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INCOMPLETE MARKETS AND AGGREGATE DEMAND

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ABSTRACT

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1 Introduction

Basic macroeconomic models are minimalist. For example, the New Keynesian model is often described as consisting of two fundamental parts,

1. the Intertemporal Euler equation or 'Demand block';

2. the Phillips Curve or 'Supply block';

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this picture may be completed by a description of monetary policy, providing a third equation. A similar picture holds outside model without nominal rigidities. For example, the real business cycle model also features the Euler equation and adopts the flexible price limit as its supply block.

Microfoundations can be provided for both blocks, but these can only be viewed as extremely simplified approximations of a deeper reality. In particular, behind the standard Euler equation lies an assumption of complete markets or the adoption of a representative agent. Similarly, behind the New Keynesian Phillips Curve lies the simplifying assumption of Calvo pricing. These assumptions are easily rejected at face value, but it is hard to deny their usefulness as tractable starting points.

For the New Keynesian model, the Phillips curve 'supply block' has undergone deep scrutiny. Empirically, much research has been devoted to documenting and interpreting facts on price and wage dynamics. Theoretically, the literature has pursued the implications of other forms of nominal rigidities, such as menu costs, incomplete information or rational inattention. Close attention has also been paid to the structure of marginal costs, market competition, strategic complementarities, or institutional features such as inflation indexation.

This paper is instead concerned with the 'demand block' and asks: What are the effects of market incompleteness on aggregate demand? Given the well-documented importance of idiosyncratic uncertainty and the lack of perfect insurance at the household level, market incompleteness hardly requires motivation. The existing literature has made important strides, especially in some particular contexts and applications, but is still far from offering a comprehensive and general answer to this important question.

I study an economy populated by a continuum of households facing idiosyncratic uncertainty and incomplete markets. Each household receives income that depends on an idiosyncratic shock (the microeconomic component) and also depends on aggregate spending (the macroeconomic component). In equilibrium, consistency requires aggregate consumption to equal aggregate income. The main goal of this paper is to solve for the path for aggregate consumption, taking into account this general-equilibrium requirement, and relate it to the path for real interest rates. In the standard complete market or representative agent case this relation boils down to the well-known intertemporal Euler equation.

When markets are incomplete, characterizing equilibria is known to be challenging. I first provide a simple result under extreme illiquidity (no borrowing and no outside assets). This case is relatively tractable thanks to the fact that, in equilibrium, no intertemporal trade is possible. However, the allocation for consumption is still endogenously determined and depends on the interest rate path. I obtain an aggregate relation involving the real interest rate, current and future aggregate consumption that fully characterizes equilibria. In general, according to this generalized Euler relation, current consumption may respond more or less than one-for-one with changes in future consumption—a departure from the standard Euler equation. However, for a benchmark case, where individual income is proportional to aggregate income, the relation specializes to that of a standard representative-agent Euler equation. This provides a useful *as if* result, one that should be interpreted with care.

This 'as if' result does *not* imply that incomplete markets are irrelevant for aggregates. Idiosyncratic uncertainty, lack of insurance and borrowing constraints all have a *level* impact on aggregate consumption. In particular, greater uncertainty or greater scarcity of liquidity tend to depress consumption, as one may expect. This level effect materializes in my result by the fact that market incompleteness affects the subjective discount factors of 'as if' representative agent. Indeed, the equilibrium consumption allocation may involve rich dynamics as in the deleveraging episodes modeled by Eggertsson and Krugman (2012) and Guerrieri and Lorenzoni (2011), which emphasize market incompleteness and borrowing constraints. As far as aggregates are concerned, all these effects can be captured by a sequence of discount factors and a representative agent representation. Indeed, my results show that these may be interpreted as providing foundations for "aggregate demand shocks" for a representative agent model, which is convenient since working with an incomplete market model directly may be onerous.

Although the 'as if' result does not imply that market incompleteness is irrelevant, it does imply is that the *sensitivity* of aggregate demand to interest rates is unaffected. Thus, when the conditions for this representation hold, aggregate consumption reacts to changes in the path of interest rates in the same manner as in representative agent models. I shall come back to this point below.

Next, I turn to the case with a positive supply of liquidity: situations where an outside asset in positive net supply is available or when households can borrow positive amounts from other households. With positive liquidity the allocation no longer coincides with financial autarky and households manage to smooth their consumption somewhat. In principle, one would expect such a scenario not to lend itself easily to analysis or aggregation. After all, the allocation is non-trivial and may depend in non-trivial ways on the path for interest rates. However, I am able to show that when utility is logarithmic the previous results carry over to the case of positive liquidity: when individual income and borrowing constraints are proportional to aggregate income, there is an aggregate representative-agent Euler equation representation.

There are important and immediate corollaries of these results for "forward guidance", a commitment to lower future interest rates. These policies have been advocated as a way to stimulate aggregate demand in "liquidity trap" situations—when current interest rates are against the zero lower bound. Applying the above results, when the conditions for the the representative-agent representation hold, forward guidance is exactly as effective as in representative agent models. This result may be contrasted with those reported in McKay et al. (2015).

I also explore situations that depart from conditions for the representative-agent representation. In the case of zero liquidity this is easily done with relative generality. The results suggest that if one departs from the assumption that income is proportional to aggregate income, adopting assumptions where uncertainty is countercyclical then this tends to make aggregate demand more sensitive to interest rates. The assumption of countercyclical risk is widely used in the asset pricing literature to help explain the high average equity premium, at least since Mankiw (1986), and has received empirical support (Storesletten et al., 2004; Guvenen et al., 2014).[1] To see this most clearly, I consider in detail an application with a varying extensive margin for employment. Aggregate income increases earnings of the fully employed, but also decreases the probability of being underemployed. In this case, aggregate demand is shown to react more strongly to current interest rates, and even more strongly to future interest rates. Intuitively, lower interest rates in the future increase future aggregate income, this then lowers uncertainty, which lowers the desire for precautionary savings. This amplifies the usual consumption smoothing effect. Note that for this result to hold, it is crucial that the probability of being underemployed varies with aggregate income.

With positive liquidity, I show that a crucial statistic is the value of assets relative to income. When utility is logarithmic this statistic is constant. When the intertemporal elasticity of substitution is less than one, I show that interest rate changes make asset value relatively more volatile than income. A lowering of the interest rate makes the ratio of asset value to aggregate income rise. This endogenously increases the availability of liquidity, which in turn stimulates consumption. Consumption in earlier periods is stimulated even further because the extra liquidity decreases the sensitivity of consumption to income shocks, lowering uncertainty, which lowers the precautionary savings motive.

These last two points suggest that if one departs in plausible directions from the as-

[1] A large literature in asset pricing investigates the implications of incomplete markets for asset pricing, as compared to a representative agent model. A partial list of papers includes Heaton and Lucas (1996), Constantinides and Duffie (1996), Krusell and Smith (1997), Alvarez and Jermann (2001), , Krueger and Lustig (2010), Krusell et al. (2011). Some of these papers approach the problem numerically or analytically by studying benchmarks, such as cases where the equilibrium involves autarky.

sumptions that deliver the representative-agent representation one may actually obtain greater sensitivity of consumption to interest rates. Moreover, since this is particularly true of future interest rate it strengthens the power of forward guidance policies. Of course these suggestive results are not definitive, and any conclusions ultimately depends on the particular assumptions one makes, especially those regarding household income and its dependence on aggregate income. The main contribution of this paper is to provide guidance regarding the different possibilities and isolate the main assumptions.

All of these results have obvious implications in models with nominal rigidities, where monetary policy may be seen as affecting the path for real interest rates. However, my methods and results transcend such a setting. To illustrate this, towards the end of the paper I consider a real economy with capital and productivity shocks, a real business cycle model. I show that the representative agent representation holds for a particular case. Admittedly, the particular case requires full depreciation which is rather special and not plausible, yet it complements existing numerical results, such as Krusell and Smith (1998), with an exact analytical aggregation result. This may help interpret prior numerical results in a new light. For example, the literature has mostly stressed that approximate aggregation holds when households are able to smooth their consumption effectively. However, my aggregation result holds no matter how large or persistent the idiosyncratic uncertainty faced by households. Thus, household consumption may experience violent fluctuations and yet my exact aggregation result holds. This points to a new rationale for aggregation, other than the notion that the incomplete markets may not bite if agents are able to smooth their consumption sufficiently and approximate the complete market allocation.

This paper belongs to a vast literature in macroeconomics exploring the implications of relaxing the representative agent assumption and adopting an incomplete markets model. A very partial list includes Krusell and Smith (1998), Heathcote et al. (2009), Guerrieri and Lorenzoni (2011), Kaplan and Violante (2011), Ravn and Sterk (2012), Sterk and Tenreyro (2013), Eggertsson and Krugman (2012), Sheedy (2014) and McKay et al. (2015). All of these efforts have contributed to our understanding, often focusing on different issues. One distinguishing feature of the present paper is the goal to provide an aggregate relation under relatively general conditions, rather than working and solving a full particular model. Another difference is the effort to provide conditions under which there exists a tractable representative-agent representation.

2 A General Incomplete Market Setting

This section introduces the incomplete market setting where households are assumed to make consumption and savings decisions. The setup is fairly general and will be specialized in different directions later on.

My main goal is to understand how incomplete markets affect the behavior of macroeconomic aggregates. Ideally, one wishes to obtain something like the standard representative Euler equation for the incomplete-market model. These goals turn out to be attainable in some interesting cases.

The focus will be on households, with firms and the government largely relegated to the background. However, my analysis is not partial equilibrium since it imposes the general equilibrium feedback between aggregate consumption and aggregate income. Of particular interest is the relationship between the paths for household spending and interest rates. To close the model, this "demand block" of the model can be combined with a "supply block", of any variety, together with a specification for government policy including monetary interest rate policy.[2] For our purposes, it is more useful not to commit to one particular way of doing this. This can be avoided by focusing on the "demand block" of the model only.

2.1 Economic Environment

To simplify and focus on idiosyncratic uncertainty, the model abstracts, for now, from aggregate uncertainty. Aggregate shocks can be included at a later stage, but they are not essential to the relationship between aggregate spending and real interest rates, which can be addressed by examining aggregate deterministic dynamics.

