$. Note that the private information constraint (55) (together with the constant continuation value over $(s^L, s^H]$) implies $\mu(s) = s + \alpha$, and thus a slack enforcement constraint, in the neighborhood of such type s'. Then we can perform an incentive feasible perturbation that drills a hole in the $\mu(s)$ schedule in this neighborhood, as that described in Case 2 in Step 2b. By the arguments in that case, this perturbation strictly increases the principal's welfare, yielding a contradiction.

We next show that a segment $(s^L, s^H]$ with $\mu(s) = \mu$ and $V(s) = \overline{V}$ for all $s \in (s^L, s^H]$ and $s^L \geq s^{**}$ cannot exist. Suppose by contradiction that it does. Take s^L to be the lowest point weakly above s^{**} at which V jumps, and take s^H to be the lowest point above s^L at which V jumps again. Note that $s^H < \infty$ must exist, since (56) cannot be satisfied for all $s > s^L$ with $\mu(s) = \mu$ and $V(s) = \overline{V}$ for all such s. Then $(s^L, s^H]$ is a stand-alone segment with constant action μ and continuation value $\overline{V}$. Note that by arguments analogous to those in Step 2c, the enforcement constraint must be slack for all $s \in (s^L, s^H)$. Moreover, observe that $\mu < s^H + \alpha$ must hold, since otherwise by Lemma 2 and the monotonicity of $\mu(s)$, (55) would be violated at s^H. It follows that we can perform an incentive feasible perturbation analogous to that used in Step 2c: for $\mu' = \mu + \varepsilon, \varepsilon > 0$ arbitrarily small, we increase μ marginally to μ' and set V slightly below $\overline{V}$ so as to keep type s^H's welfare under this allocation unchanged. Since $s^L \geq s^{**}$ implies $\int_{s^L}^{s^H} Q(s^L)ds < \int_{s^L}^{s^H} Q(s)ds$, this perturbation strictly increases the principal's welfare, yielding a contradiction.

Step 4. *We characterize the optimal allocation of actions and show that any optimal instrument-based rule specifies $s^{**} \in (\widehat{s}, \infty)$ as defined in Step 3.*

Take any solution to program (54)-(57). By Step 3, either $V(s) = \overline{V}$ for all $s \in \mathbb{R}$, or there exists some $s^{**} \in (\widehat{s}, \infty)$ such that $V(s) = \overline{V}$ for $s \leq s^{**}$ and $V(s) = \underline{V}$ for $s > s^{**}$. In the latter case, by the enforcement constraint (56), $\mu(s) = s + \alpha$ for all $s > s^{**}$, and since (56) holds with equality at s^{**}, this type's allocation satisfies

$$(s^{**} + \alpha)\mu(s^{**}) - \frac{\mu(s^{**})^2}{2} + \overline{V} = \frac{(s^{**} + \alpha)^2}{2} + \underline{V}. \tag{70}$$

These results characterize the allocation for types $s \geq s^{**}$ when there exists an interior type s^{**} as defined above. We next proceed by characterizing the allocation that corresponds either to types $s < s^{**}$ in this case, or to all types in the case that such an interior type s^{**} does not exist. The final step of the proof establishes that the latter scenario does not arise, namely any optimal instrument-based rule specifies an interior type s^{**} such that $V(s) = \overline{V}$ for $s \leq s^{**}$ and $V(s) = \underline{V}$ for $s > s^{**}$.

Step 4a. *We show $\mu(s)$ is continuous over any interval $[s^L, s^H]$ such that $V(s) = \overline{V}$ for all $s \in [s^L, s^H]$.*

There are two cases to consider:

Case 1: Suppose by contradiction that $\mu(s)$ has a point of discontinuity below $\widehat{s}$. Note that if s^{**} as defined above exists, then the assumed point of discontinuity is strictly below s^{**}. The discontinuity requires that a type $s^M < \widehat{s}$ be indifferent between choosing $\lim_{s \uparrow s^M} \mu(s)$ and $\lim_{s \downarrow s^M} \mu(s) > \lim_{s \uparrow s^M} \mu(s)$. Note that given $V(s) = \overline{V}$ around this point, there must be a hole with types $s \in [s^L, s^M)$ bunched at $\mu(s^L) = s^L + \alpha$ and types $s \in (s^M, s^H]$ bunched at $\mu(s^H) = s^H + \alpha$, for some $s^L < s^M < s^H$. Now consider perturbing this rule by marginally increasing s^L, in an effort to slightly close the hole. This perturbation leaves the welfare of types strictly above s^M unchanged and is incentive feasible. The change in the principal's welfare from the perturbation is equal to

$$\mu'(s^L) \int_{s^L}^{s^M} \left(s - \mu(s^L)\right) \phi(s|0, \Delta^2) ds$$
$$+ \frac{ds^M}{ds^L} \left(s^M \mu(s^L) - \frac{\mu(s^L)^2}{2} - s^M \mu(s^H) + \frac{\mu(s^H)^2}{2} \right) \phi(s^M|0, \Delta^2).$$

