NBER WORKING PAPER SERIES

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A MARKOV CHAIN ANALYSIS

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Working Paper 22009
http://www.nber.org/papers/w22009

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
February 2016

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NBER Working Paper No. 22009
February 2016
JEL No. E31,E32,E52,E58

ABSTRACT

A large body of recent empirical work on inflation dynamics documents that current real variables (like unemployment or output gaps) have little explanatory power for future inflation. Motivated by these findings, I explore the properties of a wide class of models in which inflation expectations respond little, if at all, to real economic conditions. In this general context, I examine Markov equilibria to games in which the relevant forcing processes are Markov chains and the central bank chooses a short-term nominal interest rate at each date subject to a lower bound. I construct a simple numerical algorithm to solve for such Markov equilibria. I apply the algorithm to a numerical example. In the example, the economy can experience long periods of what looks like secular stagnation because households believe that there is a significant risk of a crisis (that is, a sharp decline in economic activity). Within the example, there are large benefits to being able to reduce the lower bound on the short-term nominal interest rate by as little as fifty basis points.

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1 Introduction

Over the past twenty years, a large body of work has documented that in US data, real variables like unemployment and output "gaps" have little additional forecasting power for future inflation relative to current and lagged values of inflation.[1] Motivated by this line of research, I analyze a class of economic models in which inflation expectations respond sluggishly, if at all, to real economic outcomes. Within the models, a central bank chooses a short-term nominal interest rate at each date subject to a zero lower bound. I model the various forcing processes as Markov chains, and focus on Markov equilibria to the resultant dynamic game. I provide an easily verifiable sufficient condition for the existence and uniqueness of such equilibria. I construct a simple numerical algorithm, analogous to the linear algebra solution methods in Mehra-Prescott (1985), that rapidly solves for the unique Markov equilibrium, while taking full account of the zero lower bound.

I apply the algorithm to a simple numerical example. In the example, in "normal" times, households assign a low probability to a crisis (that is, a sharp downturn in real economic outcomes consistent with target inflation). The crisis is (on average) short-lived, but is followed by a sustained *fear* period. In the fear period, growth normalizes, but households remain concerned about the risk of a return to the crisis state. The Markov equilibrium in this example involves the central bank's being constrained by the zero lower bound during the crisis period *and* the fear period. The economy exhibits a pronounced shortfall in economic activity during both periods (but not during the normal growth period, when households view the risk of a crisis as small).

The focus on Markov equilibria eliminates the use of so-called "Odyssean" forward guidance (Justiniano, et. al., 2012). Within the numerical example, forward guidance of plausible average durations (less than a decade) is ineffective. However, I document that relatively small relaxations of the zero lower bound via negative nominal interest rates can enhance economic efficiency in both the crisis and fear periods greatly.

[1] See Atkeson and Ohanian (2001) and Stock and Watson (2009) for example.

The class of models that I investigate is highly general. Equilibrium allocations must satisfy two sets of restrictions (but may satisfy many others). The first set is a collection of Euler equations that restricts the stochastic evolution of the cross-sectional average of the realized marginal utility of consumption. The second is a collection of restrictions between expected future inflation and the current gap between the (average) marginal utility of consumption and its *natural* level. Both of these sets of restrictions are assumed to be structural, in the sense that they are required to be invariant to the policy choices of the central bank.

There are also three forcing processes that are invariant to the policy choices of the central bank. The first is a stochastic trend in the natural level of the average marginal utility of consumption. The evolution of this stochastic trend gives rise to variation in what is usually termed the natural or neutral real rate of interest, even though households' psychic discount factors do not fluctuate.

The second forcing process is a disturbance to the otherwise fixed relationship between current real outcomes and inflation expectations. Loosely speaking, this process can be viewed as a "Phillips curve" shifter. The third forcing process is a shock to ex-post inflation realizations. All of these shocks are potentially correlated with one another. Hence, inflation can be highly persistent and exhibit non-trivial conditional heteroskedasticity (so that the inflation risk premium varies over time).

My approach to modeling the evolution of inflation expectations is related to, but distinct from, that followed in the vast sticky price literature. The more standard approach is grounded in the twin assumptions of Calvo pricing (or Rotemberg pricing or menu costs) and rational expectations. The model of firm pricing decisions generates a connection between real conditions and inflation *outcomes*. Rational expectations about those firm pricing decisions then generates a connection between real conditions and inflation *expectations*. The models that I study in this paper abstract from the specifics of firm pricing, and so are considerably more agnostic about the source of the connection between real outcomes and inflation expectations.

Unlike Eggertsson and Woodford (2003) or Werning (2012), I do not limit attention to the behavior of the economy after it hits the zero lower bound. However, these authors are focusing on the benefits of commitment, which I ignore through my focus on Markov equilibria. In its focus on Markov equilibria, my paper is related to the work of Adam and Billi (2007) and Armenter (2015). It is also related to recent work by Hills, Nakata, and Schmidt (2015) that explores how the zero lower bound has an adverse effect on the economy even when it is not binding. Michau (2015) examines the costs and benefits of various forms of fiscal policy during long stays at the zero lower bound.

2 Model

In this section, I present a general class of models in which allocations are required to satisfy at least two sets of restrictions. The first set of restrictions are a collection of aggregate Euler equations. The second set of restrictions describes relationships between real outcomes and inflation expectations.

2.1 Model Generalities

I consider a discrete-time infinite horizon economy in which all agents are infinitely-lived, have a common utility discount factor δ, and have common beliefs. Suppose that any equilibrium allocation in the economy satisfies the aggregate Euler equations:

$$m_t = \delta R_t E_t\{m_{t+1}\Pi_{t+1}^{-1}\},\ t = 1, ..., \infty \tag{1}$$

where m_t denotes the cross-person period t average of the marginal utility of consumption associated with that allocation. (Equilibrium allocations might, of course, satisfy additional restrictions.) In (1), R_t denotes the gross nominal interest rate in period t, Π_{t+1} is the gross inflation rate from period t to period $(t + 1)$, and E_t represents the agents' common expectation of period $(t+1)$ random variables conditional on information available at date t.