The framework I develop below is based on the standard Bewley-Huggett-Aiyagari incomplete market setup. The horizon is infinite, with periods are $t = 0, 1, \ldots$ There is a single final consumption good and a unit measure of infinitely-lived households. Households are subject to idiosyncratic uncertainty, undergoing fluctuations in their labor income and suffering from shocks to their spending needs. Markets are incomplete preventing insurance of idiosyncratic shocks and credit is subject to borrowing constraints. As a result, at the household level income and consumption may fluctuate significantly.

[2] The demand block can be studied fruitfully in separation from the rest of the model because it is block recursive, in the sense that only the path for real interest rates matter. For example, the breakdown between nominal interest rates and inflation is not required.

Household heterogeneity. There is a finite set of households types $i \in I$ with fraction $\mu^i > 0$ in the population, satisfying the adding up condition $\sum_{i \in I} \mu^i = 1$. Types may differ with respect to their preferences, including discounting. They may also have different labor earnings process and different degrees of access to credit, i.e. borrowing constraints.

Consumption Preferences. Household of type $i \in I$ has preferences over consumption given by the utility function

$$\sum_{t=0}^{\infty} \beta_{i,t} \mathbb{E}_0 [u_t^i(c(s^t), s_t)] \tag{1}$$

where c_t denotes consumption of the single final good, $s_t \in S^i$ denotes an idiosyncratic state of nature that follows a stochastic process, discussed further below. Shocks to utility are included for both generality and realism. They may capture important lifetime events, such as health shocks or family size changes, that affect the relative desirability of current spending.

Because the focus on consumption and savings choices, I first directly postulate a labor income process directly, instead of deriving it from labor supply choices. This keeps us closer to the incomplete markets literature, which studies consumption and savings while taking the income process as given. With this in mind, there is no need at this stage to describe preferences over leisure or labor. However, later I will derive labor income and when I do I shall assume that utility from consumption and labor are additively separable.

Budget Constraints. Households of type i faces the budget constraints

$$c(s^t) + q_t \cdot a(s^t) + b(s^t) \leq y^i(s_t) + (q_t + d_t) a(s^{t-1}) + R_{t-1} \cdot b(s^{t-1}) \tag{2}$$

for all $t = 0, 1, \ldots$ and histories $s^t \in S^{t+1}$; here $a(s^t)$ and $b(s^t)$ denote savings in the outside asset and riskless one-period bonds, respectively; q_t is the price of the outside asset and d_t is its dividend; R_t denotes the interest rate on riskless bonds, in real terms; y_t^i is income from labor.

Labor income depends on household state and aggregate income according to

$$y^i(s_t) = \gamma_t^i(s_t, Y_t), \tag{3}$$

for some function γ_t^i. The nature of the relationship between household and aggregate income encapsulated by γ_t^i, will turn out to be crucial.

Borrowing Constraints. Households of type $i \in I$ are also subject to borrowing constraints

$$b(s^t) + q_t \cdot a(s^t) \geq -B^i(s_t, Y_t), \tag{4}$$

limiting how negative their wealth is allowed to become. Here B^i is a nonnegative borrowing limit, determined as a function of the current household state and aggregate income. The constraint on borrowing is specified in terms of total wealth; alternatively, one can impose separate constraints for assets and bonds, $b(s^t) \geq -B^i(s_t, Y_t)$ and $a(s^t) \geq 0$, and the results would be similar.

Idiosyncratic Uncertainty. For each household type $i \in I$, the exogenous state $\{s_t\}$ follows a stochastic process. An important case is when s_t follows a Markov process, although this assumption is not required for most of my analysis. Uncertainty is purely idiosyncratic: the realization of states are independent across agents and for each type $i \in I$ the probability of a certain set of histories s^t equals the fraction of agents in the cross section experiencing this history. I also assume that, for each household type $i \in I$, the stochastic process for household states $\{s_t\}$ is independent of the path for aggregate income $\{Y_t\}$. This assumption is essentially a normalization, since we have not placed restrictions on the functions γ_t^i and B_t^i.

Initial Conditions. At $t = 0$ the economy inherits, for each household type $i \in I$, a joint distribution over initial states and initial asset and bonds $\Lambda_0^i(s_0, a_0, b_0)$. The stochastic process then induces a joint distribution of histories s^t and (a_0, b_0) denoted by $\Lambda_t^i(s^t, a_0, b_0)$.

Outside Asset. The outside asset is in fixed supply, normalized to unity, providing a dividend stream that is a function of current aggregate income,

$$d_t = D_t(Y_t). \tag{5}$$

To capture the case with zero outside assets simply set dividends to zero: $D_t(Y_t) = 0$ for all t.[3]

In this formulation the outside asset is in fixed supply. However, Section 6 extends the techniques and results to consider a model with capital and investment.

[3]When $D = 0$ there always exists an equilibrium where the asset price is zero. In some cases there may be other equilibria, akin to monetary equilibria where fiat money has value, I shall not consider these equilibria.

To ensure that Y_t can be interpreted as aggregate income from labor and capital, we require that the functions γ_t and D_t satisfy the identity

$$\sum_{i \in I} \mu^i \int \gamma_t^i(s_t, Y_t)\, d\Lambda_t^i + D_t(Y_t) = Y_t, \tag{6}$$

for all Y_t. Note that given $\{\gamma_t^i\}$ the dividend function $D_t(Y_t)$ can be backed out from this identity.

2.2 Equilibrium

I now introduce a natural equilibrium concept for this framework and provide a simple characterization, reducing the conditions to a few equations.

Equilibrium Definition. An equilibrium specifies interest rates and consumption decisions that are required to be optimal as well as consistent with aggregate income. Formally, given initial conditions R_{-1} and Λ_0^i, an equilibrium is a path for aggregates

$$\{C_t, Y_t, A_t, B_t, R_t, q_t\},$$

and household choices, conditional on initial conditions,

$$\{c^i(s^t; a_0, b_0), a^i(s^t; a_0, b_0), b^i(s^t; a_0, b_0)\},$$

satisfying the following:

1. *household optimization:* taking as given the path for aggregate income and interest rates $\{Y_t, R_t\}$, household choices maximize utility (1) subject to (2), (3) (4) and (5);

2. *market clearing:* for all $t = 0, 1, \ldots$ the good, asset and bond markets clear,

$$C_t = Y_t,$$
$$A_t = 1,$$
$$B_t = 0;$$

3. *aggregation:* the aggregate quantities are consistent with household quantities,

$$C_t = \sum_{i \in I} \mu^i \int c^i(s^t; a_0, b_0)\, d\Lambda_t(s^t, a_0, b_0),$$
$$A_t = \sum_{i \in I} \mu^i \int a^i(s^t, a_0, b_0) d\Lambda_t(s^t, a_0, b_0),$$
$$B_t = \sum_{i \in I} \mu^i \int b^i(s^t, a_0, b_0) \Lambda_t(s^t, a_0, b_0).$$

Given a path for interest rates, one may seek a path for aggregate consumption that forms part of an equilibrium. It is important to understand that this aggregate perspective goes beyond the pure aggregation of household consumption choices, no small task in itself. In particular, the equilibrium notion incorporates general equilibrium feedback effects between consumption and income, through the fact that $y_t^i = \gamma_t^i(s_t, C_t)$.[4]

Implementability Conditions. The equilibrium requirements can be reduced to a small set of conditions as follows. First, the two riskless assets must satisfy a no-arbitrage condition equating returns,[5]

$$\frac{q_{t+1} + d_{t+1}}{q_t} = R_t.$$

This no-arbitrage condition is implied by imposing that the asset price equal the present value of its dividends,

$$q_t = \sum_{s=0}^{\infty} \frac{1}{R_t R_{t+1} \cdots R_{t+s}} D_{t+s}(Y_{t+s}). \tag{7}$$

Household optimality requires budget constraints to hold with equality. Defining total wealth $\hat{a}^i(s^{t-1}; a_0, b_0) \equiv q_t \cdot a^i(s^t) + b^i(s^t)$, this requires for $t \geq 1$

$$c^i(s^t; a_0, b_0) + \hat{a}^i(s^t; a_0, b_0) = \gamma_t^i(s_t, Y_t) + R_{t-1} \cdot \hat{a}^i(s^{t-1}; a_0, b_0), \tag{8a}$$

Similarly, the budget constraint at $t = 0$ requires

$$c^i(s_0; a_0, b_0) + \hat{a}^i(s_0; a_0, b_0) = \gamma_0^i(s_0, Y_0) + (q_0 + D_0(Y_0))a_0^i + R_{-1} \cdot b_0^i, \tag{8b}$$

[4]This perspective can be contrasted with some well-known aggregation exercises. For example, Huggett (1993) and Aiyagari (1994) aggregate consumption and savings for given interest rates, but taking the income process as given. They then employ this aggregate relationship graphically to determine an equilibrium that clears the market.

[5]Strictly speaking, when borrowing is completely ruled out, so that $B^i = 0$, an equilibrium requires only that $R_t^a \geq R_t$. However, in such cases, the equilibrium with $R_t^a > R_t$ is not robust to the introduction of vanishingly small amounts of borrowing.

which cannot be reduced to wealth $\hat{a}^i$ only; we must condition on initial asset and bond positions, a_0^i and b_0^i, as well as the asset price q_0. Wealth must satisfy the borrowing constraints

$$\hat{a}^i(s^t; a_0, b_0) \geq -B_t^i(s_t, Y_t). \tag{9}$$

Household optimization reduces to the Euler condition,

$$u_{c,t}^i(c^i(s^t; a_0, b_0), s_t) \geq \beta_t^i R_t \mathbb{E}_t[u_{c,t+1}^i(c^i(s^{t+1}; a_0, b_0), s_{t+1})], \tag{10}$$

with the complementary slackness requirement that this condition hold with equality in period t whenever the borrowing constraint (9) in period t holds with strict inequality. Finally, we impose the market clearing condition $C_t = Y_t$ and the aggregation condition for C_t. All the other market clearing conditions are then implied.

To summarize, an equilibrium can be reduced to aggregates

$$\{C_t, R_t\}$$

and household consumption and wealth $\{c(s^t; a_0, b_0), \hat{a}(s^t; a_0, b_0)\}$ satisfying aggregation $C_t = \sum_{i \in I} \mu^i \int c(s^t; a_0, b_0) d\Lambda(s^t, a_0, b_0)$, the budget constraints (8), borrowing constraints (9) and Euler condition (10) with complementary slackness. In these conditions we obtain q_0 by (7) and $Y_t = C_t$.

