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ABSTRACT

This paper provides a systematic analysis of identification in linear social interactions models. This is both a theoretical and an econometric exercise as the analysis is linked to a rigorously delineated model of interdependent decisions. We develop an incomplete information game that describes individual choices in the presence of social interactions. The equilibrium strategy profiles are linear. Standard models in the empirical social interactions literature are shown to be exact or approximate special cases of our general framework, which in turn provides a basis for understanding the microeconomic foundations of those models. We consider identification of both endogenous (peer) and contextual social effects under alternative assumptions on a priori information about network structure available to an analyst, and contrast the informational content of individual-level and aggregated data. Finally, we discuss potential ramifications for identification of endogenous group selection and differences between the information sets of analysts and agents.

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...it is said by some that men will think and act for themselves; that none will disuse spirits or anything else, merely because his neighbors do; and that moral influence is not the powerful engine contended for...Let me ask the man who would maintain this position most stiffly, what compensation he will accept to go to church some Sunday and sit during the sermon with his wife's bonnet upon his head? Not a trifle, I'll venture. There would be nothing irreligious in it...Then why not? Is it not because there would be something egregiously unfashionable about it? Then it is the influence of fashion; and what is the influence of fashion but the influence that other people's actions have on our own actions, the strong inclination each of us feels to do as we see our neighbors do? Nor is the influence of fashion confined to any particular thing or class of things. It is just as strong on one subject as another.

Abraham Lincoln,
February 22, 1842

Address to the Washington Temperance Society of Springfield Illinois

1. INTRODUCTION

While the proposition that individuals are subject to social influence states the obvious, the study of social influences on individual behavior in economics is a relatively recent phenomenon.¹ In the last two decades, however, a rich theoretical, econometric, and empirical literature in social economics, comprehensively overviewed in Benhabib, Bisin, and Jackson (2011a,b), has emerged. While each of these three dimensions — the theoretical, econometric, and empirical — has made important advances, it is fair to say that they are yet to be well integrated. By this we mean that the theoretical models used to study social interactions are distinct from the econometric environments in which identification is studied, while empirical work generally does not systematically exploit the implications of theory and econometrics for the formulation of data analyses.

The objective of this paper is to facilitate the integration of the theoretical, econometric, and empirical sides of the social interactions literature through a systematic investigation of linear social interactions models. Linear models are the workhorse of empirical research and have been the primary subject of econometric work on the identification of social interactions since Manski (1993). Our analysis provides rigorous microfoundations for a broad class of linear social interactions models.

The central, and fundamentally optimistic, message of this paper is that in most cases, linear social interactions models *are* identified. We employ a theoretically

¹Of course there are exceptions to this claim. Becker (1974) is an example of theoretical analysis that predate the modern literature, and Henderson, Mieszkowski, and Sauvageau (1978) and Datcher (1982) are early and seminal contributions in the empirical study of neighborhood effects. Examples of recent empirical applications of social interactions models include Conley and Udry (2010) on the diffusion of technology; Nakajima (2007) on smoking; Sirakaya (2006) on crime; Rege, Telle, and Votruba (2012) on the take up of welfare programs; and Topa (2001) and Bayer, Ross, and Topa (2008) on labor market outcomes. Another major area of social interactions work is education, which we discuss throughout the paper.

grounded model to understand the conditions under which social interaction effects are or are not identified. The identification problem is shown to depend on three factors: the prior knowledge available to an analyst on the social structure characterizing direct interactions between individuals, the type of data available to the analyst — whether aggregated or individual level — and the implications of endogenous network formation for the conditional expectations of unobserved heterogeneity given the social structure. The onus on the empiricist lies in establishing what they know about social networks *a priori* and, conditional on this information, verifying that their social interactions model satisfies the conditions needed for identification, many of which are provided in this paper. The conditions we describe do not involve adding stronger assumptions than have appeared in previous papers. Rather, we show that in some cases, non-identification results are artifices of strong assumptions and in others, we establish identification under weaker assumptions than have been previously employed.

We start by providing rigorous microfoundations that either exactly nest or approximate the many linear econometric models that have appeared in the social interactions literature. This is useful for empiricists because it permits a structural interpretation of regression parameter estimates, thereby allowing particular studies to shed light on more general contexts. Further, these microfoundations allow one to assess whether particular regression formulations that have appeared in various econometric and empirical studies are sensible when one considers them as equilibrium strategy profiles that emerge from a noncooperative game of incomplete information.

We translate this theoretical framework into an econometric one, which we use as a basis to study identification. Identification of utility parameters obviously depends on the researcher's *a priori* knowledge of social structures. Without any such prior knowledge, identification fails. This is the first basic identification result we establish. Our second set of results considers the case most commonly assumed in the applied literature, where a researcher has full prior knowledge of the social structure. In a generalization of Bramoullé, Djebbari, and Fortin (2009) we show that in this case, when the researcher has access to individual data, identification of the structural parameters is generic in a sense we make precise. This casts the non-identification results that have been at the centre of much of the conventional econometrics literature in a new light, since they pertain to non-generic models which have no obvious theoretical rationale. With full prior knowledge of social structure but access to only aggregate data, first moments do not enable identification. However, building on approaches proposed by Glaeser, Sacerdote, and Scheinkman (1996, 2003) and formalized by Graham (2008), we show that second moments do.

The assumption of full prior knowledge of social structure, although routinely imposed in empirical work, may be conceptually untenable. Our third set of identification results explore how far one can get with partial prior knowledge of social structure. These results, motivated in part by the availability of social network

data, indicate that when *a priori* information regarding the intensity of social ties between individuals is absent, prior knowledge of the mere existence (or absence) of ties between individuals enables identification. Identification for this case bears a conceptual resemblance to classical rank and order conditions for identification in linear simultaneous equations models (cf. Fisher (1966)), but the structure of the social interactions framework means that there are interesting differences from the standard results. Our results indicate that much more general models of social interactions can be employed in empirical work than has been done previously, when individual-level data are available. At the same time, we argue that there are limits to identification when data are comprised of individual observations and group level averages.

Finally, we address the issues of endogenous network formation, and the presence of public variables observable to those in the network but unobservable to the researcher. We treat endogenous network formation as the first stage of a two-stage game in which our general linear social interactions model describes payoffs from choices in the second stage. We show how the implications for identification of endogenous network formation entirely depend on the information available to agents at the time of network formation, so that for a number of interesting cases endogeneity does not matter. For the case where our results no longer apply, we indicate that the dealing with endogeneity involves the construction of the equivalent of the control function invented in Heckman (1979) and extended in Heckman and Robb (1986). This focus on information and network formations allows us to provide identification results for environments in which networks are durable in a sense we make precise. This places the endogeneity issue in the realm of a particular information asymmetry between the analyst and the population of agents

Two previous studies are relatively close to this one. (We discuss others in the context of our results later on.) Bramoullé, Djebbari, and Fortin (2009) consider identification for known social structures. We provide a mild generalization of their results by allowing for distinct social structures for contextual and endogenous effects, i.e. the effect of network members' exogenous characteristics and endogenous behaviors, respectively, on individual behavior. Further, they do not study identification when one does not know the complete social structure and when it is endogenous. Blume et al. (2011) anticipates some of our analysis. We employ a more general preference structure than that paper in that we allow for distinct social structures for different types of social interactions. Our results on identification under partial knowledge of the social structure are completely new as are our results on identification under aggregation and our discussion of endogeneity and information asymmetries between the analyst and the agents under study.

Throughout the paper, we will employ social interactions effects between students as an example in order to interpret assumptions and findings. The evidence for social interactions in education is well surveyed in Epple and Romano (2011)

and Sacerdote (2011). This empirical literature is large, exploring social influences on educational and other outcomes. It includes a range of environments that fall into the general framework we study. For example, it is common to assume that individual outcomes are determined by unweighted averages of peer outcomes and/or characteristics, with definitions of peer groups ranging from self-identified friendships (Patacchini, Rainone, and Zenou 2012) to classmates (Graham 2008) to schoolmates (Bifulco, Fletcher, and Ross 2011) to Zip Codes (Corcoran et al. 1992).

In section 2 we develop a social interactions game of incomplete information whose Bayes-Nash equilibrium produces linear strategy profiles. Section 3 introduces additional assumptions that are needed to study these equilibrium strategy profiles as econometric models of individual outcomes. Section 4 studies identification based on complete knowledge of the social structure that connects agents in the population. Section 5 provides conditions under which identification will hold for partial knowledge of social structure. Section 6 considers the implications of alternate formulations of unobserved heterogeneity due to endogenous network formation and information differences between the information sets of agents and the analyst. Section 7 concludes.

2. MICROFOUNDATIONS

In this section, we set up a theoretical model from which the econometric model we subsequently study is directly derived. We consider a Bayesian game – a social interactions game – in which the population of network members is a set V containing $N < \infty$ members. Each individual i is described by a vector of characteristics (x_i, z_i) , where $x_i \in \mathbf{R}$ is a publicly observed characteristic, and $z_i \in \mathbf{R}$ is a private characteristic observable only to individual i .² An individual's type t_i is a vector $(x, z_i) \in \mathbf{R}^{N+1}$, which details i 's observable and unobservable characteristics, and the observable characteristics of everyone else. The vector of players' types is $(x, z) \in \mathcal{T} = \mathbf{R}^{2N}$. The *a priori* distribution of types is an exogenous probability distribution ρ on \mathcal{T} . Knowledge of ρ is common to all individuals, and each individual's beliefs about the types of others is a conditional distribution of ρ given the individual's type.

Utility depends on an individual's own action and characteristics as well as network members' actions and characteristics. Individual i chooses an action

²We restrict attention to one observable and one unobservable characteristic in order to simplify notation. All of our results are easily extended to the case when each of these is a vector.

$\omega_i \in \mathbf{R}$ to maximize utility:

$$U_i(\omega_i, \omega_{-i}) = \left(\gamma x_i + z_i + \delta \sum_j c_{ij} x_j \right) \omega_i - \frac{1}{2} \omega_i^2 - \frac{\phi}{2} \left(\omega_i - \sum_j a_{ij} \omega_j \right)^2. \quad (1)$$

Utility is separable into two components. The first line is the private component of utility and the second is the social component. Both are strictly concave in individual i 's action. Marginal private utility is linear in individual i 's own observable characteristic x_i and private characteristic z_i . The term $\delta \sum_j c_{ij} x_j$ captures *contextual effects* – the direct influence of others' characteristics on i 's choices. It is a weighted average of the characteristics of neighbors in a *contextual-effects network*. In our model, *endogenous*, or *peer effects*, come from social pressure, that is, social norms. This is described in the second line as the squared distance between individual i 's behavior ω_i and the average $\sum_j a_{ij} \omega_j$ of the behaviors of his peers in an *peer effects network*. The parameter ϕ determines the marginal rate of substitution between the private and social components of utility.

The matrices A and C whose elements a_{ij} and c_{ij} determine peer and contextual effects are *weighted adjacency matrices* or *weighted sociomatrices* for the peer- and contextual-effects network, respectively. Each has dimension $N \times N$ and the magnitudes of the matrix elements measures the strength of network ties. The networks themselves can be described by graphs: the peer-effects network \mathcal{A} has vertex set V and edge set $E = \{(i, j) : a_{ij} > 0\}$. The contextual-effects network \mathcal{C} is defined similarly with C instead of A .

In the schooling example, the network is the population of N students. Each student i choses a level of effort. Observable characteristics include indicators of socio-economic status, such as family income. Contextual effects emerge because families with resources may contribute public goods such as school supplies or volunteer time to the classroom. The unobserved type z_i may capture individual characteristics such as ability and family values concerning education, and can include common influences such as teacher quality. The peer effect is understood here to be a pressure to conform. Peer- and contextual-effects networks may differ, then, because the entire classroom may benefit from such things as parent volunteers, while peer effects come only from a student's friends.

Our utility function nests examples in the literature such as Davezies, d'Haultfoeuille, and Fougère (2009) and Manski and Mayshar (2003). Our model is closest to Blume et al. (2011) but strictly nests it because we allow for distinct peer-effect and contextual-effect sociomatrices and because we work with much weaker error restrictions than the i.i.d. assumption made by the earlier paper. This weakening is important in moving from the theoretical to the econometric model.

Since all actions are chosen simultaneously, an equilibrium concept is required. We see this as an incomplete information game, and look for a Bayes-Nash equilibrium. That is, individuals choose an action to maximize the expectation of their utility given their type and the public types of others. The Bayesian game formalism assumes that the description of the game $\langle U, \gamma, \delta, \phi, A, C, \rho \rangle$ is common knowledge among individuals. Furthermore, we assume that x is common knowledge, and each individual alone observes his private z_i . Equilibrium beliefs are constructed from the individuals' strategy functions and the common prior belief. The following axioms ensure the existence of a Bayes-Nash equilibrium:

- T. 1.** $\phi \geq 0$. A and C are non-negative, for each $i \in V$, $\sum_j a_{ij}$ is either 0 or 1, and similarly for C . For all $i \in V$, $a_{ii} = 0$.
- T. 2.** Second moments of ρ exist.

Axiom T.1 has several parts. The restrictions $\phi \geq 0$ and non-negative A together impose a preference for behavioral conformity. Hence, a student is more likely to exert effort if classmates also exert effort. The analogous restriction on C means the effect of exogenous characteristics is proportionate to the strength of a tie. So the age of a student's friends may matter more than that of acquaintances and may be positive or negative, but the sign of the contextual effects will be the same as the sign of δ . The restriction that the row sums of the peer and contextual effects be either 1 or 0 means that individuals in the network either care about these variables as weighted averages, or the individual is a social isolate — a "loner". The restriction $a_{ii} = 0$ ensures that i does not affect i 's behavior via the sociomatrix. This is an obvious assumption given that this component of utility captures the effect of others' behaviors and not own behavior. Our model is not circular in that a student does not directly affect his performance via his own performance. Axiom T.2 is necessary to ensure that expected utility is well-defined for a large class of strategies.

Formally, a strategy for individual i is a function $f_i : \mathbf{R}^{N+1} \rightarrow \mathbf{R}$ that assigns a choice ω_i to each of his possible types (x, z_i) . Denote by \mathcal{F} the set of all strategy profiles $f(x, z) = (f_1(x, z_1), \dots, f_N(x, z_N))$ such that for each x and i , $f_i(x, \cdot) : z_i \mapsto \mathbf{R}$ is in L^2_ρ .³ A Bayes-Nash equilibrium of the game is a vector of strategy profiles $f(x, z)$ such that each f_i maximizes $E(U_i(\omega_i, \omega_{-i}) | x, z_i)$, where the expectation is taken with respect to the strategies f_{-i} and the common prior ρ .

Theorem 1. *If the Bayesian game satisfies axioms T.1 and T.2, it has a unique Bayes-Nash equilibrium. The equilibrium strategy profile can be written*

$$f(x, z) = \frac{1}{1 + \phi} \left(I - \frac{\phi}{1 + \phi} A \right)^{-1} (\gamma I + \delta C)x + \mu(x, z) + \frac{1}{1 + \phi} z \quad (2)$$

³This means that the squared integral of $f_i(x, \cdot)$ with respect to z_i exists and is finite, ensuring that preferences over strategies for the Bayesian game are well-defined.

where $\mu_i(x, z)$ depends only on x and z_i . If z is independent of x , then $\mu_i(x, z)$ depends only on z_i . If the elements of the z are all pairwise independent, then $\mu_i(x, z)$ depends only on x . If both are true,

$$\mu(x, z) = \frac{1}{1 + \phi} \left(\left(I - \frac{\phi}{1 + \phi} A \right)^{-1} - I \right) E(z), \quad (2a)$$

a constant vector.

Theorem 1 provides sufficient conditions for the existence of a unique Bayes-Nash equilibrium to the game, whose individual strategy profiles obey the linear structure commonly assumed in the empirical literature. The first term in (2) describes endogenous, contextual, and direct own-effects of public types x . This is the focus of empirical research. The third term expresses the direct effect on equilibrium behavior of individuals' private types z . The second term, $\mu(x, z)$, is the effect of higher-order beliefs — individuals' expectations of others' private types, their expectation of others' expectations of their beliefs, etc. In general, i 's higher order beliefs may be a function of (x, z_i) , others' characteristics and i 's private type. This may be important when network membership is endogenous and x and the z_i 's are correlated. We consider this case in section 6. When x and z are independent, however, $\mu(x, z) = \mu(z)$, i.e. each individual's expectation of others' private types depends only on his own private type. Our econometric models will make this assumption (E.4 below), and we will sweep these higher-order beliefs into the unobserved term in the regression equation.

There are generally speaking, two kinds of endogenous social interactions models. We have chosen to model social interactions as coming through social norms. We model this as a conformity effect. Another source of social interactions comes through strategic complementarities in production. In the education literature papers such as Epple and Romano (1998) and Calvó-Armengol, Patacchini, and Zenou (2009) take this approach in modeling peer effects.⁴ The interpretation of the peer effects in such papers is that the marginal cost of educational achievement is affected by peer composition. Such an approach has the utility function

$$U_i(\omega_i, \omega_{-i}) = \left(\gamma x_i + z_i + \delta \sum_j c_{ij} x_j \right) \omega_i + \phi \sum_j a_{ij} \omega_i \omega_j - \frac{1}{2} \omega_i^2 \quad (3)$$

Here the first two terms describe a production function that maps effort to an educational outcome. The second term is a strategic complementarity. The hard work of other students spills over to increase the marginal product of student i 's effort. This is plausible for all kinds of reasons (see Sacerdote (2011)). The third term is the disutility of effort. The proof of theorem 1 applies to this model too, and an equilibrium exists and is unique for $0 \leq \phi < 1$.⁵ The difference between

⁴Epple and Romano (2011, sec. 2.1) surveys different models of externalities between students.