Note that $\mu(s^L) = s^L + \alpha$, $\mu(s^H) = s^H + \alpha$, and $\mu'(s^L) = 1$. Moreover, by indifference of type s^M, we have

$$(s^M + \alpha)\mu(s^L) - \frac{\mu(s^L)^2}{2} = (s^M + \alpha)\mu(s^H) - \frac{\mu(s^H)^2}{2}. \tag{71}$$

Substituting into the expression above yields that the change in the principal's welfare is equal to

$$\int_{s^L}^{s^M} \left(s - s^L - \alpha\right) \phi(s|0, \Delta^2) ds + \frac{ds^M}{ds^L} \phi(s^M|0, \Delta^2) \alpha \left(\mu(s^H) - \mu(s^L)\right). \tag{72}$$

Note that differentiating the indifference condition (71) with respect to s^L (and substituting with $\mu(s^L) = s^L + \alpha$) yields

$$\frac{ds^M}{ds^L} = \frac{(s^M - s^L)}{\mu(s^H) - \mu(s^L)}.$$

Substituting this back into (72), we find that the change in the principal's welfare is equal to

$$B^L(s^L, s^M) = \int_{s^L}^{s^M} \left(s - s^L - \alpha\right) \phi(s|0, \Delta^2) ds + \alpha\phi(s^M|0, \Delta^2)(s^M - s^L).$$

It follows from $s^M < \widehat{s}$ and Lemma 3 that $B^L(s^L, s^M) > 0$. Thus, the perturbation strictly increases the principal's welfare, showing that $\mu(s)$ cannot jump at a point below $\widehat{s}$.

Case 2: Suppose by contradiction that $\mu(s)$ has a point of discontinuity above $\widehat{s}$. Note that if s^{**} as defined above exists, then the assumed point of discontinuity is strictly below s^{**}. By the same logic as in Step 3b, we can show that $\mu'(s) = 0$ over any continuous interval above $\widehat{s}$ over which $V(s) = \overline{V}$. It follows that there must exist a stand-alone segment $(s^L, s^H]$ with constant action μ and continuation value $V = \overline{V}$, satisfying $s^L \geq \widehat{s}$. However, using again the arguments in Step 3b, a perturbation that marginally increases μ and sets V slightly below $\overline{V}$ would then be incentive feasible and would strictly increase the principal's welfare. Therefore, $\mu(s)$ cannot jump at a point above $\widehat{s}$ around which $V(s) = \overline{V}$.

Step 4b. *We show $\mu(s) \leq s + \alpha$ for all s for which $V(s) = \overline{V}$.*

Consider types $s \leq s^{**}$ when an interior point s^{**} as described above exists, or all types $s \in \mathbb{R}$ when such a point s^{**} does not exist. Step 4a above implies that the allocation for these types must be bounded discretion, with either a minimum action level or a maximum action level or both. Note that if a minimum action level is prescribed, then there must exist some interior point $s^\star$ such that the allocation satisfies $\{\mu(s), V(s)\} = \{s^\star + \alpha, \overline{V}\}$ for all $s \leq s^\star$. However, such an allocation would violate the enforcement constraint (56) for s sufficiently low. Therefore, a minimum action level is not enforceable and only a maximum action level can be imposed, establishing the claim.

Step 4c. *We show $\mu(s) < s + \alpha$ for some s for which $V(s) = \overline{V}$. Moreover, there exists $s^{**} \in (\widehat{s}, \infty)$ as defined in Step 3.*

Suppose that an interior point s^{**} as described above exists. By Steps 4a and 4b, (70) must hold for $\mu(s^{**}) = s^* + \alpha$, where the value of s^* is unique given s^{**} and satisfies $s^{**} = s^* + \sqrt{2(\overline{V} - \underline{V})}$. In this circumstance, the optimal instrument-based rule is implemented with a strictly interior threshold $\mu^* = s^* + \alpha$, and the agent's action satisfies $\mu(s) < s + \alpha$ for all $s \in (s^*, s^{**})$.

