

NBER WORKING PAPER SERIES

THE PERMANENT INCOME
HYPOTHESIS REVISITED

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Working Paper No. 2209

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
April 1987

We have benefited from helpful discussions with S. Rao Aiyagari, Lars Hansen, and Thomas Sargent. The first two authors acknowledge the financial support of the National Science Foundation. The third author was an Alfred P. Sloan doctoral dissertation Fellow while this paper was being written. The research reported here is part of the NBER's research program in Economic Fluctuations. Any opinions expressed are those of the authors and not those of the National Bureau of Economic Research.

The Permanent Income Hypothesis Revisited

ABSTRACT

This paper investigates whether there are simple versions of the permanent income hypothesis which are consistent with the aggregate U.S. consumption and output data. Our analysis is conducted within the confines of a simple dynamic general equilibrium model of aggregate real output, investment, hours of work and consumption. We study the quantitative importance of two perturbations to the version of our model which predicts that observed consumption follows a random walk: (i) changing the production technology specification which rationalizes the random walk result, and (ii) replacing the assumption that agents' decision intervals coincide with the data sampling interval with the assumption that agents make decisions on a continuous time basis. We find substantially less evidence against the continuous time models than against their discrete time counterparts. In fact neither of the two continuous time models can be rejected at conventional significance levels. The continuous time models outperform their discrete time counterparts primarily because they explicitly account for the fact that the data used to test the models are time averaged measures of the underlying unobserved point-in-time variables. The net result is that they are better able to accommodate the degree of serial correlation present in the first difference of observed per capita U.S. consumption.

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1. Introduction

Few subjects in macroeconomics have received as much attention as the relationship between aggregate consumption and output. This attention reflects, at least in part, the belief that an understanding of the structural determinants of aggregate consumption is central to resolving many of the outstanding issues in business cycle theory. During the past decade much of the empirical literature on aggregate consumption has centered on Hall's (1978) demonstration that, under certain conditions, the permanent income hypothesis (PIH) implies that consumption is a random walk. Under this random walk hypothesis (RWH) no variable apart from current consumption should be of value in predicting future consumption.

In fact, a number of authors, including Flavin (1981) and Hayashi (1982), report statistically significant correlations between the change in consumption and lagged consumption and income. The response to these findings has generally fallen into one of two categories. First, some researchers have attributed the "excess sensitivity" of consumption to current and lagged income to the presence of a substantial number of consumers who are liquidity constrained. Under this interpretation, the PIH is fundamentally flawed as a principle for organizing the aggregate time series data (see for example Hall and Mishkin [1982] and Zeldes [1985]).

A second view of the empirical shortcomings of the RWH is that they do not reflect the failure of the PIH per se. Instead they reflect the failure of the auxiliary assumptions required to derive the RWH from the PIH (see for example Nelson [1985] or Mankiw and Shapiro [1985]). This view underlies both intertemporal capital asset pricing models (see for example Hansen and Singleton [1982, 1983], Dunn and Singleton [1986] and Eichenbaum and Hansen [1986]) and real business cycle theories (see for example Kydland and Prescott

[1982] and Long and Plosser [1983]) which abstract from liquidity constraints and other market imperfections which would prevent consumers from optimally adjusting consumption to permanent income. This view also underlies Lucas' (1985) argument that the welfare gains associated with countercyclical government policies would, at the very best, be small. Given the radically different policy implications of the two types of responses it is not surprising that the relationship between aggregate consumption and income continues to command widespread interest.

This paper pursues the second of the two responses discussed above. In particular we investigate, at a theoretical and empirical level, whether there are simple and testable perturbations of the PIH as implemented by Hall (1978) and Flavin (1981) which are consistent with the aggregate consumption and output data. Our analysis is conducted within the confines of a dynamic general equilibrium model of aggregate real output, investment, hours of work and consumption. We investigate the quantitative importance of two perturbations of the random walk version of our model: (i) changing the production technology to a specification which no longer implies the RWH, and (ii) replacing the assumption that agents' decision intervals coincide with the data sampling interval with the assumption that agents make decisions on a continuous time basis.

Our analysis uses a simple version of the Brock-Mirman growth model in which the equilibrium law of motion for consumption and output takes the form of a constrained vector ARMA. Consumers' preferences are defined over consumption and leisure in a way that nests the specification considered by Hall (1978) and Flavin (1981). We do not allow for shocks to preferences although these easily could be incorporated into our analysis. Output is produced using both labor and capital according to a Leontieff type production

function in which the labor requirement per unit of capital is allowed to be stochastic. When this labor requirement is nonstochastic our model satisfies the RWH. When agents derive disutility from working and the labor requirement per unit of capital is a non-trivial stochastic process, consumption does not follow a random walk. Aggregate income will Granger cause the first difference of consumption and current and lagged changes in consumption will be of value for predicting future changes in consumption. Consequently, this version of our model can, in principle, explain Flavin's rejection of the RWH.

A second possible explanation of these rejections is the impact of temporal aggregation bias. Sims (1971), Geweke (1978), and Marcet (1986) have shown that temporal aggregation bias can induce spurious serial correlation and Granger causality findings in observed data. Much of the empirical evidence against different versions of the PIH consists of findings that the first difference of aggregate consumption is serially correlated and is Granger caused by a variety of other variables. If agents make economic decisions at intervals of time that are finer than the data sampling interval these serial correlation and Granger causality findings could be spurious in the sense that they reflect only the effects of temporal aggregation bias.

In order to investigate this possibility we analyze the equilibrium of the continuous time analogues to our two discrete time models. The models are estimated using techniques developed by Hansen and Sargent (1980, 1981) to estimate continuous time models from discrete time data. This strategy allows us to directly address the possibility of temporal aggregation bias and to explicitly account for the fact that consumption and income data are not point-in-time sampled.

We report the results of estimating and testing four specific models. The first is a discrete time model in which the RWH holds by construc-

tion. This model serves a useful benchmark against which we can measure the empirical performance of our other models. The second model is a discrete time model in which the labor requirement per unit of capital is a serially correlated random variable. Here the RWH does not hold. The last two models we test are the continuous time analogues of the discrete time models.

Our main results can be summarized as follows. First, we find strong evidence against both discrete time models. We attribute the rejection of the discrete time random walk model to the counterfactual zero restrictions which it imposes on the law of motion for consumption and output. In contrast, the discrete time stochastic labor requirement model can in principle accommodate serial correlation in the first difference of observed consumption. However, in practice the cross equation restrictions imposed by that model do not permit a sufficient amount of serial persistence in the first difference of observed consumption to significantly improve the fit of the model.

Second, we find substantially less evidence against the continuous time models than against their discrete time counterparts. In fact, using standard likelihood ratio tests neither of the two continuous time models can be rejected at conventional significance levels. This is very encouraging given the simplicity of the models and the extensive cross equation restrictions implied by our theory. Using more informal diagnostics we argue that the continuous time models outperform the discrete time models primarily because they explicitly correct for the fact that the data used to test the models are time averaged measures of the underlying unobserved point-in-time variables. The net result is that the continuous time models are better able to accommodate the degree of serial correlation in the first difference of observed consumption.

The remainder of this paper is organized as follows. In section 2 we present the discrete time versions of our model. Empirical results for the discrete time models are presented in section 3. In section 4 we present the continuous time analogue to the models discussed in section 2. Empirical results for the continuous time models are discussed in section 5. Section 6 concludes the paper.

2. The Discrete Time Version of the Model.

This section is divided into three subsections. In subsection 2.A we present the discrete time version of our model. In addition we report the implied equilibrium laws of motion for economy-wide consumption, capital accumulation and output without being explicit about the stochastic structure of the shocks to agents' production technologies. Subsections 2.B and 2.C describe two alternative specifications of the technology shocks. The first is designed to imply the random walk hypothesis studied by Hall and Flavin. We call this the Discrete Time Random Walk (DRW) model. The second makes the capital-labor ratio stochastic. Under this assumption consumption is not a random walk. We refer to this version of the model as the Discrete Time Stochastic Labor Requirement (DSLRL) model.

2.A A Discrete Time Model of Consumption and Output

Preferences

A representative consumer ranks alternative streams of consumption and leisure according to the preference specification,

$$(2.1) \quad E_0 \sum_{t=0}^{\infty} \beta^t \left\{ -\frac{1}{2}(c_t - b_t)^2 - \alpha_t h_t \right\}$$

where $0 < \beta < 1$ is the subjective discount rate, b_t denotes the consumer's bliss point for consumption at time t , c_t denotes consumption at time t , h_t

denotes the number of hours worked at time t , α_t is the marginal disutility of work in period t and E_t is the expectations operator conditioned on the information set I_t , $t \geq 0$. Throughout this paper we assume that b_t and α_t are deterministic functions of time.^{2.1/}

Technology

There is a technology that converts time t consumption goods and labor effort into time $t + 1$ consumption goods. This technology is given by

$$(2.2) \quad \tilde{y}_t = \min\{\tilde{\delta}k_{t-1}, \tau_{t-1}h_{t-1}\} + e_t.$$

Here, \tilde{y}_t denotes economy-wide average output, k_{t-1} is the average capital stock at the end of time $t-1$ and h_{t-1} is the average number of hours worked at time $t - 1$. We think of the variable e_t either as the average endowment of consumption at time t or as an aggregate shock to the production function at time t which affects only the average productivity of labor and capital. The variable $\tau_{t-1}/\tilde{\delta}$ represents the labor requirement per unit of capital in the Leontieff type production function.

The economy-wide resource constraint is given by:^{2.2/}

$$(2.3) \quad c_t + k_t - (1-d)k_{t-1} \leq \tilde{y}_t$$

where d is the rate at which a unit of capital depreciates, $0 \leq d < 1$ and $\tilde{\delta} \geq d$. We impose the condition

$$(2.4) \quad \beta[\tilde{\delta} + (1-d)] = \beta\delta = 1, \text{ where } \delta \equiv \tilde{\delta} + 1 - d.$$

Condition (2.4) results in a unit autoregressive root in the consumption process, which is a necessary (but not sufficient) condition for the RWH. Condition (2.4) is related to a similar restriction imposed by Hall (1978), Flavin (1981), Hansen (1986), and Sargent (1986).

As in Hansen (1986) and Sargent (1986), we do not impose a nonnegativity constraint on the aggregate capital stock. Imposition of this constraint makes it difficult if not impossible to solve the model analytically. Instead we follow Hansen (1986) in imposing the requirement that^{2.3/}

$$(2.5) \quad E_0 \sum_{t=0}^{\infty} \beta^t k_t^2 < \infty.$$

Condition (2.5) implies $\beta^{1/2t} k_t \rightarrow 0$ almost surely as $t \rightarrow \infty$ so that (2.5) restricts the limiting behavior of the capital stock.

The Competitive Equilibrium

To determine the competitive equilibrium of this economy, we utilize the result that, in the absence of externalities, competitive equilibria are Pareto optimal. For our model, the relevant Pareto problem consists of choosing contingency plans for c_t , h_t , and k_t , as a function of the information set I_t to maximize (2.1) subject to (2.2) and (2.3). The set I_t is composed of all variables date t and earlier, but cannot be specified precisely until we describe the stochastic structure of the exogenous shocks.

It is convenient to derive the equilibrium decision rules subject to the restriction that capital and labor are always fully utilized:

$$(2.6) \quad \bar{\delta} k_t = \tau_t h_t \quad \text{for all } t.$$

In Appendix A we discuss conditions under which this restriction is non-binding.

Substituting (2.6) into (2.2) and (2.3), we obtain

$$(2.7) \quad c_t = \delta k_{t-1} - k_t + e_t.$$

Substituting (2.6) and (2.7) into (2.1), and defining $H_t \equiv \bar{\delta} \alpha_t / \tau_t$, we see that the representative consumer's problem is to maximize

$$(2.8) \quad E_0 \sum_{t=0}^{\infty} \beta^t \left\{ -\frac{1}{2} [\delta k_{t-1} - k_t + e_t - b_t]^2 - H_t k_t \right\}$$

by choice of a contingency plan for k_t subject to (2.5).

Before describing the solution, we introduce a useful notational device borrowed from Hansen (1986):

$$(2.9) \quad x_{pt} \equiv (1-\beta) E_t \sum_{j=0}^{\infty} \beta^j x_{t+j}.$$

$$(2.10) \quad \mu_{x_p, t} \equiv x_{pt} - E_{t-1} x_{pt}.$$

Definitions (2.9) and (2.10) apply to any random variable x_t for which the indicated conditional expectation exists. Below, we refer to objects like x_{pt} as the "permanent" value of x_t and to $\mu_{x_p, t}$ as the innovation to the permanent value of x_t . 2.4/

In Appendix B, we show that the equilibrium laws of motion for k_t and c_t are:

$$(2.11) \quad k_t - k_{t-1} = (e_t - e_{pt}) - (b_t - b_{pt}) - \beta H_{pt} / (1-\beta).$$

$$(2.12) \quad c_t = e_{pt} + (b_t - b_{pt}) + (\beta / (1-\beta)) H_{pt} + (\delta - 1) k_{t-1}.$$

Since net output, y_t , is equal to consumption plus net investment, we see that

$$(2.13) \quad c_t - y_t = e_{pt} - e_t + b_t - b_{pt} + \beta H_{pt} / (1-\beta).$$

Relation (2.11) implies that investment increases when the current value of the productivity shock exceeds its permanent value and decreases when the utility associated with a given amount of consumption is unusually high ($b_t > b_{pt}$). In addition, net investment depends negatively on H_{pt} , reflecting the utility cost of the labor input needed to make additions to the capital stock productive in the future.

