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ABSTRACT

We construct a model to capture the Keynesian idea that production and employment decisions are based on expectations of aggregate demand driven by sentiments, and that realized demand follows from the production and employment decisions of firms. We cast the Keynesian idea into a simple model with imperfect information about aggregate demand and we characterize the rational expectations equilibria of this model. We find that the equilibrium is not unique despite the absence of any non-convexities or strategic complementarity in the model. In addition to multiple fundamental equilibria, there can be serially correlated stochastic equilibria driven by self-fulfilling consumer sentiments. Furthermore, these sentiment-driven equilibria are not based on randomizations of the fundamental equilibria

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> Most, probably, our decisions to do something positive, the full consequences of which will be drawn over many days to come, can only be taken as a result of animal spirits-of a spontaneous urge to action rather than inaction, and not as the outcome of a weighted average of quantitative benefits multiplied by quantitative probabilities.
>
> J. M. Keynes, *General Theory*

1 Introduction

We construct a model to capture the Keynesian insight that employment and production decisions are based on expectations of consumer demand driven by sentiments and that realized aggregate demand follows firms' production and employment decisions. We cast the Keynesian insight in a simple model in which (i) firms must produce in advance without having perfect information about aggregate demand and (ii) realized demand and income depend on firms' output and employment decisions. We characterize the rational expectations equilibria of this model. We find that despite the lack of any non-convexities in technologies and preferences, there can be multiple rational expectations equilibria.[1] The sentiment-driven equilibria are the result of the signal extraction problem faced by firms when production decisions must be made prior to the realization of demand.

The sentiment-driven fluctuations in our model may be driven by waves of optimism or pessimism, or as in Keynes' terminology by "animal spirits," as noted in the block quote above. However unlike Keynes' contention, the sentiments in our model give rise precisely to perpetually self-fulfilling rational expectations equilibria, even though they are the outcome of the "weighted average of quantitative benefits multiplied by quantitative probabilities" that turn out to be correct.

Our model is inspired by Angeletos and La'o (2011) on sentiment-driven fluctuations as well as the Lucas (1972) island model. In our baseline model trades take place in centralized markets rather than bilaterally through random matching, and at the end of the period all trading history is public knowledge. Informational asymmetries exist only within the period as firms decide on how much to produce on the basis of the signals they receive at the beginning of the period. We extend our approach in Benhabib, Wang, and Wen (2012) into a setting with aggregate preference shocks. In the model, firms make production and employment decisions based on a signal related to consumer demand that may or may not also contain some idiosyncratic noise. Since consumer demand reflects fundamental preference shocks as well as pure sentiments, the firms face a signal extraction problem, even without the idiosyncratic noise. We show that under

[1] See also Benhabib, Wang, and Wen (2012), where the fundamental equilibrium is unique but, in contrast to models of global games as in Morris and Shin (1998), where the model has only a unique rational expectations equilibrium, an additional stochastic equilibrium emerges with the introduction of private but correlated signals for aggregate demand—an endogenous variable. For the classical work on extrinsic uncertainty and sunspots, see Cass and Shell (1983). For related work in the global games context, see Amador and Weill (2010), Angeletos and Werning (2006), Angeletos, Hellwig, and Pavan (2006), Gaballo (2012), Hellwig, Mukherji, and Tsyvinski (2006), and Hellwig and Veldkamp (2009).

reasonable conditions the signal extraction problem can lead to a continuum of sentiment-driven self-fulling equilibria. In contrast to Benhabib, Wang, and Wen (2012), sentiment-driven equilibria can arise here without any idiosyncratic noise in firms' information sets. They can be serially correlated over time, and furthermore they are not based on randomizations over the fundamental equilibria.

We describe the baseline model, the behavior of households, and the equilibria in the Sections that follow. In Section 5 we explore the persistence properties of the model when the preference shock is an autoregressive process, but we leave the exploration of Markov sunspots that randomize the fundamental and sentiment driven equilibria to the reader.[2] Finally we extend our model to the cases where consumer sentiments are heterogenous but correlated, and show that the results continue to hold.

