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DYNAMIC DEBT MATURITY

Zhiguo He
Konstantin Milbradt

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ABSTRACT

A firm chooses its debt maturity structure and default timing dynamically, both without commitment. Via the fraction of newly issued short-term bonds, equity holders control the maturity structure, which affects their endogenous default decision. A shortening equilibrium with accelerated default emerges when cash-flows deteriorate over time so that debt recovery is higher if default occurs earlier. Self-enforcing shortening and lengthening equilibria may co-exist, with the latter possibly Pareto-dominating the former. The inability to commit to issuance policies can worsen the Leland-problem of the inability to commit to a default policy—a self-fulfilling shortening spiral and adverse default policy may arise.

Zhiguo He
University of Chicago
Booth School of Business
5807 S. Woodlawn Avenue
Chicago, IL 60637
and NBER
zhiguo.he@chicagobooth.edu

Konstantin Milbradt
Kellogg School of Management
Northwestern University
2001 Sheridan Rd #401
Evanston, IL 60208
and NBER
milbradt@northwestern.edu

1 Introduction

The 2007/08 financial crisis has put debt maturity structure and its implications squarely in the focus of both policy discussions as well as the popular press. However, dynamic models of debt maturity choice are difficult to analyze, and hence academics are lagging behind in offering tractable frameworks in which a firm's debt maturity structure follows some endogenous dynamics. In fact, a widely used framework for debt maturity structure is based on Leland [1994b, 1998] and Leland and Toft [1996] who, for tractability's sake, take the frequency of refinancing/rollover as a fixed parameter. Essentially, equity holders are able to commit to a policy of constant debt maturity structure.

This stringent assumption is at odds with mounting empirical evidence that non-financial firms in aggregate tend to have pro-cyclical debt maturity structure (Chen et al. [2013]). What is more relevant to our paper is the evidence on active management of firms' debt maturity structure. In a comprehensive survey by Graham and Harvey [2001], company CFOs claim that they manage debt maturity to "reduce risk of having to borrow in bad times." Indeed, a recent paper by Xu [2014] shows that speculative-grade firms are actively lengthening their debt maturity structure—especially in *good* times—via early refinancing. On the other hand, Brunnermeier [2009], Krishnamurthy [2010], Gorton et al. [2015] document that financial firms were shortening their debt maturity structure during the 2007/08 crisis. Our paper not only provides the first dynamic model to investigate this question, but also delivers predictions that are consistent with these empirical patterns.

We remove the equity holders' ability to commit to a debt maturity structure ex-ante, allowing us to analyze how equity holders adjust the firm's debt maturity structure facing time-varying firm fundamentals and endogenous bond prices. To focus on endogenous debt maturity dynamics only, we fix the firm's book leverage policy, by following the Leland-type model assumption that the firm commits to maintaining a constant aggregate face-value of outstanding debt. As explained later, this treatment rules out *direct dilution* and gives the sharpest contrast of our paper to Brunnermeier and Oehmke [2013].[1]

In our model with a flat term structure of the risk-free rate, a firm has two kinds of debt,

[1] This assumption is also consistent with the fact that in practice, most bond covenants put restrictions on the firm's future leverage policies, but rarely on the firm's future debt maturity structure.

long and short term bonds. Equity holders control the firm's debt maturity structure by changing the maturity composition of new debt issuances: if just-matured long-term bonds are replaced by short-term bonds, then the firm's debt maturity structure shortens. In refinancing their maturing bonds, equity holders absorb the cash shortfall between the face value of matured bonds and the proceeds from selling newly issued bonds at market prices. If default is imminent, bond prices will be low and hence so-called rollover losses arise for equity holders. These rollover losses feed back to the default decision by equity holders, leading to even earlier default.[2]

Endogenous default in the Leland tradition—ex-post equity holders are more likely to default when facing low cash-flows—is widely accepted as an important mechanism to understand firm default and credit risk.[3] In our paper, it is the endogenous default that makes the endogenous debt maturity structure relevant. If the default decision were exogenously given, then a Modigliani-Miller argument would imply irrelevance of the debt maturity structure (Section 3.2). However, equity holders are more likely to default if the firm has a shorter debt maturity structure and thus needs to refinance more maturing bonds, as shown by He and Xiong [2012b] and Diamond and He [2014] in a setting where the firm can precommit to its debt maturity structure. Intuitively, the more debt has to be rolled over, the heavier the rollover losses are for the firm when fundamentals deteriorate, thereby pushing the firm closer to default.

As the equity holders' endogenous default decision is affected by the firm's debt maturity structure due to the aforementioned rollover concerns, debt maturity structure matters for both total firm value as well as how this total firm value gets split among different stakeholders including equity, short-term debt, and long-term debt. Because equity holders are choosing the fraction of newly issued short-term bonds when refinancing the firm's maturing bonds, bond valuations in turn affect the equity's endogenous choice of the firm's debt maturity structure over time in response to observable firm fundamentals. At the heart of our model is the analysis of the *joint determination* of endogenous default, endogenous dynamic maturity structure and bond prices in equilibrium.

We first focus on equilibrium behavior in the situation of imminent default. We show that, right before default, equity holders are choosing an issuance strategy that maximizes the total proceeds

[2]This rollover channel emerges in a variant of the classic Leland [1994a] model that involves finite maturity debt (Leland and Toft [1996] and Leland [1994b]).

[3]For example, the fact that firms default more often in recessions given worse economic outlooks goes a long way toward explaining the credit spreads puzzle (Huang and Huang [2012], Chen [2010], Bhamra et al. [2010]).

of newly issued bonds, knowing that their issuance strategy will delay or hasten the endogenous default timing slightly. Hence, in any conjectured shortening equilibrium in which the firm keeps issuing short-term bonds and then defaults, it must be that hastening default marginally improves the value of short-term bonds when default is imminent. As a result, when the debt recovery value in default is independent of the endogenous default timing (e.g. if the firm's cash-flows are constant), then shortening equilibria are impossible—this is because fixing the recovery value, defaulting marginally earlier always hurts bond values as we assume a coupon rate commensurate with the discount rate.

When cash-flows deteriorate over time and the debt recovery value is an increasing function of the current cash-flows, then the endogenous default timing will affect the debt recovery value. Consequently, a shortening equilibrium with earlier default can emerge. The earlier the default, the higher the defaulting cash-flows, the higher the debt recovery value. In a shortening equilibrium, right before default the value of short-term bonds gets maximized by maturity shortening, i.e., the benefit of a more favorable recovery value by taking the firm over earlier outweighs the increased expected default risk due to earlier default. Further, the equilibrium shortening strategy is indeed welfare-maximizing in our special setting if only *local* deviations, i.e. delaying or hastening default slightly, are considered.[4] This seems to be an empirical relevant force in 2007/08 crisis during which we observed debt maturity shortening together with earlier default. There, the fundamental value of collateral assets deteriorated rapidly over time, and if default was going to occur in the near term anyway, then bond holders gained significantly by taking possession of the collateral sooner.

In terms of the general statement regarding welfare, the equity holders' issuance decision in the vicinity of default only maximizes the proceeds of newly issued bonds, but not the total value of the firm. As a result, our equilibria in general feature a conflict of interest, which is best illustrated when the equilibrium issuance strategy takes some interior value (as opposed to a cornered issuance strategy in the above shortening equilibrium). In this situation, we show that short-term debt holders prefer a marginal lengthening of the maturity structure, whereas long-term debt holders

[4]In our stylized model with equal coupons for both long-term and short-term bonds, we show that in the vicinity of default the equity's shortening strategy indeed maximizes the value of the firm, in the "local" sense that delaying default a bit by slightly lengthening the maturity structure hurts the firm value slightly. However, the firm value typically is non-monotone in its survival time, so that a sufficiently long delay of default may lead to a higher firm value. This explains why the lengthening equilibrium may Pareto dominate the shortening equilibrium when we are away from default boundary. For details, see Section 6.1.

prefer a marginal shortening.

Away from imminent default, starting at some initial state, i.e., today's cash flows and debt maturity structure, there often exist two equilibrium paths toward default. One is with maturity shortening and the other with lengthening, in which the firm keeps issuing long-term bonds so its debt maturity structure grows longer and longer over time. We highlight that these two equilibria can often be even *Pareto* ranked: In our example, the lengthening equilibrium with a longer time to default has a higher overall welfare and Pareto dominates the shortening equilibrium. The multiplicity of equilibria emerges in our model without much surprise. If bond investors expect equity holders to keep shortening the firm's maturity structure in the future, then bond prices reflect this expectation, self-enforcing the optimality of issuing short-term bonds only. Similarly, the belief of always issuing long-term bonds can be self-enforcing as well.

We compare these two equilibria with the equilibrium that would result from the traditional Leland setup, in which the firm commits to an issuance strategy that keeps its debt maturity structure constant over time. A flexible issuance policy should help equity holders avoid inefficient (i.e., too early) default due to rollover pressure. However, the presence of the shortening equilibrium shows the downside of such flexibility—as equity holders cannot commit to neither any particular future issuance strategy nor default policy, equilibria can arise in which a self-fulfilling shortening spiral hurts equity holders and may even lead to the shortening equilibrium being Pareto-dominated by the Leland equilibrium.

Our results are in sharp contrast to Brunnermeier and Oehmke [2013] who show that equity holders might want to privately renegotiate the bond maturity down (toward zero) with each individual bond investor. The key difference is on who bears the rollover losses in case of unfavorable (cash-flow) news in any interim period. In Brunnermeier and Oehmke [2013], there are no covenants about the firm's aggregate face value of outstanding bonds, so that the rollover losses when refinancing the maturing short-term bonds are absorbed by promising a sufficiently high new face-value to new bond investors. This *directly dilutes* the (non-renegotiating) existing long-term bond holders, i.e., their claim on the bankruptcy recovery value of the firm is diminished. In contrast, in our model equity holders are absorbing rollover losses through their own deep pockets (or equivalently through equity issuance), while existing long-term bond holders remain undiluted in default. Nevertheless, equity holders who are protected via limited liability will refuse to absorb these losses at some point, leading to endogenous default of the firm. By shutting down the *direct dilution* channel that drives Brunnermeier and Oehmke [2013], our paper highlights a different and empirically relevant mechanism: the interaction between the endogenous debt maturity structure and endogenous default decisions which leads to what we term *indirect dilution* via the timing of default and level of recovery.

According to our model, debt maturity shortening is more likely to be observed in response to deteriorating economic conditions. This prediction is consistent with the empirical findings cited above: Xu [2014] shows that speculative-grade firms are actively lengthening their debt maturity structure in *good* times, and Krishnamurthy [2010] shows that financial firms are shortening their debt maturity right before 2007/08 crisis. We also show that shortening equilibria exist only when the existing debt maturity is sufficiently short and/or the debt burden is sufficiently high. Hence, our model suggests that conditional on deteriorating economic conditions, debt maturity shortening is more likely to be observed in firms with already short maturity structures and/or large debt burdens, straightforward empirical predictions that are readily tested.[5]

We make two key simplifying assumptions which render the tractability of our model. First, unlike typical Leland-type models, we rule out Brownian cash-flow shocks and assume a deterministically decreasing cash-flow with a possible terminal large upwards jump. In Section 6 we discuss how cash-flow volatility affects our results. Second, the firm commits to a constant aggregate amount of face-value outstanding, which rules out *directly diluting* existing bond holders by promising higher face value to new incoming bond holders following unfavorable news. As we discussed, we rule out *direct dilution* to purposefully contrast our effect to that of Brunnermeier and Oehmke [2013].[6]

Debt maturity is an active research area in corporate finance. The repricing of short-term debt given interim news in Flannery [1986], Diamond [1991] and Flannery [1994] is related to the

[5] There is no obvious reasoning to think that these predictions are implied by the mechanism in Brunnermeier and Oehmke [2013].

[6] Dynamic models of endogenous leverage decisions over time are challenging by themselves. The literature usually take the tractable framework of Fischer et al. [1989], Goldstein et al. [2001] so that the firm needs to buy back all outstanding debt if it decides to adjust aggregate debt face value. This assumption requires a strong commitment ability on the side of equity holders. Recently, Dangl and Zechner [2006] allow equity holders to adjust the firm's debt fact value downwards by issuing less bonds than the amount of bonds that are maturing, and equity holders in DeMarzo and He [2014] may either repurchase or issue more at any point of time. In contrast to our paper in which the firm commits to a constant aggregate face value but can freely adjust debt maturity structure over time, Dangl and Zechner [2006] and DeMarzo and He [2014] instead assume that the firm can commit to certain debt maturity structure but not its book leverage.

endogenous rollover losses of our paper. For dynamic corporate finance models with finite debt maturity, almost the entire existing literature is based on a Leland-type framework in which a firm commits to a constant debt maturity structure.[7] To the best of our knowledge, our model is the first that investigates endogenous debt maturity dynamics.[8]

We abstract from various mechanisms that may favor short-term debt. For instance, Calomiris and Kahn [1991] and Diamond and Rajan [2001] emphasize the disciplinary role played by short-term debt, a force not present in our model. At a higher level, this economic force originates from the firm side—rather than the investor side—just like in our model. This is because our analysis is based on the underlying debt-equity conflict of endogenous default when absorbing the firm's rollover losses. In practice, debt maturity shortening can also originate from concerns on the investor side, which is another highly relevant economic force. The best example is Diamond and Dybvig [1983] in which debt investors who suffer idiosyncratic liquidity shocks demand early consumption; He and Milbradt [2014] study its implications in a Leland framework with over-the-counter secondary bond markets.[9] Another related paper is Milbradt and Oehmke [2015] which shows how adverse on the funding side impacts a firm's debt and asset maturity choice.

Admati et al. [2015] and DeMarzo and He [2014] are two recent papers that concern themselves with the inability of a firm to commit to any future leverage policies. They show that there is a leverage ratchet effect: as the firm cares about current bond proceeds, but not about the value of bonds issued some time ago, the firm's incentives are tilted towards issuing more current debt. In equilibrium, the firm always keeps issuing more debt, thereby increasing potential bankruptcy costs. In the dynamic setting of DeMarzo and He [2014], (exogenous) debt maturity plays a role in that shorter maturity disciplines this behavior by reducing the mass of old bond holders each period.

Our paper is also related to the study of debt maturity and multiplicity of equilibria in the

[7]For more recent development, see He and Xiong [2012b], Diamond and He [2014], Chen et al. [2014], He and Milbradt [2014], and McQuade [2013]. For another closely related literature on dynamic debt runs, see He and Xiong [2012a], Cheng and Milbradt [2012], Suarez et al. [2014].

[8]Our model nests the Leland framework (without Brownian shocks) if we assume that both long-term bonds and short-term bonds have the same maturity. In Leland [1994a] the firm is unable to commit not to default. Introducing a fixed rollover term in Leland [1994b] with finite debt maturity makes the outcome of this inability to commit worse as default occurs earlier the higher the rollover. We show that introducing a flexible maturity structure with an inability to commit might further worsen this default channel, even though a priori the added flexibility would seem work in equity holder's favor to move closer to the first-best welfare maximizing strategy.

[9]Often, these models with investors' liquidity needs only establish the advantage of short-term debt unconditionally, while our model emphasizes the endogenous preference of short-term debt when closer to default.

sovereign debt literature, in which models are typically cast in a dynamic setting (e.g., Cole and Kehoe [2000]; Arellano and Ramanarayanan [2012]; Dovis [2012]; Lorenzoni and Werning [2014]).[10] Like us, Aguiar and Amador [2013] provide a transparent and tractable framework for analyzing maturity choice in a dynamic framework without commitment. They study a drastically different economic question, however: there, a sovereign needs to reduce its debt and the debt maturity choices matter for the endogenous speed of deleveraging. By making a zero-recovery in default assumption, that paper also excludes a *direction dilution* channel. In contrast, in our model the total face value of debt is fixed at a constant, and the maturity choice trades off rollover losses today versus higher rollover frequencies tomorrow.

Due to challenging identification issues, there are very few empirical papers documenting endogenous debt maturity management in a systematic way. On the bank loan market, Mian and Santos [2011] show that creditworthy firms actively manage the maturity of their syndicated loans in normal times. As a result, the liquidity demand from these creditworthy firms becomes countercyclical, as they choose not to refinance when liquidity costs rise. Xu [2014] instead focuses on the public corporate bond market, a market that our paper more readily applies to. Xu [2014] shows that speculative-grade firms are those that display a pro-cyclical pattern in early refinancing and maturity extension, which complements the findings in Mian and Santos [2011].

After laying out our model in Section 2, we study the key incentive compatibility condition and preview the economic mechanism in Section 3. We solve the model by working backwards: Section 4 characterizes the equilibria in the neighborhood of the default boundary, and discusses when shortening can arise. Section 5 characterizes the equilibria that arise further away from the default boundary. Section 6 covers welfare and provides further results, and we discuss empirical predictions and provide concluding remarks in Section 7. All proofs are in Appendix A.

[10] Arellano and Ramanarayanan [2012] provide a quantitative model where the sovereign country can actively manage its debt maturity structure and leverage, and show that maturities shorten as the probability of default increases; a similar pattern emerges in Dovis [2012]. As standard in sovereign debt literature, one key motive for the risk-averse sovereign to borrow is for risk-sharing purposes in an incomplete market. Because debt maturity plays a role in how the available assets span shocks, the equilibrium risk-sharing outcomes are affected by debt maturity. This force is absent in most corporate finance models–including this paper–that are typically cast in a risk-neutral setting with some deep-pocketed equity holders (a la Leland framework).

2 Model setup

2.1 Firm and Asset

All agents in the economy, equity and debt-holders, are risk-neutral with a constant discount rate $r \geq 0$. The firm has assets-in-place generating cash flows at a rate of y_t, with

$$dy_t = -\mu_y(y_t)\,dt, \text{ and } \mu_y(y) \geq 0. \tag{1}$$

with $\mu_y'(y) \geq 0$.[11] Thus, y_t is *weakly* decreasing over time. Our later analysis emphasizes that debt maturity shortening occurs only when y_t decreases *strictly* over time. This captures the economic scenario in which the firm is facing deteriorating fundamental, just like the episodes leading to the 2007/08 financial crisis. Because most evidence of debt maturity shortening is documented in these episodes, condition (1) is a particularly relevant scenario for our theoretical investigation of endogenous dynamic debt maturity.

There is also a Poisson event arriving with a constant intensity $\zeta > 0$; at this event, assets-in-place pay out a sufficiently large constant cash-flow $X > 0$ and the model ends. This "upside event" gives a terminal date for the model and can also be interpreted as the realization of a growth option.[12]

The above formulation allows for the cash-flow rate y_t to become negative (e.g., operating losses). Since y_t is decreasing over time, when y_t takes negative values it might be optimal to abandon the asset at some time even in the first best case. Denote this optimal abandonment time by T_a. Denote the arrival time of upside event by T_ζ. Then, given the cash-flow process y_t, the asset value is

$$A(y) = \mathbb{E}\left[\int_0^{\min(T_a, T_\zeta)} e^{-rt} y_t dt + \mathbf{1}_{\{T_\zeta < T_a\}} e^{-rT_\zeta} X\right]. \tag{2}$$

The firm is financed by debt and equity. When equity holders default, debt holders take over the firm with some bankruptcy cost, so that the asset's recovery value from bankruptcy is $B(y) < A(y)$. Throughout we assume that $B'(y) \geq 0$ as well as $B''(y) \geq 0$, i.e., the firm's liquidation value is

[11] The key results of the paper will be for the two specifications $\mu_y(y) = \mu$ and $\mu_y(y) = \mu \cdot y$.

[12] The upside event is introduced to give equity holders an incentive to keep the firm alive for some range of negative y_t. This will play a role when we link $B(y)$ to the underlying cash-flows, as in Section 4, but is not of relevance beforehand.

weakly increasing and convex in the current state of cash-flows. In Section 4, we connect $B(y)$ to the unlevered firm value $A(y)$, and the optionality of abandonment naturally gives the properties required of $B(y_b)$.

2.2 Dynamic Maturity Structure and Debt Rollover

2.2.1 Assumptions

We aim to study the dynamic maturity structure of the firm while maintaining tractability. The firm has two kinds of bonds outstanding: long-term bonds whose time-to-maturity follows an exponential distribution with mean $1/\delta_L$, and short-term bonds whose time-to-maturity follows an exponential distribution with mean $1/\delta_S$, where δ_i's are positive constants with $i \in \{S, L\}$ and $\delta_S > \delta_L$.[13]

Maturity is the only characteristic that differs across the two bonds. Both bonds have the same coupon rate c and the same principal normalized to 1. To avoid arbitrary valuation difference between two bonds, we set $c = r$ which is the discount rate. We abstract away from tax-benefits of debt in the most part of this paper, but they could be easily accommodated. This way, without default both bonds are "risk-free" and have a unit value, i.e., $D_L^{rf} = D_S^{rf} = 1$.

To focus on maturity structure only and to minimize state variables, we assume that the firm follows a constant "book leverage" policy. Specifically, following the canonical assumption in Leland [1998], the firm rolls over its bonds in such a way that the total promised face-value is kept at a constant normalized to 1. Implicitly, we assume that debt covenants, while restricting the firm's future leverage policies, do not impose restrictions on a firm's future maturity. This assumption is realistic, as debt covenants often specify restrictions on firm leverage but rarely on debt maturity. We further assume *equal seniority* in default to rule out any other *direct dilution* motives. Then, in bankruptcy, both bond holders receive, per unit of face-value, $B(y)$ as the asset's liquidation value. Throughout, we assume that

$$B(y) < D_i^{rf} = 1, \text{ for } i \in \{S, L\}. \tag{3}$$

[13]Thus, bonds mature in an i.i.d. fashion with Poisson intensity $\delta_i > 0$. An equivalent interpretation is that of a sinking-fund bond as discussed in Leland [1994b, 1998].

which implies a strictly positive loss-given-default for bond investors while equity recovers nothing in bankruptcy.

For our paper, the essence of constant "book" leverage is that it rules out *direct dilution*—more future face-value issuance reduces the recovery value in default for each unit of face-value held by bond investors. As explained later in Section 6.3, this *direct dilution* effect is the economic force behind Brunnermeier and Oehmke [2013], and by shutting this off we are highlighting a different and novel channel, something we term *indirect dilution*. We discuss the relation to Brunnermeier and Oehmke [2013] at length in Section 6.3, and show robustness of our results to potential deleveraging in Section 6.2.

2.2.2 Maturity structure and its dynamics

Let $\phi_t \in [0,1]$ be the *fraction of short-term bonds* outstanding. We also call ϕ_t the current *maturity structure* of the firm.[14] Given ϕ_t, during $[t, t+dt]$ there are $m(\phi_t)\,dt$ dollars of face-value of bonds maturing, where

$$m(\phi_t) \equiv \phi_t \delta_S + (1-\phi_t)\,\delta_L. \tag{4}$$

We have $m'(\phi) = (\delta_S - \delta_L) > 0$; intuitively, the more short-term the current maturity structure is, the more bonds are maturing each instant. We restrict the firm to have non-negative outstanding bond issues so that the maturity structure is restricted to $\phi \in [0,1]$. We discuss this assumption in Section 6.2.

Under the constant debt face value assumption, the firm is (re-)issuing $m(\phi_t)\,dt$ units of new bonds to replace its maturing bonds every instant. The main innovation of the paper is to allow equity holders to endogenously choose the proportion of newly issued short-term bonds, which we denote by $f_t \in [0,1]$,[15] so that

$$\frac{d\phi_t}{dt} = \underbrace{-\phi_t \cdot \delta_S}_{\text{Short-term maturing}} + \underbrace{m(\phi_t)\, f_t}_{\text{Newly issued short-term}}. \tag{5}$$

[14] The assumption of random exponentially distributed debt maturities rules out any "lumpiness" in debt maturing, which is termed "granularity" in Choi et al. [2015]. As another interesting dimension of corporate debt structure, debt granularity is related to but different from debt maturity structure.

[15] We assume that there is no debt buybacks, call provisions do not exist, and maturity of debt contracts cannot be changed once issued. We discuss a larger possible issuance space $f \in [-f_L, f_H]$ allowing for some debt buybacks in Section 6.2.

Consider constant issuance policies that take *corner* values 0 or 1, i.e. $f \in \{0,1\}$. Suppose that $f = 1$ always, so that the maturity structure is shortened; then $\phi_t = 1 - e^{-\delta_L t}(1-\phi_0)$, so that over time, the firm's maturity structure ϕ_t monotonically rises toward 100% of short-term debt. Similarly, if the firm were to issue only long-term bonds, i.e., $f = 0$ always, then the maturity structure ϕ_t would monotonically fall toward 0% of short-term debt.

Let $f_{ss}(\phi)$ be the issuance fraction that keeps the maturity structure constant, given by

$$f_{ss}(\phi) \equiv \frac{\phi\delta_S}{\phi\delta_S + (1-\phi)\,\delta_L} \in [\phi, 1]\,. \tag{6}$$

Then, the firm is shortening (lengthening) its maturity structure for $f > (<)\, f_{ss}(\phi)$. For later reference, this (constant) issuance policy $f_{ss}(\phi)$ also gives us the benchmark case of Leland [1998] in which equity holders commit to maintain a constant debt maturity structure.

2.3 Rollover losses and endogenous default

In Leland [1998], equity holders commit to rolling over (refinancing) the firm's maturing bonds by re-issuing bonds of the same type. In our model, the firm chooses the fraction of short-term bonds f continuously amongst newly issued bonds. Let $D_S(\phi_t, y_t)$ and $D_L(\phi_t, y_t)$ be the bond-prices offered in the competitive market. Per unit of face value, by issuing an f_t fraction of short-term bonds, the equity's net rollover cash-flows are

$$\underbrace{f_t D_S(\phi_t, y_t) + (1-f_t)\, D_L(\phi_t, y_t)}_{\text{proceeds of newly issued bonds}} - \underbrace{1}_{\text{payment to maturing bonds}}.$$

Each instant there are $m(\phi_t)\,dt$ units of face value to be rolled over, hence the instantaneous expected cash flows to equity holders are

$$\underbrace{y_t}_{\text{operating CF}} - \underbrace{c}_{\text{coupon}} + \underbrace{\zeta E^{rf}}_{\text{upside event}} + \underbrace{m(\phi_t)\,[f_t D_S(\phi_t, y_t) + (1-f_t)\, D_L(\phi_t, y_t) - 1]}_{\text{rollover losses}}. \tag{7}$$

We call the last term "rollover losses."[16] The third term "upside event" is the expected equity payoff of this event, $E^{rf} \equiv X - D^{rf} = X - 1 > 0$, multiplied by its instantaneous probability, ζ.