When these conditions hold, one can find the remaining equilibrium objects as follows. The asset price in all periods is given by (7). Asset and bond holdings, however, are indeterminate: any portfolio split satisfying $\hat{a}^i(s^{t-1}; a_0, b_0) = q_t \cdot a^i(s^t) + b^i(s^t)$ constitutes an equilibrium.

2.3 Pitfalls of Aggregation under Partial Equilibrium

Before stating my aggregation results, it is worth briefly reviewing common pitfalls of aggregation under partial equilibrium. This discussion will help underscore how my results rely on general equilibrium considerations.

The Euler condition is nonlinear, but absent uncertainty, absent borrowing constraints and assuming homogeneous discounting and power utilities ($u(c) = \frac{1}{1-\sigma} c^{1-\sigma}$) it can be transformed into a linear relationship and aggregated. To see this, note that the household Euler equation then implies

$$c^i(s^t) = (\beta R_t)^{-\frac{1}{\sigma}} c^i(s^{t+1})$$

and thus, aggregating,

$$C(s^t) = (\beta R_t)^{-\frac{1}{\sigma}} C(s^{t+1}) \tag{11}$$

Unfortunately, this aggregation is delicate and easily upset by idiosyncratic uncertainty and borrowing constraints, among other things. I discuss these two issues in turn.

In the presence of idiosyncratic uncertainty and assuming no binding borrowing constraints, we start from $c^i(s^t) = (\beta R_t)^{-\frac{1}{\sigma}} \cdot \left(\mathbb{E}_t[c^i(s^{t+1})^{-\sigma}]\right)^{-\frac{1}{\sigma}}$, so that aggregating and using Jensen's inequality gives

$$C(s^t) > (\beta R_t)^{-\frac{1}{\sigma}} C(s^{t+1}).$$

The magnitude of the departure from (11) varies with the degree of uncertainty in $c^i(s^{t+1})$.

When borrowing constraints bind and abstracting from uncertainty, we start from $c^i(s^t) < (\beta R_t)^{-\frac{1}{\sigma}} c^i(s^{t+1})$, implying

$$C(s^t) < (\beta R_t)^{-\frac{1}{\sigma}} C(s^{t+1}).$$

The departure from (11) now goes in the opposite direction. Of course, by combining uncertainty with binding borrowing constraints there is no telling in which direction the departure from (11) goes.[6]

These difficulties in aggregation are overcome below, but by taking a different route: rather than simply aggregating Euler equations, I use additional information by imposing an additional equilibrium condition, using it to pinpoint an exact aggregate equilibrium relation. In particular, an equilibrium requires the equality of aggregate income and consumption. Imposing this condition turns out to be crucial. Indeed, I do not uncover any novel aggregation result for consumption when taking the income process as given. I find tractable aggregate relations only when imposing the equality of aggregate consumption and income. The aggregate relations I obtain in this way do not always take the form in (11) and even in the cases that it does admit such a representation the discount factor β is different and possibly time varying.

To sum up, my results do not rely on cleverly overcoming standard microeconomic aggregation problems. Instead, I solve for aggregate consumption by exploiting its general equilibrium relation with aggregate income.[7]

[6] This discussion shows that one cannot expect equation (11) to hold. While it does not prove, it is suggestive that there is no other stable relationship, such as equation (11) with a modified discount factor. The reason this is generally impossible is that the departures from (11) depend on forces such as precautionary effects and binding borrowing constraints, which are not stable. They do not depend only on t and $t+1$ variables, but also on the entire future.

[7] By the same token, my results are designed for comparative static exercises, e.g. changing the interest

3 Vanishing Liquidity

A necessary condition for incomplete markets to matter at the aggregate level is that it make a difference at the household level. The market imperfection must generate significant departures from the complete market outcome. Clearly, risk sharing is impaired and the perfect insurance outcome characteristic of complete markets is generally impossible. However, as has been widely noted, due to the households' incentive to smooth consumption and their precautionary motives to accumulate assets, the outcome with incomplete markets may achieve significant improvements over autarky and in some cases approximate the complete market outcome.

The extent to which this is true depends on the capacity of the market to self insure agents. This in turn depends on the amount of liquidity, the value of available assets and the amount of borrowing permitted. When liquidity is plentiful, the outcome is closer to complete markets; when liquidity is scarce, the allocation may be greatly affected by the imperfection in financial markets.

My strongest and simplest characterization obtains under the extreme assumption of absolute illiquidity, where liquidity is completely absent: no borrowing is allowed and there are no outside assets in positive net supply. This situation is best thought of as a limiting case of extreme scarcity of liquidity, with very limited borrowing and small asset values. While extreme, economies with zero or vanishing liquidity are on the opposite side of the spectrum from complete markets. Thus, a priori, they are the poster child for studying the effects that financial market imperfections have on aggregate demand.

3.1 A General Euler Relation

I now study situations with zero liquidity, that is, assuming

$$
\begin{aligned}
D_t(Y_t) &= 0, \\
B_t^i(s_t, Y_t) &= 0,
\end{aligned}
$$

and initial conditions $b_0^i = 0$ for all agents.

Since savers can only save with borrowers and borrowing is ruled out, it follows immediately that in equilibrium no intertemporal trade is possible and the allocation coincides with autarky,

$$c^i(s^t) = y^i(s^t) = \gamma_t^i(s_t, C_t),$$

rate path. They may not resolve the empirical problem faced by microeconomic tests (e.g. Attanasio and Weber, 1993).

where I have substituted the equilibrium condition $C_t = Y_t$. The equilibrium interest rate path that sustains this equilibrium must, in each period, ensure that no household has an incentive to save. This in turn requires

$$R_t \leq R_t^* = \min_{i \in I, s^t} \left(\beta_t^i \mathbb{E} \left[\frac{u_{c,t+1}(\gamma_{t+1}^i(s_{t+1}, C_{t+1}), s_{t+1})}{u_{c,t}(\gamma_t^i(s_t, C_t), s_t)} \mid s_t \right] \right)^{-1}. \tag{12}$$

Equilibrium interest rates are not uniquely determined: interest rates below R_t^* imply that agents find the corner solution, with $b_t^i = 0$, optimal. However, interest rates strictly below R_t^* are not robust to the introduction of small amounts of liquidity. Positive but vanishing levels of liquidity require $R_t = R_t^*$, since the Euler equation must hold with equality for some agents when $D_t(Y) > 0$ and $B_t^i(s, Y) > 0$. Given to this refinement, which I henceforth adopt, the equilibrium is unique. The next proposition summarizes this characterization.

Proposition 1. *A path for aggregate consumption and interest rates $\{C_t, R_t\}$ is part of an equilibrium with vanishing liquidity if and only if*

$$g_t(R_t, C_t, C_{t+1}) = 0, \tag{13}$$

where the function g_t is given by

$$g_t(R, C, C') \equiv \log R + \log \left(\max_{i \in I, s^t} \beta_t^i \mathbb{E} \left[\frac{u_{c,t+1}^i(\gamma_{t+1}^i(s_{t+1}, C'), s_{t+1})}{u_{c,t}^i(\gamma_t^i(s_t, C), s_t)} \mid s_t \right] \right).$$

This proposition provides a simple and condensed way of exploring the aggregate consequences incomplete market. It all boils down to a single relation involving the same variables as in the standard representative-agent Euler equation; namely, the current interest rate, R_t, present and future consumption, C_t and C_{t+1}. Whatever the form taken by the function g_t, even if it does not coincide with the standard Euler equation, Proposition 1 offers a similar tractability for investigating the dynamics of aggregate consumption. In particular, a relation like (13) can be handled, alongside other equilibrium conditions, as easily as the standard Euler equation in positive or normative equilibrium analyses.

The familiar complete-market representative-agent Euler equation, with time-invariant discounting and utility function, is then a special case with

$$g(R, C, C') = \log R + \log \beta + \log u'(C') - \log u'(C).$$

While the more general Euler condition (13) does not necessarily take this particular functional form, it is nevertheless a simple condition involving only aggregates, and in that sense, much like the standard Euler condition. Due to this property, despite incomplete markets, the aggregate equilibrium conditions are tractable. Indeed, they are equally tractable to the representative agent case. In particular, computing an equilibrium does not require carrying a large endogenous state and confronting the curse of dimensionality, as Krusell and Smith (1998) did. Indeed, the simplifying assumption of zero liquidity ensures that wealth is zero for all agents, at all times, so that there is no wealth distribution to keep track of. Moreover, given this assumption, no additional simplifying assumptions are needed to keep the model tractable.

In general the function g_t varies over time. This is not surprising, since no stationarity assumptions have been placed on any primitives, such as utility functions, discounting, the stochastic process, the income function γ^i_t. As we discuss further below in the context of our next result, this may help capture interesting situations, such as temporary episodes with heightened idiosyncratic uncertainty. Even when primitives are stationary, the function g_t may vary simply because the cross-section of states s_t is not at an invariant steady-state distribution. This time dependence, however, vanishes in the long run if the Markov process is ergodic, since the cross section of states then converges to its invariant distribution in the long run.

3.2 Standard Euler Equation

I now apply the general characterization obtained above to an important benchmark specification. In the benchmark, utility functions are power function and taste shocks are multiplicative,

$$u^i_t(c,s) = \theta^i_t(s) \cdot U^i(c) \qquad \text{with} \qquad U^i(c) = \frac{c^{1-\sigma^i}}{1-\sigma^i}, \tag{14}$$

for $\sigma^i > 0$; household income is proportional to aggregate income,

$$\gamma^i_t(s,Y) = \tilde{\gamma}^i_t(s)Y, \tag{15}$$

for some function $\tilde{\gamma}^i_t$.

The next result shows that aggregate consumption and interest rates are related by a standard Euler equation, just as in complete-market or representative-agent economies.