2 Baseline Model

We consider a simple Dixit-Stiglitz model where the final consumption good is produced by a representative final-good firm from a continuum of intermediate goods. Each intermediate good is produced by a single monopolistic firm. Producers of intermediate goods must make production and employment decisions before the demand for intermediate goods is realized.[3] The output from each firm is then combined by a representative final-good producer to yield the final consumption good. Producers can perfectly observe the entire history of the economy up to the current decision period. At this stage production has not yet taken place, so households have only expectations or sentiments about their real wage and employment to guide their consumption plans, along with aggregate shocks to their preferences—which is the only fundamental shock we consider in this paper.[4]

Households can the real wage based on their sentiments about aggregate demand and their aggregate preference shock. The firms engage in market research and consumer surveys to get a sense, or a noisy signal, about consumer sentiments and aggregate demand. They then try to infer the island-specific demand for their particular intermediate goods based on their information set or imperfect signals, so they face a signal extraction problem. They hire workers from households by offering a nominal wage. As already noted, households have an expectation of the realization of output, and therefore of prices and the real wage, which is correct in equilibrium.

We show that in equilibrium firms' expectations on the sentiments of the households will be self-fulfilling, in the sense that at realized prices the goods markets and the labor market will clear, household expectations of the prices and real wages will be correct, and firms' forecasts of aggregate consumption demand will be confirmed. Furthermore the actual equilibrium distribution of output in this set-up will be consistent with the

[2] For constructing such sunspot equilibria see Benhabib, Wang, and Wen (2012).

[3] Since demand depends on income, income in turn depends on firms' production, which in turn depends on expected consumer demand, we can, in principle, make a distinction between planned demand and realized demand in this paper, following the traditional Keynesian literature.

[4] We can also interpret the preference shock as an aggregate productivity shock to the production of the aggregate final good.

distribution of consumer sentiments in a stochastic self-fulfilling equilibrium. We obtain, therefore, a stochastic rational expectations equilibrium driven by consumer sentiments. This equilibrium is not based on radomizations over multiple fundamental equilibria, in constrast to the indeterminacy literature (e.g., Benhabib and Farmer, 1994; and Farmer 2012).[5] In addition to the sentiment-driven equilibria, we can also obtain multiple fundamental rational expectations equilibria driven only by fundamental shocks but not by sentiments, despite the lack of any non-convexities in technologies and preferences.

To generate aggregate fluctuations, sentiments in our model must be correlated across households.[6] In the benchmark model, the aggregate sentiment is identical for all consumers. In the extension in Section 6 the consumer sentiments have a common as well as an i.i.d. idiosyncratic component. In this set-up we obtain essentially the same results.

To be more explicit, a representative household derives utility from a final good and leisure. The final good is produced by a representative final goods producer using a continuum of intermediate goods indexed by $j \in [0,1]$. Each intermediate good is produced using labor. We use labor as the numéraire so the wage rate is fixed at 1. The real wage (in terms of final goods) can of course fluctuate with the price of the final goods. The households are subject to aggregate preference (fundamental) shocks and sentiment (non-fundamental) shocks in each period. In the equilibrium of the benchmark model the households have perfect foresight. Namely, conditional on the aggregate shock and their sentiments, they can perfectly forecast the price level. Based on the forecasted price, and therefore the real wage, they make their consumption and labor supply decisions. The consumption decisions made by the households are the source of noisy demand signals for the intermediate goods producers. Based on their demand signals, obtained through market research, intermediate goods producers decide how much to produce, and the price of each intermediate good adjusts to equalize demand and supply on each island. These prices then determine the average cost of the final good and hence the price of the final good. In equilibrium this realized price coincides with the price expected by households based on sentiments. The results extend to the case where consumer sentiments are heterogenous but correlated.

2.1 Households

A representative household derives utility from final goods and leisure according to the utility function

$$U_t = A_t \frac{C_t^{1-\gamma}}{1-\gamma} - \frac{N_t^{1+\eta}}{1+\eta}, \tag{1}$$

where C_t is consumption of the final good, A_t is the preference shock, and N_t is labor supply. We assume that $\eta = 0$ for convenience.[7] The parameter γ is the inverse of the price elasticity of final good consumption. We normalize the nominal wage to 1 and

[5] See Cass and Shell (1983) for the classical case of sunspot equilibrium that is not based on randomizations over fundamental equilibria.

[6] The correlated sentiments induce correlated optimal choices for firms to generate additional stochastic equilibria, similar to correlated equilibria in games where the introduction of correlations in players' strategies can enlarge a game's equilibrium possibilities beyond the set of Nash equilibria.

