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ABSTRACT

In affine asset pricing models, the innovation to the pricing kernel is a function of innovations to current and expected future values of an economic state variable, for example consumption growth, aggregate market returns, or short-term interest rates. The impulse response of this priced variable to fundamental shocks has a frequency (Fourier) decomposition, which captures the fluctuations induced in the priced variable at different frequencies. We show that the price of risk for a given shock can be represented as a weighted integral over that spectral decomposition. The weight assigned to each frequency then represents the frequency-specific price of risk, and is entirely determined by the preferences of investors. For example, standard Epstein-Zin preferences imply that the weight of the pricing kernel lies almost entirely at extremely low frequencies, most of it on cycles longer than 230 years; internal habit-formation models imply that the weight is shifted to high frequencies. We estimate the frequency-specific risk prices for the equity market, focusing on economically interesting frequencies. Most of the pricing weight falls on low frequencies – corresponding to cycles longer than 8 years – broadly consistent with Epstein-Zin preferences.

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1 Introduction

This paper studies how risk prices for shocks depend on their dynamic effects on the economy. Theoretical asset pricing models have strong implications for how short- and long-term shocks should be priced, and we empirically estimate how the power of a shock at different frequencies determines its risk price.

Affine models, which model innovations to the pricing kernel as being linearly (or log-linearly) related to innovations in economic state variables, comprise the backbone of both theoretical and empirical asset pricing. This paper shows that many widely used affine frameworks can be written, estimated, and interpreted in the frequency domain. The frequency-domain decompositions give a clear and compact characterization of the precise manner in which the dynamics of the economy affect risk prices and provide sharp tests of competing asset pricing models. The decomposition is also economically intuitive in the sense that it provides a direct link between priced shocks and their dynamic effects on the economy.

Economic dynamics are a key input to many asset pricing models. Under Epstein–Zin (1989) preferences, the risk premium of an asset depends on the covariance of its return with current and expected future consumption growth. For the intertemporal CAPM (Merton, 1973; Campbell, 1993), risk premia depend on covariances with shocks to both current market returns and also expected future returns. And in affine term structure models, risk premia can be expressed as a function of the covariance with innovations to current and future short-term interest rates.

In dynamic asset pricing models, then, the price of risk for a shock depends on how it affects the state of the economy in the current period and in the future. Under the standard time-domain representation, the dynamic response of the economy to a shock is summarized by an impulse response function (IRF). Long-run shocks to consumption growth that have large risk prices under Epstein–Zin preferences, for example, those studied in Bansal and Yaron (2004), have IRFs that decay slowly.

In this paper we propose and derive a new frequency-domain representation of risk prices. First, we map the IRF of a shock into the frequency domain. A shock that has strong long-run effects has high power at low frequencies, whereas shocks that dissipate rapidly have more power at high frequencies. We refer to the frequency-domain version of the IRF as the impulse transfer function (ITF).

Our key result is that the price of risk for a shock depends on the integral of the impulse transfer function over the set of all frequencies ω, weighted by a function $Z(\omega)$. The weighting function, Z, determines how shocks are priced depending on how they affect the economy at different frequencies. In other words, $Z(\omega)$ represents the price of exposure to shocks with frequency ω: it is a frequency-specific price of risk. In this paper we derive $Z(\omega)$ in closed form for various theoretical models and

estimate it empirically in equity markets.

The advantage of studying risk prices in the frequency domain is that Z gives a compact and intuitive measure of how different shocks affect the pricing kernel. Importantly, we obtain a separation result: the total price of risk for a shock depends on the interaction of two objects: the impulse transfer function of the shock, which depends on the dynamics of the economy but not on the agent's preferences, and the Z function, which *only* depends on the preferences. We can therefore study the frequency-specific risk prices by only looking at the agents' utility functions. For example, under power utility, the only thing that determines the price of risk for a shock is how it affects consumption today. So Z is perfectly flat across frequencies because cycles of all frequencies receive identical weight in the pricing kernel. Under Epstein–Zin preferences, long-run risks matter, and Z places much more weight at low than high frequencies; in fact, the weight is focused only at the very lowest frequencies. Conversely, for an agent with internal habit formation most of the weight of Z is located at high frequencies.

The spectral representation we derive is useful for two main reasons. First, it gives us new insights about the importance of the dynamics of shocks for asset prices in different models. For example, from the spectral decomposition we learn that Epstein–Zin preferences imply that the majority of the pricing weight lies extraordinarily close to frequency zero: under a standard calibration of the model, more than half of the mass of the spectral weighting function lies on cycles longer than 230 years. Conversely, for an agent with internal habit formation most of the weight of Z is located at high frequencies. The decomposition also tells us which aspects of the consumption process one needs to focus on most when calibrating models: for example, under Epstein–Zin preferences, we find that the key statistics are the unconditional standard deviation and the long-run standard deviation of consumption growth; other aspects of consumption's dynamic behavior are unimportant.[1]

The second key reason the spectral representation is useful is that it enables more general and powerful estimation of dynamic asset pricing models. Our theoretical analysis shows that Epstein–Zin preferences put weight on mainly the lowest-frequency effects of shocks on consumption growth. But low frequencies are precisely where we have the least estimation power empirically. Our spectral decomposition makes it simple to test a more general model in which agents still care about "long-run" shocks, but where the long-run is defined as say, 10 or more years in the future, rather than 200 years. Economically, given the range of people's lifespans, it makes sense to test for the importance of fluctuations that last for decades but decay before reaching one or two hundred years. Furthermore, we obtain much more precise estimates of the weighting functions when considering this alternative definition of "long-run" because estimation power rises substantially as we move

[1] See Dew-Becker (2013) for a more extensive analysis of the issues around calibrating the long-run standard deviation.

away from the lowest frequencies. We are thus able to retain the economic intuition from tightly theorized models that long-term consumption dynamics may be important for asset pricing, but at the same time estimate a model in a more reduced form that is superior statistically in the sense that risk prices are more tightly estimated, as well as economically more plausible.

After deriving the frequency decomposition in section 2 and characterizing weighting functions theoretically for various consumption-based models in section 3, we proceed in section 4 to estimate them using the cross-section of equity prices. Since we are interested in distinguishing among theoretical models that have very stark implications for the pricing of different frequencies, we parametrize Z to be able to capture separately the price of high-frequency and low-frequency shocks, letting the data speak on which are considered the more important frequencies by investors. We do this in two ways: by restricting the weighting function Z to be the one literally implied by the various models, and, second, by focusing on the pricing of economically interesting groups of frequencies – frequencies related to the business cycle, trends lasting longer than the business cycle, and fluctuations at frequencies shorter than business cycles.

The estimation shows strong support for long-run risk models, but only when we define the long-run based on the frequency-domain interpretation of shocks with cycles longer than the business cycle. When we estimate the parameters of Epstein–Zin preferences structurally, most of them are insignificant or barely significant, which would normally be interpreted as a rejection of the model: neither short- nor long-run consumption growth seems to price equities. But that rejection would be a mistake. When we allow "long-run" to simply mean cycles longer than the business cycle, we find that covariance with long-run shocks is a statistically and economically significant determinant of average portfolio returns.

Section 5 extends the analysis to returns-based models, in which agents price equity portfolios based on their covariance with short- and long-run shocks to equity market returns. We find (consistent with Campbell and Vuolteenaho, 2004) that it is low-frequency shocks to equity market returns that drive the pricing kernel. In section 6 we show that the methodology easily generalizes to pricing kernels that depend on innovations of multiple variables. For example, stochastic volatility models (as in Campbell et al., 2013) can imply that agents care about long-run innovations in returns *and* volatility.

There is very little extant analysis of preference-based asset pricing in the frequency domain. Otrok, Ravikumar, and Whiteman (2002) and Yu (2012) are two recent examples. While these papers also present spectral decompositions of prices and consumption fluctuations, the object of the decomposition is different from ours. Instead of studying how shocks at different frequencies are priced by an agent, they ask how the price of a consumption claim depends on the spectral density of consumption and its relation with returns. Since the price of the asset reflects a combination of

preferences and dynamics, it is impossible to evaluate the relative importance of the two beyond very specific cases (in other words, no separation holds in their analysis). Relatedly, unlike this paper, they do not obtain analytic relationships between the spectrum and asset prices; their results are all generated numerically.[2]

Our paper is closely related to a vast empirical literature studying the importance of dynamics for asset pricing in the time domain. Empirically, a number of papers study the relationship between asset returns and consumption growth at long horizons as methods of testing the implications of Epstein–Zin or power utility preferences.[3] These papers test the asset pricing implications of specific models (power utility, Epstein–Zin) about the pricing of consumption risk at different horizons. By working in the frequency domain, we can allow for a much more general specification where shocks to consumption at different horizons may have different risk prices, for example a modified interpretation of "long-run risks" in which the important shocks are those with cycles longer than the business cycle.

Finally, our work is related to other important decompositions of the stochastic discount factor (SDF), most notably Alvarez and Jermann (2005), Hansen and Scheinkman (2009) and Borovicka et al. (2011). These decompositions study the dynamic effects of shocks for the evolution of the stochastic discount factor over time. Instead, we focus on decomposing how the *one-period innovation* in the stochastic discount factor depends on the way consumption responds dynamically to the shocks. In other words, these papers analyze the impulse response function of the SDF, while we study the impulse response function of consumption (and how it affects the one-period innovation in the SDF). Relative to these alternative decompositions, our approach has advantages and disadvantages. The disadvantage is that we cannot study the theoretical valuation of claims to consumption many periods from now (the focus for example of Hansen, Heaton, and Li, 2008, and Lettau and Wachter, 2007). The reason is that these depend on the evolution of the interest rate, which in turn depends on how the mean of the SDF will evolve over time. Naturally, this does not leave us unable to price assets; one-period innovations in the SDF are enough to study the cross-section of returns. The main advantage of our approach is that by focusing on the spectral decomposition of the one-period innovation, we are able to separately study the *dynamics* of the shocks from the *preferences* about the dynamics, a separation result that is not possible when studying the evolution of the whole SDF. As the next sections will show, this separation result

[2] Another related paper, Ortu et al. (2013), studies a different decomposition of the consumption growth process, based on components that operate at different time scales. That paper shows by numerical calibration that the more persistent components of the consumption growth process (as estimated from the data) could be responsible for the high equity premium in a standard Epstein–Zin model.

[3] For example, Parker and Julliard, 2005 ; Malloy, Moskowitz, and Vissing-Jorgensen, 2009; Bansal, Dittmar, and Lundblad, 2005; Yu, 2012; Daniel and Marshall, 1997; Binsbergen, Brandt and Koijen, 2012; Hansen, Heaton, and Li (2008).

will be crucial both for understanding the implications of different models for dynamics, and for allowing us to extend and generalize the models in an economically important direction.

2 Spectral decomposition and the weighting function

We derive our spectral decomposition of the pricing kernel under two main assumptions. First, the log pricing kernel, m_t, depends on the current and future values of a scalar state variable, x_t (perhaps consumption growth or market returns). Second, the dynamics of the economy are described by a vector moving average process X_t which includes x_t.

Assumption 1: Structure of the SDF.

Denote the log pricing kernel (or stochastic discount factor, SDF) $m_{t+1} = log(M_{t+1})$.[4] We assume that m_t depends on current and future values of some state variable in the economy x_t:

$$m_{t+1} = F(I_t) - \Delta E_{t+1} \sum_{k=0}^{\infty} z_k x_{t+1+k} \tag{1}$$

where x_t is the *priced variable*, $F(I_t)$ is some unspecified function of the time-t information set I_t, $\Delta E_{t+1} \equiv E_{t+1} - E_t$ denotes the innovation in expectations, and E_t is the expectation operator conditional on information available on date t, I_t. This specification is sufficiently flexible to match standard empirical applications of power utility, habit formation, Epstein–Zin preferences, the CAPM and the ICAPM (in some cases under log-linearization). Equation (1) implies that risk prices are constant, but we relax that assumption in section 6.

Assumption 2: Dynamics of the economy.

x_t is driven by an N-dimensional vector moving average process

$$x_t = B_1 X_t \tag{2}$$

$$X_t = \Gamma(L)\, \varepsilon_t \tag{3}$$

where X_t has dimension $N \times 1$, L is the lag operator, $\Gamma(L)$ is an $N \times N$ matrix lag polynomial,

$$\Gamma(L) = \sum_{k=0}^{\infty} \Gamma_k L^k \tag{4}$$

and ε_t is an $N \times 1$ vector of (potentially correlated) martingale difference sequences. We refer to ε_t as the fundamental shocks to the economy. We make no assumptions about the conditional

[4] We do not take a position on whether m_t is the pricing kernel for all markets or whether there is some sort of market segmentation. We also do not assume at this point that there is a representative investor.

distribution of ε_t except that it has a mean of zero. Throughout the paper B_j denotes a conformable (here, $1 \times N$) vector equal to 1 in element j and zero elsewhere. We assume without loss of generality that x_t is the first element of X_t. Furthermore, we require $\Gamma(L)$ to have properties sufficient to ensure that x_t is covariance stationary with a finite and continuous spectrum.

Putting together the assumptions about m_{t+1} with those about the dynamics of the economy, we can write the innovations to the pricing kernel as function of the impulse-response functions (IRFs) of x_t to each of the fundamental shocks. In particular, for the jth fundamental shock, $\varepsilon_{j,t}$, the IRF of x_t is the set of $g_{j,k}$ for all horizons k defined as:

$$g_{j,k} \equiv \begin{cases} B_1 \Gamma_k B_j' \text{ for } k \geq 0 \\ 0 \text{ otherwise} \end{cases} \tag{5}$$

We can then rewrite the innovation to the log SDF as:

$$\Delta E_{t+1} m_{t+1} = -\sum_j \left(\sum_{k=0}^{\infty} z_k \, g_{j,k} \right) \varepsilon_{j,t+1} \tag{6}$$

and we refer to $(\sum_{k=0}^{\infty} z_k g_{j,k})$ as the price of risk for shock j. In this representation, the effect of a fundamental shock $\varepsilon_{j,t+1}$ on the pricing kernel is decomposed by horizon: for every horizon k, the effect of the shock on m_{t+1} depends on the response of x at that horizon (captured by $g_{j,k}$) and on the horizon-specific price of risk z_k.

Our main result is a spectral decomposition in which the price of risk of a shock depends on the response of x to that shock at each frequency ω and on a frequency-specific price of risk, $Z(\omega)$.