These aggregate restrictions assume that agents do not face binding short-sales constraints on their holdings of short-term assets.

I model the evolution of inflation as follows:

$$\Pi_{t+1}^{-1} = \zeta(m_t/m_t^{nat}; \nu_t)^{-1}\xi_{t+1}, t \geq 1 \tag{2}$$

where $\{\nu_t, m_t^{nat}, \xi_{t+1}\}_{t=1}^{\infty}$ are a triple of exogenous positive stochastic processes. I assume that:

$$E_t\xi_{t+1} = 1 \text{ for all } t$$

This assumption implies that the agents' (harmonic) one-period-ahead expectation of (gross) inflation at date t is given by:

$$(E_t\Pi_{t+1}^{-1})^{-1} = \zeta(m_t/m_t^{nat}; \nu_t) \tag{3}$$

The process ν represents exogenous shocks to inflation expectations. The function ζ describes the influence of current real conditions, as measured by equilibrium marginal utility m_t normalized by the exogenous *natural* marginal utility process m_t^{nat}, on inflation expectations. (By exogenous, I mean that the natural marginal utility process is invariant to decisions about monetary policy by the central bank.) Going forward, I will refer to the stochastic process $\widehat{m} \equiv m/m^{nat}$ as being the equilibrium marginal utility *gap*. Along those lines, it is useful to rewrite (2) as:

$$\Pi_{t+1}^{-1} = \zeta(\widehat{m}_t; \nu_t)^{-1}\xi_{t+1}, t \geq 1 \tag{4}$$

It is helpful to substitute (4) into (1). To that end, I assume that there exists a function $\Psi(.; \nu)$ such that, for all ν, Ψ is strictly increasing in its first argument and such that:

$$\Psi(\widehat{m}\zeta(\widehat{m}; \nu); \nu) = \widehat{m} \text{ for all } \nu \text{ and all } \widehat{m} \text{ in } [0, \infty)$$

Let $\lambda_{t+1}^{nat} \equiv m_{t+1}^{nat}/m_t^{nat}$ denote the gross growth rate of natural marginal utility. Then, we can use Ψ and λ to rewrite (1) as:

$$\widehat{m}_t = \Psi(R_t \delta E_t(\widehat{m}_{t+1}\xi_{t+1}\lambda_{t+1}^{nat}); \nu_t) \tag{5}$$

This nonlinear difference equation (5) will be the foundation of the remainder of the paper. In order to simplify the analysis of the equation, I assume that Ψ is Lipschitz with modulus B so that:

$$|\Psi(y;\nu) - \Psi(y';\nu)| \ \leq B|y-y'| \text{ for all } (y,y') \text{ in } R_+^2 \text{and all } \nu$$

Notice that if ζ is constant at ζ^*, so that there is no connection between real outcomes and inflation expectations, then Ψ exists and is Lipschitz with modulus $1/\zeta^*$.[2]

2.2 Markov Models

In this subsection, I specialize the model by imposing a Markov chain structure on the various forcing processes. I then define and characterize a Markov equilibrium for two distinct central bank objectives within this model.

2.2.1 Markov Structure

Suppose that agents believe that s_t follows a Markov chain with transition matrix P and state space $S = \{1, 2, .., J\}$. Agents all observe s_t at each date t. However, their observations do not lead agents to change their beliefs that s_t evolves according to the Markov chain defined by (P, S).

I suppose that the shocks $(\nu, \lambda^{nat}, \xi)$ to inflation expectations, natural marginal utility, and inflation realizations are all governed by the underlying Markov chain s_t. More specifically,

[2]More generally, the strictly increasing function Ψ exists and is Lipschitz with modulus B if $inf_\nu inf_{\widehat{m}\geq 0}\zeta'(\widehat{m};\nu)\widehat{m} + \zeta(\widehat{m};\nu) = 1/B$.

I assume that there are sets:

$$V = \{\bar{v}_j\}_{j=1}^J; \Lambda = \{\bar{\lambda}_{ij}^{nat}\}_{i,j=1}^J; \Xi = \{\bar{\xi}_{ij}\}_{i,j=1}^J$$

such that:

$$\nu_t = \bar{\nu}_{s_t}; \lambda_t^{nat} = \bar{\lambda}_{s_{t-1}s_t}^{nat}; \xi_t = \bar{\xi}_{s_{t-1}s_t}$$

The assumption that $E_t \xi_{t+1} = 1$ implies that:

$$\sum_{j=1}^J P_{ij}\bar{\xi}_{ij} = 1 \text{ for all } i$$

2.2.2 Games and Equilibrium

In this section, I consider two distinct policy games and study the unique Markov equilibrium to each. The games share the following common structure. At each date, the central bank chooses R_t, taking as given the Euler equation restrictions (5) described above and the Markov chain law of motion for the exogenous variables. I will describe the central bank's objective in these games later. For now, it suffices to say that I focus on Markov equilibria, in which the marginal utility gap $\widehat{m}_t$ is a function only of s_t. The aggregate Euler equation (5) can then be rewritten as a system of J nonlinear equations:

$$\widehat{m}_i = \Psi(\delta R_i \sum_{j=1}^J P_{ij}\bar{\lambda}_{ij}^{nat}\bar{\xi}_{ij}\widehat{m}_j; \bar{\nu}_i), i = 1, ..., J \tag{6}$$

It will be useful to rewrite this system (6) of nonlinear equations as:

$$\widehat{m}_i = \Psi(R_i Q_i \widehat{m}; \bar{\nu}_i), i = 1, ..., J \tag{7}$$

where the $J \times J$ matrix Q is defined by:

$$Q_{ij} = \delta P_{ij}\bar{\lambda}_{ij}^{nat}\bar{\xi}_{ij}$$

The notation Q_i refers to the ith row of the matrix Q.