When the above cash flows in (7), net of the "upside event" expected flow, are negative, these losses are covered by issuing additional equity, which dilutes the value of existing shares.[17] Equity holders are willing to buy more shares as long as the equity value is still positive. When equity holders—protected by limited liability—declare bankruptcy at some time denoted by T_b, equity value drops to zero, and bond holders receive the firm's liquidation value $B\left(y_{T_b}\right)$ as their recovery value at default.

The two state variables, y_t and ϕ_t, give rise to two distinct channels that expose equity holders to heavier losses, leading to endogenous default. The first cash-flow channel has been studied extensively in the literature (Leland and Toft [1996], He and Xiong [2012b]). For a given static maturity structure ϕ_t and thus a constant rollover $m\left(\phi_t\right)$, when y_t deteriorates (say, y_t turns negative), equity holders are absorbing (i) heavier operating losses (the first term in (7)), and (ii) heavier rollover losses in the third term in (7), as bond prices D_S and D_L drop given more imminent default.

Novel to the literature, the endogenous maturity structure ϕ_t also affects the equity holders' cash-flows in (7). As indicated in the "rollover losses" term, ϕ_t enters the rollover frequency $m\left(\phi_t\right)$ as well as the endogenous bond prices $D_i\left(\phi_s, y_s\right)$'s. First, as $m'\left(\phi\right) > 0$, a shorter maturity structure today implies an instantaneously higher rollover frequency $m\left(\phi\right)$, which amplifies the rollover losses given bond prices. Second, as we will show below, a future path of increasing ϕ lowers bond prices D_S and D_L today as equity holders tend to default earlier given shorter maturity structure. These two forces jointly give rise to heavier rollover losses in (7), for a given cash-flow state y.

The above discussion suggests that there exists a default curve $\left(\Phi\left(y\right), y\right)$, where the increasing function $\Phi\left(\cdot\right)$ gives the threshold maturity structure given cash-flow y at which equity holders declare default. We will derive $\Phi\left(\cdot\right)$ shortly. In equilibrium, the firm defaults whenever the state lies in the bankruptcy (or default) region $\mathcal{B} = \left\{\left(\phi, y\right) : \phi \geq \Phi\left(y\right)\right\}$.

[16] Equity holders are always facing rollover losses as long as $c = r$ and $B\left(y_{T_b}\right) < 1$, which imply that $D_i < 1$. When $c > r$, rollover gains occur for safe firms who are far from default. As emphasized in He and Xiong [2012b], since rollover risk kicks in only when the firm is close to default, it is without loss of generality to focus on rollover losses only.

[17] The underlying assumption is that either equity holders have deep pockets or the firm faces a frictionless equity market.

2.4 Equilibrium

The equilibrium concept in this paper is that of Markov perfect equilibrium, with payoff-relevant states being $(\phi, y) \in \mathcal{S} \equiv [0,1] \times \mathbb{R}$.

2.4.1 Strategies and payoffs

The players in our game are equity and bond holders. At any given state $(\phi_t, y_t) \in \mathcal{S}$, the strategy of equity holders is given by $\{f, d\}$ where $f : \mathcal{S} \to [0,1]$ is the issuance strategy, and $d : \mathcal{S} \to \{0,1\}$ gives the default decision, with $d = 1$ indicating default. The evolution of the exogenous cash-flow state y_t is given in (1), and the evolution of the endogenous state ϕ_t is affected by the issuance strategy $f(\phi_t, y_t)$ as in (5). The default region $\mathcal{B} \subset \mathcal{S}$ is the region where $d = 1$. As default is irreversible, the default time is given by $T_b \equiv \min\{s \geq 0 : d_s = 1\}$; or, equivalently, it is the first hitting time that the state (ϕ_t, y_t) hits the default region $\mathcal{B}$. Hence, given the equity holders' strategy $\{f, d\}$, there will be an endogenous mapping from the current Markov state (ϕ_t, y_t) to the (distribution of) future default time T_b.

The strategy of bond-holders can be described by their offered competitive prices for long and short-term bonds, i.e., $D_S(\phi_t, y_t)$ and $D_L(\phi_t, y_t)$, given the state $(\phi_t, y_t) \in \mathcal{S}$. We assume here that bond-holders always offer prices for both bonds, even though there may be only one of the two bonds sold in equilibrium.

Perfect competition amongst bond-holders implies that their offered bond prices will be the discounted future bond payoffs, which is simply the coupon c until some stopping time that results in a terminal lump-sum payout. This stopping time is the lesser of the default time T_b, in which case there is a lump-sum payoff $B(y_{T_b})$, the random upside even time, T_ζ, and the random maturity time of the individual bond, T_δ, the latter two featuring a lump-sum payoff of 1. Denote $T_g \equiv \min\{T_{\delta_i}, T_\zeta\}$ where g indicates the "good outcome" with full principal repayment. Then, we have the bond holders' break-even condition for either bond $i \in \{S, L\}$:

$$\begin{aligned} D_i(\phi_t, y_t) &= \mathbb{E}_t\left[\int_t^{\min\{T_b, T_g\}} e^{-r(s-t)} c ds + \mathbb{1}_{\{T_b < T_g\}} e^{-rT_b} B(y_{T_b}) + \mathbb{1}_{\{T_b > T_g\}} e^{-rT_g}\right] \\ &= \mathbb{E}_t\left[\int_t^{T_b} e^{-(r+\zeta+\delta_i)(s-t)} (c + \zeta + \delta_i) ds + e^{-(r+\zeta+\delta_i)T_b} B(y_{T_b})\right], \end{aligned} \tag{8}$$

where the second expression just integrates out the good outcome event $T_g = \min\{T_{\delta_i}, T_\zeta\}$ occurring with intensity $\delta_i + \zeta$.

The payoffs to equity holders are given by the discounted flow payoffs in (7), until T_b when they receive nothing in default:

$$E(\phi_t, y_t) = \mathbb{E}_t\left[\int_t^{T_b} e^{-(r+\zeta)(s-t)}\Big(y - c + \zeta E^{rf} + m(\phi_s)\left[f_s D_S(\phi_s, y_s) + (1 - f_s) D_L(\phi_s, y_s) - 1\right]\Big)\, ds\right]. \tag{9}$$

Here, we have integrated out the upside event T_ζ, so that equity holders payoffs are as if receiving ζE^{rf} per unit of time until T_b. Also, notice that the bond prices offered by bond-holders (i.e., bond holders' strategies) enter the value of equity holders via the rollover term.

2.4.2 Markov perfect equilibrium

Definition 1. *A* Markov perfect equilibrium in pure strategies *of our dynamic maturity choice game is defined as a strategy profile of equity-holders and bond-holders with* $\{f, d, D_S, D_L\} : \mathcal{S}^4 \to [0,1] \times \{0,1\} \times \mathbb{R}^+ \times \mathbb{R}^+$, *so that the state evolutions are given by (1) and (5), and*

1. ***Optimality of equity holders****. The issuance strategy* $\{f\}$ *and the default decision* $\{d\}$ *maximize (9) given any state* (ϕ, y)*;*

2. ***Break even condition for bond holders.*** *Given the issuance strategy* $\{f\}$ *and the default decision* $\{d\}$ *which jointly determine the equilibrium default time* T_b*, bond prices* $\{D_S, D_L\}$ *satisfy (8).*

We pay special attention to the class of equilibria in which equity holders are taking cornered issuance strategies.

Definition 2. *A* cornered equilibrium *is an equilibrium where equity holders set either* $f_t = 0$ *or* $f_t = 1$ *always. An equilibrium with* $f_t = 1$ *always is called a* shortening equilibrium *(SE), and an equilibrium with* $f_t = 0$ *always is called a* lengthening equilibrium *(LE).*

As an example what does not constitutes an equilibrium, suppose that we conjecture an LE in which equity holders are lengthening until default, i.e. $f = 0$ until a default time T_b. Consequently, bond investors offer corresponding bond prices D_i's in (8) based on this conjectured T_b. Suppose, however, that the offered prices incentivize equity holders to instead follow an SE path, i.e. keep issuing short-term bonds with $\hat{f} = 1$ today and in the future, with a different default time $\hat{T}_b$ due to a different $\left\{\hat{\phi}_s\right\}$ evolution. If this different resulting default time $\hat{T}_b$ implies different bond prices $\hat{D}_i \neq D_i$ using (8), then the conjectured LE is not an equilibrium.

As a Markov perfect equilibrium, we need to specify strategies that are potentially off-equilibrium, which still are themselves equilibria of the resulting subgame. More specifically, on and inside the bankruptcy region $\mathcal{B}$, the bond values equal the recovery value $D_i(\phi, y) = B(y)$, while equity holders immediately default $d(\phi, y) = 1$ so that $T_b = 0$. Further, in the survival or continuation region $\mathcal{C} = \mathcal{S} \setminus \mathcal{B}$ we impose the additional refinement that given a deviation, bond investors "pick" the continuation equilibrium that is *closest* to the original equilibrium in terms of default time T_b, which allow us to use local deviations for the equilibrium construction (as we imposed gradual changes by making the adjustment rate f bounded). This refinement is similar in nature to off-equilibrium "beliefs" that treat deviations as *mistakes*, and are thus very much akin to a *trembling-hand* refinement. For example, suppose that all investors expect the firm to always shorten the maturity structure in the future and then default at a certain time. A deviation today of lengthening then does not alter the belief of bond investors that in the future the firm will always keep shortening and default, if indeed for the slightly perturbed state today always shortening in the future is still an equilibrium.

3 Incentive Compatibility Conditions and Endogenous Default

We study the key incentive compatibility condition for the endogenous issuance strategy taken by equity holders, and explain the importance of endogenous default. Throughout, we assume that the equity value function $E(\phi, y)$ and two bond value functions $D_S(\phi, y)$ and $D_L(\phi, y)$ are sufficiently smooth that they satisfy the corresponding Hamilton-Jacobi-Bellman (HJB) equations.[18]

[18]We show f has to be continuous in the state-space by the IC condition in the proof of Lemma 6, implying equity value is $\mathcal{C}^1$. Fleming and Rishel [1975] Chapter IV, Theorem 4.4 shows that the value function being a $\mathcal{C}^1$ function is a sufficient condition for the solution of the HJB equation to give the optimal control of the problem.

3.1 Valuations and incentive compatibility condition

3.1.1 Valuations

Bond values solve the following HJB equation where $i \in \{S, L\}$:

$$\begin{aligned}\underbrace{rD_i(\phi, y)}_{\text{required return}} &= \underbrace{c}_{\text{coupon}} + \underbrace{\delta_i\left[1 - D_i(\phi, y)\right]}_{\text{maturing}} + \underbrace{\zeta\left[1 - D_i(\phi, y)\right]}_{\text{upside event}} \\ &\quad + \underbrace{\left[-\phi\delta_S + m(\phi) f\right]\frac{\partial}{\partial\phi}D_i(\phi, y)}_{\text{maturity structure change}} + \underbrace{\mu_y(y)\frac{\partial}{\partial y}D_i(\phi, y)}_{\text{CF change}}, \end{aligned} \tag{10}$$

Equal seniority implies that in default $D_i(\Phi(y), y) = B(y)$. Later analysis involves the price wedge between short- and long-term bonds, which is defined as

$$\Delta(\phi, y) \equiv D_S(\phi, y) - D_L(\phi, y) \quad \text{with} \quad \Delta(\Phi(y), y) = 0.$$

We will later show that in our baseline setup we have

$$\Delta(\phi, y) > 0 \text{ for } \phi < \Phi(y), \tag{11}$$

i.e., short-term bonds have a higher price than long-term bonds away from the default boundary. Intuitively, short-term bonds are paid back sooner and hence less likely to suffer default losses compared to long-term bonds.

For equity holders who are choosing f endogenously, their valuation can be written as the following HJB equation

$$\begin{aligned}\underbrace{rE(\phi, y)}_{\text{required return}} &= \underbrace{y - c}_{\text{CF net coupon}} + \underbrace{\zeta\left[E^{rf} - E(\phi, y)\right]}_{\text{upside event}} + \underbrace{\mu_y(y)\frac{\partial}{\partial y}E(\phi, y)}_{\text{CF change}} \\ &\quad + \max_{f\in[0,1]}\left\{\begin{array}{c} \underbrace{m(\phi)\left[fD_S(\phi, y) + (1-f)D_L(\phi, y) - 1\right]}_{\text{current rollover losses}} \\ + \quad \underbrace{\left[-\phi\delta_S + m(\phi) f\right]\frac{\partial}{\partial\phi}E(\phi, y)}_{\text{maturity structure change affects future value}} \end{array}\right\}. \end{aligned} \tag{12}$$

Here, the last term uses the evolution of firm's maturity structure in (5). At default, equity is worthless, which yields the boundary condition $E(\Phi(y_b), y_b) = 0$.

3.1.2 Optimal issuance policy and incentive compatibility condition

As indicated by the optimization term in (12), equity holders are choosing the fraction f of newly issued short-term bonds to minimize the firm's current rollover losses, but taking into account any long-run effect of changing the maturity structure on their continuation value. Let $E_\phi(\phi, y) \equiv \frac{\partial}{\partial \phi} E(\phi, y)$ and define the *Incentive Compatibility* (IC) condition for equity as

$$IC(\phi, y) \equiv \Delta(\phi, y) + E_\phi(\phi, y). \tag{13}$$

Due to linearity of (12) with respect to the issuance policy f, we have the following bang-bang solution:

$$f = \begin{cases} 1 & \text{if } IC(\phi, y) > 0 \\ [0, 1] & \text{if } IC(\phi, y) = 0 \\ 0 & \text{if } IC(\phi, y) < 0 \end{cases}. \tag{14}$$

In general, issuing more short-term bonds today (say $f = 1$) lowers the firm's rollover losses today, as short-term bonds have higher prices than long-term bonds. This just says that $\Delta(\phi, y) > 0$ in (13) in general favors issuing more short-term bonds. However, issuing more short-term bonds today makes the firm's future maturity structure more short-term (higher ϕ). This has two negative effects: first, it increase the firm's future rollover losses (higher $m(\phi)$); second, as shown shortly, it drives the firm closer to its strategic default boundary. Both hurt equity holders' continuation value, leading to $E_\phi < 0$ and hence pushing the IC towards long-term bond issuance. In Section 3.2 we show that the first effect just involves value neutral transfers between equity and debt holders; it is the second effect of endogenous default that drives our analysis.

3.1.3 Endogenous default boundary

We assume that, as in Leland [1994a], equity holders choose when to default optimally in a dynamically consistent way, i.e., equity holders cannot commit ex-ante to some default policy that may violate their limited liability condition.

In our model with deterministically deteriorating cash-flows, it is easy to show that equity holders default exactly when their expected flow payoff in (7) hits zero from above.[19] Suppose that equity holders default at $\phi = \Phi$ and defaulting cash-flow $y = y_b \equiv y_{T_b}$. By equal seniority in default, we have $D_i\left(\Phi\left(y_b\right), y_b\right) = B\left(y_b\right)$ for $i \in \{S, L\}$, so that the equity's expected flow payoff (7) at default becomes *independent* of f:

$$y_b - c + \zeta E^{rf} + \underbrace{m\left(\Phi\right)\left[B\left(y_b\right) - 1\right]}_{\text{rollover losses}}. \tag{15}$$

Equating the above term to zero, and solving for the default boundary $\Phi\left(y_b\right)$, we have

$$\Phi\left(y_b\right) = \frac{1}{\delta_S - \delta_L}\left[\frac{y_b - c + \zeta E^{rf}}{1 - B\left(y_b\right)} - \delta_L\right], \text{ with } \Phi'\left(y_b\right) > 0. \tag{16}$$

Let us define y_{min} and y_{max} by $\Phi\left(y_{min}\right) = 0$ and $\Phi\left(y_{max}\right) = 1$; we have $y_{min} < y_{max}$. Then, as $\phi \in [0, 1]$, we know that all admissible bankruptcy points $\left(\Phi\left(y_b\right), y_b\right)$ have $y_b \in \left[y_{min}, y_{max}\right]$. We map an example bankruptcy boundary in the left panel of Figure 1, with y_{min} and y_{max} indicated by vertical lines.

As we will emphasize in Section 3.2, an upward sloping default boundary $\Phi'\left(y_b\right) > 0$ implies that firms are more likely to default with a shorter maturity structure. The intuition is clear in (15): The higher the Φ, the shorter the maturity structure, the heavier rollover losses the equity holders are absorbing. As a result, equity will default at a higher cash-flow state.

Finally, on the default boundary $\Phi\left(y_b\right)$, some issuance policies may pull the firm away from the boundary. We restrict our attention to the situation in which this never occurs, which requires a sufficiently large (downward) drift $\mu_y\left(y_b\right)$. The left panel of Figure 5 in the Appendix, and the related discussion in Appendix A.2, provide more details. Importantly, this assumption yields tractability and uniqueness of equilibria in the vicinity of the boundary. Intuitively, it restricts the flexibility of the firm in changing its maturity structure relative to the change in the recovery value—cash-flows or recovery value are assumed to change relatively faster than the speed at which the firm can change its debt maturity structure, a reasonable assumption in reality.

[19]For a formal argument with smooth pasting conditions, see Lemma 4 in Appendix A.2.

3.1.4 Valuations in (τ, y_b) space

For ease of analysis and comparison to the literature, we introduce a change of variables from (ϕ, y) to (τ, y_b) where τ is the firm's time-to-default, i.e., $\tau \equiv T_b - t$ where T_b is the firm's endogenous default time and y_b is the defaulting cash-flow. In Section 4 we will analyze the sign of $IC(\phi, y)$ in (13) in this new space, i.e., $IC(\tau, y_b)$, especially when τ is close to zero.

Denote y_τ and ϕ_τ as the cash-flow and the maturity structure with τ periods left until default. Given the ultimate bankruptcy state $(\phi_{\tau=0} = \Phi(y_b), y_{\tau=0} = y_b)$, the equilibrium path (ϕ_τ, y_τ) is essentially a one-dimensional object indexed by time-to-default τ, with y_b operating as a parameter. It is natural to solve the model in the state space of (τ, y_b): Working our way back from the default boundary to derive the equilibrium of the game. As we show later, the transformed state-space also greatly helps us illustrate the model intuitions in Section 4.3. For details of the one-to-one mapping between (ϕ, y) and (τ, y_b), as well as the closed-form expressions for bond values and equity value, see Appendix A.1.

3.2 Why endogenous default is important

Before we discuss the equilibrium in detail, we highlight the role played by endogenous default in the mechanism underlying our model. Endogenous default is at the heart of Leland-type models, and the key contribution of our model is to study the *joint determination* of equity holder's issuance strategy f, the maturity structure ϕ, and the default decision T_b.

3.2.1 Exogenous default – MM and irrelevance of the issuance strategy

As a benchmark, suppose that the firm defaults at an exogenously fixed time T_b (and thus a fixed y_{T_b}), so that we are switching off the impact of the maturity structure ϕ on default; the logic applies even to random T_b, as long as it is independent of ϕ. Following the Modigliani-Miller logic, we can calculate total firm value V by simply summing up the discounted expected cash-flows from t to T_b, i.e.,

$$V = \int_t^{T_b} e^{-(r+\zeta)(s-t)} (y_s + \zeta X) \, ds + e^{-(r+\zeta)(T_b - t)} B(y_{T_b}).$$

Further, from (8) we recall that bond prices are affected by T_b only (i.e., independent of f or ϕ directly), i.e.

$$D_i = \int_t^{T_b} e^{-(r+\zeta+\delta_i)(s-t)} (c + \zeta + \delta_i) \, ds + e^{-(r+\zeta+\delta_i)(T_b-t)} B(y_{T_b}).$$

As T_b (and thus y_{T_b}) is fixed exogenously, all agents are risk-neutral and share the same discount rate, we can derive equity as a residual value (recall ϕ_t is the fraction of short-term bonds; also note that future debt investors always break even when purchasing newly issued debt)

$$E = V - [\phi_t D_S + (1 - \phi_t) D_L]. \tag{17}$$

Now we can map this result back to Section 3.1.2: Taking the derivative of (17) with respect to ϕ, noting that V, D_S and D_L are independent of ϕ, yields an identically zero IC condition everywhere (13):

$$E_\phi = -[D_S - D_L] = -\Delta \Rightarrow IC = E_\phi + \Delta = 0.$$

Intuitively, as V and D_i's are independent of the debt issuance policy,[20] the residual equity value must be independent of the issuance policy as well. What is going on is that any cash-flow gain that stems from changing the maturity structure today is exactly offset by an equivalent change in rollover losses in the future, once we fix the default policy/timing.[21] This is simply a Modigliani-Miller result: if cash-flows are fixed (here, the reader should think of the recovery value $B(y_{T_b})$ simply as a fixed terminal cash-flow at $t = T_b$), then in a friction-less world, firm value is invariant to the financing (be it static or dynamic) chosen by the firm. We conclude that once the firm has an ex-ante commitment ability to a default time, the inability to ex-ante commit to a debt issuance path becomes irrelevant.

Remark 1. *The MM irrelevance result for all stakeholders of the firm continues to hold under generalizations of the setup: First, adding volatility to the cash-flow process, say in the form of a Brownian motion, can be accommodated as cash-flows to debt-holders are still fixed if the default*

[20] As the total face-value of bonds is fixed, there is no *direct dilution*. But, there is no *indirect dilution* either, because the exogenous default time and equal seniority imply that the value of each bond is fixed.

[21] Even with exogenous default timing, a higher current $\phi(t)$ indeed leads to a lower equity value today. However, equity holders are indifferent in their issuance strategies, as their trading gains by changing maturity structure (reflected by Δ) would exactly offset the equity value decrease.

timing is exogenous. Second, non-constant aggregate face-value of bonds can be accommodated as long as older bonds have seniority. This is one of the key differences to Brunnermeier and Oehmke [2013]; in that paper, the debt face-value varies over time, and expected recovery value of bonds varies with the outstanding face-value by the underlying equal seniority assumption. See Section 6.3 for a detailed discussion.

3.2.2 Endogenous default – issuance strategy interacts with default decision

We now introduce endogenous default by equity holders as summarized by the default boundary $\Phi(y)$. In contrast to the previous case with exogenous default timing, the issuance policy f affects the maturity structure ϕ, which in turn affects the default time T_b. Given that bankruptcy is costly, the economic surplus of the firm is varying, and the above irrelevance result ceases to hold.

For illustration, an always shortening path, i.e., $f = 1$, will feature default at an earlier time (recall the dynamics of y are unaffected by the choices of the firm) than say an always lengthening path, i.e., $f = 0$. This is because with $f = 1$, the firm's debt maturity structure ϕ_t grows, whereas with $f = 0$ it shrinks. Then given an upward sloping default boundary $\Phi(y)$, the state pair (ϕ_t, y_t) hits the boundary $(\Phi(y), y)$ earlier in the lengthening case. The right panel of Figure 1 illustrates these two possible paths, with "SE" showing the $f = 1$ path, and "LE" showing the $f = 0$ path.

Let us preview two economic mechanisms highlighted in our results. First of all, the issuance policy f and the resulting maturity structure ϕ not only affect the size of pie (i.e., the total firm value), but also how the pie is split (among equity, long-term bond, and short-term bond holders), all indirectly via its impact on T_b through endogenous default. Equity holders are choosing the issuance policy f to maximize their own value only, and in equilibrium it is likely that these equity-value maximizing issuance policies are adversely affecting other stake-holders, leading to a lower total surplus.

The second point is regarding the (potentially negative) value of commitment. Recall that in Leland models equity holders just follow a static maturity structure, i.e., $f = f_{ss}(\phi)$ always, so that $\phi_t = \phi_0$ for all t, as illustrated by the "Leland" path in the left panel of Figure 1. In contrast, in our model equity holders have the flexibility to choose the firm's future issuance policy, but of course cannot commit ex-ante to any issuance path—rather, issuance paths have to be dynamically optimal given the market prices of bonds. Here, flexibility and lack of commitment come as two

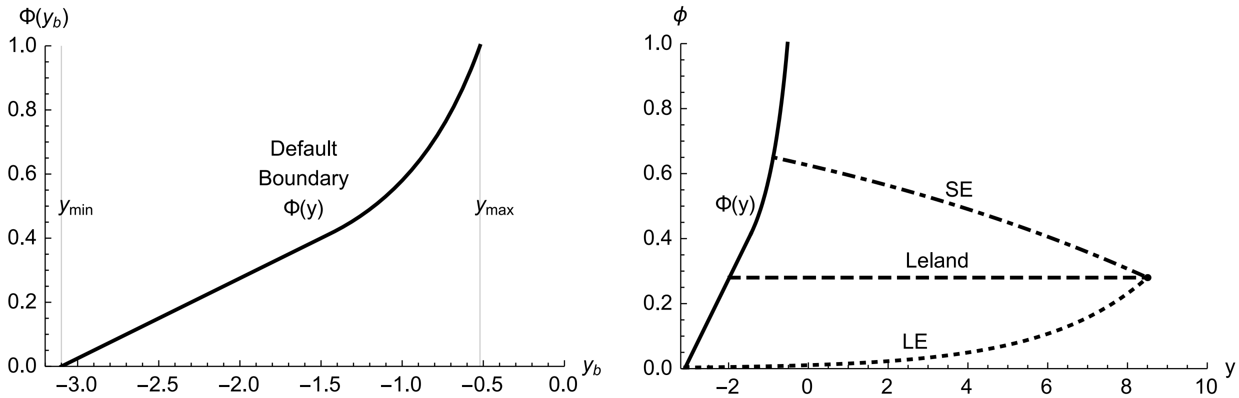

Figure 1: **Left Panel:** Default boundary $\Phi(y_b)$ as a function of the defaulting cash-flow y_b. **Right Panel:** Three sample paths for initial point $(\phi, y) = (0.28, 8.5)$: An always lengthening path (LE; dotted), an always shortening path (SE; dot-dashed), and a path with constant maturity structure (Leland; dashed). For both panels parameters are given by $c = r = 10\%$, $D^{rf} = 1$, $E^{rf} = 12$, $\mu = 13$, $\zeta = .35$, $\delta_S = 5$, $\delta_L = 1$, $\alpha_y = 3$, $\alpha_X = .95$.

sides of the same coin. Given the lack of commitment on default timing, it is unclear whether this "flexibility" is a good thing; in fact, we later show that "flexibility" in future issuance policy may exacerbate the endogenous default decision and lower firm value.