⁵This model becomes more complicated when $\phi \geq 1$, and so this case is assumed away in the literature.

this model and that of equation (1) is that here the vector x enters the equilibrium strategy profile through the term $(I - \phi A)^{-1}x$, that is, ϕ rather than $\phi/(1 + \phi)$ multiplies the matrix A . Thus the two models cannot be distinguished in the joint distribution of x and ω without further assumptions. This renders moot the issue of identifying the source of endogenous social interaction. Hence we use the theoretical model outlined in (2) as our basis for econometric analyses of linear social interactions. We make this transition in the next section.

3. FROM A THEORETICAL TO AN ECONOMETRIC MODEL

An econometric evaluation of our theoretical model requires additional assumptions. In this section we introduce these assumptions and discuss their role in identification of the utility parameters. Loosely speaking, parameters are “identified” if the map from utility parameters into the joint distribution of regressors and outcomes is one-to-one. The following definition, due to Koopmans (1953), is useful in translating our theoretical framework to an econometric one.

Definition 1. *A structure s for the linear social network model is a list $\langle \gamma, \delta, \phi, A, C, \rho \rangle$, where γ, δ and ϕ are utility parameters, A and C are peer- and contextual-effects sociomatrices, and ρ is the a priori probability distribution on $\mathbf{R} \times \mathbf{R}$. A model is a set of structures.*

The empiricist is interested in whether the utility parameters γ, δ and ϕ are identified in a model in which a number of sometimes implicit restrictions have been imposed on A, C and ρ . Here, we make these restrictions explicit by concerning ourselves with the model \mathcal{M} , all of whose structures satisfy T.1-T.2 and the following assumptions:

- E. 1.** The support of the marginal distribution of x has dimension N .
- E. 2.** For all i and j , $a_{ij} > 0$ iff $a_{ji} > 0$. For some i and j , $a_{ij} > 0$.
- E. 3.** For all i and j , $c_{ij} > 0$ iff $c_{ji} > 0$. For some i and j , $i \neq j$, $c_{ij} > 0$.
- E. 4.** For all $i, j \in V$, x_j and z_i are uncorrelated.
- E. 5.** At least one of γ and δ is nonzero.

E.1 ensures that the $N \times N$ matrix of parameters post-multiplied by x in equation (2) is unique. Assumptions E.2 and E.3 place additional restrictions on the sociomatrices. While these are not necessary conditions in our theorems, they greatly simplify derivations. Each is, in itself, weak and only serves to eliminate knife-edge cases. The first parts of E.2 and E.3 require that when j exerts social influence on i , i also exerts social influence on j . These are weak assumptions because nonzero elements of the sociomatrices are allowed to be arbitrarily small and the strength of ties between two individuals may be asymmetric. We do not require these two assumptions for most of our results, including those which rely on holes in the network for identification. However, we prefer to maintain them because they greatly simplify proofs. The second parts of E.2 and E.3 require,

respectively, that there exist at least one pair of agents who exert peer effects on each other and one pair of agents who exert contextual effects on each other. This rules out $C = I$. E.4 is a standard exogeneity condition with respect to x . In the context of our theory model, it means that higher order beliefs depend only on own-types, so $\mu(x, z) \equiv \mu(z)$. We will relax this assumption in section 6. E.5 eliminates the special case in which the x 's have no effect on the outcomes. Lemma 1 in the appendix shows that in this special case identification fails, so we ignore it in subsequent discussion.

For what follows, it will be convenient to define

$$\begin{aligned}\mu &= \mathbb{E} \left(\mu(z) + \frac{1}{1+\phi} z \right), \\ \varepsilon &= \mu(z) + \frac{1}{1+\phi} z - \mu, \\ B_\phi(s) &= \frac{1}{1+\phi} \left(I - \frac{\phi}{1+\phi} A \right)^{-1},\end{aligned}$$

and

$$B(s) = B_\phi(s)(\gamma I + \delta C). \quad (4)$$

For structures in models satisfying assumptions T.1-T.2 and E.1-E.4, the equilibrium equation system of theorem 1 becomes

$$\omega = \mu + B(s)x + \varepsilon, \quad (5)$$

With this change of variables, the residual term ε has unconditional mean 0. Although it should not be forgotten that both μ and the distribution of ε depend upon ρ , we shall not be using either for identification (except in section 4.2). Instead, we will identify parameters through the matrices $B(s)$. We index these matrices by s to emphasize that it is from the structure that we will recover utility parameters.

Equation (5) may be contrasted with a purely statistical model of the form

$$\omega = \alpha + Bx + e, \quad (6)$$

in which e is constructed to be orthogonal to $(1, x)$. Viewing this statistical model through the prism of the game of section 2 and the econometric assumptions of this section has three advantages. First, it imposes some parameter restrictions on the model (e.g. the row sums of B will be identical). Second, it facilitates the interpretation of parameter values in terms of commonly accepted models of interactive decision making. Third, it allows for causal conclusions from parameter estimates because it makes clear what environmental perturbations leave the structure unchanged.

From E.1 it is immediate that α, B and $\text{Var}(e)$, the covariance matrix of the reduced form errors from equation (6), summarize the relevant information for identification via the first and second moments of the data, and that each is unique. For

purposes of identification, these moments are the objects that the data provide to the analyst.

Most of this paper is concerned with identification of the utility parameters of structures from the matrices $B(s)$ of equation (5), which describe how equilibrium strategy profiles vary with characteristics x .

Definition 2. *Utility parameters γ , δ and ϕ are identified in a model \mathcal{M} by B if for all $s, s' \in \mathcal{M}$, if $B(s) = B(s')$ then $(\gamma, \delta, \phi) = (\gamma', \delta', \phi')$.*

Our identification definition ignores the constant term because, in comparing the equilibrium strategy profile (5) with the statistical model (6), without restrictions on $\rho_{\varepsilon|x}$ (the marginal prior of ε given x), the individual-specific constant terms cannot provide additional information on γ , δ and ϕ .

Since B depends on the structure only through γ , δ , ϕ , A and C , identification of the utility parameters will obviously depend on what is known about A and C *a priori*. Without *a priori* information about structures identification will fail since the inverse image of a matrix B under the map $s \mapsto B(s)$ could contain structures with very different sociomatrices. In section 4.1 we assume that the pair (A, C) is known *a priori*, while in section 5.1, *a priori* knowledge only pertains to C . Let $\mathcal{M}(A, C)$ and $\mathcal{M}(C)$ denote the sets of all $s \in \mathcal{M}$ with fixed sociomatrices A and C , and with fixed contextual-effects sociomatrix C , respectively. These should be thought of as *sub-models* of \mathcal{M} . For instance, when A and C are known *a priori*, the identification exercise is that of identifying the utility parameters in the set of structures $\mathcal{M}(A, C)$. It follows from these definitions that anything identified in \mathcal{M} is identified in $\mathcal{M}(C)$ for every contextual-effects matrix sociomatrix C , and anything identified in $\mathcal{M}(C)$ is identified in $\mathcal{M}(A, C)$ for every peer-effects sociomatrix A . Let M_A and M_C denote the set of all peer- and contextual-effects sociomatrices, respectively, that satisfy the relevant parts of T.1–2 and E.1–5.

We first establish a basic identification result. This result supposes the following structure on observations:

K. 1. For all i , the analyst observes (ω_i, x_i)

Without any *a priori* knowledge other than T.1–2, E.1–E.4 and K.1, the reduced form parameters $B = B(s)$ and the sum $\beta = \gamma + \delta$ are nonetheless identified; that is, they are identified in \mathcal{M} . This is critical. The remainder of the paper is concerned with the unpacking of $B = B(s)$ to recover utility and social interactions parameters. The theorem goes on to state that the parameter set $\gamma = \delta = 0$ is identified in \mathcal{M} . A third result states that with an additional piece of *a priori* information, the set $\delta = \phi = 0$ is identified, and in this case, by virtue of the first result, γ is identified as well.

Theorem 2. *Let \mathcal{M} denote the model satisfying axioms T.1–T.2, E.1–E.4 and K.1.*

- i. The matrix $B(s)$, μ , and the sum $\beta = \gamma + \delta$ are identified in \mathcal{M} from the joint distribution of ω and x without any additional a priori information.
- ii. $E(\omega|x)$ is independent of x if and only if $\delta = \gamma = 0$.
- iii. If C is known a priori and it is known a priori that there are two individuals i and j for whom $c_{ii} \neq c_{jj}$, then for all i , $E(\omega_i|x) = E(\omega_i|x_i)$ if and only if $\delta = \phi = 0$. In this case, γ is identified as well.

Otherwise, the parameters $\gamma, \delta, \phi, A, C$ are not identified without additional a priori information.

These results do not require E.5. But having dispensed with this case, it is convenient for expository purposes to maintain E.5 for the remainder of the paper. Note too that in case *iii*. it is not necessary to know the values of any elements of C , but merely that two diagonal values are distinct.

Theorem 2 is a negative result from the perspective of identifying social interactions. The non-identifiability of δ and ϕ means that the structural parameters that characterize the presence of the two possible social effects, peer and contextual, cannot be recovered given the assumptions we have made so far. To understand why, consider the following econometric specification, which is delivered from the first-order conditions for expected utility maximization in the Bayes-Nash equilibrium:

$$\omega_i = \frac{\gamma}{1+\phi}x_i + \frac{\delta}{1+\phi}\sum_j c_{ij}x_j + \frac{\phi}{1+\phi}\sum_j a_{ij}E(\omega_j|x) + \frac{1}{1+\phi}\varepsilon_i \quad (7)$$

This system of N equations is just a classic simultaneous equations system except that expectations of endogenous variables appear on the right hand side of the equation rather than realizations. The non-identification of this simultaneous equations system is a classical result — one that is unaffected by the row summability of A and C . In fact one can interpret two stage least squares as making exactly this substitution. From this vantage point, identification failure stems from the absence of exclusion restrictions in the system. (See Bramoullé, Djebbari, and Fortin (2009) for elaboration of this intuition.)

We close this section by showing how a number of existing models constitute special cases of our general framework. The social interactions literature has focused on equation (7), the first order conditions for expected utility maximization, rather than the equilibrium strategy profiles. Hence our first two examples focus on econometric models that may be interpreted as special cases of (7). Our third example illustrates how our models instantiates the idea of weak versus strong ties, a sociological distinction which is important for a variety of economic network analyses.

Example 1: linear-in-means models. In many social networks models, individuals are partitioned into non-overlapping groups g . Let n^g denote the size of group g . In the linear-in-means model, an individual's behavior depends on his average

group characteristics and average group behavior. This amounts to imposing the following constraints on the sociomatrices:

$$\begin{aligned} c_{ij} &= \frac{1}{n^g} \text{ if } i, j \in g, \\ a_{ij} &= \frac{1}{n^g - 1} \text{ if } i, j \in g, \\ c_{ij} &= a_{ij} = 0 \text{ if } i \in g, j \notin g \end{aligned} \quad (8)$$

Combined with the assumption that $E(\varepsilon|x) = 0$, the first order conditions (7) may be rewritten:

$$\begin{aligned} \omega_i &= \frac{\gamma}{1 + \phi} x_i + \frac{\delta}{(1 + \phi)(n^g - 1)} \sum_j x_j \\ &\quad + \frac{\phi}{(1 + \phi)(n^g - 1)} \sum_{j \neq i} E(\omega_j|x) + \frac{1}{1 + \phi} \varepsilon_i \end{aligned} \quad (9)$$

Manski's (1993) study of identification of social effects is based upon a large sample approximation of this model, in which for all i , $n^g \rightarrow \infty$. In the limit

$$\omega_i = \frac{\gamma}{1 + \phi} x_i + \frac{\delta}{1 + \phi} \bar{x}^g + \frac{\phi}{1 + \phi} E(\bar{\omega}^g|x) + \frac{1}{1 + \phi} \varepsilon_i \quad (10)$$

where \bar{x}^g and $\bar{\omega}^g$ are group-level averages of the respective variables.

The unweighted averaging assumed in the linear-in-means model does not have a theoretical justification but rather reflects a modeling choice made for simplicity, or because of limits on what is observable about the groups. It is trivial to think of contexts in which weights will not be equal. For high school students, one could easily imagine differences in sociomatrix elements that reflect relative popularity, strong versus weak friendships, and the like. One message of this paper will be that it is not necessary to rely on the simplification of unweighted averaging. While some prior information on the sociomatrices A and C is necessary for identification of the utility parameters, the necessary information is less than that assumed in the linear-in-means model.

Our framework can also be used to assess the interpretability of different variations of (7) with respect to rigorous microfoundations. For example, a major empirical study of educational peer effects is Sacerdote (2001), which examines roommate pair interactions at Dartmouth. Sacerdote assumes that each student i 's grade point average depends on his own ability and the ability and grade point average of his roommate j . Sacerdote is careful to allow for measurement error in ability. We ignore this for simplicity, since in its absence his model reduces to

$$\omega_i = d_0 + d_1 x_i + d_2 x_j + d_3 \omega_j + \zeta_i.$$

Sacerdote follows the theoretically appropriate formulation of endogenous social effects by employing the average of each individual's roommates, which for pairs is simply the outcome of the other roommate. Further, given that there is only a

single roommate, there is no issue of the restriction of the linear-in-means model A matrix. On the other hand, as in Lee (2007), the inclusion of ω_j rather than $E(\omega_j)$ begs the question of what information sets are available to agents, since a roommate's grades are not observable contemporaneously. That said, there is a simple reinterpretation of this model as

$$\omega_i = d_0 + d_1 x_i + d_2 x_j + d_3 E(\omega_j) + d_3 (\omega_j - E(\omega_j)) + \zeta_i$$

which is isomorphic to our equilibrium best response function when roommates are playing a Bayes-Nash game. Does this do violence to Sacerdote's analysis? We argue that it does not, since instrumenting for ω_j is equivalent to replacing this variable with $E(\omega_j)$. Note that in this specification, d_2 and d_3 are not identified if $E(\omega_j)$ is determined by a linear combination of x_i , x_j and a constant. (See corollary 1 for a further discussion.) Identification of $d_2 + d_3$ holds and is a special case of theorem 2.⁶

Example 2: linear-in-means models based on neighborhoods. A second common approach to analyzing social effects has extended the linear-in-means model by exploiting observed network data to locate individuals in neighborhoods, and using these neighborhoods to generate sociomatrices. One example of this strategy is De Giorgi, Pellizari, and Redaelli (2010) who employ administrative data from university students to explore peer effects among classmates, where interactions are determined by overlapping classroom enrollments. A similar approach is employed in Calvó-Armengol, Patacchini, and Zenou (2009), using AddHealth data.

Formally, let i 's neighborhood h be the set of other agents to whom he is connected and let n^h be the number of agents in this set. Note that $i \notin h$. The weights associated with a linear-in-means model based on neighborhoods correspond to

$$\begin{aligned} c_{ij} &= \frac{1}{n^h} \text{ if } j \in h, \\ a_{ij} &= \frac{1}{n^h} \text{ if } j \in h, \\ c_{ij} &= a_{ij} = 0 \text{ if } j \notin h \end{aligned} \tag{11}$$

The reduced form regression that is generated by the addition of these assumptions to our framework is

$$\omega_i = d_0 + d_1 x_i + d_2 \bar{x}^h + d_3 E(\bar{\omega}^h) + \zeta_i \tag{12}$$

where \bar{x}^h and $\bar{\omega}^h$ denote averages for neighborhood h . While equation (12) may resemble equation (7), it in fact implies a much richer structure for social interactions. Unlike the linear-in-means model, agents are no longer partitioned into

⁶Sacerdote (2001) shows that identification can hold under restrictions on the unobservables in his model. In our formulation of his model, this would require that $\omega_j - E(\omega_j)$ and ζ_i are uncorrelated, which Sacerdote (2001) notes involves the very stringent and arguably non-credible requirement that ζ_i and ζ_j be uncorrelated.

non-overlapping groups; an agent to whom many are connected has a larger influence than one to whom few are connected because of differences in the number of neighborhoods the respective agents inhabit.

However, the formulation is still restrictive relative to our general C and A sociomatrix formulation as (12)'s generalization of (7) involves the relaxation of the block diagonality assumption of the linear-in-means model but retains equal values of the non-zero elements of each row of the implied sociomatrix. This could be an inaccurate, not to mention excessively restrictive, representation of social interactions.

In this example too, our microfoundations can be used to evaluate the statistical formulations of (12). For example, De Giorgi, Pellizari, and Redaelli (2010) use $\bar{\omega}^h$ rather than $E(\bar{\omega}^h)$. As argued above, our Bayes-Nash formulation is more natural and does no violence if employed to interpret their regression. Further, the assumption $c_{ii} = 0$ may not be natural.