We end the proof by showing that an interior point s^{**} as described above must indeed exist in any optimal instrument-based rule. Suppose by contradiction that this is not the case. Then by the steps above, there must be an optimal instrument-based rule prescribing $\{\mu(s), V(s)\} = \{s+\alpha, \overline{V}\}$ for all $s \in \mathbb{R}$. Using the representation in (59), the principal's welfare under this rule is equal to

$$\lim_{\underline{s} \to -\infty} \left\{ \frac{(\underline{s}+\alpha)^2}{2} + \overline{V} + \int_{\underline{s}}^{\infty} (s+\alpha) Q(s) ds \right\}. \tag{73}$$

Consider the principal's welfare under a maximally enforced threshold $\mu^* = s^* + \alpha$:

$$\lim_{\underline{s} \to -\infty} \left\{ \frac{(\underline{s}+\alpha)^2}{2} + \overline{V} + \int_{\underline{s}}^{\infty} (s+\alpha) Q(s) ds + \int_{s^*}^{s^{**}} [(s^*+\alpha) - (s+\alpha)] Q(s) ds \right\}, \tag{74}$$

where $s^{**} = s^* + \sqrt{2(\overline{V} - \underline{V})}$. The contradiction assumption requires that (73) weakly exceed (74) for all strictly interior s^*. That is, for all $s^* \in (-\infty, \infty)$ and $s^{**} = s^* + \sqrt{2(\overline{V} - \underline{V})}$, the following condition must hold:

$$\int_{s^*}^{s^{**}} (s^* - s)\, Q(s) ds \leq 0. \tag{75}$$

Note that $\lim_{s \to \infty} Q(s) = 0$ and $Q'(s) > 0$ for all $s > \widehat{s}$. Thus, setting $s^* \geq \widehat{s}$ yields $Q(s) < 0$ for all $s \in [s^*, s^{**}]$. This implies that the left-hand side of (75) is an integral over the product of two negative terms, and thus strictly positive, yielding a contradiction.

B.1.2 Optimal Target-Based Rule

The characterization of the optimal target-based rule under a continuum of types follows the same steps as in our baseline model. Letting $\mu(s)$ denote the action of type s, condition (12) characterizes the value of $\kappa = \mu(s) - s$. The optimal target-based rule can then be represented as the solution to (26)-(28), where it is clear that this solution is independent of the distribution of types. As such, the arguments in the proof of Proposition 2 apply directly to this setting with a continuum of types.

B.1.3 Optimal Class of Rule

Since the optimal target-based rule, and the principal's welfare under this rule, are identical to those in our baseline model, the same arguments as in the proof of Lemma 1 (in the proof of Proposition 3) apply. Those arguments imply that the principal's welfare is strictly decreasing in σ under the optimal target-based rule. In fact, the proof of Lemma 1 shows that the derivative of the principal's welfare with respect to σ is strictly lower than $-\sigma$ under this rule.

To compare with the optimal instrument-based rule, we prove the following lemma:

Lemma 4. *Consider changing σ while keeping $Var(\theta)$ unchanged. The change in the principal's welfare from a marginal increase in σ is strictly higher than $-\sigma$ under the optimal instrument-based rule.*

Proof. The principal's welfare under the optimal instrument-based rule can be written as

$$-\frac{\sigma^2}{2} + \int_{-\infty}^{\infty} \left[-\frac{(s-\mu(s))^2}{2} + V(s) \right] \phi(s|0,\Delta^2) ds. \tag{76}$$

Substituting with the structure of the optimal rule yields

$$-\frac{\sigma^2}{2} + \left\{ \int_{-\infty}^{s^*} \left(-\frac{\alpha^2}{2} + \overline{V} \right) \phi(s|0,\Delta^2) ds + \int_{s^*}^{s^{**}} \left[-\frac{(s-s^*-\alpha)^2}{2} + \overline{V} \right] \phi(s|0,\Delta^2) ds \right.$$
$$\left. + \int_{s^{**}}^{\infty} \left(-\frac{\alpha^2}{2} + \underline{V} \right) \phi(s|0,\Delta^2) ds \right\}, \tag{77}$$