Proposition 2. *Suppose utilities satisfy (14) and household income satisfy (15). Then a sequence*

$\{C_t, R_t\}$ is part of a equilibrium with vanishing liquidity if and only if

$$U'(C_t) = \beta_t R_t U'(C_{t+1}),$$

with the discount factors given by

$$\beta_t = \max_{i \in I, s^t} \beta_t^i \cdot \mathbb{E}\left[U'\left(\frac{\tilde{\gamma}_{t+1}^i(s_{t+1})\theta_{t+1}^i(s_{t+1})}{\tilde{\gamma}_t^i(s_t)\theta_t^i(s_t)} \right) \mid s^t \right]. \tag{16}$$

According to this proposition, aggregate demand is determined *as if* the economy were populated by a single representative agent with discount factor β_t. By implication, changes in the path for interest rates have the same effect on the aggregate consumption path as they do in the representative-agent benchmark. In this sense, the response of consumption to interest rates, is not affected by incomplete markets.

Are Incomplete Markets Irrelevant? No, not at all. In fact, Proposition 2 not only implies that market incompleteness is *not* irrelevant, it identifies very clearly the influence. Due to market incompleteness the discount factor β_t is a function of idiosyncratic uncertainty, as shown in (16). For instance, in periods with greater uncertainty or downward tail risk (for the growth rate of household income) we can expect the discount factor to be higher; given interest rate and future consumption, R_t and C_{t+1}, this implies lower current consumption, C_t. In contrast, with complete markets the aggregate Euler equation holds with a discount factor that does not depend on idiosyncratic uncertainty (just as in (11)) implying that idiosyncratic uncertainty has no effect on current consumption.

This discussion underscores that incomplete markets matters, affecting the *level* of demand, even if according to Proposition it does not affect the responsiveness of demand to current and future interest rates.

3.3 Departures from Standard Euler Equation

It is useful to recast Proposition 2 as providing conditions for

$$g_t(R_t, C_t, C_{t+1}) = \log R_t + \log \beta_t + \sigma \log C_t - \sigma \log C_{t+1},$$

so that g_t is exactly log linear, with equal coefficients in absolute value on $\log C_t$ and $\log C_{t+1}$ given by σ, the reciprocal of the intertemporal elasticity of substitution. This implies the exact relationship for the log changes,

$$d\log R_t + \sigma d\log C_t - \sigma d\log C_{t+1} = 0.$$

Below I investigate departures from this representation by characterizing the analog first-order expansion.

I now depart from this baseline and characterize g_t by considering the coefficients in the expansion

$$d\log R_t + \alpha_{C,t} d\log C_t - \alpha_{C',t} d\log C_{t+1} = 0.$$

I will not study this as an approximation, but rather as an exact relation characterizing the derivatives of g_t; thus, the coefficients $\alpha_{C',t}$ depend on the position where they are evaluated.

Define the elasticity of γ_t^i by

$$\varepsilon_t^i(s,Y) \equiv \frac{\gamma_t^i(s,Y)Y}{\gamma_t^i(s,Y)}.$$

This elasticity measures the responsiveness of individual income to aggregate income. Then next result shows that the coefficients depend can be expressed in terms of this elasticity.

Proposition 3. *Suppose utilities satisfy (14) without taste shocks, i.e. $\theta_t^i(s) = 1$. Then*

$$\alpha_{C,t} = Cg_{C,t}(R,C,C') = \sigma\epsilon_t^i,$$
$$\alpha_{C',t} = -C'g_{C',t}(R,C,C') = \sigma\frac{\mathbb{E}_t[u_{c,t+1}^i\epsilon_{t+1}^i]}{\mathbb{E}_t[u_{c,t+1}^i]}.$$

for the household type $i \in I$ and history s^t that attains the maximum in (16).

Note that

$$\left.\frac{d\log C_t}{d\log C_{t+1}}\right|_{d\log R_t=0} = \frac{\alpha_{C',t}}{\alpha_{C,t}} = \mathbb{E}_t\left[\frac{u_{c,t+1}^i}{\mathbb{E}_t[u_{c,t+1}^i]}\cdot\frac{\varepsilon_{t+1}^i}{\varepsilon_t^i}\right]$$
$$= \mathbb{E}_t\left[\frac{\varepsilon_{t+1}^i}{\varepsilon_t^i}\right] + Cov_t\left[\frac{\lambda_{t+1}^i}{\mathbb{E}\left[\lambda_{t+1}^i \mid s_t\right]}, \frac{\varepsilon_{t+1}^i}{\varepsilon_t^i}\right].$$

Under the conditions for Proposition 2 this ratio equals 1, so that current consumption, C_t, varies proportionally with future consumption, C_{t+1}. One can see this by observing that these conditions ensure that $\varepsilon_t^i = 1$, so that the first term is 1 while the covariance term is zero.

Implications for Forward Guidance. Away from this neutrality case, what are the plausible possibilities? Consider a simple, but telling, example. Assume γ_t^i is independent of the period t and s_t is a Markov chain, with finitely many values for s_t, that is mean reverting. Suppose further that γ^i is an increasing function of s_t, so that a higher state s_t is associated with a higher income level. Suppose, as is plausible, that the agent most willing to save, the agent pinning down the interest rate, the household type $i \in I$ and history s^t that attains the maximum in (16), has the highest value of s_t. This implies that s_{t+1} is expected to be lower, $s_{t+1} < s_t$, due to mean reversion.

Now suppose ϵ_{t+1}^i is a decreasing function. Then the first term must be be greater than 1, due to the mean reversion. The second, covariance term, is positive. Together, this implies that the ratio is strictly greater than 1. The converse is also true. Thus, in this example, when income at the bottom is more sensitive to aggregate income, we find that current consumption is more sensitive to future consumption than one-for-one.

This implies that changes in interest rates into the future have *stronger* effects on current consumption than changes in current interest rates. Thus, in this sense, forward guidance is more powerful.

Income Growth Rate Perspective. The characterization above was performed using the income function γ and its elasticity ϵ. In other words, the perspective was a description of primitives in terms of levels of income. Another useful perspective is to think in terms of growth rates of income. After all, it is the growth rate of income that enters individual household Euler equations and the formula for β_t in (16).

To pursue this define the growth rate

$$\Gamma_t^i(s, s', Y, Y') = \frac{\gamma_{t+1}^i(s', Y')}{\gamma_t^i(s, Y)}.$$

Then we obtain

$$\alpha_{C,t} = -\sigma \frac{\mathbb{E}\left[u_{c,t}^i \frac{\Gamma_C}{\Gamma} C\right]}{\mathbb{E}\left[u_{c,t}^i\right]},$$
$$\alpha_{C',t} = \sigma \frac{\mathbb{E}\left[u_{c,t}^i \frac{\Gamma_{C'}}{\Gamma} C'\right]}{\mathbb{E}\left[u_{c,t}^i\right]}.$$

for the household type $i \in I$ and history s^t that attains the maximum in (16).

By implication

$$\alpha_{C',t} - \alpha_{C,t} = -\sigma \mathbb{E}\left[u^i_{c,t}\left(\frac{\Gamma_{C'}}{\Gamma}C' + \frac{\Gamma_C}{\Gamma}C\right)\right].$$

This last expression says that what matters is the sum of the two elasticities (which are, naturally, of opposite signs). In words, suppose we increase consumption in the current and next period proportionally. Then we ask, how does this affect the growth rate for consumption of households that price bonds? If higher aggregate income in both periods lowers the mean reversion, uncertainty and downside risk in consumption, as seems natural, then we should expect

$$\left.\frac{d\log C_t}{d\log C_{t+1}}\right|_{d\log R_t=0} = \frac{\alpha_{C',t}}{\alpha_{C,t}} > 1,$$

so that, in proportional terms, current consumption reacts more than one-to-one to increases in future consumption.

3.4 An Example Based with Varying Employment Probabilities

To illustrate the previous results, consider the following example, where the income process is motivated by employment risk, intensive and extensive margins. The probability of being employed potentially varies with the level of aggregate demand, Y.

All workers are identical ex ante, so there is a single household type. There are no taste shocks and utility is iso-elastic $U(c) = c^{1-\sigma}/(1-\sigma)$. Income shocks are due to employment fluctuations. Each period households may or may not be fully employed. When they do, they earn $\bar{y}Y^\gamma$; otherwise, they earn $\underline{y}Y^\gamma < \bar{y}Y^\gamma$. We assume underemployed households earn nonzero, to avoid zero consumption. This can be motivated by supposing unemployment shocks hit only a fraction of household members, or by extending the model to include an unemployment insurance payment.[8]

The parameter γ controls how much of the adjustment in income takes place along the intensive margin; if $\gamma = 1$ then all the adjustment is along the intensive margin; if $\gamma = 0$ then all adjustment is along the extensive margin.[9]

[8]Indeed, in this zero liquidity environment introducing an unemployment insurance system requires balancing the budget: taxing employed workers and rebating the proceeds to unemployed workers. One can then consider a tax system where income of each unemployed is guaranteed to be a proportion o the income of each employed household.

[9]We have assumed the intensive margin sensitivity γ is the same for employed and underemployed workers. Similar results obtain if we assume instead that the underemployed earn a fixed income, independent of aggregate income Y, as long as the intensive margin sensitivity of the fully employed is not

Define the fraction of underemployed $\lambda(Y)$ households so that we satisfy the income identity

$$Y \equiv (1-\lambda(Y))\bar{y}Y^{\gamma} + \lambda(Y)\underline{y}Y^{\gamma}.$$

We limit attention to values of Y that imply $\lambda \in (0,1)$.[10] If $\gamma = 1$ then λ is constant, set so that $1 = (1-\lambda)\bar{y} + \lambda\underline{y}$. As long as $\gamma < 1$ then $\lambda(Y)$ is a strictly decreasing function of Y. We assume that $\lambda(Y)$ also represents the probability each household faces of being unemployed.[11]

The bond is priced by those currently employed, i.e. the optimum in (12) is attained by any household that is employed. Their Euler equation can be written as

$$U'(\bar{y}Y^{\gamma}) = \beta R\left((1-\lambda(Y'))U'(\bar{y}Y'^{\gamma}) + \lambda(Y')U'(\underline{y}Y'^{\gamma})\right).$$

This reflects the fact that their current consumption and income is $c = \bar{y}Y^{\gamma}$ while their consumption next period is either $\bar{y}Y'^{\gamma}$, with probability $1-\lambda(Y')$, or $\underline{y}Y'^{\gamma}$, with probability $\lambda(Y')$. Using the fact that $U'(c) = c^{-\sigma}$ this condition can be rewritten more usefully as follows.