3.3 Roadmap of results

We will use Figure 1 as the springboard for our analysis. First, in the next section, we establish that there exists a unique equilibrium in the vicinity of the bankruptcy boundary. More specifically, Proposition 1 shows that we can partition the bankruptcy boundary into an SE, Interior, and LE region, as show in the left panel of Figure 2. Importantly, the set of equilibria on the boundary then significantly restricts the possible equilibria away from the boundary, as we show in Proposition 3 in Section 5. Figure 4 illustrates the proposition and the state-space partition. For example, any candidate SE has to hit the SE region on the boundary. Then, by our assumption that the state (ϕ, y) changes gradually, we can rule out what equilibria can arise for any point (ϕ, y) by checking whether candidate maximal shortening and maximal lengthening paths lead to an inconsistency on the boundary.

4 Equilibria in the neighborhood of the default boundary

We start by analyzing equilibria in the neighborhood of the default boundary. Then in the next section, we solve the model away from the default boundary by working backwards and imposing the equilibrium restrictions derived in the neighborhood of the default boundary.

Throughout the rest of paper, we will use the following setting for our numerical examples. We assume that the cash-flow drift is a negative constant, i.e., $\mu_y(y) = \mu > 0$. Debt holders are less efficient in running the liquidated firm (relative to equity holders), so that post-default the upside payoff X becomes $\alpha_X X$ with $\alpha_X \in (0,1)$; and, given defaulting cash-flow y_b, the current cash-flow post-default becomes $\alpha_y y_b$. Since in our numerical examples the defaulting cash-flows $y_b < 0$, to capture the inefficiency we set $\alpha_y > 1$.[22] For simplicity, the liquidated firm is assumed to be unlevered, so that $B(y) = A(\alpha_y y; \alpha_X X)$ where $A(\cdot)$ is the first-best firm value given in (2).

4.1 Incentive compatibility condition and intuition

Consider the equilibrium issuance policy in the vicinity of default, i.e., in the neighborhood of the default boundary. To this end, let us define $f_b \equiv f_{\tau\to 0}$; recall $\tau \geq 0$ is the time-to-default. Exactly at default, smooth pasting $E_\phi = 0$ and equal seniority $\Delta = D_S - D_L = 0$ imply we have $IC(0, y_b) = E_\phi + \Delta = 0$. However, the equilibrium issuance policy in the immediate vicinity of default, i.e., $f_{\tau\to 0}$, is determinate and a function of how $IC(\tau, y_b)$ approaches $IC(0, y_b) = 0$—from above, which implies $IC(\tau, y_b) > 0$ so that $f = 1$, or from below, which implies $IC(\tau, y_b) < 0$ so that $f = 0$.

As both E_ϕ and Δ are analytical in their arguments, we analyze the sign of $IC(\tau, y_b)$ slightly away from $\tau = 0$ by considering the Taylor expansion in the τ-dimension of $IC(\tau, y_b)$:

$$IC(\tau, y_b) = IC(0, y_b) + IC_\tau(0, y_b)\,\tau + o(\tau)\,. \tag{18}$$

Taking the limit, the sign of the derivative $IC_\tau(0, y_b)$ determines how $IC(\tau, y_b)$ approaches $IC(0, y_b) = 0$ as $\tau \to 0$. The next lemma shows that $IC_\tau(0, y_b)$ is driven by the derivatives of the issuance proceeds of newly issued bonds, with respect to the maturity structure ϕ.

[22]Thus, the cash-flows are assumed worse under the management of debt holders. This specification is similar to the one found in Mella-Barral and Perraudin [1997].

Lemma 1. *The* τ *derivative of* $IC(\tau, y_b)$ *at default, which gives the sign of* $IC(\tau, y_b)$ *for sufficiently small* τ*, is given by*

$$IC_\tau(0, y_b) = m(\Phi(y_b))\left[f_b \frac{\partial}{\partial \phi} D_S(0, y_b) + (1 - f_b)\frac{\partial}{\partial \phi} D_L(0, y_b)\right]. \tag{19}$$

Recall $m(\phi) > 0$. The term in the bracket in (19), which can be rewritten for a fixed f_b as

$$\frac{\partial}{\partial \phi}\left[f_b D_S(0, y_b) + (1 - f_b) D_L(0, y_b)\right],$$

is the impact of maturity shortening on the issuance proceeds of newly issued bonds (f_b of short-term and $(1 - f_b)$ of long-term), which gives the sign of the equity holders' incentive compatibility condition near the default boundary. Thus, equity holders act as if maximizing the issuance proceeds of newly issued bonds. This result has implications on the welfare discussion in Section 6, as issuance proceeds in general differ from the value of existing securities.

This result echoes the Modigliani-Miller irrelevance result established in Section 3.2.1. There, we showed that if the firm's default policy is exogenously given, then the values of both bonds are independent of ϕ so that $\frac{\partial}{\partial \phi} D_S = \frac{\partial}{\partial \phi} D_L = 0$; as a result, we have $IC_\tau = 0$ always. With endogenous default, equity holders control the maturity structure dynamically, taking into account the fact that maturity structure affects the firm's endogenous default policy and thus impacts bond valuations, i.e. $\frac{\partial}{\partial \phi} D_i(0, y_b) \neq 0$. Section 4.3 shows how this gives rise to a non-trivial incentive compatibility condition for equity holders. Essentially, right before default, equity holders are choosing the firm's debt maturity structure to maximize their flow payoffs, in which the proceeds of newly issued bonds play a key role.

4.2 Equilibrium uniqueness in the neighborhood of the default boundary

We first analyze "cornered" equilibria, i.e., equilibria with $f = 0$ or $f = 1$; recall Definition 2. Consider an SE just before default, i.e., $\lim_{\tau \to 0} f(\tau, y_b) = 1$. For this issuance policy to be optimal, (14) says that $IC(\tau, y_b)$ has to be approaching $IC(0, y_b) = 0$ from above. Plugging the conjectured strategy $f(0, y_b) = 1$ into (18), for the conjectured strategy to be an equilibrium we require $IC_\tau(0, y_b) > 0$, which in turn requires by (19) (noting that $m(\phi) > 0$)

$$\frac{\partial}{\partial\phi} D_S\left(\Phi\left(y_b\right), y_b\right) \geq 0. \tag{20}$$

This condition says that an SE can only arise if the shortening strategy of equity holders locally increases the value of short-term bonds, as equity holders are maximizing the total issuance proceeds (in SE, only short-term bonds are issued). A similar derivation holds for a lengthening equilibrium (LE) just before default, i.e., $\lim_{\tau\to 0} f(\tau, y_b) = 0$, with

$$\frac{\partial}{\partial\phi} D_L\left(\Phi\left(y_b\right), y_b\right) \leq 0. \tag{21}$$

An LE can only arise if the lengthening strategy of the equity holders locally increases the value of long-term bonds.

Now consider an interior equilibrium just before default, i.e., $\lim_{\tau\to 0} f(\tau, y_b) \in (0,1)$. For such an interior issuance policy to be an equilibrium, $f_b \in (0,1)$ must be such that $IC_\tau(0, y_b) = 0$, i.e., the IC condition is equal to 0 even in the vicinity of default. Setting (19) equal to 0, after some algebra detailed in Appendix A.3, we find the unique candidate issuance strategy (nc stands for non-constrained)

$$f_b^{nc}\left(y_b\right) = \frac{\mu_y\left(y_b\right) B'\left(y_b\right) - \left[r + \zeta + \delta_L\right]\left[1 - B\left(y_b\right)\right]}{\left(\delta_S - \delta_L\right)\left[1 - B\left(y_b\right)\right]}, \tag{22}$$

with $f_b^{nc}(y_b)$ weakly increasing in y_b. Of course, f_b^{nc} is only an equilibrium strategy if it is interior, i.e., $f_b^{nc} \in (0,1)$. When f_b^{nc} exceeds the feasible issuance space $[0,1]$, the corresponding cornered equilibrium arises, as shown by the next proposition. In the following proposition we show that this uniqueness extend to the neighborhood of the default boundary:

Proposition 1. *Define* $y_0 = \sup\{y : f_b^{nc}(y) = 0\}$ *and* $y_1 = \inf\{y : f_b^{nc}(y) = 1\}$. *Then the unique equilibrium issuance policy in the neighborhood of the bankruptcy boundary is given by*

$$f_b\left(y_b\right) = \begin{cases} 0, & y < y_0 \\ f_b^{nc}\left(y_b\right), & y \in \left[y_0, y_1\right] \\ 1, & y > y_1 \end{cases}.$$

Consider a recovery function with $B(y_{min}) = B'(y_{min}) = 0$. Then an LE always exists for some part of the state-space as $f_b^{nc}(y_{min}) < 0$. An SE exists for some part of the state space if and only if $f_b^{nc}(y_{max}) \geq 1 \iff \frac{B'(y_{max})}{1-B(y_{max})} \geq \frac{r+\zeta+\delta_L}{\mu_y(y_{max})}$.

The first part of the proposition discuss the optimal issuance policy in the vicinity of default. The left panel of Figure 2 illustrates the different equilibria regions in the neighborhood of the default boundary. On the far left, i.e. for low y_b, we have the lengthening equilibria region labeled "LE". Next, we have the region labeled "Interior Equ."—this region has $f_b(y_b) \in (0,1)$. Finally, we have the shortening equilibria region labeled "SE" to the far right of the bankruptcy boundary, i.e., for high y_b. As $f_b^{nc}(y_b)$ is increasing in y_b, we have the following ordering of the equilibria: "LE" always lies to the left of "Interior Equ." which lies to the left of "SE" if we line them up according to y_b. The right panel of Figure 2 depicts the equilibrium issuance strategy by mapping $f_b(y_b)$ as a function of y_b. The curve kinks exactly at the points of transition from LE to interior equilibria, y_0, and from interior equilibria to SE, y_1.

The second part of the proposition gives conditions when different equilibria exist. For $B(y_{min}) = B'(y_{min}) = 0$, we can show that an LE always exists on some part of the state-space. Next, for an SE to exists on at least some part of the state-space we need to check $\frac{B'(y_{max})}{1-B(y_{max})} \geq \frac{r+\zeta+\delta_L}{\mu_y(y_{max})}$. Thus, we need a large slope of the recovery value function, $B'(y_{max})$, relative to the loss-given-default $[1-B(y_{max})]$, for an SE to exist.[23] Later on, when we link $B(y_b)$ to the cash-flow process, we will naturally have a flat part of the recovery function around y_{min}, thus immediately giving us the existence of an LE.

4.3 When can a shortening equilibrium arise?

We are now ready to discuss the above results in more depth, with a special focus on shortening equilibria.

We use the chain rule to rewrite the derivative of a generic bond $D_i(\phi, y), i \in \{S, L\}$ with respect to ϕ at the time of default; this derivative shows up in the equilibrium conditions (20) and

[23]It is straightforward to show that in our setting of inefficient default, $A(y) > B(y)$, and coupon equal to the discount rate, $c = r$, we must have $B(y_{max}) < 1$, as otherwise equity holders would leave money on the table by defaulting early.

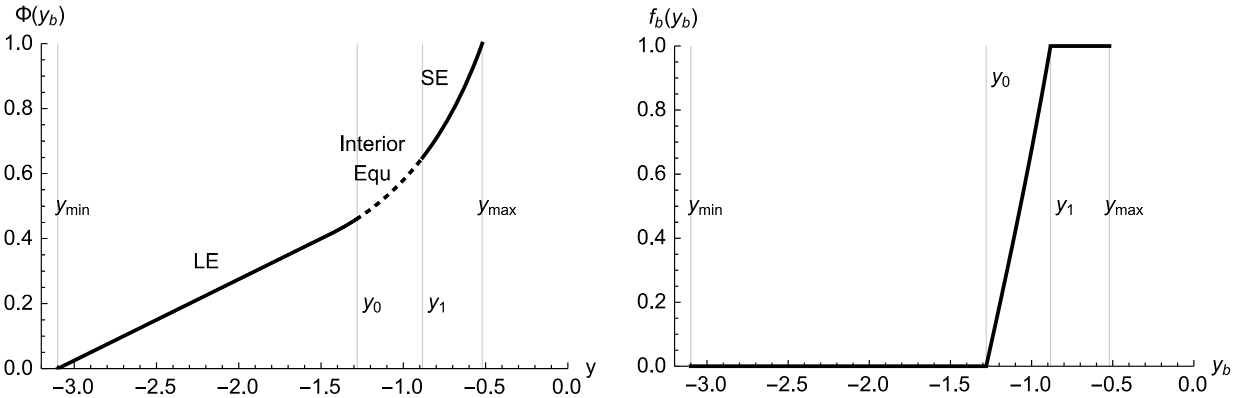

Figure 2: **Left Panel:** The regions of default boundary $\Phi(y_b)$: LE marks boundary regions with $f_b(y_b) = 0$, SE marks regions with $f_b(y_b) = 1$, and the dotted part indicated by Interior Equ marks the boundary region with $f_b(y_b) \in (0,1)$. **Right Panel:** Equilibrium boundary issuance policy $f_b(y_b)$. The kinks in the curve correspond to the transition from LE to Interior Equ, and from Interior Equ to SE in the left panel. For both panels parameters are given by $c = r = 10\%$, $D^{rf} = 1$, $E^{rf} = 12$, $\mu = 13$, $\zeta = .35$, $\delta_S = 5$, $\delta_L = 1$, $\alpha_y = 3$, $\alpha_X = .95$.

(21):

$$\left.\frac{\partial D_i(\tau(\phi,y), y_b(\phi,y))}{\partial\phi}\right|_{\tau=0} = \underbrace{\frac{\partial D_i(\tau,y_b)}{\partial\tau}\left.\frac{\partial\tau}{\partial\phi}\right|_{\tau=0}}_{\text{time-to-default, }(-)} + \underbrace{\frac{\partial D_i(\tau,y_b)}{\partial y_b}\left.\frac{\partial y_b}{\partial\phi}\right|_{\tau=0}}_{\text{default CF level, }(0/+)}. \tag{23}$$

The first partial derivative is with respect to the time-to-default τ, while holding the defaulting cash-flow y_b (and thus the recovery value $B(y_b)$) fixed. The second partial derivative is with respect to the defaulting cash-flow y_b, while holding the time-to-default τ fixed.

We now sign each term in (23). Given a positive loss-given-default and $c = r$, a longer time to default increases the value of both bonds, all else equal. Similarly, a higher recovery in default, all else equal, leads to a higher bond value. Stated formally, we have

$$\left.\frac{\partial D_i(\tau,y_b)}{\partial\tau}\right|_{\tau=0} = (r+\delta_i+\zeta)\left[1-B(y_b)\right] > 0, \quad \text{and} \quad \left.\frac{\partial D_i(\tau,y_b)}{\partial y_b}\right|_{\tau=0} = B'(y_b) \geq 0. \tag{24}$$

Further, $\Phi(y)$ being upward sloping implies that increasing the maturity structure marginally leads to slightly earlier default in the vicinity of the default boundary. Similarly, hitting $\Phi(y)$ earlier leads to a higher defaulting cash-flow, as cash-flows are deteriorating over time. This is illustrated in the schematic drawing in Figure 3—as we shift up ϕ slightly, the distance to the default boundary (and thus τ) shrinks, whereas the cash-flow level at which the boundary is hit increases. The mathematical expressions for these observations, proved in Appendix A.1, are

$$\left.\frac{\partial \tau}{\partial \phi}\right|_{\tau=0} < 0, \quad \text{and} \quad \left.\frac{\partial y_b}{\partial \phi}\right|_{\tau=0} > 0. \tag{25}$$

We will now consider two setups to highlight the intuition behind the results.

Constant recovery at default. Assume first that bond recovery value is fixed at $B(y_b) = cst$ regardless of y_b. This implies that the second term in (23) is zero, so that the bond values decrease as the maturity structure shortens on the default boundary. Then, by (21) the *unique* equilibrium in the neighborhood of the boundary is a lengthening equilibrium. For any equilibrium other than LE, i.e., for an interior or SE, there has to be a local gain for at least one of the bond holders when increasing ϕ, but this cannot be the case according to (23).

Variable recovery at default. Recall that in our model the recovery value is an increasing function of the defaulting cash-flow of the firm y_b, i.e., $B'(y_b) \geq 0$, with the inequality being strict for at least some y_b on the default boundary. Inspecting (23), we can see that the derivative of the short-term bond with respect to ϕ may be positive on the default boundary for sufficiently large $B'(y_b)$, in which case an SE by (20) can arise.

Why can a bond gain from a shortening of the maturity structure and an effective earlier default time? This can happen when there is sufficient slope in the recovery value function with respect to the cash-flow state, i.e. $B'(y_b) \gg 0$, so that it can outweigh the effect of the earlier default time leading to a cessation of coupon flows. Thus, the recovery value function $B(y)$ with $B'(y) > 0$ is the key driver of our result. Economically, if default in the near future is unavoidable, then short-term bond holders may prefer taking possession of the collateral early before it has lost most of its value. This seems to be an empirical relevant force during 2007/08 crisis during which we observed debt maturity shortening together with fundamental values of collateral assets deteriorating rapidly over time (Krishnamurthy [2010]). We will come back to discuss its welfare implication in Section 6.1.

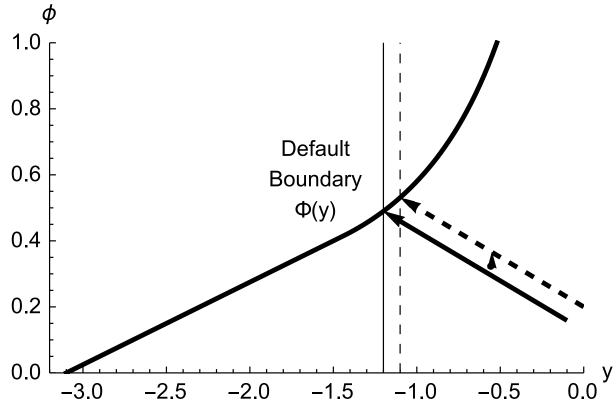

Figure 3: Default boundary $\Phi(y)$ and two *schematic* shortening paths: the solid path is the original path, and the dashed path is the path after a deviation in ϕ today. Parameters are given by $c = r = 10\%$, $D^{rf} = 1$, $E^{rf} = 12$, $\mu = 13$, $\zeta = .35$, $\delta_S = 5$, $\delta_L = 1$, $\alpha_y = 3$, $\alpha_X = .95$.

5 Equilibria away from Boundary

Building on the result of the uniqueness of equilibria in the neighborhood of the bankruptcy boundary, we now work backward to characterize equilibria further away from the boundary. Our results roughly show that SE are more likely "close" to the SE part of the boundary, and vice-versa for LE. Recall that if an SE exists on the boundary, it occurs for relatively high levels of short-term debt, i.e., for high ϕ. Consequently, as the state (ϕ, y) adjusts gradually, an SE will also be more likely to occur away from the bankruptcy boundary for high levels of short-term debt.

5.1 Equilibrium Regions

We first analyze cornered equilibria, for which we have sharper theoretical results. Lemma 6 in Appendix A.4 significantly simplifies the subsequent analysis by showing that any equilibrium issuance path $\{f\}_\tau$ has to be continuous with respect to τ, which rules out jumps in f.

5.1.1 Cornered equilibria

For any point (ϕ, y), define τ^i to be the time-to-default given the cornered strategy of either always shortening ($i = S$) or always lengthening ($i = L$). Define y_b^i analogously as the cash-flow at default. Thus, in the (τ, y_b) space, we have

$$y\left(\tau^i, y_b^i\right) = y \text{ and } \phi\left(\tau^i, y_b^i\right) = \phi.$$

In Figure 1, we graph the two cornered candidate equilibrium paths associated with the initial state $(\phi, y) = (0.28, 8.5)$, indicated by SE and LE respectively. In the SE, the firm keeps issuing short-term bonds and defaults at $\left(\phi_b^S = \Phi\left(y_b^S\right), y_b^S\right)$, while in the LE the firms keeps issuing long-term bonds and defaults at $\left(\phi_b^L = \Phi\left(y_b^L\right), y_b^L\right)$. The times to default differ substantially across these two equilibria: $\tau^S = 0.7217$ for the SE while $\tau^L = 0.8913$ for the LE.

Next, observe that for any type of cornered equilibrium, any two distinct paths never cross. This is simply a result of the exponential nature of ϕ_t in case of always $f = 1$ or $f = 0$. As Lemma 6 implies there is no jumps in f, SE and LE represent all possible cornered strategies. Next, to derive sharper analytical results, let us consider drift specifications $\mu_y(y) = \mu$ or $\mu_y(y) = \mu \cdot y$. Under these specifications, we can prove the optimality of a cornered issuance strategies along the whole path, *if indeed such a strategy is optimal in the neighborhood of the default boundary.* This implies that for cornered equilibria, we only need to check the IC condition at the defaulting boundary.

Proposition 2. *Let $\mu_y(y) = \mu$ or $\mu_y(y) = \mu \cdot y$. Then, given any initial starting value (ϕ, y), there exist (at most) two cornered equilibria: one with shortening always $f_s = 1$ until default, i.e., $s \in \left[0, \tau^S\right]$, and the other with lengthening always $f_s = 0$ until default, i.e., for $s \in \left[0, \tau^L\right]$. Moreover, for the IC condition along the whole path $s \in [0, \tau^i]$ for $i \in \{S, L\}$, it is sufficient to check the IC condition at default, i.e., for $\tau \to 0$, given by either (20) or (21), respectively.*

Going back to Figure 1, we see that the conjectured SE and LE paths are actual equilibria, because they end in the appropriate regions of the bankruptcy boundary given in the left panel of Figure 2. Since the defaulting cash-flow y_b^S is negative, $\alpha_y > 1$ (see footnote 22) says that the firm is experiencing even worse (negative) cash-flows under the debt holders' management post-default. From (23) and the discussion afterward, a relatively high α_y—which implies a greater $B'(y_b)$—helps satisfy the *IC* condition in SE. The lengthening path is also an equilibrium given the same initial state, in which equity holders find it optimal to keep issuing long-term bonds and default at $\left(\phi_b^L = \Phi\left(y_b^L\right), y_b^L\right)$. We compare the welfare of these two equilibria in Section 6.1.

This multiplicity of either SE or LE echoes the intuition of self-enforcing default in the sovereign debt literature (e.g., Cole and Kehoe [2000]). If bond investors expect equity holders to keep shortening the firm's maturity structure in the future and default early, then bond investors price this expectation in the bond's market valuation, inducing bond prices which can self-enforce the optimality of issuing short-term bonds via (14). Similarly, the belief of always issuing long-term bonds can be also self-enforcing.

A special situation can arise in which this multiplicity disappears:

Corollary 1. *Suppose that all points in the neighborhood of the default boundary have the same cornered equilibrium. Then, the unique equilibrium for any* (ϕ, y) *has to be that same cornered equilibrium.*

To see this, we know that any path has to end up, at least in the neighborhood of the default boundary, with $f = 0$ or $f = 1$ always. Working backwards, we see that the IC condition for the same cornered equilibrium also holds any distance away from the boundary. Finally, as cornered paths do not cross, uniqueness naturally follows. The corollary applies, for example, to the specification of the model with constant recovery, so that $B'(y_b) = 0$ everywhere as discussed in Section 4.3. There, we can only have LE on the default boundary. This in turn implies that–regardless of the initial point–LE is the the only equilibrium of the game, and the firm which keeps lengthening its maturity structure survives as long as possible subject to the endogenous default decision of equity holders.

5.1.2 Equilibrium regions

Due to $f_b^{nc}(y_b)$ being increasing in y_b, the set of shortening equilibria in the neighborhood of the bankruptcy boundary (if it exists) must take the form $[y_1, y_{max}]$ (recall that y_1 satisfies $f_b^{nc}(y_1) = 1$, i.e. the point y at which $f \leq 1$ starts binding, and y_{max} satisfies $\Phi(y_{max}) = 1$), and the set of lengthening equilibria must take the form $[y_{min}, y_0]$. Then, by our assumption $f \in [0, 1]$ (which bounds the rate of change of ϕ) and Proposition 2, we can partially characterize the equilibrium in different regions of the state space:

Proposition 3. *There exist at most six equilibrium regions:*

(I) Any initial point (ϕ, y) *above the lengthening path emanating from* $(\Phi(y_0), y_0)$ *and below the shortening path emanating from* $(\Phi(y_1), y_1)$ *can only feature interior equilibria.*

(S) Any initial point (ϕ, y) *above the lengthening path emanating from* $(\Phi(y_1), y_1)$ *has a unique equilibrium—a SE.*

(SI) Any initial point (ϕ, y) below the lengthening path emanating from $(\Phi(y_1), y_1)$, above lengthening path emanating from $(\Phi(y_0), y_0)$, and above the shortening path emanating from $(\Phi(y_1), y_1)$ has a SE and possible interior equilibria.

(SLI) Any initial point (ϕ, y) above the shortening path emanating from $(\Phi(y_1), y_1)$ and below the lengthening path emanating from $(\Phi(y_0), y_0)$ has both an SE and an LE, and possible interior equilibria.

(LI) Any initial point (ϕ, y) below the shortening path emanating from $(\Phi(y_1), y_1)$, below the lengthening path emanating from $(\Phi(y_0), y_0)$, and above the shortening path emanating from $(\Phi(y_0), y_0)$ has a LE and possible interior equilibria.