Relation (2.12) implies that consumption increases when the utility associated with consumption is unusually high ($b_t > b_{pt}$) and depends positively on permanent endowment income and the capital stock. In addition consumption depends positively on H_{pt} . This is because high values of H_{pt} signify a low opportunity cost of consuming goods at time t as opposed to combining them with labor in order to produce future consumption goods. According to (2.13) unusually high levels of utility associated with consumption, high levels of H_{pt} or unusually low levels of endowment income ($e_t < e_{pt}$) cause consumption to exceed current period income.

We now formally define the random walk hypothesis:

Definition: We say that consumption satisfies the random walk hypothesis (RWH) if and only if

$$(2.14). \quad E_{t-1} c_t = c_{t-1} + f_t$$

where f_t is a deterministic, but possibly trivial, function of time.

From relations (2.11) and (2.12) we see that

$$(2.15) \quad \Delta c_t = \mu_{e_p, t} - \mu_{b_p, t} + \Delta b_t + (\beta/(1-\beta)) \mu_{H_p, t} - H_{t-1}.$$

Since b_t is by assumption deterministic, the RWH will be satisfied if and only if H_t is deterministic. Since a_t is by assumption deterministic, we conclude that RWH will hold if and only if the time t labor requirement per unit of capital, τ_t , is deterministic.

It is worth contrasting our derivation of the RWH with the derivation in Hall (1978). While both derivations impose strong restrictions on the underlying economic model there is at least one important difference. Hall derives the RWH by restricting directly the stochastic structure of the risk

free real interest rate, r_t , which he assumes to be constant. In contrast the risk free real rate of interest in our analysis is stochastic even under those circumstances for which the RWH is satisfied. To see this recall that $b_t - c_t$ is the marginal utility of time t consumption. Consequently, the representative consumer's intertemporal Euler equation for one period risk free consumption loans can be written as: $(b_t - c_t) = \beta r_t E_t(b_{t+1} - c_{t+1})$. Rewriting (2.15), applying the conditional expectation operator to evaluate $E_t(b_{t+1} - c_{t+1}) = (b_t - c_t) + H_t$ and solving for r_t we obtain^{2.5/}

$$(2.16) \quad r_t = \beta^{-1} [1 + H_t / (b_t - c_t)]^{-1}.$$

It follows that r_t will be stochastic as long as the ratio of H_t to the marginal utility of consumption is a nontrivial random variable. This condition will be satisfied even if H_t is deterministic (so that the RWH is satisfied) provided that e_t is stochastic. Thus a constant risk free real interest is neither a necessary nor a sufficient condition for the RWH to hold.

In summary, our model has the following four key characteristics. First, the model nests, as a special case, the benchmark random walk model considered by Hall (1978) and Flavin (1981). Second, the random walk property of consumption depends sensitively on the stochastic structure of the unobserved shocks to technology. Third, in our model, the RWH does not require that the risk free real interest rate be constant. Fourth, our model implies that when the shocks to technology, e_t and H_t , are covariance stationary processes, consumption minus output is a covariance stationary process (see relation [2.13]).

It is important to note that our model, like the models in Hall (1978), Flavin (1981), and Campbell (1986), delivers relationships in terms of the levels, not the log levels, of the variables of interest. Because of

this, property four applies to levels, as opposed to log levels, of consumption and output, and is clearly counterfactual for post war U.S. data. In order to avoid this implication we parameterize the law of motion of the shocks to technology, b_t and α_t so as to imply that consumption and output grow at the same geometric rate over time. While the parameterizations that we adopt are very restrictive, they do have an important compensating advantage: they imply that the models of sections 2.B and 2.C apply to consumption and output data which have been detrended assuming a common geometric trend. This result has two useful consequences. First, it allows us to reinterpret property four so that it is no longer counterfactual. Secondly, it allows us to accommodate growth in an internally consistent manner while preserving the applicability of a set of econometric tools developed for nongrowing time series processes.

2.B Parameterizing the Discrete Time Random Walk Model (DRW)

This subsection describes a set of assumptions on the technology shocks that are consistent with the RWH. We then display the resulting reduced forms for consumption and output.

Throughout sections 2.B and 2.C we assume that $b_t = b\phi^t$ and $\alpha_t = \alpha\phi^{2t}$ where $\phi > 1$, $b > 0$ and $\alpha > 0$. By allowing b_t to grow over time we are able to avoid the implication that consumers become satiated. The fact that α grows at the geometric rate ϕ^2 implies, in conjunction with the other assumptions in our model, that neither leisure nor labor's share of net output exhibit any trend. In order for the representative consumer's problem to be well defined we require that $\beta\phi^2 < 1$.

Recall that our model satisfies the RWH if and only if H_t is deterministic. Accordingly we assume that

$$(2.17) \quad H_t = H\phi^t, \quad 0 < H < b(\phi-1).$$

The lower bound on H guarantees that the drift in the marginal utility of consumption, H_t , is positive. The upper bound on H ensures that the drift in consumption, $\Delta b_t - H_{t-1}$, is positive. We suppose that

$$(2.18) \quad e_t = e_{1t} + e_{2t},$$

where e_{1t} and e_{2t} are in I_t , and

$$(2.19) \quad (1-L)e_{it} = e_i \phi^t + \eta_{it}/(1-a_i L), \quad i = 1, 2 \quad \phi(e_1+e_2) < b(\phi-1) - H.$$

The condition on $e_1 + e_2$ guarantees that the deterministic component of savings, $y_t - c_t$, is positive. Let $x_t = [\eta_{1t}, \eta_{2t}]'$. The vector x_t is white noise, orthogonal to $I_t = \{k_{t-1-s}, h_{t-1-s}, e_{t-1-s} : s \geq 0\}$, and satisfies

$$(2.20) \quad E x_t x_t' = V \phi^{2t},$$

where V is a two by two positive definite symmetric matrix of constants. In addition we assume that $a_1 \neq a_2$, and $|a_i| < 1$, $i = 1, 2$. The reason for assuming that the endowment process is the sum of two stochastic processes, the realizations of which are separately observed by agents, is to guarantee that the bivariate consumption and income process is of full spectral rank.

According to our specification all deterministic terms and innovation standard deviations in the DRW model grow at the rate ϕ . Thus, it is not surprising that we can "detrend" Δc_t and $c_t - y_t$ by ϕ^t to obtain a stationary stochastic process. Define,

$$(2.21) \quad c_t^* \equiv \phi^{-t} c_t, \quad y_t^* \equiv \phi^{-t} y_t, \quad \eta_{it}^* \equiv \phi^{-t} \eta_{it}, \quad i = 1, 2$$

and

$$(2.22) \quad q_t^{*'} = [c_t^* - y_t^*, c_t^* - \phi^{-1} c_{t-1}^*].$$

Relations (2.13), (2.15), and (2.17) - (2.19) imply that q_t^* has the constrained VAR(2) representation:

$$(2.23) \quad A(\phi^{-1}L)q_t^* = T + X_t$$

where,

$$(2.24) \quad A(L) = A_0 + A_1L + A_2L^2, \quad A_0 = I,$$

$$A_1 = \begin{vmatrix} -(a_1 + a_2) & \beta a_1 a_2 \\ 0 & 0 \end{vmatrix}, \quad A_2 = \begin{vmatrix} a_1 a_2 & 0 \\ 0 & 0 \end{vmatrix}$$

$$X_t = \begin{vmatrix} \beta a_1 & \beta a_2 \\ 1 & 1 \end{vmatrix} \begin{vmatrix} \eta_{1t}^* / (1 - a_1 \beta) \\ \eta_{2t}^* / (1 - a_2 \beta) \end{vmatrix}$$

$$E[X_t X_t'] = V_d.$$

In (2.23) T is a two dimensional column of positive constants and in (2.24) V_d is a two by two positive definite symmetric matrix of constants.

Notice that the second rows of A_1 and A_2 are identically equal to zero. This reflects two basic properties of the DRW model. First, the fact that the (2,1) elements of A_1 and A_2 equal to zero implies that $\phi^{-t} \Delta c_t$ is not Granger caused by $\phi^{-t}(c_t - y_t)$. Second, the fact that the (2,2) element of $A(z)$ is equal to zero implies that lagged values of $\phi^{-t} \Delta c_t$ should not be useful for predicting $\phi^{-t} \Delta c_t$. These two properties summarize the implication of the DRW model that c_t is a random walk with potentially time varying drift.

2.C The Discrete Time Stochastic Labor Requirement Model (DSLRL).

This section describes a parameterization of our model in which consumption does not satisfy the RWH. In particular, this specification implies that $\phi^{-t}\Delta c_t$ is correlated with lagged values of itself and lagged values of $\phi^{-t}(c_t - y_t)$. Here, we allow H_t to be stochastic by adding an AR(1) random variable to (2.17):

$$(2.25) \quad H_t = H\phi^t + \varepsilon_t/(1-fL) \quad |f| < 1, \quad 0 < H < b(\phi-1).$$

Since $H_t = \bar{\delta}\alpha_t/\tau_t$ condition (2.25) implies that the labor requirement per unit of capital is stochastic. Also, we replace (2.18) - (2.19) by

$$(2.26) \quad (1-L)e_t = e\phi^t + \eta_t/(1-aL), \quad \phi e \leq b(\phi-1) - H, \quad |a| < 1.$$

Let $\mathbf{x}_t = [\varepsilon_t \ \eta_t]'$. The vector \mathbf{x}_t is white noise, orthogonal to $I_t = \{k_{t-1-s}, h_{t-1-s}, e_{t-s}, s \geq 0\}$, and satisfies

$$(2.27) \quad E\mathbf{x}_t\mathbf{x}_t' = V\phi^{2t},$$

where V is a two by two positive definite symmetric matrix of constants.

Relations (2.13), (2.15), and (2.25) - (2.27) imply that q_t^* has the VAR(2) representation:

$$(2.28) \quad A(\phi^{-1}L)q_t^* = T + \mathbf{x}_t$$

where,

$$(2.29) \quad A(L) = A_0 + A_1L + A_2L^2, \quad A_0 = I,$$

$$A_1 = \begin{vmatrix} \beta a^2 - f & -\beta a(a-f) \\ \beta^{-1} - f & -a(1-f\beta) \end{vmatrix} / (1-\lambda a) \quad A_2 = \begin{vmatrix} -a(a-f) & 0 \\ -a(\beta^{-1}-f) & 0 \end{vmatrix} / (1-\lambda a)$$

$$X_t = \begin{vmatrix} \beta a & 1 \\ 1 & 1 \end{vmatrix} \begin{vmatrix} \eta_t^*/(1-\beta a) \\ \beta \varepsilon_t^*/(1-f\beta) \end{vmatrix}$$

$$\eta_t^* = \phi^{-t} \eta_t \text{ and } \varepsilon_t^* = \phi^{-t} \varepsilon_t$$

and

$$E[X_t X_t'] = V_d.$$

In (2.28) T is a two dimensional vector of positive constants and in (2.29) V_d is a two by two positive definite symmetric matrix of constants.

Relations (2.28) and (2.29) display the basic properties of the DSLR model. First, the fact the (2,1) elements of A_1 and A_2 are not equal to zero implies that $\phi^{-t} \Delta c_t$ is Granger caused by $\phi^{-t} (c_t - y_t)$. Second, the fact that the (2,2) element of A_1 is not equal to zero implies that $\phi^{-(t-1)} \Delta c_{t-1}$ should be useful for predicting $\phi^{-t} \Delta c_t$. Both these results reflect the fact that, in the DSLR model, c_t does not satisfy the RWH.

3. Empirical Results for the Discrete Time Models

In this section we report the results of estimating the DRW model of section 2.B and the DSLR model of section 2.C. In subsection 3.A we briefly describe our estimation methodology. In subsection 3.B we report our empirical results.

3.A Estimation Strategy

In section 2 we derived the implications of our model for the vector q_t^* which is defined in terms of detrended aggregate consumption and detrended net output, y_t^* (see [2.22]). Unfortunately the data on aggregate depreciation is not particularly reliable. In addition there is little reason to believe that our model of depreciation is consistent with the model of depreciation used by the Department of Commerce to construct NNP from GNP

data. Consequently, we implement our model using data on GNP rather than NNP. In order to do this we exploit relations (2.23) and (2.28) to derive the implications of our model for the analogue to the vector q_t^* defined in terms of gross, rather than net, output. Not surprisingly, this mapping involves the depreciation rate d , which we view as a structural parameter to be estimated.