Proposition 4. *In the intensive-extensive margin example economy with varying underemployment we have*

$$U'(Y') = \hat{\beta}(Y')R^{\frac{1}{\gamma}}U'(Y'),$$

where the discount rate function is decreasing in Y' and given by

$$\hat{\beta}(C') \equiv \left(\beta\left(1-\lambda(C') + \lambda(C')U'(\underline{y}/\bar{y})\right)\right)^{\frac{1}{\gamma}}.$$

There are two implications of Proposition 4. We achieve a relation similar to a standard Euler equation, but with two important differences. First, the power on the interest rate is $\frac{1}{\gamma}$ instead of 1. As a result, whenever $\gamma < 1$ consumption is *more* sensitive to changes in the current interest rate, for a given value for the intertemporal elasticity of substitution $\frac{1}{\sigma}$. Second, note that when $\gamma = 1$ the conditions for Proposition 2 are met and so the discount factor $\hat{\beta}$ comes out to be constant; however, as long as $\gamma < 1$ the discount factor $\hat{\beta}(C')$ is strictly decreasing. By implication, current aggregate consumption C becomes more sensitive to changes in future aggregate consumption C'.

Intuitively, when $\gamma < 1$ we are departing from the neutrality result by assuming that

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strong.

[10] For high enough values $\lambda = 1$ and household income must be proportional to aggregate income Y.

[11] To fit this into our notation, we may assume $s \in [0,1]$ is uniformly distributed and that γ is a step function with an upward discontinuity at $s = \lambda(Y)$.

lower aggregate income increases individual income risk. For a given interest rate, heightened uncertainty then leads households to desire precautionary savings for any given current income, depressing aggregate consumption and income (so that in equilibrium savings are zero). The assumption that labor income risk is countercyclical is standard in the asset pricing literature seeking to explain high values of the equity premium puzzle (Constantinides and Duffie, 1996; Alvarez and Jermann, 2001) and has been supported by empirical studies (Storesletten et al., 2004; Guvenen et al., 2014).[12]

This same feedback between the level of spending and uncertainty lies at the heart of Ravn and Sterk (2012). They outline a full macroeconomic model, calibrate it to the US and show that this mechanism is capable of inducing deep recessions. Notably, their paper also exploits the tractability afforded by the zero liquidity assumption. However, their approach is quantitative and requires working with a fuller model. As a result, they do not represent this feedback mechanism in terms of a general aggregate Euler relation, as I have done here.

4 Positive Liquidity

Incomplete markets are likely to have more bite when liquidity is scarce, since this makes it harder for households to smooth transitory income fluctuations or cushion permanent labor income shocks using their savings. Thus, the extreme case without liquidity studied in the previous section may isolate the strongest case for incomplete markets. Nevertheless, it is interesting to investigate situations with positive liquidity.

4.1 Standard Euler Equation

With positive liquidity obtaining sharp results is more challenging, because the allocation does not coincide with autarky. Liquidity allows agents to smooth their consumption and the resulting equilibrium allocation departures nontrivially from autarky. Despite these challenges, I now show that with logarithmic utility aggregate implications can be worked out. Indeed, I obtain a standard representative agent Euler equation condition.

[12] Earlier literature specified and found support for income processes with a countercyclical variance for the shocks e.g. Storesletten et al. (2004). Recent work with administrative data has instead found support for a different specification. In particular, Guvenen et al. (2014) find that it is the left-skewness of shocks that is strongly cyclical. For our purposes downside risk is likely to be what matters most and their evidence supports strong countercyclicality of this risk.

Utility is given (14) with $\sigma^i = 1$ so that the utility function is logarithmic

$$U^i(c) = \log(c).$$

As before household labor income satisfies (15), so that it is proportional to aggregate income. It then follows from identity (6) that

$$D_t(Y_t) = d_t \cdot Y_t,$$

for some $\{d_t\}$, so that dividends are also proportional to total income. Finally, we also assume that borrowing constraints are proportional to aggregate income

$$B^i_t(s, Y) = \tilde{B}^i_t(s)Y, \tag{17}$$

for some function $\tilde{B}^i_t(s)$.

When $d_t = 0$ and $\tilde{B}^i_t(s) = 0$ we are back to the case zero liquidity studied in the previous section. We say there is positive liquidity if the asset's dividend is positive or if borrowing is allowed, if $d_t > 0$ or $\tilde{B}^i_t(s) > 0$ for some t and s.[13]

Revaluation Effects. As it turns out, when studying comparative statics of equilibria with given initial conditions, the initial portfolio households hold matters due to revaluation effects. Recall, at $t = 0$ the budget constraint features initial wealth $q_0 a_0 + R_- b_0$; this is the only period where the asset price enters the budget constraint. The interest rate R_- is predetermined and fixed, but q_0 is endogenous. Different interest rate paths $\{R_t\}$ imply different q_0 affecting the value of initial wealth. The redistribution channel is the focus of Auclert (2015), who gives a detailed analysis of the different possibilities. In the present paper, these revaluation effects are present, but will play out in the background. I will consider two cases depending on the initial asset and bond holdings. In the first case, the revaluation effect is proportional to output, leading to the standard Euler representation.

4.1.1 Zero Initial Bond Holdings

It is convenient to first consider the case where initial bond holdings are zero for all households,

$$b^i_0 = 0.$$

[13]Strictly speaking, when $d_t = 0$ and $\tilde{B}^i_t(s) > 0$ for some s, then in some cases the equilibrium coincides with autarky. This is the case if borrowing is only allowed for the agent setting the interest rate.

Formally, the initial distribution Λ_0^i has full mass over $b_0^i = 0$. Note that this initial condition does *not* rule out or even constrain borrowing. It restricts initial bonds holdings to be zero, but as long as $\tilde{B}_t^i(s) > 0$ future borrowing and saving in bonds, $b_t^i \neq 0$ for $t > 0$, is permitted. Indeed, even initial indebtedness is possible if it takes the form of negative positions in the asset, so that $a_0^i < 0$. Recall that, along an equilibrium, bonds and assets are perfect substitutes and households are indifferent to borrowing and saving in one or the other. As a result, the equilibrium is indeterminate and for any equilibrium with $b_t^i \neq 0$ there is another equilibrium with $b_t^i = 0$ and identical interest rates and allocations.

The next result shows that under these conditions we recover the standard Euler equation.

Proposition 5. *Suppose utilities satisfy (14), household income satisfies (15) and borrowing constraints satisfy (17). In addition, suppose initial bond holdings are zero $b_0^i = 0$ for all households. Then $\{C_t, R_t\}$ is part of an equilibrium if and only if*

$$U'(C_t) = \beta_t R_t U'(C_{t+1}) \tag{18}$$

for some given sequence of discount factors $\{\beta_t\}$, independent of both $\{R_t\}$ and $\{C_t\}$.

Proposition 2 obtained with zero liquidity and the analysis was simplified because the equilibrium allocation was autarkic. In contrast, Proposition 5 applies to situations with positive liquidity with nontrivial allocations that are not autarky. In equilibrium, households smooth their consumption by saving and borrowing, to an extent that depends on the availability of liquidity. Indeed, the consumption and saving allocation may involve rich dynamics as in the deleveraging episodes modeled by Eggertsson and Krugman (2012) and Guerrieri and Lorenzoni (2011).

How is it that all these nontrivial individual decisions under uncertainty can be subsumed in as simple an aggregate relation as (18)? Indeed, the underlying allocation is not simple and cannot generally be solved in closed form—it is well known that Bewley-Aiyagari models quickly require a numerical approach for their solution. The key is that Proposition 5 does not rely on solving for the entire equilibrium. Instead, I obtain relation (18) by studying the more manageable problem of understand how a given equilibrium adjusts to changes in interest rates. In a nutshell, although individual household allocations are nontrivial, as in any Bewley-Aiygari model, I show that across equilibrium with different interest rate paths these allocations scale up and down in a simple manner with aggregate consumption.

To see this more clearly, it is useful to spell out the argument behind the result in some detail. The discount factors β_t and the household allocation behind this result are obtained as follows. Consider a normalized economy with constant income $\tilde{Y}_t = 1$ and its associated equilibrium, including the interest rate $\{\tilde{R}_t\}$, household consumption and wealth $\{\tilde{c}(s^t; a_0), \tilde{a}(s^t; a_0)\}$ and asset prices $\tilde{q}_t = \sum_{s=0}^{\infty} (\tilde{R}_t \tilde{R}_{t+1} \cdots \tilde{R}_{t+s})^{-1} d_{t+s}$. Define $\beta_t \equiv \frac{1}{\tilde{R}_t}$, then by construction, the aggregate Euler equation 18 holds for $C_t = Y_t = 1$.

Now, for any other sequence $\{C_t, R_t\}$ satisfying the aggregate Euler equation (18), construct the rest of the equilibrium objects as follows. Renormalize household consumption and wealth proportionally

$$c(s^t; a_0) = \tilde{c}(s^t; a_0) C_t \qquad \text{and} \qquad \hat{a}(s^t; a_0) = \tilde{a}(s^t; a_0) C_t$$

and adjust the interest rate and asset prices by

$$q_t R_t = \tilde{R}_t \frac{C_{t+1}}{C_t} \qquad \text{and} \qquad q_t = \tilde{q}_t C_t.$$

With this guess, one can verify all equilibrium conditions. The crucial observation is that the Euler equation, the budget constraints and borrowing constraints are all linear homogeneous in $Y_t = C_t$. Notably, the $t = 0$ budget constraint is

$$c^i(s_0; a_0) + \hat{a}^i(s_0; a_0) = \tilde{\gamma}_0^i(s_0) C_0 + (\tilde{q}_0 + d_0) a_0^i C_0,$$

which is homogenous in C_0 by virtue of the auxiliary assumption that $b_0^i = 0$.

Does the Level of Liquidity Matter? Yes and no. As Proposition 5 makes clear, the amount of liquidity has absolutely no effect on the response of consumption to current and future interest rates. Indeed, the response is identical to that of a representative agent with the same preferences. The response of current consumption C_t to changes in the interest rate R_t or changes in future consumption is identical to those implied by a representative agent.