(L) Any initial point (ϕ, y) below the shortening path emanating from $(\Phi(y_0), y_0)$ has a unique equilibrium—a LE.

Proposition 3 combines three previously established results: the insight from Proposition 2 that checking the IC condition at default is sufficient for the whole path of any cornered equilibrium; that SE and LE paths do not cross; and that adjustments to the state-variables have to be gradual. As a result, there exists regions of no return, indicated by regions S and L in the left panel of Figure 4, for which a unique equilibrium exists. Intuitively, given gradual adjustments to the state variables, any maturity profile in region S (L), regardless of the future issuance profile $\{f_\tau\}$, cannot change fast enough to avoid hitting the bankruptcy boundary in the SE (LE) region. But then we know by Proposition 2 that LE (SE) must be the unique equilibrium in the whole region S (E).

A similar logic implies that in region I we can only have interior equilibria; it is because any cornered strategy would nevertheless hit the bankruptcy boundary in the interior region, invalidating the conjectured cornered strategy. Finally, regions SI, LI, and SLI are simply statements about what cornered equilibria can exist, besides the possibility of multiple interior equilibria.

5.2 Interior equilibria

The last objects to analyze are paths that originate in the interior region of the bankruptcy boundary, as indicated by the dashed region in the left panel of Figure 2. In Appendix A.4 we show that any point on the boundary $(\Phi(y_b), y_b)$ has a unique equilibrium issuance path $\{f\}_\tau$ leading to it that is defined via backward induction, i.e., no two distinct paths end at the same default

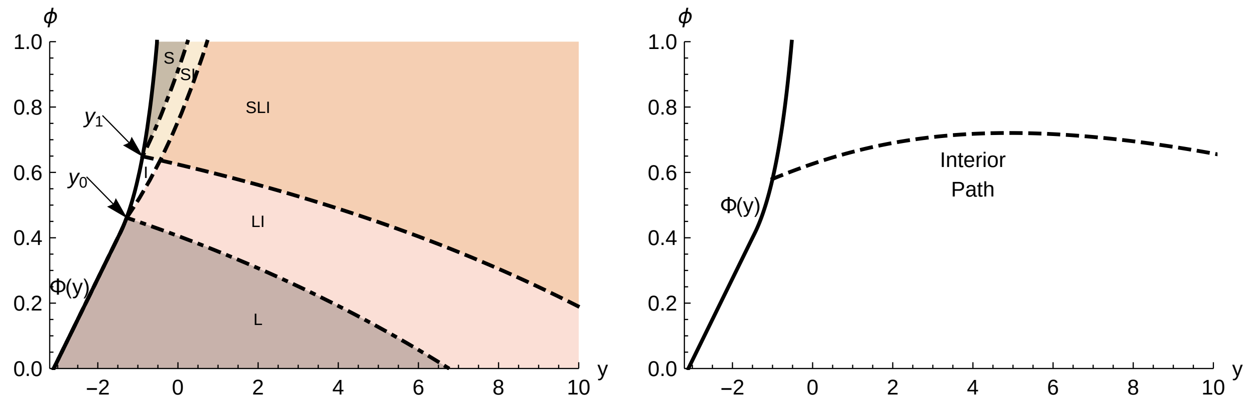

Figure 4: **Left Panel:** The six possible equilibrium regions I, S, L, SI, LI, and SLI as described in Proposition 3. **Right Panel:** An interior issuance path (dashed line) that fulfills the IC conditions at all of its points. For both panels parameters are given by $c = r = 10\%$, $D^{rf} = 1$, $E^{rf} = 12$, $\mu = 13$, $\zeta = .35$, $\delta_S = 5$, $\delta_L = 1$, $\alpha_y = 3$, $\alpha_X = .95$.

point. Further, we show that along any path with $IC(\tau, y_b) = 0$, we can derive the unique interior issuance policy f_τ at τ explicitly given the forward-looking endogenous equilibrium objects.[24] However, unlike in Proposition 2, we cannot prove uniqueness of interior equilibria at a distance from the default boundary. This is because at a sufficient distance, equilibrium paths may cross due to the non-constant nature of f.

To illustrate one interesting property of an interior equilibrium, we pick an ultimate default point on the boundary that lies in the region of interior equilibria (the dashed region in the left panel of Figure 2 which corresponds to region I in the left panel of Figure 4) and work our way backward to trace out the equilibrium path. The right panel of Figure 4 maps out one such path. We see that the firm's maturity structure along the shown path is no longer monotone in time (as opposed to the monotonicity in the cornered equilibria analyzed above), i.e., $\frac{d\phi(t)}{dt}$ switches signs (see earlier discussion around (6)). For large y's the firm is shortening its maturity structure with $1 > f_\tau > f_{ss}(\phi_\tau)$, leading to a slow rise in ϕ. However, once the firm is getting close to the default boundary $\Phi(y)$, it reverses course and starts lengthening its maturity structure $0 < f_\tau < f_{ss}(\phi_\tau)$, leading to a fall in ϕ.

[24] Forward looking here refers to natural time t, i.e., incorporating all times from today until the default time.

6 Welfare, Robustness, and Discussion

We first discuss the welfare implications of our equilibria identified in previous sections. We then discuss how robust our results are to the restrictive assumptions that we impose in this paper, and then compare our model to Brunnermeier and Oehmke [2013].

6.1 Welfare analysis

Because equity holders in our model only maximize the value of their stake in the firm, and are unable to commit to a default time ex-ante, their default decision and their issuance decisions leading up to default are not be maximizing firm value in general. This section analyzes the detailed welfare implications for various equilibria in our paper.

For subsequent analysis, let us take the derivative of firm value $V = E + \phi D_S + (1-\phi) D_L$ with respect to the maturity structure ϕ, which decomposes its impact on the firm value into three components:

$$V_\phi (\phi, y) = \underbrace{\Delta (\phi, y) + E_\phi (\phi, y)}_{\text{Incentive Compatibility } IC} + \underbrace{\phi \frac{\partial}{\partial \phi} D_S (\phi, y)}_{\text{Impact on ST bonds}} + \underbrace{(1-\phi) \frac{\partial}{\partial \phi} D_L (\phi, y)}_{\text{Impact on LT bonds}}. \tag{26}$$

Recall (19) showed that in the neighborhood of the default boundary equity holders are maximizing the proceeds of newly issued bonds.

6.1.1 Conflicts of interest in the neighborhood of default

We analyze (26) in the neighborhood of the default boundary $(\Phi(y_b), y_b)$ in this subsection. We have $IC = 0$ at default, hence the impact of maturity shortening on firm value is given by the last two terms in (26), which are the derivatives of the bond valuations with respect to the maturity structure, weighted at the current maturity structure.

In our stylized model, given the same coupon rate for both bonds, one can show that long-term bonds put more weight on the bankruptcy recovery $B(y_b)$ than short-term bonds.[25]

Lemma 2. *At the neighborhood of default boundary, the sensitivity of long-term debt with respect*

[25] Intuitively, long-term bonds have a higher discounted (future) default probability than short-term bonds.

to the maturity structure ϕ is greater than that of short-term debt:

$$\frac{\partial}{\partial \phi} D_L\left(\Phi\left(y_b\right), y_b\right) > \frac{\partial}{\partial \phi} D_S\left(\Phi\left(y_b\right), y_b\right). \tag{27}$$

Cornered equilibria. From Proposition 1, we have an SE, i.e., $f_b = 1$, if $\frac{\partial}{\partial \phi} D_S > 0$ according to (20). But then Lemma 2 implies that $\frac{\partial}{\partial \phi} D_L > \frac{\partial}{\partial \phi} D_S > 0$, i.e., shortening improves the values of both long-term and short-term bonds. Similarly, for an LE with $f_b = 0$, (21) and Lemma 2 imply $\frac{\partial}{\partial \phi} D_S < \frac{\partial}{\partial \phi} D_L < 0$, i.e., lengthening improves the values of both bonds as well. Thus, by inspecting (26), in any cornered equilibrium equity holders are taking the *locally* firm-value maximizing action, and everybody is better off by the equity holders' self-interested policy of hastening or delaying default slightly.

There are two points worth making about this stark local-efficiency result. First of all, in next subsection we will show that this *local* efficiency result at the neighborhood of default boundary fails when the firm is away from boundary. In general, as a common result in Leland-type models with costly default and debt-equity conflicts, there exists a *global* deviation featuring a sufficiently large delay in default to improve the total firm value. This discrepancy implies that firm value can be non-monotone in the firm's default time.[26] This non-monotonicity is behind the result that away from default boundary, the LE with a longer time to default can Pareto dominate the SE, as shown in the next section.

Second, the local-efficiency result is more or less coincidence, and relies heavily on Lemma 2. In fact, (27) might reverse if long-term bonds put *less* weight on the bankruptcy recovery $B\left(y_b\right)$ than short-term bonds, which is empirically relevant if short-term bonds are zero-coupon discount bonds in the form of say commercial paper while long-term bonds are traditional coupon bonds.[27] Perhaps a more empirically relevant situation is one where there are also other stakeholders present who may suffer from earlier default. In Appendix A.5.3, we imagine that the firm has another group of debt holders holding consol bonds whose valuation does not enter the equity holders' rollover

[26] Intuitively, we can understand the non-monotonicity through state-dependence of bankruptcy cost. In our example, the bankruptcy cost, which is measured by $A\left(y\right) - B\left(y\right)$, is endogenously determined via the optimal stopping problem (see the beginning of Section 4). Although it is always the case that $A\left(y\right) > B\left(y\right)$ with a positive bankruptcy cost, for sufficiently large α_y the slopes might reverse with $A'\left(y\right) < B'\left(y\right)$ for some y, i.e., the higher the defaulting cash-flows, the smaller the bankruptcy cost. Since earlier default leads to a higher defaulting cash-flows, this force naturally gives rise to the non-monotonicity result.

[27] To see this, with short-term debt coupon c_S less than long-term debt coupon c_L, at the default boundary we have $\frac{\partial D_S}{\partial \phi} - \frac{\partial D_L}{\partial \phi} = \left[c_S - c_L + \left(\delta_S - \delta_L\right)\left(1 - B\left(y_b\right)\right)\right] \frac{\partial \tau}{\partial \phi}$, which might turn negative if $c_S < c_L$.

decisions at all. As earlier default leads to value losses to consol bonds (another form of dilution), the maturity-shortening equilibrium may become locally inefficient, in the sense that right before default the firm value is improved by marginally lengthening the firm's maturity structure.

Interior equilibria. An interior equilibrium $f_b \in (0, 1)$ arises if and only $IC = 0$ holds in the vicinity of $\tau = 0$, i.e., $IC_\tau = 0$. From (19), Proposition 1 and Lemma 2, $IC_\tau = 0$ if and only if $\frac{\partial}{\partial \phi} D_S(\phi, y) < 0 < \frac{\partial}{\partial \phi} D_L(\phi, y)$ in the vicinity of the default boundary—short-term bond holders prefer a marginal lengthening of the maturity structure, whereas long-term bond holders prefer a marginal shortening. The issuance policy $f_b(y_b) \in (0, 1)$ in general does not keep the maturity structure constant, i.e., $f_b(y_b) \neq f_{ss}(\Phi(y_b))$, so that *local* conflicts of interest arise—equity holders and one part of the debt holders gain, while the other part of the debt holders looses. Exactly which bond holders get hurt depends on the characteristics of the equilibrium.

Regarding total firm value, we know that (19) suggests that the issuance policy maximizes the value of the currently issued short- and long-term bonds, which are issued in proportions $f_b(y_b)$ and $[1 - f_b(y_b)]$, respectively. However, the total value of the firm at default in (26) stems from the value of the stock of short- and long-term bonds outstanding, which are in proportion $\Phi(y_b)$ and $[1 - \Phi(y_b)]$, respectively. In general, $f_b(y_b) \neq \Phi(y_b)$, so that the equity's issuance policy does not maximize total firm value. Formally, we set condition (19) equal to 0 for interior equilibria, divide it by $m(\Phi(y_b))$ and subtract it from (26) evaluated at default, to finally get

$$V_\phi|_{\tau=0} = -[f_b(y_b) - \Phi(y_b)]\, \Delta_\phi(\Phi(y_b), y_b).$$

As the change of ϕ is proportional to $f_b(y_b) - f_{ss}(\Phi(y_b))$, the total firm value decreases when $V_\phi|_{\tau=0}\, d\phi \propto V_\phi|_{\tau=0}\, [f_b(y_b) - f_{ss}(\Phi(y_b))] < 0$.

At its heart, the conflict of interest among various stake holders (equity, long-term bonds, and short-term bonds) is driving our results. The value gains from issuance and default policies do not accrue to equity holders in a sufficient measure to incentivize them to undertake the socially optimal policy. On the boundary, equity holders are maximizing the value of debt issuance proceeds, instead of total firm value. In contrast to Brunnermeier and Oehmke [2013], the conflict of interest is not centered around direct redistribution (by expanding or shrinking the aggregate face-value of bonds) of a fixed amount of recovery value (*direct dilution*), but rather around the impact of the maturity structure ϕ on the timing of default and thus the changing size of the recovery value (*indirect dilution*). The bond price expressions show that long-term bonds load on the recovery value $B(y_b)$ and coupon c at a different rate than short-term bonds, and the former derive a larger percentage of their value from default recovery than the latter. Therefore, in the interior equilibrium, a conflict of interest between bond-holders arises with respect to the default timing.

6.1.2 Conflicts of interests away from default

The conflicts of interests we discussed above were of a *local* nature, i.e., we considered the impact of a slight change in ϕ on the different stakeholders of the company in the neighborhood of default. Away from the default boundary, the local derivatives we were considering above cannot be studied analytically anymore. We will rather concentrate on comparing the outcomes *across* different equilibria.

Suppose we are at a point away from the boundary at which there exist both a shortening and a lengthening equilibrium, as considered in Section 5 and depicted in Figure 1. Table 1 shows firm, equity, and debt valuations for several different equilibria for a starting value of $(\phi, y) = (0.28, 8.5)$. The first-best case is the one without debt-equity conflict, i.e., equity holders can commit to the best policy available. The lengthening and shortening equilibria are as discussed in Section 4. Finally, "Leland equilibrium" describes the equilibrium in which equity holders have committed to keeping ϕ constant at its current level via issuance policy $f = f_{ss}(\phi)$ until $\Phi(y_b) = \phi$ is hit (note that the default boundary $\Phi(y_b)$ is still the same even for Leland models), an assumption made in Leland-type models.

We highlight several interesting results in Table 1. First, the LE *Pareto dominates* the SE; the shortening strategy of always $f = 1$ in the SE—by adversely affecting the endogenous default policy-–hurts short-term and long-term debt holders, together with equity holders.[28] In other words, it is because equity holders, given the debt prices prevails in the shortening equilibrium, find that it is privately optimal to keep issuing short-term debt and eventually default earlier, which hurts everybody in equilibrium.

[28] The property of Pareto dominance may not hold generally, and we find other numerical examples in which relative to the shortening equilibrium, equity and short-term bond holders gain in the lengthening equilibrium while long-term bond holders lose strictly.

Initial $(\phi, y) = (0.28, 8.5)$	Time-to-default τ	V	E	D_S	D_L
First-best case	1.0039	5.6658	4.8350	0.9958	0.7667
Lengthening equilibrium	0.8913	5.6116	4.8115	0.9922	0.7254
Leland equilibrium	0.8062	5.4942	4.7214	0.9876	0.6893
Shortening equilibrium	0.7217	5.3839	4.6148	0.9825	0.6861

Table 1: Firm, equity, long-term bond, and short-term bond values for parameters $c = r = 10\%$, $D^{rf} = 1$, $E^{rf} = 12$, $\mu = 13$, $\zeta = .35$, $\delta_S = 5$, $\delta_L = 1$, $\alpha_y = 3$, $\alpha_X = .95$, and initial point $(\phi, y) = (0.28, 8.5)$.

Second, the introduction of flexible issuance strategies $f \neq f_{ss}(\phi)$ can be both beneficial (in case of the LE) and detrimental (in case of the SE) when compared to the Leland equilibrium with an inflexible issuance strategy. As may appear intuitive, having flexibility in ones issuance policy should help equity holders avoid inefficient (i.e., too early) default due to rollover pressure. However, the presence of the SE shows the downside of such flexibility—as equity holders cannot commit to any particular path of f and the resulting default policy, a worse equilibrium can arise with a self-fulfilling shortening spiral. Oftentimes, the SE arising from this added flexibility hurts total firm value. Furthermore, as Table 1 shows, the Leland equilibrium may even *Pareto dominate* SE.

6.2 Robustness

We have adopted a stylized setting in order to deliver the main economic mechanism in a transparent way. However, this simplification may come at some cost, as our framework misses at least two important empirically relevant features: cash-flow volatility, and potentially endogenous (de)leveraging policies. We discuss each in turn, as well as the restricted issuance space.

6.2.1 Stochastic cash flows

We work in a setting without cash-flow volatility. Mathematically, adding volatility to y leads to some second-order derivative terms in (10) and (12), and we are unable to recover the clean expression (A.26) for equity holders' IC condition right before default. We cannot think of any obvious link between fundamental volatility and endogenous debt maturity structure. Nevertheless, there is one interesting observation hinting that volatility might help the existence of shortening equilibria. Note that (20) requires debt values to go down if the firm marginally lengthens its maturity structure and hence delays default. With positive cash-flow volatility, if the firm survives longer, the additional volatility is likely to cause debt values to go down, as debt values are concave in the firm fundamental with capped upside—this exactly helps the existence of shortening equilibria. We await future research to explore this possibility.

6.2.2 Potential deleveraging

To isolate debt maturity from leverage decisions, we follow the Leland tradition in assuming that the firm commits to a constant aggregate face value (normalized to 1). Note, even with constant-face value, market leverage is not fixed as bond and equity values fluctuate with distance to default. Dynamic leverage decisions without commitment are a challenging research question itself,[29] and no doubt in practice firms have certain flexibilities in simultaneously adjusting their leverage and debt maturity structure.

In our setting, there is a strong force pushing firms to lengthen its debt maturity structure when it is cutting its debt face value (either voluntarily or involuntarily). Appendix A.5.1 gives the argument in details, but the intuition turns out to be quite simple. With a changing debt burden with face-value F_t, as a standard technique in this literature (Goldstein et al. [2001], Fischer et al. [1989], Dangl and Zechner [2006], DeMarzo and He [2014]), the *effective* recovery value per bond is $B(y_b)/F_t$, and the debt price in (23) is generally increasing in recovery value. A time-decreasing F_t thus translates into time-increasing recovery value for fixed y_b, which runs counter to the intuitive requirement that shortening equilibria need rapidly decreasing recovery values, and thus the firm will be more likely to issue long-term bonds. Conversely, firms would like to shorten their debt maturity structure if they are leveraging up toward bankruptcy (in fact, this observation is consistent with Brunnermeier and Oehmke [2013]).

[29]The literature usually takes the tractable framework of Fischer et al. [1989], Goldstein et al. [2001] so that the firm needs to buy back all outstanding debt if it decides to adjust aggregate debt face value. Apparently, this assumption requires a strong commitment ability on the side of equity holders. Recently, Dangl and Zechner [2006] study the setting where a firm can freely adjust its aggregate debt fact value downwards by issuing less bonds than the amount of bonds that are maturing. DeMarzo and He [2014] study the setting without any commitment on outstanding debt face value so that equity holders may either repurchase or issue more at any point of time; it is shown that equity holders always would like to issue more. In sharp contrast to our paper in which firms who commit to a constant aggregate face value but can freely adjust debt maturity structure over time, Dangl and Zechner [2006] and DeMarzo and He [2014] instead assume that the firm can change its book leverage, but is able to commit to certain debt maturity structure (parametrized by some exogenous rollover frequency).

6.2.3 Large issuance space $f \in [-f_L, f_H]$

Motivated by realistic trading frictions and institutional restrictions in buying back corporate bonds (Xu [2014]), we impose the exogenous bounds on the firm's bond issuance strategy. The restriction $f \in [0, 1]$ does not affect the underlying trade-off present in the model, but it allows for sharper analytical results especially in regards to uniqueness (see Proposition 1). In Appendix A.5.2, with f only restricted to a large issuance space $f \in [-f_L, f_H]$, for large enough but still finite f_L and f_H the equilibrium outcome in the neighborhood of the default boundary is uniquely $f_b^{nc}(y_b)$ given in (22), as it is easy to show that $f_b^{nc}(y_b)$ is finite for any y_b such that $\Phi(y_b) \in [0, 1]$. By continuity of all functions involved, any interior issuance path $\{f\}_\tau$ is well defined and finite, and again, for sufficiently large f_L and f_H is also interior.

6.2.4 Comparative statics with respect to outstanding face-value

Consider increasing the total amount of face-value outstanding F from $F = 1$. Suppose that the recovery function $B(y)$ satisfies the following condition (recall that y_1 is the point at which $f_b^{nc}(y_1) = 1$):

$$Q(y_1) > 0, \text{ where } Q(y) \equiv \left[(r+\zeta) B'(y) - 1\right] + \left[y + X\zeta - (r+\zeta) B(y)\right] \left[\frac{B''(y)}{B'(y)} + \frac{\mu_y'(y)}{\mu_y(y)}\right]. \quad (28)$$

Then, we can establish the following comparative static with respect to total debt burden F:

Proposition 4. *When face-value F increases, y_1 increases, i.e., $\frac{dy_1}{dF} > 0$. Further, when $B(\cdot)$ fulfills the condition (28), then as F increases, $\Phi(y_1)$ decreases, i.e., $\frac{d\Phi(y_1)}{dF} < 0$. Thus, under condition (28), the point $(y_1, \Phi(y_1))$ shifts in a south-east direction. If additionally $B(\cdot)$ is such that $Q(y_1) > [F - B(y_1)](\delta_S - \delta_L)\frac{\delta_L[1-\Phi(y_1)]}{\mu_y(y_1)}$, then increasing F expands the set of points (ϕ, y) that fulfill the conditions for a shortening equilibrium.*

The first result that $\frac{dy_1}{dF} > 0$ is intuitive: As in Leland [1994a], the greater the debt burden, the earlier the default. It is also intuitive to have $\frac{d\Phi(y_1)}{dF} < 0$ so that the defaulting debt maturity becomes longer, because a lower rollover frequency but a greater refinancing shortfall $F - B(y_1)$ (due to a larger F) can deliver a similar rollover loss which pushes equity holders to default.[30]

[30] The extra condition in (28) guarantees that the endogenous increase of y_1 does not overturn the direct effect of F, so that the refinancing shortfall $F - B(y_1)$ increases with F overall.

The second part of Proposition 4 delivers an empirical prediction: Conditional on $Q(y_1) > [F - B(y_1)](\delta_S - \delta_L)\frac{\delta_L[1-\Phi(y_1)]}{\mu_y(y_1)}$, larger outstanding face-value, all else equal, makes shortening equilibria more likely to arise. As F increases the dividing point between the interior and SE part of the bankruptcy boundary, $(y_1, \Phi(y_1))$, shifts in a south-east direction, and does so below the (original) shortening path emanating from $(\Phi(y_1), y_1)$ (in Figure 4 this is the path that separates regions S, SI and SLI (all the regions for which an SE exists) from I and LI).[31] Consequently, the boundary of points (y, ϕ) in which an SE exists also expands in a south-east direction. Thus, higher face-value makes a firm more prone to shortening equilibria. Our numerical example easily fulfills both conditions given in Proposition 4.

6.3 Comparison to Brunnermeier and Oehmke (2013)

Our analysis highlights an economic mechanism that is different from Brunnermeier and Oehmke [2013]. In that paper, the firm with a long-term asset is borrowing from a continuum of identical creditors. Only standard debt contracts are considered with promised face value and maturity, and covenants are not allowed. News about the long-term asset arrives at interim periods, so that a debt contract maturing on that date will be repriced accordingly, as in Diamond [1991]. The key assumption that Brunnermeier and Oehmke [2013] make is that the rollover for each period is cash-neutral, i.e., the aggregate face-value of bonds is adjusted so as to exactly give a zero rollover loss. For certain types of interim uncertainty resolutions (about profitability rather than recovery value), Brunnermeier and Oehmke [2013] show that, given other creditors' debt contracts, equity holders find it optimal to deviate by offering any individual creditor a debt contract that matures one period earlier, so that it gets repriced sooner. In equilibrium, equity holders offer the same deal to every creditor, and the firm's maturity will be "rat raced" to zero.

The repricing mechanism constitutes the key difference between Brunnermeier and Oehmke [2013] and our model. In their model, after negative interim news, a short-term bond gets repriced by adjusting up the promised face value to renegotiating bond holders. Importantly, all bonds are assumed to have equal seniority, so that interim expansion of aggregate face-value to new bond-holders *directly dilutes* existing bond-holders without repricing opportunities by lowering their

[31] Here, the slope of the shortening path (when working backwards in time) emanating from $(\Phi(y_1), y_1)$ is given by $-\frac{\delta_L[1-\Phi(y_1)]}{\mu_y(y_1)}$, whereas the slope of the movement in $(\Phi(y_1), y_1)$ itself when F changes is given by $-\frac{Q(y_1)}{[F-B(y_1)](\delta_S-\delta_L)}$.

proportional claim on the recovery value.

As emphasized in Section 2.2.1, to preclude *direct dilution* we follow Leland and Toft [1996] in assuming that the firm commits to maintain a constant total outstanding face value when refinancing its maturing bonds, which represents the minimum departure from the dynamic structural corporate finance literature. Besides, in practice, most bonds feature covenants with restrictions regarding the firm's future leverage policies, but rarely on the firm's future maturity structures. This empirical observation lends support to our premise of a full commitment on the firm's book leverage policy but no commitment on its debt maturity structure policy.