Calvó-Armengol, Patacchini, and Zenou (2009) make a more substantial deviation from our framework. They develop a complete-information social interaction game where individuals respond to the choices of peers in their social network. Individuals utilities are additively separable in two choice variables, private effort and peer-induced effort. These are perfect substitutes in the production of observed output (which is not an argument of the utility function). Their model differs from ours in two important way. First, they assume that individual characteristics (including contextual effects) affect only the utility of private effort, and so they elide the identification problem since individual characteristics do not feed back into peer effects. So, for instance, if a student in a classroom had an exogenous improvement in health status that reduced his cost of effort, and so he chose to work more, this would have no effect on others' effort levels or outcomes. Second, although individual characteristics are not an argument of peer effort utility, there is heterogeneity in equilibrium peer effort nonetheless, because *ceteris paribus* the marginal utility of peer effort is assumed to scale linearly with the number of connections one has — individuals with more contacts are assumed to be more susceptible to peer pressure. These unusual modeling choices serve a purpose. Calvó-Armengol, Patacchini, and Zenou make a direct connection between the peer-effort choice and sociological measures of centrality. They derive that the equilibrium peer effect equals the Katz-Bonacich centrality vector.

This and other measures can be derived from our A matrix, and both models have the advantage of deriving the rate of attenuation of influence that is a key parameter of these measures from the marginal rate of substitution between private and social components of utility.

Example 3: strong and weak ties. Empirical work by economists on networks has largely concerned networks with only one type of connection between agents. Sociologists, on the other hand, have recognized that social connections may

have different manifestations, and that the distribution of different kinds of connections in a social network has an impact on network outcomes. Perhaps the most well-known distinction among connections is that of strong and weak social ties. Granovetter (1973) argued that weak ties play an important role in job search because they relay useful job information more frequently. Lin (2002) suggests that weak ties are useful because weak-tie job referrals are drawn from a different and often better distribution of openings. Montgomery (1994) has embedded simple two-edge-type social networks into job search models to investigate the impact of the distribution of weak versus strong ties on employment rates and wage distributions. While the labor-market literature extolls the virtues of weak ties, in other aspects of economic life strong ties may be more important. Some ethnographic work suggests that strong ties have more value to poor individuals than weak ties. The suggestion is that the poor, lacking access to markets, rely more on reciprocity in their social networks for the provision of credit and a variety of commodity flows (see Granovetter (1973, pp. 209–13).)

The flexibility of weighted sociomatrices allows for the empirical distinction between strong and weak ties. Suppose that an individual i has n_{iS} strong ties and n_{iW} weak ties. Suppose too that the ratio of the strength of strong to weak ties is θ . Define elements of the peer-effects sociomatrix as

$$a_{ij} = \begin{cases} 1/(n_{iW} + \theta n_{iS}) & \text{if } j \text{ is weakly tied to } i; \\ \theta/(n_{iW} + \theta n_{iS}) & \text{if } j \text{ is strongly tied to } i; \\ 0 & \text{otherwise.} \end{cases}$$

A statistical model with this kind of network structure can be estimated from survey data that includes information on tie strength or data on ties that would allow a researcher to infer the nature of the tie. The AddHealth data set is an obvious candidate for constructing weak versus strong ties. Patacchini, Rainone, and Zenou (2012) in fact explore this possibility by assuming that the weights on friends report in the data are linearly declining in the order listed by each student and by studying the differences between friendships report in two survey waves versus one.

4. IDENTIFICATION WITH KNOWN SOCIOMATRICES

In this section, we describe identification of the primitive utility parameters γ , δ and ϕ when the sociomatrices A and C are known to the analyst. We do not take a stance on the source of this *a priori* knowledge. It may be the case that the matrices are empirical constructions or chosen for theoretical reasons. Formally, we augment the assumptions made in sections 2 and 3 with

K. 2. A and C are exogenous and known to the analyst *a priori*.

K.1 defines the information assumption on the sociomatrices maintained in this section. The assumption that the analyst knows the values of the sociomatrices is a strong one, and we believe that standard approaches to generating *a priori* values of A and C are often theoretically unjustified. However, since this is in fact how the bulk of the social interactions and networks literatures has proceeded, it is important to understand identification for such contexts. K.2 says that the analyst observes both the outcomes and characteristics of each member of the population. We will relax this assumption when we consider identification with aggregated data in Section 4.2.

One major result of this section is that when individual data are available, there is a precise sense in which identification of the primitive utility parameters of the linear social networks models is “typically” the case. In this section we say that “parameters are identified for generic pairs of sociomatrices” if the set of sociomatrix pairs (A, C) for which utility parameters are not identified is a lower-dimensional subset of $M_A \times M_C$. We show this below when A and C are known *a priori*. Our results indicate that concerns over simultaneity as a source of non-identification of social effects are misplaced when A and C are known, unless one has a justification for working with a model from the non-generic (small) set of linear social network models in which identification fails.

We further consider identification when data are aggregated. Consideration of this case was initiated by Glaeser, Sacerdote, and Scheinkman (2003) and formalized in Graham (2008). They employ versions of the linear-in-means model and focus on identifying the equivalent of the contextual-effects parameter δ under the assumption that $\phi = 0$, i.e. peer effects are not present. Here we provide identification results that generalize the cases these authors studied. We show that identification is possible when both effects are present and for social structures other than the linear-in-means specification.

4.1. Individual-level data. We begin by generalizing an important result due to Bramoullé, Djebbari, and Fortin (2009), which places conditions on the sociomatrices that are sufficient for identification of the parameters. Our generalization accounts for distinct peer- and contextual-effects sociomatrices. The result provides conditions such that the matrix B in the statistical model (6), when identified with $B(s)$ in the structural equation (5), can be used to back out the values of γ , δ and ϕ . Since A and C are known *a priori*, “is identified” means in this subsection “is identified in $\mathcal{M}(A, C)$ ”. The conditions, however, are abstract, and so we elaborate on them. We show next that the set of (A, C) pairs for which these conditions fail is a lower-dimensional subset of $M_A \times M_C$. Thus identification is the “typical case”. We provide conditions on the structure of the underlying networks that are sufficient to guarantee identification of the utility parameters. These results can be seen as demonstrating how exclusion restrictions are sufficient for identification.

Theorem 3. *Suppose T.1–T.2, E.1–E.5, and K.1–K.2.*

- i. If A , C and AC are distinct, then linear independence of the matrices I , A , C and AC is necessary and sufficient for identification of γ , δ and ϕ . The parameter γ is identified if and only if δ is identified.⁷
- ii. If $A \neq C$ and $AC = C$, then linear independence of I , A and C is necessary for identification of γ , δ and ϕ , and sufficient to identify γ and δ . If $\gamma \neq 0$, then independence is also sufficient to identify ϕ .

Case *i* of theorem 3 says that the failure of identification implies the existence of a non-zero solution in $\lambda_1, \lambda_2, \lambda_3$ and λ_4 of the following equation system:

$$\begin{aligned} \lambda_1 + \lambda_2 c_{ij} + \lambda_3 \sum_j a_{ij} c_{ji} &= 0 \text{ for all } i, \\ \lambda_2 c_{ij} + \lambda_4 a_{ij} + \lambda_3 \sum_j a_{ij} c_{ji} &= 0 \text{ for all } i \neq j \end{aligned} \quad (13)$$

These linear dependence conditions implicitly define the set of matrices A and C such that identification fails. Our interest in theorem 3 lies in understanding whether sociomatrices A and C “almost always”, “sometimes”, or “almost never” satisfy these restrictions. Genericity of identification captures the idea of “almost always”.⁸ The following definition explains what we mean when we say that utility parameters are generically identified.

Definition 3. *The utility parameters ϕ , γ and δ are identified in a generic model with a priori known A and C if there is a closed subset K of $M_A \times M_C$ of dimension less than that of $M_A \times M_C$ such that for all $(A, C) \in (M_A \times M_C)$ that are not in K , utility parameters γ , δ and ϕ are identified in $\mathcal{M}(A, C)$.⁹*

It will be useful to know that the dimensions of the set of allowable A and C sociomatrices are $\dim M_A = N(N - 2)$ and $\dim M_C = N(N - 1)$, respectively.¹⁰ Identification in a generic model means that even when identification in \mathcal{M} is impossible, we can identify utility parameters if we restrict the set of structures by discarding those with sociomatrices in a lower-dimensional subset of the allowed sociomatrices. The next result says that this is indeed the case when A and C are known *a priori*.

⁷Additional conditions on A and C imply that if any one parameter is identified, so are the other two. These conditions hold generically in C for each A and vice versa.

⁸Our results may be seen as a complement to McManus (1992) who established generic identification for parametric nonlinear models. He studies a space of nonlinear functions each of which is indexed by a parameter vector and employs a slightly different notion of genericity.

⁹The dimensions of sets in this definition are well-defined because these are semi-algebraic sets. Each is thus the finite union of manifolds, and the dimension is defined to be the largest dimension of these manifolds.

¹⁰Each row of C has one restriction, that the sum be 1 or 0, while each row of A has two: that the sum be 1 or 0, and that every diagonal element of the matrix equal 0.

Corollary 1. *For each peer-effects sociomatrix A such that the network contains a component¹¹ of size at least 3, there is generic subset of contextual-effects sociomatrices C such that γ, δ and ϕ are identified in $\mathcal{M}(A, C)$.*

This corollary states that utility parameters are identified in a generic model. To place this corollary in the education context, suppose that social interactions are confined to students in a given classroom. Interpret nonzero entries in the peer-effects sociomatrix A as indicating direct friendship ties between students. Corollary 1 says that if there are three or more students in this classroom who are friends or friends-of-friends, for a generic contextual-effects sociomatrix C the utility parameters are identified.

The proof of the corollary indicates that there are two ways that a pair (A, C) of sociomatrices can cause a failure of identification. First, every individual is either isolated in the peer-effects network or is influenced by only one person. Second, for each A there is a low-dimensional set of contextual-effects matrices C for which identification can fail. This second collection of pairs has the larger dimension, but the dimension of this set is no more than $\dim M_A(\dim M_C - 1)$. This is less than $\dim M_A \dim M_C$, the dimension of the set of all allowable sociomatrix pairs.

Corollary 1, while striking, follows from the natural intuition that *a priori* knowledge of A and C radically decreases the number of unknown parameters in the best response function in equation (2). With N equations describing individual behavior, we have N exogenous variables. One would intuitively expect that the triple (γ, δ, ϕ) be identified for $N \geq 3$. Identification fails when the sociomatrices are so symmetric that distinct relationships within the network become redundant. As lemma 3 in the appendix shows, this happens most prominently when $C = A$ and $A^2 = A$.

Our finding that identification is generic for known sociomatrices contrasts with much of the conventional wisdom in the econometric literature on the identification of social effects. In particular, since Manski (1993), there has been a recognition that for certain models in which individual choices are a linear function of the averages of individual characteristics, identification can fail. Manski's (1993) demonstration that the utility parameters (γ, δ, ϕ) are not identified for the large sample approximation (10) immediately follows from the fact that $E(\bar{\omega}^g|x)$ is linearly dependent on 1 and \bar{x}^g . Manski dubbed this identification failure "the reflection problem", and it has dominated econometric work on social effects ever since. Note that as the size n^g of each group g becomes large, $|(A^2)_{ij} - A_{ij}| \rightarrow 0$ and $|A_{ij} - C_{ij}| \rightarrow 0$. The large sample approximation imposes the extreme symmetry condition described above, resulting in non-identification.

¹¹A component is a maximal connected sub-network.

Corollary 2 gives a sufficient condition that rests on information regarding the network rather than the specific values of the weights. Formally, this kind of information can be represented by a graph for each network with vertex set V and a set of edges corresponding to non-zero elements in the relevant weighted sociomatrix. For instance, there is an edge from i to j in the peer-effects network \mathcal{A} if and only if $a_{ij} > 0$, and similarly for the contextual-effects network \mathcal{C} . Axioms E.2 and E.3 mean that we can represent these networks by undirected graphs. We write $i \sim_A j$ if there is an edge between i and j in the peer-effects network, $i \sim_C j$ if the edge exists in the contextual-effects network.

Corollary 2. *Suppose there are distinct individuals i and j who are connected by a sequence of edges, some in the peer-effects network and some in the contextual-effects network, and who are not connected by a path in either the peer-effects network or the contextual-effects network alone. Then γ , δ and ϕ are identified.*

This corollary demonstrates how restrictions on merely the existence of social interactions imply identification. Its conditions are satisfied whenever components of the networks overlap.¹² For instance, if families attending a given school deliver contextual effects at the school level (e.g. they provide public goods) but peer effects do not cross classrooms or grade levels, then the utility parameters are identified. This condition cannot arise when one assume that $A = C$, and illustrates how the existence of richer peer and contextual social structures can facilitate rather than hamper identification.

Taken as whole, theorem 3 and corollaries 1 and 2 show, in our judgment, that concerns about non-identification with *a priori* knowledge of A and C are misplaced. Of course, our genericity finding in corollary 1 does not mean that a given model is identified. Rather, it says that if the utility parameters are not identified, then the researcher's choice of A and C is a very special case relative to the set of matrices that are consistent with the behavioral model we have described. It is always possible that a researcher has a principled reason for choosing sociomatrices under which identification fails. Our message is simply that such a reason needs to be present to conclude that the presence of social effects in preferences cannot be uncovered by the data.

Corollaries 1 and 2 are instances of case *i.* of theorem 3, and comparing them to previous work on identification and the linear-in-means model highlights the fragility of the reflection problem. Previous work has already produced variations of the linear-in-means model in which, unlike Manski's formulation, identification holds. Lee (2007), Lee, Liu, and Lin (2010), Bramoullé, Djebbari, and Fortin (2009) and Davezies, d'Haultfoeuille, and Fougère (2009) provide positive identification results based on (9), the exact linear-in-means model as opposed to the large sample approximation. Bramoullé, Djebbari, and Fortin (2009) and

¹²The exception is when the components of the peer-effects network are unions of components of the contextual-effects network.

Davezies, d'Haultfoeuille, and Fougère (2009) study a version of Lee (2007), and find that if there are at least 3 groups of different size, identification holds for the exact model. Further, Bramoullé, Djebbari, and Fortin (2009) and Blume et al. (2011) show that the Manski non-identification result will hold if, contrary to our theoretical reasoning, a_{ii} is non-zero, which in the linear-in-means case implies that if each agent reacts to an unweighted average of the expected choice, the reflection problem reemerges even if groups are finite – a conclusion that was anticipated in Moffitt (2001).

The following corollary is a mild generalization of previous results and is an instance of case *ii* of theorem 3.

Corollary 3. *Suppose there is a component of the contextual-effects sociomatrix C such that all c_{ij} are equal and assume that any component of the peer-effects network is either a subset of or disjoint from the contextual-effects component. Suppose also that there exist two pairs of individuals $i \neq j$ and $k \neq l$ in the contextual-effects component such that $a_{ij} \neq a_{kl}$. Then γ , δ and ϕ are identified.*

This corollary says that if there is a subset of the network such that the linear-in-means assumption applies with respect to contextual effects and components associated with peer effects are either subsets or disjoint from this component, then identification will hold if two distinct pairs of individuals have different peer-effects sociomatrix values. This illustrates how fragile the reflection problem is. In a classroom setting, for instance, if one takes the peer- and contextual-effects components to be the entire classroom as the linear-in-means model does, then a *single* inequality in A involving non-overlapping pairs of students is sufficient for identification.

4.2. Aggregate Data. Individual-level data on social networks are often unavailable, or are incomplete due to sampling gaps. However, aggregate statistics are widely available (e.g.: average standardized test scores at the school-level, city-level crime incidence, county-level unemployment rates, etc.). One approach that takes advantage of such data, originating with Glaeser, Sacerdote, and Scheinkman (1996, 2003) and later extended by Graham (2008), focuses on the informational content of cross-sectional data on group level averages.

To see how such data can be related to the linear social interactions model we have developed, consider data drawn from $G + 1$ non-overlapping groups numbered $g = 0, \dots, G$. Each group g contains n^g members. We assume that the primitive utility parameters γ , δ and ϕ are constant across the groups in order to render the use of aggregate data interpretable, but that sociomatrices are group-specific, so each group g is associated with a distinct set of sociomatrices A^g and C^g . For many contexts, heterogeneity in social structure seems natural across groups, even when populations are of the same size. One example is school classrooms, where one would naturally expect different social structures, even for classrooms of a given size.

Denote by ω_i^g, x_i^g and ε_i^g the outcomes, observed characteristics, and unobserved characteristics, respectively, of individual i in group g and let $\bar{\omega}^g, \bar{x}^g$ and $\bar{\varepsilon}^g$ be the group-level averages of these variables. The model is such that assumptions T.1–T.2, E.1–E.5 hold at the group level. In addition, we assume:

- K. 1'**. For all g , A^g and C^g are exogenous and known to the analyst *a priori*.
K. 2'. For all g , the analyst observes $(\bar{\omega}^g, \bar{x}^g)$.