Note that in a Markov equilibrium, the (logged) equilibrium level of marginal utility is given by the sum $(ln(\widehat{m}) + ln(m_{nat}))$ of two processes. The first process is endogenous and stationary. The second process is exogenous and non-stationary. Hence, logged equilibrium marginal utility shares a common, exogenous, stochastic trend with logged natural marginal utility.

In the next two subsections, I describe two dynamic stochastic policy games, and their Markov equilibria.

2.2.3 Game 1: Targeting A Desired Real Outcome

In Game 1, the central bank seeks to target a desired real outcome that is uniquely identified with the marginal utility process m^{des}. I define $\widehat{m}^{des} = m^{des}/m^{nat}$ to be the desired marginal utility gap (more precisely, the level of marginal utility associated with the desired allocation, relative to the natural level of marginal utility). I assume that the process $\widehat{m}^{des}$ is governed by the underlying Markov chain s_t, so that there exists a set $M^{des} = \{\bar{m}_j^{des}\}_{j=1}^J$ and $\widehat{m}_t^{des} = \bar{m}_{s_t}^{des}$. This assumption implies that the central bank's desired level of marginal utility shares a common, exogenous, stochastic trend with logged natural marginal utility. This assumption is consistent with the central bank does not seek to offset *permanent* shocks to the level of natural marginal utility as being desirable.

Then, at date t, the central bank chooses the nominal interest rate R_t so as to minimize a loss function $\Gamma_1(\widehat{m}_t - \widehat{m}_t^{des})$, where Γ_1 has a global minimum when its argument is zero. It faces the constraint that the gross nominal interest rate R_t is required to be no smaller than one.

I define a Markov equilibrium of this game to be a vector $(\widehat{m}_i^*, R_i^*)_{i=1}^J$ such that:

$$\begin{aligned} \text{For all } i = 1, ..., J, \; R_i^* &\in \arg\min_R \Gamma_1(\Psi(RQ_i\widehat{m}^*; \bar{\nu}_i) - \bar{m}_i^{des}) \\ & s.t. \; R \geq 1 \end{aligned}$$

and:

$$\widehat{m}_i^* = \Psi(R_i^* Q_i \widehat{m}^*) \text{ for all } i = 1, ..., J$$

The main restriction in a Markov equilibrium is that, when choosing at date t, the central bank treats the future evolution of marginal utility gaps as outside of its control.

2.2.4 Game 2: Targeting Expected Inflation

In the second game, the central bank's loss function Γ_2 is defined over the difference between the expected inflation rate and a target inflation rate Π^*. The loss function is at a global minimum when its argument is zero. In this game, I define a Markov equilibrium to be a vector $(\widehat{m}_i^*, R_i^*)_{i=1}^J$ such that:

$$\begin{aligned} \text{For all } i,\ R_i^* &\in \arg\min_R \Gamma_2(\zeta(\Psi(RQ_i\widehat{m}^*; \bar{\nu}_i) - \Pi^*) \\ & s.t.\ R \geq 1 \end{aligned}$$

and:

$$\widehat{m}_i^* = \Psi(R_i^* Q_i \widehat{m}^*) \text{ for all } i = 1, ..., J$$

3 Results

In this section, I prove three results about Markov equilibria in each of the two games. The first result is that there is a unique equilibrium. The second result is that, in the unique equilibrium, there is at least one state in which the zero lower bound does not bind. The third result provides a condition such that the zero lower bound does in fact bind in a given state.

All three of these results will hinge on the following assumption. Recall that, in a Markov equilibrium, we can write the aggregate Euler equation as:

$$\widehat{m}_i^* = \Psi(R_i^* Q_i \widehat{m}^*; \bar{\nu}_i) \text{ for all } i = 1, ..., J$$

In order to prove the relevant results, we need an assumption that limits the dependence of the current marginal utility gap on the future marginal utility gap vector. This assumption takes the following form. Recall that the function Ψ is Lipschitz with modulus B. Let $||Q||$ be the norm of Q associated with the Euclidean norm on R^J. (This matrix norm is equal to the square root of the maximal eigenvalue of $Q'Q$.) Then, I impose the following restriction on B and Q.

Assumption 1. $B||Q|| < 1$

This assumption is the foundation for all of the analysis that follows.[3]

At the end of the section, I connect the characterizations of equilibria in the two games to the notions of divine coincidence and the natural rate of interest often used in the New Keynesian literature.

3.1 Results for Game 1

I begin by establishing uniqueness of Markov equilibrium in Game 1.

Proposition 1. *Suppose Assumption 1 is satisfied, and that the central bank's loss function* Γ_1 *is strictly quasi-convex. Then there is a unique Markov equilibrium* $(\widehat{m}^*, R^*)$ *to game 1 in which the central bank targets its desired allocation. The Markov equilibrium is characterized*

[3] In Kocherlakota (2016), I examine finite-horizon versions of models with near-vertical Phillips curves. I show that, given a lower bound on the nominal interest rate, these models exhibit a wide class of sunspot equilibria. These sunspot equilibria feature extreme dependence of current outcomes on long-run outcomes when the Phillips curve is close to vertical. Assumption 1 rules out the possibility that the Phillips curve is near-vertical in the class of models under study in this paper.

as the solution to the equations:

$$\begin{aligned}\widehat{m}_i^* &= \max(\bar{m}_i^{des}, \Psi(Q_i\widehat{m}^*;\bar{\nu}_i)), i=1,...,J \\ \widehat{m}_i^* &= \Psi(R_i^* Q_i\widehat{m}^*;\bar{\nu}_i), i=1,...J\end{aligned}$$

Proof. I first show that any solution to the central bank's decision problem when $s_t = i$ satisfies the two sets of restrictions. Because Γ_1 is strictly quasi-convex, there is a unique solution to the central bank's decision problem for any $s_t = i$. There are two possibilities. First, suppose:

$$\Psi(Q_i\widehat{m}^*;\bar{\nu}_i) \leq \bar{m}_i^{des}$$

Then, since Ψ is strictly increasing, we know that the zero lower bound does not bind. As a result, the central bank can achieve a global minimum by setting $\widehat{m}_i^* = 1$. Alternatively, suppose:

$$\Psi(Q_i\widehat{m}^*;\bar{\nu}_i) > \bar{m}_i^{des}$$

Then, the zero lower bound binds, and the central bank's solution is to set $R = 1$. Since Ψ is strictly increasing, it follows that the unique solution to the central bank's decision problem when $s_t = i$ satisfies the system of nonlinear equations in the Proposition.