[7] The quasi-linear utility function is assumed for simplicity without loss of generality.

write the household's budget constraint as $P_t C_t \leq N_t + \Pi_t$, where P_t is the price of the final good and Π_t is the aggregate profit income from all intermediate firms. Define $\frac{1}{P_t}$ as the real wage, then the budget constraint becomes

$$C_t \leq \frac{1}{P_t} N_t + \frac{\Pi_t}{P_t}. \tag{2}$$

Note that the real incomes of households fluctuate with P_t. The first-order condition for C_t is

$$A_t C_t^{-\gamma} = P_t. \tag{3}$$

A conjectured decrease in the price level P_t will induce the household to consume more. Households observe the aggregate preference shock A_t and an aggregate sentiment ("sunspot") shock Z_t and conjecture that the equilibrium aggregate price is given by $P_t = P(A_t, Z_t)$ and therefore that the real wage is $(P_t)^{-1}$. We assume $z_t \equiv \log(Z_t)$ is normally distributed with zero mean and unit variance. An equilibrium is a "fundamental equilibrium" if it is not affected by z_t. Otherwise we call the equilibrium a "sentiment-driven equilibrium".

2.2 Firms

The supply side has a representative final good producer and a continuum of intermediate goods producers indexed by $j \in [0, 1]$. The final good producer serves as an aggregator of all intermediate goods, and it does not play an active role in the model. We assume the final good producer makes decisions after all shocks are realized, so its decisions are not subject to any uncertainty.

The final good firm. The final good firm solves

$$\max_{C_{jt}} P_t C_t - \int P_{jt} C_{jt} dj, \tag{4}$$

where C_t is produced by a continuum of intermediate goods according to the Dixit-Stiglitz production function,

$$C_t = \left[\int_0^1 C_{jt}^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}}. \tag{5}$$

The final goods producer's profit maximization problem yields the inverse demand function for each intermediate good,

$$\frac{P_{jt}}{P_t} = C_{jt}^{-\frac{1}{\theta}} C_t^{\frac{1}{\theta}}, \tag{6}$$

and the aggregate price index,

$$P_t = \left[\int_0^1 P_{jt}^{1-\sigma} \right]^{\frac{1}{1-\sigma}}. \tag{7}$$

The intermediate goods firms. The intermediate goods firms use labor as the only input to produce output according to

$$C_{jt} = N_{jt}. \tag{8}$$

Unlike the households and the final good producer, the intermediate goods producers face uncertainty in making their production decisions: they do not have full information regarding the aggregate demand shock A_t and the aggregate price P_t. We assume that intermediate firm j has to choose its production based on a noisy signal about aggregate demand. Denote the signal as S_{jt}. The intermediate good firm j solves

$$\max_{C_{jt}} E[(P_{jt}C_{jt} - C_{jt})|S_{jt}], \tag{9}$$

with the constraint (6). Substituting out P_{jt}, the first-order condition for C_{jt} is

$$C_{jt} = \left\{ E[P_t C_t^{\frac{1}{\theta}} |S_{jt}] \left(1 - \frac{1}{\theta}\right) \right\}^{\theta}. \tag{10}$$

Using the first-order condition of the household in equation (3), we then have

$$C_{jt} = \left(1 - \frac{1}{\theta}\right)^{\theta} \left\{ E[A_t C_t^{\frac{1}{\theta}-\gamma} |S_{jt}] \right\}^{\theta}. \tag{11}$$

We assume that the signal is a mixture of aggregate demand (C_t) and idiosyncratic noise (v_{jt}) given by

$$s_{jt} \equiv \log S_{jt} = \log C_t + v_{jt} \equiv c_t + v_{jt}, \tag{12}$$

where v_{jt} is normally distributed with mean of 0 and variance of σ_v^2. For notational convenience we will re-scale the aggregate preference shock A_t as $A_t = \left(\frac{\theta}{\theta-1}\right)\exp(a_t/\theta)$, where a_t is normally distributed with mean 0 and variance σ_a^2.

We note that in what follows, the noise v_{jt} will not be essential for our results: we could have set $\sigma_v^2 = 0$. In that case the signal s_{jt} would fully reveal aggregate consumption c_t to the intermediate goods firms; but, as we will see in Section 4, sentiment-driven rational expectations equilibria would still exist.