In essence, the commitment of maintaining a constant total outstanding face value amounts to a bond covenant about the firm's "book leverage,"[32] so that equity holders cannot simply issue more bonds to cover the firm's rollover losses as in Brunnermeier and Oehmke [2013]. Instead, in our model equity holders are absorbing these losses through their own deep pockets (or through equity issuance), and thus existing long-term bonds are insulated from *direct dilution*. However, equity holders are protected via the limited liability provision, and thus at some point will refuse to absorb these losses, leading to endogenous default. As highlighted in Section 3.2, the key driver of our model is exactly the interaction between the endogenous debt maturity structure and endogenous default decisions. In sum, once we shut down the interim *direct dilution* channel that drives the result in Brunnermeier and Oehmke [2013], we identify a new empirically relevant force—which operates through endogenous default timing, something we term *indirect dilution*—in our paper.

We have several empirical predictions that are unique to our model. First, we show that following an economic downturn, the likelihood of observing a shortening of the maturity structure increases—the economic force for shortening, *indirect dilution*, is present only in bad aggregate states as it operates via a shrinking recovery value. In contrast, in Brunnermeier and Oehmke [2013], *direct dilution* is the driving force of the shortening result, and is present irrespective of the aggregate state. Second, as illustrated by the left panel of Figure 4, shortening equilibria exists only when the existing debt maturity is sufficiently short (ϕ is sufficiently high). Hence, our model suggests that conditional on deteriorating economic conditions, debt maturity shortening is more likely to be observed in firms with already short maturity structures. Finally, the analysis

[32]The covenant specifies a constant "book leverage." However, the firm's market leverage, which is defined as the market value of equity divided by the firm's market value, varies with the cash-flow state y_t.

in Section 6.2.4 suggests that firms with greater debt burden are more likely have debt maturity shortening. There is no obvious reasoning to think that Brunnermeier and Oehmke [2013] has the same empirical prediction.[33]

7 Empirical Predictions and Concluding Remarks

Our model with endogenous dynamic debt maturity structure is based on a Leland framework where the basic agency conflict is the equity holders' endogenous default decision at the expense of debt holders. Dynamic debt maturity choice affects the endogenous default decision though rollover concerns, leading to *indirect dilution* of existing debt holders. In the meantime, the endogenous default decision affects bond valuations, feeding back to the endogenous debt maturity structure.

We show that when cash-flows deteriorate over time so that the debt recovery value is affected by the endogenous default timing, a shortening equilibrium with earlier default can emerge. The short-term debt holders gain from the earlier default: The benefit of a more favorable recovery value by taking the firm over earlier outweighs the increased expected default risk due to earlier default. This seems to be an empirical relevant force during 2007/08 crisis during which we observed debt maturity shortening together with earlier default, as the fundamental values of collateral assets deteriorated rapidly over time and bond holders gained by taking possession of the collateral sooner. We further show that the shortening equilibrium can be *locally* efficient while being *globally* inefficient (in fact, it could be Pareto dominated), relative to the lengthening equilibrium which features a much longer time to default.

Though highly stylized, our model yields the following empirical predictions that are not implied by the *direct dilution* mechanism of Brunnermeier and Oehmke [2013]. First, one is more likely to observe debt maturity shortening in response to worsening economic conditions. This is consistent with the empirical findings cited at the beginning of the Introduction: that speculative-grade firms are actively *lengthening* their debt maturity structure in *good* times as shown by Xu [2014], and that financial firms are shortening their debt maturity shortening right before 2007/08 crisis when the subprime mortgage market is worsening, as documented by Brunnermeier [2009], Krishnamurthy

[33] The logic of Brunnermeier and Oehmke [2013] seems to suggest the opposite, as the *direct dilution* motive seems to be the strongest when the existing debt contracts are relatively long-term. Of course, this conjecture require a rigorous analysis to confirm.

[2010], Gorton et al. [2015]. Second, our model suggests that conditional on worsening economic conditions, debt maturity shortening is more likely to be observed in firms with already short debt maturity structure and greater debt burden, an empirical prediction that is readily tested. Finally, Garcia-Appendini and Montoriol-Garriga [2014] present some evidence that when approaching default firms start by issuing more short-term debt, but stop issuing short-term debt right before default. This is somewhat consistent with the non-monotone interior equilibrium found in Figure 4 in Section 5.2.

We obtain great tractability and hence sharp analytical results by assuming that the firm commits to a constant debt face value over time, and there is no volatility in cash-flows. As we discussed in Section 6.2, relaxing either of them is a nontrivial task, and will be an interesting direction for future research.

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A Appendix

A.1 Change of variables

We will solve the model in terms of (τ, y_b), that is time to maturity and cash-flow at time of bankruptcy. This change of variables transforms the PDEs in the main part of the paper into ODEs on the equilibrium path. We then calculate separately the IC conditions via the derivatives of E_ϕ under different assumptions of the issuance strategies. For the moment, fix the issuance strategy $f(\tau)$.

The proportion of short-term debt $\phi(\tau, y_b)$. Recall that we have $\phi \equiv \frac{S}{P}$ is the proportion of short-term debt. Consider an arbitrary path $f(\tau) \in [0,1]$ for the issuance strategy. Then, we have

$$\phi'(\tau) = \phi(\tau)\left[\delta_S(1-f(\tau)) + f(\tau)\delta_L\right] - \delta_L f(\tau)$$

Integrating up, imposing $\phi(0) = \phi_b = \Phi(y_b)$, we have

$$\phi(\tau, y_b) = e^{\int_0^\tau [\delta_S(1-f_s)+f_s\delta_L]ds}\left[\Phi(y_b) - \delta_L \int_0^\tau e^{-\int_0^s [\delta_S(1-f_u)+f_u\delta_L]du} f_s ds\right] \tag{A.1}$$

Suppose that $f(\tau) = f$ is constant throughout. Then we can solve to get

$$\begin{aligned}
\phi(\tau, y_b)_{f(\tau)=f,\forall\tau} &= e^{\tau[\delta_S(1-f)+f\delta_L]}\left[\Phi(y_b) - \delta_L f \frac{1-e^{-\tau[\delta_S(1-f)+f\delta_L]}}{\delta_S(1-f)+f\delta_L}\right] \\
&= e^{\tau[\delta_S(1-f)+f\delta_L]}\left[\Phi(y_b) - \frac{\delta_L f}{\delta_S(1-f)+f\delta_L}\right] + \frac{\delta_L f}{\delta_S(1-f)+f\delta_L}
\end{aligned}$$

In the general setting, taking derivatives, while keeping $f(\tau)$ fixed, we have

$$\frac{\partial h_1}{\partial \tau} = \frac{\partial \phi(\tau, y_b)}{\partial \tau} = \phi(\tau, y_b)\left[\delta_S(1-f(\tau)) + f(\tau)\delta_L\right] - \delta_L f(\tau) \tag{A.2}$$

$$\frac{\partial h_1}{\partial y_b} = \frac{\partial \phi(\tau, y_b)}{\partial y_b} = \Phi'(y_b)\, e^{\int_0^\tau [\delta_S(1-f_s)+f_s\delta_L]ds} \tag{A.3}$$

The current cash-flow state $y(\tau, y_b)$. The differential equation for y gives $\frac{\partial y(\tau, y_b)}{\partial y_b} > 0$.

Derivatives w.r.t. ϕ. The ODEs are solved in terms of

$$\mathbf{z} = (\tau, y_b) \tag{A.4}$$

However, the incentives of the equity holders are derived from the Markov system

$$\mathbf{x} = (\phi, y), \tag{A.5}$$

as the optimal f requires the derivative E_ϕ. We are looking for points $\mathbf{z} = \mathbf{g}(\mathbf{x})$ such that $\mathbf{h}(\mathbf{x}, \mathbf{z}) = \mathbf{h}(\mathbf{x}, \mathbf{g}(\mathbf{x})) = \mathbf{0}$ where

$$\mathbf{h}(\mathbf{x}, \mathbf{z}) = \begin{bmatrix} h_1(\mathbf{x}, \mathbf{z}) \\ h_2(\mathbf{x}, \mathbf{z}) \end{bmatrix} = \begin{bmatrix} -\phi + \phi(\tau, y_b) \\ -y + y(\tau, y_b) \end{bmatrix} = \mathbf{0} \tag{A.6}$$

and where

$$\mathbf{g}(\mathbf{x}) = \begin{bmatrix} \tau(\phi, y) \\ y_b(\phi, y) \end{bmatrix} \tag{A.7}$$

To calculate the derivative of of for example$E(\tau, y_b) = E(\mathbf{z})$ w.r.t. ϕ, we have to use

$$\frac{\partial}{\partial \phi} E(\tau, y_b) = E_\tau(\tau, y_b)\frac{\partial \tau}{\partial \phi} + E_{y_b}(\tau, y_b)\frac{\partial y_b}{\partial \phi} = \left[\frac{\partial}{\partial \mathbf{z}} E(\mathbf{z})\right] \cdot \left[\frac{\partial \mathbf{z}}{\partial \phi}\right] \tag{A.8}$$

The Jacobian matrix is given by

$$\mathbf{J} = \frac{\partial \mathbf{h}(\mathbf{x},\mathbf{z})}{\partial \mathbf{z}} = \begin{bmatrix} \frac{\partial h_1}{\partial \tau} & \frac{\partial h_1}{\partial y_b} \\ \frac{\partial h_2}{\partial \tau} & \frac{\partial h_2}{\partial y_b} \end{bmatrix} \tag{A.9}$$

Then, applying the chain rule when taking the derivative w.r.t. x_i, $\frac{\partial \mathbf{h}}{\partial x_i} + \frac{\partial \mathbf{h}}{\partial \mathbf{z}}\frac{\partial \mathbf{z}}{\partial x_i} = \mathbf{0}$, we have for $x_i = \phi$,

$$\frac{\partial \mathbf{z}}{\partial \phi} = \frac{\partial \mathbf{g}(\mathbf{x})}{\partial \phi} = \frac{\partial}{\partial \phi}\begin{bmatrix} \tau(\phi, y) \\ y_b(\phi, y) \end{bmatrix} = -\mathbf{J}^{-1}\frac{\partial}{\partial \phi}\mathbf{h}(\mathbf{x},\mathbf{z}) \tag{A.10}$$

Let us calculate the different derivatives. First, we have

$$\frac{\partial h_1}{\partial \phi} = -1 \tag{A.11}$$

$$\frac{\partial h_2}{\partial \phi} = 0 \tag{A.12}$$

so that $\frac{\partial}{\partial \phi}\mathbf{h}(\mathbf{x},\mathbf{z}) = -[1,0]^{\top}$. Then, we have

$$\begin{aligned} \frac{\partial \mathbf{z}}{\partial \phi} = \begin{bmatrix} \frac{\partial \tau(\phi,y)}{\partial \phi} \\ \frac{\partial y_b(\phi,y)}{\partial \phi} \end{bmatrix} &= -\begin{bmatrix} \frac{\partial h_1}{\partial \tau} & \frac{\partial h_1}{\partial y_b} \\ \frac{\partial h_2}{\partial \tau} & \frac{\partial h_2}{\partial y_b} \end{bmatrix}^{-1}\begin{bmatrix} \frac{\partial h_1}{\partial \phi} \\ \frac{\partial h_2}{\partial \phi} \end{bmatrix} \\ &= \frac{1}{\frac{\partial h_1}{\partial \tau}\frac{\partial h_2}{\partial y_b} - \frac{\partial h_1}{\partial y_b}\frac{\partial h_2}{\partial \tau}}\begin{bmatrix} \frac{\partial h_2}{\partial y_b} & -\frac{\partial h_1}{\partial y_b} \\ -\frac{\partial h_2}{\partial \tau} & \frac{\partial h_1}{\partial \tau} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{\frac{\partial h_1}{\partial \tau}\frac{\partial h_2}{\partial y_b} - \frac{\partial h_1}{\partial y_b}\frac{\partial h_2}{\partial \tau}}\begin{bmatrix} \frac{\partial h_2}{\partial y_b} \\ -\frac{\partial h_2}{\partial \tau} \end{bmatrix} \\ &= \frac{1}{\frac{\partial y(\tau,y_b)}{\partial y_b}\frac{\partial \phi(\tau,y_b)}{\partial \tau} - \frac{\partial y(\tau,y_b)}{\partial \tau}\frac{\partial \phi(\tau,y_b)}{\partial y_b}}\begin{bmatrix} \frac{\partial y(\tau,y_b)}{\partial y_b} \\ -\frac{\partial y(\tau,y_b)}{\partial \tau} \end{bmatrix} \end{aligned} \tag{A.13}$$

Thus, we ultimately have

$$\begin{bmatrix} \frac{\partial \tau(\phi,y)}{\partial \phi} \\ \frac{\partial y_b(\phi,y)}{\partial \phi} \end{bmatrix} = \frac{1}{\frac{\partial y(\tau,y_b)}{\partial y_b}\left\{\phi(\tau,y_b)\left[\delta_S(1-f(\tau)) + f(\tau)\delta_L\right] - \delta_L f(\tau)\right\} - \mu_y(y(\tau,y_b))\Phi'(y_b)e^{\int_0^{\tau}[\delta_S(1-f_s)+f_s\delta_L]ds}}\begin{bmatrix} \frac{\partial y(\tau,y_b)}{\partial y_b} \\ -\mu_y(y(\tau,y_b)) \end{bmatrix} \tag{A.14}$$

A.2 Proofs of Section 3

First, an issuance policy f_b is called *admissible* as a defaulting strategy on the boundary if it leads to immediate default, i.e., the trajectory of $(\Phi(y_b), y_b)$ points *into* the bankruptcy region $\mathcal{B}$. The left panel of Figure 5 illustrates the intuition: When we are at a point $(\Phi(y_b), y_b)$, we want to ensure that any admissible strategy $f \in [0,1]$ still points into the bankruptcy region. For paths $f = 0$ and $f = 1$ this is the case. As an example of a path that violates admissibility, we graph a path for $f = -1$ which points inside $\mathcal{C}$, the continuation region, and not inside $\mathcal{B}$, the bankruptcy region. Thus, the point $(\Phi(y_b), y_b)$ would not be a defaulting point for all strategies f.

Lemma 3. *An issuance strategy f_b is admissible as a defaulting strategy on the boundary if*

$$f_b > f_{admissible}(y_b) \equiv \frac{\Phi(y_b)\delta_S - \mu_y(y_b)\Phi'(y_b)}{\Phi(y_b)\delta_S + [1-\Phi(y_b)]\delta_L} \tag{A.15}$$

with $f_{admissible}(y_b) < 1$.

We assume throughout the paper that every point on the bankruptcy boundary is admissible, that is $f_{admissible}(y_b) \leq 0$ for all $y_b \in [y_{min}, y_{max}]$ where $\Phi(y_{min}) = 0$ and $\Phi(y_{max}) = 1$.

Proof of Lemma 3. We cannot allow such f's that have (ϕ, τ) pointing inside the bankruptcy region $\mathcal{B}$ when increasing the time to maturity τ. Thus, we need to impose

$$\Phi'(y_b) > \frac{\left.\frac{d\phi(\tau,y_b)}{d\tau}\right|_{\tau=0}}{\left.\frac{dy(\tau,y_b)}{d\tau}\right|_{\tau=0}} = \frac{\Phi(y_b)\left[\delta_S(1-f) + f\delta_L\right] - \delta_L f}{\mu_y(y_b)}. \tag{A.16}$$

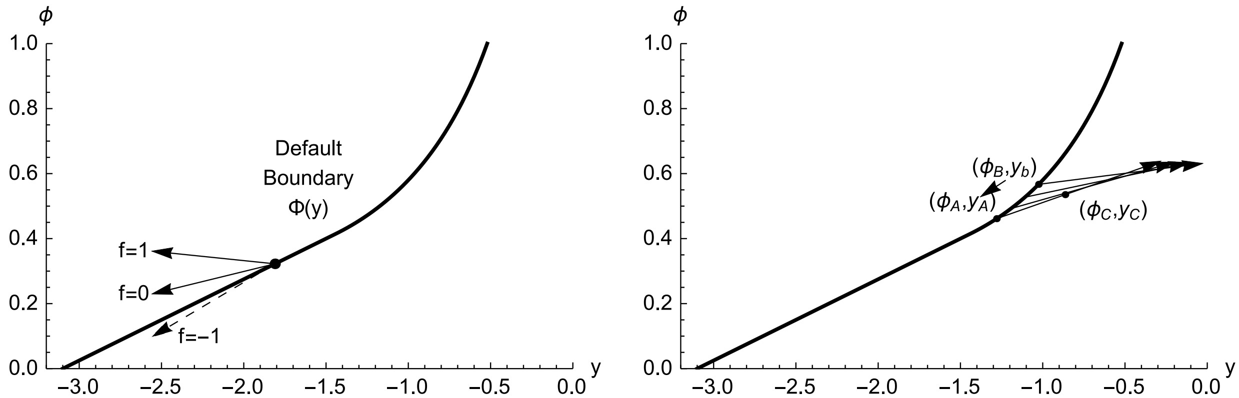

Figure 5: **Left Panel:** Linearized forward (in time) trajectory from point $(\Phi(y_b), y_b)$ for different arbitrary issuance strategies f. **Right Panel:** Linearized equilibrium backward (in time) trajectories from different boundary points $(\Phi(y_b), y_b)$ with equilibrium issuance strategy $f(y_b)$. The point (ϕ_C, y_C) is the closest intersection point of the (linearized) equilibrium path emanating from $(\Phi(y_A), y_A)$ to any neighboring (linearized) equilibrium path.

Rearranging, we have the following inequality that defines admissible f:

$$0 \quad > \quad \{\Phi(y_b)[\delta_S(1-f)+f\delta_L]-\delta_L f\}-\mu_y(y_b)\Phi'(y_b). \tag{A.17}$$

and setting this equal to 0 and solving for f, we get (A.15). □

Next, let us derive debt and equity values for given paths of f for (τ, y_b).

Debt. Debt has an ODE

$$(r+\delta_i+\zeta)D_i(\tau,y_b)=(c_i+\delta_i+\zeta)-\frac{\partial}{\partial\tau}D_i(\tau,y_b) \tag{A.18}$$

that is solved by

$$D_S(\tau,y_b) \quad = \quad \frac{c_S+\delta_S+\zeta}{r+\delta_S+\zeta}+e^{-(r+\delta_S+\zeta)\tau}\left[B(y_b)-\frac{c_S+\delta_S+\zeta}{r+\delta_S+\zeta}\right] \tag{A.19}$$

$$D_L(\tau,y_b) \quad = \quad \frac{c_S+\delta_L+\zeta}{r+\delta_L+\zeta}+e^{-(r+\delta_L+\zeta)\tau}\left[B(y_b)-\frac{c_S+\delta_L+\zeta}{r+\delta_L+\zeta}\right] \tag{A.20}$$

Importantly, for a given (τ, y_b) debt values are independent of the path of f. Imposing $c_i = r$ for $i \in \{S, L\}$ we get the result that $\Delta > 0$.

Equity. Equity solves the ODE where $y = y(\tau, y_b)$ and $\phi = \phi(\tau, y_b)$

$$(r+\zeta)E(\tau,y_b)=y+\zeta E^{rf}-c+m(\phi)[fD_S(\tau,y_b)+(1-f)D_L(\tau,y_b)-1]-E_\tau(\tau,y_b) \tag{A.21}$$

with boundary condition $\frac{\partial}{\partial\tau}E(\tau,y_b)\big|_{\tau=0}=0$. Integrating up for a given path of f, we have

$$\begin{aligned} E(\tau,y_b) \quad = \quad & \int_0^\tau e^{(r+\zeta)(u-\tau)}\left\{y(u,y_b)+\zeta E^{rf}-c\right. \\ & \left.+m(\phi(u,y_b))[f_u D_S(u,y_b)+(1-f_u)D_L(u,y_b)-1]\right\}du \end{aligned} \tag{A.22}$$

Here, we can see how f affects the value of equity even for a given (τ, y_b).

Lemma 4. *At the endogenous default boundary, we have smooth-pasting conditions in each dimension:* $E_\phi(\Phi(y_b), y_b) = 0$ *and* $E_y(\Phi(y_b), y_b) = 0$.

Proof of Lemma 4. $E(\Phi(y_b), y_b) = 0$ at default is obvious as equity defaults when their cash-flows turn exactly zero in our deterministic setting. Plugging in $E(\Phi(y_b), y_b) = 0$ into the ODE for equity valuation, we see that

$E_\tau(\tau, y_b)|_{\tau=0} = 0$. Change the coordinates of the state space to (ϕ, y), we have

$$E_\phi(\Phi(y_b), y_b)|_{\tau=0} = E_\tau(\tau, y_b)\left.\frac{\partial \tau}{\partial \phi}\right|_{\tau=0} + E_{y_b}(\tau, y_b)\left.\frac{\partial y_b}{\partial \phi}\right|_{\tau=0} = E_{y_b}(\tau, y_b)\left.\frac{\partial y_b}{\partial \phi}\right|_{\tau=0}, \tag{A.23}$$

where we use $E_\tau(\tau, y_b)|_{\tau=0} = 0$. However, we have $E_{y_b}(\tau, y_b)|_{\tau=0} = 0$, because equity defaults at $\tau = 0$ and y_b only affects the recovery value that bond holders receives and equity holders get nothing. (Note that $E_{y_b}(\tau, y_b)$ fixes τ while changes y_b; it differs from $E_y(\phi, y)$). Similarly we can show $E_y(\Phi(y_b), y_b) = 0$. □

Because the recovery value $B(y_b)$ is weakly increasing in y_b, i.e., $B'(y_b) \geq 0$, one can verify that $\Phi(y_b)$ is strictly increasing in y_b. Note that $y_b - c + \zeta E^{rf} + m(\Phi)[B(y_b) - 1] = 0 \iff y_b - c + \zeta E^{rf} = m(\Phi)[1 - B(y_b)] > 0$, so that

$$\Phi'(y_b) = \frac{1}{\delta_S - \delta_L} \times \frac{[1 - B(y_b)] + B'(y_b)\left(y_b - c + \zeta E^{rf}\right)}{[1 - B(y_b)]^2} = \frac{1}{\delta_S - \delta_L} \times \frac{1 + m(\Phi) B'(y_b)}{[1 - B(y_b)]} > 0.$$

Evaluating the derivatives of time-to-default and defaulting cash-flows w.r.t. the maturity structure at $\tau = 0$, $\left.\frac{\partial y(\tau, y_b)}{\partial y_b}\right|_{\tau=0} = 1$, we have

$$\begin{bmatrix} \frac{\partial \tau(\phi, y)}{\partial \phi} \\ \frac{\partial y_b(\phi, y)}{\partial \phi} \end{bmatrix}_{\tau=0} = \frac{1}{\{\Phi(y_b)[\delta_S(1 - f(0)) + f(0)\delta_L] - \delta_L f(0)\} - \mu_y(y_b)\Phi'(y_b)} \begin{bmatrix} 1 \\ -\mu_y(y_b) \end{bmatrix}$$

Thus, for any admissible $f(0)$ on the default boundary, the denominator is negative by (A.17), so that the time-to-default shrinks and the defaulting cash-flow increases as the debt maturity shortens, i.e., $\left.\frac{\partial \tau}{\partial \phi}\right|_{\tau=0} < 0$ and $\left.\frac{\partial y_b}{\partial \phi}\right|_{\tau=0} > 0$.

A.3 Proofs of Section 4

Proof of Lemma 1. Differentiating $IC(\tau, y_b) = \Delta(\tau, y_b) + E_\phi(\tau, y_b)$ w.r.t. τ, we have $IC_\tau(\tau, y_b) = \Delta_\tau(\tau, y_b) + E_{\phi\tau}(\tau, y_b)$. Next, we differentiate (A.21) w.r.t. ϕ to get

$$E_{\tau\phi}(\tau, y_b) = m'(\phi)[f D_S(\tau, y_b) + (1 - f) D_L(\tau, y_b) - 1] + m(\phi)\left[f \frac{\partial D_S(\tau, y_b)}{\partial \phi} + (1 - f)\frac{\partial D_L(\tau, y_b)}{\partial \phi}\right] - (r + \zeta) E_\phi(\tau, y_b) \tag{A.24}$$

where $E_{\tau\phi}$ (with slightly abused notation) is defined as

$$E_{\tau\phi} = \frac{\partial}{\partial \phi} E_\tau(\tau, y_b) = E_{\tau\tau}(\tau, y_b)\frac{\partial \tau}{\partial \phi} + E_{\tau y_b}(\tau, y_b)\frac{\partial y_b}{\partial \phi}$$

Note here that we are using the envelope theorem with regards to derivatives of f w.r.t. ϕ—as equity is optimizing, marginal changes in ϕ on f can be ignored.

Plugging in for $E_{\phi\tau}(\tau, y_b)$ from (A.24), we have

$$\begin{aligned} IC_\tau(\tau, y_b) = & \overbrace{\Delta_\tau(\tau, y_b)}^{(1)} + \overbrace{m'(\phi)[f D_S(\tau, y_b) + (1 - f) D_L(\tau, y_b) - 1]}^{(2)} \\ & \underbrace{+ m(\phi)\left[f \frac{\partial D_S(\tau, y_b)}{\partial \phi} + (1 - f)\frac{\partial D_L(\tau, y_b)}{\partial \phi}\right]}_{(3)} - \underbrace{(r + \zeta) E_\phi(\tau, y_b)}_{(4)}. \end{aligned}$$

Interpreting the terms, we have the following terms:

1. Change in the bond price wedge purely from time-to-default (keeping y_b fixed).
2. Change in rollover speed multiplied by the rollover loss.
3. Change in value of newly issued bonds multiplied by rollover speed.
4. Change in equity continuation value multiplied by discounting terms.