I next show that there exists a unique vector $\widehat{m}$ that solves this system of nonlinear equations. Define a nonlinear operator T from $\mathbb{R}_+^J$ into $\mathbb{R}_+^J$ by

$$T(\widehat{m})_i = \max(\bar{m}_i^{des}, \Psi(Q_i\widehat{m};\bar{\nu}_i))$$

The marginal utility gap vector in a Markov equilibrium is a fixed point of T. I complete the proof of uniqueness by showing that T is a contraction with respect to the Euclidean norm in $\mathbb{R}^J_+$. Pick any $(\widehat{m}, \widehat{m}')$ in $\mathbb{R}^J_+$. Then:

$$
\begin{aligned}
& [\sum_{i=1}^{J} |T(\widehat{m})_i - T(\widehat{m}')_i|^2]^{1/2} \\
= \quad & [\sum_{i=1}^{J} max(\bar{m}_i^{des}, \Psi(Q_i\widehat{m}; \bar{\nu}_i)) - max(\bar{m}_i^{des}, \Psi(Q_i\widehat{m}'; \bar{\nu}_i))|^2]^{1/2} \\
\leq \quad & [\sum_{i=1}^{J} |\Psi(Q_i\widehat{m}; \bar{\nu}_i) - \Psi(Q_i\widehat{m}'; \bar{\nu}_i)|^2]^{1/2} \\
\leq \quad & B[\sum_{i=1}^{J} |Q_i\widehat{m} - Q_i\widehat{m}'|^2]^{1/2}, \text{ since } \Psi \text{ is Lipschitz with modulus } B \\
\leq \quad & B||Q||[\sum_{j=1}^{J} |\widehat{m}_j - \widehat{m}'_j|^2]^{1/2}, \text{from the definition of the norm of } Q
\end{aligned}
$$

The proposition assumes that $B||Q|| < 1$. It follows that T is a contraction with respect to the Euclidean norm, and we can conclude that there is a unique fixed point to T. □

Under Assumption 1, we can prove that, in any Markov equilibrium, there exists some state in which the central bank is able to achieve its desired outcome.

Proposition 2. *Suppose Assumption 1 is satisfied and that the central bank's loss function Γ_1 is strictly quasi-convex. In the unique Markov equilibrium $(\widehat{m}^*, R^*)$ to game 1, there exists some i such that $R_i^* > 1$.*

Proof. Suppose not. Then, it must be true that:

$$\widehat{m}_i^* = \Psi(Q_i\widehat{m}^*; \bar{\nu}_i) \text{ for all } i = 1, .., J$$

Note that $\Psi(0) = 0$. Hence, $\Psi(Q_i m; \bar{\nu}_i) \leq B|Q_i m|$. This implies that:

$$\begin{aligned}
[\sum_{i=1}^{J} |\widehat{m}_i^*|^2]^{1/2} &\leq B[\sum_{i=1}^{J} |Q_i \widehat{m}^*|^2]^{1/2} \\
&= B||Q||[\sum_{i=1}^{J} |\widehat{m}_i^*|^2]^{1/2} \\
&< [\sum_{i=1}^{J} |\widehat{m}_i^*|^2]^{1/2}
\end{aligned}$$

This is a contradiction. □

Proposition 2 rules out the possibility that the economy is stuck forever at the zero lower bound in a Markov equilibrium. This result stands in contrast with the findings of Benhabib, Schmitt-Grohe, and Uribe (2001). They study a class of models in which there is a steady-state equilibrium in which the nominal interest rate is always zero. The difference in results is attributable to differences in the treatments of inflation expectations. In Benhabib, Schmitt-Grohe, and Uribe (2001), the zero lower bound binds in the long run because expected inflation converges to a low level. In contrast, in this paper, Assumption 1 ensures that expected inflation remains sufficiently high so that the zero lower bound can't always bind.

The next proposition provides a sufficient condition on exogenous parameters that ensures that, in a Markov equilibrium, there is a state in which the central bank is at the zero lower bound.

Proposition 3. *Suppose that the loss function Γ_1 is strictly quasi-convex and there exists k such that:*

$$\bar{m}_k^{des} < \Psi(Q_k \bar{m}^{des}; \bar{\nu}_k)$$

Then, in any Markov equilibrium $(\widehat{m}^*, R^*)$ of game 1, $R_k^* = 1$.

Proof. Since the loss function Γ_1 is strictly quasi-convex, a Markov equilibrium is characterized as the solution to the equations:

$$\begin{aligned}
\widehat{m}_i^* &= \max(\bar{m}_k^{des}, \Psi(Q_i \widehat{m}^*; \bar{\nu}_i)), i = 1, ..., J \\
\widehat{m}_i^* &= \Psi(R_i^* Q_i \widehat{m}^*; \bar{\nu}_i), i = 1, ...J
\end{aligned}$$

These equations imply that if the assumption in the proposition is satisfied :

$$\begin{aligned}
\widehat{m}_k^* &= \max(\bar{m}_k^{des}, \Psi(Q_k \widehat{m}^*; \bar{\nu}_k)) \\
&\geq \max(\bar{m}_k^{des}, \Psi(Q_k \bar{m}^{des}; \bar{\nu}_k)) \text{ because } \widehat{m}_j^* \geq \bar{m}^{des} \text{ for all } j \\
&> \bar{m}_k^{des}
\end{aligned}$$

and so R_k^* must equal one. □

As we shall see later, the converse to Proposition 3 may not be true.