2.3 General Equilibrium

We define the general equilibrium recursively as follows:

- Based on the preference shock A_t and sentiment Z_t, households conjecture that the aggregate price is $P_t = P(A_t, Z_t)$,and real wage is $(P_t)^{-1}$;
- Based on the conjectured price P_t and real wage is $(P_t)^{-1}$, the households choose their consumption plan $C_t = C(A_t, Z_t)$ according to (3) to maximize their utility;
- The consumption decisions create signals to firms j as $\log S_{jt} = c_t + v_{jt}$;

- Based on the signal S_{jt}, firm j hires workers and produces C_{jt} according to (11) to maximize its expected profit;
- Given the production of C_{jt}, price P_{jt} adjusts to equate demand and supply according to equation (6);
- The total production of final good C_t, according to (5), equals the households' planned consumption. Hence the realized aggregate price is equal to the conjectured price P_t and the realized real wage is the conjectured real wage is $(P_t)^{-1}$.

It turns out that equations (5), (11), and (12) are sufficient to characterize the general equilibrium. We conjecture that the equilibrium production (in logarithm) can be written as $\log C_{jt} = \tilde{c} + c_{jt}$ and $\log C_t = \bar{c} + c_t$ and that c_{jt} and c_t are solutions to the following systems of equations:

$$c_{jt} = E\{[a_t + \beta c_t]|s_{jt}\}, \tag{13}$$

$$c_t = \int_0^1 c_{jt} dj, \tag{14}$$

$$s_{jt} = c_t + v_{jt}, \tag{15}$$

where

$$\beta \equiv 1 - \gamma\theta. \tag{16}$$

Notice that $\theta > 1$ and $\gamma > 0$; hence, we have $\beta \in (-\infty, 1)$. The intermediate firm's output c_{it} would decrease with aggregate demand c_t if $\beta < 0$, which implies that intermediate goods are strategic substitutes; whereas $\beta > 0$ would correspond to the case of strategic complementarity among intermediate goods. Hence, our model is flexible enough to characterize both strategic complementarity and strategic substitutability in production, and our results hold true even if $\beta < 0$. Equilibrium in the model is then fully characterized by $\{\tilde{c}, \bar{c}\}$ and two mappings, $c_{jt} = c_{jt}(s_{jt})$ and $c_t = c(a_t, z_t)$ that solve equations (13) for all j and equation (14). We are now ready to characterize all the possible equilibria.

3 Fundamental Equilibria

We first study the equilibria driven only by fundamentals, in particular by preference shocks a_t. In a fundamental equilibrium neither aggregate consumption c_t nor the production of each intermediate good c_{jt} is affected by consumer sentiments. We show that this simple model permits multiple fundamental equilibria.

We use a conjecture-and-verification strategy to find the equilibria. A guess for the solution to the system of equations (13) to (15) is

$$c_t = \phi a_t, \tag{17}$$

where ϕ is an undetermined coefficient. Fnding equilibrium is then equivalent to determining the coefficient ϕ.

3.1 A Constant Output Equilibrium

Proposition 1 *The allocation with $p_t \equiv \log P_t - \bar{p} = a_t/\theta$ and $c_{jt} = c_t = 0$ is always an equilibrium.*

Proof: See Appendix A.1 ■

In this case consumption does not respond to preference shock a_t; namely, the demand elasticity $\phi = 0$ and the equilibrium aggregate price $P_t = \exp(\bar{p} + a_t/\theta)$. To see the intuition, suppose that there is an increase in a_t, which makes households want to spend more if all else is equal. But whether households actually spend more also depends on their expectation of the aggregate price (or real wage). In the above equilibrium, households conjecture that the price will rise exactly in proportion to preference shocks so their incentive to consume more is completely curbed.

3.2 Stochastic Fundamental Equilibria

In this case consumption responds to the preference shock a_t with the demand elasticity $\phi \in \left(0, \frac{1}{\gamma\theta}\right)$. We show that there can be two such equilibria in the model under certain conditions. Suppose household consumption in logarithm is given by $\log C_t - \bar{c} = c_t = \phi a_t$, with $\phi > 0$. Households conjecture that price is given by

$$\log P_t - \bar{p} = \phi_p a_t = \left(\frac{1}{\theta} - \gamma\phi\right) a_t. \tag{18}$$

Note that equation (3) is satisfied, implying that the households' consumption is optimal. In what follows we use the method of undetermined coefficients to determine the coefficient ϕ and the constants $\bar{c}$ and $\bar{p}$.