Substituting in $\Delta_\tau = (\delta_S - \delta_L) - (r + \zeta)\Delta - [\delta_S D_S - \delta_L D_L]$ from (A.18), and use $m'(\phi) = \delta_S - \delta_L$ from (4), we have

$$IC_\tau(\tau, y_b) = \underbrace{-[r + \zeta + \delta_S(1 - f) + \delta_L f]\Delta(\tau, y_b)}_{(1)+(2)} + \underbrace{m(\phi)\left[f \frac{\partial D_S(\tau, y_b)}{\partial \phi} + (1 - f)\frac{\partial D_L(\tau, y_b)}{\partial \phi}\right]}_{(3)} - \underbrace{(r + \zeta) E_\phi(\tau, y_b)}_{(4)}$$

and we see that (1) and (2) combine to yield an expression that involves the price-wedge itself times an issuance weighted discounting term. At default, the terms (1) and (2) vanish as the price wedge between the bonds vanishes by equal seniority, and the change in the continuation value term (4) is zero by optimality. This is linked to the fact that equity defaults at a point at which its expected cash-flows, as well as its default payoffs, are approximately zero. Formally, at $\tau \to 0$ we have $IC = \Delta = E_\phi = 0$, so we are left with term (3). Q.E.D. □

Proof of Proposition 1. First, let us concentrate on f_b on the boundary. Taking derivatives of (A.19) and (A.20) w.r.t. ϕ via (A.14), and evaluating at $\tau = 0$, we have

$$\left.\frac{\partial D_i}{\partial \phi}\right|_{\tau=0} = \frac{(r+\zeta+\delta_i)\left[1-B(y_b)\right]-\mu_y(y_b)\,B'(y_b)}{\left\{\Phi(y_b)\left[\delta_S(1-f_b)+f_b\delta_L\right]-\delta_L f_b\right\}-\mu_y(y_b)\,\Phi'(y_b)} \tag{A.25}$$

Using Lemma 1, i.e. (19), and plugging in, we have

$$\begin{aligned}\left.\frac{\partial IC(\tau,y_b)}{\partial \tau}\right|_{\tau=0} &= m(\Phi(y_b))\left[f_b\frac{\partial D_S(\tau,y_b)}{\partial\phi}+(1-f_b)\frac{\partial D_L(\tau,y_b)}{\partial\phi}\right]_{\tau=0}. && \text{(A.26)}\\ &= m(\Phi(y_b))\frac{\left[r+\zeta+f_b\delta_S+(1-f_b)\,\delta_L\right]\left[1-B(y_b)\right]-\mu_y(y_b)\,B'(y_b)}{\left\{\Phi(y_b)\left[\delta_S(1-f_b)+f_b\delta_L\right]-\delta_L f_b\right\}-\mu_y(y_b)\,\Phi'(y_b)} && \text{(A.27)}\end{aligned}$$

As $m(\phi) \geq \delta_L > 0$, we can ignore this term for determining the sign. Next, let us collect all terms in the numerator multiplying f_b, which are given by $(\delta_S - \delta_L)\left[1 - B(y_b)\right] > 0$. Further, we know from condition (A.17) that for all admissible f_b the denominator has to be negative. Thus, we can concentrate on the numerator to determine the optimal f_b. We have

$$\begin{aligned}&\left[r+\zeta+f_b\delta_S+(1-f_b)\,\delta_L\right]\left[1-B(y_b)\right]-\mu_y(y_b)\,B'(y_b)\\ =\;&(r+\zeta+\delta_L)\left[1-B(y_b)\right]-\mu_y(y_b)\,B'(y_b)\\ &+f_b\cdot(\delta_S-\delta_L)\left[1-B(y_b)\right]\end{aligned} \tag{A.28}$$

and we see that we have a linear increasing function in f_b. Thus, we have at most one unique root in (A.28), given by $f_b^{nc}(y_b)$. Importantly, we also know from (A.28) and the admissibility condition that IC_τ crosses 0 from above if at all. As the numerator is monotone, this implies a unique equilibrium. Since we have the restriction $f_b \in [0,1]$, if the numerator is everywhere negative for $f_b \in [0,1]$, then $f_b = 1$ if admissible. If the numerator is everywhere positive for $f_b \in [0,1]$, then $f_b = 0$ if this is admissible. Lastly, if there exits an admissible

$$f_b^{nc}(y_b) = \frac{\mu_y(y_b)\,B'(y_b)-(r+\zeta+\delta_L)\left[1-B(y_b)\right]}{(\delta_S-\delta_L)\left[1-B(y_b)\right]} \in (0,1) \tag{A.29}$$

then this is the unique equilibrium. Note that

$$\frac{d}{dy_b}f_b^{nc}(y_b) = \frac{1}{(\delta_S-\delta_L)}\left(\frac{\mu_y(y_b)\,B'(y_b)}{\left[1-B(y_b)\right]}\right)' = \frac{\left[\mu_y(y_b)\,B''(y_b)+\mu_y'(y_b)\,B'(y_b)\right]\left[1-B(y_b)\right]+\mu_y(y_b)\left(B'(y_b)\right)^2}{(\delta_S-\delta_L)\left[1-B(y_b)\right]^2} > 0 \tag{A.30}$$

as $B'(y_b) \geq 0$ and $B''(y_b) \geq 0$.We thus have

$$f_b(y_b) = \min\left[1, \max\left[f_b^{nc}(y_b), 0\right]\right] \tag{A.31}$$

as the unique equilibrium subject to admissibility on the bankruptcy boundary. As we assumed $f_{admissible}(y_b) < 0$ for all points on the bankruptcy boundary this concludes the proof.

We now prove equilibrium uniqueness in the neighborhood of the default boundary. What we need to show is that even though every point on the bankruptcy boundary has a unique path leading to it, that for points close to the boundary this statement can be inverted—for an arbitrary point in the neighborhood of the bankruptcy boundary there exists a unique equilibrium path to the boundary. In essence, we need to show that we can invert the problem. The right panel of 5 illustrates the intuition: the intersection of the backward linearized equilibrium paths for different points on the boundary stays bounded away from the boundary.

What we need to show that for any two points, say $A = (y_A, \Phi(y_A))$ and $B = (y_B, \Phi(y_b))$, the backward linearized paths originating from these points cross at a distance from the boundary. For $f_b(y_i) \in \{0,1\}$ this is straightforward—paths are essentially parallel and do not cross as f is held constant—this is true for the actual paths, and not just for the linearized paths. However, for $y_b \in (y_0, y_1)$ such that $f_b(y_b) \in (0,1)$, moving along the boundary, i.e. changing y_b, changes f_b and thus changes the direction of the path.

First, the linearized path from A is described by $\phi_A(y) = \Phi(y_A) + m_A(y - y_A)$ where the slope is given by

$$m_A = \frac{d\phi}{dy} = \frac{\Phi(y_A)\,\delta_S - [\Phi(y_A)(\delta_S - \delta_L) + \delta_L]\, f_b(y_A)}{\mu_y(y_A)}$$

and similarly, we have $\phi_B(y) = \Phi(y_B) + m_B(y - y_B)$ where

$$m_B = \frac{\Phi(y_B)\,\delta_S - [\Phi(y_B)(\delta_S - \delta_L) + \delta_L]\, f_b(y_B)}{\mu_y(y_B)}$$

We are looking for an intersect y_C that defines a point $C = (\phi_C, y_C)$ where the two lines meet, that is,

$$\phi_C = \phi_A(y_C) = \phi_B(y_C) \iff y_C = \frac{\Phi(y_A) - \Phi(y_B) + m_B y_B - m_A y_A}{m_B - m_A}$$

as $f_b'(y_b) > 0$ this point exists and is bounded away from the boundary for any two points on the boundary that are at a distance from each other. We next shrink the distance between the two points, A and B. To this end, suppose that $y_B = y_A + \varepsilon$. Then the distance along the curve between A and B, which we call c, is approximately

$$c = \sqrt{(y_B - y_A)^2 + (\Phi(y_B) - \Phi(y_A))^2} \approx \varepsilon\sqrt{1 + (\Phi'(y_A))^2}$$

Let us assume for the moment that $\mu_y(y) = \mu$. Then, we approximate the slope m_B for small ε by

$$\begin{aligned}
m_B &= \frac{\Phi(y_A+\varepsilon)\,\delta_S - [\Phi(y_A+\varepsilon)(\delta_S - \delta_L) + \delta_L]\, f_b(y_A+\varepsilon)}{\mu} \\
&= m_A + \frac{\Phi'(y_A)\,\delta_S - \{\Phi'(y_A)(\delta_S - \delta_L)\, f_b(y_A) + [\Phi(y_A)(\delta_S - \delta_L) + \delta_L]\, f_b'(y_A)\}}{\mu}\varepsilon \\
&= m_A + m_A'\varepsilon
\end{aligned}$$

Plugging into our equation for y_C, and again approximating around small ε, we have

$$\begin{aligned}
y_C &= \frac{\Phi(y_A) - \Phi(y_B) + m_B y_B - m_A y_A}{m_B - m_A} \\
&= \frac{\Phi(y_A) - [\Phi(y_A) + \Phi'(y_A)\,\varepsilon] + (m_A + m_A'\varepsilon)(y_A + \varepsilon) - m_A y_A}{(m_A + m_A'\varepsilon) - m_A} \\
&= \frac{-\Phi'(y_A)\,\varepsilon + m_A\varepsilon + m_A' y_A\varepsilon}{m_A'\varepsilon} \\
&= \frac{-\Phi'(y_A) + m_A + m_A' y_A}{m_A'} = y_A + \frac{-\Phi'(y_A) + m_A}{m_A'}
\end{aligned}$$

Thus, the distance of y_C from y_A is bounded away from zero, even as $\varepsilon \to 0$. Note here that we are using $|\Phi'(y_A)| < \infty$ and $|f_b'(y_A)| < \infty$ (see equation A.30), which itself comes from $B(y_b)$ being well behaved and uniformly below 1 for $y \in [y_{min}, y_{max}]$ by the endogenous default decision of the equity holders, to get $m_A' < \infty$. Thus, as $\varepsilon \to 0$, every point in the neighborhood of the boundary has a unique path leading to the boundary defined by f_b. A similar proof can be constructed for $\mu_y(y) = \mu \cdot y$, as for small enough ε we have $\mu_y(y+\varepsilon) = \mu_y(y) + \mu_y'(y)\,\varepsilon$ with $\left|\mu_y'(y)\right| < \infty$ by assumption on $\mu_y(\cdot)$.

As an aside, if f_b did have a jump say at y_A, then $f_b'(y_A) = \infty$ and thus $m_A' = \infty$, which results in $\lim_{\varepsilon\to 0} y_C(\varepsilon) = y_A$. This would imply that in the neighborhood of point A we cannot rule out multiple equilibria. Or in other words, the distance of the intersection point y_C to y_A shrinks to zero at the same speed as the distance between y_A and y_B. □

Next, we want to answer the question if the first-order condition on default really implies optimality? We have $E = E_\tau = 0$ or $E = E_\phi = E_y = 0$.

Lemma 5. *For any admissible $f_b = \min[1, \max[f_b^{nc}(y_b), 0]]$, immediate default is indeed optimal.*

Proof of Lemma 5. To show optimality of default, we show that for any admissible equilibrium defaulting boundary strategy f, $E_{\tau\tau}(\tau, y_b)|_{\tau=0} > 0$. Differentiate (A.21) w.r.t. τ to get

$$\begin{aligned}
E_{\tau\tau}(\tau, y_b) &= y_\tau(\tau, y_b) + m'(\phi(\tau, y_b))\,\phi_\tau(\tau, y_b)\,[f D_S(\tau, y_b) + (1-f) D_L(\tau, y_b) - 1] \\
&\quad + m(\phi(\tau, y_b))\left[f\frac{\partial}{\partial\tau}D_S(\tau, y_b) + (1-f)\frac{\partial}{\partial\tau}D_L(\tau, y_b)\right] - (r+\zeta) E_\tau(\tau, y_b) \\
&= \mu_y(y(\tau, y_b)) + (\delta_S - \delta_L)[\phi(\tau, y_b)\delta_S - m(\phi(\tau, y_b)) f][f D_S(\tau, y_b) + (1-f) D_L(\tau, y_b) - 1] \\
&\quad + m(\phi(\tau, y_b))\left[f\frac{\partial}{\partial\tau}D_S(\tau, y_b) + (1-f)\frac{\partial}{\partial\tau}D_L(\tau, y_b)\right] - (r+\zeta) E_\tau(\tau, y_b)
\end{aligned}$$

Evaluating at $\tau = 0$, we have

$$\begin{aligned}
E_{\tau\tau}(\tau, y_b)|_{\tau=0} &= \mu_y(y_b) + (\delta_S - \delta_L)[\Phi(y_b)\delta_S - m(\Phi(y_b)) f][B(y_b) - 1] \\
&\quad + m(\Phi(y_b))[r + \zeta + \delta_L + f(\delta_S - \delta_L)][1 - B(y_b)] \\
&= \mu_y(y_b) + 2m(\Phi(y_b))(\delta_S - \delta_L)[1 - B(y_b)] f \\
&\quad - (\delta_S - \delta_L)\Phi(y_b)\delta_S[1 - B(y_b)] \\
&\quad + [\delta_L + \Phi(y_b)(\delta_S - \delta_L)](r + \zeta + \delta_L)[1 - B(y_b)] \\
&= \mu_y(y_b) + 2m(\Phi(y_b))(\delta_S - \delta_L)[1 - B(y_b)] f \\
&\quad + \delta_L(r + \zeta + \delta_L)[1 - B(y_b)] \\
&\quad - (\delta_S - \delta_L)\Phi(y_b)[1 - B(y_b)]\delta_S \\
&\quad + (\delta_S - \delta_L)\Phi(y_b)[1 - B(y_b)](r + \zeta + \delta_L) \\
&= \mu_y(y_b) + 2m(\Phi(y_b))(\delta_S - \delta_L)[1 - B(y_b)] f \\
&\quad + \delta_L(r + \zeta + \delta_L)[1 - B(y_b)] \\
&\quad + (\delta_S - \delta_L)\Phi(y_b)[1 - B(y_b)](r + \zeta + \delta_L - \delta_S)
\end{aligned}$$

Suppose first we have an SE, i.e., $f = 1$. Then we have

$$\begin{aligned}
E_{\tau\tau}(\tau, y_b)|_{\tau=0} &= \mu_y(y_b) + 2[\Phi(y_b)(\delta_S - \delta_L) + \delta_L](\delta_S - \delta_L)[1 - B(y_b)] \\
&\quad + \delta_L(r + \zeta + \delta_L)[1 - B(y_b)] \\
&\quad + (\delta_S - \delta_L)\Phi(y_b)[1 - B(y_b)](r + \zeta + \delta_L - \delta_S) \\
&= \mu_y(y_b) + 2\Phi(y_b)(\delta_S - \delta_L)^2[1 - B(y_b)] \\
&\quad + 2\delta_L(\delta_S - \delta_L)[1 - B(y_b)] \\
&\quad + \delta_L(r + \zeta + \delta_L)[1 - B(y_b)] \\
&\quad + (\delta_S - \delta_L)\Phi(y_b)[1 - B(y_b)](r + \zeta) \\
&\quad - (\delta_S - \delta_L)^2\Phi(y_b)[1 - B(y_b)] \\
&= \mu_y(y_b) + \Phi(y_b)(\delta_S - \delta_L)^2[1 - B(y_b)] \\
&\quad + 2\delta_L(\delta_S - \delta_L)[1 - B(y_b)] \\
&\quad + \delta_L(r + \zeta + \delta_L)[1 - B(y_b)] \\
&\quad + (\delta_S - \delta_L)\Phi(y_b)[1 - B(y_b)](r + \zeta)
\end{aligned}$$

which is always positive.

For interior $f_b(y_b) = f_b^{nc}(y_b) = \frac{\mu_y(y_b)B'(y_b)-(r+\zeta+\delta_L)[1-B(y_b)]}{(\delta_S-\delta_L)[1-B(y_b)]}$ from (14), we have

$$\begin{aligned}
E_{\tau\tau}(\tau, y_b)\big|_{\tau=0} &= \mu_y(y_b) \\
&\quad +2m(\Phi(y_b))(\delta_S-\delta_L)[1-B(y_b)]f \\
&\quad -(\delta_S-\delta_L)\Phi(y_b)\delta_S[1-B(y_b)] \\
&\quad +m(\Phi(y_b))(r+\zeta+\delta_L)[1-B(y_b)] \\
&= \mu_y(y_b) \\
&\quad +2m(\Phi(y_b))\left\{\mu_y(y_b)B'(y_b)-(r+\zeta+\delta_L)[1-B(y_b)]\right\} \\
&\quad -(\delta_S-\delta_L)\Phi(y_b)\delta_S[1-B(y_b)] \\
&\quad +m(\Phi(y_b))(r+\zeta+\delta_L)[1-B(y_b)] \\
&= \mu_y(y_b)\left[1+2m(\Phi(y_b))B'(y_b)\right] \\
&\quad -(\delta_S-\delta_L)\Phi(y_b)\delta_S[1-B(y_b)] \\
&\quad -m(\Phi(y_b))(r+\zeta+\delta_L)[1-B(y_b)]
\end{aligned}$$

But we also know that for any equilibrium we must have $f_b(y_b) = f_b^{nc}(y_b) \geq f_{admissible}$, where

$$f_{admissible}(y_b) \equiv \frac{\Phi(y_b)\delta_S - \mu_y(y_b)\Phi'(y_b)}{\Phi(y_b)\delta_S + [1-\Phi(y_b)]\delta_L} = \frac{\Phi(y_b)\delta_S - \mu_y(y_b)\frac{1+m(\Phi(y_b))B'(y_b)}{(\delta_S-\delta_L)[1-B(y_b)]}}{m(\Phi(y_b))}$$

Let us rewrite $f_b^{nc}(y_b) \geq f_{admissible}$ as

$$\begin{aligned}
f_b^{nc}(y_b) &\geq f_{admissible}(y_b) \\
\Longleftrightarrow \frac{\mu_y(y_b)B'(y_b)-(r+\zeta+\delta_L)[1-B(y_b)]}{(\delta_S-\delta_L)[1-B(y_b)]} &\geq \frac{\Phi(y_b)\delta_S - \mu_y(y_b)\frac{1+m(\Phi(y_b))B'(y_b)}{(\delta_S-\delta_L)[1-B(y_b)]}}{m(\Phi(y_b))} \\
\Longleftrightarrow m(\Phi(y_b))\left\{\mu_y(y_b)B'(y_b)-(r+\zeta+\delta_L)[1-B(y_b)]\right\} &\geq \Phi(y_b)\delta_S(\delta_S-\delta_L)[1-B(y_b)] - \mu_y(y_b)\left[1+m(\Phi(y_b))B'(y_b)\right] \\
\Longleftrightarrow \mu_y(y_b)\left[1+2m(\Phi(y_b))B'(y_b)\right] &\geq \Phi(y_b)\delta_S(\delta_S-\delta_L)[1-B(y_b)] + m(\Phi(y_b))(r+\zeta+\delta_L)[1-B(y_b)]
\end{aligned}$$

Plugging this into $E_{\tau\tau}(0, y_b)$ above, we see that $E_{\tau\tau}(\tau, y_b)\big|_{\tau=0} \geq 0$.

Lastly, suppose we have an LE with $f_b(y_b) = 0 \geq f_b^{nc}(y_b)$. The proof above covers any $f = f_b(y_b) \geq f_b^{nc}(y_b) \geq f_{admissible}$, so the only case left is the case in which $f_b(y_b) = 0 \geq f_{admissible} \geq f_b^{nc}(y_b)$. Then, plugging in $f_b(y_b) = 0$, we get the following inequality

$$\begin{aligned}
E_{\tau\tau}(\tau, y_b)\big|_{\tau=0} &= \mu_y(y_b) + m(\Phi(y_b))(r+\zeta+\delta_L)[1-B(y_b)] \\
&\quad -\delta_S(\delta_S-\delta_L)\Phi(y_b)[1-B(y_b)]
\end{aligned}$$

Let us consider the information

$$\begin{aligned}
0 &\geq f_{admissible}(y_b) = \frac{\Phi(y_b)\delta_S - \mu_y(y_b)\frac{1+m(\Phi(y_b))B'(y_b)}{(\delta_S-\delta_L)[1-B(y_b)]}}{m(\Phi(y_b))} \\
\Longleftrightarrow 0 &\geq \Phi(y_b)\delta_S - \mu_y(y_b)\frac{1+m(\Phi(y_b))B'(y_b)}{(\delta_S-\delta_L)[1-B(y_b)]} \\
\Longleftrightarrow \mu_y(y_b)\left[1+m(\Phi(y_b))B'(y_b)\right] &\geq \Phi(y_b)\delta_S(\delta_S-\delta_L)[1-B(y_b)] \\
\Longleftrightarrow \mu_y(y_b) &\geq \frac{\Phi(y_b)\delta_S(\delta_S-\delta_L)[1-B(y_b)]}{[1+m(\Phi(y_b))B'(y_b)]}
\end{aligned}$$

as well as the information

$$\begin{aligned}
0 &\geq f_b^{nc}(y_b) = \frac{\mu_y(y_b)B'(y_b)-(r+\zeta+\delta_L)[1-B(y_b)]}{(\delta_S-\delta_L)[1-B(y_b)]} \\
\Longleftrightarrow 0 &\geq \mu_y(y_b)B'(y_b)-(r+\zeta+\delta_L)[1-B(y_b)] \\
\Longleftrightarrow (r+\zeta+\delta_L)[1-B(y_b)] &\geq \mu_y(y_b)B'(y_b)
\end{aligned}$$

Thus, we have

$$\begin{aligned}\mu_y(y_b) &\geq \Phi(y_b)\,\delta_S(\delta_S-\delta_L)\,[1-B(y_b)] - m(\Phi(y_b))\,\mu_y(y_b)\,B'(y_b)\\ &\geq \Phi(y_b)\,\delta_S(\delta_S-\delta_L)\,[1-B(y_b)] - m(\Phi(y_b))\,(r+\zeta+\delta_L)\,[1-B(y_b)]\end{aligned}$$

The result follows immediately. Thus, $E_{\tau\tau}(\tau, y_b)|_{\tau=0} > 0$. □

A.4 Proofs of Section 5

Lemma 6. *There is no discontinuities in f on any equilibrium path, i.e. $\left|\frac{f_{t+dt}-f_t}{dt}\right| < \infty$ everywhere.*

Proof of Lemma 6. $E_\phi(\phi, y)$ can be calculated as

$$\frac{\partial}{\partial\phi}E(\phi,y) = \frac{\partial}{\partial\phi}E(\tau,y_b) = E_\tau(\tau,y_b)\frac{\partial\tau}{\partial\phi} + E_{y_b}(\tau,y_b)\frac{\partial y_b}{\partial\phi}. \tag{A.32}$$

As before, the first term captures the effect of time-to-default τ, while the second term captures the effect of defaulting cash-flows y_b. Suppose now there exists a time-to-default $\hat\tau$ at which there is a jump in f, i.e., $f_{\hat\tau-} \neq f_{\hat\tau+}$. Equity values and debt values (and thus the bond value wedge Δ) are continuous across $\hat\tau$ along the path (ϕ_τ, y_τ) by inspection of (A.19), (A.20) and (A.22). However, equity's derivative with respect to τ, i.e., E_τ, displays a discontinuity at the policy switching point $\hat\tau$. Plugging into (A.21), we have

$$E_{\hat\tau-} - E_{\hat\tau+} = m(\phi)\,\Delta\cdot(f_{\hat\tau-} - f_{\hat\tau+}) = m(\phi)\,\Delta. \tag{A.33}$$

Since $m(\phi)\,\Delta > 0$, it implies that when equity switches to issuing more short-term bonds at $\hat\tau$, i.e., $f_{\hat\tau-} - f_{\hat\tau+}$, the equity value's derivative with respect to τ jumps up, i.e., the benefit of surviving longer goes up.