Result 1. *Under Assumptions 1 and 2, we can write the innovations to the log SDF as,*

$$\Delta E_{t+1} m_{t+1} = -\sum_j \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} Z(\omega) \, G_j(\omega) \, d\omega \right) \varepsilon_{j,t+1} \tag{7}$$

where $Z(\omega)$ is a weighting function depending on the risk prices $\{z_k\}$ and $G_j(\omega)$ measures the dynamic effects of $\varepsilon_{j,t}$ on x in the frequency domain,

$$Z(\omega) \equiv z_0 + 2 \sum_{k=1}^{\infty} z_k \cos(\omega k) \tag{8}$$

$$G_j(\omega) \equiv \sum_{k=0}^{\infty} \cos(\omega k) \, g_{j,k} \tag{9}$$

Equivalently, the price of risk for a shock ε_j can be written as

$$\sum_{k=0}^{\infty} z_k g_{j,k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} Z(\omega) G_j(\omega) d\omega \tag{10}$$

Derivation and discussion

For each shock $\varepsilon_{j,t}$, the set of coefficients $\{g_{j,k}\}$ is the impulse-response function of x_t at different horizons k. Moving into the frequency domain, the first step is to decompose the effects of each shock $\varepsilon_{j,t}$ on the future values of x_t into cycles of different frequencies. To do this, we use the discrete Fourier transform to define

$$\tilde{G}_j(\omega) \equiv \sum_{k=0}^{\infty} e^{-i\omega k} g_{j,k} \tag{11}$$

If $\varepsilon_{j,t}$ has very long-lasting effects on x, then most of the mass of $\tilde{G}_j(\omega)$ will lie at low frequencies, while if $\varepsilon_{j,t}$ induces mainly transitory dynamics in x, then $\tilde{G}_j(\omega)$ will isolate high frequencies.[5] We refer to $\tilde{G}_j$ as the *impulse transfer function* (ITF) of shock j since it is the transfer function associated with the filter $\sum_{k=0}^{\infty} g_{j,k} L^k$.

Using the inverse Fourier transform, the price of risk for shock j is

$$\sum_{k=0}^{\infty} z_k g_{j,k} = \sum_{k=0}^{\infty} \left(z_k \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{G}_j(\omega) e^{i\omega k} d\omega \right) \tag{12}$$

Now note that $g_{j,k} = 0$ for all $k < 0$. In the Appendix we show that, as a consequence, for any

[5] To be more rigorous about the sense in which $\tilde{G}$ gives weights in terms of cycles of different frequencies, we refer to the spectral representation theorem. Specifically, denote $\bar{x}_{k,t}$ the process induced in x_t if the only shock realizations were for ε_k. That is,

$$\bar{x}_{k,t} = \sum_{j=0}^{\infty} g_{k,j} \varepsilon_{k,t-k}$$

$\varepsilon_{k,t}$ has a spectral representation

$$\varepsilon_{k,t} = \int_{-\pi}^{\pi} e^{it\omega} dZ(\omega)$$

where $dZ(\omega)$ is an orthogonal increment process with constant variance (see, e.g., Priestley, 1981, for a textbook statement and proof of the spectral representation theorem). The spectral representation of $x_{k,t}$ is then

$$\bar{x}_{k,t} = \int_{-\pi}^{\pi} e^{it\omega} \sum_{j=0}^{\infty} g_{k,j} e^{-ij\omega} dZ(\omega) = \int_{-\pi}^{\pi} e^{it\omega} \tilde{G}_k(\omega) dZ(\omega)$$

$\tilde{G}_k$ thus determines the magnitude of fluctuations in $\bar{x}_{k,t}$ at frequency ω.

$k > 0$ we can rewrite equation (12) as:

$$\sum_{k=0}^{\infty} z_k g_{j,k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} G_j(\omega) \left(z_0 + 2 \sum_{k=1}^{\infty} z_k \cos(\omega k) \right) d\omega \tag{13}$$

where $G_j(\omega)$ is the real part of $\tilde{G}_j(\omega)$,

$$G(\omega) = re\left(\tilde{G}_j(\omega)\right) = \sum_{k=0}^{\infty} \cos(\omega k) g_{j,k} \tag{14}$$

In other words, the price of risk for any shock depends on the integral of its response in the frequency domain, $G_j(\omega)$, weighted by a real-valued function $Z(\omega)$, where

$$Z(\omega) \equiv z_0 + 2 \sum_{k=1}^{\infty} z_k \cos(\omega k) \tag{15}$$

We thus have

$$\sum_{k=0}^{\infty} z_k g_{j,k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} G_j(\omega) Z(\omega) d\omega \tag{16}$$

This equation maps an element-by-element product of the infinite collections $\{g_{j,k}\}$ and $\{z_k\}$ into a simple integral over a finite range in the frequency domain. This result is closely related to the convolution and Parseval's theorems, but is not identical because we take advantage of the fact that $g_{j,k} = 0$ for $k < 0$ to ensure that $Z(\omega)$ is real-valued.[6]

The price of risk for shock ε_j thus depends on an integral over the function $G_j(\omega)$, with weights $Z(\omega)$. Recall that for each frequency ω, $G_j(\omega)$ tells us the effect of ε_j on x at frequency ω. $Z(\omega)$ therefore determines the price of risk for any shock to the variable x at frequency ω.

Three final points are worth making. First, in our derivation we assume that the *log* SDF, m_{t+1}, is linear in the news about future values of x_t. We present the result in this form because the most widely used models (including log-linearized version of non-affine models) usually specify an affine form for the log SDF. The same decomposition holds if, instead of assuming that m_{t+1} is affine, we assume that the *level* of the SDF M_{t+1} is linear in the news terms. For the remainder of the paper, we focus on the decomposition of the innovations in the log SDF.

Second, the result does not hinge on any particular assumption about the conditional distribution of the shocks ε_{t+1} (e.g., normality). Third, we are also not making any assumptions about whether the conditional distribution of ε_{t+1} is identical across periods or varies over time. It could be

[6] We thank Urban Jermann for pointing out an alternative derivation of our result based on Parseval's theorem.

heteroskedastic, for example.

2.1 Examples of impulse transfer functions $G_j(\omega)$

Before proceeding further, it is helpful to see examples of what the impulse transfer function looks like for some simple impulse response functions. For the sake of concreteness, suppose for the moment that the priced variable x_t is log consumption growth, Δc_t.

Figure 1 plots the impulse response and impulse transfer functions for four different hypothetical shocks. Note that while we are ultimately interested in the effects of the shocks on consumption growth, Δc_t (since this is what enters the log SDF), we plot the IRF in terms of consumption levels, c_t, as they are the more natural way to think about consumption.

The first shock is a simple one-time increase in the level of consumption. This shock has a flat impulse transfer function on consumption growth, indicating it has power at all frequencies. The second shock is a long-run-risk type shock, inducing persistently positive consumption growth, with the level of consumption ultimately reaching the same level as that induced by the first shock. In this case, there is much less power at high frequencies, but the power at frequency zero is identical, since $G(0)$ depends only on the long-run effect of the shock on the level of consumption ($G_j(0) = \sum_{k=0}^{\infty} g_{j,k}$).

The next two shocks have purely transitory effects. The third shock raises consumption for just a single period, and we see now zero power at frequency zero and positive power at high frequencies. The fourth shock is more interesting. Consumption rises initially, turns negative in the second period, and returns to its initial level in the third period. The transfer function is again equal to zero at $\omega = 0$, but it now actually has negative power at low and middle frequencies. This is a result of the fact that the impulse response of consumption is actually negative in some periods. The sign of G reflects the direction in which the shock drives consumption. If we had reversed the signs of the impulse responses for the first three shocks, their transfer functions would all have been negative.

3 Weighting functions in consumption-based models

This section applies the analysis above to a range of standard utility functions for which m_{t+1} can be written as a linear function of innovations to consumption growth. We analyze power utility, models of internal and external habit formation, and Epstein–Zin preferences.[7] We then estimate

[7] While these models of preferences are often applied under the assumption of the existence of a representative agent, note that that assumption is not strictly necessary. In particular, the pricing kernel generated by an agent's Euler equation will hold for any market in which he participates. We thus do not concern ourselves, for now, with

weighting functions empirically using data on equity returns.

3.1 Weighting functions in theoretical models

3.1.1 Power utility

Under power utility, the log pricing kernel is

$$m_{t+1} = \log \beta - \alpha \Delta c_{t+1} \tag{17}$$

where c_t denotes the log of an agent's consumption, α is the coefficient of relative risk aversion, and $-\log \beta$ is the rate of pure time preference. (17) implies that $z_0 = \alpha$ and $z_k = 0$ for all $k > 0$, so the weighting function under power utility is simply

$$Z^{power}(\omega) = \alpha \tag{18}$$

Z^{power} is flat and exactly equal to the coefficient of relative risk aversion. Z^{power} is constant because the only determinant of the innovation to the SDF is the innovation to consumption on date $t+1$. A shock to consumption growth has the same effect on the pricing kernel regardless of how long the innovation is expected to last.

3.1.2 Habits

Adding an internal habit to the preferences yields the lifetime utility function

$$V_t = \sum_{j=0}^{\infty} \beta^j \frac{(C_{t+j} - bC_{t+j-1})^{1-\alpha}}{1-\alpha} \tag{19}$$

where $C_t = \exp(c_t)$ is the level of consumption and $0 \leq b < 1$ is a parameter determining the importance of the habit. The pricing kernel is

$$\exp(m_{t+1}) = \beta \frac{(C_{t+1} - bC_t)^{-\alpha} - E_{t+1} b (C_{t+2} - bC_{t+1})^{-\alpha}}{(C_t - bC_{t-1})^{-\alpha} - E_t b (C_{t+1} - bC_t)^{-\alpha}} \tag{20}$$

If we log-linearize the pricing kernel in terms of Δc_{t+1} and Δc_{t+2} around a zero-growth steady-state, we obtain

$$\Delta E_{t+1} m_{t+1} \approx -\alpha \left(b (1-b)^{-2} + 1 \right) \Delta E_{t+1} \Delta c_{t+1} + \alpha b (1-b)^{-2} \Delta E_{t+1} \Delta c_{t+2} \tag{21}$$

issues of market completeness or the existence of a representative agent. Of course, we *will* assume a representative agent when testing the model using data on aggregate consumption.

With internal habits the pricing kernel depends on both the innovation to current consumption growth and also the change in consumption growth between dates $t+1$ and $t+2$. The spectral weighting function under habit formation is

$$Z^{internal}(\omega) = \alpha\left(1 + b(1-b)^{-2}\right) - \alpha b(1-b)^{-2} 2\cos(\omega) \tag{22}$$

The weighting function with habits is equal to a constant plus a negative multiple of $\cos(\omega)$. As we would expect, $Z^{internal}(\omega) = Z^{power}(\omega)$ when $b = 0$.

The left panel of Figure 2 plots $Z^{internal}(\omega)$ for various values of b. Here and in all cases below we only plot Z between 0 and π as is standard, since Z is even and periodic. The x-axis lists the wavelength of the cycles, as opposed to the frequency ω. Given a frequency of ω, the corresponding cycle has length $2\pi/\omega$ periods (the smallest cycle we can discern lasts two periods).

As b rises, there are two effects. First, the integral over Z gets larger, and second, its mass shifts to higher frequencies. The latter effect is consistent with the usual intuition about internal habit formation that households prefer to smooth consumption growth and avoid high-frequency fluctuations to a greater extent than they would under power utility.[8]

One lesson from the equation for $Z^{internal}$ is that as long as b is the only parameter we can vary, there is little flexibility in controlling preferences over different frequencies. $\cos(\omega)$ always crosses zero at $\pi/2$, so the pricing kernel will always place higher weight on cycles of frequency greater than $\pi/2$ and relatively less weight on cycles with frequency less than $\pi/2$. Furthermore, $Z^{internal}$ is monotone, regardless of the value of b.[9]

Under external habit formation, the SDF is

$$\exp(m_{t+1}) = \beta \frac{\left(C_{t+1} - b\bar{C}_t\right)^{-\alpha}}{\left(C_t - b\bar{C}_{t-1}\right)^{-\alpha}} \tag{23}$$

where $\bar{C}$ denotes some external measure of consumption (e.g. aggregate consumption or that of an agent's neighbors). In this case, the innovation to the SDF depends only on the innovation to C_{t+1}. So the weighting function with an external habit will be completely flat. Otrok, Ravikumar, and Whiteman (2002) show that the external habit has a strong effect on what weights *utility* places on consumption cycles of different frequencies, but what we show here is the SDF is driven entirely by

[8] Otrok, Ravikumar, and Whiteman (2002) obtain a similar result, but in a different manner. Rather than characterize the volatility of the pricing kernel, they characterize the price of a Lucas tree, which is equivalent to simply characterizing lifetime utility as a function of the spectral density of consumption growth. While lifetime utility is important, it is not the same as the price of risk in the economy. Our results are therefore complements rather than substitutes.

[9] One potential way to enrich preferences to allow preferences to isolate smaller ranges of the spectrum may be to allow for more lags of consumption to enter the utility function.

one-period innovations, so all cycles receive the same weight in pricing assets. The pricing kernel in models with external habit formation, e.g. Campbell and Cochrane (1999), places equal weight on all frequencies. On the other hand, the internal habit models of Constantinides (1990) and Abel (1990) are heavily weighted towards high-frequency fluctuations.

3.1.3 Epstein–Zin preferences

An alternative way of incorporating non-separabilities over time is Epstein and Zin's (1991) formulation of recursive preferences. In general, under recursive preferences, anything that affects an agent's welfare affects the pricing kernel. So not only shocks to current and future consumption growth, but also innovations to higher moments will be priced. We begin by focusing on the case where consumption growth is log-normal and homoskedastic. Section 5 considers models with stochastic volatility.

Suppose an agent has lifetime utility

$$V_t = \left\{ (1-\beta) C_t^{1-\rho} + \beta \left(E_t \left[V_{t+1}^{1-\alpha} \right] \right)^{1-\rho} \right\}^{\frac{1}{1-\rho}} \tag{24}$$

Campbell (1993) and Restoy and Weil (1998) show that if consumption growth is log-normal and homoskedastic, the stochastic discount factor for these preferences can be log-linearized as

$$\Delta E_{t+1} m_{t+1} \approx -\left(\rho \Delta E_{t+1} \Delta c_{t+1} + (\alpha - \rho) \Delta E_{t+1} \sum_{j=0}^{\infty} \theta^j \Delta c_{t+1+j} \right) \tag{25}$$

where ρ is the inverse elasticity of intertemporal substitution (EIS), and α is the coefficient of relative risk aversion. θ is a parameter (generally close to 1) that comes from the log-linearization of the return on the agent's wealth portfolio (Campbell and Shiller, 1988).[10] θ is a measure of impatience: if the agent is highly impatient, then he consumes a large fraction of his wealth in each period and θ is small. In the case where $\rho = 1$, equation (25) is exact.