3.2 Results about Game 2

In this section, I prove analogous results about Game 2, in which the central bank targets an expected inflation rate of Π^*. The results are analogous because Game 2, is in some sense, simply a special case of Game 1 in which the central bank's desired marginal utility process is given by $\zeta^{-1}(\Pi^*; \nu)$.

I begin by establishing existence and uniqueness of Markov equilibrium.

Proposition 4. *Suppose Assumption 1 is satisfied, that the loss function Γ_2 is quasi-convex, and the function ζ (that maps marginal utility into expected inflation) is strictly increasing or strictly decreasing with respect to marginal utility. Then there is a unique Markov equilibrium $(\widehat{m}^*, R^*)$ to Game 2. The Markov equilibrium is the unique solution to the system of nonlinear* *equations:*

$$\begin{aligned}
\widehat{m}_i^* &= \max(\zeta^{-1}(\Pi^*;\bar{\nu}_i), \Psi(Q_i\widehat{m}^*;\bar{\nu}_i)), i = 1, ..., J \\
\widehat{m}_i^* &= \Psi(R_i^* Q_i\widehat{m}^*;\bar{\nu}_i)
\end{aligned}$$

Proof. The function ζ can be strictly increasing or strictly decreasing in its first argument. Suppose first that ζ is strictly decreasing. Then, since Γ_2 is strictly quasi-convex, the function $\Gamma_2(\zeta(m;\bar{\nu}_i) - \Pi^*)$ is strictly quasi-convex as a function of m. It follows that at any date t, the central bank's decision problem has a unique solution. There are two possibilities. The first is that:

$$\zeta(\Psi(Q_i\widehat{m}^*;\bar{\nu}_i);\bar{\nu}_i) \geq \Pi^*$$

Then, since Ψ is strictly increasing and ζ is strictly decreasing, we know that the zero lower bound does not bind and the central bank can achieve a global minimum by setting R_i^* sufficiently high so that:

$$\zeta(\Psi(R_i^* Q_i\widehat{m}^*;\bar{\nu}_i);\bar{\nu}_i) = \Pi^*$$

and:

$$m_i^* = \zeta^{-1}(\Pi^*;\bar{\nu}_i)$$

Alternatively, suppose:

$$\zeta(\Psi(Q_i\widehat{m}^*;\bar{\nu}_i);\bar{\nu}_i) < \Pi^*$$

Then, higher values of R_i than one will increase the central bank's loss, and the central bank should set $R_i^* = 1$. It follows that any Markov equilibrium $(\widehat{m}_i^*, R_i^*)_{i=1}^J$ is fully characterized by the nonlinear equations:

$$\begin{aligned}
\widehat{m}_i^* &= \max(\zeta^{-1}(\Pi^*;\bar{\nu}_i), \Psi(Q_i\widehat{m}^*;\bar{\nu}_i)) \\
\widehat{m}_i^* &= \Psi(R_i^* Q_i\widehat{m}^*;\bar{\nu}_i)
\end{aligned}$$

Suppose next that ζ is strictly increasing. Again, because Γ_2 is strictly quasi-convex, there is a unique solution to the central bank's decision problem. There are two possibilities. The first is that:

$$\zeta(\Psi(Q_i\widehat{m}^*);\bar{\nu}_i) \leq \Pi^*$$

The central bank can achieve a global minimum by setting R_i^* sufficiently high that $\zeta(\Psi(R_i^*Q_i\widehat{m}^*;\bar{\nu}_i);\bar{\nu}_i) = \Pi^*$ and $m_i^* = \zeta^{-1}(\Pi^*;\bar{\nu}_i)$. Alternatively, suppose:

$$\zeta(\Psi(Q_i\widehat{m}^*;\bar{\nu}_i);\bar{\nu}_i) > \Pi^*$$

Then, higher values of R_i than one will increase the central bank's loss, and the central bank should set $R_i^* = 1$. It follows that any Markov equilibrium $(\widehat{m}_i^*, R_i^*)_{i=1}^J$ is fully characterized by the nonlinear equations:

$$\begin{aligned} \widehat{m}_i^* &= \max(\zeta^{-1}(\Pi^*;\bar{\nu}_i), \Psi(Q_i\widehat{m}^*;\bar{\nu}_i)) \\ \widehat{m}_i^* &= \Psi(R_i^*Q_i\widehat{m}^*;\bar{\nu}_i) \end{aligned}$$

Given the restriction on Q assumed in the proposition, we can use the contraction mapping logic from the proof of Proposition 1 to establish that there is a unique solution to these equations. □

Note that Proposition 4 is agnostic about whether ζ - the function that links the equilibrium marginal utility gap to inflation expectations - is strictly increasing or decreasing. Hence, higher inflation expectations can be associated with higher or lower levels of real economic activity.

The next proposition establishes that, as in game 1, there is no Markov equilibrium in which the zero lower bound always binds.

Proposition 5. *Suppose Assumption 1 is satisfied and suppose too that the central bank's loss function Γ_2 is strictly quasi-convex. In the unique Markov equilibrium $(\widehat{m}^*, R^*)$, there* *exists some i such that $R^*_i > 1$.*

Proof. Same as the proof of Proposition 2. $\square$

The next proposition provides a necessary condition on exogenous parameters that ensures that the central bank is at the zero lower bound in a given state in a Markov equilibrium to game 2.