Axioms K.1' establishes what, in this section, is assumed to be the analyst's *a priori* knowledge. Axiom K.2' modifies K.2 so that the analyst only observes group-level averages of x and ω . For each group, let $(\omega^g, x^g, \varepsilon^g)$ denote the vectors of outcomes and observed and unobserved group characteristics for group g , respectively, and let \tilde{n}^g denote the vector each of whose elements is $1/n^g$. Given equations (5) and (4), $\bar{\omega}^g, \bar{x}^g$, and $\bar{\varepsilon}^g$ are related to the individual-level variables by

$$\bar{\omega}^g \equiv \tilde{n}^g \omega^g = \tilde{n}^g \mu^\varepsilon + \tilde{n}^g B_\phi^g (\gamma x^g + \delta C^g x^g) + \frac{1}{1 + \phi} \tilde{n}^g \varepsilon^g \quad (14)$$

A natural starting point for many empiricists would be to estimate a linear-in-means model, which amounts to imposing the restriction on (14) that the rows and columns of A^g and C^g each sum to 1. This yields

$$\bar{\omega}^g = \mu^\varepsilon + (\gamma + \delta) \bar{x}^g + \frac{1}{1 + \phi} \bar{\varepsilon}^g \quad (15)$$

It is easy to see from this equation that separate identification of the structural parameters from the joint distribution of $\bar{\omega}^g$ and \bar{x}^g is not possible: δ and γ enter the joint distribution only through the sum, and ϕ cannot be untangled from the variance of $\bar{\omega}$ under our current assumptions.

While the first moments do not permit identification, the key insight of Glaeser, Sacerdote, and Scheinkman and Graham is that, under further assumptions, second moments may. Glaeser, Sacerdote, and Scheinkman pointed out that conditional on \bar{x}^g , variation in $\bar{\omega}^g$ is consistent with the variation that would be predicted in averaging i.i.d. random variables. Their argument, which is heuristic, is that $\text{Var}(\bar{\omega}^g | \bar{x}^g)$ will reveal social interactions by comparing the sample variances for different group sizes to one in which $\omega_i - E(\omega_i | \bar{x}^g)$ is i.i.d. within and across groups.

Following Glaeser, Sacerdote, and Scheinkman (1996) and the case in Graham (2008) where group level effects are absent, we add the following additional constraint on the model in this section.¹³

- E. 6.** For all i, j x_i^g and x_j^g are i.i.d. and ε_i^g and ε_j^g are i.i.d.

¹³Graham (2008) has additional identification results in the case where random group effects are present.

This is a stronger i.i.d. assumption than we have had so far, namely that for all i, j, x_i^g and ε_i^g are i.i.d. Together with E.6, equation (14) yields:¹⁴

$$\text{Var}(\bar{\omega}^g) = \frac{1}{(n^g)^2} \sum_{j \in g} \left(\sum_{i \in g} B_{ij}^g \right)^2 \sigma_x^2 + \left(\frac{1}{1 + \phi} \right)^2 \frac{1}{n^g} \sigma_\varepsilon^2 \quad (16)$$

Setting one group, $g = 0$, as the baseline, define the following statistic for the remaining groups $g = 1, \dots, G$, where $\text{Var}(x) = \sigma_x^2$ is observed

$$v^g = \frac{n^g \text{Var}(\bar{\omega}^g)}{\sigma_x^2} - \frac{n^0 \text{Var}(\bar{\omega}^0)}{\sigma_x^2}$$

The following theorem shows that v^g can be used for identification. Define M_A^g and M_C^g to be the set of peer- and contextual-effects sociomatrices for group g ; that is, they satisfy the matrix assumptions T.1, E.2 and E.3.

Theorem 4. *Assume T.1–T.2, E.1–E.6 and K.1'–K.2'. Suppose δ, ϕ , and $\beta = (\gamma + \delta) \neq 0$. Suppose there are groups $0, \dots, G$, that $G \geq 5$, and that for each group $g = 1, \dots, 5$, $n^g \geq 3$ and A^g is not bistochastic. Then the set J of matrices (C^1, \dots, C^5) such that v^1, \dots, v^5 does not identify the utility parameters γ, δ and ϕ is a closed lower-dimensional subset of $\prod_{g=1}^5 M_C^g$.*

Theorem 4 says when data are in the form of group averages, second moments can be used to identify the utility parameters provided that the (A^g, C^g) pairs fulfill a condition on C^g that, given A^g , holds generically. In addition, one needs a certain degree of variation across the sociomatrices to allow for the different groups to provide distinct second moments from which the utility parameters can be backed out.

The theorem builds on Graham (2008), who explores the case in which peer effects are effectively absent (for all $g, A^g = 0$), and contextual effects are characterized by a linear-in-means structure where for all $g, c_{ii} = 0$ (all of which are allowed by T.1–T.2 and E.1–E.5). The linear-in-means assumption reduces the number of required groups with distinct C^g 's to 3 relative to the 5 in theorem 4. Our result indicates that the logic of Graham's analysis extends beyond the linear-in-means model. It does not require the absence of a peer effect. Neither does it require different group sizes, which in his analysis generates the necessary variation for identification. However, in our formulation, this comes at a cost: data loss in moving from individual to group-average observations necessitates *a priori* information on characteristic covariances. Although we conjecture that similar

¹⁴Equation (16) follows directly from

$$\bar{\omega}^g = \mu^\varepsilon + \frac{1}{n^g} \sum_{j \in g} \left(\sum_{i \in g} B_{ij}^g \right) x_j^g + \frac{1}{1 + \phi} \frac{1}{n^g} \sum_{i \in g} (\varepsilon_i^g - \mu^\varepsilon).$$

statistics can be constructed with more complicated covariance structures, here, we have assumed independence.

4.3. Mixed individual and aggregate data. We conclude this section by considering linear social interactions models that are based on a combination of individual-level and aggregate data. A number of studies, including many in the important first generation of empirical social interactions research, combine individual-data from the Panel Study of Income Dynamics (PSID) with averages of individual outcomes measured at the Zip Code level (Datcher 1982, Corcoran et al. 1992) or the census-tract level (Sharkey and Elwert 2011). This combination exploits the geographic identifiers in the PSID to construct measures that are interpreted as social influences. The sampling scheme for the PSID, when combined with aggregate information, produces regressions of the form

$$\omega_i = b_0 + b_1 x_i + b_2 \bar{x}^g + \eta_i \quad (17)$$

where g denotes the relevant level of aggregation. This regression, to be interpretable as an equilibrium strategy profile, implies assumptions of the form (8). Since the sampling scheme we describe provides no information on A^g and C^g , this equation represents an information reduction relative to the row describing ω_i in equation (6), which we have already showed is not identified when these matrices are unknown. Relative to the ω_i row found in equation (6), equation (17) represents a misspecified regression so the parameters in (17) will depend on the underlying parameters $\gamma, \delta, \phi, A^g$ and C^g . The one positive use of (17) is that if $b_2 = 0$, then neither peer nor contextual effects are present in the preferences of agents.

5. IDENTIFICATION WITH PARTIAL INFORMATION ON SOCIOMATRICES

Theorem 2 states that without prior knowledge on the sociomatrices beyond what is necessary for the existence of a Bayes-Nash equilibrium in the quadratic-payoff game, there is little that can be learned about the preference parameters which constitute the primitives of the behavioral model. Section 4 explored the polar opposite case that is employed in most empirical applications, namely identification when these matrices are (assumed to be) known. We now explore the degree to which parameters can be identified with only partial knowledge of social interactions.

There are many ways in which one can model partial knowledge of A and C . Two forms of partial knowledge are, in our view, particularly salient. First, the analyst may have *a priori* knowledge about C without any *a priori* knowledge about A beyond T.1 and E.2. This is a natural case to consider because peer effects embodied in A represent a primitive psychological proclivity to behave similarly to others, for which theory provides no guidance. Such guidance may, however, exist for C . Classrooms provide a simple example. If students supply some goods

which are partially public, for example musical instruments, then average parental income may be plausibly assumed to determine the level of such goods, which constitute a contextual effect in our model. Alternatively, data sets exist in which parental involvement in a classroom is measured (e.g. Bassani (2008) and Sui-Chu and Willms (1996)). The total level of parental involvement can represent a public good analogous to the musical instrument expenditures example. If social units produce public goods, the decision mechanism will implicitly define contextual effects; this occurs in Calabrese et al. (2006).

A second type of partial knowledge of sociomatrices may come from data sets in which individuals are asked to identify those to whom they are connected. These data sets, leaving aside imperfections such as limits on the number of friends that can be named, represent cases in which the analyst has information about connections between individuals but not the sociomatrices themselves. In such contexts, a researcher needs to make a judgment as to the interpretation of the data on direct connections in term of the sociomatrices. Knowledge of the presence or absence of ties between individuals in the network creates a close parallel between identification of social interaction parameters and classical results on the identification of simultaneous equations systems, since holes in the network (i.e. the absence of edges) in essence provide exclusion restrictions that can be exploited.

Throughout subsection 5.1 we impose two additional constraint on models.

- E. 6. $-\gamma/\delta$ is not an eigenvalue of C .
- E. 7. $\phi > 0$.

These constraints differ from those in section 4 since they do not pertain to objects observed *a priori*. Assumption E.6 ensures that B is non-singular. Assumption E.7 is largely for convenience. If $\phi = 0$, then the identified matrix B is a linear combination of I and C . The converse is true if I , A , C and AC are independent. This is generically true, but we would like to do better since A is unobservable. Results such as corollary 2 show that this can be guaranteed with modest additional *a priori* information about the peer-effects social network. Such is the case, for example, in theorem 5 below.

5.1. Unknown peer-effects sociomatrix A . Peer networks are notoriously hard to measure, and so here we investigate identification when the contextual-effects sociomatrix C is known, but the analyst has either partial information, or none at all, on the peer-effects sociomatrix A . For the case in which the researcher knows C and the topology of the peer-effects network, one can establish identification under weak conditions, as seen in theorem 5.

Theorem 5. *Assume that contextual-effects sociomatrix C is known a priori. Assume too that the peer-effects sociomatrix A is unknown but the peer-effects network is known a priori. Suppose that $N \geq 3$. Suppose that there are two distinct*

individuals j and i who are known to be unconnected in the peer-effects network. If $B_{ji}^{-1} \neq 0$, then the utility parameters γ , δ and ϕ are identified from the conditional mean of ω given x .

Theorem 5 demonstrates that even when the peer-effects sociomatrix A is unknown, all three utility parameters can still be separately identified from first moments under certain conditions. The theorem presents an instance where the analyst has partial knowledge of A . Even without knowledge of A , however, identification is possible. Denote by $\mathcal{M}'(C)$ the set of models $\mathcal{M}(C)$ satisfying, T.1–T.2 and E.1–E.7. Theorem 6 states that the phenomena of which theorem 5 is a specific instance, is generic.

Theorem 6. *Assume T.1–T.2 and E.1–E.7. Suppose only the contextual-effects sociomatrix C is known.*

- i. *If $N \geq 3$, then for each $C \in M_C$ there is a generic subset $S_A \subset M_A$ such that if $A \in S_A$, then γ , δ and ϕ are also identified in \mathcal{M}'_C by the conditional mean of outcomes given characteristics.*
- ii. *If E.7 is relaxed by assuming that $\phi = 0$, then result i. holds for all but a generic subset of \mathcal{M}'_C .*

The theorem specifies conditions under which there exist two distinct (i.e. non-identical) diagonal elements of B^{-1} , which as the proof establishes, is sufficient for identification of all utility parameters. Indeed, under these conditions, the peer-effects sociomatrix is also identified. While this may seem surprising, the intuition can be understood by considering equation (4). The dimension of the set of peer-effects matrices is $N(N - 2)$. The dimension of the set of all $B(s)$ matrices that come from a (known) common contextual-effects matrix C^* is no more than $N(N - 1) + 1$, but we can show it to be no less than $N(N - 2)$. We need to recover $N(N - 2) + 3$ parameters from $B(s)$. Corollary 4 states that this recovery is possible.¹⁵

Corollary 4. *Under the conditions of theorem 6, the peer-effects sociomatrix A is identified.*

5.2. Identification with *a priori* qualitative network knowledge. As noted in section 2, data sets with network data, such as AddHealth do not furnish the sociomatrices A and C : survey respondents indicate to whom they are connected, but not the weights. We can interpret such data as providing information about exclusion restrictions, the location of 0's in the A and C matrices. No survey we know of distinguishes between peer- and contextual-effects networks. We suggest that data collection, even if measures of interaction intensity cannot be constructed, allow for distinct sociomatrices.

¹⁵Drton, Foygel, and Sullivant (2011) provide similar results to ours on the recovery of A . However, that paper assumes the unobserved heterogeneity is jointly normal and it assumes that contextual effects are absent.

How can such knowledge facilitate identification? In this section we show that if the networks are sufficiently sparse, there is a path to identification which is analogous to classical simultaneous equations results. Without loss of generality, assume that $\mu = 0$, since the more general case involves constant terms that are irrelevant to the identification of parameters of interest. The matrix B , which characterizes the equilibrium strategy profiles, is identified by theorem 2, and this identifies the sum $\gamma + \delta$.

Let b_i denote the i th row of B , and similarly for A and C . Consider the i th equation of the system

$$\omega_i = \frac{\gamma + \delta c_{ii}}{1 + \phi} x_i + \frac{\delta}{1 + \phi} \sum_{j \sim_C i} c_{ij} x_j + \frac{\phi}{1 + \phi} \sum_{j \sim_A i} a_{ij} b_j \cdot x + \frac{1}{1 + \phi} \varepsilon_i \quad (18)$$

$$= \pi_1 x_i + \sum_{\substack{j \sim_C i \\ j \neq i}} \pi_{2j} x_j + \sum_{j \sim_A i} \pi_{3j} b_j \cdot x + \frac{1}{1 + \phi} \varepsilon_i \quad (19)$$

These equations suggest regressing ω_i on the the vectors x and Bx , and the following theorem describes what can be learned from this procedure. Let B_{A-C} denote the submatrix of B whose rows correspond to individuals connected to i through the peer-effects network and whose columns correspond to individuals not connected to i through the contextual-effects network. Recall that $i \sim_A j$ indicates that there is an edge between i and j in the peer-effects network, and $i \sim_C j$ that an edge exists in the contextual-effects network.

Theorem 7. *Assume T.1–T.2 and E.1–E.4. Assume that for some i , B_{A-C} has full row rank equal to the number of individuals j connected in the peer-effects network to i , denoted as $\#\{j : j \sim_A i\}$. Then*

- i. *The weights a_{ij} for $j \sim_A i$ are identified.*
- ii. *ϕ is identified.*
- iii. *If there is a $j \neq i$ such that $i \sim_C j$, then the point (γ, δ) lies on the line $\gamma + \delta = \beta$ at the point $(\gamma, \delta) = (1 - c_{ii})^{-1}(\beta - \pi_1, \pi_1 - \beta c_{ii})$, for $0 \leq c_{ii} < 1$.*
- iv. *If there is a $w \neq i$ such that $i \sim_C j$ and $\pi_{2j} = 0$ for some $j \sim_C i$, then this is true for all $j \sim_C i$, $\delta = 0$ and $\gamma = \pi_1$.*
- v. *If $k, j \sim_C i$ then c_{ik}/c_{ij} is identified.*

If C is known a priori then a_i (the i th row of A) and all utility parameters are identified. A necessary condition for these conclusions is that $1 + \#\{j : j \sim_C i, j \neq i\} + \#\{j : j \sim_A i\} \leq N$.

Theorem 7 says that under certain conditions, holes in the network can enable identification: they serve as exclusion restrictions, which can be exploited to back out the utility parameters. Theorem 7's necessary condition is an order condition, and the sufficient condition is a rank condition (see Hsiao (1983, Theorem 3.3.1) for a clear exposition). One way to understand this theorem is to recall that under our assumptions, $Bx = E(\omega|x)$. Thus the regression of equation (19) looks

like the second stage of a two-stage least-squares estimation procedure. The differences between the classical results and ours lay in the fact that in the classical case identification of the second-stage parameter estimates is not an issue, and the rank and order conditions have to do with backing structural parameter estimates out of the second-stage estimates. Here, in contrast, the issue is first identifying the parameters π_i , then using these to recover structural parameters. The utility of part *iii.* of the theorem is that information about c_{ii} may partially identify γ and δ . In particular, if c_{ii} is known *a priori*, then γ and δ are identified. Finally, note that typically $i \sim_C i$, and in this case the necessary condition is that the total number of edges emanating from i in either network not exceed N .

The results in sections 5.1 and 5.2 have important implications in terms of the interpretation of surveys that measure social networks. In terms of interpretation, Theorems 5–7 demonstrate the importance of network structure in generating identification. The key conditions across our results is *a priori* knowledge of 0's in the sociomatrices. Survey data on social networks do not provide information on the intensity of bilateral interactions. Rather they provide information on whether or not a bilateral interaction is present. Our emphasis on the importance of “holes” in the social structure extends the argument in Bramoullé, Djebbari, and Fortin (2009) that 0's in a known sociomatrix allow for instruments. Our results show that these 0's can facilitate identification even when, unlike Bramoullé, Djebbari and Fortin, the sociomatrices are unknown.

Our results also suggest a potentially serious limitation in current surveys, specifically Addhealth, which is arguably the most popular data set for the study of social network effects. Its main draw is that high school students in its nationally representative sample are interviewed not only about the usual demographic and outcome variables of interest, but also about who their friends are. Unfortunately, the data set's friendship questions are restricted in that each student is allowed to name up to 5 friends of each gender. This has important ramifications in view of the result in theorem 5, which indicates that it is more useful to know who is *not* someone's friend rather than who *is*. Moreover, the restriction on the number of friends means that the failure to identify someone as a friend does not mean that there is a corresponding zero in the associated sociomatrices. While the limitation on the number of friends that could be named in the interviews has long been understood as inducing measurement error in network structure, as far as we know, the effects of this limitation on identification per se have not been recognized.¹⁶

Our results provide an important generalization of Lee, Liu, and Lin (2010) as we do not need to assume that each agent equally weights others to whom he is directly connected. Lee, Liu, and Lin assume that the sociomatrices are functions of the common adjacency matrix, which clearly does not need to be the case.