For the case of equation (25), the weighting function is

$$Z^{EZ}(\omega) \equiv \alpha + (\alpha - \rho) \sum_{j=1}^{\infty} \theta^j 2 \cos(\omega j) \tag{26}$$

[10] Specifically, $\theta = \left(1 + \overline{DP}\right)^{-1}$, where $\overline{DP}$ is the dividend-price ratio for the wealth portfolio (i.e. the consumption-wealth ratio) around which we approximate. θ generalizes the rate of pure time preference somewhat because it also depends on discounting due to uncertainty about future consumption.

which can be further simplified using Euler's formula as

$$Z^{EZ}(\omega) = \rho + (\alpha - \rho)\left(\frac{1-\theta^2}{1-2\theta\cos(\omega)+\theta^2}\right) \tag{27}$$

Under power utility, $\alpha = \rho$ and $Z^{EZ}(\omega) = \alpha$ is flat, so all frequencies receive equal weight, as discussed above. On the other hand, if $\alpha \neq \rho$, then weights can vary across frequencies due to the second term.

Z^{EZ} is much richer than what we obtain in the case of power utility and it has a number of interesting properties. First, as with power utility, its average value is exactly equal to the coefficient of relative risk aversion,

$$\frac{1}{\pi}\int_0^{\pi} Z^{EZ}(\omega)\, d\omega = \alpha \tag{28}$$

Therefore, the total weight placed on the spectrum depends only on risk aversion. Another way to see this is that α is the price of a shock that affects all frequencies equally (and therefore has impulse transfer function $G_j(\omega)$ equal to 1, like shock 1 in Figure 1). To the extent that the volatility of the pricing kernel depends on the EIS, it is due only to how $\alpha - \rho$ affects which frequencies receive weight.

An obvious question is how rapidly Z^{EZ} falls as ω rises above zero. That is, how much of the mass of Z^{EZ} is concentrated at very low frequencies? In the limit as $\theta \to 1$, i.e. the case where households are indifferent about when consumption occurs, $Z^{EZ}(\omega)$ approaches

$$Z^{EZ}(\omega) = (\alpha - \rho) D_{\infty}(\omega) + \rho \tag{29}$$

where D_{∞} is the limit of the Dirichlet kernel (closely related to the Dirac delta function), with the key properties

$$D_{\infty}(\omega) = 0 \text{ for } \omega \neq 0 \tag{30}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_{\infty}(\omega)\, d\omega = 1 \tag{31}$$

for ω in the interval $(-\pi, \pi)$. $Z^{EZ}(\omega)$ can thus be thought of as roughly a point mass weighted by $(\alpha - \rho)$ plus a constant ρ. For an agent who is effectively infinitely patient, then, two features of the consumption process matter: the permanent innovations at $\omega = 0$ ($\lim_{j\to\infty} \Delta E_{t+1} c_{t+j}$), which are weighted by α, and all transitory innovations, which have no effect on $\lim_{j\to\infty} \Delta E_{t+1} c_{t+j}$, and are weighted by ρ.

Moving away from the limiting case, the right-hand panel of figure 2 plots Z^{EZ} for a variety of parameterizations. The parameterizations are meant to correspond to annual data, so we take

$\theta = 0.975$ as our benchmark, which corresponds to a 2.5 percent annual dividend yield. For $\alpha = 5$ and $\rho = 0.5$ (an EIS of 2), we see a large peak near frequency zero, with little weight elsewhere. In fact, half the mass of Z^{EZ} in this case lies on cycles with length of 230 *years* or more, and 75 percent lies on cycles with length 72 years or more. In this parameterization, it is effectively only the very longest cycles in consumption (up to permanent shocks) that carry any substantial weight in the pricing kernel. Purely temporary shocks to the level of consumption (which is what are induced by shocks to monetary policy in standard models, for example) are essentially unpriced. One way to interpret these numbers is the following. Take a permanent shock to consumption (like shock 1 in Figure 1). We have seen above that it will have price of risk of α. Now suppose we eliminate all fluctuations induced by this shock that are longer than 230 years. The price of this new shock will be one half that of the permanent shock. Then suppose we further remove all fluctuations with cycles longer than 72 years. The price of the remaining shock drops to one quarter of that of the permanent shock. A very large part of the risk premium comes from the longest fluctuations (of hundreds of years) in this model.

The line that is highly negative near $\omega = 0$ is for $\alpha = 0.5$ and $\rho = 5$, where households prefer a late resolution of uncertainty. In this case, the mass of Z^{EZ} is still effectively isolated near zero, but because households now prefer an early resolution of uncertainty, Z^{EZ} is negative at that point. The integral of Z^{EZ} is still equal to α, though, so it turns positive at higher frequencies.[11]

3.1.4 Ambiguity aversion interpretation

As usual, the analysis of Epstein–Zin preferences naturally also applies to the preferences of an ambiguity averse agent (e.g. Hansen and Sargent, 2001; Barillas, Hansen, and Sargent, 2009). When the agent has a preference for robustness, he can be viewed as having a reference distribution (the true distribution) and a worst-case distribution, which is what he uses to actually price assets. Under the reference distribution, the agent simply has power utility, so his weighting function would be flat. Under the worst-case distribution, though, he places relatively more weight on certain "bad" states of the world (based on a joint entropy condition on the two distributions). Our weighting function shows the effect of that reweighing in the frequency domain. Agents essentially place more weight on the possibility of the occurrence of low-frequency fluctuations, which gives them a relatively high weight in the function Z^{EZ}.[12]

Ambiguity aversion also gives a convenient way to motivate a more general spectral weighting function than those derived above for standard models. Specifically, suppose an agent is ambiguity

[11] Note, though, that the case where $\rho > \alpha$ is not taken as a benchmark and is not widely viewed as empirically relevant (see, e.g., Bansal and Yaron, 2004).

[12] Hansen and Sargent (2007) provide a similar interpretation of their multiplier preferences in the frequency domain for a linear-quadratic control problem.

averse in the sense that he maximizes

$$E_t^* \left[\sum_{j=0}^{\infty} z_j \log C_{t+j} \right] \tag{32}$$

where E^* is an expectation taken under some subjective probability measure. The set of parameters z_j determines how much weight consumption receives at various dates in the future. The appendix shows that in a standard implementation of Hansen and Sargent's constraint preferences, i.e. where E^* is taken under a distribution that differs from the true distribution by some constrained entropy distance, the agent will have a spectral weighting function equal to $Z(\omega) \equiv 1 + 2\lambda^{-1} \sum_{k=1}^{\infty} z_k \cos(\omega k)$, where λ^{-1} is the Lagrange multiplier on the entropy constraint, which essentially determines risk aversion. Ambiguity averse preferences thus give a way to understand where an arbitrary spectral weighting function might come from, and what it would imply for an agent's underlying utility.

4 Estimates of weighting functions

We now estimate the weighting function $Z(\omega)$ for consumption growth using the cross-section of equity prices. Estimating Z involves three main steps. First, we need to estimate the dynamics of the economy and identify the fundamental shocks ε_{t+1} and the dynamic response of consumption growth to these shocks. Second, we parametrize the function $Z(\omega)$. Third, we estimate the parameters of Z using the cross-section of equity returns.

4.1 Step 1: Estimation of the dynamics

We assume the process driving the priced variable, x_t, follows a finite-order VAR,

$$\bar{X}_t = \bar{\Phi}(L) \bar{X}_{t-1} + \varepsilon_t \tag{33}$$

where $x_t = B_1 \bar{X}_t$ is the first element of $\bar{X}_t$ and $\bar{X}_t$ has dimension $N \times 1$.[13] In our benchmark results, x_t is log consumption growth, Δc_t. If the lag polynomial $\bar{\Phi}(L)$ has order K, then we can stack K consecutive observations of $\bar{X}_t$ so that $X_t \equiv \left[\bar{X}_t', \bar{X}_{t-1}', ...\right]'$ follows a VAR(1)

$$X_t = \Phi X_{t-1} + \varepsilon_t \tag{34}$$

and $x_t = B_1 X_t$. We estimate this VAR using OLS, yielding estimates of Φ and ε_t.

[13] Recall that B_j represents a conformable selection vector equal to 1 in element j and 0 elsewhere.

4.2 Step 2: Parametrization of the spectral weighting function

The weighting function that we want to estimate, $Z(\omega)$, is potentially infinite-dimensional, but we only have a finite number of risk prices (one for each estimated shock in ε_t) with which to estimate it. We therefore need to choose a functional form to approximate Z with a finite number of parameters. We consider two specifications, a flexible function motivated by the utility functions discussed above, and a step function.

4.2.1 The utility basis

The analysis of the utility functions in the previous sections suggests modeling Z as:

$$Z^U(\omega) = q_1 \sum_{j=1}^{\infty} \theta^j \cos(\omega j) + q_2 + q_3 \cos(\omega) \tag{35}$$

where q_1, q_2, and q_3 are unknown coefficients. We call (35) the *utility basis* because it nests the weighting functions derived from utility-based models. If $q_3 = 0$, (35) matches the weighting function for Epstein–Zin preferences in (26). If $q_1 = 0$, the long-run component that is crucial in the Epstein–Zin case is shut off, and we obtain the weighting function for internal habit formation in (22). Finally, if both $q_1 = 0$ and $q_3 = 0$, then we have the weighting function for power utility. Note that we have an extra parameter θ here. Following the most common calibration of the Epstein–Zin model that motivated our specification of the Z function, we choose $\theta = 0.975^{1/4}$ for quarterly data, corresponding to a 2.5 percent annual consumption/wealth ratio.[14]

Because the utility basis is so closely related to the weighting functions we derived under various preference specifications, the constituent functions are already plotted in Figure 2. In particular, the lines in the right-hand panel represent the first function, $\sum_{j=1}^{\infty} \theta^j \cos(\omega j)$, shifted upward by a constant. This function clearly isolates very low frequencies, and the extent to which the lowest frequencies are isolated depends on the parameter θ.

4.2.2 The bandpass basis

One advantage of working in the frequency domain is that it is straightforward to estimate risk prices for ranges of frequencies of interest. In particular, we can model $Z(\omega)$ directly in a way that captures the preferences of agents for economically interesting frequencies, without mapping literally to any of the models presented above. This generalizes the various models by focusing on the main intuitions about the preferences over dynamics. To do this, we simply break the

[14] In theory, we could estimate θ. However, we find that it is poorly identified in the data, so we proceed to calibrate it to the value most commonly used in the literature.

interval $[0, \pi]$ into three economically motivated intervals, corresponding to business-cycle length fluctuations with wavelength between 6 and 32 quarters (as is standard in the macro literature, e.g. Christiano and Fitzgerald, 2003), and frequencies above and below that window. If agents dislike long-run risks, we would expect most of the weight of Z to lie in the range of frequencies below the business cycle, while habit formation-type preferences imply that the mass should lie at higher frequencies. At the same time, what is considered "long-run risk" here is not the literal interpretation of the Epstein–Zin calibration (230 years): we generalize that model by considering as "long-run risks" any shocks that induce cycles longer than the business cycle.

We refer to the set of three step functions as the *bandpass basis*, since $Z(\omega)$ is composed of the sum of three bandpass filters. Specifically, we define

$$Z^{(a,b)}(\omega) \equiv \begin{cases} 1 \text{ if } a < |\omega| \leq b \\ 0 \text{ otherwise} \end{cases} \tag{36}$$

For quarterly data, our three basis functions are then $Z^{(0,2\pi/32)}(\omega)$, $Z^{(2\pi/32,2\pi/6)}(\omega)$, and $Z^{(2\pi/6,\pi)}(\omega)$. We therefore estimate the function

$$Z^{BP}(\omega) = q_1 Z^{(0,2\pi/32)}(\omega) + q_2 Z^{(2\pi/32,2\pi/6)}(\omega) + q_3 Z^{(2\pi/6,\pi)}(\omega) \tag{37}$$

4.3 Step 3: Estimation of the spectral weighting function

Result 1 and the estimated VAR imply that the innovations to the log SDF are:

$$\Delta E_{t+1} m_{t+1} = -W(\bar{q}) \varepsilon_{t+1} \tag{38}$$

for a $1 \times N$ vector W that depends on the parameters $\bar{q} \equiv [q_1 \ q_2 \ q_3]'$. We then estimate the vector $\bar{q}$ using the cross-section of asset prices.

To find $W(\bar{q})$ for a given basis (either utility or bandpass), we go back to the VAR representation and write:

$$\Delta E_{t+1} m_{t+1} = -\sum_{k=0}^{\infty} z_k B_1 \Phi^k \varepsilon_{t+1} \tag{39}$$

According to Result 1, the time-domain weights $\{z_k\}$ are transformations of the weighting function,

$$z_k = \begin{cases} \frac{1}{2\pi} \int_{-\pi}^{\pi} Z(\omega) \, d\omega \text{ for } k = 0 \\ \frac{1}{\pi} \int_{-\pi}^{\pi} Z(\omega) \cos(\omega k) \, d\omega \text{ for } k > 0 \end{cases} \tag{40}$$

For both the utility and bandpass basis, $Z(\omega)$ is linear in the coefficients $\bar{q}$, which implies that z_k

is also linear in $\bar{q}$. Specifically,

$$z_k = \bar{q}' H_k \tag{41}$$

where H_k contains the integrals of the basis functions for Z. Importantly, these H_k vectors are completely known (they don't need to be estimated) because they only depend on the choice of the set of basis functions. For the utility basis,

$$H_0 = \begin{bmatrix} \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{i=1}^{\infty} \theta^i \cos(\omega i)\, d\omega \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 d\omega \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\omega)\, d\omega \end{bmatrix} \quad H_{k>0} = \begin{bmatrix} \frac{1}{\pi}\int_{-\pi}^{\pi} \sum_{i=1}^{\infty} \theta^i \cos(\omega i)\cos(\omega k)\, d\omega \\ \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(\omega k)\, d\omega \\ \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(\omega)\cos(\omega k)\, d\omega \end{bmatrix} \tag{42}$$

For the bandpass basis, we obtain:

$$H_0 = \begin{bmatrix} \frac{1}{2\pi}\int_{-\pi}^{\pi} Z^{(0,2\pi/32)}(\omega)\, d\omega \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} Z^{(2\pi/32,2\pi/6)}(\omega)\, d\omega \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} Z^{(2\pi/6,\pi)}(\omega)\, d\omega \end{bmatrix} \quad H_{k>0} = \begin{bmatrix} \frac{1}{\pi}\int_{-\pi}^{\pi} Z^{(0,2\pi/32)}(\omega)\cos(\omega k)\, d\omega \\ \frac{1}{\pi}\int_{-\pi}^{\pi} Z^{(2\pi/32,2\pi/6)}(\omega)\cos(\omega k)\, d\omega \\ \frac{1}{\pi}\int_{-\pi}^{\pi} Z^{(2\pi/6,\pi)}(\omega)\cos(\omega k)\, d\omega \end{bmatrix} \tag{43}$$

which can be further simplified as a function of sines (without integrals) and then computed numerically.