To solve for ϕ, we utilize equation (13). Using the above conjectured equilibrium for c_t, we express the production of each intermediate goods firm as

$$\log C_{jt} - \tilde{c} = c_{jt} = E(a_t + \beta c_t)|(\phi a_t + v_{jt}) = \frac{(\phi + \beta\phi^2)\sigma_a^2}{\phi^2\sigma_a^2 + \sigma_v^2}(\phi a_t + v_{jt}). \tag{19}$$

Aggregating all firms' output across j gives aggregate output c_t; then, by matching the coefficient of a_t, we obtain $\frac{(\phi+\beta\phi^2)\phi\sigma_a^2}{\phi^2\sigma_a^2+\sigma_v^2} = \phi$. Rearranging terms leads to a quadratic equation in ϕ:

$$(\phi + \beta\phi^2)\sigma_a^2 = \phi^2\sigma_a^2 + \sigma_v^2. \tag{20}$$

Notice that in general, there is no guarantee that the solution to the above equation is unique. Denoting

$$\mu = \frac{\sigma_v^2}{\sigma_a^2} \tag{21}$$

as the noise ratio, we have the following Proposition:

Proposition 2 *Suppose* $0 < \mu < \frac{1}{4(1-\beta)}$ *and let* $c_t = \log C_t - \bar{c} = \phi a_t$. *In a rational expectations equilibrium the aggregate price is*

$$p_t = \log P_t - \bar{p} = \left(\frac{1}{\theta} - \gamma\phi\right) a_t, \tag{22}$$

each firm j *produces*

$$c_{jt} = \phi a_t + v_{jt}, \tag{23}$$

where ϕ *is given by*

$$\phi = \frac{1}{2(1-\beta)} \pm \sqrt{\frac{1}{4(1-\beta)^2} - \frac{\mu}{1-\beta}} \in \left(0, \frac{1}{\gamma\theta}\right), \tag{24}$$

and $\{\bar{c}, \bar{p}, \tilde{c}\}$ *are given by*

$$\bar{c} = \frac{1}{2}\frac{1}{\theta^2\gamma}[(\theta-1)\sigma_v^2 + (1+\beta\phi)(1-(1-\beta)\phi\sigma_a^2)], \tag{25}$$

$$\bar{p} = \log\left(\frac{\theta}{\theta-1}\right) - \gamma\bar{c}, \tag{26}$$

$$\tilde{c} = (1-\theta\gamma)\bar{c} + \frac{1}{2}\frac{(1+\beta\phi)(1-(1-\beta)\phi)\sigma_a^2}{\theta} \tag{27}$$

Proof: *See Appendix A.2* ■

This proposition shows that for any given value of the noise ratio $\mu \in \left(0, \frac{1}{4(1-\beta)}\right)$, there exist two additional fundamental equilibria: each corresponds to a particular value of ϕ. In the special case where $v_{jt} \equiv 0$ so that $\sigma_v^2 = 0$ and the signal fully reveals aggregate demand c_t to the firms, it is easy to see from equation (20) that one of the equilibria now coincides with the constant output equilibrium with $\phi = 0$. In the second equilibrium we have $\phi = (1-\beta)^{-1}$. Since the signal reveals c_t fully, by equation (23) a_t is also fully revealed to firms in equilibrium, so we may call this type of equilibrium (with $\sigma_v^2 = 0$) the full information equilibrium.

In the two additional fundamental equilibria of Proposition 2, the equilibrium price does not respond fully to preference shocks. If the households think price will respond to the preference shocks less strongly, they will consume more in the aggregate when a_t increases. This then sends a more precise signal to the intermediate goods producers, as consumption volatility would be relatively larger relative to the noise in the signal. As a result, the firms produce more and, indeed, the aggregate market clearing price rises less, confirming the initial belief of the households.[8]

[8] In Benhabib, Wang, and Wen (2012), where firms that produce intermediate goods face idiosyncratic demand shocks as opposed to aggregate demand shocks, there also are sentiment-driven stochastic equilibria, but the fundamental equilibrium is always unique.

4 Sentiment-Driven Equilibria

Now we consider another type of equilibrium in which consumption responds to a pure sentiment variable z_t that is completely unrelated to the fundamental shock a_t. More importantly, the variance (uncertainty) of sentiment is itself a self-fulfilling object. We note that the existence of sentiment-driven equilibria is not based on randomizations over the fundamental equilibria studied above.