In the original (ϕ, y) state space, denote the corresponding switching points to be $(\hat\phi_-, \hat y_-)$ and $(\hat\phi_+, \hat y_+)$. Equity's incentive compatibility condition depends on $\frac{\partial}{\partial\phi}E_\phi(\phi,y)$ at these two points. By writing out the terms in integral form, and noting that any f are bounded, we can show that in (A.32), both the $\frac{\partial\tau}{\partial\phi}$ in the first term, and the entire second term related to y_b, i.e., $E_{y_b}(\tau, y_b)\frac{\partial y_b}{\partial\phi}$, are continuous at the switching point. Hence, equation (A.33) implies that

$$E_\phi(\hat\phi_-,\hat y_-) - E_\phi(\hat\phi_+,\hat y_+) = (E_{\tau-} - E_{\tau+})\frac{\partial\tau}{\partial\phi} = m(\phi)\,\Delta\,(f_{\hat\tau-} - f_{\hat\tau+})\cdot\frac{\partial\tau}{\partial\phi}.$$

Next, note that $\frac{\partial\tau}{\partial\phi} < 0$, i.e., shortening maturity gives rise to a shorter time-to-default. Following the intuition right after (A.33), when equity switches to issuing short-term bonds, the benefit of surviving longer going up implies that marginal negative impact of shortening maturity is more severe. To make the general point, let us write

$$\begin{aligned}IC(\hat\phi_+,\hat y_+) &= \Delta(\hat\phi,\hat y) + E_\phi(\hat\phi_+,\hat y_+)\\ &= \Delta(\hat\phi,\hat y) + E_\phi(\hat\phi_-,\hat y_-) + \left[-m(\phi)\,\Delta\,(f_{\hat\tau-}-f_{\hat\tau+})\frac{\partial\tau}{\partial\phi}\right]\\ &= IC(\hat\phi_-,\hat y_-) + \left[m(\phi)\,\Delta\,(f_{\hat\tau-}-f_{\hat\tau+})\left(-\frac{\partial\tau}{\partial\phi}\right)\right]\end{aligned}$$

Consider first the case when $f_{\hat\tau-} = 1$ and $f_{\hat\tau+} < 1$. This implies that $\left[m(\phi)\,\Delta\,(f_{\hat\tau-}-f_{\hat\tau+})\left(-\frac{\partial\tau}{\partial\phi}\right)\right] > 0$ and we immediately have a violation: if $f_{\hat\tau-} = 1$was optimal, then $IC(\hat\phi_+,\hat y_+) > IC(\hat\phi_-,\hat y_-) \geq 0$ and thus $f_{\hat\tau+} < 1$ violates the IC condition. Next, consider the case when $f_{\hat\tau-} = 0$ and $f_{\hat\tau+} > 0$. This implies that $\left[m(\phi)\,\Delta\,(f_{\hat\tau-}-f_{\hat\tau+})\left(-\frac{\partial\tau}{\partial\phi}\right)\right] < 0$, which implies $IC(\hat\phi_+,\hat y_+) < IC(\hat\phi_-,\hat y_-) \leq 0$ and thus invalidates $f_{\hat\tau+} > 0$. Lastly, consider the case when $f_{\hat\tau-} \in [0,1]$ such that $IC(\hat\phi_-,\hat y_-) = 0$. Then we immediately see that any $f_{\hat\tau+} \neq f_{\hat\tau-}$ violates IC: (i) if $f_{\hat\tau-} \in (0,1)$, then we must have $IC(\hat\phi_+,\hat y_+) = 0$ as well, which is violated by $\left[m(\phi)\,\Delta\,(f_{\hat\tau-}-f_{\hat\tau+})\left(-\frac{\partial\tau}{\partial\phi}\right)\right] \neq 0$. (ii) if $f_{\hat\tau-} \in \{0,1\}$, then we are in the above proofs, and see that the violation exactly runs counter to the IC condition. □

Proof of Proposition 2. We start with the following observation. Suppose the current state of the system is given by (ϕ, y). Firm-value is then given by

$$V(\phi,y) = E(\phi,y) + \phi D_S(\phi,y) + (1-\phi)\,D_L(\phi,y)$$

Suppose we consider an arbitrary equilibrium path $(\phi, y) \to (\Phi(y_b), y_b)$ where default occurs at the point $(\Phi(y_b), y_b)$. We know that the default time is deterministic given the equilibrium strategy $\{f_\tau\}$. That is, we fix the starting point and the end-point of the path, and thereby the time to default, but leave the actual issuance strategy $\{f_\tau\}$ and thus the actual path taken by ϕ undefined. Let us sum up all the cash-flows to get an alternate expression for firm value,

$$\begin{aligned} V(\tau, y_b) &= \int_0^\tau e^{(r+\zeta)(s-\tau)}\left[y(s, y_b) + \zeta X\right] ds + e^{-(r+\zeta)\tau} B(y_b) \\ &= \int_0^\tau e^{(r+\zeta)(s-\tau)} y(s, y_b)\, ds + \zeta X \frac{1 - e^{-(r+\zeta)\tau}}{r+\zeta} + e^{-(r+\zeta)\tau} B(y_b) \end{aligned}$$

and we can thus define equity as a residual,

$$E(\tau, y_b) = V(\tau, y_b) - \phi(\tau, y_b) D_S(\tau, y_b) - [1 - \phi(\tau, y_b)] D_L(\tau, y_b)$$

Importantly, equity value is invariant to the specific future path of ϕ taken as long as y_b and thus τ are held fixed. However, incentives are not invariant to the path taken, as we will show below. Consider now

$$\begin{aligned} E_\phi &= \frac{\partial}{\partial \phi}\left[V(\phi, y) - \phi D_S(\phi, y) - (1-\phi) D_L(\phi, y)\right] \\ &= \left[V_\phi(\phi, y) - \phi \frac{\partial}{\partial \phi} D_S(\phi, y) - (1-\phi)\frac{\partial}{\partial \phi} D_L(\phi, y)\right] - \left[D_S(\phi, y) - D_L(\phi, y)\right] \end{aligned}$$

so that we have, after rearranging

$$\begin{aligned} IC(\tau, y_b) &= E_\phi(\tau, y_b) + \Delta(\tau, y_b) \\ &= V_\phi(\phi, y) - \phi \frac{\partial}{\partial \phi} D_S(\phi, y) - (1-\phi)\frac{\partial}{\partial \phi} D_L(\phi, y) \\ &= \left\{\frac{\partial}{\partial \tau} V(\tau, y_b) - \phi(\tau, y_b)\frac{\partial}{\partial \tau} D_S(\tau, y_b) - [1-\phi(\tau, y_b)]\frac{\partial}{\partial \tau} D_L(\tau, y_b)\right\}\frac{\partial \tau}{\partial \phi} \\ &\quad + \left\{\frac{\partial}{\partial y_b} V(\tau, y_b) - \phi(\tau, y_b)\frac{\partial}{\partial y_b} D_S(\tau, y_b) - [1-\phi(\tau, y_b)]\frac{\partial}{\partial y_b} D_L(\tau, y_b)\right\}\frac{\partial y_b}{\partial \phi} \end{aligned}$$

Thus, we notice that since $V(\tau, y_b)$ as well as $D_S(\tau, y_b)$ and $D_L(\tau, y_b)$ are independent of the path of f for a given (τ, y_b), we see that f is only reflected in change-of-variables $\frac{\partial \tau}{\partial \phi}$ and $\frac{\partial y_b}{\partial \phi}$. We know that $IC(0, y_b) = 0$ by boundary conditions.

As the $IC(\tau, y_b)$ condition is not monotone in τ, we use a scaled up version $e^{k\tau} IC(\tau, y_b)$ with $k = r + \zeta + f\delta_L + (1-f)\delta_S$.

For $\mu_y(y) = \mu$, we can show that for shortening equilibria (i.e. $f = 1$)

$$\begin{aligned} \frac{\partial}{\partial \tau}\left[e^{(r+\zeta+\delta_L)\tau} IC\left(\tau, y_b^S\right)\right] &= \frac{e^{-\delta_S \tau}\left\{\delta_S - (\delta_S - \delta_L)\left[1 - \Phi\left(y_b^S\right)\right] e^{\delta_L \tau}\right\}\left\{(\delta_S + r + \zeta)\left[1 - B\left(y_b^S\right)\right] - \mu B'\left(y_b^S\right)\right\}}{\left[\delta_L \Phi\left(y_b^S\right) - \delta_L\right] - \mu \Phi'\left(y_b^S\right)} \\ &= \frac{e^{-\delta_S \tau}\left\{\delta_S - (\delta_S - \delta_L)\left[1 - \phi\left(\tau, y_b^S\right)\right]\right\}\left\{(\delta_S + r + \zeta)\left[1 - B\left(y_b^S\right)\right] - \mu B'\left(y_b^S\right)\right\}}{\left[\delta_L \Phi\left(y_b^S\right) - \delta_L\right] - \mu \Phi'\left(y_b^S\right)} \\ &= e^{-\delta_S \tau} m\left(\phi\left(\tau, y_b^S\right)\right)\frac{(\delta_S + r + \zeta)\left[1 - B\left(y_b^S\right)\right] - \mu B'\left(y_b^S\right)}{\left[\delta_L \Phi\left(y_b^S\right) - \delta_L\right] - \mu \Phi'\left(y_b^S\right)} \end{aligned}$$

and for lengthening equilibria (i.e. $f = 0$) we have

$$\begin{aligned} \frac{\partial}{\partial \tau}\left[e^{(r+\zeta+\delta_S)\tau} IC\left(\tau, y_b^L\right)\right] &= \frac{e^{-\delta_L \tau}\left[\delta_L + (\delta_S - \delta_L)\Phi\left(y_b^L\right) e^{\delta_S \tau}\right]\left\{(\delta_L + r + \zeta)\left[1 - B\left(y_b^L\right)\right] - \mu B'\left(y_b^L\right)\right\}}{\delta_S \Phi\left(y_b^L\right) - \mu \Phi'\left(y_b^L\right)} \\ &= \frac{e^{-\delta_L \tau}\left[\delta_L + (\delta_S - \delta_L)\phi\left(\tau, y_b^L\right)\right]\left\{(\delta_L + r + \zeta)\left[1 - B\left(y_b^L\right)\right] - \mu B'\left(y_b^L\right)\right\}}{\delta_S \Phi\left(y_b^L\right) - \mu \Phi'\left(y_b^L\right)} \\ &= e^{-\delta_L \tau} m\left(\phi\left(\tau, y_b^L\right)\right)\frac{(\delta_L + r + \zeta)\left[1 - B\left(y_b^L\right)\right] - \mu B'\left(y_b^L\right)}{\delta_S \Phi\left(y_b^L\right) - \mu \Phi'\left(y_b^L\right)} \end{aligned}$$

Next, for $\mu_y(y) = \mu y$, we can show that for shortening equilibria (i.e. $f = 1$)

$$\frac{\partial}{\partial \tau}\left[e^{(r+\zeta+\delta_L)\tau} IC\left(\tau, y_b^S\right)\right] = e^{-\delta_S \tau} m\left(\phi\left(\tau, y_b^S\right)\right)\frac{(\delta_S + r + \zeta)\left[1 - B\left(y_b^S\right)\right] - \mu y_b^S B'\left(y_b^S\right)}{\left[\delta_L \Phi\left(y_b^S\right) - \delta_L\right] - \mu \Phi'\left(y_b^S\right)}$$

and for lengthening equilibria (i.e. $f=0$) we have

$$\frac{\partial}{\partial\tau}\left[e^{(r+\zeta+\delta_S)\tau}IC\left(\tau,y_b^L\right)\right] = e^{-\delta_L\tau}m\left(\phi\left(\tau,y_b^L\right)\right)\frac{\left(\delta_L+r+\zeta\right)\left[1-B\left(y_b^L\right)\right]-\mu y_b^S B'\left(y_b^L\right)}{\delta_S\Phi\left(y_b^L\right)-\mu\Phi'\left(y_b^L\right)}$$

As $e^{-[f\delta_S+(1-f)\delta_L]\tau}m\left(\phi\left(\tau,y_b\right)\right)>0$, we notice that the remaining term is exactly condition (22) evaluated at the appropriate f. Thus, the IC condition at 0 is sufficient for all cornered paths.

Then, by the fact that LE paths never cross other LE paths, and SE paths never cross other SE paths, there can at most be one LE and at most one SE equilibrium for any point (ϕ,y). □

Proof of Proposition 3. By $f_b^{nc}\left(y_b\right)$ increasing and the fact that any $f_b\left(y_b\right)=1$ implies admissibility we know that if an SE exists, it has to be of the form $[y_1,y_{max}]$ where $f_b^{nc}\left(y_1\right)=1$ and $\Phi\left(y_{max}\right)=1$ with $y_1\leq y_{max}$. By Proposition 2 then we know that there is at most one LE and one SE at any point (ϕ,y). We further know that any point on the SE region of the boundary has paths that also fulfills the SE incentive conditions. As paths do not cross, the lowest point $\left(\Phi\left(y_1\right),y_1\right)$ and the shortening path emanating from it describe the boundary of the SE possible set—any point above this path features an SE equilibrium, as the SE paths are dense in the (ϕ,y) space. For the second part, we note that the fastest rate of change for ϕ to decrease is given by $f=0$. Thus, extending a lengthening path out form $\left(\Phi\left(y_1\right),y_1\right)$, any point between the boundary and this path cannot escape hitting the boundary in the SE region. But that implies that the only equilibrium in this region is the SE equilibrium. A similar argument holds for the LE regions. □

Proposition 5. *Any point on the bankruptcy boundary $\left(\Phi\left(y_b\right),y_b\right)$ has a unique path leading to it.*

Proof of Proposition 5. By the fact that cornered paths do not cross, any cornered equilibrium on the boundary clearly has a unique path leading up to it. What remains to show is that interior paths defined by $IC_\tau=0$ also have a unique path leading up to it, i.e. a unique sequence of issuance decisions f. We now show that any interior path features a sequence of uniquely determined f when working back from the boundary. Writing out $IC\left(\tau,y_b\right)$, we have

$$\begin{aligned}IC\left(\tau,y_b\right) &= \Delta\left(\tau,y_b\right)+E_\phi\left(\tau,y_b\right)\\ &= D_S\left(\tau,y_b\right)-D_L\left(\tau,y_b\right)+\frac{\partial y_b}{\partial\phi}\left\{\frac{\partial}{\partial y_b}E\left(\tau,y_b\right)\right\}\\ &\quad+\frac{\partial\tau}{\partial\phi}\left\{\begin{array}{c}y\left(\tau,y_b\right)-c+\zeta E^{rf}-\left(r+\zeta\right)E\left(\tau,y_b\right)\\ +m\left(\phi\left(\tau,y_b\right)\right)\left[fD_S\left(\tau,y_b\right)+\left(1-f\right)D_L\left(\tau,y_b\right)-1\right]\end{array}\right\}\end{aligned}\tag{A.34}$$

Let us move things under the common denominator $\frac{\partial y(\tau,y_b)}{\partial y_b}\left\{\phi\left(\tau,y_b\right)\left[\delta_S\left(1-f\right)+f\delta_L\right]-\delta_L f\right\}-\mu_y\left(y\left(\tau,y_b\right)\right)\frac{\partial}{\partial y_b}\phi\left(\tau,y_b\right)$ that comes from $\frac{\partial y_b}{\partial\phi}$ and $\frac{\partial\tau}{\partial\phi}$. Plugging in for $\frac{\partial y_b}{\partial\phi}$ and $\frac{\partial\tau}{\partial\phi}$, we have

$$\begin{aligned}IC\left(\tau,y_b\right) &= \frac{1}{\frac{\partial y(\tau,y_b)}{\partial y_b}\left\{\phi\left(\tau,y_b\right)\left[\delta_S\left(1-f\right)+f\delta_L\right]-\delta_L f\right\}-\mu_y\left(y\left(\tau,y_b\right)\right)\frac{\partial}{\partial y_b}\phi\left(\tau,y_b\right)}\\ &\quad\times\left\{\begin{array}{c}\Delta\left(\tau,y_b\right)\left[\frac{\partial y(\tau,y_b)}{\partial y_b}\left\{\phi\left(\tau,y_b\right)\left[\delta_S\left(1-f\right)+f\delta_L\right]-\delta_L f\right\}-\frac{\partial y(\tau,y_b)}{\partial y_b}\frac{\partial}{\partial y_b}\phi\left(\tau,y_b\right)\right]-\mu_y\left(y\left(\tau,y_b\right)\right)\frac{\partial}{\partial y_b}E\left(\tau,y_b\right)\\ +\frac{\partial y(\tau,y_b)}{\partial y_b}\left[\begin{array}{c}y\left(\tau,y_b\right)-c+\zeta E^{rf}-\left(r+\zeta\right)E\left(\tau,y_b\right)\\ +m\left(\phi\left(\tau,y_b\right)\right)\left[fD_S\left(\tau,y_b\right)+\left(1-f\right)D_L\left(\tau,y_b\right)-1\right]\end{array}\right]\end{array}\right\}\end{aligned}\tag{A.35}$$

Suppose we have an interior equilibrium. For interior equilibria we have $IC\left(\tau,y_b\right)=0$, so that for non-zero denominators, we must have

$$\begin{aligned}0 &= \Delta\left(\tau,y_b\right)\left[\frac{\partial y\left(\tau,y_b\right)}{\partial y_b}\left\{\phi\left(\tau,y_b\right)\left[\delta_S-f\left(\delta_S-\delta_L\right)\right]-\delta_L f\right\}-\mu_y\left(y\left(\tau,y_b\right)\right)\frac{\partial}{\partial y_b}\phi_0\left(\tau,y_b\right)\right]-\mu_y\left(y\left(\tau,y_b\right)\right)\frac{\partial}{\partial y_b}E\left(\tau,y_b\right)\\ &\quad+\frac{\partial y\left(\tau,y_b\right)}{\partial y_b}\left\{\begin{array}{c}y\left(\tau,y_b\right)-c+\zeta E^{rf}-\left(r+\zeta\right)E\left(\tau,y_b\right)\\ +m\left(\phi\left(\tau,y_b\right)\right)\left[f\Delta\left(\tau,y_b\right)+D_L\left(\tau,y_b\right)-1\right]\end{array}\right\}\end{aligned}\tag{A.36}$$

Collecting powers of f on the LHS, we see that f cancels out:

$$\overbrace{\left\{\frac{\partial y(\tau,y_b)}{\partial y_b}\left[\phi(\tau,y_b)(\delta_S-\delta_L)+\delta_L\right]-\frac{\partial y(\tau,y_b)}{\partial y_b}\cdot m(\phi(\tau,y_b))\right\}}^{=\frac{\partial y(\tau,y_b)}{\partial y_b}\left[\phi(\delta_S-\delta_L)+\delta_L-m(\phi)\right]=0}\Delta(\tau,y_b)f$$

$$\begin{aligned}
= \quad & \Delta(\tau,y_b)\left[\frac{\partial y(\tau,y_b)}{\partial y_b}\delta_S\phi(\tau,y_b)-\mu_y(y(\tau,y_b))\frac{\partial}{\partial y_b}\phi(\tau,y_b)\right]-\mu_y(y(\tau,y_b))\frac{\partial}{\partial y_b}E(\tau,y_b)\\
& +\frac{\partial y(\tau,y_b)}{\partial y_b}\left\{\begin{array}{c} y(\tau,y_b)-c+\zeta E^{rf}-(r+\zeta)E(\tau,y_b)\\ +m(\phi(\tau,y_b))\left[D_L(\tau,y_b)-1\right]\end{array}\right\}
\end{aligned} \tag{A.37}$$

Let us take the derivative with respect to τ of the RHS only, noting that the LHS is identically 0 across τ as long as we have an interior equilibrium.

For future reference, differentiating (A.21) w.r.t. y_b and using the envelope theorem, we have

$$\begin{aligned}
(r+\zeta)\frac{\partial E(\tau,y_b)}{\partial y_b} \quad = \quad & \frac{\partial y(\tau,y_b)}{\partial y_b}+(\delta_S-\delta_L)\left[fD_S(\tau,y_b)+(1-f)D_L(\tau,y_b)-1\right]\frac{\partial\phi(\tau,y_b)}{\partial y_b}\\
& +m(\phi)\left[f\frac{\partial D_S(\tau,y_b)}{\partial y_b}+(1-f)\frac{\partial D_L(\tau,y_b)}{\partial y_b}\right]-\frac{\partial}{\partial\tau}\left(\frac{\partial E(\tau,y_b)}{\partial y_b}\right)
\end{aligned} \tag{A.38}$$

with boundary condition $\frac{\partial}{\partial\tau}\left(\frac{\partial E(\tau,y_b)}{\partial y_b}\right)\Big|_{\tau=0}=0$ and where we used $m'(\phi)=\delta_S-\delta_L$. Integrating up $\frac{\partial E(\tau,y_b)}{\partial y_b}$, we have

$$\begin{aligned}
\frac{\partial E(\tau,y_b)}{\partial y_b} \quad = \quad & \int_0^\tau e^{(r+\zeta)(u-\tau)}\left\{\frac{\partial y(u,y_b)}{\partial y_b}\right.\\
& +(\delta_S-\delta_L)\left[f_uD_S(u,y_b)+(1-f_u)D_L(u,y_b)-1\right]\frac{\partial\phi(u,y_b)}{\partial y_b}\\
& \left.+m(\phi(u,y_b))\left[f_u\frac{\partial D_S(u,y_b)}{\partial y_b}+(1-f_u)\frac{\partial D_L(u,y_b)}{\partial y_b}\right]\right\}du
\end{aligned} \tag{A.39}$$

Coming back to the IC condition, we then have

$$\begin{aligned}
0 \quad = \quad & \left[\frac{\partial y(\tau,y_b)}{\partial y_b}\delta_S\phi(\tau,y_b)-\mu_y(y(\tau,y_b))\frac{\partial\phi(\tau,y_b)}{\partial y_b}\right]\frac{\partial\Delta(\tau,y_b)}{\partial\tau}\\
& +\Delta(\tau,y_b)\left[\frac{\partial y(\tau,y_b)}{\partial y_b}\delta_S\boldsymbol{\frac{\partial\phi(\tau,y_b)}{\partial\tau}}-\mu_y(y(\tau,y_b))\boldsymbol{\frac{\partial^2\phi(\tau,y_b)}{\partial y_b\partial\tau}}\right]\\
& -\mu_y(y(\tau,y_b))\boldsymbol{\frac{\partial^2 E(\tau,y_b)}{\partial y_b\partial\tau}} && \text{(A.40)}\\
& +\frac{\partial y(\tau,y_b)}{\partial y_b}\left\{\begin{array}{c}\frac{\partial y(\tau,y_b)}{\partial\tau}+m'(\phi(\tau,y_b))\left[D_L(\tau,y_b)-1\right]\boldsymbol{\frac{\partial\phi(\tau,y_b)}{\partial\tau}}\\ +m(\phi(\tau,y_b))\frac{\partial D_L(\tau,y_b)}{\partial\tau}-(r+\zeta)\boldsymbol{\frac{\partial E(\tau,y_b)}{\partial\tau}}\end{array}\right\} && \text{(A.41)}\\
& +\Delta(\tau,y_b)\left[\frac{\partial^2 y(\tau,y_b)}{\partial y_b\partial\tau}\delta_S\phi(\tau,y_b)-\frac{\partial\mu_y(y(\tau,y_b))}{\partial\tau}\frac{\partial\phi(\tau,y_b)}{\partial y_b}\right] && \text{(A.42)}\\
& -\frac{\partial\mu_y(y(\tau,y_b))}{\partial\tau}\frac{\partial E(\tau,y_b)}{\partial y_b} && \text{(A.43)}\\
& +\frac{\partial^2 y(\tau,y_b)}{\partial y_b\partial\tau}\left\{\begin{array}{c} y(\tau,y_b)-c+\zeta E^{rf}-(r+\zeta)E(\tau,y_b)\\ +m(\phi(\tau,y_b))\left[D_L(\tau,y_b)-1\right]\end{array}\right\} && \text{(A.44)}
\end{aligned}$$

where bold-face functions indicate (linear) functions of **contemporaneous** f. For $\mu_y(y)=\mu$, we have $y(\tau,y_b)=y_b+\mu\tau$ and thus $\frac{\partial^2 y(\tau,y_b)}{\partial y_b\partial\tau}=\frac{\partial\mu_y(y(\tau,y_b))}{\partial\tau}=0$, so the last three lines are identically zero. Similarly, for $\mu_y(y)=\mu y$, we have $y(\tau,y_b)=y_be^{\mu\tau}$ and thus $\frac{\partial^2 y(\tau,y_b)}{\partial y_b\partial\tau}=\mu\frac{\partial y(\tau,y_b)}{\partial y_b}=\mu\mu_\tau(y(\tau,y_b))=\frac{\partial\mu_y(y(\tau,y_b))}{\partial\tau}$, and the last three lines are identically zero.

Plugging in for the bold-face functions, dropping (τ, y_b) for brevity, we have

$$
\begin{aligned}
0 \quad = \quad & \left[\frac{\partial y}{\partial y_b}\delta_S\phi - \mu_y(y)\frac{\partial \phi}{\partial y_b}\right]\frac{\partial \Delta}{\partial \tau} \\
& +\Delta\left[\frac{\partial y}{\partial y_b}\delta_S\left[f\boldsymbol{\{-m(\phi)\}}+\delta_S\phi\right]-\mu_y(y)\left[f\left\{\boldsymbol{(\delta_L-\delta_S)\frac{\partial\phi}{\partial y_b}}\right\}+\delta_S\frac{\partial}{\partial y_b}\phi\right]\right] \\
& -\mu_y(y)\left[\begin{array}{c} \frac{\partial y}{\partial y_b}+(\delta_S-\delta_L)\left[D_L-1\right]\frac{\partial\phi}{\partial y_b} \\ +f\left\{\boldsymbol{(\delta_S-\delta_L)\Delta\frac{\partial\phi}{\partial y_b}+m(\phi)\frac{\partial\Delta}{\partial y_b}}\right\} \\ +m(\phi)\frac{\partial D_L}{\partial y_b}-(r+\zeta)\frac{\partial E}{\partial y_b}\end{array}\right] \\
& +\frac{\partial y}{\partial y_b}\left\{\begin{array}{c} \frac{\partial y}{\partial \tau}+m'(\phi)\left[D_L-1\right]\left[f\boldsymbol{\{-m(\phi)\}}+\delta_S\phi\right] \\ +m(\phi)\frac{\partial D_L}{\partial\tau}-(r+\zeta)\left[\begin{array}{c} y+\zeta E^{rf}-c+m(\phi)\left[D_L-1\right] \\ +f\cdot\boldsymbol{\{m(\phi)\Delta\}}-(r+\zeta)E\end{array}\right]\end{array}\right\}
\end{aligned}
\tag{A.45}
$$

where we left terms multiplying f bold-face. Gathering terms as

$$
0=(numerator)-(denominator)\,f \iff f=\frac{(numerator)}{(denominator)}
\tag{A.46}
$$

we have

$$
\begin{aligned}
denominator \quad = \quad & m(\phi)\left[\frac{\partial y}{\partial y_b}\left\{\Delta\left[\delta_S+(r+\zeta)\right]-(\delta_S-\delta_L)(1-D_L)\right\}+\mu_y(y)\frac{\partial\Delta}{\partial y_b}\right] \\
numerator \quad = \quad & \left[\frac{\partial y}{\partial y_b}\delta_S\phi-\mu_y(y)\frac{\partial\phi}{\partial y_b}\right]\frac{\partial\Delta}{\partial\tau}+\Delta\delta_S\left[\frac{\partial y}{\partial y_b}\delta_S\phi-\mu_y(y)\frac{\partial\phi}{\partial y_b}\right] \\
& -\mu_y(y)\left[\begin{array}{c}\frac{\partial y}{\partial y_b}+(\delta_S-\delta_L)\left[D_L-1\right]\frac{\partial\phi}{\partial y_b} \\ +m(\phi)\frac{\partial D_L}{\partial y_b}-(r+\zeta)\frac{\partial E}{\partial y_b}\end{array}\right] \\
& +\frac{\partial y}{\partial y_b}\left\{\begin{array}{c}\frac{\partial y}{\partial\tau}+m'(\phi)\left[D_L-1\right]\delta_S\phi+m(\phi)\frac{\partial D_L}{\partial\tau} \\ -(r+\zeta)\left[y+\zeta E^{rf}-c+m(\phi)\left[D_L-1\right]-(r+\zeta)E\right]\end{array}\right\}
\end{aligned}
\tag{A.47}
$$