Liquidity is still important at the microeconomic level, with greater liquidity allowing greater consumption smoothing. In the representative-agent Euler equation representation, this shows up by affecting discounting. In this way, the level of liquidity does have a macroeconomic effect in levels: it affects interest rates for a given consumption path or consumption for a given interest rate path. However, these level effects do not affect the response of aggregate consumption to changes in the interest rate path, which is independent of the amount of liquidity.

Discounting and Natural Interest Rates. When primitives are stationary (i.e. s_t is a Markov process) and the economy is initialized with an invariant distribution for (s_0, a_0) then the discount factor is constant $\beta_t = \frac{1}{\tilde{R}}$, where $\tilde{R}$ is the steady state interest rate. As shown by Huggett (1993) and Aiyagari (1994) steady states require $\beta\tilde{R} < 1$, implying that a higher discount factor appears in the representation, compared to the true subjective discount factor.

In general, the discount factors β_t may not be constant, but this is a feature, not a bug. For example, periods of higher idiosyncratic uncertainty are likely to increase β_t temporarily. Likewise, deleveraging episodes—situation where some household start with high initial debt but must lower their debt over time—as modeled, for example, by Eggertsson and Krugman (2012) and Guerrieri and Lorenzoni (2011), may also increase the discount factor temporarily. A higher discount factor β_t is associated with a lower natural interest rate, in the New Keynesian model. Indeed, the economy may be pushed up against the zero interest rate bound if β_t is high enough, as stressed by the liquidity trap literature.

4.1.2 Arbitrary Initial Bond Holdings

I now allow any initial distribution for initial bonds holdings, b_0^i. The main result is a similar characterization to that obtained when initial bonds holdings are zero, except for an adjustment to the discount factors that depends on initial consumption.

Proposition 6. *Suppose utilities satisfy (14), household income satisfies (15) and borrowing constraints satisfy (17). Then $\{C_t, R_t\}$ is part of an equilibrium if and only if*

$$U'(C_t) = \beta_t R_t U'(C_{t+1}) \tag{19}$$

for some a sequence of discount factors $\{\beta_t\}$ that depends on C_0, i.e. $\beta_t = \hat{\beta}_t(C_0)$.

According to this result, a standard Euler equation continues to hold, except that now the relevant discount factors are endogenous. Conveniently, these new effects are entirely summarized by the initial aggregate consumption level, C_0. Thus, the aggregate dynamics for consumption remains tractable and tied to a standard Euler equation.

Why are the discount factors dependent on initial consumption? When output expands in the first period, this diminishes the relative value of bonds. In this relative sense, the distribution of bond holdings contracts towards zero. In contrast, the asset's price and dividends expand proportionally with output, and that wealth held in the form of assets remains in proportion to output. This explains why Proposition 6 requires an adjustment relative to Proposition 5.

Indeed, by a simple extension of the argument provided above, the discount factors are now precisely $\beta_t = \frac{1}{\tilde{R}_t}$, the reciprocal of the equilibrium interest rates $\tilde{R}_t$ for a normalized economy featuring constant output, but with initial bond holdings rescaled to $\tilde{b}^i_0 = \frac{1}{C_0} b^i_0$. This works since then the budget constraint at $t = 0$ becomes

$$c^i(s_0; a_0) + \hat{a}^i(s_0; a_0) = \tilde{\gamma}^i_0(s_0) C_0 + (\tilde{q}_0 + d_0) a^i_0 C_0 + R_- \tilde{b}^i_0 C_0,$$

which is homogeneous in C_0.

Implications. How is the dependence of β_t on C_0 likely to play out? What are the implications of this adjustment? I offer a few tentative, but informed, speculations.

As I just argued, a higher C_0 diminishes the dispersion in bond holding levels relative to output, scaling it down towards zero. For given interest rates, it is reasonable to expect lower dispersion in initial bond holdings to increase consumption (since consumption functions are concave as a function of wealth), at least in earlier periods. This pushes the interest rates $\tilde{R}_t$ upward to reestablish equilibrium with constant unitary output. This argument suggests that the discount factors $\beta_t = \frac{1}{\tilde{R}_t}$ decrease with C_0. The effects on β_t are likely to die out in later time periods t, since $\tilde{R}_t$ is likely to converge to a common steady state $\tilde{R}$, regardless of initial conditions.

If, as seems likely, discount factors are decreasing in consumption, then interest rate paths that increase consumption, e.g. lower interest rates, are likely to have an additional effect, especially in earlier periods. In other words, non-zero initial bond holdings are likely to amplify the effects of interest rate changes.

4.2 Departures from Standard Euler Equation

With positive liquidity, away from the baseline case studied in the previous subsection no aggregation result appears immediately available. This makes studying the effects of interest rate changes more challenging. For each interest rate path, one must solve the individual allocations that are part of an equilibrium and, unlike the previous section, there is no obvious relation across these equilibria. One possibility is to turn to a numerical analysis, but this may not uncover the forces at work transparently. Fortunately, one can work out a few simple cases analytically that shed light on the mechanism and help understand the expected direction of departure from the previous results.

Consider a three period economy, with $t = 0, 1, 2$; one should think of period $t = 2$ as collapsing the entire future in the infinite horizon setting. Agents are ex ante identical. There is no uncertainty at $t = 0$ nor $t = 2$, so that $y_0 = (1 - d_0)Y_0$ and $y_2 = (1 - d_2)Y_2$. At $t = 1$ households experience an idiosyncratic shock s_1 with c.d.f. F and collect labor income $y_1 = s_1(1 - d_1)Y_1$. Agents cannot borrow: $B_t(s) = 0$. The asset pays $D_t(Y_t) = d_t Y_t$ and for simplicity we set $d_0 = d_1 = 0$. All agents start with one unit of the asset.

In equilibrium, households are identical at $t = 0$, so they all carry the asset into $t = 1$. In that period, households are hit with a temporary income shock. Those with a negative enough shock sell the asset, while those with a positive enough shock buy the asset. Intuitively, this helps households smooth consumption between $t = 1$ and $t = 2$; this mitigates the dispersion or uncertainty in consumption at $t = 1$.

I now consider the sensitivity of aggregate consumption to changes in R_0 and R_1. The experiment fixes Y_2 at a constant and solves for $C_1 = Y_1$ and $C_0 = Y_0$.[14]

Slack Liquidity Constraint Case. If first consider the case where no agent sells off their entire asset holdings, so that no agent is liquidity constrained.

Proposition 7. *Consider the three-period economy with uncertainty at $t = 1$. Suppose borrowing constraints are never binding, so that the Euler condition holds with equality for all agents. If $\sigma > 1$,*

$$\frac{d \log C_0}{d \log R_1} < \frac{d \log C_1}{d \log R_1} = \frac{d \log C_0}{d \log R_0} = -\frac{1}{\sigma};$$

the inequality is reversed if $\sigma < 1$. Finally, we have $\frac{d \log C_1}{d \log R_0} = 0$.

The effect of the interest rate on current spending at $t = 1$ is standard, exactly as in the representative agent case: a one percent drop in the interest rate R_1 leads to a rise in consumption of $\frac{1}{\sigma}$ percent. The effects on earlier periods, however, depends on the value of σ. When $\sigma > 1$ this interest rate drop makes consumption at $t = 0$ rise by a greater percentage; when $\sigma < 1$ the effect is smaller. The borderline logarithmic case, $\sigma = 1$, was covered earlier and boils down to the standard representative agent case, with consumption rising by the same percentage in both periods, i.e. the standard consumption smoothing property.

Underlying these results and their dependence on σ is the asset price relative to income, which affects the amount of liquidity, which in turn affects how much agents are

[14] Consumption in all periods scales proportionally with Y_2, given interest rates R_0 and R_1. One interpretation for fixing Y_2 in a monetary economy is when prices are assumed flexible at $t = 2$ or when monetary policy is able to achieve the flexible price equilibrium; in this case, output is at the flexible price at $t = 2$, but we are interested in aggregate demand in the short and medium run, $t = 0, 1$.

able to smooth consumption in the intermediate period. To see this, note that the asset price at $t = 1$ rises in proportion to the fall in R_1, yet, output rises by $\frac{1}{\sigma} < 1$. Thus, if $\sigma > 1$, the asset value rises relative to output relative to output, increasing the supply of liquidity in this sense. This lowers the dependence of consumption on the current income shock. Consumption at $t = 0$ then rises for two reasons: the higher level of consumption at $t = 1$ and the lower uncertainty at $t = 1$. When $\sigma < 1$ this effect is reversed, since consumption at $t = 1$ expands more than the asset value.

This result underscores the importance of the response of asset prices to interest rate changes. It is worth clarifying that there are revaluation effects from the change in the asset value for any value of σ, regardless of whether $\sigma > 1$, $\sigma < 1$ or $\sigma = 1$. Instead, whether or not these revaluations effects increases or decreases the sensitivity of consumption to interest rate changes depends on σ because this determines the relative strength of the standard substitution channel. In effect, what is relevant is whether the asset value rises *relative* to output. Finally, the revaluation effects I focus on here work through outside assets that are in net positive supply with their own stream of dividends. Thus, these effects are distinct from the redistribution effects between creditor and debtors focused on in Auclert (2015) or Sheedy (2014).

Binding Liquidity Constraint Case. Next I consider the case where some agents sell off all their assets at $t = 0$. These agents find themselves liquidity constrained and their Euler equation hold with strict inequality.

Proposition 8. *Consider the three-period economy with uncertainty at $t = 1$. Suppose borrowing constraints bind at $t = 1$ for some households. equality for all agents. Then if $\sigma > 1$ we have that*

$$\frac{d \log C_0}{d \log R_1} < \frac{d \log C_1}{d \log R_1} < \frac{d \log C_0}{d \log R_0} = -\frac{1}{\sigma};$$

the inequalities are reversed if $\sigma < 1$. Finally, we have $\frac{d \log C_1}{d \log R_0} = 0$.

When households find themselves liquidity constrained, the asset value relative to income determines the degree to which this constraint binds. When $\sigma > 1$, the asset value rises relative to output. The increase in the asset value has a direct one-for-one impact on the consumption at $t = 1$ for those households that are constrained; these households have a marginal propensity to consume out of wealth of unity. This amplifies the effect of R_1 on C_1.