Recall that we denote gross output by \tilde{y}_t . Relations (2.2) and (2.3) imply:

$$(3.1) \quad c_t - \tilde{y}_t = -[k_t - (1-d)k_{t-1}].$$

Let \tilde{y}_t^* denote detrended gross output:

$$(3.2) \quad \tilde{y}_t^* \equiv \phi^{-t} \tilde{y}_t$$

and define

$$(3.3) \quad \tilde{q}_t' \equiv [c_t^* - \tilde{y}_t^*, c_t^* - \phi^{-1} c_{t-1}^*].$$

It follows that ^{3.1/}

$$(3.4) \quad q_t^* = H(\phi^{-1}L)\tilde{q}_t$$

where

$$(3.5) \quad H(z) \equiv \frac{1}{1 - (1-d)z} \begin{vmatrix} 1-z & 0 \\ 0 & 1-(1-d)z \end{vmatrix}.$$

By substituting (3.4) into (2.23) and (2.28), we obtain the implications of the DRW and DSLR models respectively for \tilde{q}_t , which involves consumption and gross output. Specifically, the DRW model implies:

$$(3.6) \quad A(\phi^{-1}L)H(\phi^{-1}L)\tilde{q}_t = T + X_t$$

and the DSLR model implies

$$(3.7) \quad A(\phi^{-1}L)H(\phi^{-1}L)\tilde{q}_t = T + X_t.$$

Both discrete time models imply that \tilde{q}_t is a constrained ARMA(3,1) representation. The assumptions which we have imposed on the structural parameters of our model are sufficient to guarantee that the roots of $A(\phi^{-1}z)$, $A(\phi^{-1}z)$, and $H(\phi^{-1}L)$, defined in (2.24), (2.29), and (3.5), lie outside the unit circle. It follows that \tilde{q}_t is a covariance stationary stochastic process with conditionally homoscedastic disturbances. Since $|\phi^{-1}| < 1$ both the DRW and DLSR model imply that c_t and y_t have unconditional growth rates equal to ϕ . When $d = 0$, (3.6) collapses to (2.23) and (3.7) collapses to (2.28). This is because with $d=0$ depreciation is identically equal to zero, so gross output equals net output and $q_t^* = \tilde{q}_t$.

We now describe the procedure used to estimate systems (3.6) and (3.7). Define,

$$(3.8) \quad Q_t = \tilde{q}_t - E\tilde{q}_t.$$

Suppose that we have a sample on Q_t for $t = 1, 2, \dots, T$. Let $Q = [Q_1, Q_2, \dots, Q_T]'$ and define

$$(3.9) \quad \Lambda = E[QQ'].$$

The normal log likelihood function of Q is given by

$$(3.10) \quad \ell_T = -.5 \log [2\pi] - .5 \log \det \Lambda_T - .5 Q' \Lambda_T^{-1} Q.$$

The theoretical spectral density of the $Q(t)$ process is given by

$$(3.11) \quad Z(\omega) = H(\phi^{-1}e^{-i\omega})^{-1} A(\phi^{-1}e^{-i\omega})^{-1} v_d A(\phi^{-1}e^{i\omega})^{-1} H(\phi^{-1}e^{i\omega})^{-1},$$

when the data is generated by the DRW model, and is given by,

$$(3.11)' \quad Z(\omega) = H(\phi^{-1}e^{-i\omega})^{-1} A(\phi^{-1}e^{-i\omega})^{-1} v_d A(\phi^{-1}e^{i\omega})^{-1} H(\phi^{-1}e^{i\omega})^{-1},$$

when the data is generated by the DSLR model. In (3.11) and (3.11)', ω is defined over the interval $(0, 2\pi)$. Let $I(\omega_j)$ be the periodogram of the Q_t process at frequency $\omega_j = 2\pi j/T$, $j = 1, 2, \dots, T$. Based on results in Hannan (1970), Hansen and Sargent (1981a) suggest approximating the log likelihood function (3.10) by

$$(3.12) \quad \ell_T = -.5 \log[2\pi] - .5 \sum_{j=1}^{T-1} \log\{\det[Z(\omega_j)]\} - .5 \sum_{j=1}^{T-1} \text{tr}[Z(\omega_j)^{-1} I(\omega_j)].$$

Our estimates of the structural parameters of the DRW and DSLR models are the argmax of ℓ_T when $Z(\omega)$ is given by (3.11) and (3.11)' respectively. Since we fix the value of β and ϕ a priori, the free parameters of the DRW model are a_1 , a_2 , d , and the three independent elements of V_d . In the case of the DSLR model the free parameters are a , f , d , and the three independent elements of V_d .

We used as a measure of consumption real quarterly expenditures on nondurable consumption goods and services, plus the imputed rental value of the stock of consumer durables, plus real government consumption expenditures. All of these measures except the last two were taken from the Survey of Current Business. A measure of the imputed rental value of consumer durables was obtained from the data base documented in Brayton and Mankopf (1985). Government consumption was measured by real government purchases of goods and services minus real government (federal, state, and local) investment. A measure of government investment was provided to us by John Musgrave of the Bureau of Economic Analysis.^{3.2/} Output was measured by real quarterly GNP plus the imputed quarterly rental value of the stock of consumer durables. All series cover the period 1950:2 - 1985:3 and are expressed in per capita terms.

3.B Empirical Results

When $d > 0$, both the DRW and DSLR models lead to constrained infinite ordered vector autoregressive representations for the stochastic process Q_t . Given the results of section 2 it is not surprising that when d is close to zero, these infinite ordered VAR's are well approximated, in a sense to be made precise below, by finite ordered VARs. As it turns out, these finite ordered VARs are more revealing for diagnostic purposes than the corresponding ARMA representations. It is therefore useful to consider the statistical properties of the data as summarized by the unconstrained VAR's for Q_t and the vector stochastic process, Z_t , consisting of demeaned and detrended aggregate consumption and gross output.

To determine the appropriate lag lengths of the unconstrained VARs, we estimated VARs of lag length 1 through 6 and performed likelihood ratio tests sequentially. Our results, displayed in Table 3.1, indicate that for both Q_t and Z_t the second lag is significantly different from zero. On the other hand the hypotheses that lags 3, 4, 5, and 6 are zero cannot be rejected at the five percent significance level.

Using the second order VAR for Q_t , we examined the Granger causality patterns in the data and tested the hypothesis that $\phi^{-t}\Delta c_t$ is a white noise stochastic process. The results, which are reported in Table 3.2, provide useful diagnostic devices for evaluating the empirical performance of the DRW and DSLR models. The hypothesis that $\phi^{-t}\Delta c_t$ does not Granger cause $\phi^{-t}(c_t - y_t)$ can be rejected at the ten percent significance level, but not at the five percent level. The hypothesis that $\phi^{-t}(c_t - y_t)$ does not Granger cause $\phi^{-t}\Delta c_t$ cannot be rejected at the ten percent significance level. On the other hand the null hypothesis that $\phi^{-t}\Delta c_t$ is serially uncorrelated is decisively rejected. This last result is consistent with results in Flavin (1981) and

Nelson (1985) indicating that the first difference of consumption can be predicted from past information. There is, however, an important difference between our results and those obtained by Flavin (1981). In implementing her model, Flavin imposes, a priori, the restriction that the first difference of consumption does not depend on its own lagged values. At least for our measure of consumption, this restriction is counterfactual. When such a restriction is imposed, lagged output plays a useful role in predicting the first difference of consumption primarily because it is proxying for lagged values of the first difference of consumption.

We now report the results of estimating the DRW and DSLR models. In implementing these models we set the parameters β and ϕ equal to .990 and $\exp[.00456]$, respectively.^{3.3/} This value of ϕ implies a quarterly per capita growth rate in consumption and income of approximately 1/2 percent. Given this value of ϕ , we can form a time series on \tilde{q}_t . By using the sample mean of \tilde{q}_t as a proxy for $E\tilde{q}_t$, we can then form a time series on Q_t .

The left hand column of Table 3.3 reports the results for the DRW model. The parameters a_1 and a_2 , which are the AR coefficients in the laws of motion for the two endowment shocks, are not globally identified, since we can simply reverse the labels on these two parameters and obtain precisely the same value of λ_T . Notice that the estimated depreciation rate is small and insignificant. The estimates for the DSLR model are reported in the right hand column of Table 3.3. Here, the depreciation rate is estimated to be approximately seven percent per quarter. While the estimated standard error of d is fairly large the hypothesis that $d = 0$ can be rejected at the ten percent significance level (but not at the five percent level).

We adopt two methods for assessing the overall performance of the DRW and DSLR models: (1) a formal statistical test of the overidentifying

restrictions imposed by the models, and (2) an informal comparison of the constrained VAR's implied by the models with the unconstrained VAR representation of the data. Our formal statistical test is based on the fact that all of the models in this paper are nested within scalar autoregressive vector moving average (SARMA) representations for Q_t . Both the DRW and DSLR models imply a constrained SARMA(3,3) representation for Q_t .^{3.4/} However, the continuous time models of section 4 imply a constrained SARMA(3,4) representation for Q_t . In order to allow all of the structural models to be nested within a common unconstrained specification we use the SARMA(3,4) as our unconstrained model. It should be noted, however, that the zero restrictions implicit in the lower order SARMA specifications cannot be rejected at conventional significance levels. This can be seen from Table 3.4, where we report goodness of fit tests for SARMA (p,q), (p = 2, 3; q = 1, 2, 3, 4) models of the data.

Let J_T denote twice the difference between the maximized value of f_T for the unconstrained SARMA specification minus the maximized value of f_T for the constrained SARMA specification. Then J_T is asymptotically distributed as a Chi-square random variable with degrees of freedom equal to the number of restrictions imposed in the constrained specification. The test statistic J_T can be multiplied by an adjustment factor suggested by Whittle (1953), Lissitz (1972), and Sims (1980) designed to correct for small sample bias.^{3.5/} We denote the resulting test statistic by J_T .

The values of J_T and J_T for both discrete time models are reported in Table 3.3. The DRW model is easily rejected at the one percent significance level.^{3.6/} The DSLR model is rejected at the five percent significance level, but not at the one percent level, when compared to the unconstrained SARMA(3,4). However, this last result reflects in part the overparameterization of our unconstrained specification. When the DSLR model is compared to

an unconstrained SARMA(3,3) the unadjusted J_T statistic is 30.36, with probability value of .998. The adjusted J_T statistic is 28.75, which has a probability value of .996. Thus, when compared to a more parsimoniously parameterized alternative, the DSLR model is also rejected at the one percent level.

The fact that we reject the random walk model is not very surprising given our rejection of the hypothesis that $\phi^{-t}\Delta c_t$ is a white noise process. However rejection of the white noise hypothesis cannot be viewed as evidence against the DSLR model. In order to more fully understand the empirical shortcomings of both these models, we now compare the constrained VAR representations for Q_t and Z_t implied by the structural models with the corresponding unconstrained VARs. Tables 3.5 and 3.6 display the unconstrained VAR(2) and constrained VARs implied by the DRW and DSLR models for Q_t and Z_t respectively. The constrained VARs must be truncated because both structural models imply infinite ordered VARs for Q_t and Z_t . We use the truncation rule of not reporting matrix coefficients whose maximal element are smaller than .02 in absolute value. In all cases this results in truncation after the second lag.

First consider our results for the DRW model. For both $Q(t)$ and $Z(t)$ the major discrepancy between the constrained and unconstrained representations appears to be in the second row of coefficients matrices. Specifically, the DRW model imposes the restriction that all coefficients in the equation for $\phi^{-t}\Delta c_t$ are zero and that all coefficients after lag 1 in the equation for c_t^* are zero. In contrast the unconstrained estimate of the coefficient on $\phi^{-(t-1)}\Delta c_{t-1}$ in the equation for $\phi^{-t}\Delta c_t$ (see Table 3.5) is .313, and the unconstrained estimate of the coefficient on c_{t-2}^* in the equation for c_t^* (see Table 3.6) is -0.321. In both cases the unconstrained point estimates are more than three standard deviations away from zero. We conclude

that the DRW model fails because the zero restrictions that it imposes are simply incompatible with the data.

In contrast, the DSLR model does not impose any zero restrictions a priori. Unlike the DRW model, it can in principle accommodate serial persistence in $\phi^{-t}\Delta c_t$ and any pattern of Granger causality between the elements of Q_t and Z_t . However in practice it seems that the cross equation restrictions imposed by the DSLR model prevent it from fitting the degree of serial correlation observed in $\phi^{-t}\Delta c_t$. This can be seen by comparing the VAR implied by the DSLR model, reported at the bottom of Table 3.5, with the corresponding unconstrained VAR(2). Notice that the first row of the coefficient matrices in the constrained VAR closely resembles the corresponding row in the unconstrained VAR. This suggests that the DSLR model fits the $\phi^{-t}(c_t - y_t)$ process fairly well. However, the (2,2) element of the coefficient matrix on the first lagged value in the constrained VAR is .026, more than three standard deviations from the unconstrained estimate of .313. A similar story emerges from Table 3.6. There, the (2,2) element of the coefficient matrix on the second lagged value is .008 in the constrained VAR, more than three standard deviations smaller than the absolute value of the unconstrained point estimate. This suggests that the cross equation restrictions imposed by the DSLR model do not allow the model to simultaneously fit the $\phi^{-t}(c_t - y_t)$ process and allow a sufficient degree of serial correlation in the $\phi^{-t}\Delta c_t$ process.

In summary, our results indicate substantial evidence against both versions of the models discussed in section 2. We attribute the empirical shortcomings of both models to their inability to accommodate the degree of serial persistence in the quasi first difference of consumption. In the DRW model, this failure is due to the zero restrictions which constrain the quasi first difference of consumption to be white noise, while in the DSLR model this failure is due to the nature of the cross equation restrictions.

4. The Continuous Time Version of the Model.

This section formulates continuous time versions of our random walk (CRW) and stochastic labor requirement (CSLR) models. In section 3 we presented evidence detailing the empirical shortcomings of the DRW and DSLR models. There are, however, a priori reasons for believing that reformulating the models in continuous time may improve their empirical performance.