where $s^{**} = s^* + \sqrt{2(\overline{V} - \underline{V})}$. Since Δ declines as σ rises, it is sufficient to show that the term in curly brackets in (77) is decreasing in Δ. To evaluate the derivative of this term, define $\widetilde{s} = s/\Delta$, with $\widetilde{s}^* = s^*/\Delta$ and $\widetilde{s}^{**} = s^{**}/\Delta$, where $\widetilde{s}^{**} = \widetilde{s}^* + \frac{1}{\Delta}\sqrt{2(\overline{V} - \underline{V})}$. Using integration by substitution, the term in curly brackets in (77) can be written as

$$\int_{-\infty}^{\widetilde{s}^*} \left(-\frac{\alpha^2}{2} + \overline{V} \right) \phi(\widetilde{s}|0,1) d\widetilde{s} + \int_{\widetilde{s}^*}^{\widetilde{s}^{**}} \left[-\frac{\left(\Delta(\widetilde{s} - \widetilde{s}^{**}) + \sqrt{2(\overline{V} - \underline{V})} - \alpha \right)^2}{2} + \overline{V} \right] \phi(\widetilde{s}|0,1) d\widetilde{s}$$
$$+ \int_{\widetilde{s}^{**}}^{\infty} \left(-\frac{\alpha^2}{2} + \underline{V} \right) \phi(\widetilde{s}|0,1) d\widetilde{s}. \tag{78}$$

Since the optimal instrument-based rule selects values for $\widetilde{s}^*$ and $\widetilde{s}^{**}$ to maximize (78), this rule necessarily satisfies the following first-order condition:

$$\int_{\widetilde{s}^*}^{\widetilde{s}^{**}} \left(\Delta(\widetilde{s} - \widetilde{s}^{**}) + \sqrt{2(\overline{V} - \underline{V})} - \alpha \right) \phi(\widetilde{s}|0,1) ds = -\alpha(\widetilde{s}^{**} - \widetilde{s}^*) \phi(\widetilde{s}^{**}|0,1) < 0. \tag{79}$$

The derivative of (77) with respect to Δ, taking into account the Envelope condition, is thus equal to

$$-\int_{\widetilde{s}^{*}}^{\widetilde{s}^{**}} (\widetilde{s}-\widetilde{s}^{**})\Big(\Delta(\widetilde{s}-\widetilde{s}^{**})+\sqrt{2(\overline{V}-\underline{V})}-\alpha\Big)\phi(\widetilde{s}|0,1)ds. \tag{80}$$

Both terms in the integral are increasing in $\widetilde{s}$, which means that (80) takes the same sign as

$$-\int_{\widetilde{s}^{*}}^{\widetilde{s}^{**}} (\widetilde{s}-\widetilde{s}^{**})\phi(\widetilde{s}|0,1)ds \int_{\widetilde{s}^{*}}^{\widetilde{s}^{**}} \Big(\Delta(\widetilde{s}-\widetilde{s}^{**})+\sqrt{2(\overline{V}-\underline{V})}-\alpha\Big)\phi(\widetilde{s}|0,1)ds. \tag{81}$$

The first integral in (81) is negative since $\widetilde{s} < \widetilde{s}^{**}$ for all $\widetilde{s} \in (\widetilde{s}^{*}, \widetilde{s}^{**})$. The second integral is also negative by (79). Therefore, (81) is strictly negative. It follows that the second term in (77) is decreasing in Δ, establishing the claim. □

We now proceed with the proof of Proposition 5. By Lemma 4, when σ increases, the principal's welfare under the optimal instrument-based rule declines by less than her welfare under the optimal target-based rule. To prove the claim in Proposition 5, it is thus sufficient to show that, among these two rule classes, a target-based rule is optimal at one extreme, for $\sigma \to 0$, whereas an instrument-based rule is optimal at the other extreme, for $\sigma \to \sqrt{Var(\theta)}$. This is what we prove next.

Take first $\sigma \to 0$ and thus $\Delta \to \sqrt{Var(\theta)}$. The same arguments as those used in the proof of Proposition 3 imply that the principal's welfare under the optimal target-based rule approaches $\overline{V}$ in this limit. Using the representation in (76), the principal's welfare under the optimal instrument-based rule approaches

$$\lim_{\Delta \to \sqrt{Var(\theta)}} \left\{ \int_{-\infty}^{\infty} \left[-\frac{(s-\mu(s))^2}{2} + V(s) \right] \phi(s|0,\Delta^2)ds \right\}.$$

Note that this expression can only exceed $\overline{V}$ if $\mu(s) = s$ and $V(s) = \overline{V}$ for all $s \in \mathbb{R}$, but such an allocation would violate the private information constraint (55). Therefore, this expression must be strictly lower than $\overline{V}$. It follows that the optimal target-based rule dominates the optimal instrument-based rule for $\sigma \to 0$.