¹⁶Another concern is that the failure to identify someone as a friend is consistent with a negative entry in one or both of the sociomatrices we have employed. While we have assumed that all elements of A and C are non-negative (axiom T.1) negative values are certainly empirically plausible. We thank Jesse Naidoo for this observation

5.3. Identification with aggregated social network data. We conclude this section with an analysis of a different type of partial knowledge, namely partial knowledge that reflects the absence of individual level data with which to evaluate social effects. Section 5.2 provided some positive results on inference of structural parameters when social interaction effects data are aggregated. Here we show that these effects disappear when individuals are sampled across groups and paired with group level averages. For data sets employing the PSID, for example, it is common to see models in which individual outcomes are assumed to depend on individual characteristics and certain census tract aggregates. We provide a link to this type of empirical analysis by considering the case where data are of the form $\omega_i, x_i, \bar{\omega}^g, \bar{x}^g$, where $\bar{\omega}^g$ and \bar{x}^g denote group level average outcomes and characteristics, respectively, of i 's group g . We make two knowledge assumptions. First, we assume that C is known because otherwise information would be lost relative to theorem 2 and identification would obviously fail. Second, the analyst only observes one individual per group, whom we denote as 1.

- K. 1''.** C is exogenous and known to the analyst.
- K. 2''.** For all g , the analyst observes $(\bar{\omega}^g, \bar{x}^g, \omega_1, x_1)$.

Finally, we place a restriction on the nature of observed heterogeneity, namely, that it is i.i.d. across members of the same group.

- E. 8.** For each g , x_i is i.i.d. within g .

For each individual i in the sample we observe that individual's record and his group averages. The presumption is that the individual's social network is confined to the group. The individual can be netted out of the group average, so from equation (5) we derive two relationships: one for the behavior of everyone but individual i , and one for the behavior of individual i .

$$\begin{aligned} E(\bar{\omega}^g | x_i, \bar{x}^g) &= \mu^g + b^g E(x | x_i, \bar{x}^g) + b_{gi} x_i \\ E(\omega_i | x_i, \bar{x}^g) &= \mu_i + b_{-i} E(x | x_i, \bar{x}^g) + b_{ii} x_i \end{aligned} \quad (20)$$

where the bars denote group averages exclusive of individual i , and variables with a i subscript refer to individual i . The coefficients $b_{gi} = \sum_{k \in g, k \neq i} b_{ki}$, $b^g = (1/(N-1)) \sum_{j, k \in g, k \neq i} B_{kj}$, and $b_{-i} = \sum_{j \in g, j \neq i} b_{ij}$ are all sums of terms in the matrix B .

Theorem 8. *Assume T.1–T.2, E.1–E.4 and E.8, and suppose that $E(\bar{\omega}^g | x_i)$ and $E(\omega_i | x_i, \bar{x}^g)$ are known. Then $\beta = \gamma + \delta$ is identified, and γ, δ, ϕ are not identified.*

Theorem 8 shows that the assumption of a linear-in-means structure entails too great a loss of information to allow for identification of the utility parameters. As in other cases, if the projection of ω_i onto x_i and x_{-i} differs from the projection of ω_i onto x_i , then all one can say is that some sort of social interaction is present.

This is a cautionary message given the ubiquity of these models in empirical practice. Our results provide a complement to Davezies, d'Haultfoeuille, and Fougère (2009) who consider the problem of identification for a linear-in-means model in which the analyst does not have data on the group aggregate variables, but does know the group sizes. Identification is shown to hold when there are groups of at least 3 distinct sizes. Our relatively negative result stems from the heterogeneity in sociomatrices across groups. This precludes our use of that paper's approach, whereby observed means of others can be treated as mismeasured true means. For our context, the mismeasurement involves loss of information on the weights of the sociomatrix as well as the values of x and ω .

6. ENDOGENEITY OF SOCIAL STRUCTURE

In this section we explore the consequences of network endogeneity. We first discuss the self-selection problem. Second, we examine how durable networks, in which unobserved (to the analyst) determinants of social interactions arise after the network is formed, create a distinct identification issue.

6.1. The selection problem. A standard concern in uncovering social interactions is the endogeneity of the social structure. The issue is straightforward: does a correlation between high ability friends and an individual student's educational performance reflect a social interaction of the type we have modeled or does it occur because the student's unobserved type is correlated with his friendship choices? This concern has generated interest in randomized assignment to groups, as in Sacerdote (2001), as well as cases in which a "natural experiment" alters group composition, e.g. Cipollone and Rosolia (2007).¹⁷ A focus on data in which exogenous social structure is present delimits the domain of environments that may be studied, so it is important to understand how endogeneity should be understood and accounted for in more general settings.

A natural way to extend our model of social interactions to network formation is to formulate a two-stage game, in which networks are formed in the first stage and actions are determined in the second. For each possible network there is a unique second-stage equilibrium, and each individual's expected utility of this second-stage equilibrium is a value function for the network which gives payoffs for the first-stage game.

While this abstract conceptualization is useful in understanding the implications of endogeneity, it is not one that can be directly implemented in the context of an econometric model of network formation and subsequent choices. The reason for this is that there simply does not exist a viable general theoretical model of network formation. Networks for business relations, job search and classroom

¹⁷Blume et al. (2011) discuss how quasi-experiments may not satisfactorily resolve self-selection problems in identifying social interactions.

friendships are formed according to very different rules, and vary greatly in the degree to which they are instrumental for the second-stage game. While network formation games have been formulated for particular contexts, they do not even include pair-specific weights in the decision process.

An alternative approach is to imagine conditions that should be properties of equilibrium outcomes for many different games. This path, first travelled by Gale and Shapley (1962), leads to network stability concepts such as pairwise stability (Jackson and Wolinsky 1996) and pairwise-Nash stability (Calvó-Amrengol and Ilkiliç 2009). A network is pairwise-Nash stable if and only if a) no individual wants to drop any edges, and b) there is no missing edge that if added would, *ceteris paribus*, be a Pareto improvement for the individuals it connects. It is neither a strictly cooperative nor a strictly non-cooperative concept. Stability expresses the idea that breaking relations is a non-cooperative activity while forming new relations involves mutual consent.

While this approach has been employed in a few recent studies¹⁸ it is not a panacea. The basic problems are threefold. First, stable networks may not exist. Non-existence, however, can be circumvented by introducing random stable networks, that is, probability distributions on graphs which satisfy an expectation based concept of stability — that is, one can imagine a probability distribution on graphs for which the inequalities in the stability definition are satisfied in expectation. Existence can easily be shown in two cases: if ε is observed only just prior to the second stage or the support of the marginal distribution of ε is finite.¹⁹ In the first case, selection is not an issue because private types are not observed until after the network is formed. In the second case, discreteness of the set of possible z 's rules out many common econometric models. It is quite possible, however, that an existence proof can be provided for more general classes of models.

A second problem for both pairwise-stable and random pairwise-stable networks is that factors other than the utility of second-stage choices may play a role in determining the utility of a given network. The sociology literature is replete with descriptions of such payoffs. For instance, there might be an independent value to homophily — associating with people similar to oneself — which is distinct from

¹⁸Badev (2013) studies the coevolution of friendship networks and smoking behaviors in an environment where agents make myopic friendship decisions among $k - 1$ randomly selected others. This model is shown to converge to a k -stable Nash network, which means that no agent wishes to deviate by simultaneously altering $k - 1$ friendship statuses as well as his choice. Sheng (2012) uses pairwise stability in the context of identification of a network formation game but omits choices that are affected by network structure. Hsieh and Lee (2012) follow a strategy related to ours but with perfect information, considering a two-stage game in which social structure is formed in stage 1 and choices are made in stage 2.

¹⁹For both cases, consider Myerson's (1991) network formation game. In the first case, the first-stage game is a complete-information game, and a correlated equilibrium will satisfy the needed inequalities. In the second case, a perfect direct correlated equilibrium (Dhillon and Mertens 1996) of Myerson's (1991) network formation game is a pairwise-Nash stable random graph, and since Myerson's game is finite, these equilibria exist.

the value of the game outcome. Structural estimation of these models, then, involves specifying these additional factors. This requirement may be impossible to realize.

A third problem is that the set of pairwise stable random graphs will typically not be a singleton. Thus partial-identification techniques will come into play, and it may be that the set of pairwise stable random graphs is too large to impose useful first-stage restrictions.

For these reasons, we believe it makes more sense to address endogeneity by considering its effects on inference from data on the second stage of the game. This involves returning to our model and asking how endogeneity can invalidate our assumptions. From this vantage point, the implications of endogeneity depend on the information available to agents when networks form.

If either the public types or the private types relevant for the second-stage choice are not observed at the time the network is formed, then the missing variable cannot enter into the first-stage interim payoff functions. In this case, the linear structure of the second stage is maintained and endogeneity is not an issue. By contrast, suppose that x and z_i are available to agent i at the outset of the first stage. The expected second state payoff will depend upon both of these variables, and so both will influence individuals' first-stage choices. Consequently, an individual i , observing that he is connected to j , can, with knowledge of x_j , make an inference about the value of z_j that is dependent on x_j . Thus E.4 is violated. In this case $\mu(x, z)$ is not independent of x , and second-stage equilibrium strategy profiles are no longer linear in x , except for special cases. (They are, however, still described by theorem 1.) This is the selection problem. It is not just a statistical issue. It affects the basic structure of equilibrium, because it affects inference not only of the econometrician but of individuals constructing the network.

How can one proceed? From the perspective of the reduced form model (5) coefficients B , the only effect that endogeneity can have under the information regime we have described is through $E(\varepsilon|x) = E(\mu(x, z)|x)$. This expression is in fact nothing more than Heckman's classic control function (e.g. Heckman (1979), Heckman and Robb (1986)). So long as $\mu(x, z)$ does not depend linearly on x , identification will still hold. To be clear, the robustness of identification to endogenous network formation exploits the quadratic game structure that leads to linear equilibrium strategy profiles. But this is true for general control function approaches; they break down when $E(\varepsilon|x)$ is linear in x . Hence Heckman's fundamental idea that self-selection can be addressed by incorporating self-selection into the analysis, rather than using instrumental variables, applies to social interactions contexts.²⁰

²⁰The idea that selection on unobservables can aid in identification of social effects via control functions was first shown in Brock and Durlauf (2001). Brock and Durlauf (2006) provide a more general treatment when agents select into non-overlapping groups and the sociomatrix weights are required to be equal as occurs for the linear-in-means model. Our current discussion makes two

Where would instrumental variables approaches come into play in this setting? Suppose that the researcher has available a vector of observable individual attributes v . From the vantage point of this two-stage game, the critical question involves the timing by which this information is revealed. If agents observe v by the outset of the second stage, then endogenous network formation means that one needs to analyze $\mathbb{E}(z_i|x, v)$. But this means that v no longer constitutes an instrument, since it is correlated with the errors in the regressions that emerge in the second stage of the game. In this sense, the pro forma use of instruments on the grounds that they are associated with the payoffs of network formation and not behaviors conditional on the network is invalid. Once one introduces instruments to account for network heterogeneity, one needs to account for their implications for the second stage regression errors, which will, outside of special cases, be present even if the payoff in the second stage is independent of the instrument.

6.2. Durable Networks. The preceding discussion took the point of view that network formation transpired prior to, and only for the purpose of, second-stage strategic interaction. An alternative (and sociologically richer) view is that a given social network serves many purposes, and co-evolves continuously with the behaviors it facilitates. We submit that for many of the subjects to which social interaction models have been applied, this is more descriptively accurate than the preceding two-stage game. Suppose, then, that social networks are durable informal institutions, and that relations are ongoing. It is reasonable to suppose that as a result of this process, variables that are unobserved at the time of network formation are revealed over time to network members and generate social interactions, but nonetheless may remain invisible to the researcher. Such variables may include things like generalized ability, or personality traits that the researcher cannot measure but may nonetheless theorize about. These variables – public to network members but unobservable to the analyst – leave traces in the correlation of individuals' choices. In our view, the issue of unobserved carriers of social interactions represents the complement to the standard concern about self-selection that is the focus of Section 6.1. Endogeneity of networks, we argue, can occur with respect to one set of variables, while social interactions occur with respect to another. Endogeneity, from this vantage point, alters the relationship between the variables relevant to the second stage of the game and the variables relevant to the first. In this subsection we investigate what can be learned about utility parameters for such variables from second moments.

We suppose that there is a characteristic u which is unobservable to the econometrician, but observable to individuals in the networks. We amend the theory

important extensions of this earlier work. First, an explicit game for the sequential formation of social networks and the subsequent choices of actors in the network are described. Second, the analysis indicates that the control function approach applies to a much wider class of environments than had previously been established.

accordingly, so the utility function is now²¹

$$U_i(\omega_i, \omega_{-i}) = \left(\gamma x_i + \delta \sum_j c_{ij} x_j + \gamma_u u_i + \delta_u \sum_j c_{ij} u_j + z_i \right) \omega_i - \frac{1}{2} \omega_i^2 - \frac{\phi}{2} \left(\omega_i - \sum_j a_{ij} \omega_j \right)^2 \quad (21)$$

The equilibrium analysis is unchanged, and the equilibrium strategy profile is

$$f(x, u, z) = \frac{1}{1+\phi} \left(I - \frac{\phi}{1+\phi} A \right)^{-1} (\gamma I + \delta C) x + \frac{1}{1+\phi} \left(I - \frac{\phi}{1+\phi} A \right)^{-1} (\gamma_u I + \delta_u C) u + \mu(x, u, z) + \frac{1}{1+\phi} z \quad (22)$$

where $\mu(x, u, z)$ is derived in a fashion analogous to $\mu(x, z)$. We make the following assumptions about observations:

- K. 1'''**. A and C are exogenous and known to the analyst *a priori*.
- K. 2'''**. For all i , the analyst observes (ω_i, x_i) , but not u_i .

Under these assumptions, the econometric model becomes

$$\omega = \mu + B(s)x + B_u(s)u + \varepsilon \quad (5')$$

where

$$B_u(s) = B_\phi(s)(\gamma_u I + \delta_u C).$$

The unobserved component ε is now the entire second line of equation (22).

We ask two questions in this framework: when are the utility parameters γ , δ and ϕ identified from $B(s)$, and when are the utility parameters corresponding to the unobserved public variables γ_u and δ_u identified? The answer to the first question is straightforward: if (u, z) is independent of x , then $\mu(x, u, z)$ will be independent of x , and so all the theorems above not requiring independent errors still apply.

More interesting is the second question, what can be identified concerning utility parameters for the unobservable public type. While social interactions effects pertaining to unobservables have yet to be systematically investigated, there are good reasons to think that such effects are of interest. For example, in the modern work on cognitive and noncognitive skills, e.g. Cunha and Heckman (2007) and Cunha, Heckman, and Schennach (2010), these skills are explicitly interpreted as latent variables in a measurement system. It is natural to think that the skills of others affect a student in a classroom, for example. An important extension of our

²¹We have assumed the same sociomatrix describes contextual effects for both x and u . The cost of relaxing this assumption is that the econometrician is required to know *a priori* both sociomatrices.

result would involve an explicit consideration of unobservables when a measurement system exists that relates the unobservables to observables such as test scores. Identification will have to come from the covariance matrix, and this will require strong independence hypotheses. We extend our econometric assumptions as follows:

E. 4'. For all $k, i, j \in V$, x_k , u_i and ε_j are pairwise independent.

It follows from E.4' that equation (2a) still holds, and that the u_i will be independent. We have the following result:

Theorem 9. *Suppose that the ε_i are i.i.d. with variance σ_ε^2 , that the u_i are i.i.d. with variance σ_u^2 , and that assumptions T.1–T.2, E.1–E.3, E.4', E.5, and K.1'''–K.2''' . For generic pairs (A, C) of peer- and contextual-effects sociomatrices, the following are identified from the covariance matrix of ω : γ , δ , ϕ , and σ_ε^2 , and the sets $\pm(1/\sigma_u^2)(\gamma_u, \delta_u)$. In particular, the sets $\gamma_u = 0$ and $\delta_u = 0$, and the ratio γ_u/δ_u (when it is defined) are identified.*

The parameters γ_u and δ_u are identified only up to a non-zero multiple, $\pm\sigma_\varepsilon^2$, but this nonetheless identifies the ratio γ_u/δ_u . Identifying this ratio is useful because we learn the relative significance of own versus contextual effects. Identification up to a sign is also interesting because we may have *a priori* information that signs one of the terms. For instance, in studies of earnings in workplace networks, ability has both an own effect and a contextual effect. While the sign of the own effect may be obvious, the sign of the contextual effect will depend upon whether workers and worker tasks are substitutes or complements in the production process.