We have argued that the most important evidence against the DRW model is the fact that $\phi^{-t} \Delta c_t$ is positively autocorrelated. A possible explanation is that this autocorrelation does not reflect the failure of the RWH per se, but the failure of one of the maintained assumptions of the DRW model, namely that the decision interval of private agents coincides with the data sampling interval. This explanation is suggested by Working's (1960) observation that the first difference of a time averaged and sampled continuous time random walk process is positively autocorrelated. In particular, Working showed that when $\phi = 1$, this process has a univariate MA(1) representation with MA coefficient .268. Christiano and Marshall (1986) show that, after rounding to three digits, Working's result is also valid for the case $\phi = \exp(.00456)$. One simple test of the continuous time random walk hypothesis, which does not require that we specify the underlying shocks to the model, is to see whether our measure of consumption has such a statistical representation. To this end we estimated univariate MAR representations of the $\phi^{-t} \Delta c_t$ process of lag lengths 1 through 4 and performed likelihood ratio tests sequentially. The hypothesis that lag lengths 2, 3, and 4 are zero cannot be rejected at even the fifty percent significance level. Moreover, the point estimate of the MA coefficient in the MA(1) representation is .284 with standard error .08. Consequently, this test yields almost no evidence against the continuous time random walk hypothesis.

Temporal aggregation also has potentially important implications for the dynamic correlations between consumption and income. For example the results of Sims (1971), Geweke (1978), and Marcet (1986) imply that unit averaged sampled lagged income may Granger cause the first difference of unit averaged sampled consumption even if the derivative of continuous time consumption is uncorrelated with lagged income.

We also expect the empirical performance of the CSLR model to differ from that of the DSLR model. Recall that we attributed the empirical shortcomings of the DSLR model to the empirical implausibility of the cross equation restrictions imposed by that model. However, Hansen, and Sargent (1981, 1983) and Christiano (1984, 1985), among others, have noted that temporal aggregation bias can lead to spurious rejection of cross equation constraints in dynamic rational expectations models.

The remainder of this section is organized as follows. In subsection 4.A we present the continuous time analogue to the discrete time model of section 2. In subsection 4.B we present the continuous time random walk (CRW) model. In subsection 4.C we present the continuous time stochastic labor requirement (CSLR) model.

4.A The Model and Its Equilibrium Decision Rules

In this subsection we present the continuous time analogue of the discrete time model of consumption and output discussed in section 2. Our notation is the same as that used in sections 2 and 3 except that all random variables are assumed to evolve in continuous rather than discrete time. In addition, we adopt the convention of placing the time index of a continuous time random variable in parentheses.

The preferences of the representative consumer are given by:

$$(4.1) \quad E_0 \int_0^{\infty} e^{-rt} \left\{ -\frac{1}{2} (c(t) - b(t))^2 - \alpha(t)h(t) \right\} dt,$$

where $b(t) = b \exp(\theta t)$, $\alpha(t) = \alpha \exp(2\theta t)$, $r, b, \alpha, \theta > 0$, and $r - 2\theta > 0$. In (4.1) $b(t)$ is the representative consumer's time t bliss point for consumption, $\alpha(t)$ measures the disutility of work at time t , $c(t)$ is consumption at time t and $h(t)$ is hours worked at time t .

As before, there is an aggregate technology that converts capital, $k(t)$, and labor effort into consumption goods:^{4.1/}

$$(4.2) \quad \tilde{y}(t) = \min\{\tilde{\delta}k(t), \tau(t)h(t)\} + e(t).$$

Here, $\tau(t)$ represents the (possibly) stochastic labor requirement per unit of capital, $\tilde{\delta} > 0$ is a parameter in the production function, and $e(t)$ is an aggregate shock to the time t production function. We impose the continuous time analogue to condition (2.4).

$$(4.3) \quad r = \delta, \text{ where } \delta \equiv \tilde{\delta} - d.$$

In (4.3), $d > 0$ is the depreciation rate on capital.

The economy-wide resource constraint is given by

$$(4.4) \quad \tilde{y}(t) = c(t) + Dk(t) + dk(t),$$

where D denotes the time derivative operator.

The representative consumer's problem is to maximize (4.1) over contingency plans for setting $Dc(t)$, $Dk(t)$, $Dh(t)$, and $Dy(t)$ as a function of $I(t)$, subject to (4.2) and (4.4) and the constraint

$$(4.5) \quad E_0 \int_0^{\infty} e^{-rt} k(t)^2 dt < \infty.$$

The set $I(t)$ is composed of all model variables dated t and earlier, but cannot be specified precisely until we describe the stochastic structure of the exogenous shocks. We assume that

$$(4.6) \quad \tilde{\delta}k(t) = \tau(t)h(t).$$

Relations (4.2), (4.4), and (4.6) imply

$$(4.7) \quad c(t) = \delta k(t) - Dk(t) + e(t).$$

Substituting (4.6) and (4.7) into (4.1) we see that the representative consumer's problem is to maximize:

$$(4.8) \quad E_0 \int_0^{\infty} e^{-rt} \left\{ -\frac{1}{2} [\delta k(t) - Dk(t) + e(t) - b(t)]^2 - H(t)k(t) \right\} dt$$

by choice of a contingency plan for $Dk(t)$. In (4.8), $H(t) \equiv \tilde{\delta}a(t)/\tau(t)$. Notice that, as in the discrete time model, the principle of certainty equivalence applies to the representative consumer's problem.

It is convenient to define

$$(4.9) \quad x_p(t) = \delta \int_0^{\infty} e^{-\delta\tau} E_t x(t+\tau) d\tau,$$

for any x process such that (4.9) converges. In Appendix C we show that the equilibrium laws of motion for $Dk(t)$ and $c(t)$ can be written as

$$(4.10) \quad Dk(t) = e(t) - e_p(t) + b_p(t) - b(t) - H_p(t)/\delta$$

$$c(t) = e_p(t) + b(t) - b_p(t) + \delta k(t) + H_p(t)/\delta.$$

As before we let $y(t)$ denote net output: $y(t) = c(t) + Dk(t)$. Then

$$(4.11) \quad c(t) - y(t) = e_p(t) - e(t) + [b(t) - b_p(t)] + H_p(t)/\delta$$

and

$$(4.12) \quad Dc(t) = \mu_{e_p}(t) - \mu_{b_p}(t) + Db(t) + \mu_{H_p}(t)/\delta - H(t)$$

where $\mu_{x_p}(t)$ is the change in the value of $x_p(t)$ due to a disturbance in $x(t)$ that is unpredictable on the basis of $I(t-\tau)$, for all $\tau > 0$. (See Appendix C for a more careful discussion of this point.) Relations (4.11) and (4.12) are analogous to their discrete time counterparts, (2.13) and (2.15), so that the intuition underlying our results for the discrete time economy remains valid.

4.B The Continuous Time Random Walk Model (CRW).

In this subsection we display a set of restrictions on the technology shocks that give rise to the continuous time RWH. We then display the resulting reduced form representations for consumption and output.

Given our assumptions on the $b(t)$ process, relation (4.12) implies that $c(t)$ is a random walk with deterministic drift if, and only if, $H(t)$ is deterministic. Accordingly, we assume

$$(4.13) \quad H(t) = H \exp(\theta t),$$

where $H > 0$. The shock to endowment income is assumed to satisfy,

$$(4.14) \quad e(t) = e_1(t) + e_2(t),$$

where

$$De_i(t) = \frac{\eta_i(t)}{(a_i + D)} + e_i \exp(\theta t),$$

where $a_i > 0$, $i = 1, 2$, and a_1 is not equal to a_2 . Let $x(t) = [\eta_1(t) \eta_2(t)]'$. The vector $x(t)$ is the continuous time linear least squares innovation to the joint $[e_1(t)e_2(t)]$ process and satisfies,

$$(4.16) \quad E[x(t)x(t-u)'] = \exp(2\theta t)\xi(u)\bar{V},$$

where $\xi(u)$ is the Dirac delta generalized function and \bar{V} is a two by two positive definite symmetric matrix of constants. Thus, $e(t)$ is the sum of two stochastic processes whose first derivatives are AR(1) continuous time stochastic processes.

Substituting (4.13) - (4.15) into (4.12), we see that

$$(4.17) \quad Dc(t) = \eta_1(t)/(a_1+\delta) + \eta_2(t)/(a_2+\delta) + T_{c1} \exp(\theta t)$$

where T_{c1} is a positive scalar constant. According to (4.17), the derivative of consumption is a serially uncorrelated continuous time white noise process. While $Dc(t)$ is not a physically realizable process, its average over any discrete interval of time is physically realizable.

Substituting (4.13) - (4.15) into (4.11) we obtain,

$$(4.18) \quad c(t) - y(t) = De_1(t)/(a_1+\delta) + De_2(t)/(a_2+\delta) + T_{c2} \exp(\theta t)$$

where T_{c2} is a positive scalar. We define the vector

$$q^*(t) = [c^*(t) - y^*(t), (D+\theta)c^*(t)]',$$

where $c^*(t) \equiv \exp[-\theta t]c(t)$, $y^*(t) \equiv \exp[-\theta t]y(t)$. Relations (4.15), (4.17), and (4.18) imply that $q^*(t)$ has a continuous time VAR(2) representation:

$$(4.19) \quad A_c(D+\theta)q^*(t) = X(t) + T_c$$

where

$$(4.20) \quad A_c(D) = I + A_{c1}D + A_{c2}D^2,$$

$$A_{c1} = \begin{vmatrix} (a_1+a_2)/(a_1a_2) & -1/(a_1a_2) \\ 0 & 0 \end{vmatrix}, \quad A_{c2} = \begin{vmatrix} 1/(a_1a_2) & 0 \\ 0 & 0 \end{vmatrix},$$

$$X(t) = \begin{vmatrix} 1/a_1 & 1/a_2 \\ 1 & 1 \end{vmatrix} \begin{vmatrix} \eta_1^*(t)/(a_1+\delta) \\ \eta_2^*(t)/(a_2+\delta) \end{vmatrix},$$

$$\eta_i^*(t) = \exp[-\theta t] \eta_i(t), \quad i = 1, 2,$$

and

$$E[X(t)X(t-u)'] = \xi(u)V_c.$$

In (4.19) T_c is a two dimensional vector of positive constants and in (4.20) V_c is a two by two positive definite symmetric matrix of constants. An implication of (4.19) and (4.20) is that $c(t)$ satisfies the continuous time random walk hypothesis. This does not imply that the quasi first difference of unit averaged, sampled geometrically detrended consumption will satisfy the discrete time random walk hypothesis.

4.C The Continuous Time Stochastic Labor Requirement Model (CSLR)

In this subsection we display the continuous time analogue to the discrete time model of section 2.C. Our specification of $b(t)$ and $a(t)$ is the same as that given in section 4.B. However, we abandon the assumption that the labor requirement per unit of capital is nonstochastic. Instead, we assume

$$(4.21) \quad (f+D)H(t) = H \exp(\theta t) + \alpha \varepsilon(t),$$

where $E[\varepsilon(t)\varepsilon(t-u)'] = \exp(2\theta t)\xi(u)\sigma_\varepsilon^2$ and $\sigma_\varepsilon^2 > 0$. The shock to endowment income, $e(t)$, is assumed to satisfy

$$(4.22) \quad De(t) = \eta(t)/[a+D] + e \exp(\theta t)$$

where $E[\eta(t)\eta(t-u)] = \exp(2\theta t)\xi(u)\sigma_\eta^2$ and $\sigma_\eta^2 \geq 0$. Let $\mathbf{x}(t) = [\varepsilon(t)\eta(t)]'$. The vector $\mathbf{x}(t)$ is the continuous time linear least squares innovation to the joint $[H(t), e(t)]$ process and satisfies

$$(4.23) \quad E[\mathbf{x}(t)\mathbf{x}(t-u)'] = \exp(2\theta t)\xi(u)\bar{V}$$

where \bar{V} is a two by two positive definite symmetric matrix of constants.

Substituting (4.21) - (4.22) into (4.11) and (4.12) we obtain

$$(4.24) \quad c(t) - y(t) = Dc(t)/(a+\delta) + \delta H(t)/(f+\delta) + T_{c1} \exp(\theta t)$$

$$Dc(t) = \eta(t)/(a+\delta) + [D-\delta]\delta H(t)/(f+\delta) + T_{c2} \exp(\theta t)$$

where T_{c1} and T_{c2} are positive scalar constants. Relations (4.21) - (4.24)

imply that $q^*(t)$ has the continuous time VAR(2) representation:

$$(4.25) \quad A_c(D+\theta)q^*(t) = X(t) + T_c$$

where

$$(4.26) \quad A_c(D) = I + A_{c1}D + A_{c2}D^2,$$

$$A_{c1} = \begin{vmatrix} a+f\delta/a & f/a-1 \\ -a(f+\delta) & f+\delta \end{vmatrix} / f(a+\delta), \quad A_{c2} = \begin{vmatrix} 1-f/a & 0 \\ -(f+\delta) & 0 \end{vmatrix} / f(a+\delta),$$

$$X(t) = \begin{vmatrix} 1/a_1 & 1/f \\ 1 & -\delta/f \end{vmatrix} \begin{vmatrix} \eta^*(t)/(a+\delta) \\ \alpha \epsilon^*(t)/(f+\delta) \end{vmatrix},$$

$$\eta^*(t) = \exp[-\theta t]\eta(t), \quad \epsilon^*(t) = \exp[-\theta t]\epsilon(t),$$

and

$$E[X(t)X(t-u)'] = \xi(u)V_c.$$

In (4.25) T_c is a two dimensional vector of positive constants and in (4.26) V_c is a two by two dimensional positive definite symmetric matrix of constants. Relations (4.25) and (4.26) imply that $c(t)$ does not satisfy the continuous time RWH.