Suppose households incur a sentiment shock called z_t. After knowing the sentiment shock and observing aggregate preference shock a_t, households choose their optimal consumption based on their conjecture of the price level. Let us conjecture an equilibrium in which household consumption takes the form

$$c_t = \phi a_t + \sigma_z z_t, \tag{28}$$

along with the conjectured price $p_t = \alpha_a a_t + \alpha_z z_t$. For notational convenience, we have normalize the variance of z_t to unity, so the scaler σ_z represents the standard deviation of the sentiment shock (i.e., $var(\sigma_z z_t) = \sigma_z^2$). Given aggregate consumption demand, the production of the individual firm j is

$$\begin{aligned} c_{jt} &= E(a_t + \beta c_t)|(c_t + v_{jt}) \qquad (29)\\ &= E(a_t + \beta\phi a_t + \beta\sigma_z z_t)|(\phi a_t + \sigma_z z_t + v_{jt}) \qquad (30)\\ &= \frac{(\phi + \beta\phi^2)\sigma_a^2 + \beta\sigma_z^2}{\phi^2\sigma_a^2 + \sigma_v^2 + \sigma_z^2}(\phi a_t + \sigma_z z_t + v_{jt}). \end{aligned}$$

Aggregating firm-level production across j and comparing coefficients of a_t and z_t between this aggregated equation and equation (28) gives

$$\phi = \frac{(\phi + \beta\phi^2)\sigma_a^2 + \beta\sigma_z^2}{\phi^2\sigma_a^2 + \sigma_v^2 + \sigma_z^2}\phi \tag{31}$$

and

$$\frac{(\phi + \beta\phi^2)\sigma_a^2 + \beta\sigma_z^2}{\phi^2\sigma_a^2 + \sigma_v^2 + \sigma_z^2} = 1. \tag{32}$$

Notice that equations (31) and (32) are identical as long as $\phi \neq 0$.

Lemma 1 *If $\phi = 0$, then there is no sentiment-driven equilibrium.*
Proof: *The proof is straightforward. If $\phi = 0$ and a sentiment-driven equilibrium exists, then (32) becomes*

$$(\beta - 1)\sigma_z^2 = \sigma_v^2 \geq 0 \tag{33}$$

Since $\beta < 1$, we have a contradiction. ■

Proposition 3 *Suppose $\phi > 0$ and $\sigma_v^2 < \frac{1}{4(1-\beta)}\sigma_a^2$. There exists a continuum of sentiment-driven equilibria indexed by variance of sentiments in the interval $\sigma_z^2 \in \left(0, \frac{1}{4(1-\beta)^2}\sigma_a^2 - \frac{\sigma_v^2}{1-\beta}\right)$.*

At each sentiment-driven equilibrium within this interval, the equilibrium price is given by

$$p_t = \log P_t - \bar{p} = \left(\frac{1}{\theta} - \gamma\phi\right) a_t - \gamma\sigma_z z_t, \tag{34}$$

and the optimal consumption level is given by

$$c_t = \log C_t - \bar{c} = \phi a_t + \sigma_z z_t, \tag{35}$$

where ϕ is given by

$$\phi = \frac{1}{2(1-\beta)} \pm \sqrt{\frac{1}{4(1-\beta)^2} - \frac{\tilde{\mu}}{1-\beta}} > 0 \tag{36}$$

and $\tilde{\mu} = \frac{\sigma_v^2 + \sigma_z^2(1-\beta)}{\sigma_a^2}$. The constants in the price and consumption rules are given by

$$\bar{c} = \frac{1}{2\gamma\theta^2}[(\theta-1)\sigma_v^2 + (1+\beta\phi)(1-(1-\beta)\phi)\sigma_a^2 - \beta\sigma_z^2(1-\beta)]. \tag{37}$$

$$\bar{p} = \log\left(\frac{\theta}{\theta-1}\right) - \gamma\bar{c}. \tag{38}$$

Proof: *See Appendix 3.* ■

Re-arranging equation (32) yields

$$\sigma_z^2 = \frac{\phi(1-(1-\beta)\phi)\sigma_a^2 - \sigma_v^2}{1-\beta}. \tag{39}$$

Hence, the equilibrium aggregate demand (production) is determined by

$$c_t = \phi a_t + \sqrt{\frac{\phi(1-(1-\beta)\phi)\sigma_a^2 - \sigma_v^2}{1-\beta}} z_t, \tag{40}$$

which shows that not only the level of sentiments z_t matters but the variance of sentiments also matters (since ϕ depends on σ_z by equation (36)). More importantly, the degree of uncertainty, that is σ_z^2, is itself self-fulfilling in a sentiment-driven equilibrium.