Proposition 6. *Suppose that the loss function Γ_2 is strictly quasi-convex and the function ζ is invertible. Define the vector $\bar{\mu} \equiv (\zeta^{-1}(\Pi^*; \bar{\nu}_k))_{k=1}^{J},$; and suppose there exists some k such that:*

$$\bar{\mu}_k < \Psi(Q_k \bar{\mu}; \bar{\nu}_k)$$

Then, in any Markov equilibrium $(\widehat{m}^*, R^*)$ of game 2, $R^*_k = 1$.

Proof. Since the loss function Γ_2 is strictly quasi-convex, a Markov equilibrium is characterized as the solution to the equations:

$$\begin{aligned}
\widehat{m}^*_i &= \max(\bar{\mu}_i, \Psi(Q_i \widehat{m}^*; \bar{\nu}_i)), i = 1, ..., J \\
\widehat{m}^*_i &= \Psi(R^*_i Q_i \widehat{m}^*; \bar{\nu}_i), i = 1, ...J
\end{aligned}$$

In state k:

$$\begin{aligned}
\widehat{m}^*_k &= \max(\bar{\mu}_k, \Psi(Q_k \widehat{m}^*; \bar{\nu}_k)) \\
&\geq \max(\bar{\mu}_k, \Psi(Q_k \bar{\mu}; \bar{\nu}_k)) \text{ because } \widehat{m}^*_j \geq \bar{\mu}_j \text{ for all } j \\
&> \bar{\mu}_k
\end{aligned}$$

and so R^*_k must equal one. $\square$

3.3 Divine Coincidence

In simple New Keynesian models, the equilibrium allocation is efficient when the central bank achieves its inflation target in all dates and states. This situation has been termed a divine coincidence. Along those same lines, I say that there is a *divine coincidence* in this class of models if:

$$\zeta(\bar{m}_i^{des}; \bar{\nu}_i) = \Pi^* \text{ for all } i = 1, ..., J$$

so that the central bank's desired allocation is consistent with achieving its inflation objective (in an expected sense) in every date and state.

The equilibria of the two games are identical if there is a divine coincidence.

Proposition 7. *Suppose that ζ is invertible and that Assumption 1 is satisfied. Then, the unique Markov equilibria of game 1 (real outcome targeting) and game 2 (inflation targeting) are the same in a divine coincidence.*

Proof. The equilibrium outcomes are characterized by the same nonlinear equations, because $\bar{\mu}_i = \zeta^{-1}(\Pi^*; \bar{\nu}_i) = \bar{m}_i^{des}$ for all $i = 1, ..., J$. □

In a divine coincidence, the conditions in Propositions 3 and 6 become the same. We can connect this common condition to a more familiar one in the literature. Define the (gross) natural real rate of interest in state k to be:

$$r_k^{nat} = \frac{1}{\delta \sum_{j=1}^{J} P_{kj} \bar{\lambda}_{kj}^{nat}}$$

Note that the natural real rate of interest depends on the current state k only through households' beliefs $(P_{kj})_{j=1}^{J}$ about future realizations of the natural growth rate of marginal utility.

The following proposition supposes that ex-post inflation is (conditionally) independent of natural marginal utility growth, that the central bank's desired marginal utility process is in fact the same as the natural marginal utility process, and that there is a divine coincidence.

It shows that under this assumption, the sufficient conditions in Propositions 3 and 6 are equivalent to the natural real rate of interest being higher than $1/\Pi^*$- that is, equivalent to the natural *nominal* rate of interest being larger than 1.

Proposition 8. *Suppose that for all i,* $\bar{m}_i^{des} = 1$ *and:*

$$\delta \sum_{j=1}^{J} P_{ij} \bar{\lambda}_{ij}^{nat} \bar{\xi}_{ij} = \delta \sum_{j=1}^{J} P_{ij} \bar{\lambda}_{ij}^{nat}$$

In a divine coincidence, $r_k^{nat} \Pi^* < 1$ *if and only if* $\bar{m}_k^{des} < \Psi(Q_k \bar{m}^{des}; \bar{\nu}_k)$.

Proof. We can readily show that:

$$r_k^{nat} \Pi^* = \frac{\Pi^*}{\delta \sum_{j=1}^{J} P_{kj} \bar{\lambda}_{kj}^{nat}} = \frac{\zeta(\bar{m}_k^{des}; \bar{\nu}_k)}{\delta \sum_{j=1}^{J} P_{kj} \bar{\lambda}_{kj}^{nat} \bar{\xi}_{kj}} = \frac{\Psi^{-1}(\bar{m}_k^{des}; \bar{\nu}_k)}{Q_k \bar{m}^{des}}$$

which proves the proposition. □

Gali (2008) shows that in a simple New Keynesian model characterized by the divine coincidence, monetary policy can only be optimal if the real interest rate is equal to the natural real rate of interest in every date and state. However, Gali's analysis assumes that there is no lower bound on the nominal interest rate. Of course, when the lower bound binds, the central bank is forced to set the short-term interest *above* its natural level (as in Propositions 3 and 6). The following proposition uses the conditions in Proposition 8 to show that, if there is some risk of the lower bound's binding in the future, a currently unconstrained central bank will set the real interest rate *below* its natural level.