5. Empirical Results for the Continuous Time Models

In this section we report the results of estimating the CRW model of section 4.B and the CSLR model of section 4.C. In subsection 5.A we briefly

describe our estimation method. Empirical results are reported in subsection 5.B.

5.A Estimation Strategy

In section 4 we derived the constrained continuous time VAR representations for $q^*(t)$ implied by the CRW and CSLR models [see (4.19)-(4.20) and (4.25)-(4.26) respectively]. In order to proceed with estimation we must deduce the implications of these VARs for the probability law of the vector of observable variables, which we denote by $\tilde{q}(t)$. We define $\tilde{q}(t)$ to be the 2x1 continuous time stochastic process whose first element is the difference between detrended quarterly averaged consumption and gross output, and whose second element is the detrended first difference of quarterly averaged consumption. The vectors $q^*(t)$ and $\tilde{q}(t)$ differ in two important respects. First, $q^*(t)$ involves a measure of detrended NNP, whereas $\tilde{q}(t)$ involves a measure of detrended GNP. Second, $q^*(t)$ represents point in time measured variables, whereas $\tilde{q}(t)$ represents variables which have been averaged over the discrete data sampling interval.

Our strategy for obtaining the probability law for $\tilde{q}(t)$ is to derive the linear mapping relating $q^*(t)$ and $\tilde{q}(t)$, and then to use this expression to substitute out for $q^*(t)$ in terms of $\tilde{q}(t)$ in (4.19) and (4.25). We proceed by first obtaining the linear mapping between undetrended $q^*(t)$ and undetrended $\tilde{q}(t)$. Let $z(t)$ denote the undetrended, point-in-time sampled data underlying $\tilde{q}(t)$, i.e., $z(t) \equiv [c(t)-\bar{y}(t), Dc(t)]'$. Let $\bar{z}(t)$ denote the undetrended, averaged data underlying $\tilde{q}(t)$, i.e., $\bar{z}(t) = \exp[\theta t]\tilde{q}(t)$. Formally,

$$(5.1) \quad \bar{z}(t) = \begin{vmatrix} \int_0^1 [c(t-\tau)-\bar{y}(t-\tau)]d\tau \\ \int_0^1 [c(t-\tau)-c(t-1-\tau)]d\tau \end{vmatrix} = \begin{vmatrix} \int_0^1 [c(t-\tau)-\bar{y}(t-\tau)]d\tau \\ \int_0^1 \left[\int_0^1 Dc(t-\tau-\mu)d\mu \right]d\tau \end{vmatrix}.$$

Here we have used the fact that $\int_0^1 Dc(t-\mu)d\mu = c(t) - c(t-1)$.^{5.1/} In operator notation:^{5.2/}

$$(5.2) \quad \bar{z}(t) = G(D)z(t),$$

where

$$(5.3) \quad G(D) = \begin{bmatrix} 1 & 0 \\ 0 & [(1-e^{-D})/D] \end{bmatrix}.$$

Let $q(t)$ denote the undetrended value of $q^*(t)$, i.e., $q(t) = \exp(\theta t) q^*(t) = [c(t)-y(t), Dc(t)]$. In operator notation, the link between $q(t)$ and $z(t)$ is given by^{5.3/}

$$(5.4) \quad q(t) = H(D)z(t)$$

where

$$(5.5) \quad H(D) = \frac{1}{D+d} \begin{bmatrix} D & 0 \\ 0 & D+d \end{bmatrix}.$$

Substituting (5.4) into (5.2), we obtain

$$(5.6) \quad q(t) = H(D)G(D)^{-1}\bar{z}(t)$$

which provides a mapping between the continuous time processes $q(t)$ and $\bar{z}(t)$ i.e., between undetrended $q^*(t)$ and undetrended $\bar{q}(t)$. Finally, the link between $\bar{q}(t)$ and $q^*(t)$ is obtained by multiplying both sides of (5.6) by $\exp(-\theta t)$:

$$(5.6) \quad q^*(t) = H(D+\theta)G(D+\theta)^{-1}\bar{q}(t).$$

We substitute (5.6) into (4.19) and (4.25) to obtain the time series representations for $\bar{q}(t)$ implied by the CRW and CSLR models, respectively:

$$(5.7) \quad A_c(D+\theta)H(D+\theta)G(D+\theta)^{-1}\bar{q}(t) = X(t) + T_c$$

$$(5.8) \quad A_c(D+\theta)H(D+\theta)G(D+\theta)^{-1}\tilde{q}(t) = X(t) + T_c.$$

These are the continuous time analogues of (3.6) and (3.7). An important difference between the two sets of representations is $G(D+\theta)^{-1}$, which appears in (5.7) and (5.8), but has no analogue in (3.6) and (3.7). This reflects the fact that time averaging of the data is taken into account in the continuous time models, but not in the discrete time models.

We now describe the procedure used to estimate systems (5.7) and (5.8). Define

$$(5.9) \quad Q(t) \equiv \tilde{q}(t) - E\tilde{q}(t).$$

Suppose we have a sample on $Q(t)$, $t = 1, 2, 3, \dots, T$. Our estimation criterion is the frequency domain approximation to the Gaussian density function described in subsection 3.A. As before, we denote the periodogram of the data by $I(\omega_j)$, $\omega_j = 2\pi j/T$, $j = 1, 2, \dots, T$, while $Z(\omega)$ denotes the theoretical spectral density of the discrete process $\{Q(t), t \text{ integer}\}$ at frequency ω implied by (5.7) or (5.8).

The matrix function $Z(\omega)$ is derived as follows. Using results in Phillips (1958) it can be shown that the spectral density of $\{Q(t), t \text{ real}\}$ generated by (5.7), is given by:

$$(5.10) \quad Z^c(\omega) = \psi(i\omega+\theta)A_c(i\omega+\theta)^{-1}V_c[A_c(-i\omega+\theta)']^{-1}\psi(-i\omega+\theta),$$

for $-\infty \leq \omega \leq \infty$, where $\psi(s) \equiv G(s)H(s)^{-1}$. The corresponding spectral density generated by (5.8) is:

$$(5.11) \quad Z^c(\omega) = \psi(i\omega+\theta)A_c(i\omega+\theta)^{-1}V_c[A_c(-i\omega+\theta)']^{-1}\psi(-i\omega+\theta).$$

Hannan [1970, p. 45] shows that the following "folding operator" links $Z(\omega)$ and $Z^c(\omega)$:

$$(5.12) \quad Z(\omega) = \sum_{k=-\infty}^{\infty} Z^c(\omega+2\pi k).$$

Equations (5.10) or (5.11) and (5.12) provide a computationally feasible algorithm for obtaining $Z(\omega)$ for a given ω from $[\psi, A, V_c]$ or $[\psi, A, V_c]$. Because this algorithm is relatively slow, we used an alternative method based on a partial fractions decomposition of Z^c (see Durbin [1961], Hannan [1970, pp. 405-407] and Hansen and Sargent [1981]).

The preceding estimation strategy assumes that the values of θ and $E\tilde{q}(t)$ are known. We proceeded as in the discrete time case, by replacing $E\tilde{q}(t)$ by its sample mean and setting θ to .004568.5.4/

We conclude this subsection by noting that an implication of results in Christiano and Marshall (1986), is that both the CRW and CSLR models give rise to constrained SARMA(3,4) representations for $Q(t)$.5.5/

5.B Empirical Results

Our estimates of the CRW and CSLR models are reported in Table 5.1. The parameter r is set equal to .0098, which implies an annual discount rate of four percent.5.6/ According to both the unadjusted and adjusted likelihood ratio statistics (J_T and J_T respectively) neither the CRW nor the CSLR model can be rejected, at the five percent significance level. This is to be contrasted with our findings that the DRW model can be rejected at close to the one percent level. Thus for both models there is some evidence that the continuous time formulations are in greater conformity with the data than their discrete time counterparts.

The large number of parameters in the unconstrained SARMA used to construct the likelihood ratio tests raises questions regarding the power of our specification tests. Since the SARMA(3,4) is the most parsimoniously parameterized unconstrained model that nests the continuous time structural models, we cannot formally compare the performance of these models with a more tightly parameterized alternative. However, the point estimates reported in

Table 5.1 suggest a way of reformulating the continuous time models so that they are nested in an unconstrained SARMA(2,3). The point estimates of a_2 and a are extremely large so that the $e_2(t)$ and $e(t)$ processes are virtually indistinguishable from continuous time random walks.^{5.7/} It follows that in the SARMA(3,4) representations implied by both the CRW and CSLR models, the MA matrix coefficient in the fourth lag and the AR coefficient on the third lag are approximately zero.^{5.8/} Hence we can compute likelihood ratio statistics by comparing the likelihood values given in Table 5.2 with the value obtained for the unconstrained SARMA(2,3) (reported in Table 3.5). The resulting test statistics, which are distributed asymptotically with 12 degrees of freedom, are given below. (Probability values are in parentheses.)

	CRW Model	CSLR Model
J_T	22.74 (.970)	22.20 (.965)
J_T	21.61 (.958)	21.10 (.951)

According to these results, neither model is rejected at the three percent significance level.

The dimensions along which the CRW model appears to be in greater conformity with the data than the DRW model can be seen by comparing their reduced forms to each other and to the unconstrained VAR(2). In Tables 5.2 and 5.3 we report the constrained VARs for Q_t and Z_t implied by our estimates of the CRW model. These VARs are in principle infinite ordered, so we use the truncation rule of not reporting matrix coefficients whose maximal element are smaller than .02 in absolute value. Comparing Tables 3.5 and 5.2 we see that the DRW and CRW models capture the dynamics in $\phi^{-t}(c_t - y_t)$ about equally well. However the CRW and DRW models differ substantially in their implica-

tions for $\phi^{-t}\Delta c_t$. By construction the DRW model implies that $\phi^{-t}\Delta c_t$ is uncorrelated with lagged values of both $\phi^{-t}\Delta c_t$ and $\phi^{-t}(c_t - y_t)$. While the CRW model embodies this restriction for the continuous time point-in-time sampled data, it does not imply this restriction for the actual measured, discrete time data. This follows for two reasons. First, as Working (1960) showed, a time averaged random walk is serially correlated. Second, results in Sims (1971) imply that the cross dynamics between $\phi^{-t}\Delta c_t$ and $\phi^{-t}(c_t - y_t)$ will be distorted by time averaging. For example, time averaged measures of $\phi^{-t}(c_t - y_t)$ should Granger cause $\phi^{-t}\Delta c_t$ even if this is not the case for continuous point-in-time sampled measures of $\phi^{-t}\Delta c_t$ and $\phi^{-t}(c_t - y_t)$.

Our evidence suggests that the effect discussed by Working (1960) is the major factor accounting for the improved fit of the CRW model relative to the DRW model. In particular, this effect accounts for the coefficient 0.27 that appears on the first own lag of $\phi^{-t}\Delta c_t$ in the constrained VAR. This is within one standard error of the point estimate (.313) of the corresponding coefficient in the unconstrained VAR. Moreover the first own lag on $\phi^{-t}\Delta c_t$ in the unconstrained VAR is more than three standard deviations away from zero. Taken together these observations suggest that the change in the value of the coefficient of once lagged $\phi^{-t}\Delta c_t$ on $\phi^{-t}\Delta c_t$ from 0 in the DRW model to 0.27 in the CRW model has a substantial effect on the likelihood ratio statistic.

A similar picture emerges from examining the constrained VAR for Z_t reported in Table 5.3. Comparing this representation with the unconstrained VAR(2) for Z_t in Table 3.6, we see that the value of the coefficient on the own second lag of c_t^* implied by the CRW model is -.357, which is within a half standard deviation of the unconstrained estimate of -.321. In contrast, the DRW model constrains this coefficient to equal zero, a value which is more than three standard deviations away from the corresponding point estimate.

The effect of time averaging discussed by Sims (1971) is also present. This can be seen by the nonzero values of the coefficients on lagged values of $\phi^{-t}(c_t - y_t)$ in the second row of the VAR in Table 5.2 corresponding to the CRW model. However, when we compare these point estimates to the corresponding entries in the unconstrained VAR we see that this effect may be harmful with regards to the overall fit of the CRW model. This is because the sign on $\phi^{-(t-1)}(c_{t-1} - y_{t-1})$ in the $\phi^{-t}\Delta c_t$ equation of the constrained VAR is positive, in contrast to the negative sign of the corresponding term in the unconstrained VAR. Since the latter coefficient is not precisely estimated, this effect is not sufficiently important to negate the favorable impact of the effects suggested by Working (1960).