Take next $\sigma \to \sqrt{Var(\theta)}$ and thus $\Delta \to 0$. The same arguments as those used in the proof of Proposition 3 imply that the principal's welfare under the optimal target-based rule approaches a value strictly lower than $-\frac{Var(\theta)}{2} + \overline{V}$ in this limit. The principal's welfare under the optimal instrument-based rule approaches

$$-\frac{Var(\theta)}{2}+\lim_{\Delta\to 0}\left\{\int_{-\infty}^{\infty}\left[-\frac{(s-\mu(s))^2}{2}+V(s)\right]\phi(s|0,\Delta^2)ds\right\}. \tag{82}$$

In the limit as $\Delta \to 0$, $\phi(s|0,\Delta^2)$ corresponds to a Dirac's delta function, with cumulative distribution function $\Phi(s|0,\Delta^2)=0$ if $s<0$ and $\Phi(s|0,\Delta^2)=1$ if $s\geq 0$. Consider the principal's limiting welfare under an instrument-based threshold specifying $s^*=-\alpha<0$ and $s^{**}=-\alpha+\sqrt{2(\overline{V}-\underline{V})}$, where $s^{**}>0$ by Assumption 2. Under this rule, $\mu(s)=0$ if $s\in[s^*,s^{**}]$ and $\mu(s)=s+\alpha$ otherwise, and $V(s)=\overline{V}$ if $s\leq s^{**}$ and $V(s)=\underline{V}$ otherwise. Therefore, (82) under this rule becomes equal to $-\frac{Var(\theta)}{2}+\overline{V}$. Since the principal's welfare under the optimal instrument-based rule must weakly exceed this value, it follows that the optimal instrument-based rule dominates the optimal target-based rule for $\sigma\to\sqrt{Var(\theta)}$.

B.2 Proof of Proposition 6

We first study the optimal instrument-based and target-based rules separately and then compare them.

B.2.1 Optimal Instrument-Based Rule

The program that solves for the optimal instrument-based rule under asymmetric punishments is analogous to that in (7), with constraint (4) replaced by

$$\int_{-\infty}^{\infty}\left[-\frac{\left(\mu^i-\theta-\alpha\right)^2}{2}+cV^i\right]\phi(\theta|s^i,\sigma^2)d\theta\geq\int_{-\infty}^{\infty}\left[-\frac{\left(\mu^{-i}-\theta-\alpha\right)^2}{2}+cV^{-i}\right]\phi(\theta|s^i,\sigma^2)d\theta, \tag{83}$$

and constraint (5) replaced by

$$\int_{-\infty}^{\infty}\left[-\frac{\left(\mu^i-\theta-\alpha\right)^2}{2}+cV^i\right]\phi(\theta|s^i,\sigma^2)d\theta\geq\int_{-\infty}^{\infty}\left[-\frac{\left(s^i-\theta\right)^2}{2}+c\underline{V}\right]\phi(\theta|s^i,\sigma^2)d\theta, \tag{84}$$

for $c>1$. To prove that the optimal instrument-based rule is as described in Proposition 6, we follow the same steps as in the proof of Proposition 1:

Step 1. We solve a relaxed program which ignores (83) for $i=H$ and (84) for $i=L,H$. Step 2 verifies that the solution to this relaxed program satisfies these constraints.

Step 1a. The analog of Step 1a in the proof of Proposition 1 applies directly without change and establishes that (83) for $i=L$ must hold as an equality.

Step 1b. The analog of Step 1b in the proof of Proposition 1 applies directly without change and establishes that $\mu^H \geq \mu^L$.