Given that $z_k = \bar{q}' H_k$, (39) becomes

$$\Delta E_{t+1} m_{t+1} = -\bar{q}' \left(\sum_{j=0}^{\infty} H_j B_1 \Phi^j \right) \varepsilon_{t+1} \tag{44}$$

$\Delta E_{t+1} m_{t+1}$ is thus a function of the VAR parameters Φ, the innovations $\varepsilon_t = X_t - \Phi X_{t-1}$, and the three parameters q_1,q_2 and q_3.

Equation (44) suggests an alternative interpretation of the decomposition we propose, most immediate for the case of the bandpass basis. By regrouping the right hand side of that equation, we can write:

$$\Delta E_{t+1} m_{t+1} = -\bar{q}' u_{t+1}$$

with

$$u_{t+1} = \left(\sum_{j=0}^{\infty} H_j B_1 \Phi^j \right) \varepsilon_{t+1}$$

It is then clear that the linearity of $Z(\omega)$ with respect to the basis functions gives us a linear factor model: the factors will be the shocks u_{t+1}, obtained by rotating the fundamental shocks ε_{t+1} into the three frequency-determined directions: a "long-run" direction, a "business cycle" direction, and a "high frequency" direction. The parameters q_1, q_2 and q_3 can then be directly interpreted as the

prices of these three types of risk.

We can now proceed and estimate the model by specifying the moment conditions. Following a large empirical literature (for example Campbell and Vuolteenaho, 2004; Campbell et al., 2013; Bansal et al., 2013) we assume joint lognormality of the shocks and the returns. As we show below, this assumption yields a linear factor model, which is easy to estimate and interpret. However, similar results are obtained by directly using the nonlinear moment condition $E[exp(\Delta E_{t+1} m_{t+1})(R_{t+1} - R^f_{t+1})] = 0$, which does not require the assumption of lognormality.[15] Under the assumption of log-normality of the shocks, the risk prices can be estimated from the asset pricing condition[16]

$$E[R_{it+1} - R^f_{t+1}] = -Cov(m_{t+1}, r_{it+1}) \quad (45)$$

$$= E\left[\bar{q}' \left(\sum_{j=0}^{\infty} H_j B_1 \Phi^j\right) \varepsilon_{t+1} r_{it+1}\right] \quad (46)$$

which, as mentioned above, is a linear factor model.

[15] To derive that equation, consider that for any excess return $R_{t+1} - R^f_{t+1}$, we have $E_t[exp(m_{t+1})(R_{t+1} - R^f_{t+1})] = 0$. Premultiplying by $exp(-E_t m_{t+1})$ we obtain: $E_t[exp(m_{t+1} - E_t m_{t+1})(R_{t+1} - R^f_{t+1})] = 0$. The moment condition is then obtained by conditioning down.

[16] The derivation of this equation follows Campbell and Vuolteenaho (2004). Given the assumption of lognormality of all shocks, we can write:

$$E_t r_{it+1} - r^f_{t+1} + \frac{1}{2}\sigma^2_{it} = -Cov_t(m_{t+1}, r_{it+1})$$

where $r_{it+1} = log(1 + R_{it+1})$, $r^f_{t+1} = log(1 + R^f_{t+1})$, and $\sigma^2_{it} = Var_t(r_{it+1})$. We then note that

$$Cov_t(m_{t+1}, r_{it+1}) = Cov_t(\Delta E_{t+1} m_{t+1}, r_{it+1}) = E_t(\Delta E_{t+1} m_{t+1} r_{it+1}) = E_t(-\bar{q}' \left(\sum_{j=0}^{\infty} H_j B_1 \Phi^j\right) \varepsilon_{t+1} r_{it+1})$$

Therefore, we obtain:

$$E_t r_{it+1} - r^f_{t+1} + \frac{1}{2}\sigma^2_{it} = E_t(\bar{q}' \left(\sum_{j=0}^{\infty} H_j B_1 \Phi^j\right) \varepsilon_{t+1} r_{it+1})$$

Since $E_t r_{it+1} - r^f_{t+1} + \frac{1}{2}\sigma^2_{it} \simeq E_t[R_{it+1} - R^f_t]$, and taking unconditional expectations, we obtain

$$E[R_{it+1} - R^f_t] = E\left[\bar{q}' \left(\sum_{j=0}^{\infty} H_j B_1 \Phi^j\right) \varepsilon_{t+1} r_{it+1}\right]$$

Our full set of moment conditions identifying the parameters of the model is

$$G_{t+1}\left(\Phi,\bar{q}\right)=\left[\underbrace{\left(X_{t+1}-\Phi X_{t}\right)\otimes X_{t}}_{\text{VAR moments}}\;,\;\underbrace{\mathbf{R}_{t+1}-R_{t+1}^{f}-\left(\overbrace{\overline{q}'\left(\sum_{j=0}^{\infty}H_{j}B_{1}\Phi^{j}\right)}^{\text{Mapping into frequency domain}}\left(X_{t+1}-\Phi X_{t}\right)\right)\mathbf{r}_{t+1}}_{\text{Asset pricing moments}}\right] \tag{47}$$

where $\mathbf{R}_t$ is the vector of test asset returns, R_t^f is the risk-free rate and $\mathbf{r}_{t+1}$ is the vector of log test asset returns.

While we could in principle minimize the GMM objective function for all the parameters simultaneously, that method has the drawbacks that the optimization is difficult to perform (due to the large number of parameters) and that it allows errors in the asset pricing model to affect the VAR estimates. We therefore construct estimates of Φ and $\bar{q}$ by minimizing the two moment conditions separately. That is, Φ is simply estimated through OLS and then $\bar{q}$ is estimated taking Φ as given, using GMM.[17] Given estimates $\hat{\Phi}$ and $\widehat{\bar{q}}$, we construct standard errors using the *full* set of moments, $G\left(\hat{\Phi},\widehat{\bar{q}}\right)$. The standard errors we report for the risk prices $\bar{q}$ therefore always incorporate uncertainty about the dynamics of the economy through Φ.

We perform the GMM estimation of $\bar{q}$, taking Φ as given, using either one-step GMM (using the identity matrix to weight the asset pricing moments) or two-step GMM (using the estimated variance-covariance matrix of the moment residuals to construct the weighting matrix for the second step), and report the results separately.[18]

4.4 Empirical results

4.4.1 Data

The most natural choice for the priced variable, x_t, is consumption growth, but we also explore using other variables: GDP, durable consumption, and investment growth. The rationale for using variables other than consumption, even though we are motivated by consumption-based models, is

[17] The same methodology is used in Campbell and Vuolteenaho (2004) and Campbell et al. (2013). Optimizing the full GMM objective function (or even using two-stage GMM) would be more efficient, so our standard errors will in general be larger than if we used a fully efficient method.

[18] When computing the standard errors incorporating the full estimation uncertainty (according to eq. 47), we take into account the weighting matrix we have used to estimate $\bar{q}$. We construct the full weighting matrix in the following way. We assign equal weight to all VAR moment conditions (i.e. we use the identity matrix for the block of the weighting matrix that corresponds to the VAR moment conditions). For the block that corresponds to the asset pricing moments, we use the same weighting matrix we used in the estimation of $\overline{q}$. We set to zero the weight on the cross-product between VAR and asset pricing moment conditions. Finally, we scale the VAR moment conditions by a (common) constant such that on average the block of VAR moments and the block of asset pricing moments get the same weight.

that to the extent that the pricing kernel is driven by permanent shocks to consumption, permanent shocks to any variable that is cointegrated with consumption should also proxy for the pricing kernel, since the permanent shocks to consumption and any variable it is cointegrated with must be perfectly correlated. That said, households want to smooth consumption compared to income, so we cannot view estimates of the spectral weighting function for aggregates other than consumption as yielding direct tests comparing utility functions. Rather, we interpret them as simply illustrating how the dynamics of the economy are priced. Furthermore, Cochrane (1996) argues that investment growth should price the cross-section of asset returns. Our results on investment are a generalization of his analysis that asks whether and how future dynamics of investment growth are priced.

For the vector of state variables $\bar{X}_t$, we want variables that are both priced in the cross-section and can forecast our priced variable x_t. Since the number of parameters of the VAR increases quadratically with the dimension of $\bar{X}_t$, we look for a parsimonious representation with few state variables. The first element of $\bar{X}_t$ is the priced variable, and the other elements are the first two principal components of a set of 9 real and financial variables: the aggregate price/earnings and price/dividend ratios; the 10 year/3 month term spread; the Aaa–Baa corporate yield spread (default spread); the small-stock value spread; the unemployment rate minus its 8-year moving average; RREL, the detrended version of the short-term interest rate that Campbell (1991) finds forecasts market returns; the three-month Treasury yield rate; and Lettau and Ludvigson's (2001) *cay*. The results are robust to the choice of variables from which the principal components are extracted. Because some of the variables used are only available after 1952, in the analysis that follows we use the quarterly data over the period 1952–2011. Finally, in the analysis that follows we use 3 lags of quarterly data, as suggested by the Akaike information criterion, but results are robust to the choice of the number of lags. Table 1 reports the VAR coefficients of the consumption equation (using consumption growth as a priced variable).

4.4.2 Parameter estimates

Table 2 reports the estimation results using two-step GMM to estimate the risk prices. The left-hand side uses the set of 25 size- and book/market-sorted portfolios, while the right-hand side adds in a set of 49 industry portfolios (both sets of portfolios are obtained from Ken French's website; we drop six industry portfolios that have missing data in the period considered). For both portfolio sets we estimate both the bandpass basis and the utility basis. For the bandpass basis, q_1 corresponds to the price of lower-than-business cycle risks, q_2 to business cycle risks, and q_3 to higher-than-business cycle risks. For the utility basis, q_1 is price of the long-run component, q_2 is the constant, and q_3 is the high-frequency component (coefficient on $cos(\omega)$).

The first set of rows in table 2 reports results obtained using consumption growth as priced

variable in the SDF. We find that long-run shocks to consumption are strongly priced both in the 25 Fama–French portfolios and in the industry portfolios, while business-cycle frequency shocks and high frequency shocks do not seem to be priced.

The result can be seen much more clearly when using the bandpass basis. When looking at the utility basis, we can barely reject at the 10 percent level that long-run consumption shocks are not priced, and the coefficient on the long-run (Epstein–Zin) shock is not statistically different from 0 when industry portfolios are included. In other words, when estimating structural preference-based models, we find no significant parameters, implying that neither short- nor long-run shocks to consumption growth are priced in equity returns. On the other hand, with the bandpass basis we find that long-run shocks are significantly priced.

We obtain similar results for the other priced variables: with the bandpass basis we find strongly significant low-frequency risk prices in almost all of the variables (GDP growth, durable consumption growth, and the various measures of investment growth). On the contrary, we find almost no significance when using the utility basis.

The appendix reports two main robustness tests: it shows that we obtain similar results when we bootstrap the p-values instead of using the asymptotic approximation, and also when we include a set of risk-sorted portfolios (obtained sorting stocks by their loadings on the shocks in the different frequency ranges). The appendix also reports the factor loadings of the size and book/market sorted portfolios and the risk-sorted portfolios.[19]

Table 3 repeats the analysis using one-step GMM (i.e. using the identity matrix as a weighting matrix when estimating the risk prices). Using the utility basis, we can only distinguish the price of long-run risk from zero in a single case (residential investment, only for the cross-section of the 25 FF portfolios). Using the bandpass basis, we find several cases in which the long-run risk price is significant: consumption growth, durables growth, fixed investment and residential investment. In any case, the bandpass basis yields results on the price of long-run risks that are much stronger than the ones indicated by the utility basis.

We interpret these results in two ways. First, it is possible that agents do care about long-run shocks, but their definition of "long-run" is closer to the one captured by the bandpass basis (cycles longer than the business cycle) rather than that captured by the utility basis (where more than half of the pricing weight falls on cycles longer than 230 years). Second, the bandpass basis loads on frequencies that can be economically considered "long-run" but are much easier to detect empirically than frequencies close to 0 for which we have little empirical power.

Using the frequency-domain decomposition leads us to very different conclusions about the

[19] We have also looked at whether our factors can explain the average returns on short-term dividend strips documented by van Binsbergen, Brandt, and Koijen (2012), but find inconclusive results because of the brevity of the available sample, which only begins in 1996.

underlying theories than standard time-domain techniques would have. The results that employ the utility basis show essentially no support for the long-run risk model. Looking at the problem using the bandpass filter and targeting the economically relevant set of frequencies instead yields strong and robust support for the idea that low-frequency shocks to the economy are priced in equity markets.

4.4.3 Impulse transfer and weighting functions

The left-hand panel of figure 3 plots the estimated impulse transfer functions, G_j, of the three shocks ε_j for consumption growth. To help show the behavior of the functions near frequency zero, we plot them from $-\pi$ to π, instead of beginning at zero as elsewhere. Note that the functions are all symmetrical across the vertical axis (since they are linear combinations of cosines). The shaded regions in each figure are 95-percent confidence intervals.

There are two key features of the transfer functions to note. First, there are meaningful qualitative and quantitative differences across the functions in how power is distributed, which helps identify the underlying risk prices. If the transfer functions were all highly similar, then we would not expect to be able to distinguish risk prices across frequencies very well. Looking at the confidence bands, though, it is clear that the transfer functions are poorly estimated near frequency zero. $\omega = 0$ corresponds to the the infinite-horizon response to each shock, so it is not surprising that it is most difficult to estimate. Nevertheless, the fact that the uncertainty rises so much at very low frequencies helps explain why we have trouble estimating the coefficient on the low-frequency component of the utility basis.

The right-hand set of plots in each figure zooms in on frequencies corresponding to cycles longer than 5 years. In each of those right-hand-side figures, the vertical lines demarcate the set of frequencies that receive half the weight under our benchmark calibration of Epstein–Zin preferences (i.e. cycles longer than 230 years). In all the cases, it is clear that the confidence bands are far larger in the region where the mass of the Epstein–Zin weighting function is focused than elsewhere.