The dimensions along which the CSLR model outperforms the DSLR model can also be seen by comparing their reduced forms to each other and to the unconstrained VARs. The constrained VAR(2)s implied by the DSLR model for Q_t and Z_t are reported in Tables 3.5 and 3.6. The corresponding VARs implied by the CSLR model are reported in Tables 5.2 and 5.3. Because the constrained VAR's are in principle infinite ordered we again use the truncation rule of not reporting matrices whose maximal element is smaller than .02 in absolute value. Comparing Tables 3.5 and 5.2 we see that the DSLR and CSLR models do not differ in any substantial way regarding the dynamics of $\phi^{-t}(c_t - y_t)$. However they do differ substantially in their implications for $\phi^{-t}\Delta c_t$. In section 3.B we attributed the rejection of the DSLR model to the failure of the cross equation restrictions. In particular, the DSLR model succeeds in fitting the $\phi^{-t}(c_t - y_t)$ process fairly well, but is prevented by the cross equation restrictions from matching the serial correlation properties of $\phi^{-t}\Delta c_t$. In the VAR corresponding to the CSLR model the coefficient on $\phi^{-(t-1)}\Delta c_{t-1}$ in the $\phi^{-t}\Delta c_t$ equation is approximately .27. In contrast the

value of corresponding coefficient in the DSLR model is approximately .03. Similarly, in the VAR implied by the CSLR model for Z_t , the coefficient on $\phi^{-(t-2)}\Delta c_{t-2}$ in the $\phi^{-t}c_t$ equation is .340 (within one half standard deviation of the unconstrained point estimate) while the corresponding coefficient in the DSLR model is .008. Thus, the principal difference between the DSLR and CSLR models is that the latter model is able to handle substantially more serial correlation in $\phi^{-t}\Delta c_t$. This difference is presumably attributable in part to the effects of temporal averaging discussed by Working (1960) but could also be due in part to the effects of temporal aggregation on cross equation restrictions discussed by Hansen and Sargent (1981, 1983).

In summary we find that the CRW and CSLR models appear to be empirically more plausible than the DRW and DSLR models, respectively. In our view the evidence against the CRW model and the CSLR model is far from overwhelming. This is surprising given the simplicity and parsimonious parameterization of both these models. In both instances the impact of moving to a continuous time model is an enhanced ability to mimic the serial correlation properties of the quasi first difference of consumption.

6. Conclusion

This paper develops and tests models of consumption and output which are consistent with the fact that measured aggregate consumption does not behave as a random walk. It is not particularly challenging to develop theories which can explain this fact in principle. The random walk hypothesis is clearly a special case of the permanent income hypothesis. However, as much of the recent literature on the macroeconomics of consumption reveals, it is quite challenging to develop empirically plausible models of the comovements in aggregate consumption and output.

We investigated two possible reasons why the change in consumption fails to behave like a white noise. The first possibility is that exogenous shocks to the economic system generate serial persistence in the first difference of consumption. We modeled this shock as a stochastic perturbation to the amount of labor required to make capital productive. As it turns out, there is a great deal of evidence against this version of our model when it is implemented under the assumption that agents' decision intervals coincide with the data sampling interval. However, there is surprisingly little evidence against the continuous time version of this model.

The second possibility is that the RWH holds in the (unobserved) continuous consumption process, with serial persistence in measured consumption being an artifact of temporal aggregation. Our results indicate that when temporal aggregation bias is taken into account, the fit of the random walk model improves substantially. This suggests that the random walk hypothesis may yet be a useful way to conceptualize the relation between aggregate consumption and output.

While both of the continuous time models that we tested outperform their discrete time counterparts, it is very difficult, at least on the basis of aggregate consumption and output data, to distinguish between the two continuous time models. However the CRW model does have a number of implications which we did not test in this paper but which call its plausibility into question. One such implication is that the capital-labor ratio is deterministic. This implication is obviously counterfactual. While this could be remedied by allowing for measurement error, we regard the CLSR model as a more promising starting point for future research.

A different set of implications which were not explored in this paper concern the equilibrium wage rate and real interest rate. Unlike the

quantity variables, our models imply that these price processes are nonlinear functions of the state variables in the system. Consequently, deriving the laws of motion for measured wages and interest rates that are implied by our continuous time models involves technical difficulties not encountered in this paper. Nonetheless, we believe that our results for the consumption and output are sufficiently encouraging to warrant an empirical investigation of the model's implications for relative prices.

Footnotes

2.1/In Christiano, Eichenbaum, and Marshall (1986), it is shown that if the econometrician uses only consumption and output data then the DSLR model is observationally equivalent to a model in which b_t is stochastic but the labor requirement per unit of capital is deterministic.

2.2/In a formulation which allows for positive population growth. The expression on the left hand side of (2.3) must be replaced by $c_t + k_t - [(1-d)/n]k_{t-1}$, where n denotes the gross growth rate of the population.

2.3/Hansen's (1986) model sets $\alpha_t \equiv 0$.

2.4/This terminology is slightly unconventional since β^{-1} is not the gross rate interest in our model economy.

2.5/With positive population growth the right hand side of (2.17) must be scaled up by n , the gross growth rate in the population.

3.1/Equation (3.4) can be seen as follows:

$$q_t^* = \begin{vmatrix} c_t^* - y_t^* \\ c_t^* - \phi^{-1}c_{t-1}^* \end{vmatrix} = \begin{vmatrix} -(k_t^* - \phi^{-1}k_{t-1}^*) \\ c_t^* - \phi^{-1}c_{t-1}^* \end{vmatrix}$$

$$= \begin{vmatrix} \frac{(1-\phi^{-1}L)}{1-(1-d)\phi^{-1}L} \{-[k_t^* - (1-d)\phi^{-1}k_{t-1}^*]\} \\ c_t^* - \phi c_{t-1}^{*-1} \end{vmatrix} = H(\phi^{-1}L)q_t$$

where the last equality follows from (3.1) and (3.2) and $k_t^* = \phi^{-t}k_t$.

3.2/Our measure of government investment is a revised and updated version of the measure discussed in Musgrave (1980).

3.3/We obtained this value of ϕ by regressing $\log c_t$ and $\log \tilde{y}_t$ on a linear time trend subject to the restrictions that the growth rates in con-

sumption and output are equal. When we did not impose this restriction, we found that the growth rates in consumption and output were $\exp[.004582]$ and $\exp[.004606]$, respectively.

3.4/ That Q_t is predicted to be SARMA(3,3) is proved in Christiano, Eichenbaum, and Marshall (1986).

3.5/ Whittle's (1953) correction for small sample bias is as follows: Let N equal the total number of parameters under the alternative hypothesis (excluding the covariance matrix of the observables), M = number of equations, and T = number of observations. Then $J_T = J_T(1-N/MT)$ where J_T is the unadjusted likelihood ratio statistic and J_T is the adjusted likelihood ratio statistic. When the unconstrained alternative is a SARMA(3,4), $N=19$, $M=2$, and $T=141$, so $J_T = 0.9326241J_T$.

3.6/ When compared with the SARMA(3,3), the DRW model is rejected even more strongly. In that test the unadjusted J_T statistic is 35.49 (probability value .9996) and the adjusted J_T statistic is 33.59 (probability value .9992).

4.1/ Equation (4.2) differs from (2.2) in the timing of the productive inputs. (4.2) results as a limiting case of (2.2) if we rewrite the latter as $\tilde{y}_t = \min\{\tilde{\delta}_{t-\epsilon}, \tau_{t-\epsilon} h_{t-\epsilon}\} + e_t$ and let $\epsilon \rightarrow 0$ from above.

5.1/ Treating measured consumption and income as unit integrals of the underlying instantaneous quantities is a rough approximation to the methods used by the Department of Commerce to gather data.

5.2/ In deriving (5.3) we use the fact that

$$\int_0^1 x(t-\tau) d\tau = \int_0^1 e^{-\tau D} x(t) d\tau = [(1-e^{-D})/D] x(t).$$

5.3/ Equation (5.5) can be derived as follows:

$$q(t) = \begin{vmatrix} c(t) - y(t) \\ Dc(t) \end{vmatrix} = \begin{vmatrix} -Dk(t) \\ Dc(t) \end{vmatrix} = H(D) \begin{vmatrix} -(D+d)k(t) \\ Dc(t) \end{vmatrix} = H(D)\tilde{q}(t)$$

where the last equality follows from (4.4) and (5.2).

5.4/ The relationship between θ and ϕ is $\phi = e^\theta$.

5.5/ The particular SARMA representation corresponding to a given continuous time model is characterized by a third order scalar polynomial, $E^d(\cdot)$, the two by two fourth order matrix polynomial, $C^d(\cdot)$, and the two by two positive semidefinite matrix V^d which satisfy:

$$Z(\omega) = C^d(e^{-i\omega})V^dC^d(e^{i\omega})/[E^d(e^{-i\omega})E^d(e^{i\omega})].$$

Here, we impose the normalizations $C^d(0) = I$, $\det[C^d(z)] = 0$ implies $|z| \geq 1$, and $E^d(0) = 1$. The algorithm we used to calculate E^d , C^d , and V^d is the one described in Rozanov (1967, chapter I, section 10). Thus both continuous time models are nested within the SARMA specification:

$$E^d(L)Q_t = C^d(L)X_{ct},$$

where X_{ct} is the serially uncorrelated innovation in $Q(t)$, with variance V^d , and

$$E^d(L) = 1 + E_1^d L + E_2^d L^2 + E_3^d L^3,$$

$$C^d(L) = I + C_1^d L + C_2^d L^2 + C_3^d L^3 + C_4^d L^4.$$

5.6/ The relationship between β in the discrete formulations and r in the continuous formulations is $\beta = e^{-r}$.

5.7/ That the reported point estimates imply that $e_2(t)$ and $e(t)$ behave essentially as continuous time random walks can be seen from the following argument. If $x(t)$ is a continuous time first order autoregression:

$x(t) = \epsilon(t)/(a+D)$, $\epsilon(t)$ continuous time white noise, then $x(t)$ has an exponentially declining impulse response function:

$$x(t) = \int_0^{\infty} e^{-a\tau} \epsilon(t-\tau) d\tau.$$

Therefore, the impulse response function for $De_2(t)$ is $e^{-28\tau}$ and for $De(t)$ is $e^{-12\tau}$. These functions decline so steeply that past impulses have negligible effect on current values of $De_2(t)$ and $De(t)$.

5.8/In the SARMA(3,4) implied by the CRW model, the MA matrix coefficient on the fourth lag consists entirely of zeroes and the AR coefficient on the third lag equals 0.9×10^{-12} . In the case of the CSLR model, the MA matrix coefficient on the fourth lag has no element greater than 3×10^{-7} in absolute value and the AR coefficient on the third lag is -7×10^{-6} .

Table 3.1

Sequential Likelihood Ratio Tests for VAR Lag Length

VARs Tested	Degrees Freedom	Likelihood Ratio Statistic* For Q_t^{**}	For Z_t^{***}
VAR(1) against VAR(2)	4	12.60 (.987)	21.72 (.999)
VAR(2) against VAR(3)	4	6.085 (.807)	5.231 (.756)
VAR(3) against VAR(4)	4	3.807 (.567)	6.944 (.861)
VAR(4) against VAR(5)	4	5.246 (.730)	2.838 (.415)
VAR(5) against VAR(6)	4	6.587 (.838)	5.977 (.799)

*Reported likelihood ratio statistic is twice the difference between the log likelihood of the VAR(n) and the VAR(n-1), for n = 2, ..., 6. Probability values in parentheses.

** Q_t is demeaned $[\phi^{-t}(c_t - \tilde{y}_t), \phi^{-t} \Delta c_t]$.

*** Z_t is demeaned $[\phi^{-t} \tilde{y}_t, \phi^{-t} c_t]$.

Table 3.2

Null Hypothesis	Degrees Freedom	J_T^*
$\phi^{-t}\Delta c_t$ does not Granger cause $\phi^{-t}(c_t - \bar{y}_t)$	2	5.08 (.921)
$\phi^{-t}(c_t - \bar{y}_t)$ does not Granger cause $\phi^{-t}\Delta c_t$	2	0.84 (.343)
$\phi^{-t}\Delta c_t$ is white noise	4	24.98 (.9999)

* J_T is the likelihood ratio statistic comparing an unrestricted VAR(2) with a VAR(2) estimated subject to the restrictions imposed by the stated null hypothesis. Probability values of test statistics in parentheses.

Table 3.3

The Discrete Time Random Walk Model		The Discrete Time Stochastic Labor Requirement Model	
Parameter	Point Estimate*	Parameter	Point Estimate
a_1	0.212 (.094)	f	.915 (.024)
a_2	.849 (.072)	a	.215 (.085)
d	.007 (.062)	d	.066 (.035)
Estimated Covariance Matrix** of V_t^*		Estimated Covariance Matrix** of V_t^*	
$\begin{vmatrix} 887.85 & -712.78 \\ (404.39) & (372.08) \\ -712.78 & 651.48 \\ & (342.14) \end{vmatrix}$		$\begin{vmatrix} 656.25 & -495.49 \\ (156.99) & (135.81) \\ -495.49 & 442.58 \\ & (118.17) \end{vmatrix}$	
$\hat{\epsilon}_T = -849.00$		$\hat{\epsilon}_T = 846.44$	
$J_T^{***} = 37.26$ (.998)		$J_T = 32.14$ (.990)	
$J_T^{***} = 34.75$ (.996)		$J_T = 29.97$ (.982)	

*Standard errors in parentheses.

** V_t^* and V_t^* are the innovations in the SARMA representations implied by the DRW model and the DSLR model respectively.