Step 1c. The analog of Step 1c in the proof of Proposition 1 applies, with $V^L = \overline{V}$ and (23) now replaced by:

$$\left(s^L + \alpha\right)\mu^L - \frac{(\mu^L)^2}{2} + c\overline{V} = \left(s^L + \alpha\right)\mu^H - \frac{(\mu^H)^2}{2} + cV^H. \tag{85}$$

This equation implies that, up to an additive constant independent of the allocation, the high type's welfare satisfies the analog of (24), given by:

$$\left(s^H + \alpha\right)\mu^H - \frac{(\mu^H)^2}{2} + cV^H = \left(s^L + \alpha\right)\mu^L - \frac{(\mu^L)^2}{2} + c\overline{V} + \left(s^H - s^L\right)\mu^H. \tag{86}$$

We proceed as in Step 1c in the proof of Proposition 1 to establish that $V^H = \overline{V}$. Suppose by contradiction that $V^H < \overline{V}$. Then it follows from (85) and Step 1b that $\mu^H > \mu^L$. Substituting (86) into the objective in (7), the principal's welfare up to an additive constant independent of the allocation is equal to

$$\left(s^L + \frac{1}{2}\alpha\right)\mu^L - \frac{(\mu^L)^2}{2} - \frac{1}{2}\left[\alpha - \left(s^H - s^L\right)\right]\mu^H + \overline{V} + (\overline{V} - V^H)\frac{(c-1)}{2}. \tag{87}$$

Consider a perturbation that changes μ^H by $d\mu^H < 0$ arbitrarily small and changes V^H so as to keep (85) unchanged:

$$\frac{dV^H}{d\mu^H} = \frac{\mu^H - (s^L + \alpha)}{c}.$$

Using the representation in (87), the change in the principal's welfare from this perturbation is equal to

$$\frac{1}{2}\left[\alpha - \left(s^H - s^L\right)\right] + \frac{dV^H}{d\mu^H}\frac{(c-1)}{2} = \frac{1}{2}\left[\alpha - \left(s^H - s^L\right)\right] + \left[\mu^H - (s^L + \alpha)\right]\frac{(c-1)}{2c}. \tag{88}$$

We show that this change in welfare is strictly positive. Suppose for the purpose of contradiction that this is not the case. Then the right-hand side of (88) must be weakly negative, which is equivalent to

$$\frac{\alpha}{c} - 2\Delta + (\mu^H - s^L)\frac{(c-1)}{c} \leq 0. \tag{89}$$

Recall that we have assumed $1 < c < \alpha/2\Delta$. The above inequality therefore requires $\mu^H - s^L < 0$. However, by Step 1b, if $\mu^H - s^L < 0$, then $\mu^L < s^L$. In this case, we can perform a perturbation that changes μ^L by $d\mu^L > 0$ and changes V^H so as to keep (85) unchanged:

$$\frac{dV^H}{d\mu^L} = \frac{s^L + \alpha - \mu^L}{c}.$$

Using the representation in (87) and given $\mu^L < s^L$, the change in the principal's welfare from this perturbation is equal to

$$\left(s^L + \frac{1}{2}\alpha\right) - \mu^L - \frac{dV^H}{d\mu^L}\frac{(c-1)}{2} = (s^L - \mu^L)\frac{(c+1)}{2c} + \frac{\alpha}{2c} > 0. \tag{90}$$

It follows that (89) cannot hold. Therefore, the change in the principal's welfare in (88) is strictly positive, establishing that $V^H = \overline{V}$ in any optimal instrument-based rule.

Step 1d. The analog of Step 1d in the proof of Proposition 1 applies directly without change and establishes that $\mu^H = \mu^L = 0$.

Step 2. The analog of Step 2 in the proof of Proposition 1 applies directly, and in fact it is strengthened by the fact that $c > 1$. This step verifies that the solution to the relaxed problem satisfies the constraints of the original problem.

Step 3. The analog of Step 3 in the proof of Proposition 1 applies directly, and in fact it is strengthened by the fact that $c > 1$. Therefore, the optimal instrument-based rule can be implemented with a maximally-enforced instrument threshold.

B.2.2 Optimal Target-Based Rule

To prove that the optimal target-based rule is as described in Proposition 6, we follow the same steps as in the proof of Proposition 2:

Step 1. The program that solves for the optimal target-based rule under asymmetric punishments is analogous to that in (10), with constraint (8) replaced by

$$\mu^i \in \arg\max_{\mu} \left\{ \int_{-\infty}^{\infty} \left[-\frac{(\mu - \theta - \alpha)^2}{2} + cV(\mu - \theta) \right] \phi\left(\theta | s^i, \sigma^2\right) d\theta \right\}. \tag{91}$$

We follow the same first-order approach as in Step 1 in the proof of Proposition 2, where equation (27) is replaced by

$$\alpha - \kappa + \int_{-\infty}^{\infty} cV(\pi)\, \phi'(\kappa - \pi | 0, \sigma^2) d\pi = 0. \tag{92}$$

Step 1a. Letting λ represent the Lagrange multiplier on (92), the same logic as in Step 1a in the proof of Proposition 2 yields that $\lambda < 0$.