Figure 4 plots the estimated spectral weighting functions for consumption growth and 95-percent confidence intervals, obtained using the 25 Fama–French portfolios, for the bandpass basis (darker shaded area) and the utility basis (lighter shaded area). The left panel plots all frequencies, while the right panel zooms in on the cycles longer than 5 years. The figure shows significant weight at low frequencies. The price of long-run risks is quite precisely estimated using the bandpass basis (and significantly different from zero), while the standard errors of the utility basis estimates diverge quickly as we look at frequencies closer to zero, confirming the huge amount of statistical uncertainty exactly in the frequency range most important for the Epstein–Zin model.

5 Weighting functions in returns-based models

5.1 Theoretical models

5.1.1 The CAPM

Under the CAPM, innovations to the SDF are proportional to innovations to the market return,

$$m_{t+1} - E_t m_{t+1} = -\frac{E\left[r_{m,t+1} - r_{f,t+1}\right]}{Var\left(r_{m,t+1} - r_{f,t+1}\right)}\left(r_{m,t+1} - Er_{m,t+1}\right) \tag{48}$$

where $r_{m,t+1}$ is the market return. The weighting function under the CAPM is thus simply

$$Z^{CAPM}(\omega) = \frac{E\left[r_{m,t+1} - r_{f,t+1}\right]}{Var\left(r_{m,t+1} - r_{f,t+1}\right)} \tag{49}$$

Z^{CAPM} is flat, and its level depends on the price of risk in the market, just like we obtain with power utility (though obviously with a different priced variable).

5.1.2 Epstein–Zin and power utility

In a model with a representative agent with Epstein–Zin preferences (with power utility as a special case) and where consumption growth is log-normal and homoskedastic, Campbell (1993) shows that innovations to the pricing kernel can be written purely in terms of returns on the representative agent's wealth portfolio,

$$m_{t+1} - E_t m_{t+1} = -\alpha \Delta E_{t+1} r_{w,t+1} - (\alpha - 1)\,\Delta E_{t+1} \sum_{j=1}^{\infty} \theta^j r_{w,t+1+j} \tag{50}$$

where r_w is the log return of the wealth portfolio of the representative agent. θ is the same log-linearization parameter as in the previous section. Campbell (1993) interprets (50) as a version of Merton's (1973) intertemporal CAPM because both current returns and changes in the investment opportunity set are priced risk factors.

The weighting function for (50) is

$$Z^{EZ-returns}(\omega) = \alpha + (\alpha - 1)\sum_{j=1}^{\infty} \theta^j 2cos(\omega j) \tag{51}$$

As $\theta \to 1$ we obtain the limit

$$Z(\omega) = (\alpha - 1)\, D_\infty(\omega) + 1 \tag{52}$$

with D_∞ the limit of the Dirichlet kernel, which essentially corresponds to a point mass at 0. All agents, then, regardless of ρ (i.e., regardless of whether they have power utility or more general recursive preferences) place high weight on low-frequency fluctuations in equity returns.

5.1.3 Returns-based asset pricing when we can forecast returns but not consumption

Campbell's (1993) analysis, and that used in Campbell and Vuolteenaho (2004), assumes that risk premia are constant and that consumption growth is potentially forecastable. Suppose, alternatively, that we cannot forecast consumption growth at all, and that when we forecast asset returns we are simply forecasting risk premia. For example, return predictability might arise from stochastic volatility (as in Bansal and Yaron, 2004 and Campbell, Giglio, Polk and Turley, 2012) or time-varying risk aversion (Campbell and Cochrane, 1999; Dew-Becker, 2012). The Campbell–Shiller approximation when consumption is unpredictable reduces to

$$\Delta E_{t+1} r_{w,t+1} = \Delta E_{t+1} \Delta c_{t+1} - \Delta E_{t+1} \sum_{j=1}^{\infty} \theta^j r_{w,t+j+1} \tag{53}$$

and the pricing kernel is

$$\Delta E_{t+1} m_{t+1} = -\alpha \Delta E_{t+1} r_{w,t+1} - \rho \frac{1-\alpha}{1-\rho} \Delta E_{t+1} \sum_{j=1}^{\infty} \theta^j r_{w,t+j+1} \tag{54}$$

This result is notably different from that of Campbell (1993) and equation (50) above, which are derived assuming risk premia are constant. Specifically, if the EIS is greater than 1 ($\rho < 1$), then the coefficient on expected future returns becomes proportional to $-(1-\alpha)$: it has the *opposite* sign from the one found in Campbell (1993) and equation (50). The intuition for this result is as follows. In Campbell (1993), news about high future returns corresponds to an improvement in future expected consumption growth (or, in the language of ICAPM, the investment opportunity set), which is unambiguously good. If, however, high expected returns are due to high future risk aversion or volatility, then there is only a discounting effect: high future expected returns are associated with low lifetime utility. An increase in risk aversion or volatility is purely bad news.

5.2 Estimation of the weighting function

5.2.1 Methods

Given that the weighting function presented above can be decomposed in two of the three constituent functions that we saw for the case of consumption (and that are plotted in Figure 2), the utility

basis representation in the case of returns will simply be:

$$Z^U(\omega) = q_1 \sum_{j=1}^{\infty} \theta^j cos(\omega j) + q_2 \tag{55}$$

Since we are mostly interested in estimating the pricing of long-run discount rate news, we parametrize the bandpass basis to only include a constant and a generalized long-run component,

$$Z^{BP}(\omega) = q_1 Z^{(0,2\pi/32)}(\omega) + q_2 \tag{56}$$

again capturing frequencies lower than the business cycle.

As in Campbell and Vuolteenaho (2004), we use a VAR(1) with state vector composed of log excess returns, the price/earnings ratio, term spread and default spread. We use quarterly data from 1926q3 to 2011q2. We estimate the VAR using OLS, and set $\theta = 0.95$ per year as in Campbell and Vuolteenaho (2004).

We then use GMM as above to estimate the two parameters q_1 and q_2 using the cross-section of 25 Fama-French portfolios or the combination of those assets and the 49 industry portfolios. As before, we estimate Φ and $\bar{q} \equiv [q_1, q_2]$ separately. Again, we report results using both one-step and the two-step GMM to estimate $\overline{q}$, and compute standard errors for $\overline{q}$ taking into account the uncertainty related to the estimation of the VAR parameters as explained in Section 4. For robustness, we also compute the results using the three-parameters bandpass and utility basis we presented in Section 4.

5.2.2 Results

Table 4 shows the results using only the 25 Fama–French portfolios (left columns) or adding the 49 industry portfolios (right columns). The top panel reports the version with two parameters (where the first one captures the long-run risks) discussed in the previous section. For both the bandpass basis and the utility basis, we find evidence that the long-run component of return news is priced, at least when using only two parameters and using the efficient matrix to estimate q. Consistent with equation (51), when we use the utility basis we find that both the constant and the discount-rate news (long-run shock to expected returns) are priced, and that q_1 is approximately equal to $q_2 - 1$. Similarly, for the bandpass basis, the price for frequencies below the business cycle is positive and significant. The bottom panel of Table 4 reports estimates of the three-parameter version described in Section 4. Here we find weaker evidence that long-run innovations in returns are priced. Overall, though, looking at returns we find some evidence that news about the long-term expected returns carry a positive risk price, though the results seem to be more sensitive to the specification used

than in the previous section. Furthermore, the results are more consistent with a model in which the main source of news is about future expected consumption rather than future expected volatility or risk aversion.

6 Multiple priced variables and stochastic volatility

So far the analysis has focused only on the case where there is a single priced variable. In some models, though, the dynamics of multiple variables matter for asset pricing. For example, in many applications with Epstein–Zin preferences, both consumption growth and variation in volatility or disaster risk are priced (e.g. Bansal and Yaron, 2004; Campbell et al., 2013; Gourio, 2012; Constantinides and Ghosh, 2013, study a model with time-varying cross-sectional skewness with similar results). It turns out that the results above map easily into a multivariate setting.

Assumption 1a: Structure of the SDF

Instead of there being a single priced variable x_t, suppose there is an $M \times 1$ vector of priced variables, $\vec{x}_t$, with

$$m_{t+1} = F(I_t) - \Delta E_{t+1} \sum_{k=0}^{\infty} Z_k \vec{x}_{t+1+k} \tag{57}$$

where Z_k is a $1 \times M$ vector of weights and $F(I_t)$ is a scalar valued function.

Assumption 2a: Dynamics of the economy

We assume that $\vec{x}_t$ is driven by a vector moving average process as before,

$$\vec{x}_t = JX_t \tag{58}$$

$$X_t = \Gamma(L)\varepsilon_t \tag{59}$$

for some matrix J of dimension $M \times N$.

The appendix derives the following extension to Result 1,

Result 2. *Under Assumptions 1a and 2a, we can write the innovations to the SDF as,*

$$\Delta E_{t+1} M_{t+1} = -\sum_j \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \vec{Z}(\omega) G(\omega)\, d\omega \right) \varepsilon_{j,t+1} \tag{60}$$

where $\vec{Z}(\omega)$ is a $(1 \times M)$ vector-valued weighting function and $G(\omega)$ is an $(M \times N)$ transfer

function that measures the dynamic effects of ε_t *on* $\vec{x}$ *in the frequency domain,*

$$\vec{Z}(\omega) \equiv Z_0 + 2\sum_{k=1}^{\infty} Z_k \cos(\omega k) \tag{61}$$

$$G(\omega) \equiv \sum_{k=0}^{\infty} \cos(\omega k)\, \bar{g}_k \tag{62}$$

and $\bar{g}_k$ *is the vector of impulse response functions,*

$$\bar{g}_k \equiv J\Gamma_k$$

In this case, then, we have multiple variables whose impulse responses we track in G, and each of the priced variables has its own weighting function, represented as one of the elements of $\vec{Z}(\omega)$.

6.1 Epstein–Zin with stochastic volatility

Using Result 2, we now extend the results on Epstein–Zin preferences to also allow for stochastic volatility, similar to Campbell et al. (2013) and Bansal and Yaron (2004). We use the same log-normal and log-linear framework as above. The log stochastic discount factor under Epstein–Zin preferences is,

$$m_{t+1} = -\rho\frac{1-\alpha}{1-\rho}\Delta c_{t+1} + \frac{\rho-\alpha}{1-\rho} r_{w,t+1} \tag{63}$$

where $r_{w,t+1}$ is the return on a consumption claim on date $t+1$. Whereas we previously assumed that consumption growth was log-normal and homoskedastic, we now allow for time-varying volatility driven by a variable σ_t^2. We assume that σ_t^2 follows a linear, homoskedastic, and stationary process. We assume that log consumption growth is driven by a VMA process as in assumption 1, but that now the shocks ε_t have variances that scale linearly with σ_t^2.

It is then straightforward to show that expected returns on a consumption claim will follow

$$E_t r_{w,t+1} = k_0 + \rho E_t \Delta c_{t+1} + k_1 \sigma_t^2 \tag{64}$$

where k_0 and k_1 are constants that depend on the underlying process driving consumption growth. Using the Campbell–Shiller approximation, we can then write the innovation to the SDF as

$$\Delta E_{t+1} m_{t+1} = -\alpha \Delta c_{t+1} - (\alpha-\rho)\,\Delta E_{t+1} \sum_{j=1}^{\infty} \theta^j \Delta c_{t+1+j} \tag{65}$$

$$-\frac{\rho-\alpha}{1-\rho}\Delta E_{t+1}\theta k_1 \sigma_{t+1}^2 - \frac{\rho-\alpha}{1-\rho}\Delta E_{t+1}\sum_{j=1}^{\infty}\theta^j \theta k_1 \sigma_{t+j+1}^2 \tag{66}$$

The weighting functions for consumption growth and volatility are now

$$Z_C^{EZ-SV}(\omega) = \alpha + (\alpha - \rho)\sum_{j=1}^{\infty}\theta^j 2\cos(\omega j) \tag{67}$$

$$Z_{\sigma^2}^{EZ-SV}(\omega) = \theta k_1 \frac{\rho - \alpha}{1-\rho}\left(1 + \sum_{j=1}^{\infty}\theta^j 2\cos(\omega j)\right) \tag{68}$$

So the price of risk for a shock depends on its ITFs for both consumption growth and volatility. In the case where $\rho = 0$, Z_C^{EZ-SV} is exactly proportional to $Z_{\sigma^2}^{EZ-SV}$. In any case, even for $\rho > 1$ they are highly similar. Z_C^{EZ-SV} is in fact the same we obtained in the homoskedastic case. Both weighting functions have a constant and also allow for a point mass near zero. $Z_{\sigma^2}^{EZ-SV}$ always has the same basic shape regardless of the value of ρ: unless we are in the particular case $\rho = \alpha$ in which $Z_{\sigma^2}^{EZ-SV}(\omega) = 0$, agents always place high weight on the low-frequency features of volatility.

Alternatively, the weighting functions can be written in terms of returns and their volatility,

$$Z_R^{EZ-SV-R}(\omega) = \alpha - (1-\alpha)\sum_{j=1}^{\infty}\theta^j 2\cos(\omega j) \tag{69}$$

$$Z_{\sigma^2}^{EZ-SV-R}(\omega) = \theta k_1 \frac{1-\alpha}{1-\rho}\left(1 + \sum_{j=1}^{\infty}\theta^j 2\cos(\omega j)\right) \tag{70}$$

which yields conceptually similar results.

7 Conclusion

This paper studies risk prices in the frequency domain. The impulse response of consumption growth to a given shock to the economy can be decomposed into components of varying frequencies. In a model where innovations to current and expected future consumption growth drive the pricing kernel, the price of risk for a given shock then depends on a weighted integral over the frequency-domain representation of the impulse response function. The weights assigned to each frequency represent frequency-specific prices of risk. They can be characterized in closed form and only depend on the agents' preferences. We study this weighting function both theoretically and empirically. Theoretically, we find that the weighting function helps us gain a deeper understanding of the behavior of asset pricing models. Empirically, our estimates of the weighting function are consistent with the idea of long-run risk models. Estimating a standard version of Epstein–Zin preferences yields statistically weak results, but using our spectral decomposition to target economically meaningful "long-run" frequencies (specifically, below-business-cycle frequencies) yields

strong support for the importance of long-run risks for asset prices.