Proposition 9. *Suppose that there is a divine coincidence, and that for all i,* $\bar{m}_i^{des} = 1$. *Suppose too that inflation is conditionally uncorrelated with the natural level of marginal utility:*

$$\delta \sum_{j=1}^{J} P_{ij} \bar{\lambda}_{ij}^{nat} \bar{\xi}_{ij} = \delta \sum_{j=1}^{J} P_{ij} \bar{\lambda}_{ij}^{nat}$$

Consider a Markov equilibrium with states (j,k) such that $R_j^* > 1$, $\widehat{m}_k^* > 1$, and $Q_{jk} > 0$. Then:

$$R_j^* < \Pi^* r_j^{nat}$$

Proof. The proof of Proposition 8 shows that, under the conditions of Proposition 9, it is true that for all i:

$$\Psi(r_i^{nat}\Pi^* Q_j \bar{m}^{des}) = \bar{m}_i^{des}$$

If $R_j^* > 1$ in a Markov equilibrium, then it must be true that $\widehat{m}_j^* = 1$. In that case, we know that:

$$1 = \Psi(R_j^* Q_j \widehat{m}^*; \bar{\nu}_j)$$

Since $\widehat{m}_k^* > 1$ and $Q_{jk} > 0$, we know also that:

$$Q_j \widehat{m}^* > Q_j \bar{m}^{des}$$

It follows that:

$$R_j^* < r_j^{nat}\Pi^*$$

$\square$

Suppose that there is a state k such that the natural (gross) nominal interest rate is less than one. Proposition 9 implies that if the central bank is unconstrained in state j, and there is a positive probability of transiting from state j to state k in finite time, then the nominal interest rate in state j is below its natural level.

4 Secular Stagnation: A Numerical Example

In this section, I first describe how to solve numerically for Markov equilibria under Assumption 1. I then discuss the properties of a particular numerical example. I argue that these properties correspond to what some observers[4] have termed "secular stagnation". More specifically, in the (unique) Markov equilibrium, the economy spends long stretches at the zero lower bound with large marginal utility gaps. During these periods, the natural nominal interest rate is actually positive most of the time.

4.1 Solution Method

In this subsection, I describe how to solve numerically for the set of Markov equilibria in games 1 and 2 under the assumptions made in Propositions 1 and 4. Recall that we defined the matrix Q by:

$$Q_{ij} = \delta P_{ij} \bar{\lambda}_{ij}^{nat} \bar{\xi}_{ij}$$

and the function Ψ is restricted to be Lipschitz with modulus B. Assume that Assumption 1 is satisfied, so that $B||Q|| < 1$.

In game 1, consider the operator $T_1 : \mathbb{R}_+^J \to \mathbb{R}_+^J$ defined by:

$$(T_1(\widehat{m}))_i = \max(\bar{m}_i^{des}, \Psi(Q_i \widehat{m}; \bar{\nu}_i)), i = 1, ..., J$$

The proof of Proposition 1 shows that T_1 is a contraction. It follows that we can find the unique equilibrium to game 1 by iterating on T_1 from an arbitrary positive J-dimensional vector:

$$\widehat{m}^* = \lim_{N \to \infty} (T_1)^N (\widehat{m}^0)$$

In game 2, consider the operator $T_2 : \mathbb{R}_+^J \to \mathbb{R}_+^J$ defined by:

$$(T_2(\widehat{m}))_i = \max(\zeta^{-1}(\Pi^*; \bar{\nu}_i), \Psi(Q_i \widehat{m}; \bar{\nu}_i))$$

Under assumption 1, T_2 is a contraction. It follows that we can find the unique equilibrium

[4]See, among others, Summers (2014).

to game 2 by iterating on T_2 from an arbitrary positive J-dimensional vector:

$$\widehat{m}^* = \lim_{N \to \infty} T_2^N(\widehat{m}^0)$$

In the next section, I apply these solution methods to a particular numerical example to illustrate how fear of a sharp crisis can lead to secular stagnation.

4.2 A Numerical Example

In this subsection, I describe, and solve, a simple numerical example in which the number of states $J = 3$.

4.2.1 Secular Stagnation in Markov Equilibrium

The example features fixed inflation expectations, so that $\Pi^* = 1.02$ and $\zeta(\widehat{m}; i) = 1.02$ for all $\widehat{m}$ and i. I set $\delta = 0.97$ (the settings for ζ and δ indicate that this is intended to be an example in which the length of a period is a year). The state space for natural marginal utility growth is $(\bar{\lambda}_{ij})_{j=1}^3$ =$(0.98, 0.98, 1.23)$ for all i. This state space indicates that the realization of natural marginal utility growth in a given period is independent of information received prior to that period. The state space for inflation realizations is given by $(1, 1, 1)$. The transition matrix P is:

$$\begin{matrix} 0.975 & 0 & 0.025 \\ 0.1 & 0.7 & 0.2 \\ 0 & 0.5 & 0.5 \end{matrix}$$

The stationary probability vector associated with P is $(0.71, 0.18, 0.11)$.

In this example, state 1 is relatively safe, because average marginal utility is shrinking over time, and agents believe that there is a high probability of staying in state 1. However, state 1 does admit a small probability of exiting to state 3, in which the the growth rate of average marginal utility is very high. In state 3, the probability of immediately returning to the (good) state 1 is zero. However, there is a chance of returning to state 2. In some sense, state 2 is a good state because marginal utility growth is low (just as low as in state 1). But state 2 is a delicate one, because there is a relatively high probability of switching back to state 3.

I focus on Game 1 and assume that $\bar{m}_j^{des} = 1$ for $j = 1, 2, 3$, so that the central bank's desired real outcome is to keep equilibrium marginal utility equal to its natural level.[5] Given the central bank's objective, the vector of natural nominal interest rates is given by:

$$(\frac{1}{\sum_{j=1}^{J} Q_{ij}})_{i=1}^{3} = (1.067, 1.021, 0.95)$$

We know, therefore, that the zero lower bound has to bind in state 3 in a Markov equilibrium. However, in the unique Markov equilibrium, the marginal utility gap vector is given by:

$$(1, 1.096, 1.23)$$

The central bank achieves this outcome by setting the (gross) nominal interest rate vector equal to:

$$(1.058, 1, 1)$$

In this example, the zero lower bound binds in both states 2 and 3. The marginal utility gap in state 2 is very large, even though the realized decline of natural marginal utility in state 2 is the same as in state 1. Note that the central bank sets the nominal interest rate *lower* than its natural level in both states 1 and 2. The natural interest rate in a given state is defined as the interest rate that delivers the best outcome in that state, *conditional* on the central bank's achieving the best outcome in all other states. Since the central bank is unable to achieve the best outcome in state 3 (because the zero lower bound binds), it is desirable to set the nominal interest rate below its natural level in all other states.