Step 1b. We show that the solution satisfies $V(\pi) = \overline{V}$ if $\pi \leq \pi^*$ and $V(\pi) = \underline{V}$ if $\pi > \pi^*$, for some $\pi^* \in (-\infty, \infty)$. The argument is analogous to that in Step 1b in the proof of Proposition 2, where equations (30) and (31) are now given by

$$\phi(\kappa - \pi|0, \sigma^2) + \lambda c \phi'(\kappa - \pi|0, \sigma^2) + \underline{\psi}(\pi) - \overline{\psi}(\pi) = 0, \text{ and} \tag{93}$$

$$-\frac{1}{\lambda c} = \frac{\phi'(\kappa - \pi|0, \sigma^2)}{\phi(\kappa - \pi|0, \sigma^2)} = \frac{\pi - \kappa}{\sigma^2}. \tag{94}$$

Step 1c. We show that $\pi^* > \kappa$ and $\kappa \in (0, \alpha)$. The argument is analogous to that in Step 1c in the proof of Proposition 2, where (32) is now given by

$$\alpha - \kappa - \phi\left(\kappa - \pi^*|0, \sigma^2\right) c \left(\overline{V} - \underline{V}\right) = 0, \tag{95}$$

and the Lagrangian in (33) becomes

$$\begin{aligned} -\frac{\kappa^2}{2} + \left(1 - \Phi(\kappa - \pi^*|0, \sigma^2)\right)\overline{V} + \Phi(\kappa - \pi^*|0, \sigma^2)\underline{V} \\ + \lambda\left[\alpha - \kappa - \phi(\kappa - \pi^*|0, \sigma^2) c \left(\overline{V} - \underline{V}\right)\right]. \end{aligned} \tag{96}$$

Following analogous arguments as in Step 1c in the proof of Proposition 2, we can prove the claim by obtaining equation (36).

Step 2. We verify the validity of the first-order approach: we establish that the choice of κ in the relaxed problem satisfies (91) and therefore corresponds to the agent's global optimum.

Step 2a. We show that the agent has no incentive to choose some $\kappa' \neq \kappa$, $\kappa' \leq \pi^*$. The argument is analogous to that in Step 2a in the proof of Proposition 2.

Step 2b. We show that the agent has no incentive to choose some $\kappa' \neq \kappa$, $\kappa' > \pi^*$. The argument proceeds analogously to Step 2b in the proof of Proposition 2. We first establish that in the solution to the relaxed problem, given π^*, κ satisfies $\kappa - \pi^* \leq -\sigma$. Suppose by contradiction that $\kappa - \pi^* > -\sigma$. Note that by (94) and (36), $\kappa - \pi^* = -\frac{\sigma^2}{\kappa c}$. Hence, the contradiction assumption implies $\kappa c > \sigma$. Substituting $\kappa - \pi^* = -\frac{\sigma^2}{\kappa c}$ into (95) yields

$$\alpha - \kappa - \phi\left(-\frac{\sigma^2}{\kappa c}|0, \sigma^2\right) c \left(\overline{V} - \underline{V}\right) = 0. \tag{97}$$

Since the left-hand side of (97) is decreasing in κ and (by the contradiction assumption)

$\kappa c > \sigma$, (97) requires

$$\alpha - \frac{\sigma}{c} - \phi\left(-\sigma|0, \sigma^2\right) c\left(\overline{V} - \underline{V}\right) > 0.$$

Multiply both sides of this equation by $\frac{\sigma}{c} > 0$ to obtain:

$$\frac{\sigma}{c}\left(\alpha - \frac{\sigma}{c}\right) - \sigma\phi\left(-\sigma|0, \sigma^2\right)\left(\overline{V} - \underline{V}\right) > 0. \tag{98}$$

Note that since $0 < \sigma < \sqrt{Var(\theta)}$, $c > 1$, and, by Assumption 1, $\sqrt{Var(\theta)} \leq \alpha/2$, we have $\frac{\sigma}{c}\left(\alpha - \frac{\sigma}{c}\right) < \frac{\alpha^2}{2}$. Hence, (98) yields

$$\frac{\alpha^2}{2\sigma\phi\left(-\sigma|0, \sigma^2\right)} > \overline{V} - \underline{V}.$$

However, this inequality violates Assumption 2 since $\sigma\phi\left(-\sigma|0, \sigma^2\right) = \phi\left(1|0, 1\right)$. Thus, given π^*, κ satisfies $\kappa - \pi^* \leq -\sigma$.