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A Derivation of equation (13)

For any $g_{j,k}$, we have

$$g_{j,k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{G}_j(\omega)\left(\cos(\omega k) + i\sin(\omega k)\right) d\omega \tag{71}$$

Now since $g_{j,k} = 0$ for $k < 0$, for any $k > 0$ we have

$$\begin{aligned} g_{j,k} &= g_{j,k} + g_{j,-k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{G}_j(\omega) \begin{pmatrix} \cos(\omega k) + i\sin(\omega k) \\ \cos(-\omega k) + i\sin(-\omega k) \end{pmatrix} d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{G}_j(\omega)\, 2\cos(\omega k)\, d\omega \end{aligned}$$

Furthermore, note that the complex part of $\tilde{G}(\omega)$ multiplied by any $\cos(\omega k)$ for integer k integrates to zero, which is why we can just study $G \equiv re\left(\tilde{G}\right)$. We thus have

$$\sum_{k=0}^{\infty} z_k g_{j,k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} G_j(\omega)\left(z_0 + 2\sum_{k=1}^{\infty} z_k \cos(\omega k)\right) d\omega \tag{72}$$

B Ambiguity aversion and generalized time discounting

Suppose the utility an agent gets from a realization of her consumption stream is

$$U_0 = c_0 + \sum_{j=1}^{\infty} z_j c_j \tag{73}$$

U_0 represents the total utility that the agent gets from consumption between dates 0 and ∞, discounted from the perspective of date 0 in the absence of any uncertainty. We assume that consumption follows an MA(∞) process,

$$c_t = b(L)\,\varepsilon_t \tag{74}$$

We do not make any assumptions about the distribution of ε_t other than that it is not serially correlated. It may be non-normal and heteroskedastic. We assume that $\varepsilon_j = 0$ for $j \leq 0$.

Now suppose the agent solves the minimax problem,

$$V_0 = c_0 + \min_{G\left(\varepsilon_1^{\infty}\right)} E\left[G\left(\varepsilon_1^{\infty}\right)\sum_{j=1}^{\infty} z_j b(L)\,\varepsilon_{j+1}\right] \tag{75}$$

$$+\lambda_{\varepsilon} E\left[G\left(\varepsilon_1^{\infty}\right)\log G\left(\varepsilon_1^{\infty}\right) - K\right] - \phi E\left[G\left(\varepsilon_1^{\infty}\right) - 1\right] \tag{76}$$

where ε_1^∞ denotes a particular history of ε_j for $j = 1$ to ∞. $G(\varepsilon_1^\infty)$ is the extra weight placed on a given history under the agent's subjective measure. λ multiplies the constraint on the entropy of the shift in the distribution (with K being the bound), and ϕ multiplies the constraint that the subjective distribution must integrate to 1.

The first-order condition for G yields

$$0 = \sum_{j=1}^{\infty} z_j b(L) \varepsilon_j + \lambda(1 + \log G) - \phi \tag{77}$$

$$G = \exp\left(\frac{-\sum_{j=1}^{\infty} z_j b(L) \varepsilon_j + \phi - \lambda}{\lambda}\right) \tag{78}$$

$$= \frac{\exp\left(-\sum_{j=1}^{\infty} z_j b(L) \varepsilon_j / \lambda\right)}{E_{0-\infty}\left[\exp\left(-\sum_{j=1}^{\infty} z_j b(L) \varepsilon_j / \lambda\right)\right]} \tag{79}$$

The pricing kernel between dates 0 and 1 is then proportional to

$$\frac{1}{\exp(c_1)} G(\varepsilon_1^\infty) \propto \exp\left(-\left(1 + \lambda^{-1} \sum_{j=1}^{\infty} z_j b(L)\right) \varepsilon_1\right) \tag{80}$$

So the price of risk for a shock ε_1 is then $\left(1 + \lambda^{-1} \sum_{j=1}^{\infty} z_j b(L)\right)$, and thus the spectral weighting function is $Z(\omega) = 1 + 2\lambda^{-1} \sum_{j=1}^{\infty} \cos(\omega j) z_j$, as in the text.

C Derivation of weighting function with multiple priced variables

The impulse response function is denoted

$$\bar{g}_k \equiv J \Gamma_k \tag{81}$$

where $\bar{g}_k$ is an $M \times N$ matrix whose $\{m, n\}$ element determines the effect of a shock to the nth element of ε_t on the mth element of $\bar{X}_{t+k}$. The innovation to the SDF is then

$$\Delta E_{t+1} m_{t+1} = -\left(\sum_{k=0}^{\infty} Z_k \bar{g}_k\right) \varepsilon_{t+1} \tag{82}$$

The price of risk for the jth element of ε is simply the jth element of $\sum_{k=0}^{\infty} Z_k \bar{g}_k$.

As before, we take the discrete Fourier transform of $\{\bar{g}_k\}$, defining

$$\tilde{G}(\omega) \equiv \sum_{k=0}^{\infty} e^{-i\omega k} \bar{g}_k \tag{83}$$

Following the same steps as in section 2 and defining $G(\omega) \equiv re\left(\tilde{G}(\omega)\right)$, we arrive at

$$\begin{aligned} \sum_{k=0}^{\infty} Z_k \bar{g}_k B_j &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \vec{Z}(\omega) G(\omega) B_j d\omega & (84) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_m \vec{Z}_m(\omega) G_{m,j}(\omega) d\omega & (85) \end{aligned}$$

where

$$\vec{Z}(\omega) \equiv Z_0 + 2 \sum_{k=1}^{\infty} Z_k \cos(\omega k) \tag{86}$$

and where $\vec{Z}_m(\omega)$ denotes the mth element of $\vec{Z}(\omega)$ and $G_{m,j}(\omega)$ denotes the m,jth element of $G(\omega)$. We thus have M different weighting functions, one for each of the priced variables. The M weighting functions each multiply N different impulse transfer functions, $G_{m,j}(\omega)$. The price of risk for shock j depends on how it affects the various priced variables at all horizons.

D Robustness tests for the empirical analysis and factor loadings

This section discusses three issues related to the robustness of the main results. First, we bootstrap the t-statistics to account for the possibility that the GMM asymptotics provide a poor small-sample approximation. Second, we augment our set of test assets with nine risk-sorted portfolios and find similar results to what is in the main text. Finally, we report the factor loadings of the individual portfolios to emphasize the fact that there large and statistically significant difference in the factor loadings across portfolios.

D.1 Bootstrapped t-statistics

We compute bootstrapped t-statistics following suggestions in Efron and Tibshirani (1994). Specifically, in every bootstrap sample we calculate the t-statistic for each coefficient and then use the simulated distribution of the t-statistics to construct p-values for the test of whether the coefficients are different from zero.

The estimation has two separate parts: the VAR and the asset pricing equations. For the VAR, we bootstrap the residuals, and then use the simulated innovations to construct a new time series of the state variables (based on the estimated feedback matrix). For the test assets, we draw the returns for the same dates for which we drew the VAR residuals. More concretely, given a sample size of N, we take (discrete) uniformly distributed draws from the interval $[1, N]$ with replacement. The jth draw in bootstrap simulation i is denoted b_j^i (that is, each b_j^i is a random draw from the discrete uniform distribution on $[1, N]$). The ith simulated dataset is then the set of VAR residuals and test asset returns for observations $\left\{b_j^i\right\}_{j=1}^N$. To construct the set of state variables, we draw an initial value of the state variables randomly from the set of observations and then use the drawn innovations to construct the full sample.

The estimation then proceeds on the simulated dataset exactly as it does on the true dataset. For each simulated sample we form t-statistics for the difference between the bootstrapped estimate of the coefficient and the original point estimate. Suppose the empirically observed t-statistic in the main estimate for some coefficient k is equal to $\hat{t}_k > 0$. Then the bootstrapped p-value is twice the fraction of the simulated t-statistics at least as high as $\hat{t}_k$ (for a full description of the procedure, see Efron and Tibshirani, 1994)

Table A1 replicates table 2 but using bootstrapped p-values instead of the asymptotic values reported in table 2. We now obtain far more significant coefficients. The reason is that according to the bootstrap, many of the estimators are substantially biased (which should not be too surprising, since it is well known that GMM can be poorly behaved in small samples). For example, when we simulate the model using consumption growth as the priced risk factor, the coefficient estimates are in general substantially smaller than we observe in the empirical sample, which implies that the estimator is biased downwards. By bootstrapping the t-statistics, we are implicitly taking this bias into account when forming p-values (though note that to be conservative we do not adjust the point estimate to account for the bias as this would generally increase the variance of the estimate). The basic result in table A1 is that the results we obtain are if anything stronger when we use bootstrapped confidence intervals rather than those obtained from the asymptotic distribution.

D.2 Risk-sorted portfolios

The 25 Fama–French portfolios were originally constructed because their returns spanned a number of observed anomalies in the cross-section of excess returns. We would not necessarily expect them to have large spreads in their loadings on shocks to consumption growth at different horizons. In this section we therefore construct portfolios that are specifically designed to have a large spread in factor loadings.

In every quarter, we estimate factor loadings with respect to the low- and business-cycle fre-

quency shocks (we refrain from also sorting on the high-frequency shocks to keep the portfolios relatively large and well diversified). The loadings are estimated on quarterly data over the previous 10 years. The loadings on each factor are split into terciles and we construct 3×3 portfolios. The low- and business-cycle frequency shocks are constructed using the bandpass basis and equation (44). Specifically, we have

$$\Delta E_{t+1} M_{t+1} = -\vec{q}' \left(\sum_{j=0}^{\infty} H_j B_1 \Phi^j \right) \varepsilon_{t+1} \tag{87}$$

The rotated shocks are thus,

$$u_{t+1} = \left(\sum_{j=0}^{\infty} H_j B_1 \Phi^j \right) \varepsilon_{t+1} \tag{88}$$

And the low- and business-cycle frequency components are the first two elements of this vector.

Table A1 reports results using the risk-sorted portfolios in addition to the size- and book/market-sorted portfolios. The results correspond to those in table 2 in that we use the efficient GMM weighting matrix. The left-hand set of columns combines the 9 risk-sorted portfolios with the 25 Fama–French portfolios used in the main text, while the right-hand side uses only 6 size- and book/market-sorted portfolios (two size categories and three book/market categories) to put relatively more weight on the risk-sorted portfolios.

In both sets of columns we replicate our main results that low-frequency components of consumption growth and other real variables are significantly priced under the bandpass basis, and that few if any coefficients are significant with the utility basis. The coefficients are also of a similar magnitude to those in table 2.

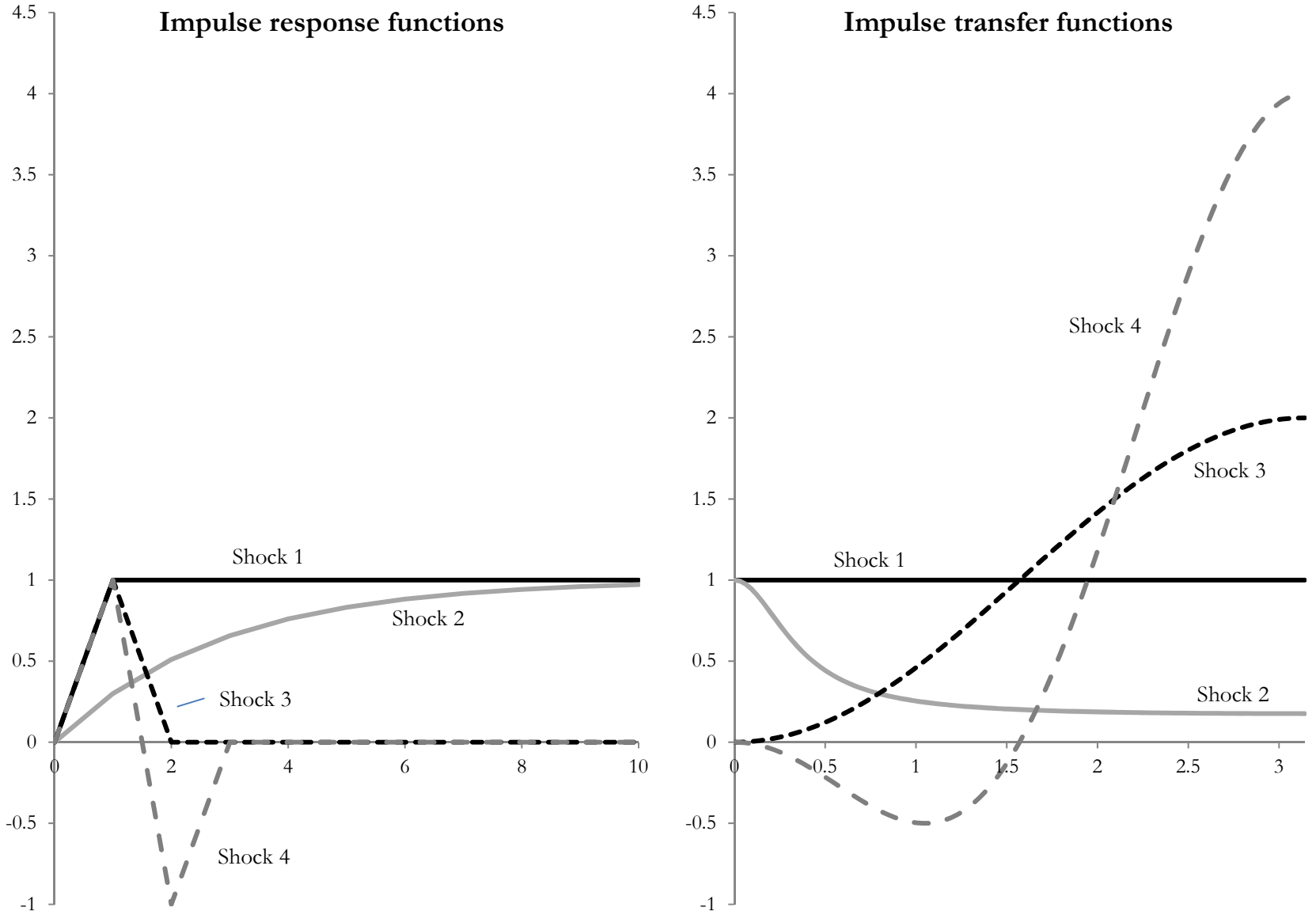
D.3 Factor loadings

Table A3 reports the factor loadings and standard errors for the 25 Fama–French portfolios and our nine risk-sorted portfolios. For the Fama–French portfolios, the differences in loadings on both the low- and business-cycle-frequency shocks are large and statistically significant for all the small versus large comparisons. Small firms appear to be robustly more exposed to long-run and business-cycle shocks than large firms, and this wide spread in factor loadings is what helps us identify the risk prices in tables 2 and 3. On the value-growth dimension we find a much smaller spread, most of the time statistically insignificant. In fact, we find no significant spread between value and growth stocks if one excludes the extreme growth portfolios (especially the small growth portfolio, typically difficult to price), which display a modestly higher exposure to long-run and business-cycle shocks. We conclude that most of the identification within the 25 Fama-French portfolio comes from the heterogeneity in loadings between small and large firms, though we note that it is not the main

purpose of this paper to explain the small-large or value-growth puzzle.