It is possible to give conditions on edges in the network that imply the u_i -independence condition, just as was done for the ε_i -independence condition in section 4. We also conjecture that one can go further in identifying the magnitudes of γ_u and δ_u . Let T denote the transpose of a matrix. Since $\phi/(1+\phi)$ and σ_ε^2 are identified, the econometrician can observe

$$(1+\phi)^2 B_u u \cdot (B_u u)^T = \sigma_u^2 (I - \phi(1+\phi)^{-1}A)^{-1} (\gamma_u I + \delta_u C) \\ \cdot (\gamma_u I + \delta_u C^T) (I - \phi(1+\phi)^{-1}A^T)^{-1}.$$

The difficulty in pinning down magnitudes of γ_u and δ_u is the presence of σ_u^2 . Suppose, however, that the components of the contextual-effects network \mathcal{C} are unions of components of the peer-effects network \mathcal{A} . This is the setting of section 4.2. Suppose \mathcal{C} has H such components. Index all symbols by h , the component to which they apply, and let \tilde{n}^h denote the row vector with all terms $1/n^h$ and of length n^h , the size of component h . Then following Graham (2008), ratios of the form

$$\tilde{n}^h B_u^h u^h \cdot (B_u^h u^h)^T (\tilde{n}^h)^T / e^1 B_u^1 u^1 \cdot (B_u^1 u^1)^T (e^1)^T$$

are independent of the variance of ω . This is essentially a variance contrast method. One can show with a transversality argument like that in the proof of theorem 4, that if \mathcal{C} has at least 4 components, then for any A and generic C , $\pm(\delta_u, \sigma_u)$ will be identified. The sign ambiguity remains because both the numerator and denominator are linear combinations of products of degree 2.

Of course, unobserved public variables are not the only source of correlated shocks across individuals. Bramoullé, Djebbari, and Fortin (2009) consider the problem of group shocks, a common shock that effects i and everyone connected to her. They provide an identification criterion like our theorem 3 when $A = C$ involving independence of matrix products up to third order, which comes from a procedure in which each individual i 's behavior is differenced from the average behavior of her neighbors. Bramoullé, Djebbari, and Fortin's insight that de-meaning individual choices using group level averages can eliminate group-specific fixed effects, is complementary to other forms of differencing. As a whole, appropriate choice of differencing can eliminate the role of group effects, which is why their presence has not been a focus of this paper.²²

7. CONCLUSION

In this paper, we have provided a theoretical and econometric characterization of linear social interactions models. Our analysis provides both a clear description of the behavioral assumptions needed to employ these models as well as the conditions under which the primitive utility parameters that characterize social influences may be recovered. Our results demonstrate the possibilities and limits to identification as determined by the degree of prior information on the sociomatrices that determine how the characteristics and behaviors of others affect each individual's utility. We show that for the most common case in the empirical literature, namely when these matrices are known *a priori* identification holds generically. The absence of any *a priori* knowledge on these matrices unsurprisingly means that identification fails. The known sociomatrices results demonstrates that variants of the workhorse linear-in-means model for which identification fails are in fact knife edge cases.

We further explore a range of possible forms of *a priori* knowledge that represent intermediate cases compared to these two extreme information assumptions. These intermediate cases correspond to plausible source of *a priori* information as derived from economic theory and/or empirical social structure measurement. We also address the identification question when a researcher is limited to aggregated data of various types. Finally, we demonstrate that endogenous network formation does not constitute an unbridgeable impediment to identification.

²²Kwok (2012) explores the effects of alternate differencing schemes on identification. He shows that different schemes can affect the information in the de-meaned data. Hence the choice of de-meaning procedure matters.

In terms of future research, we see a number of important directions. First, our findings may be understood as fleshing out parts of the “assumptions/possibilities” frontier in terms of the edges between different types of *a priori* information on social structure and identification. There is no reason to believe that the cases we have examined span the possible types of information that may be available to a researcher, so there is certainly more work to be done in fully characterizing the environments in which identification does or does not hold.

Second, the operationalization of the control function approach to addressing network endogeneity needs to be developed. Third, we have not addressed issues of estimation. This suggests a necessary complementary paper to this one if one wishes to make our results operational. For example, the positive identification results we provide on endogenous social structure formation need to be supplemented with procedures for constructing the analog to control functions whose existence are implicit in the identification theorem.

Fourth, while we have addressed the question of how our identification results are affected by endogenous social structure, we have not addressed how this endogeneity can, when explicitly modeled, facilitate identification. For example, if group memberships are associated with prices, then prices can help to uncover social effects, as demonstrated in recent advances in the econometrics of hedonic models. (See Ekeland, Heckman, and Nesheim (2004)). As discussed above, the control functions associated with the changes in conditional error distributions conditional on group membership may be able to facilitate identification.

Finally, information on social interactions may be encoded in the composition of the groups themselves. Becker’s (1957) model of taste-based discrimination implies that information on the presence of discriminatory preferences is embodied both in any black/white wage gap and in the degree of segregation of workers across firms. All these directions emphasize the importance of extending the theoretical and econometric arguments developed here in directions that fully exploit the codetermination of social structure and associated behavioral outcomes.

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APPENDIX

In fact we prove a more general theorem. Suppose that each individual i has his or her own ϕ_i . Define the matrices

$$\Phi_{ij} = \begin{cases} \frac{1}{1 + \phi_i} & \text{if } i = j, \\ 0 & \text{otherwise;} \end{cases} \quad \hat{\Phi}_{ij} = \begin{cases} \frac{\phi_i}{1 + \phi_i} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Assumption T.1 is modified appropriately:

T. 1'. For all i , $\phi_i \geq 0$. A and C are non-negative, for each $i \in V$, $\sum_j a_{ij}$ is either 0 or 1, and similarly for C . For all $i \in V$, $a_{ii} = 0$.

Theorem A1. *If the Bayesian game satisfies axioms T.1' and T.2, then the game has a unique Bayes-Nash equilibrium. The equilibrium strategy profile is*

$$f(x, z) = \Phi (I - \hat{\Phi}A)^{-1} (\gamma I + \delta C)x + g(x, z),$$

where $g(x, z)$ satisfies, for each i , the relation

$$g_i(x, z_i) = \frac{1}{1 + \phi_i} z_i + \frac{\phi_i}{1 + \phi_i} \sum_j a_{ij} E (g_j(x, z_j) | x, z_i).$$

If z is independent of x , then each $g(x, z_i)$ depends only on z_i . If the elements of z are pairwise independent, then $g_i(x, z_i) = (1 + \phi_i)^{-1} z_i + \mu_i(x)$.

This theorem breaks the strategy profile into two pieces. The first measures direct and contextual effects of the public type x , and the feedback through their peer effects. The second term measures the effects of each individual's private type and is estimate of the private types of others.

Proof of Theorem A1. Suppose that in the payoff function (1) the parameter ϕ is indexed by i . Give \mathcal{F} the L^2_ρ max norm; $\|f\| = \max_i \|f_i\|_2$. Let

$$\psi_i = \gamma x_i + \delta \sum_j c_{ij} x_j + z_i,$$

so that

$$u_i = \psi_i \omega_i - \frac{1}{2} \omega_i^2 - \frac{\phi_i}{2} (\omega_i - \sum_j a_{ij} \omega_j)^2 \quad (23)$$

Since the strategies are in L^2_ρ , the expected payoff to any i of any strategy profile f is finite, so preferences over strategies for the Bayesian game are well-defined.

The first-order conditions for expected utility maximization are that for each i , and given the strategy profile f_{-i} of the other individuals and type $t \in \mathcal{T}$,

$$\psi_i + \phi_i \sum_j a_{ij} E (f_j(\psi_j, z_j) | x, z_i) - (1 + \phi_i) \omega_i = 0. \quad (24)$$

Since the problem is concave in ω_i , the first-order conditions are sufficient.

Define the operator $T : \mathcal{F} \rightarrow \mathcal{F}$ such that

$$(Tf)_i(\psi_i, z_i) = \frac{1}{1 + \phi_i} \psi_i + \frac{\phi_i}{1 + \phi_i} \sum_j a_{ij} \mathbb{E} (f_j(\psi_j, z_j) | x, z_i) .$$

A fixed point of T satisfies the first-order condition for all i , ψ_i and z_i ; thus it will be a Bayes-Nash equilibrium profile. Assumption T.1 and a computation shows that this map is a contraction in the norm topology with contraction constant $\phi = \max_i \phi_i / (1 + \phi_i)$, and so a fixed point exists, and is unique. The fixed-point strategy profile satisfies the sufficient first-order optimality conditions, and so it is a Bayes-Nash equilibrium.

Any strategy profile can be written in the form

$$f(x, z) = \Phi (I - \hat{\Phi}A)^{-1} (\gamma I + \delta C)x + g(x, z).$$

where $g_i(x, z)$ depends on z through z_i alone. Apply the operator T to see that f will be an equilibrium if and only if $g(x, z)$ satisfies, for each i ,

$$g_i(x, z) = \frac{1}{1 + \phi_i} z_i + \frac{\phi_i}{1 + \phi_i} \sum_j a_{ij} \mathbb{E} (g_j(x, z_j) | x, z_i) . \quad (25)$$

Thus each g_i depends upon z only through z_i . From now on we take the arguments of each g_i to be x and z_i . Take

$$\mu(x, z)_i = \sum_j a_{ij} \mathbb{E} (g_j(x, z_j) | x, z_i) .$$

This proves the general characterization of equilibrium strategy profiles.

For the characterizations of the $g_i(x, z_i)$, define the operator T_g such that

$$(T_g h)_i = \frac{1}{1 + \phi_i} z_i + \frac{\phi_i}{1 + \phi_i} \sum_j a_{ij} \mathbb{E} (h_j(x, z_j) | x, z_i) .$$

This operator too is a contraction on L_ρ^2 , and so it has a unique fixed point, which is clearly g . The characterizations are proven by showing that the different assumptions imply that sets of g with given properties are invariant under T , and so the fixed point must be in this set.

To prove the second claim, suppose now that x and z are independent. Then for any function $h_j : z_j \mapsto \mathbf{R}$,

$$\mathbb{E} (h_j(z_j) | x, z_i) = \mathbb{E} (h_j(z_j) | z_i) .$$

Consequently, the set of functions $h : (z) \mapsto \mathbf{R}^N$ is invariant under T_g . Thus each g_i depends only on z_i .

For the third claim, observe that if the private types are independent, then if $h_i(x, z_i)$ is of the form $(1 + \phi_i)^{-1}z_i + \mu_i(x)$, then

$$T_g(h)_i(x, z) = \frac{1}{1 + \phi_i}z_i + \frac{\phi}{1 + \phi_i} \sum_j a_{ij} \left(\frac{1}{1 + \phi_j} E(z_j|x) + \mu_j(x) \right),$$

since $E(z_j|x, z_i) = E(z_j|x)$. The sum over j is a function only of x , and so the set of all functions of this form is invariant under T_g . Thus the fixed point has this property too. This proves theorem A1.

To complete the proof of theorem 1, observe that if for all i and j , $\phi_i = \phi_j$, and if the z_i are independent of each other and of x , the fixed point of T_g can be computed directly, and gives equation (2a). \square

The remainder term $\mu(x, z)$ has to do with higher-order beliefs. Suppose, to simplify the exposition, that all the ϕ_i are identical. Equation (25) contains a recursion, and by iterating it, one sees that

$$\begin{aligned} \mu_i(x, z_i) = \frac{1}{1 + \phi} & \left(\frac{\phi}{1 + \phi} \sum_{i'} a_{ii'} E(z_{i'}|x, z_i) \right. \\ & \left. + \left(\frac{\phi}{1 + \phi} \right)^2 \sum_{i'} \sum_{i''} a_{ii'} a_{i'i''} E(E(z_{i''}|x, z_{i'}) | x, z_i) + \dots \right) \end{aligned}$$

The second term contains expressions whose meanings are, “ i ’s expectation of j ’s expectation of $z_k \dots$ ”.

Now we take up identification questions. We have assumed for convenience (E.5) that γ and δ are not both 0. Lemma 1 settles the question of identification when the true γ and δ are both 0.

Lemma 1. *Assume T.1–3 and E.1–5. The set of parameters $\{(\gamma, \delta, \phi) : \gamma = \delta = 0, \phi \geq 0\}$ is weakly identified from the conditional mean of ω . No parameter vector $(0, 0, \phi)$ is identified.*

Proof of lemma 1. Let γ^* , δ^* and ϕ^* denote the true values of the un-starred parameters. If $\gamma^* = \delta^* = 0$, then $B(s) = 0$. If $B(s) = 0$, then since $B_\phi(s)$ is non-singular, $\gamma^*I + \delta^*C = 0$. The sociomatrix C has some positive off-diagonal element (E.3), so the unique solution to the equation $\gamma I + \delta C = 0$ is $\gamma = \gamma^*$ and $\delta = \delta^*$.

If $\gamma^* = \delta^* = 0$, then $B(s) = 0$, and ϕ affects ω only through its effect on ε . Since $E(\varepsilon|x) = 0$, $E(\omega|x)$ is independent of the parameter ϕ . \square

Proof of Theorem 2. Identification of μ follows from E.1 and equation (5). $E(\omega|x)$ is an affine function whose behavior on an open set is observed. Of course $E(\omega|x)$ is identified; this, the spanning assumption E.1 and the orthogonality

assumption E.4 identify $B(s)$. Since C is stochastic, $B(s)e = (\gamma + \delta)B_\phi(s)e = (\gamma + \delta)e$.

If $E(\omega|x)$ is independent of x , then $B(s) = 0$. Since $B_\phi(s)$ is always non-singular, $\gamma I + \delta C = 0$. E.3 implies that C is not a multiple of I , so $\gamma = \delta = 0$. Conversely, if $\gamma = \delta = 0$, then $B(s) = 0$ and ω is independent of x .

If for all i , $E(\omega_i|x) = E(\omega_i|x_i)$, then $B(s) = \alpha I$, and α is identified. It follows that

$$\left(\gamma - \frac{\alpha}{1 + \phi}\right) I + \delta C + \frac{\alpha\phi}{(1 + \phi)^2} A = 0.$$

If the *a priori* hypothesis is satisfied, T.1 implies that I , C and A are linearly independent, and it follows that $\delta = \phi = 0$ and $\gamma = \alpha$. The converse is obvious. \square

Proof of Theorem 3. Part i: Suppose that s' and s'' are two structures in \mathcal{M} that are identical except perhaps for the values they assign to the parameters γ , δ and ϕ . Then $B(s') = B(s'')$ if and only if

$$B_\phi(s')(\gamma' I + \delta' C) = B_\phi(s'')(\gamma'' I + \delta'' C). \quad (26)$$

Multiply this out:

$$((1 + \phi'')I - \phi'' A)(\gamma' I + \delta' C) = ((1 + \phi')I - \phi' A)(\gamma'' I + \delta'' C),$$

since $B_\phi(s')$, $B_\phi(s'')$, and their inverses, all being power series in A , all commute. Consequently, $B(s') = B(s'')$ if and only if

$$\begin{aligned} ((1 + \phi'')\gamma' - (1 + \phi')\gamma'')I + ((1 + \phi'')\delta' - (1 + \phi')\delta'')C \\ + (\phi'\gamma'' - \phi''\gamma')A + (\phi'\delta'' - \phi''\delta')AC = 0 \end{aligned} \quad (27)$$

Independence of I , A , C and AC implies that each coefficient is 0. Now enumerate cases. First, suppose that one of ϕ' and ϕ'' is 0. Wlog assume $\phi' = 0$. Looking at the coefficients of A and AC , independence implies that either $\phi'' = 0$ or $\gamma' = \delta' = 0$. The second possibility has been dispensed with in lemma 1, so conclude that $\phi'' = 0$. It follows immediately from the independence hypothesis that $\gamma' = \gamma''$ and $\delta' = \delta''$.

Next suppose that neither ϕ' nor ϕ'' is 0. If either γ' or $\gamma'' = 0$, then the other must be 0 for the coefficient of A to equal 0. Similarly, if either δ' or $\delta'' = 0$, then the other must be 0 for the coefficient of AC to equal 0. Suppose that $\gamma' \neq 0$. Then from the coefficients for I and A deduce that $(1 + \phi'')/(1 + \phi') = \phi''/\phi'$, and so $\phi' = \phi''$. Conclude from the coefficient of I that $\gamma' = \gamma''$. A similar argument applies to the coefficient of C if $\delta' \neq 0$.

To establish the other direction, suppose that $\alpha I + \beta C + \theta A + \tau AC = 0$ for $\alpha, \beta, \theta, \tau$ not all 0. The parameters of the structures s' and s'' will satisfy equation

(27) if and only if they satisfy the following matrix equation:

$$\begin{pmatrix} 1 + \phi'' & -1 - \phi' & 0 & 0 \\ -\phi'' & \phi' & 0 & 0 \\ 0 & 0 & 1 + \phi'' & -1 - \phi' \\ 0 & 0 & -\phi'' & \phi' \end{pmatrix} \begin{pmatrix} \gamma' \\ \gamma'' \\ \delta' \\ \delta'' \end{pmatrix} = \begin{pmatrix} \alpha \\ \theta \\ \beta \\ \tau \end{pmatrix}.$$

The determinant of the matrix is $(\phi' - \phi'')^2$. Choose any $\phi' \neq \phi''$, both non-negative. Then there will be parameters $\gamma', \gamma'', \delta'$ and δ'' which solve the equation. Consequently, for these parameter values $B(s') = B(s'')$, and the two parameter vectors cannot be distinguished.