Thus, by linearity we have a unique candidate f_τ. The bold terms feature **contemporaneous** f that is linear in all cases:

$$\begin{aligned}
m(\phi) &= \delta_L + \phi(\delta_S - \delta_L) && \text{(A.48)}\\
y(\tau, y_b) &= \begin{cases} y_b + \mu\tau & \text{linear} \\ y_b e^{\mu\tau} & \text{exponential} \end{cases} && \text{(A.49)}\\
\frac{\partial}{\partial y_b} y(\tau, y_b) &= \begin{cases} 1 & \text{linear} \\ e^{\mu\tau} & \text{exponential} \end{cases} && \text{(A.50)}\\
\frac{\partial}{\partial \tau} y(\tau, y_b) &= \begin{cases} \mu & \text{linear} \\ \mu y_b e^{\mu\tau} & \text{exponential} \end{cases} && \text{(A.51)}\\
\phi(\tau, y_b) &= e^{\int_0^\tau [\delta_S(1-f_s)+f_s\delta_L]ds}\left[\Phi(y_b) - \delta_L \int_0^\tau e^{-\int_0^s [\delta_S(1-f_u)+f_u\delta_L]du} f_s ds\right] && \text{(A.52)}\\
\frac{\partial}{\partial y_b}\phi(\tau, y_b) &= e^{\int_0^\tau [\delta_S(1-f_s)+f_s\delta_L]ds}\Phi'(y_b) && \text{(A.53)}\\
\boldsymbol{\frac{\partial}{\partial \tau}\phi(\tau, y_b)} &= f\boldsymbol{\{-m(\phi(\tau, y_b))\}} + \delta_S\phi(\tau, y_b) && \text{(A.54)}\\
\boldsymbol{\frac{\partial}{\partial \tau}\frac{\partial}{\partial y_b}\phi(\tau, y_b)} &= f\left\{\boldsymbol{(\delta_L - \delta_S)\frac{\partial}{\partial y_b}\phi(\tau, y_b)}\right\} + \delta_S\frac{\partial}{\partial y_b}\phi(\tau, y_b) && \text{(A.55)}\\
D_S(\tau, y_b) &= \frac{c+\delta_S+\zeta}{r+\delta_S+\zeta} + e^{-(r+\delta_S+\zeta)\tau}\left[B(y_b) - \frac{c+\delta_S+\zeta}{r+\delta_S+\zeta}\right] && \text{(A.56)}\\
\frac{\partial}{\partial \tau} D_S(\tau, y_b) &= -(r+\delta_S+\zeta)e^{-(r+\delta_S+\zeta)\tau}\left[B(y_b) - \frac{c+\delta_S+\zeta}{r+\delta_S+\zeta}\right] && \text{(A.57)}\\
\frac{\partial}{\partial y_b} D_S(\tau, y_b) &= e^{-(r+\delta_S+\zeta)\tau}B'(y_b) && \text{(A.58)}\\
E(\tau, y_b) &= \int_0^\tau e^{(r+\zeta)(u-\tau)}\left\{y(u, y_b) + \zeta E^{rf} - c\right. && \text{(A.59)}\\
&\quad \left. + m(\phi(u, y_b))\left[f_u D_S(u, y_b) + (1-f_u)D_L(u, y_b) - 1\right]\right\} du && \text{(A.60)}\\
\boldsymbol{\frac{\partial}{\partial \tau}E(\tau, y_b)} &= y + \zeta E^{rf} - c + m(\phi(\tau, y_b))\left[D_L(\tau, y_b) - 1\right] && \text{(A.61)}\\
&\quad + f\cdot\boldsymbol{\{m(\phi(\tau, y_b))[D_S(\tau, y_b) - D_L(\tau, y_b)]\}} - (r+\zeta)E(\tau, y_b) && \text{(A.62)}\\
\frac{\partial}{\partial y_b}E(\tau, y_b) &= \int_0^\tau e^{(r+\zeta)(u-\tau)}\left\{\frac{\partial}{\partial y_b}y(u, y_b)\right. && \text{(A.63)}\\
&\quad + (\delta_S - \delta_L)\left[f_u D_S(u, y_b) + (1-f_u)D_L(u, y_b) - 1\right]\frac{\partial}{\partial y_b}\phi(u, y_b) && \text{(A.64)}\\
&\quad \left. + m(\phi(u, y_b))\left[f_u\frac{\partial}{\partial y_b}D_S(u, y_b) + (1-f_u)\frac{\partial}{\partial y_b}D_L(u, y_b)\right]\right\} du && \text{(A.65)}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\frac{\partial}{\partial \tau}\frac{\partial}{\partial y_b}E(\tau, y_b)} &= \frac{\partial y(\tau, y_b)}{\partial y_b} + (\delta_S - \delta_L)\left[D_L(\tau, y_b) - 1\right]\frac{\partial \phi(\tau, y_b)}{\partial y_b} \\
&\quad + f\left\{\begin{array}{c} \boldsymbol{(\delta_S - \delta_L)[D_S(\tau, y_b) - D_L(\tau, y_b)]\frac{\partial\phi(\tau, y_b)}{\partial y_b}} \\ \boldsymbol{+m(\phi(\tau, y_b))\left[\frac{\partial D_S(\tau, y_b)}{\partial y_b} - \frac{\partial D_L(\tau, y_b)}{\partial y_b}\right]} \end{array}\right\} \\
&\quad + m(\phi(\tau, y_b))\frac{\partial D_L(\tau, y_b)}{\partial y_b} - (r+\zeta)\frac{\partial E(\tau, y_b)}{\partial y_b} && \text{(A.66)}
\end{aligned}$$

The interior equilibrium path is unique for any *ultimate* bankruptcy state $(\Phi(y_b), y_b)$ as it is stems from a linear equation. Thus, suppose that $f_{\tau=0} \in \{0, 1\}$. Then we know that $IC_\tau(0, y_b) \gtreqless 0$ and $f_{\tau=0}$ stays cornered until a time τ at which $IC(\tau, y_b) = 0$. Suppose $f_{\tau=0} \in (0, 1)$. Then immediately we have, by Lemma 6, as f is continuous that the above determines the path of f uniquely as it is a linear equation, until a time τ at which f becomes cornered. In this case, then, IC starts diverging from 0 and again f is uniquely determined by the sign of IC. They key step here is to note that IC is continuous by the functions involved and by the continuity of f. □

A.5 Further results for Section 6

Proof of Lemma 2. Imposing $c_i = r$ for $i \in \{S, L\}$, writing out the derivatives of D_S and D_L w.r.t. ϕ at $\tau = 0$ by using (23), (24), and (25) , we have

$$\begin{aligned}
\frac{\partial D_L}{\partial \phi} - \frac{\partial D_S}{\partial \phi} &= \left[(r + \delta_L + \zeta) \left[1 - B(y_b)\right] \left.\frac{\partial \tau}{\partial \phi}\right|_{\tau=0} + B'(y_b) \left.\frac{\partial y_b}{\partial \phi}\right|_{\tau=0} \right] \\
&\quad - \left[(r + \delta_S + \zeta) \left[1 - B(y_b)\right] \left.\frac{\partial \tau}{\partial \phi}\right|_{\tau=0} + B'(y_b) \left.\frac{\partial y_b}{\partial \phi}\right|_{\tau=0} \right] \\
&= -(\delta_S - \delta_L) \left[1 - B(y_b)\right] \left.\frac{\partial \tau}{\partial \phi}\right|_{\tau=0} > 0.
\end{aligned}$$

□

A.5.1 Deleveraging

We know that a proportion $m(\phi_t)$ of bond is maturing every instant. Suppose that a portion of $(1 - \alpha)$ of maturing debt is (forcibly) retired, so that $\alpha \in (0, 1)$ implies deleveraging and $\alpha > 1$ implies re-leveraging. Then overall face value F_t is dynamically changing according to

$$dF_t = -m(\phi_t)(1 - \alpha) F_t dt$$

whereas the amount of short-term debt changes according to

$$dS_t = \left[-\delta_S S_t + f \cdot \alpha m(\phi_t) F_t\right] dt$$

Thus, the maturity structure changes according to

$$\begin{aligned}
d\phi &= \frac{dS}{F} - \frac{S}{F}\frac{dF}{F} \\
&= \left[-\phi\delta_S + f \cdot \alpha m(\phi) - \phi\left(-m(\phi)(1 - \alpha)\right)\right] dt \\
&= \left[-\phi\delta_S + \left[f \cdot \alpha + \phi(1 - \alpha)\right] m(\phi)\right] dt \\
&= \mu_\phi(f) dt
\end{aligned}$$

which is not a function of F, but only of ϕ and α.

As we have three state variables, (y, ϕ, F), we write

$$(r + \zeta) E = y - cF + \zeta(X - F) + m(\phi) F \{\alpha[f D_S + (1 - f) D_L] - 1\} - \mu_y(y) E_y + \mu_\phi(f, \phi) E_\phi - m(\phi)(1 - \alpha) F \cdot E_F$$

The FOC w.r.t. f is then given by

$$\alpha \cdot m(\phi) \max_{f \in [0,1]} f \{F(D_S - D_L) + E_\phi\} = \alpha \cdot m(\phi) \max_{f \in [0,1]} f \{F \cdot \Delta + E_\phi\}$$

so that $IC = F \cdot \Delta + E_\phi$. We can also write everything in terms of (τ, y_b, F_b):

$$(r + \zeta) E = y - cF + \zeta(X - F) + m(\phi) F \{\alpha[f D_S + (1 - f) D_L] - 1\} - E_\tau$$

Taking derivatives w.r.t. ϕ, we have

$$(r + \zeta) E_\phi = m'(\phi) F \{\alpha[f D_S + (1 - f) D_L] - 1\} + m(\phi) F \alpha \left[f \frac{\partial D_S}{\partial \phi} + (1 - f) \frac{\partial D_L}{\partial \phi}\right] - E_{\tau\phi}$$

Evaluated at $\tau = 0$ and imposing $\left.E_\phi\right|_{\tau=0} = 0$ we have

$$\left.E_{\tau\phi}\right|_{\tau=0} = (\delta_S - \delta_L) F_b \left[\alpha \frac{B(y_b)}{F_b} - 1\right] + m(\phi) F_b \alpha \left[f \frac{\partial D_S}{\partial \phi} + (1 - f) \frac{\partial D_L}{\partial \phi}\right]$$

Further, we have for $c = 1$

$$D_i(\tau, y_b, F_b) = 1 + e^{-(r + \zeta + \delta_i)\tau} \left[\frac{B(y_b)}{F_b} - 1\right]$$

At default $\tau = 0$, we have

$$IC(0, y_b, F_b) = F_b \cdot 0 + 0 = 0$$

so that we again have to look at

$$IC_\tau(\tau, y_b, F_b) = F_\tau \Delta + F \cdot \Delta_\tau + E_{\tau\phi}$$

Evaluate at $\tau = 0$ to get

$$\begin{aligned}
IC_\tau(0, y_b, F_b) &= F_\tau 0 + F_b(\delta_S - \delta_L)\left[1 - \frac{B(y_b)}{F_b}\right] + (\delta_S - \delta_L) F_b\left[\alpha \frac{B(y_b)}{F_b} - 1\right] + m(\phi) F_b \alpha \left[f \frac{\partial D_S}{\partial \phi} + (1-f)\frac{\partial D_L}{\partial \phi}\right] \\
&= -(1-\alpha)(\delta_S - \delta_L) B(y_b) + m(\phi) F_b \alpha \left[f \frac{\partial D_S}{\partial \phi} + (1-f)\frac{\partial D_L}{\partial \phi}\right]
\end{aligned}$$

Writing out

$$\frac{\partial}{\partial \phi} D_i(\tau, y_b, F_b) = \frac{\partial}{\partial \tau} D_i(\tau, y_b, F_b)\frac{\partial \tau}{\partial \phi} + \frac{\partial}{\partial y_b} D_i(\tau, y_b, F_b)\frac{\partial y_b}{\partial \phi} + \frac{\partial}{\partial F_b} D_i(\tau, y_b, F_b)\frac{\partial F_b}{\partial \phi}$$

and we have

$$\begin{aligned}
\frac{\partial}{\partial \tau} D_i(0, y_b, F_b) &= (r + \zeta + \delta_i)\left[1 - \frac{B(y_b)}{F_b}\right] \\
\frac{\partial}{\partial y_b} D_i(0, y_b, F_b) &= \frac{B'(y_b)}{F_b} \\
\frac{\partial}{\partial F_b} D_i(0, y_b, F_b) &= -\frac{B(y_b)}{F_b^2}
\end{aligned}$$

When does shortening arise? Suppose we have $f = 1$. Then we must have

$$IC_\tau > 0 \iff (1-\alpha)(\delta_S - \delta_L) B(y_b) < m(\phi) F_b \alpha f \frac{\partial D_S}{\partial \phi}$$

In general, we have $\frac{\partial \tau}{\partial \phi} < 0$ (the higher ϕ, the earlier the default), and $\frac{\partial y_b}{\partial \phi} > 0$ (the earlier the default, the higher the defaulting cash-flow), but $\frac{\partial F_b}{\partial \phi}$ has a sign that is determined by α. In case of deleveraging at default, i.e., $\alpha < 1$, then we have $\frac{\partial F_b}{\partial \phi} > 0$, and shortening at default becomes more difficult as $\frac{\partial}{\partial F_b} D_i(\tau, y_b, F_b)\frac{\partial F_b}{\partial \phi} < 0$. However, in case of releveraging at default, i.e., $\alpha > 1$, then we have $\frac{\partial F_b}{\partial \phi} < 0$, and shortening at default becomes less difficult as $\frac{\partial}{\partial F_b} D_i(\tau, y_b, F_b)\frac{\partial F_b}{\partial \phi} > 0$.

A.5.2 Larger issuance space

For large issuance space, we note that $\min_{y_b \in [y_{min}, y_{max}]} f_b^{nc}(y_b) = -\frac{r+\zeta+\delta_L}{(\delta_S - \delta_L)}$ as in the vicinity of y_{min} we have $B'(y_b) = 0$. Further, if

$$\min_{y_b \in [y_{min}, y_{max}]} f_b^{nc}(y_b) = -\frac{r + \zeta + \delta_L}{(\delta_S - \delta_L)} > -f_L > \max_{y_b \in [y_{min}, y_{max}]} f_{admissible}(y_b)$$

then all results of the main part of the paper go through. This is the case for $f_L = -0.4$ in our numerical examples.

A.5.3 Consol bonds

Suppose that the firm borrows from another group of debt holders holding consol bonds with coupon c_{consol} as in Leland [1994a] that do not feature any rollover. To make the analysis stark and simple, we assume that these consol bonds get zero payment in both the upper and the default events.[34] Further, we assume that there is a tax advantage of bond-holders receiving ρc_i while equity holders are only paying out c_i, with $\rho > 1$. As a result, the valuation formula for the long-term and short-term bonds remain identical if we assume $\rho c_i = r$ for $i \in \{S, L\}$. The equity holder's problem remains almost the same, with the only adjustment of an additional coupon outflow of c_{consol}. The default boundary becomes

$$\Phi(y_b) = \frac{1}{\delta_S - \delta_L}\left[\frac{y_b - c - c_{consol} + \zeta E^{rf}}{1 - B(y_b)} - \delta_L\right],$$

[34] Zero recovery in the default event can be justified by the assumption that the consol bonds are junior to the term bonds we analyzed so far.

which affects the endogenous time-to-default τ. The value of consol bonds, denoted by D_{consol}, is given by

$$D_{consol}(\tau, y_b) = \frac{\rho c_{consol}}{r+\zeta}\left[1 - e^{-(r+\zeta)\tau}\right],$$

with $\frac{\partial}{\partial\phi}D_{consol}(\phi, y)\Big|_{\tau=0} = \rho c_{consol}\frac{\partial\tau}{\partial\phi} < 0$. Intuitively, shortening maturity structure leads to an earlier default and hence a lower value of consol bonds.

Now the firm value includes the value of consol bonds. As before, we can decompose the local effect of maturity shortening on the firm value, i.e., $V_\phi(\phi, y)$, into

$$V_\phi(\phi, y) = \underbrace{E_\phi(\phi, y) + \Delta(\phi, y)}_{\text{Incentive compatibility}} + \underbrace{\phi\frac{\partial}{\partial\phi}D_S(\phi, y) + (1-\phi)\frac{\partial}{\partial\phi}D_L(\phi, y)}_{\text{Impact on ST \& LT bonds}} + \underbrace{\frac{\partial}{\partial\phi}D_{consol}(\phi, y)}_{\text{Impact on consol bonds}}. \tag{A.67}$$

At default, the last term is negative, increasing in c_{consol} and may dominate the second positive term in a maturity-shortening equilibrium, leading to $V_\phi(\Phi(y_b), y_b) < 0$.

At $\tau = 0$, we have

$$V_\phi(\Phi(y_b), y_b) = \underbrace{\phi\frac{\partial}{\partial\phi}D_S(\phi, y) + (1-\phi)\frac{\partial}{\partial\phi}D_L(\phi, y)}_{\text{Impact on ST \& LT bonds}} + \underbrace{\frac{\partial}{\partial\phi}D_{consol}(\phi, y)}_{\text{Impact on consol bonds}}$$

as the IC condition vanishes.

We will concentrate on the equilibrium in which $f = 1$ is just binding, which is given by a point y_b. For this to hold, wee need

$$\frac{\partial}{\partial\phi}D_S = 0$$

Writing out the derivative of total firm value evaluated at the default boundary at point y_1, we have

$$V_\phi = [1-\Phi(y_1)]\frac{\partial}{\partial\phi}D_L + \frac{\partial}{\partial\phi}D_{consol} = [1-\Phi(y_1)]\left\{(r+\delta_L+\zeta)[1-B(y_1)]\frac{\partial\tau}{\partial\phi} + B'(y_1)\frac{\partial y_b}{\partial\phi}\right\} + \rho c_{consol}\frac{d\tau}{d\phi}$$

Let us note that

$$\frac{\partial}{\partial\phi}D_S = 0 \iff (r+\delta_S+\zeta)[1-B(y_1)]\frac{\partial\tau}{\partial\phi} + B'(y_1)\frac{\partial y_b}{\partial\phi} = 0$$

So that

$$\begin{aligned}
\frac{\partial}{\partial\phi}D_L &= (r+\delta_L+\zeta)[1-B(y_1)]\frac{\partial\tau}{\partial\phi} + B'(y_1)\frac{\partial y_b}{\partial\phi} \\
&= -(\delta_S-\delta_L)[1-B(y_1)]\frac{d\tau}{d\phi} + (r+\delta_S+\zeta)[1-B(y_1)]\frac{\partial\tau}{\partial\phi} + B'(y_1)\frac{\partial y_b}{\partial\phi} \\
&= -(\delta_S-\delta_L)[1-B(y_1)]\frac{d\tau}{d\phi} > 0
\end{aligned}$$

and thus $f = 1$ is socially inefficient locally if and only if

$$\rho c_{consol} - [1-\Phi(y_1)](\delta_S-\delta_L)[1-B(y_1)] > 0 \tag{A.68}$$

Plugging in for $\Phi(y_1)$, we have

$$\begin{aligned}
0 &< \rho c_{consol} - \left(1 - \frac{1}{\delta_S-\delta_L}\left[\frac{y_1 - c - c_{consol} + \zeta E^{rf}}{1-B(y_1)} - \delta_L\right]\right)(\delta_S-\delta_L)[1-B(y_1)] \\
&= \rho c_{consol} - \left(\delta_S[1-B(y_1)] - \left[y_1 - c - c_{consol} + \zeta E^{rf}\right]\right) \\
&= (\rho-1)c_{consol} - \left(\delta_S[1-B(y_1)] - \left[y_1 - c + \zeta E^{rf}\right]\right)
\end{aligned}$$

so we have an inefficient shortening equilibrium near the bankruptcy boundary if

$$c_{consol} > \frac{\delta_S[1-B(y_1)] - \left[y_1 - c + \zeta E^{rf}\right]}{\rho - 1}$$

A.5.4 Comparative statics w.r.t. aggregate face-value F

Proof of Proposition 4. Let F be the total face-value outstanding. For the main part of the paper, we have $F = 1$. Note that for recovery value $B(y)$, each individual bond recovers $\frac{B(y)}{F}$. Let us assume $\rho c = c = r$ throughout. First, note that

$$\Phi(y_b; F) = \frac{1}{\delta_S - \delta_L}\left(\frac{y_b - rF + \zeta(X - F)}{F - B(y_b)} - \delta_L\right)$$

and

$$f_b^{nc}(y_b) = \frac{\mu_y(y_b) B'(y_b) - (r + \zeta + \delta_L)[F - B(y_b)]}{(\delta_S - \delta_L)[F - B(y_b)]}$$

The value y_1 is defined by $f_b^{nc}(y_1) = 1$, which is equivalent to

$$f_b^{nc}(y_1) = 1 \iff \frac{\mu_y(y_1) B'(y_1)}{F - B(y_1)} = r + \zeta + \delta_S$$

Finally, note that

$$\frac{dy_1}{dF} = -\frac{\frac{\partial f_b^{nc}(y_1)}{\partial F}}{\frac{\partial f_b^{nc}(y_1)}{\partial y_1}} = \frac{\frac{\mu_y(y_1) B'(y_1)}{(\delta_S - \delta_L)[F - B(y_1)]^2}}{\frac{\partial f_b^{nc}(y_1)}{\partial y_1}} > 0$$

which implies that raising the face-value shifts up the cash-flow state y_1 at which $f = 1$, the shortening constraint, starts binding. Plugging in for y_1 and $\frac{\partial f_b^{nc}(y_1)}{\partial y_1}$, we can simplify to

$$\frac{dy_1}{dF} = \frac{\mu_y(y_1) B'(y_1)}{\mu_y(y_1)\left[B'(y_1)^2 + [F - B(y_1)] B''(y_1)\right] + \mu_y'(y_1) B'(y_1)[F - B(y_1)]}$$

Next, note that

$$\frac{\partial \Phi(y_b; F)}{\partial F} = -\frac{y_b + X\zeta - (r + \zeta) B(y_b)}{(\delta_S - \delta_L)[F - B(y_b)]^2}$$

Noting that $y_b \in [y_{min}, y_{max}]$, we have $\Phi(y_b) > 0 \iff \delta_L[F - B(y_b)] + (r + \zeta) F < y_b + X\zeta$, so that

$$y_b + X\zeta - (r + \zeta) B(y_b) > (\delta_L + r + \zeta)[F - B(y_b)] > 0$$

We thus have

$$\frac{\partial \Phi(y_b; F)}{\partial F} = -\frac{y_b + X\zeta - (r + \zeta) B(y_b)}{(\delta_S - \delta_L)[F - B(y_b)]^2} < -\frac{(\delta_L + r + \zeta)[F - B(y_b)]}{(\delta_S - \delta_L)[F - B(y_b)]^2} < 0$$

so that the bankruptcy boundary shifts outwards — default occurs at an earlier cash-flow state for any given ϕ.

Finally, we note that

$$\begin{aligned}
\frac{d\Phi(y_1; F)}{dF} &= \Phi_y(y_1; F)\frac{dy_1}{dF} + \Phi_F(y_1; F) \\
&= -\frac{B'(y_1)\mu_y'(y_1)[y_1 + \zeta X - (r + \zeta) B(y_1)] + \mu_y(y_1)\left[-B'(y_1) + (r + \zeta) B'(y_1)^2 + [y_1 + X\zeta - (r + \zeta) B(y_1)] B''(y_1)\right]}{(\delta_S - \delta_L)[F - B(y_1)]\left\{\mu_y(y_1)\left[B'(y_1)^2 + [F - B(y_1)] B''(y_1)\right] + \mu_y'(y_1) B'(y_1)[F - B(y_1)]\right\}}
\end{aligned}$$

By assumption $\mu_y'(y) \geq 0$, $B''(\cdot) \geq 0$ as well as $B'(\cdot) \geq 0$, and using $B(y_1) < F$, the denominator is positive. Thus, we are left with investigation of the numerator. After dividing the numerator by $F - B(y_1) > 0$ and by $(r + \zeta + \delta_S)$, and then using the definition of y_1, we can write the transformed numerator (without the negative sign) as

$$Q(y) \equiv \left[(r + \zeta) B'(y) - 1\right] + [y + X\zeta - (r + \zeta) B(y)]\left[\frac{B''(y)}{B'(y)} + \frac{\mu_y'(y)}{\mu_y(y)}\right]$$

If $Q(y_1) > 0$, then $\frac{d\Phi(y_1;F)}{dF} < 0$ follows immediately. Note that by assumptions on $B(\cdot)$ and $\mu_y(\cdot)$ as well as $y \geq y_{min}$ the second term is always positive.

Next, to show that the set of firms (ϕ, y) that fulfill the SE criteria expands, we need to show that

$$\frac{\frac{d\Phi(y_1;F)}{dF}}{\frac{dy_1}{dF}} < \frac{\frac{\partial \phi}{\partial \tau}}{\frac{\partial y}{\partial \tau}} = \frac{\delta_L[\Phi(y_1) - 1]}{\mu_y(y_1)}$$

Plugging in for the LHS, simplifying and switching signs, we have

$$\frac{B'(y_1)\,\mu_y'(y_1)\,[y_1+\zeta X-(r+\zeta)\,B(y_1)]+\mu_y(y_1)\left[-B'(y_1)+(r+\zeta)\,B'(y_1)^2+[y_1+X\zeta-(r+\zeta)\,B(y_1)]\,B''(y_1)\right]}{(\delta_S-\delta_L)\,[F-B(y_1)]\,\mu_y(y_1)\,B'(y_1)} > \frac{\delta_L\,[1-\Phi(y_1)]}{\mu_y(y_1)}$$

which we can rewrite as

$$\begin{aligned} Q(y_1) = \left[(r+\zeta)\,B'(y_1)-1\right]+[y+X\zeta-(r+\zeta)\,B(y_1)]\left[\frac{B''(y_1)}{B'(y_1)}+\frac{\mu_y'(y_1)}{\mu_y(y_1)}\right] &> \frac{(\delta_S-\delta_L)\,B'(y_1)\,\delta_L\,[1-\Phi(y_1)]}{(r+\zeta+\delta_S)} \\ &= [F-B(y_1)]\,(\delta_S-\delta_L)\,\frac{\delta_L\,[1-\Phi(y_1)]}{\mu_y(y_1)} \end{aligned}$$

If this holds, then the set of shortening equilibria expands away from the bankruptcy boundary. □