*** J_T as defined in footnote 3.5. Probability value of J_T and J_T in parentheses.

Table 3.4

Diagnostics For SARMA (p,q): p = 1, 2; q = 1, 1, 2, 3, 4

p	q	ℓ_T^*	J_T^{**}	J_T^{***}	Degrees of Freedom
2	1	-837.33	13.92 (.620)	12.98 (.551)	13
2	2	-834.66	8.58 (.523)	8.00 (.466)	9
2	3	-831.88	3.02 (.303)	2.82 (.272)	5
2	4	-830.70	0.66 (.583)	0.62 (.569)	1
3	1	-836.79	12.84 (.619)	11.97 (.552)	12
3	2	-834.53	8.32 (.597)	7.76 (.543)	8
3	3	-831.26	1.78 (.224)	1.66 (.202)	4
3	4	-830.37			

*Value of the log likelihood function.

**Likelihood ratio statistic testing the SARMA(p,q) (p = 2, 3; q = 1, 2, 3, 4) specification against the SARMA(3,4) specification. Figures in parentheses refer to probability value.

***Likelihood ratio statistic corrected for small sample bias using Whittle's (1953) procedure (see footnote 3.5).

Table 3.5

Comparison of VAR(2) Representations for Q_t^*

Unconstrained VAR(2)

$$Q_t = \begin{bmatrix} 1.070 & -.136 \\ (.084) & (.132) \\ -.074 & .313 \\ (.055) & (.085) \end{bmatrix} Q_{t-1} + \begin{bmatrix} -.204 & -.206 \\ (.084) & (.130) \\ .065 & .125 \\ (.054) & (.085) \end{bmatrix} Q_{t-2} + x_t^{**}$$

$$EX_t X_t' = \begin{bmatrix} 217.02 & -0.867 \\ -0.867 & 92.74 \end{bmatrix}$$

Constrained Model (Truncated) Implied
by the Discrete Random Walk Model

$$Q_t = \begin{bmatrix} 1.063 & -.178 \\ 0 & 0 \end{bmatrix} Q_{t-1} + \begin{bmatrix} -.179 & 0 \\ 0 & 0 \end{bmatrix} Q_{t-2} + x_t$$

$$EX_t X_t' = \begin{bmatrix} 248.10 & -14.78 \\ -14.78 & 113.77 \end{bmatrix}$$

Constrained VAR(2) (Truncated) Implied by the
Discrete Stochastic Labor Requirement Model

$$Q_t = \begin{bmatrix} 1.166 & -.189 \\ -.120 & .026 \end{bmatrix} Q_{t-1} + \begin{bmatrix} -.201 & 0 \\ .034 & 0 \end{bmatrix} Q_{t-2} + x_t$$

$$EX_t X_t' = \begin{bmatrix} 261.14 & -18.64 \\ -18.64 & 107.85 \end{bmatrix}$$

* Q_t is demeaned $[\phi^{-t}(c_t - \bar{y}_t), \phi^{-t} \Delta c_t]$.

** x_t is the disturbance term in the unconstrained VAR(2).

Table 3.6

Comparison of VAR(2) Representations for Z_t^*

Unconstrained VAR(2)

$$Z_t = \begin{bmatrix} 1.175 & .359 \\ (.101) & (.193) \end{bmatrix} Z_{t-1} + \begin{bmatrix} -.270 & -.293 \\ (.100) & (.182) \end{bmatrix} Z_{t-2} + x_t^{**}$$

$$EX_t X_t' = \begin{bmatrix} 320.78 & 94.95 \\ 94.95 & 87.63 \end{bmatrix}$$

Constrained Model (Truncated) Implied by the Discrete Random Walk Model

$$Z_t = \begin{bmatrix} 1.063 & .110 \\ 0 & .995 \end{bmatrix} Z_{t-1} + \begin{bmatrix} -.179 & .002 \\ 0 & 0 \end{bmatrix} Z_{t-2} + x_t^z$$

$$EX_t^z X_t^{z'} = \begin{bmatrix} 391.43 & 128.55 \\ 128.55 & 113.77 \end{bmatrix}$$

Constrained VAR(2) (Truncated) Implied by the Discrete Stochastic Labor Requirement Model

$$Z_t = \begin{bmatrix} 1.286 & -.076 \\ .120 & .901 \end{bmatrix} Z_{t-1} + \begin{bmatrix} -.235 & .021 \\ -.034 & .008 \end{bmatrix} Z_{t-2} + x_t^z$$

$$EX_t^z X_t^{z'} = \begin{bmatrix} 406.3 & 126.5 \\ 126.5 & 107.9 \end{bmatrix}$$

* Z_t is demeaned $[\phi^{-t} \tilde{y}_t, \phi^{-t} c_t]$.

** x_t is the disturbance in the unconstrained VAR(2).

Table 5.1

The Continuous Time Random Walk Model		The Continuous Time Stochastic Labor Requirement Model	
Parameter	Point Estimate*	Parameter	Point Estimate
a ₁	.152 (.060)	f	.089 (.035)
a ₂	27.57 (81.81)	a	11.75 (11.29)
d	.0032 (.058)	d	.058 (.042)
Estimated Covariance Matrix ** W _c		Estimated Covariance Matrix** W _c	
$\begin{vmatrix} 488.51 & -521.63 \\ -521.63 & 719.52 \end{vmatrix}$		$\begin{vmatrix} 659.82 & 483.60 \\ 483.60 & 465.28 \end{vmatrix}$	
E _T = -843.249		E _T = -842.982	
J _T ** = 25.76 (.942)		J _T ** = 25.22 (.934)	
J _T *** = 24.02 (.911)		J _T *** = 23.52 (.900)	

*Standard errors in parentheses.

**W_c is the covariance matrix of the vector $[\eta_1^*(t)/(a_1+\delta), \eta_2^*(t)/(a_2+\delta)]$. W_c is the covariance matrix of the vector $[\eta^*(t)/(a+\delta), \alpha\delta e^*(t)/(f+\delta)]$.

***Probability value of J_T and J_T in parentheses.

Table 5.2

Comparison of VAR(2) Representations for Q_t^*

Truncated VAR Implied by the Continuous Time Random Walk Model

$$Q_t = \begin{vmatrix} 1.126 & -.033 \\ .011 & .270 \end{vmatrix} Q_{t-1} + \begin{vmatrix} -.300 & .018 \\ -.016 & -.073 \end{vmatrix} Q_{t-2} \\ + \begin{vmatrix} .080 & -.007 \\ .008 & .019 \end{vmatrix} Q_{t-3} + \begin{vmatrix} -.021 & .003 \\ -.003 & -.005 \end{vmatrix} Q_{t-4} + X_{ct}^{**}$$

$$EX_{ct} X_{ct}' = \begin{vmatrix} 255.87 & -17.93 \\ -17.93 & 101.52 \end{vmatrix}.$$

Truncated VAR Implied by the Continuous Time Stochastic Labor Requirement Model

$$Q_t = \begin{vmatrix} 1.241 & -.072 \\ -.078 & .277 \end{vmatrix} Q_{t-1} + \begin{vmatrix} -.343 & .038 \\ -.010 & -.075 \end{vmatrix} Q_{t-2} \\ + \begin{vmatrix} .096 & -.016 \\ .007 & .020 \end{vmatrix} Q_{t-3} + \begin{vmatrix} -.024 & .006 \\ -.001 & -.005 \end{vmatrix} Q_{t-4} + X_{ct}^{**}$$

$$EX_{ct} X_{ct}' = \begin{vmatrix} 268.25 & -22.90 \\ -22.90 & 100.13 \end{vmatrix}.$$

$$*Q_t \equiv [\phi^{-t}(c_t - \bar{y}_t), \phi^{-t} \Delta c_t].$$

** $X_{ct} = Q_t - E[Q_t | Q_{t-s}; s = 1, 2, \dots$ under the null hypothesis of the CRW model].

$X_{ct} = Q_t - E[Q_t | Q_{t-s}; s = 1, 2, \dots$ under the null hypothesis of the CSLR model].

Table 5.3

Comparison of VAR(2) Representations for Z_t^*

Truncated VAR Implied by the Continuous
Time Random Walk Model

$$Z_t = \begin{bmatrix} 1.114 & .184 \\ -.113 & 1.277 \end{bmatrix} Z_{t-1} + \begin{bmatrix} -.285 & -.107 \\ .016 & -.357 \end{bmatrix} Z_{t-2} \\ + \begin{bmatrix} .073 & .044 \\ -.008 & .099 \end{bmatrix} Z_{t-3} + \begin{bmatrix} -.018 & -.016 \\ .003 & -.027 \end{bmatrix} Z_{t-4} + X_{ct}^Z ** \\ EX_{ct}^Z X_{ct}^{Z'} = \begin{bmatrix} 393.25 & 119.45 \\ 119.45 & 101.52 \end{bmatrix}.$$

Truncated VAR Implied by the Continuous Time
Stochastic Labor Requirement Model

$$Z_t = \begin{bmatrix} 1.318 & .025 \\ .078 & 1.194 \end{bmatrix} Z_{t-1} + \begin{bmatrix} -.353 & -.108 \\ -.010 & -.340 \end{bmatrix} Z_{t-2} \\ + \begin{bmatrix} .089 & .060 \\ -.007 & .102 \end{bmatrix} Z_{t-3} + \begin{bmatrix} -.026 & -.021 \\ -.001 & -.024 \end{bmatrix} Z_{t-4} + X_{ct}^Z ** \\ EX_{ct}^Z X_{ct}^{Z'} = \begin{bmatrix} 414.18 & 123.03 \\ 123.03 & 100.13 \end{bmatrix}.$$

$$*Z_t \equiv [\phi^{-t} \tilde{y}_t, \phi^{-t} c_t].$$

** $X_{ct}^Z = Z_t - E[Z_t | Z_{t-s}; s = 1, 2, 3, \dots \text{ under the null hypothesis of the CRW model}]$.

$X_{ct}^Z = Z_t - E[Z_t | Z_{t-s}; s = 1, 2, 3, \dots \text{ under the null hypothesis of the CSLR model}]$.

Appendix A

Our derivation of the decision rules in section 2 assumes that capital and labor are never underutilized under the optimal plan (see equation [2.6]). This appendix discusses conditions under which this restriction is not binding.

The fact that labor is never underutilized follows trivially from the fact that k_t and h_t are chosen after τ_t is observed and that $\alpha_t > 0$. Thus, $\bar{\delta}k_t \geq \tau_t h_t$ for all t . Deriving conditions on the distribution of the shock terms and parameters which guarantee that capital is always fully utilized is much more difficult. This problem is analogous to the problem of deriving the conditions which guarantee that nonnegativity constraints on endogenous variables in linear quadratic models are not binding. Instead, we provide conditions that make capital under-utilization very unlikely. The following Proposition is useful for this purpose.

Proposition 1:

Suppose we have a unique interior maximum to (2.1) subject to (2.3) and (2.5). Then

$$(A.1) \quad \bar{\delta}k_t \leq \tau_t h_t \quad \text{for all } t$$

is equivalent with

$$(A.2) \quad H_t / (\bar{\delta}\beta) < E_t(b_{t+1} - c_{t+1}) < (b_t - c_t) / [\beta(1-d)] \quad \text{for all } t.$$

Proof

First we prove that (A.2) implies (A.1). If (A.2) is true, then any plan that does not satisfy (A.1) is suboptimal. To see this, consider a plan for which $\bar{\delta}k_t > \tau_t h_t$ for some t . Consider the following two feasible devia-

tions from this plan: (i) increase h_t by the amount $dh_t > 0$, and consume the proceeds, $\tau_t dh_t$, in period $t + 1$, and (ii) decrease k_t by $dk_t < 0$, i.e., increase c_t by $-dk_t$, and increase c_{t+1} by $-(1-d)dk_t$. The change in utility associated with (i) is $-\alpha_t dh_t + \beta E_t(b_{t+1} - c_{t+1})\tau_t dh_t$, which is positive by (A.2). The change in utility associated with (ii) is $-(b_t - c_t)dk_t + \beta E_t(b_{t+1} - c_{t+1})(1-d)dk_t$. This is also positive by (A.2). We conclude that if (A.2) is true and an interior optimum exists, then it must satisfy (A.1).

Next we show that (A.1) implies (A.2). First, $\alpha_t > 0$ implies that if (A.1) is true, then $\delta k_t = \tau_t h_t$. By the assumption of a unique interior maximum, utility must fall upon reversing strategy (i) with $dh_t < 0$, or reversing strategy (ii) with $dk_t > 0$. These imply the left and right inequalities, respectively, in (A.2). Q.E.D.

Proposition 1 informs us that (A.2) is necessary and sufficient to rule out capital underutilization on the optimal path. While we cannot guarantee (A.2) with probability 1 in the stochastic version of the model, a suitable choice of initial conditions and constant terms will make it highly unlikely that (A.2) is violated, for sufficiently small shocks. For example, in the DRW model, a sufficient condition for (A.2) is that $B_t > 0$ for all t , where

$$(A.3) \quad B_t \equiv b_0 - c_0 - H/(\phi-1) + u_1 + u_2 + \dots + u_t, \\ = \frac{(1-\beta)b}{(1-\beta\phi)} - e_{p0} + (1-\delta)k_{-1} - \frac{1-\beta}{(\phi-1)(1-\beta\phi)} H + u_1 + u_2 + \dots + u_t$$

for all t . Here, $u_t \equiv -u_{e_p, t}$. Also, the second equality in (A.3) uses (2.12) and $b_t = \phi^t b$. In the deterministic version of the model (i.e., $u_t \equiv 0$), the condition $B_t > 0$ is satisfied for large enough b and k_{-1} , and small enough H . We can make the probability of $B_t < 0$ arbitrarily small in the stochastic version of the model by making the variance of the shocks sufficiently close to zero.