5 Labor Markets, Nominal Rigidities and Supply Side

Up to this point I have worked with the incomplete market setting underlying the consumption and savings problem faced by households. I have taken household labor income as well as asset dividends as given functions of aggregate income Y_t. This way of compartmentalizing proved fruitful, since it allowed us to focus on aggregate demand and to judge the assumptions needed for various results more easily, without the details. In this section I now briefly sketch how one can fill in the supply side of the model. I consider a few variants of the New Keynesian model. This is a well known model, so I will describe the main elements and omit the details.

I start with the common elements across the different variants. There is a single final good that is produced by a continuum of varieties according to a Dixit-Stiglitz aggregator

$$Y_t = A_t \left(\int B_t(i) Y_t(i)^{1-\frac{1}{\varepsilon}} \right)^{\frac{1-\alpha_t}{1-\frac{1}{\varepsilon}}} K_t^{\alpha_t}$$

where A_t is total factor productivity, $B_t(i)$ are taste shifters across varieties,, and $\varepsilon > 1$ is the elasticity and α is the capital share. Capital is in fixed supply, representing the outside asset held by households. The Cobb-Douglas specification implies that capital's rents are proportional to output, with $d_t = \alpha_t$.

Each variety i is produced one-for-one from labor

$$y_t(i) = N_t(i)$$

Utility is additively separable between consumption and labor

$$\sum_{t=0}^{\infty} \beta_{i,t} \mathbb{E}_0 [u_t^i(c(s^t), s_t) - v(n(s^t))].$$

Households are subject to shocks to the labor productivity: if they work $n(s^t)$ then they supply $N(s^t) = s_t n(s^t)$ units of quality adjusted labor.

The market is organized as follows. The final good is produced competitively, earning zero profits. The varieties are produced by monopolists and will generally earn positive profits. Nominal rigidities may prevent the immediate adjustment of prices or wages. The exact nature of this rigidity will not be described here, as it is not crucial.

The different variants make different assumptions regarding the functioning of the goods and labor market.

Yeomen Farmers: Collapsing the Goods and Labor Market. In the first variant, intermediate goods are produced by households themselves. Each household owns a particular variety. In other words, there is no labor market; or alternatively, the labor market coincides with the goods market. Households set the price for their variety (or their wage in the alternative labor-market interpretation).

With zero liquidity households will consume their income. As a result, their optimization will lead to income that depends on their productivity shock as well as the taste shock for their variety. These are the idiosyncratic shocks.

Sticky Prices and Flexible Labor Market. The standard New Keynesian model instead assumes the existence of firms that set prices and hire labor in a competitive labor market, with a flexible wage. Assume profits of these firms are taxed 100% with the proceeds rebated by way of a labor subsidy to households. This assumption plays two roles. First, it is a standard way to obtain the efficient allocation, by undoing the monopolistic markup. Second, it implies that profits and dividends from the ownership of firms is zero. As a result, it has the convenient property that the only outside asset with positive income flow is capital.[15]

Sticky Wages and Rationing in the Labor Market. Assume now that varieties are produced competitively, earnings zero profits. However, the nominal wage is rigid and may require rationing. Prices are flexible and set to marginal cost. Note that, since the wage is rigid, marginal costs are rigid and so prices inherit this rigidity. We assume wages are set above the market clearing wage, implying that labor equals labor demand, which in turn equals the demand for goods.

When aggregate demand is low, labor demand is low and employment is rationed. One may make various assumptions about this rationing. But it may also include an extensive and intensive margin, lending an interpretation to the specification in Section 4.2.

[15] Relaxing this assumption lead to profits from monopolistic producers that endogenously vary with output, but this does not necessarily affect the results in any one direction. First one must decide whether firm ownership is transferable, whether firms are modeled as public or private firms; in the former case the stream of profits is effectively part of the outside asset in our notation; in the latter case, profits are just a component of labor income γ_t in our notation. Profits in the standard New Keynesian may be countercyclical (due to the procyclical wage) or procyclical (due to the scale effects) depending on the calibration.

6 Exact Aggregation in a Real Business Cycle Model

Up to this point, I have considered economies with a fixed or mechanical supply of liquidity.[16] In addition, consumption has been assumed to be the only component of demand, since the model lacked investment. Finally, although the results apply more generally and make no explicit assumptions regarding nominal rigidities, their natural application is the study of monetary policy in the presence of nominal rigidities.

The purpose of this section is to show that the results can be generalized and applied in a very different setting. To this end, I now consider a real business cycle model where capital is accumulated by investment and as the outside asset. This real economy is assumed to operate under flexible prices and perfect competition.

My main result assumes utility is logarithmic and supposes full depreciation of capital, a well-known case referred to as Brock-Mirman. Under these conditions, aggregate dynamics behave exactly as those obtained in representative agent or complete market version of the model. This occurs despite potentially arbitrarily large departures at the microeconomic household level in the allocation.

Krusell and Smith (1998) studied a similar real business cycle model numerically, but without the restriction to full depreciation. Their main conclusion was approximate aggregation. My result complements theirs, providing conditions for exact aggregation.

6.1 Economic Environment

The economic environment is a standard real business cycle model, augmented to include idiosyncratic uncertainty and incomplete markets. Because most of it is well known, I will keep the description brief.

Preferences. All households have utility

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\log c_t - v(n_t)\right)$$

where

$$v(n) = \bar{v}\frac{n^{1+\gamma}}{1+\gamma}$$

with $\bar{v} > 0$ and $\gamma \geq 0$.

[16] Even with a fixed supply of assets, liquidity is arguably not exogenous, since dividends are a function on aggregate income, an endogenous variable, and the asset price is determined endogenously.

Uncertainty. Agents are subject to idiosyncratic productivity shocks so that if they work n_t^i they supply $z_t^i \cdot n_t^i$ to the market. Denote the history of idiosyncratic shocks by z^t, and the history of aggregate shocks by s^t. Let Λ_t denote the cross sectional distribution of history of shocks for households of type i.

Technology. Output is given by a constant returns to scale Cobb-Douglas production function,

$$Y(s^t) = A(s_t)F(K(s^{t-1}), N(s^t)) = A(s_t)K(s^{t-1})^{\alpha}N(s^t)^{1-\alpha}.$$

The resource constraint is

$$C(s^t) + K_{t+1}(s^t) \leq A(s_t)F(K_t(s^{t-1}), N_t(s^t)) + (1-\delta)K(s^t).$$

We shall focus on the case with $\delta = 1$.

Budget Constraints. Households are subject to the budget constraints

$$c^i(z^t, s^t) + k^i(z^t, s^t) \leq W_t(s^t)n^i(z^t, s^t) + k^i(z^{t-1}, s^{t-1})R_t(s^t)$$

To simplify, I have assumed no borrowing. The results are robust to the introduction of borrowing, as long as the borrowing constraints are proportional to output.

Equilibrium Conditions. The equilibrium conditions are standard and we relegate the details to the appendix. Households maximize utility subject to the budget constraints. Aggregate variables must be consistent with individual household choices. Firms maximize. Finally, the market for goods (consumption and investment), the market for labor and the market for capital must clear.

6.2 Main Result: Exact Aggregation

The next proposition states the main result of this section.

Proposition 9. *Consider the Real Business Cycle model with $\delta = 1$. Then the aggregate dynamics of capital and labor are equivalent to their counterparts in the complete market, or representative* *agent economy. Namely,*

$$C(s^t) = (1-\omega_t)\,Y(s^t)$$
$$K(s^t) = \omega_t Y(s^t)$$
$$N(s^t) = \bar{N}_t,$$

for some deterministic sequence of saving rates $\{\omega_t\}$ *and labor* $\{\bar{N}_t\}$. *Saving rates* ω_t *are constant if the initial distribution of wealth is at an invariant steady state.*

Individual household consumption and labor are given by

$$c^i(z^t, s^t) = c^i(z^t)C(s^t),$$
$$n^i(z^t, s^t) = n^i(z^t)N(s^t),$$

where $c^i(z^t)$ *and* $n^i(s^t)$ *is computed from an incomplete market equilibrium for a normalized economy with* $C(s^t) = 1$ *and* $N(s^t) = 1$.

7 Conclusion

This paper studies the effect of financial market imperfections on aggregate consumption. Idiosyncratic uncertainty, incomplete markets and borrowing constraints may affect the level of aggregate demand as well as its sensitivity to the path of current and future interest rate. In terms of the level, greater uncertainty or more stringent borrowing constraints typically depress consumption, as one would expect. In terms of sensitivity, I provided a benchmark cases where the response of aggregate consumption to interest rates is exactly identical to that of a standard representative-agent model. Outside these benchmark cases there are many other possibilities, but consumption becomes more sensitive to current and future interest rate changes in plausible cases. Indeed, I discussed a number of cases where consumption may react more to future interest rate changes, relative to current interest rate changes.

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A Proof of Proposition 7

The economy is described by the following primitives. At $t = 0$

$$\gamma_0(Y) = Y$$

$$d_0 = 0$$

At $t = 1$

$$\gamma_1(Y) = y(s)Y$$

$$d_1 = 0$$

At $t = 2$

$$\gamma_2(Y) = (1 - d_2)Y$$

$$d_2 > 0$$

Finally, no borrowing is allowed: $B_t(s) = 0$ for all s and t. Thus, all liquidity is provided by the outside asset. Fix Y_2 at a constant.