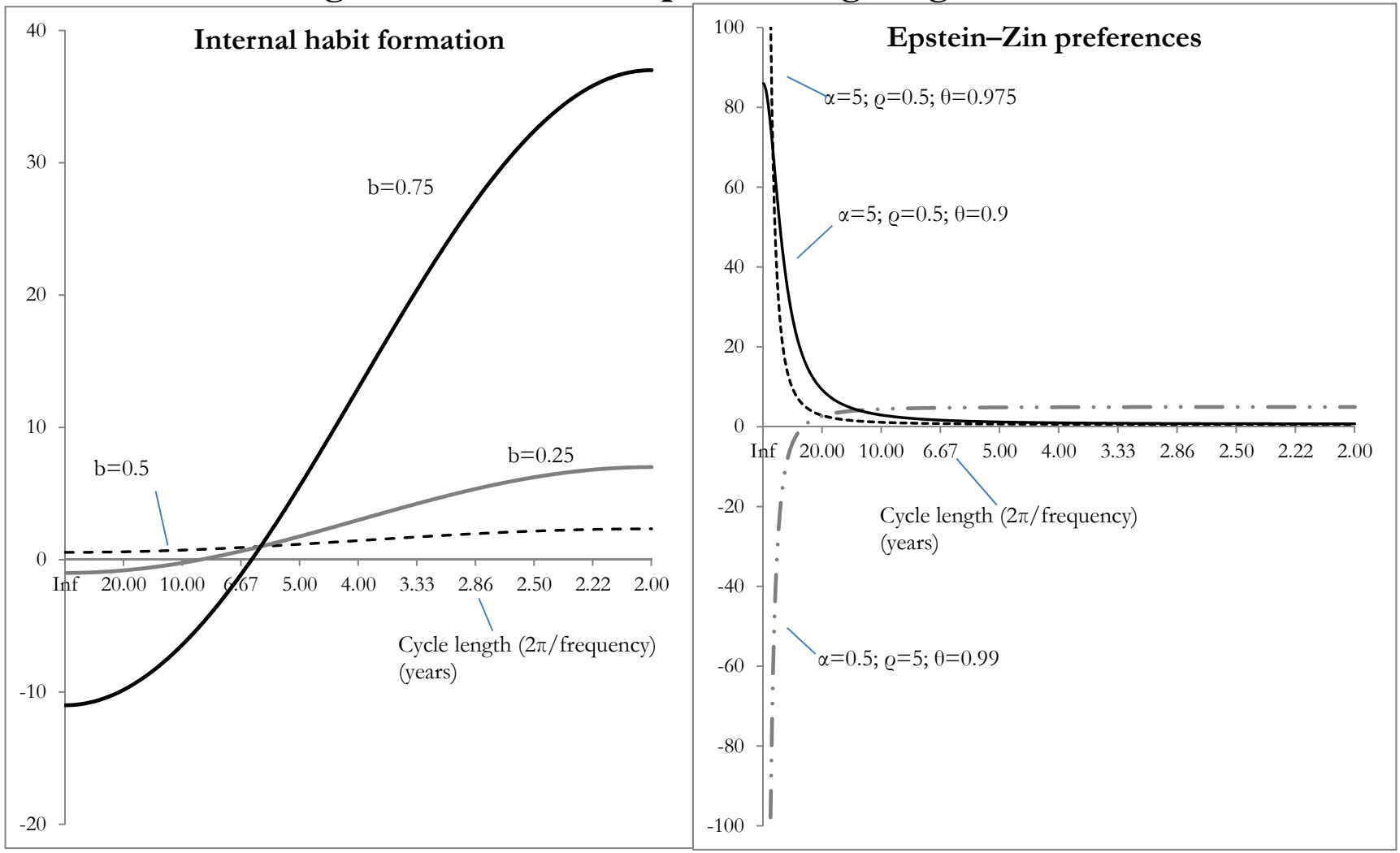
We see similarly large variation in the factor loadings for the risk-sorted portfolios. Interestingly, it seems that the Fama–French portfolios actually have a slightly wider degree of variation in their loadings. This is due to two factors. First, there are simply more of the Fama–French portfolios, so we are more likely to find large differences. Second, factor loadings for individual firms are not particularly persistent (especially since we estimate them using quarterly data, so the loadings used for portfolio formation may be somewhat imprecise). The resulting risk-sorted portfolios thus have a much smaller spread in post-formation loadings than they do in their pre-formation loadings.

Figure 1. Impulse response functions and impulse transfer functions



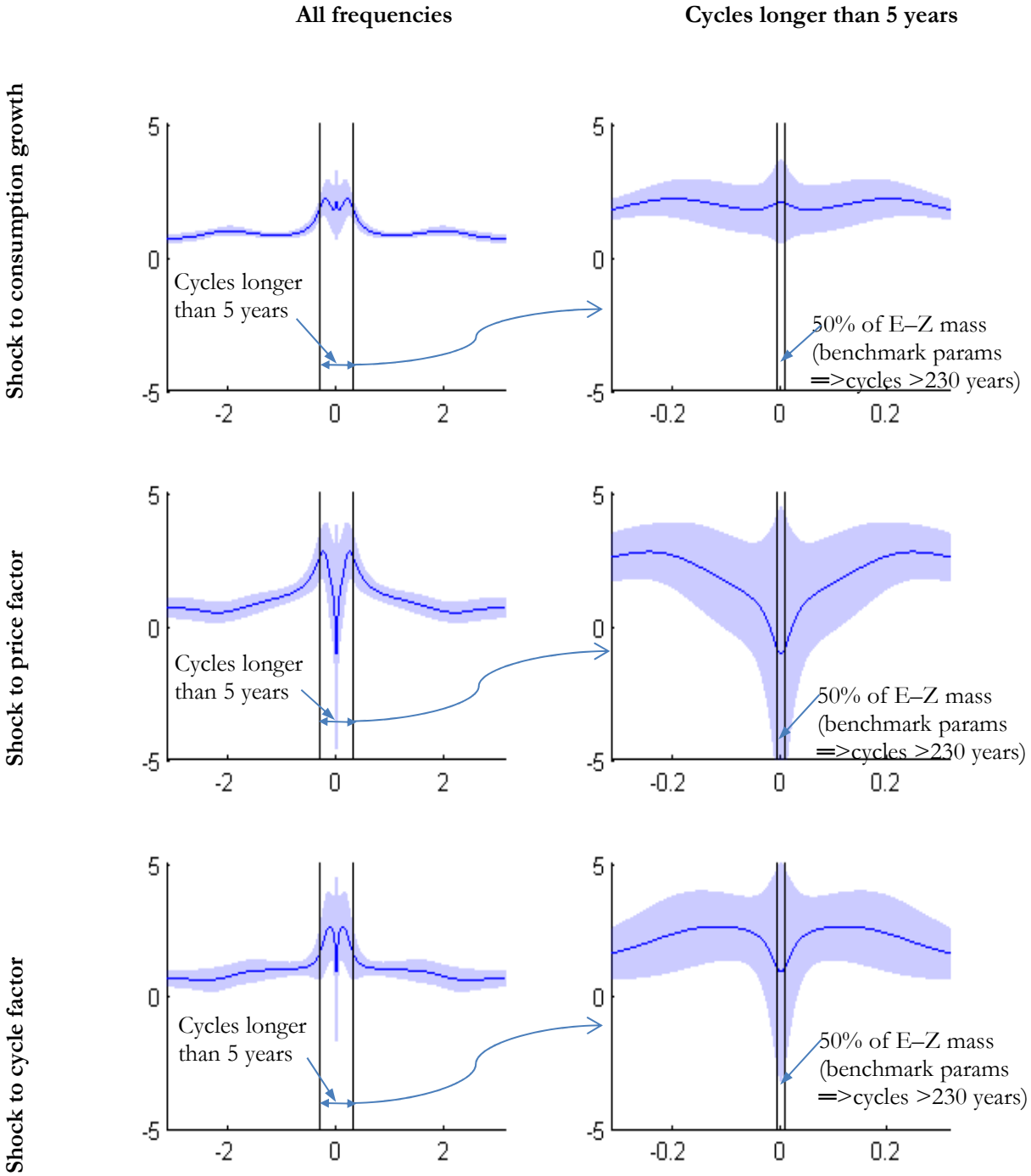
Notes: The left panel plots responses of the level of consumption to four hypothetical shocks. The right-hand panel plots the fourier transforms of the shocks to consumption *growth* , which we refer to as the impulse transfer functions

Figure 2. Theoretical spectral weighting functions



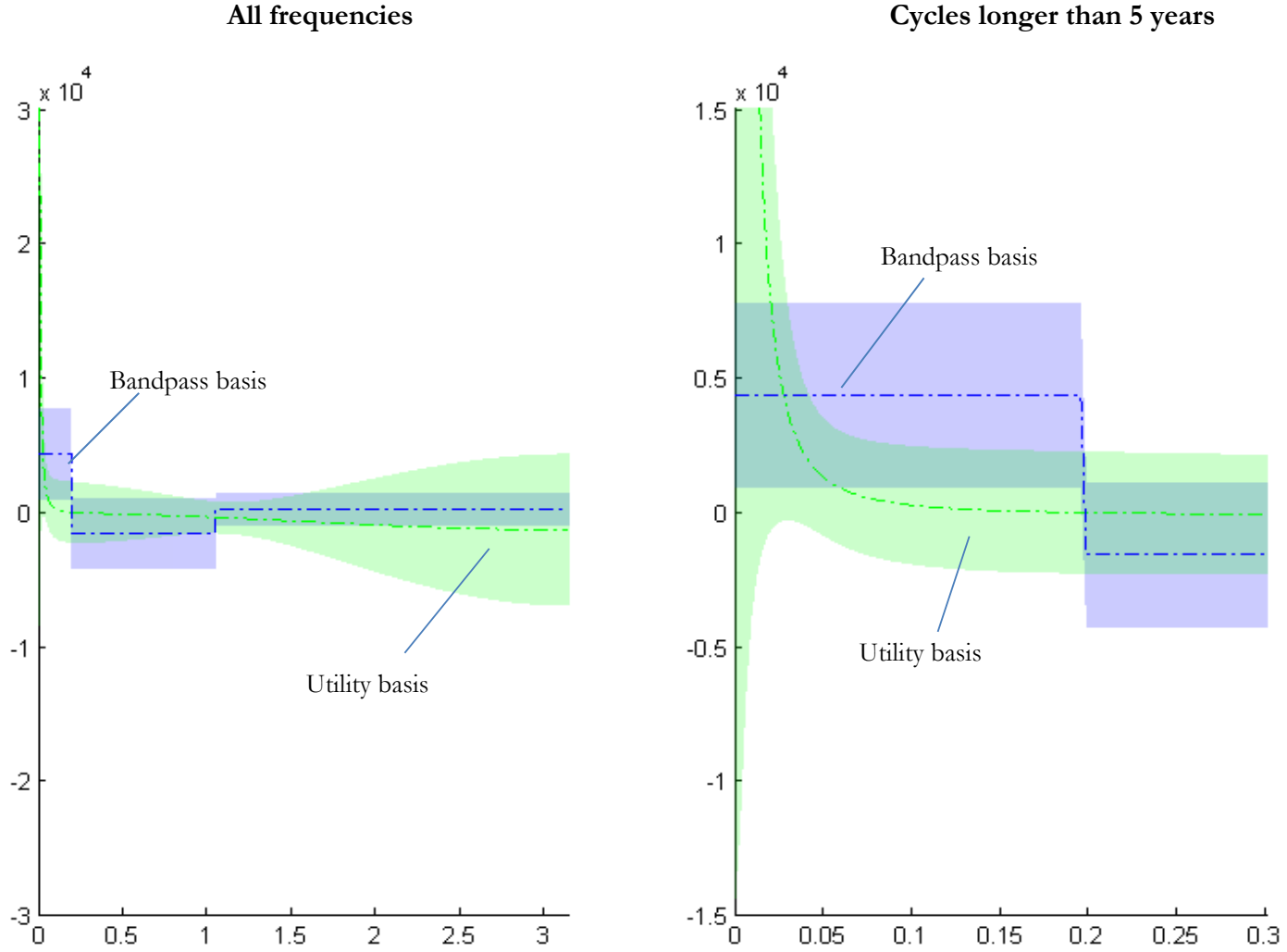
Notes: Plots of the spectral weighting function Z for various utility functions. The x-axis is the cycle length. In the left-hand panel, the parameter b determines the importance of the internal habit in the agent's utility function. In the right-hand panel, α is the coefficient of relative risk aversion; ϱ is the inverse elasticity of intertemporal substitution; and θ is the discount factor.

Figure 3. Estimated impulse transfer functions for consumption VAR



Notes: Impulse transfer functions estimated from a VAR in consumption growth and the two principal components. Shaded regions represent 95-percent confidence intervals. The left-hand plots are for all frequencies, while the right-hand plots zoom in on cycles longer than 5 years. The range between the lines on the right-hand side contains 50 percent of the mass of the weighting function for Epstein–Zin preferences with RRA=5 and EIS=2 (cycles longer than 230 years). The x-axis gives frequencies in terms of quarters. Shocks are not orthogonalized.

Figure 4. Estimated spectral weighting functions for equities



Notes: Estimated weighting functions for consumption growth as the priced variable. Risk prices are estimated using the 25 Fama–French portfolios with the efficient weighting matrix for GMM. Shaded areas denote 95-percent confidence regions. The utility basis uses a discount factor of 0.975 at the annual horizon. The x-axis gives frequencies in quarters.

Table 1. Regression coefficients from VARs

	Lag 1			Lag 2			Lag 3		
	Cons.	Price	Cycle	Cons.	Price	Cycle	Cons.	Price	Cycle
Cons.	0.322 ***	0.629 ***	0.4935 **	0.1459 **	-0.485 *	-0.516 **	0.1658 **	-0.207	0.129
se	(0.08)	(0.23)	(0.14)	(0.07)	(0.29)	(0.21)	(0.07)	(0.15)	(0.17)

Notes: VAR results for consumption growth and the two principal components. The table reports the regression of consumption growth on its own lags and those of the two pricipal components. The sample is 1952:1–2011:2, quarterly. Standard errors are reported in brackets. * indicates significance at the 10-percent level, ** the 5-percent level, and *** the 1-percent level.