To complete the proof we must show that even when ϕ cannot be identified, if either one of γ and δ is identified, so is the other. A necessary and sufficient condition for $\gamma' = \gamma''$ is that $\theta = -\alpha$. Thus $\alpha(I - A) + (\beta I + \tau A)C = 0$. For each peer-effects component b (a component is a maximal connected sub-network), denote by π_b the left Perron eigenvector of A_b and let C_b denote the $|b| \times N$ submatrix of C consisting of the rows of b and all columns. Then $(\beta + \tau)\pi_b C_b = 0$ for all components b . Each π_b is strictly positive, and some C_b does not equal 0, so conclude that $\beta = -\tau$. This implies from the equation system that $\delta' = \delta''$. Conversely, if $\delta' = \delta''$, then $\beta = -\tau$, and so $(\alpha + \theta)\pi_b = 0$. hence $\theta = -\alpha$.

Part ii. $B(s') = B(s'')$ if and only if

$$((1 + \phi'')\gamma' - (1 + \phi')\gamma'')I + (\delta' - \delta'')C + (\phi'\gamma'' - \phi''\gamma')A = 0. \quad (28)$$

Suppose that I , A and C are linearly independent. Then each coefficient in equation (28) is 0. In particular, the coefficient on C is 0, and so $\delta' = \delta''$; δ is identified.

The remaining coefficients are 0 if and only if

$$\begin{aligned} \phi'\gamma'' &= \phi''\gamma', \text{ and} \\ (1 + \phi')\gamma'' &= (1 + \phi'')\gamma'. \end{aligned}$$

We need to show that $\gamma' = \gamma''$ and $\phi' = \phi''$. Observe first that, from the second equation, $\gamma'' = 0$ if and only if $\gamma' = 0$. Next observe that if $\gamma' \neq 0$, $\phi'/(1 + \phi') = \phi''/(1 + \phi'')$. Thus $\phi' = \phi''$, and it follows that $\gamma' = \gamma''$.

For the converse, observe that if $\alpha I + \beta C + \theta A = 0$ and any coefficient is not 0, then $\beta \neq 0$ because no single coefficient can be non-zero, and if $\beta = 0$ then A would be a multiple of I , contradicting T.1. It follows from equation (28) that δ cannot be identified. Choose $\phi' \neq \phi''$, and consider the equation

$$\begin{pmatrix} 1 + \phi'' & -(1 + \phi') \\ -\phi'' & \phi' \end{pmatrix} \begin{pmatrix} \gamma' \\ \gamma'' \end{pmatrix} = \begin{pmatrix} \alpha \\ \theta \end{pmatrix}.$$

This equation says that the coefficient of I in equation (28) equals α and the coefficient of A in equation (28) equals θ . The matrix is non-singular, and so there will be γ' and γ'' which solve it. Consequently, ϕ is not identified. Finally, We

show by contradiction that γ is not identified. if γ is identified, then the equation system is solved, for all $\phi' \neq \phi''$, by some $\gamma' = \gamma''$. For this to be true, for all $\phi' \neq \phi''$, $(\phi'' - \phi')\gamma' = \alpha$ and $(\phi' - \phi'')\gamma' = \theta$, so $\theta = -\alpha$. If so, then $\alpha(I - A) + \beta C = 0$. If the coefficient α equals 0, then β equals 0 as well, which is a contradiction, so α must not equal 0. But α must equal 0, because the matrix $\alpha(I - A)$ has both positive and negative elements (by T.1, the diagonal and off-diagonal elements have different signs), while the non-zero elements of βC all have the same sign. This establishes the contradiction. \square

Proof of Corollary 1. The result follows directly from lemma 2. \square

Lemma 2. *Suppose that A is a peer-effects matrix with $N \geq 3$. Suppose C is a contextual-effects sociomatrix such that for some peer-effects component b with subgraph sociomatrix C_{bb} :*

- (1) *The invariant distribution π_b of A_b is not an eigenvector of C_{bb} .*
- (2) *C_{bb} has no eigenspace of dimension at least $|b| - 1$, and*
- (3) *$\text{rank } C_{bb} \geq 2$.*

Then the utility parameters are identified.

Proof of Lemma 2. It suffices to show that, generically, I , A , C and AC are independent. If so, then $C \neq AC$, and the criterion of the first case of theorem 3 applies. So suppose that $\alpha I + \beta A + \theta C + \tau AC = 0$. Post-multiply the sum by the vector of 1s, to conclude that $\alpha + \beta + \theta + \tau = 0$.

Next, observe that for each pair of component b of the peer-effects network,

$$\alpha I + \beta A_b + \theta C_{bb} + \tau A_b C_{bb} = 0,$$

Let π_b be a left Perron eigenvector for A_b . Then $(\alpha + \beta)\pi_b + (\theta + \tau)\pi_b C_{bb} = 0$. By hypothesis, neither vector in the sum is 0, and π_b and $\pi_b C_{bb}$ are linearly independent, so $\alpha + \beta = \theta + \tau = 0$.

Rewriting the linear combination of matrices, for all components b ,

$$\alpha(I - A_b)(I + \theta C_{bb}) = 0.$$

Suppose that $\alpha \neq 0$. The null space of $I - A_b$ is spanned by e , so the range of $I + \theta C_{bb}$ is the vector space spanned by e . Consequently, $\theta \neq 0$. If $\text{rank } I + \theta C_{bb} = 0$, then $C_{bb} = -\theta^{-1}I$, which contradicts the hypothesis.

If $\text{rank } I + \theta C_{bb}$ is 1, then the null space of $I + \theta C$ has rank equal to $|b| - 1$, where $|b|$ is the number of nodes in the component. Hence $-\theta^{-1}$ is an eigenvalue of C_{bb} , of geometric multiplicity $|b| - 1$. This too contradicts the hypothesis. Conclude that $\alpha = 0$.

If $\alpha = 0$, then $\theta(I - A_b)C_{bb} = 0$. Again from the Perron-Frobenius Theorem, this can only hold if $\text{rank } C_{bb} = 1$, which our hypothesis rules out. \square

- Lemma 3.**
- i. For all peer-effects sociomatrices A containing a component b of size $n^b \geq 3$, the set of contextual-effects sociomatrices C for which I , A , C and AC are not independent is a closed-semi-algebraic set of lower dimension in \mathcal{M}_C .
 - ii. If $N \geq 3$ there is a closed semi-algebraic set $\mathcal{G}_C \subset \mathcal{M}_C$ which is the complement of a lower-dimensional in \mathcal{M}_C such that for each C in \mathcal{G}_C the subset S_C of \mathcal{M}_A containing those peer-effects sociomatrices A for which I , A , C and AC are not linearly independent is semi-algebraic, closed, and lower-dimensional in \mathcal{M}_A .

Proof of Lemma 3. The result follows immediately from lemma 2. Part *i* is the corollary. \square

Proof of Corollary 2. Consider the equation system

$$wI + xA + yC + zAC = 0. \quad (29)$$

We will show that linear independence of all four matrices is the right condition to check, and then that the only solution to equation (29) is $w = x = y = z = 0$. The result then follows from theorem 3. Let q_0, \dots, q_n be nodes such that i) $q_0 = j$ and $q_n = i$; ii) each q_v is linked to q_{v-1} and q_{v+1} (whenever they exist), by an edge in the peer-effects network and/or an edge in the contextual-effects network; and iii) this path is shortest possible among all such paths connecting a and z . Denote by \sim_A and \sim_C peer- and contextual-effects edges. Thus we have $q_0 \sim_{X_0} q_1 \sim_{X_1} \dots \sim_{X_{n-1}} q_n$, where each X_v is either a C or an A . For any particular v it may be possible to take either C or A for X_v , but by hypothesis there is at least one X_v which must be a C and another which must be an A . Furthermore, minimality of the length implies that for all $w \neq v$, $q_w \neq q_v$, and that there is no A or C edge between q_v and q_{v+u} for $u \geq 2$.

Without loss of generality (run the chain backwards if necessary) there exists a triple $q_v \sim_A q_w \sim_C q_u$ where $u = w + 1 = v + 2$. Then

$$AC_{q_v q_u} \geq a_{q_v q_w} c_{q_w q_u} > 0 \text{ but } A_{q_v q_u} = C_{q_v q_u} = I_{q_v q_u} = 0.$$

This shows two things: First, I , A and C are already known to be distinct by virtue of axioms T.1, E.2 and E.3. This calculation shows AC to be distinct from these matrices. Thus, part i) of theorem 3 applies. Second, from the calculation infer that in any solution to equation (29), $z = 0$.

Next, create a “ C -maximal path” by choosing X_v equal to C wherever possible in the chain. By hypothesis, some A -edge must remain. Again without loss of generality choose a triple $q_v \sim_A q_w \sim_C q_u$ where $u = w + 1 = v + 2$. Then $a_{q_v q_w} > 0$ while $c_{q_v q_w} = I_{q_v q_w} = 0$. Since $z = 0$, this implies that $x = 0$ in any solution to equation (29). Finally, create an “ A -maximal path” and use the same argument to infer that $y = 0$. If $x = y = z = 0$, then $w = 0$ and we are done. \square

Proof of Corollary 3. Let c denote a component of the contextual-effects network satisfying the hypotheses. Then $AC = C$, which is case *ii*) of theorem 3. If I , A and C are linearly dependent, there are coefficients α , β and θ not all 0 such that $\alpha I + \beta A + \theta C = 0$. The coefficient α is not 0 because A is not a scalar multiple of C . The coefficient β is not 0 because C has off-diagonal positive elements, and the coefficient θ is not 0 because A has diagonal elements all 0 (T.1). So C is a linear combination of I and A . But since all off-diagonal elements of C are equal, the hypothesis on A implies this cannot be the case. \square

The proof of theorem 4 is long, tedious, and without merit beyond its existence. It will be useful to rewrite the social interaction effects with the parameter $r = \phi / (1 + \phi)$, with $r \in [0, 1)$.

Define $a(r)^T = (1 - r)e^T(I - rA)^{-1}$, where e is a vector of suitable length and T denotes transpose. The effect of r is isolated in the column-sum vector a . We need these facts:

Lemma 4. (a) *For every sociomatrix A which is not bistochastic, the map $r \mapsto a(r)$ is an injection.*

(b) *For all A and $r \in [0, 1)$, $a(r) \gg 0$.*

(c) $\sum_x a_x(r) = N$.

Proof. We use the relationship $a^T = (1 - r)e^T + ra^T A$ which is easily derived from the definition of a .

(a) If $a(r') = a(r'') = a$ for $r' \neq r''$, then

$$(r'' - r')e^T + (r' - r'')a^T A = 0,$$

so $a^T A = e^T$ and therefore $a = e$, and A is bistochastic.

(b) Without loss of generality, suppose that A is irreducible. (Otherwise consider each component of the peer-effects network \mathcal{A} separately.) $(1 - r)e^T(I - rA)^{-1}$ equals $(1 - r)(e + reA + r^2eA^2 + \dots)$. This is the sum of nonnegative vectors, and some eA^T is strictly positive.

(c) The row sums of A are 1. \square

Another technical lemma we need is this: Fix N , and let S denote the set of all triples (α, r, C) which solve the equation system

$$\alpha(a_i(r) - a_j(r)) - \beta \sum_{x \in V} a_x(r)(c_{xi} - c_{xj}) = 0 \text{ for } i \neq j,$$

and let $S_C \subset \mathcal{M}_C$ denote its projection onto the set of all contextual-effects sociomatrices.

Lemma 5. *The set S_C is closed and has dimension at most $2 + (N - 1)^2$, which is less than $\dim \mathcal{M}_C$ for $N \geq 3$.*

The set of stochastic matrices had dimension $N(N - 1)$, which exceeds $2 + (N - 1)^2$ when $N \geq 3$. The proof involves facts about semi-algebraic sets — sets defined by finite numbers of polynomial inequalities, which can be found, for instance, in Bochnak, Coste, and Roy (1987).

Proof of lemma 5. This system contains $N - 1$ equations. According to lemma 4, $a(r) \gg 0$, and so the derivative with respect to C of the left-hand side is surjective onto \mathbf{R}^{N-1} . Consequently the solution set is a semi-algebraic set of co-dimension $N - 1$, which is to say, of dimension $2 + (N - 1)^2$. It is also compact. The projection of S onto S_C is compact and has dimension at most $2 + (N - 1)^2$ since semi-algebraic functions (projection, in this case) cannot increase the dimension of their domains. \square

Proof of Theorem 4. Since \bar{x}^g is observed, σ_x^2 is identified. It is convenient to define, for the g th group,

$$f^g(\gamma, \phi, \beta, A^g, C^g) = \frac{1}{n^g} e \cdot B_\phi^g (\gamma I + (\beta - \gamma) C^g) (\gamma I + (\beta - \gamma) C^{gT}) B_\phi^{gT} e^T.$$

Then $v^g = f^g(\gamma, \phi, \beta, A^g, C^g) - f^0(\gamma, \phi, \beta, A^0, C^0)$.

Define $F = F^1, \dots, F^g$ such that

$$F^g(\gamma', r', \gamma'', r'', \beta, A^g, C^g) = f^g(\gamma', r', \beta, A^g, C^g) - f^g(\gamma', r', \beta, A^0, C^0) - f^g(\gamma'', r'', \beta, A^g, C^g) + f^g(\gamma'', r'', \beta, A^0, C^0)$$

The domain of F is taken to be $\mathbf{R}^4 / \Delta^2 \times \mathbf{R} \times M_{\mathcal{A}} \times M_{\mathcal{C}}$ where \mathcal{A} and \mathcal{C} denote the peer- and contextual-effects networks, respectively and the first set is that of all quadruples $(\gamma', r', \gamma'', r'')$ such that not both $\gamma' = \gamma''$ and $r' = r''$.

Fix the A^g, C^g and s , and consider the equation

$$F(\gamma', r', \gamma'', r'', \beta, A^0, \dots, A^g, C^0, \dots, C^g) = 0.$$

Since β is already identified, we need only to identify γ and r . The statistic (v^1, \dots, v^g) does not distinguish γ', r' from γ'', r'' (given β) if and only if $(\gamma', r', \gamma'', r'')$ solves the equation, and $(\gamma', r') \neq (\gamma'', r'')$. Thus we must show that for generic C^1, \dots, C^g ,

$$F(\gamma', r', \gamma'', r'', \beta, A^0, \dots, A^g, C^0, \dots, C^g) = 0$$

has no solution in \mathbf{R}^4 / Δ^2 .

We will show that if $F^g = 0$, then $D_{C^g} F^g$ is surjective onto \mathbf{R} . If so, it follows that DF_{C^1, \dots, C^g} is surjective onto \mathbf{R}^g . Consequently, 0 is a regular value of F , and we conclude from the transversality theorem that for almost all C^1, \dots, C^g , 0 is a regular value of the map $F(\cdot, \beta, A^1, \dots, A^g, C^1, \dots, C^g) : \mathbf{R}^4 / \Delta^2 \rightarrow \mathbf{R}^g$.

Because F is semi-algebraic, the set of critical C^1, \dots, C^g for which this may fail is closed and lower-dimensional in \mathcal{M}'_C . When the map has 0 as a regular value, the inverse image of 0 is a manifold of co-dimension G . For $G \geq 5$, this implies that the solution set in \mathbf{R}^4/Δ^2 has negative dimension, that is, it is empty.

It remains only to show that if $F^g = 0$, then $D_{C^g}F^g$ is surjective onto \mathbf{R} . Observe first that

$$D_{C^g}F^g = D_{C^g}f^g(\gamma', r', \beta, A^g, C^g) - D_{C^g}f^g(\gamma'', r'', \beta, A^g, C^g).$$

The derivative $D_{C^g}f^g$ is a linear map from the tangent space of M_C to \mathbf{R} . That tangent space is spanned by the set of all matrices H_{kij} whose vw th entry is 1 if $v = k$ and $w = i$, -1 if $v = k$ and $w = j$, and 0 otherwise. In words, H_{kij} shifts a little bit of j 's influence on k to i .

A calculation shows that

$$\begin{aligned} D_{C^g}f^g H_{kij} &= a_k(r)(s - \gamma) \left(\gamma(a_i(r) - a_j(r)) + (s - \gamma) \sum_x a_x(r)(c_{ix} - c_{jx}) \right) \\ &\equiv a_k(r) q_{ij}(\gamma, r) \end{aligned}$$

where $a = eB_\phi$. Thus if $D_{C^g}F^g$ is not surjective at $(\gamma', r', \gamma'', r'')$, then

$$a_k(r') q_{ij}(\gamma', r') = a_k(r'') q_{ij}(\gamma'', r''). \quad (30)$$

First we show that for generic C and all γ and r , there is a pair $i \neq j$ such that $q_{ij}(\gamma, r) \neq 0$. Suppose not. Since $\gamma \neq s$, for all i, j pairs

$$\gamma(a_i(r) - a_j(r)) = (\gamma - s) \sum_x a_x(r)(c_{xi} - c_{xj}).$$

Fix j . Then for all $i \neq j$, the equation system

$$\alpha(a_i(r) - a_j(r)) = \sum_x a_x(r)(c_{xi} - c_{xj})$$

has a solution. From lemma 5, the set of matrices S_C for which this system has a solution is a closed and lower-dimensional subset of \mathcal{M}'_C . So for $C \in S_C^c$, the equation system has no solution, and hence some $q_{ij} \neq 0$.