The asset price is given by

$$q_1 = \frac{1}{R_1} d_2 Y_2. \tag{20}$$

The present value budget constraint at $t = 1$ is

$$c_1(s) + \frac{1}{R_1}(c_2(s) - (1 - d_2)Y_2) = sY_1 + q_1. \tag{21}$$

Assuming borrowing constraints do not bind, the Euler equation between $t = 1$ and $t = 2$ gives

$$c_1(s) = (\beta R_1)^{-\frac{1}{\sigma}} c_2(s) \tag{22}$$

Substituting (20) and (22) into (21) gives

$$c_1(s) + \beta^{\frac{1}{\sigma}} R_1^{\frac{1}{\sigma}-1} c_1(s) = sY_1 + \frac{1}{R_1} Y_2$$

Solving gives

$$c_1(s) = \frac{sY_1 + \frac{1}{R_1} Y_2}{1 + \beta^{\frac{1}{\sigma}} R_1^{\frac{1}{\sigma}-1}}$$

Setting $C_1 = Y_1$ gives the relation

$$Y_1 = \int c_1(s) dF(s) = \frac{Y_1 + \frac{1}{R_1} Y_2}{1 + \beta^{\frac{1}{\sigma}} R_1^{\frac{1}{\sigma}-1}}$$

and solving for Y_1 gives

$$Y_1 = R^{-\frac{1}{\sigma}} \beta^{-\frac{1}{\sigma}} Y_2$$

Returning to individual household consumption we write

$$c_1(s) = \hat{c}(s; R) Y_1$$

where

$$\hat{c}(s; R) = \omega(R)s + 1 - \omega(R)$$

$$\omega(R) = \frac{\beta^{-\frac{1}{\sigma}} R^{1-\frac{1}{\sigma}}}{\beta^{-\frac{1}{\sigma}} R^{1-\frac{1}{\sigma}} + 1}$$

for $\sigma = 1$ we have that $c(s)$ is a proportion $\hat{c}(s; R)$ of aggregate income Y_1 that varies with s but is independent of the interest rate R. However, for $\sigma > 1$ this proportion does vary with R. In particular, an increase in R leads to an increase in $\hat{c}$ for $y > 1$ and a decrease in $\hat{c}$ for $y < 1$. This increases the relative spread of consumption. Conversely, a lower R decreases the spread. For $\sigma < 1$ the reverse is true.

Turning to $t = 0$ we have

$$c_0 = Y_0 = (\beta R_0)^{-\frac{1}{\sigma}} \cdot \left[\int \hat{c}(s; R)^{-\sigma} dF(s)\right]^{-\frac{1}{\sigma}}$$

and by Jensen's inequality

$$\left(\int \hat{c}(s; R_1)^{-\sigma} dF(s)\right)^{-\frac{1}{\sigma}}$$

is decreasing in R if $\sigma > 1$. This implies that Y_0 increases proportionally more than Y_1.

B Proof of Proposition 8

The argument is similar to proof of Proposition 7, so I only sketch a proof of the arguments that are different.

We now impose the borrowing constraint

$$c_1(s; Y_1) \leq sY_1 + q_1$$

and suppose it is binding for some households, but not everyone. It follows that

$$c_1(s) = \min\left\{\frac{sY_1 + \frac{1}{R_1}Y_2}{1 + \beta^{\frac{1}{\sigma}} R_1^{\frac{1}{\sigma}-1}}, sY_1 + \frac{1}{R_1} d_2 Y_2\right\} \tag{23}$$

or equivalently

$$c_1(s; Y_1, \hat{s}) = \begin{cases} sY_1 + \frac{1}{R_1} d_2 Y_2 & s \leq \hat{s} \\ \frac{sY_1 + \frac{1}{R_1}Y_2}{1 + \beta^{\frac{1}{\sigma}} R_1^{\frac{1}{\sigma}-1}} & s > \hat{s} \end{cases}$$

where the cutoff $\hat{s}$ is defined by the equality

$$\frac{\hat{s}Y_1 + \frac{1}{R}Y_2}{1 + \beta^{\frac{1}{\sigma}} {R_1}^{\frac{1}{\sigma}-1}} = \hat{s}Y_1 + \frac{1}{R_1} d_2 Y_2,$$

This provides a strictly decreasing relation between Y_1 and $\hat{s}$, given R_1.

Aggregating,

$$
\begin{aligned}
Y_1 &= \min_{\tilde{s}} \int c_1(s; Y_1; \tilde{s}) dF(s) \\
&= \int_{\hat{s}} \left(\frac{sY_1 + \frac{1}{R_1} Y_2}{1 + \beta^{\frac{1}{\sigma}} R_1{}^{\frac{1}{\sigma}-1}} \right) dF(s) + \int^{\hat{s}} \left(sY_1 + \frac{1}{R_1} d_2 Y_2 \right) dF(s) \\
&= \frac{Y_1 \int_{\hat{s}} s dF(s) + \frac{1}{R_1} Y_2 \left(1 - F(\hat{s})\right)}{1 + \beta^{\frac{1}{\sigma}} R_1{}^{\frac{1}{\sigma}-1}} + Y_1 \int^{\hat{s}} s dF(s) + \frac{1}{R_1} d_2 Y_2 F(\hat{s})
\end{aligned}
$$

The first equality follows from (23). It implies, by an Envelope condition argument, that we can compute the equilibrium derivative of Y_1 and R_1 without considering the change in $\hat{s}$. Rearranging, for given $\hat{s}$, we obtain

$$
Y_1 = Y_2 \frac{\beta^{-\frac{1}{\sigma}} R_1{}^{-\frac{1}{\sigma}} \left(1 - F(\hat{s}) + d_2 F(\hat{s})\right) + \frac{1}{R_1} d_2 F(\hat{s})}{\int_{\hat{s}} s \, dF(s)}
$$

For fixed $\hat{s}$, this implies that Y_1 is proportional to Y_2. In addition, because of the second term in the numerator we see that when $\sigma > 1$ the response to changes in R_1 are larger when $F(\hat{s}) > 0$. This implies that aggregate consumption at $t = 1$ is more sensitive to change in R_1.

The implications for C_0 are again amplified relative to the effects on C_1 for the same reason as before, due to the reduction in consumption spread at $t = 1$.

C Proof of Proposition 9

The required equilibrium conditions are

$$
\begin{gathered}
c^i(z^t, s^t) + k^i(z^t, s^t) = W_t(s^t) n^i(z^t, s^t) + k^i(z^{t-1}, s^{t-1}) R_t(s^t) \\
\frac{v'(n^i(z^t, s^t))}{u'(c^i(z^t, s^t))} = z_t W_t \\
u'(c^i(z^t, s^t)) \geq \beta \mathbb{E}_t \left[R_{t+1} u'(c^i(z^t, s^t)) \mid z^t, s^t \right]
\end{gathered}
$$

with equality whenever $k^i(z^t, s^t) > 0$. The aggregate conditions are

$$
C(s^t) + K_{t+1}(s^t) = A(s_t) F(K_t(s^{t-1}), N_t(s^t)) + (1 - \delta) K(s^t)
$$

$$R(s^t) = A(s_t)F_k(K(s^{t-1}), N(s^t)) + 1 - \delta$$
$$W(s^t) = A(s_t)F_N(K(s^{t-1}), N(s^t))$$

with aggregates consistent with household choices

$$C(s^t) = \sum \mu^i \int c^i(z^t, s^t) d\Lambda(z^t),$$
$$N(s^t) = \sum \mu^i \int n^i(z^t, s^t) d\Lambda(z^t),$$
$$K(s^t) = \sum \mu^i \int k^i(z^t, s^t) d\Lambda(z^t).$$

Guess and verify that the equilibrium satisfies

$$c^i(z^t, s^t) = c^i(z^t)C(s^t)$$

$$n^i(z^t, s^t) = n^i(z^t)N(s^t)$$

For each period t and aggregate history s^t such a decomposition is without loss of generality; define $c^i(z^t)$ and $n^i(z^t)$ to be the household decomposition of the allocation associated with some aggregate history s^t for each period. I now verify that this decomposition works for all other histories.

Substituting we obtain

$$c^i(z^t)C(s^t) + k^i(z^t)K(s^t) = W(s^t)N(s^t)n^i(z^t) + k^i(z^{t-1})K(s^{t-1})R(s^t)$$

$$\frac{c^i(z^t)}{\bar{v}z_t} v'\left(n^i(z^t)\right) v'(N(s^t))N(s^t)C(z^t) = W(s^t)N(s^t)$$

$$u'(C(s^t)) \geq \left(\frac{\beta \mathbb{E}\left[u'(c^i(z^{t+1})) \mid z^t\right]}{u'(c^i(z^{t+1}))}\right) \cdot \mathbb{E}[R(s^{t+1})u'(C(s^{t+1})) \mid s^t]$$

By definition these equations hold for one history s^t; to check whether these conditions hold for other histories, rewrite them as

$$c^i(z^t)\frac{C(s^t)}{Y(s^t)} + k^i(z^t)\frac{K(s^t)}{Y(s^t)} = \frac{W(s^t)N(s^t)}{Y(s^t)}n^i(z^t) + \frac{K(s^{t-1})R(s^t)}{Y(s^t)} \cdot k^i(z^{t-1})$$

$$\hat{v}v'(N(s^t))N(s^t)\frac{C(s^t)}{Y(s^t)} = \frac{W(s^t)N(s^t)}{Y(s^t)}$$
$$u'(C(s^t)) = \hat{\beta}_t \cdot \mathbb{E}[R(s^{t+1})u'(C(s^t)) \mid s^t]$$

where

$$\hat{v}_t \equiv \int \frac{c^i(z^t)}{\bar{v} z_t} v'\left(n^i(z^t)\right)$$

$$\hat{\beta}_t \equiv \max_{z^t} \frac{\beta \mathbb{E}\left[u'(c^i(z^t)) \mid z^t\right]}{u'(c^i(z^t))}$$

The first equation will hold as long as the terms

$$\frac{C(s^t)}{Y(s^t)} \qquad \frac{K(s^t)}{Y(s^t)} \qquad \frac{W(s^t)N(s^t)}{Y(s^t)} \qquad \frac{K(s^{t-1})R(s^t)}{Y(s^t)}$$

are independent of history s^t; the last two ratios are guaranteed to be constant by the Cobb-Douglas assumption. The second ratio is implied by the first. Thus, we require

$$C(s^t) = (1 - \omega_t)\, Y(s^t)$$

for some saving rate ω_t that does not depend on the history s^t. Turning to the second equation, we determine $N(s^t)$,

$$\hat{v} v'(N(s^t))N(s^t) = \frac{1-\alpha}{1-\omega_t}$$

Finally

$$\frac{1}{C(s^t)} = \hat{\beta}_t \mathbb{E}\left[\frac{\alpha Y(s^{t+1})}{k(s^t)} \frac{1}{C(s^{t+1})} \mid s^t\right]$$

$$\frac{1}{(1-\omega_t)Y(s^t)} = \hat{\beta}_t \frac{1}{k(s^t)} \frac{\alpha}{1-\omega_t}$$

$$\frac{\omega_t}{1-\omega_t} = \hat{\beta}_t \frac{\alpha}{1-\omega_t}$$

Note that at with a steady state invariant distribution we have $\hat{\beta}_t$ constant, so we obtain $\omega_t = \alpha\hat{\beta}$ as in the standard Brock-Mirman solution.