We next establish that the agent has no incentive to deviate to $\kappa' \neq \kappa$, $\kappa' > \pi^*$. Consider some $\kappa' > \pi^*$ that is a local maximum for the agent. By the same reasoning as in Step 2b in the proof of Proposition 2, the difference in welfare for the agent from choosing the value of κ given by the solution to the relaxed problem versus κ' is no smaller than

$$-\frac{\alpha^2}{2} + \left(\Phi\left(-\left(\kappa - \pi^*\right)|0, \sigma^2\right) - \Phi\left(\kappa - \pi^*|0, \sigma^2\right)\right) c\left(\overline{V} - \underline{V}\right).$$

Since $c > 1$ and we have shown that $\kappa - \pi^* \leq -\sigma$, Step 2b in the proof of Proposition 2 implies that the value of this expression is bounded from below by the term in (42) which is strictly positive. It follows that the agent strictly prefers κ over κ'.

B.2.3 Optimal Class of Rule

We begin by proving the following lemma, which is analogous to Lemma 1 in the proof of Proposition 3:

Lemma 5. *Consider changing σ while keeping $Var(\theta)$ unchanged. The principal's welfare is independent of σ under the optimal instrument-based rule and it is strictly decreasing in σ under the optimal target-based rule.*

Proof. Our characterization of the optimal instrument-based rule yields that the principal's welfare under this rule is equal to $-\frac{Var(\theta)}{2} + \overline{V}$, which is independent of σ.

To evaluate the principal's welfare under the optimal target-based rule, consider the Lagrangian taking into account the conditional variance term (which is exogenous and thus excluded from (96)):

$$-\frac{\sigma^2}{2}-\frac{\kappa^2}{2}+\left(1-\Phi(\kappa-\pi^*|0,\sigma^2)\right)\overline{V}+\Phi(\kappa-\pi^*|0,\sigma^2)\underline{V}$$
$$+\lambda\left[\alpha-\kappa-\phi(\kappa-\pi^*|0,\sigma^2)c\left(\overline{V}-\underline{V}\right)\right].$$

The derivative with respect to σ is:

$$-\sigma+\left(\overline{V}-\underline{V}\right)\left[\int_{\kappa-\pi^*}^{\infty}\left(-\frac{\sigma^2-z^2}{\sigma^3}\right)\phi(z|0,\sigma^2)dz+\lambda c\frac{\sigma^2-(\kappa-\pi^*)^2}{\sigma^3}\phi(\kappa-\pi^*|0,\sigma^2)\right].$$

The first term is strictly negative. Using (94) and (36) to substitute in for λ and $\kappa-\pi^*$, the sign of the second term is the same as the sign of

$$-\int_{-\frac{\sigma^2}{\kappa c}}^{\infty}\left(\sigma^2-z^2\right)\phi(z|0,\sigma^2)dz-\kappa c\left[\sigma^2-\left(\frac{\sigma^2}{\kappa c}\right)^2\right]\phi\left(-\frac{\sigma^2}{\kappa c}|0,\sigma^2\right). \qquad (99)$$

This expression is identical to that in equation (43) in the proof of Lemma 1, except that κ has been replaced with κc. Analogous steps as in that proof then yield that the sign of (99) is strictly negative, completing the argument. □

We now proceed with the proof of Proposition 6. By Lemma 5, welfare under the optimal instrument-based rule is invariant to σ, whereas welfare under the optimal target-based rule is decreasing in σ. To prove the claim in Proposition 6, it thus suffices to show that, among these two rule classes, a target-based rule is optimal at one extreme, for $\sigma\to 0$, and an instrument-based rule is optimal at the other extreme, for $\sigma\to\sqrt{Var(\theta)}$. This can be established using the same arguments as in the proof of Proposition 3.

B.3 Proof of Proposition 7

As shown in the paper, the environment of Section 5.3 is mathematically equivalent to our baseline environment. Consequently, Proposition 7 follows directly from Propositions 1-3 in Section 3.