We now show that $B_t > 0$ guarantees (A.2) in the DRW model. By recursively solving (2.15), we see that

$$(A.4) \quad b_t - c_t = B_t + H_t/(\phi-1).$$

Also, it is easy to verify that $\beta\phi < 1$ implies

$$(A.5) \quad \phi/(\phi-1) > 1/(\beta\bar{\delta}).$$

By (2.15) and (A.4),

$$(A.6) \quad \begin{aligned} E_t(b_{t+1}-c_{t+1}) - H_t/(\beta\bar{\delta}) &= b_t - c_t + H_t - H_t/(\beta\bar{\delta}) \\ &= B_t + \{[\phi/(\phi-1)]-1/(\beta\bar{\delta})\}H_t. \end{aligned}$$

Also,

$$(A.7) \quad \begin{aligned} (b_t - c_t)/[\beta(1-d)] - E_t(b_{t+1}-c_{t+1}) \\ = \{1/[\beta(1-d)]-1\}B_t + [\phi/(\phi-1)]\{1/[\phi\beta(1-d)]-1\}H_t. \end{aligned}$$

Combining (A.6) and (A.7), and using (A.5), we obtain (A.2), if $B_t > 0$.

Appendix B
Derivation of (2.11) and (2.12)

The social planner chooses a contingency plan for capital to maximize (2.8) subject to (2.5). The resulting stochastic Euler equation is:

$$E_t \{ [1 - \delta\beta L^{-1}] [1 - \delta L] k_t \} = E_t \{ [1 - \delta\beta L^{-1}] [e_t - b_t] - H_t \}$$

or, since $\delta\beta = 1$,

$$(B.1) \quad E_t \{ [1 - L^{-1}] [1 - \delta L] k_t \} = E_t \{ [1 - L^{-1}] [e_t - b_t] - H_t \}.$$

Note that we can rewrite the characteristic polynomial of (B.1) as

$$[1 - Z^{-1}] [1 - \delta Z] = [\delta - Z^{-1}] [1 - Z].$$

The condition $\delta\beta = 1$ implies that constraint (2.5) is binding (see Hansen [1986]), so (B.1) can be solved by applying the forward operator $[\delta - L^{-1}]^{-1}$ to both sides of the equation, yielding:

$$\begin{aligned} k_t - k_{t-1} &= E_t \{ [\delta - L^{-1}]^{-1} [1 - L^{-1}] (e_t - b_t) - [\delta - L^{-1}]^{-1} H_t \} \\ &= E_t \frac{\beta}{1 - \beta L^{-1}} \{ (e_t - b_t) - (e_{t+1} - b_{t+1}) - H_t \} \\ &= \beta E_t \left\{ \sum_{j=0}^{\infty} \beta^j (e_{t+j} - b_{t+j}) - \sum_{j=1}^{\infty} \beta^{j-1} (e_{t+j} - b_{t+j}) - \sum_{j=0}^{\infty} \beta^j H_{t+j} \right\}. \end{aligned}$$

Rearranging terms,

$$(B.2) \quad k_t - k_{t-1} = (\beta - 1) E_t \sum_{j=0}^{\infty} \beta^j (e_{t+j} - b_{t+j}) + e_t - b_t - \beta E_t \sum_{j=0}^{\infty} \beta^j H_{t+j}.$$

Rewriting (B.2) using the notation defined in (2.9) results in (2.11). Equation (2.12) is obtained by substituting for k_t in (2.7), using (2.11).

Appendix C

Derivation of Decision Rules for the Continuous Time Model

This appendix provides an informal derivation of the decision rules, (4.10) - (4.12) for the continuous time planning problem, (4.8), and the identity, (4.7). Proceeding as in Hansen and Sargent (1980) we can show that the Euler equation for the social planner's problem is

$$(C.1) \quad D(D-\delta)k(t) = D[e(t)-b(t)] + H(t).$$

The unique solution to this problem which satisfies (4.5) is

$$(C.2) \quad Dk(t) = e(t) - b(t) - \delta E_t \int_0^{\infty} e^{-\delta\tau} \{e(t+\tau) - b(t+\tau)\} d\tau \\ - \delta E_t \int_0^{\infty} e^{-\delta\tau} H(t+\tau) d\tau.$$

This is easily shown to equal the first equation in (4.10) after the definition (4.9) is taken into account. The second equation in (4.10) is obtained by substituting the first into (4.7). Equation (4.11) is just the sum of the two equations in (4.10).

To derive (4.12), we first present some preliminary results regarding $x_p(t)$. Suppose the fundamental representation for $x(t)$ is $x(t) = C(D)\epsilon(t)$. Here, $C(s) = 0$ implies $\text{Real}(s) \leq 0$ and the poles of $C(s)$ lie in the closure of the left side of the complex plane (see Sargent [1982] for a discussion of the link between these conditions and $\epsilon(t)$ being fundamental for $x(t)$). Then,

$$(C.3) \quad x_p(t) = -\delta E_t \frac{1}{D - \delta} x(t) = -\delta E_t \frac{C(D)}{D - \delta} \epsilon(t) \\ = -\delta \frac{C(D) - C(\delta)}{D - \delta} \epsilon(t),$$

by a formula due to Hansen and Sargent (1980). Multiply both sides of (C.3) by $D - \delta$ and rearrange, to obtain

$$(C.4) \quad (D-\delta)x_p(t) + \delta x(t) = \delta C(\delta)\epsilon(t).$$

We identify $\mu_{x_p}(t)$ with $\delta C(\delta)\epsilon(t)$. To see why, first write,

$$(C.5) \quad x(t) = C(D)\epsilon(t) = \int_0^{\infty} c(\tau)\epsilon(t-\tau)d\tau,$$

where the function $c(\tau)$, $\tau \geq 0$ is uniquely defined by

$$(C.6) \quad C(s) = \int_0^{\infty} c(\tau)e^{-s\tau}d\tau, \text{ Real}(s) > 0.$$

Equation (C.5) defines c as the impulse response function of $x(t)$ to $\epsilon(t)$.

Consider the effect of a disturbance in $x(t)$ that is uncorrelated with $I(t-\tau)$, $\tau > 0$. This arises from a pulse in $\epsilon(t)$, which leads to a revision in the forecast of $x(t+\tau)$ in the amount $c(\tau)\epsilon(t)$ $\tau \geq 0$. The permanent value of this revision is $\delta C(\delta)\epsilon(t)$. Thus the effect of the pulse in $\epsilon(t)$ is to disturb $x_p(t)$ by $\delta C(\delta)\epsilon(t)$, which is why we identify $\delta C(\delta)\epsilon(t)$ with $\mu_{x_p}(t)$. We conclude that

$$(C.7) \quad (D-\delta)x_p(t) + \delta x(t) = \mu_{x_p}(t).$$

From (4.10),

$$(C.8) \quad Dc(t) = De_p(t) + Db(t) - Db_p(t) + \delta Dk(t) + DH_p(t)/\delta.$$

Equation (4.12) is obtained by first substituting the first equation in (4.10) into (C.8) and then making use of (C.7).

References

- Brayton, F., and Mankopf, E. (1985) "The MPS Model of the United States Economy," Board of Governors of the Federal Reserve System, Division of Research and Statistics, Washington, D.C., February.
- Campbell, J. Y. (1986) "Does Saving Anticipate Declining Labor Income? An Alternative Test of the Permanent Income Hypothesis," NBER Working Paper No. 1085.
- Christiano, L. (1984) "The Effects of Aggregation Over Time on Tests of the Representative Agent Model of Consumption," manuscript.
- Christiano, L. (1985) "A Critique of Conventional Treatments of the Model Timing Interval in Applied Econometrics," unpublished manuscript.
- Christiano, L., Eichenbaum, M., and Marshall, D. (1986) "The Permanent Income Hypothesis Revisited Technical Appendix," manuscript, 1986.
- Christiano, L., and Marshall, D. (1986) "A Note on the Estimation of Continuous Time Models from Temporally Averaged Data," manuscript.
- Dunn, K. B., and Singleton, K. J. (1986) "Modeling the Term Structure of Interest Rates Under Nonseparable Utility Functions and Durability of Goods," Journal of Financial Economics Vol. 17; 27-55.
- Durbin, J. (1961) "Efficient Fitting of Linear Models for Continuous Stationary Time Series from Discrete Data," Bulletin of the International Institute of Statistics, 38: 273-82.
- Eichenbaum, M., and Hansen, L. P. (1986) "Estimating Models with Intertemporal Substitution Using Aggregate Time Series Data," manuscript.
- Flavin, M. A. (1981) "The Adjustment of Consumption to Changing Expectations about Future Income," Journal of Political Economy, 89: 974-1009.
- Geweke, J. (1978) "Temporal Aggregation in the Multivariate Regression Model," Econometrica, 46: 643-62.

- Hall, R. E. (1978) "Stochastic Implications of the Life Cycle-Permanent Income Hypothesis: Theory and Evidence," Journal of Political Economy, 86: 971-87.
- Hall, R. E., and Mishkin, F. S. (1982) "The Sensitivity of Consumption to Transitory Income: Estimates from Panel Data on Households," Econometrica 50: 461-81.
- Hannan, E. J. (1970) Multiple Time Series, New York: Wiley.
- Hansen, L. P. (1986) "Econometric Modeling of Asset Pricing Under Rational Expectations," manuscript.
- Hansen, L. P., and Sargent, T. J. (1980) "Methods for Estimating Continuous Time Rational Expectations Models From Discrete Time Data," Federal Reserve Bank of Minneapolis, Staff Report No. 59.
- Hansen, L. P., and Sargent, T. J. (1981) "Formulating and Estimating Continuous Time Rational Expectations Models from Discrete Data," manuscript.
- Hansen, L. P., and Sargent, T. J. (1981a) "Exact Linear Rational Expectations Models: Specification and Estimation," Federal Reserve Bank of Minneapolis Staff Report 71, September.
- Hansen, L. P., and Sargent, T. J. (1983) "The Dimensionality of the Aliasing Problem in Models with Rational Spectral Densities," Econometrica, 51: 377-88.
- Hansen, L. P., and Singleton, K. J. (1982) "Generalized Instrumental Variables Estimation of Nonlinear Rational Expectations Models," Econometrica, 50: 1029-54.
- Hansen, L. P., and Singleton, K. J., (1983) "Stochastic Consumption, Risk Aversion and the Temporal Behavior of Asset Returns," Journal of Political Economy, 91: 249-65.

- Hayashi, F. (1982) "The Permanent Income Hypothesis: Estimation and Testing by Instrumental Variables," Journal of Political Economy, 90: 895-916.
- Kydland, F. E., and Prescott, E. C. (1982) "Time to Build and Aggregate Fluctuations," Econometrica, 51: 1821-37.
- Lissitz, R. W. (1972) "Comparison of the Small Sample Power of the Chi-Square and Likelihood Ratio Tests on the Assumptions for Stochastic Models," Journal of the American Statistical Association, September, Vol. 67; 574-577.
- Long, J., and Plosser, C. (1983) "Real Business Cycles," Journal of Political Economy, 91: 39-69.
- Lucas, R. E. Jr. (1985) "Models of Business Cycles," manuscript.
- Lucas, R. E. Jr., and Prescott, E. C. (1971) "Investment Under Uncertainty," Econometrica 39: 659-81.
- Mankiw, N., and Shapiro, M. D. (1985) "Trends, Random Walks, and Tests of the Permanent Income Hypothesis," Journal of Monetary Economics, 16: 165-174.
- Marcet, A. (1986) "Temporal Aggregation of Economic Time Series," manuscript.
- Musgrave, J. (1980) "Government Owned Fixed Capital in the United States," Survey of Current Business, March: 33-43.
- Nelson, C. R. (1985) "A Reappraisal of Recent Tests of the Permanent Income Hypothesis," NBER Working Paper No. 1687.
- Phillips, A. W. (1958) "The Estimation of Parameters in Systems of Stochastic Differential Equations," Biometrika, 59: 657-65.
- Rozanov, Y. A. (1967) Stationary Random Processes, Holden-Day.
- Sargent, T. J. (1982) "Preliminary Introduction to Continuous Time Stochastic Processes," manuscript.
- Sargent, T. J. (1986) "Equilibrium Investment Under Uncertainty, Measurement Errors, and the Investment Accelerator," manuscript.

- Sims, C. A. (1971) "Discrete Time Approximations to Continuous Time Lag Distributions in Econometrics," Econometrica, 39: 545-564.
- Sims, C. A. (1980) "Macroeconomics and Reality," Econometrica, 48: 1-48.
- Whittle, P. (1953) "The Analysis of Multiple Stationary Time Series," Journal of the Royal Statistical Society, series B, 15: 125-39.
- Working, H. (1960) "Note on the Correlation of First Differences of Averages in a Random Chain," Econometrica, 28: 374-401.
- Zeldes, S. (1985) "Consumption and Liquidity Constraints: An Empirical Investigation," Rodney L. White Center for Financial Research Paper No. 24-85.