[5] This assumption means that, in states 1 and 2, the central bank's desired level of marginal utility is falling steadily. However, if state 3 occurs, there is permanent upward shock to the natural level of marginal utility, which is inherited by the central bank's desired level of marginal utility.

One useful way to summarize the properties of the Markov equilibrium is through average transition times. Suppose that the economy starts in state 1. Then, on average, it enters state 3 in forty years. State 3 is a crisis state, in which the marginal utility of consumption in the central bank's desired allocation is rising rapidly. The central bank responds by lowering the gross nominal interest rate to one. The economy exits state 3, on average, in 2 years. However, even after the economy returns to state 2, the central bank keeps the nominal interest rate at the zero lower bound. It stays at the zero lower bound until the economy returns to state 1 - which only takes place, on average, sixteen years after the initial entry into state 3.

4.2.2 An Enhanced Set of Policy Instruments

Central banks have used a variety of policy tools at the zero lower bound, including quantitative easing, forward guidance, and negative interest rates. In this subsection, I discuss each of these briefly. The first is the simplest: in this simple framework, quantitative easing would have no effect (just as in Eggertsson and Woodford (2003)). In order to change this result, we would need to enhance the model to incorporate effects of quantitative easing on the central bank's objective, on the evolution of inflation expectations, or on the natural level of marginal utility.

There is some room for effective forward guidance in the example. I have used the concept of Markov equilibrium to model the outcomes of central bank choices over time. This concept eliminates the ability of the central bank to commit when at the zero lower bound to future choices. In particular, the central bank could improve outcomes in states 2 or 3 by committing to deliver higher inflation and lower marginal utility than are achieved in the Markov equilibrium upon re-entry into state 1. However, recall that the central bank stays in state 3 for, on average, two years, and then stays in state 2 for, on average, fourteen years. In this example, effective forward guidance requires very long periods of commitment - lasting well over a decade on average. (Of course, as is true in any context, effective forward guidance actually requires the commitment to be of arbitrary duration.)

In contrast, negative nominal interest rates are extremely powerful within this model. Suppose that the lower bound on nominal interest rates is -0.5%, rather than zero. Then, we can use the proof in Proposition 1 (or 4) to show that the the unique Markov equilibrium in either game 1 or 2 is characterized as the fixed point to the nonlinear operator:

$$\widehat{m}_i^* = \max(1, 0.995\frac{\delta}{\Pi^*}\sum_{j=1}^{3} P_{ij}\bar{\lambda}_j^{des}\bar{\xi}_j\widehat{m}_j^*), i = 1,2,3 \tag{8}$$

The factor 0.995 reflects the reduction in the lower bound on gross nominal interest rates.

The solution to (8) is given by:

$$(1, 1, 1.11)$$

The zero lower bound only binds in state 3. The marginal utility gap in state 3 is now only 11%, not 23%. The central bank achieves this outcome by setting the gross nominal interest rates in the three states equal to:

$$(1.063, 0.995, 0.995)$$

Hence, the central bank makes use of its new tool by lowering the gross nominal interest rate to 0.995 in both states[6] 2 and 3. The better outcomes in states 2 and 3 mean that it is that it is desirable for the central bank to set the nominal interest rate equal to a higher level in state 1.

Why does a seemingly small fifty basis point reduction in the lower bound have such a big impact on real outcomes in this example? The intuition is that the fifty basis point reduction is expected to last, on average, for a long period of time. Thus, in state 2, the fifty basis point reduction is expected to last, on average, for fourteen years. The reduction in the lower bound is much less effective if the central bank perceives it as an “emergency” measure that

[6]More precisely, the gross nominal interest rate is 0.99501 in state 2. The lower bound is not binding in that state.

would be applied only in state 3. Such a transitory reduction in the lower bound results in a Markov equilibrium marginal utility gap vector of the form:

$$(1, 1.057, 1.18)$$

This represents a noticeably smaller improvement over the original Markov equilibrium.

5 Conclusions

The term "nominal rigidities" in macroeconomics is generally used to refer to frictions in the adjustment of prices and wages. However, from the point of view of monetary policy, these frictions are really just one way to motivate why *anticipated* inflation seems to adjust only slowly in response to macroeconomic shocks and policy adjustments. In this paper, I proceed more directly and simply assume that one-period-ahead inflation expectations adjust sluggishly, if at all, to real economic conditions. This assumption is consistent with a wide range of recent empirical work. I focus on the properties of Markov equilibria in a dynamic stochastic game in which exogenous variables evolve according to a Markov chain and a central bank chooses a short-term nominal interest rate subject to a zero lower bound. I allow for two distinct objectives for the central bank: targeting a desired level of economic activity or a desired level of inflation.

In this context, I make two substantive contributions. First, I develop a numerical solution algorithm to solve for the unique Markov equilibrium. The solution procedure is simple but still takes full account of the presence of the lower bound on the short-term nominal interest rate. Given its simplicity, the algorithm could readily be used by first-year Ph. D. or advanced undergraduate students. Instructors in these kinds of courses could use the framework in this paper and the associated solution procedure, as a way to help their students understand the basic consequences of the zero lower bound.

Second, I use a numerical example to show that, after a crisis event in which economic activity declines rapidly, the central bank may be constrained by the zero lower bound because the economy has entered a "fear" period in which households assign a relatively high probability to another crisis taking place. In the example, the after any crisis, the economy typically endures long stays at the zero lower bound that are associated with large efficiency losses. The extended duration of these "fear" periods imply that there are large gains to relaxing the zero lower bound by being able to lower the short-term nominal interest rate below zero by relatively small amounts (as little as 50 basis points). The bulk of these gains are only achievable if the central bank is willing to use negative nominal interest rates during periods in which households are afraid of a crisis (and not just during crises themselves).

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