Now suppose that $D_{C^g}F^g$ is not surjective at $(\gamma', r', \gamma'', r'')$, so that equation (30) holds. Lemma 4 states that the sum over u of $a_k(r)$ is N , independent of r . Consequently, summing over u in equation (30), we see that for all i and j , $q_{ij}(\gamma', r') = q_{ij}(\gamma'', r'')$. Since for at least one i, j pair, $q_{ij}(\gamma', r') \neq 0$, it follows that for all u , $a(r') = a(r'')$. Conclude from lemma 4 that $r' = r''$. Thus $D_{C^g}F^g$ can only fail to be surjective at points $(\gamma', r', \gamma'', r')$.

A calculation now shows that for all $i \neq j$,

$$(s - \gamma' - \gamma'')(a_i(r') - a_j(r')) = (2s - \gamma' - \gamma'') \sum_x a_x(r')(c_{xi} - c_{xj}),$$

and by hypothesis $\gamma' \neq \gamma''$. If $s \neq 0$, then at least one of $s - \gamma' - \gamma''$ and $2s - \gamma' - \gamma''$ must not be 0. That is, the equation system

$$\alpha(a_i(r') - a_j(r')) = \beta \sum_x a_x(r')(c_{xi} - c_{xj})$$

has a solution (α, β, r', C) has a solution with not both $\alpha = \beta = 0$. It cannot be the case that $\beta = 0$, for if so, then $\alpha \neq 0$, and $a_i(r') = a_j(r')$ for all i and j . But if this were the case, then $a(r') = e$ and it follows that $r' = 0$ or that A is bistochastic. We have ruled out both cases by assumption. Since $\beta \neq 0$, it follows that if $D_{C^s} F^s$ is not surjective, then

$$\alpha(a_i(r') - a_j(r')) = \sum_x a_x(r')(c_{xi} - c_{xj}),$$

and again from lemma 5, this can only happen for $C^s \in S_C$. \square

If $-\gamma^*/\delta^*$ is not an eigenvalue of C , then B will be non-singular, and γ^* , δ^* and ϕ^* solve the equation

$$(1 + \phi)I - \phi A = \gamma B^{-1} + \delta C B^{-1}. \quad (31)$$

This fact is used in the proofs of theorems 5 and 6 below.

Proof of Theorem 5. Axiom E.2 and the hypothesis imply that both a_{ji} and a_{ij} are 0, so we have two locations with 0's in the peer-effects sociomatrix. Let γ^* , δ^* and ϕ^* denote the true parameter values, and recall that $B = B(s)$ is identified (by theorem 2). Since it is nonsingular, they solve equation (31).

Although we do not know the matrix A , we do know that both the ji and the ij terms are 0. We also know that $\beta^* = \delta^* + \gamma^*$ is identified (by theorem 2). Thus the following two-by-two equation system in γ and δ has as one solution $\delta = \delta^*$ and $\gamma = \gamma^*$.

$$\begin{aligned} \gamma B_{ji}^{-1} + \delta \sum_k c_{jk} B_{ki}^{-1} &= 0 \\ \gamma + \delta &= \beta^*. \end{aligned}$$

Simplifying,

$$\gamma B_{ji}^{-1} + (\beta^* - \gamma) \sum_k c_{jk} B_{ki}^{-1} = 0. \quad (32)$$

The system is degenerate if and only if $B_{ji}^{-1} = \sum_k c_{jk} B_{ki}^{-1} = 0$. When $B_{ji}^{-1} \neq 0$, equation (32) can be solved for γ^* , and since β^* , this gives δ^* . Finally, identify ϕ from the diagonal of equation (31) — the equation

$$1 + \phi = \gamma^* B_{jj}^{-1} + \delta^* c_{jk} B_{kj}^{-1}$$

has ϕ^* as its unique solution. \square

Proof of theorem 6. Let γ^* , δ^* and ϕ^* denote the true utility parameter values. Since A has 0's on the diagonal, the right hand side of equation (31) has identical diagonal elements when $\gamma = \gamma^*$ and $\delta = \delta^*$. Consider the equation system

$$\gamma(B_{ii}^{-1} - B_{11}^{-1}) + \delta(B^{-1}C_{ii} - B^{-1}C_{11}) = 0, \quad (33)$$

for $i = 2, \dots, N$, with unknowns γ and δ . The system has $\gamma = \gamma^*$ and $\delta = \delta^*$ as a solution. The coefficients of this system are either observable or known *a priori*. This system has rank at most 1. Obviously the rank cannot be more than 2. If it is rank 2, the unique solution would be $\gamma = \delta = 0$, contradicting the hypothesis.

There are two possibilities, then: The system (33) has rank 0 or rank 1. If the system has rank 0, then in particular $\gamma = 1$ and $\delta = 0$ solve the system. Hence all the diagonal elements of B^{-1} are identical. A derivative calculation and a transversality argument shows that for the complement of a closed, lower-dimensional subset of \mathcal{M}_A , however, the system has rank 1.

If the system (33) has rank 1, then the solution set is either $\delta = 0$ or $\gamma = z\delta$ for some $z \neq 0$. In the first case, we have $\delta = \delta^*$, and so this case can only arise when $\delta^* = 0$ and $\gamma^* = \beta^*$. Since β^* is identified, we can infer $\gamma = \gamma^* = \beta^*$. In the second case, δ^* and γ^* solve

$$\begin{aligned} \gamma + \delta &= \beta^* \\ \gamma - z\delta &= 0. \end{aligned}$$

Notice that β^* cannot equal 0 by assumption — otherwise $-\gamma^*/\delta^*$ is an eigenvalue of C . If $z = -1$ this equation system is inconsistent, which cannot be the case. If $z \neq -1$, then it has a unique solution, which must be $\gamma = \gamma^*$ and $\delta = \delta^*$. \square

Proof of Corollary 4. Once ϕ , γ and δ are identified, A is recovered from equation (31). \square

Proof of theorem 7. First we show that the conditions of the theorem are sufficient and necessary, respectively, for identifying the parameter vectors π_1 , π_2 and π_3 . Expand equation (19):

$$\begin{aligned} \omega_i &= \pi_1 x_i + \sum_{\substack{j \sim_{C^i} \\ j \neq i}} \pi_2 x_j \\ &+ \sum_{k \sim_{A^i}} \pi_3 k \left(\sum_{\substack{j \sim_{C^i} \\ j \neq i}} b_{kj} x_j + b_{ki} x_i + \sum_{j \not\sim_{C^i}} b_{kj} x_j \right) + \frac{1}{1 + \phi} \varepsilon_i \quad (34) \end{aligned}$$

Reorder the elements of B , and partition it as follows:

$$j \sim_C i \quad j \not\sim_C i$$

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$$B = \begin{array}{c} i, j \sim_A i \\ j \not\sim_A i \end{array} \begin{pmatrix} b_{11} & \dots & \dots & b_{1N} \\ & \vdots & \vdots & \\ b_{N1} & \dots & \dots & b_{NN} \end{pmatrix} = \begin{pmatrix} \mathbf{B}_{AC} & \mathbf{B}_{A \rightarrow C} \\ \mathbf{B}_{\neg AC} & \mathbf{B}_{\neg A \rightarrow C} \end{pmatrix}.$$

Then the first term on the second line of equation (34) is

$$\begin{aligned} (\pi_3 \quad 0) \begin{pmatrix} \mathbf{B}_{AC} & \mathbf{B}_{A \rightarrow C} \\ \mathbf{B}_{\neg AC} & \mathbf{B}_{\neg A \rightarrow C} \end{pmatrix} \begin{pmatrix} x_{j \sim_C i} \\ x_{j \not\sim_C i} \end{pmatrix} \\ = \pi_3 \mathbf{B}_{AC} x_{j \sim_C i} + \pi_3 \mathbf{B}_{A \rightarrow C} x_{j \not\sim_C i}. \end{aligned}$$

Identification of the parameters requires that the dimension of the set

$$\left\{ x_i, (x_j)_{\substack{j \sim_C i \\ j \neq i}}, (x_j)_{\substack{j \not\sim_C i \\ j \neq i}} : x \in \text{supp}_x \rho \right\}$$

be at least as big as the number of parameters to be estimated, where $\text{supp}_x \rho$ denotes the support of the marginal distribution of x . The necessary condition follows from the fact that N is an upper bound on this dimension. The sufficient condition comes from observing that the dimension of this set is $1 + \#\{j : j \sim_C i, j \neq i\} + \dim \text{rank } B_{A \rightarrow C}$. and the needed dimension is $1 + \#\{j : j \sim_C i, j \neq i\} + \#\{j : j \sim_A i\}$.

Next we show how to recover the parameters from the π_i . $\sum_j \pi_{3j} = (\phi / (1 + \phi)) \sum_j a_{ij}$. Since that sum is 1, ϕ is identified. With ϕ identified, each π_{3j} identifies a_{ij} . If any $\pi_{2j} = 0$ then $\delta = 0$ and γ is identified from β . Otherwise, $\pi_{2k} / \pi_{2j} = c_{ik} / c_{ij}$. Since ϕ is identified, the sum $\sum_j \pi_{2j} = \delta(1 - c_{ii})$. This, π_1 and β deliver the one-dimensional curve described in the theorem. Finally, it is quick to check that failure to identify any π_{ij} causes a consequent drop in information: Of ρ and a_i if a component of π_3 is unidentified, c_{ik} / c_{ij} for some pair (k, j) if one of those components of π_2 are not identified, and the curve containing γ and δ is not identified if a component of π_2 or π_1 is not identified. \square

Proof of Theorem 8. Let \hat{x}^g denote the mean of the x_j for $j \in g$, including individual i . Equations (20) become

$$\begin{aligned} E(\bar{\omega} | x_i, \bar{x}^g) &= \mu_g + b_g \hat{x}^g + b_{gi} x_i \\ E(\omega_i | x_i, \bar{x}^g) &= \mu_i + b_{-i} \hat{x}^g + b_{ii} x_i \end{aligned} \tag{35}$$

The four coefficients are identified, and the problem is to determine the values of the utility parameters from these four values without knowing A . Observe that $b_{-i} + b_{ii} = \gamma + \delta$, a row-sum of B . Thus $\gamma + \delta$ is identified. Furthermore, $b_g + b_{gi} = (N - 1)(\gamma + \delta)$. Consequently there are only three independent values among these four coefficients. For fixed C , let F_C denote the map that takes quadruples $(\gamma, \delta, \phi, A)$ to triples (b_{ii}, b_{-i}, b_{gi}) with the given C matrix. This map is smooth, and so the implicit function theorem can be used to study solutions of the equation

$$F_C(\gamma, \delta, \phi, A) = (b_{ii}, b_{-i}, b_{gi}).$$

It will be convenient to take $i = 1$ for the calculations, to reparametrize with $r = \phi/(1 + \phi)$, and to work with the map $G_V : (\gamma, \delta, r, \epsilon) \mapsto (b_{11}, \sum_k b_{k1}, \sum_i b_{1i})$. This function is a non-singular linear transformation of F_C , and so we can identify utility parameters from b_{ii} , b_{gi} and b_{-i} if and only if we can identify them from these sums of elements of B as well.

We will show that there is a particular direction H for a perturbation of peer-effects matrices A such that for generic A and any $\phi > 0$, the derivative of the map $(\gamma, \delta, \phi, \epsilon) \mapsto G_C(\gamma, \delta, \phi, A + \epsilon H)$ has full-rank at $(\gamma, \delta, \phi, 0)$ and the partial derivative $\partial_\epsilon \neq 0$. Choose now γ^* , δ^* and ϕ^* , and an A for which the preceding statement holds, and denote the corresponding statistics $(s_1, s_2, s_3) \equiv (b_{11}^*, \sum_k b_{k1}^*, \sum_i b_{1i}^*)$. The derivative map is surjective in a neighborhood $J \times I$ of $(\gamma^*, \delta^*, \phi^*, 0)$ where I is an open interval around $\epsilon = 0$ and J is an open rectangle in \mathbf{R}^3 containing $(\gamma^*, \delta^*, r^*)$, and so the intersection of the inverse image of $(b_{11}^*, \sum_k b_{k1}^*, \sum_i b_{1i}^*)$ with $J \times I$ is a manifold of dimension 1. In fact, we show that $J \times I$, the partial derivative $\partial_{\gamma\delta\epsilon} G_C : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is surjective. This immediately implies that r is not identified. Suppose we parametrize the manifold locally in a neighborhood $(\gamma^*, \delta^*, \phi^*, 0)$ as $\psi(\lambda) = (\gamma(\lambda), \delta(\lambda), r(\lambda), \epsilon(\lambda))$ where $\psi(0) = (\gamma^*, \delta^*, \phi^*, 0)$. Suppose $D\psi(0) = (x_\gamma, x_\delta, x_r, x_\epsilon)$. Then $D\psi(0) \neq 0$, and

$$0 = \partial_r G_C(\psi(0)) \cdot x_r + \partial_{\gamma\delta\epsilon} G_C(\psi(0)) \cdot (x_\gamma, x_\delta, x_\epsilon).$$

To show that r is not identified, it suffices to show that $x_r \neq 0$. Suppose $x_r = 0$. Since $D\psi(0) \neq 0$, $x_r = 0$ requires that $(x_\gamma, x_\delta, x_\epsilon) \neq 0$. If so, then $DG_C(\psi(0)) \cdot D\psi(0) \neq 0$, which is a contradiction.

Now we calculate. First observe that for generic A , there will exist i and j such that $b_{1i} \neq b_{1j}$, and $a_{1i}, a_{1j} > 0$. Next, observe that $D_A B \cdot H = -r(1 - rA)^{-1}HB$, where $r = \phi/(1 + \phi)$. Observe, too, that $s_2 = \gamma + \delta$. To show that $DG_C(\gamma^*, \delta^*, \phi^*, A)$ is surjective, it suffices to show that $\partial_{\gamma\delta\epsilon} G_C(\gamma^*, \delta^*, \phi^*, A + \epsilon H)|_{\epsilon=0}$ has rank 3. Choose H to be the matrix H_{kij} where i and j are as above, $u \neq 1$, $h_{ki} = -h_{kj} = 1$, and all other elements of H are 0. Computing,

$$\begin{aligned} \partial_{\gamma\delta\epsilon} G_C(\gamma^*, \delta^*, \phi^*, A + 0H) = \\ \left(\begin{array}{ccc} (1 - r^*)d_{1,1} & (1 - r^*) \sum_u d_{1u} c_{u1} & -r^*(b_{i1} - b_{j1})d_{1k} \\ 1 & 1 & 0 \\ (1 - r^*) \sum_v d_{v1} & (1 - r^*) \sum_v \sum_u d_{vk} c_{u1} & -r^*(b_{i1} - b_{j1}) \sum_u d_{uk} \end{array} \right) \end{aligned}$$

where d_{ul} is the ul th element of $(I - r^*A)^{-1}$. Since $b_{i1} - b_{j1} \neq 0$ and $0 < r^* < 1$, this matrix is non-singular if and only if the following matrix is non-singular:

$$\left(\begin{array}{ccc} d_{1,1} & \sum_u d_{1u} c_{u1} & d_{1k} \\ 1 + \phi^* & 1 + \phi^* & 0 \\ \sum_v d_{v1} & \sum_v \sum_u d_{vk} c_{u1} & \sum_u d_{uk} \end{array} \right)$$

Notice that δ^* and γ^* have disappeared. For fixed r^* and C it is generic in A that this matrix is non-singular. \square

Proof of Theorem 9. The following conditions are generic: I , A , C and AC are independent, and I , $A + A^T$, $C + C^T$, AA^T and CC^T are independent. We show that these independence conditions are sufficient for the conclusions of the theorem.

Identification of γ , δ and ϕ comes from theorem 3. The covariance matrix of ω is

$$\begin{aligned}\Omega &= (1-r)^2\sigma_\epsilon^2 I + B_u u \cdot (B_u u)^T \\ &= (1-r)^2\sigma_\epsilon^2 I + (1-r)^2\sigma_u^2 (I - rA)^{-1} (\gamma_u I + \delta_u C) (\gamma_u I + \delta_u C^T) (I - rA^T)^{-1},\end{aligned}$$

and so

$$\begin{aligned}\hat{\Omega} &\equiv (1-r)^{-2} (I - rA) \Omega (I - rA^T) \\ &= \sigma_\epsilon^2 (I - r(A + A^T) + r^2 AA^T) + \sigma_u^2 (\gamma_u^2 I + \gamma_u \delta_u (C + C^T) + \delta_u^2 CC^T) \\ &= (\sigma_\epsilon^2 + \sigma_u^2 \gamma_u^2) I - \sigma_\epsilon^2 r (A + A^T) + \sigma_u^2 \gamma_u \delta_u (C + C^T) + \sigma_\epsilon^2 r^2 AA^T + \sigma_u^2 \delta_u^2 CC^T.\end{aligned}$$

Since r is identified and A is known *a priori*, $\hat{\Omega}$ is observable. Suppose now that σ_ϵ^2 , γ_u and δ_u , and $\sigma_\epsilon^{2'}$, γ_u' and δ_u' generate the same $\hat{\Omega}$. As in the proof of theorem 3, independence implies here that there is a z such that $z = \sigma_u^2 / \sigma_u^{2'} > 0$, $\delta_u'^2 = z \delta_u^2$, $\gamma_u' \delta_u' = z \gamma_u \delta_u$, $\sigma_\epsilon^{2'} = \sigma_\epsilon^2$, and finally, $\gamma_u'^2 = z \gamma_u^2$. This is enough to establish that $(\gamma', \delta') = \pm \sqrt{z} (\gamma, \delta)$. \square