Table 2. Parameter estimates for the spectral weighting function (efficient matrix for GMM)

Portfolios:		FF25				FF25+IND			
Basis:		Bandpass	t-stat	Utility	t-stat	Bandpass	t-stat	Utility	t-stat
Consumption growth	q1	269	2.47 **	555.47	1.66 *	112	1.95 *	197.52	1.35
	q2	-431	-1.17	-442.65	-0.44	-116	-0.87	-279.30	-0.65
	q3	138	0.33	616.12	0.32	-134	-0.70	504.32	0.63
GDP	q1	124	1.85 *	231.42	0.69	91	1.64 *	186.58	0.69
	q2	-106	-1.29	119.67	0.87	-72	-1.13	26.70	0.24
	q3	127	1.33	-217.42	-1.04	22	0.40	-77.25	-0.49
Durables	q1	49	2.66 ***	75.62	1.69 *	15	2.30 **	19.96	1.58
	q2	-38	-1.26	44.87	2.29 **	-2	-0.30	10.79	2.37 **
	q3	33	1.70 *	-86.66	-0.63	-5	-0.83	19.62	0.55
Investment	q1	12	2.03 **	29.25	1.03	14	2.26 **	31.87	1.07
	q2	-7	-1.18	-0.22	-0.03	-7	-1.22	4.93	0.88
	q3	-7	-1.12	5.17	0.36	-3	-0.66	2.11	0.15
Fixed Investment	q1	27	2.16 **	39.33	0.81	15	1.54	26.42	0.82
	q2	-25	-1.11	67.18	2.59 ***	-20	-1.26	-5.05	-0.41
	q3	61	3.10 ***	-90.96	-1.66 *	-4	-0.45	-20.23	-0.63
Residential Investment	q1	16	3.52 ***	27.00	2.44 **	3	1.96 **	5.38	1.40
	q2	-3	-0.45	-4.74	-0.12	2	1.36	-11.73	-1.19
	q3	4	0.24	55.19	0.85	-14	-2.26 **	33.03	1.97 **

Notes: Risk price estimates for the period 1952:1–2011:2 using quarterly data. The priced variable is listed in the left-hand column. The left-hand set of columns uses the Fama–French portfolios as the test assets; the right-hand columns add 49 industry portfolios from Ken French's website. For the bandpass basis, q1 is the price of low-frequency risk, q2 business-cycle frequency, and q3 high frequency. For the utility basis, q1 is the low-frequency component, q2 the constant, and q3 the coefficient on cos(w). The asset pricing moments are estimated using two-step GMM. The "t-stat" column gives the t statistics for the risk prices. * indicates significance at the 10-percent level, ** the 5-percent level, and *** the 1-percent level. t-stats take into account VAR estimation uncertainty, using GMM. The weighting matrix is constructed using the variance-covariance matrix of the asset pricing moment residuals.

Table 3. Parameter estimates for the spectral weighting function (identity matrix for GMM)

Portfolios:		FF25				FF25+IND			
Basis:		Bandpass	t-stat	Utility	t-stat	Bandpass	t-stat	Utility	t-stat
Consumption growth	q1	336	1.73 *	703.66	1.62	112	1.22	198.11	1.02
	q2	-541	-1.14	-340.19	-0.24	-116	-0.54	-277.49	-0.58
	q3	401	0.72	558.13	0.21	-131	-0.52	502.47	0.60
GDP	q1	138	1.55	258.39	0.62	91	1.38	186.71	0.62
	q2	-117	-1.11	133.31	0.80	-71	-0.85	25.95	0.22
	q3	139	1.10	-237.80	-0.95	21	0.18	-75.69	-0.37
Durables	q1	54	1.79 *	82.87	1.43	15	0.95	19.96	0.72
	q2	-40	-0.98	52.91	1.82 *	-2	-0.14	10.77	0.84
	q3	37	1.31	-91.29	-0.48	-5	-0.37	19.62	0.40
Investment	q1	13	1.44	30.07	0.86	14	1.55	31.64	0.94
	q2	-7	-0.80	0.04	0.00	-7	-0.86	4.74	0.63
	q3	-7	-0.73	4.15	0.24	-3	-0.40	2.20	0.13
Fixed Investment	q1	37	1.70 *	54.66	0.77	15	0.92	26.06	0.68
	q2	-36	-1.03	84.97	2.03 *	-19	-0.82	-5.58	-0.29
	q3	77	2.15 **	-118.72	-1.35	-4	-0.25	-19.59	-0.34
Residential Investment	q1	18	2.49 **	30.60	1.98 **	3	0.75	5.44	0.66
	q2	-3	-0.36	2.03	0.03	2	0.84	-11.63	-0.70
	q3	11	0.37	57.95	0.59	-13	-1.14	33.05	1.25

Notes: See table 2. These estimates differ only in that they use the identity matrix for the GMM weighting matrix.

Table 4. Parameter estimates with returns as priced variable

		FF25				FF25 + Industry			
		Weighting: S		Weighting: I		Weighting: S		Weighting: I	
		coeff	t-stat	coeff	t-stat	coeff	t-stat	coeff	t-stat
Utility basis	q1 Long-run	8.36	2.19 **	7.99	1.65 *	7.01	3.06 ***	7.07	1.57
	q2 Constant	9.99	3.34 ***	9.56	1.81 *	8.60	6.25 ***	8.63	2.09 **
Bandpass basis	q1 Long-run	13.10	2.36 **	13.65	1.53	9.65	2.62 ***	9.65	1.39
	q2 Constant	-2.36	-0.65	-2.76	-0.72	-1.31	-0.41	-1.36	-0.48

		FF25				FF25 + Industry			
		Weighting: S		Weighting: I		Weighting: S		Weighting: I	
		coeff	t-stat	coeff	t-stat	coeff	t-stat	coeff	t-stat
Utility basis	q1 Long-run	1.98	0.29	2.61	0.26	6.36	1.33	6.59	0.83
	q2 Constant	10.40	0.95	10.57	0.90	8.69	4.79 ***	8.75	2.09 **
	q3 High Freq	114.20	1.05	108.64	1.01	12.58	0.17	10.07	0.07
Bandpass basis	q1 Long-run	9.22	0.95	9.26	0.44	15.53	2.53 **	15.60	1.45
	q2 Business cycle	46.38	0.28	52.58	0.20	-76.23	-0.82	-77.02	-0.43
	q3 Short-run	-45.62	-0.30	-51.75	-0.22	70.11	0.81	70.80	0.42

Notes: Risk price estimates for the period 1926:3 - 2011:2, using quarterly data. The top panel uses two parameters for the weighting function (a long-run component and a constant), the bottom panel uses three parameters corresponding to the decomposition of Table 2. t-statistics take into account VAR estimation uncertainty using GMM. The weighting matrix used is either the inverse of the variance-covariance matrix of the moment residuals (Weighting: S) or the identity matrix (Weighting: I).

Table A1. Bootstrapped p-values for table 2

Portfolios:		FF25				FF25+IND			
Basis:		Bandpass	p-value	Utility	p-value	Bandpass	p-value	Utility	p-value
Consumption growth	q1	269	0.00 ***	555.47	0.00 ***	112	0.00 ***	197.52	0.01 **
	q2	-431	0.00 ***	-442.65	0.10 *	-116	0.04 **	-279.30	0.05 *
	q3	138	0.88	616.12	0.29	-134	0.62	504.32	0.15
GDP	q1	124	0.00 ***	231.42	0.01 **	91	0.00 ***	186.58	0.00 ***
	q2	-106	0.04 **	119.67	0.31	-72	0.01 **	26.70	0.60
	q3	127	0.12	-217.42	0.17	22	0.32	-77.25	0.31
Durables	q1	49	0.00 ***	75.62	0.00 ***	15	0.04 **	19.96	0.05 *
	q2	-38	0.01 ***	44.87	0.01 ***	-2	0.38	10.79	0.09 *
	q3	33	0.01 **	-86.66	0.10 *	-5	1.05	19.62	1.12
Investment	q1	12	0.03 **	29.25	0.02 **	14	0.00 ***	31.87	0.00 ***
	q2	-7	0.41	-0.22	1.00	-7	0.08 *	4.93	0.45
	q3	-7	0.77	5.17	0.74	-3	1.10	2.11	1.07
Fixed Investment	q1	27	0.00 ***	39.33	0.03 **	15	0.01 **	26.42	0.02 **
	q2	-25	0.17	67.18	0.00 ***	-20	0.03 **	-5.05	1.05
	q3	61	0.00 ***	-90.96	0.01 **	-4	1.33	-20.23	0.19
Residential Investment	q1	16	0.00 ***	27.00	0.00 ***	3	0.30	5.38	0.25
	q2	-3	0.25	-4.74	0.43	2	0.79	-11.73	0.26
	q3	4	1.12	55.19	0.07 *	-14	0.21	33.03	0.02 **

Notes: See table 2. These results use the efficient weighting matrix for the GMM estimation, but compute the standard errors using bootstrap (2500 draws).

Table A2. Results using risk-sorted portfolios

Portfolios:		FF25+9 risk sorted				FF6+9 risk sorted			
Basis:		Bandpass	t-stat	Utility	t-stat	Bandpass	t-stat	Utility	t-stat
Consumption growth	q1	275	1.97 **	382.08	1.29	276	2.04 **	328.92	1.13
	q2	-531	-1.20	-798.65	-0.75	-565	-1.14	-813.39	-0.86
	q3	-167	-0.29	1370.61	0.62	-271	-0.47	1300.84	0.66
GDP	q1	194	1.64	414.23	0.37	148	1.59	285.22	0.36
	q2	-248	-1.64	87.77	0.18	-203	-1.73 *	48.45	0.13
	q3	191	1.20	-369.39	-0.75	146	1.28	-187.73	-0.53
Durables	q1	39	3.15 ***	35.72	1.95 *	18	2.03 ***	-1.94	-0.12
	q2	-33	-1.14	30.17	2.11 **	-6	-0.32	25.59	3.71 ***
	q3	24	1.39	-42.52	-0.55	12	1.17	34.60	0.87
Investment	q1	12	2.16 **	13.91	0.86	14	1.50	10.87	0.72
	q2	-15	-2.48 **	-26.27	-5.33 ***	-11	-1.18	-12.77	-2.06 **
	q3	-16	-3.35 ***	10.27	1.41	-9	-1.57	13.08	1.10
Fixed Investment	q1	36	2.23 **	66.46	0.91	26	1.94 *	49.26	0.86
	q2	-52	-1.63	76.73	1.21	-36	-1.39	30.96	0.62
	q3	95	2.42 **	-170.89	-2.35 **	48	1.35	-101.63	-2.14 **
Residential Investment	q1	7	3.24 ***	17.83	2.40 **	0	0.07	14.01	1.86 *
	q2	1	0.46	-4.32	-0.14	6	2.24 **	-1.12	-0.04
	q3	2	0.21	40.91	0.81	4	0.49	35.89	0.71

Notes: See table 2. These results use the efficient weighting matrix for GMM. The FF6 portfolios are based on sorts into two bins by size and three by book/market. The 9 risk-sorted portfolios are based on three bins each for loadings on the low- and business-cycle frequency shocks. The loadings are measured using the previous 20 quarters of data.

Table A3. Factor loadings for test portfolios

Low-frequency loadings:

	Growth		2		3		4		Value		*Difference*	
Small	72.4	(12.3)	68.0	(10.1)	53.0	(9.0)	49.9	(8.6)	53.9	(9.8)	-18.5	(6.7)
2	64.6	(10.9)	55.2	(9.0)	48.5	(8.1)	48.1	(8.0)	51.7	(9.0)	-12.9	(6.7)
3	62.5	(9.7)	46.9	(8.1)	41.6	(7.5)	41.8	(7.5)	43.5	(8.3)	-18.9	(6.7)
4	54.0	(8.8)	45.2	(7.6)	41.3	(7.4)	41.4	(7.3)	48.7	(8.4)	-5.3	(6.5)
Large	41.6	(7.1)	32.6	(6.5)	29.3	(6.1)	31.3	(6.5)	38.2	(7.4)	-3.4	(5.9)
Difference	-30.8	(8.9)	-35.4	(7.0)	-23.7	(6.4)	-18.6	(5.9)	-15.7	(6.8)		

Business-cycle frequency loadings:

	Growth		2		3		4		Value		*Difference*	
Small	39.1	(5.6)	35.6	(4.6)	26.9	(4.1)	25.1	(4.0)	28.5	(4.5)	-10.6	(3.1)
2	33.7	(5.0)	28.0	(4.1)	24.4	(3.7)	23.3	(3.7)	26.1	(4.1)	-7.6	(3.1)
3	32.3	(4.4)	23.9	(3.7)	21.0	(3.5)	21.1	(3.5)	21.9	(3.8)	-10.5	(3.1)
4	27.3	(4.0)	23.1	(3.5)	21.4	(3.4)	20.3	(3.4)	24.8	(3.9)	-2.4	(3.0)
Large	22.5	(3.2)	17.3	(3.0)	16.4	(2.8)	17.2	(3.0)	20.5	(3.4)	-2.0	(2.7)
Difference	-16.6	(4.1)	-18.3	(3.2)	-10.6	(3.0)	-7.8	(2.8)	-8.0	(3.2)		

Low-frequency loadings:

	BC beta low		2		BC beta high		*Difference*	
LF beta low	23.5	(6.0)	23.8	(5.7)	26.7	(6.4)	3.2	(3.8)
2	26.3	(6.0)	27.9	(6.4)	35.8	(7.3)	9.5	(3.7)
LF beta high	35.6	(7.8)	46.7	(8.8)	57.8	(11.1)	22.2	(5.5)
Difference	12.1	(4.9)	22.9	(5.8)	31.2	(7.6)		

Business-cycle frequency loadings:

	BC beta low		2		BC beta high		*Difference*	
LF beta low	16.6	(3.6)	17.0	(3.5)	19.7	(3.9)	3.1	(2.3)
2	18.5	(3.7)	20.7	(3.9)	27.1	(4.4)	8.5	(2.3)
LF beta high	26.4	(4.7)	33.8	(5.3)	42.0	(6.7)	15.6	(3.4)
Difference	9.7	(3.0)	16.8	(3.5)	22.2	(4.6)		

Notes: Each cell of each table is a factor loading for one of the portfolio returns with respect to either the low- or business-cycle frequency shock. The top two panels report results for the 25 Fama–French portfolios, while the bottom two panels are for the risk-sorted portfolios used in table A2. The numbers in parentheses are standard errors for the estimated factor loadings.