We can use equation (39) to rewrite Equation (37) as

$$\bar{c} = \frac{1}{2\gamma\theta^2}[(\theta-1+\beta)\sigma_v^2 + (1-(1-\beta)\phi)\sigma_a^2)], \tag{41}$$

where ϕ is given by (36). It is evident that the effect of σ_z^2 on the mean consumption depends on the value of ϕ. For the equilibrium with $\phi = \frac{1}{2(1-\beta)} + \sqrt{\frac{1}{4(1-\beta)^2} - \frac{\tilde{\mu}}{1-\beta}}$, an increase in σ_z^2 reduces mean consumption while for the equilibrium with $\phi = \frac{1}{2(1-\beta)} - \sqrt{\frac{1}{4(1-\beta)^2} - \frac{\tilde{\mu}}{1-\beta}}$, an increase in σ_z^2 will increase mean consumption.

The intuition for the sentiment-driven equilibria is similar to the intuition for multiple fundamental equilibria. Which equilibrium prevails depends on consumer's expectation of the aggregate price level, which depends negatively on the sentiments (equation (34)). If consumers are optimistic, they would anticipate a lower aggregate price level (cheaper consumption goods, higher real wage), so they choose to consume more. Since firms cannot distinguish the fundamental shock a_t from the sentiment shock z_t, they choose to produce more to meet the higher expected consumption demand indicated by the signal, which fulfills the consumer sentiments. On the other hand, if demand is more volatile due to more variable sentiments, firms would opt to attach less weight to fundamentals (preferences) in signal extraction, rendering production more volatile. Namely, at each sentiment-driven equilibrium (indexed by σ_z), the intermediate goods firms produce exactly the amount of goods, aggregated into the final good, that the households want to consume, and markets clear.

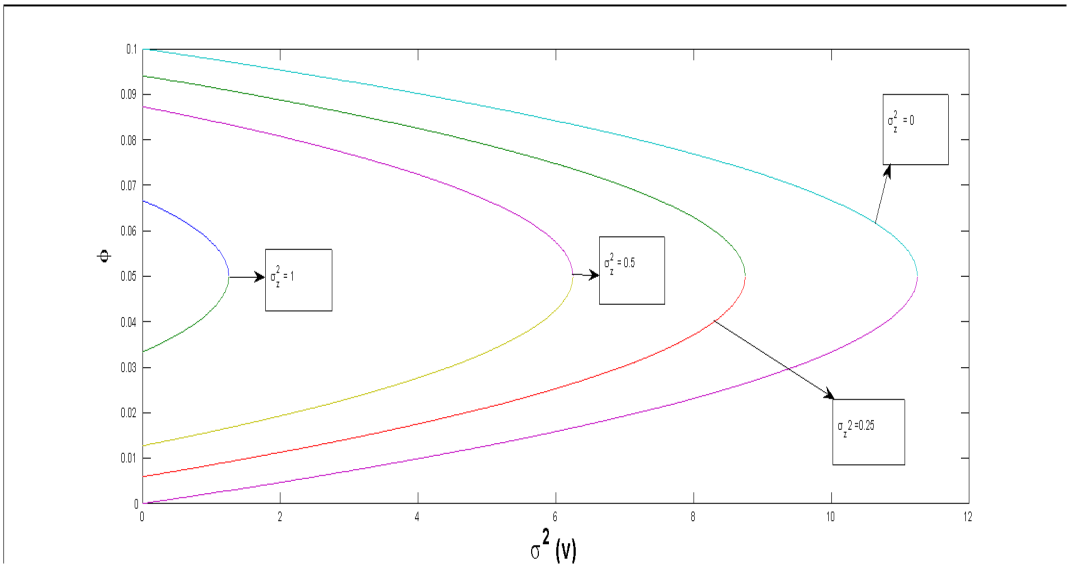

Figure 1. Contour of the relationship between ϕ and σ_v^2 for any given σ_z.

The results of Proposition 3 still hold even if we set $\sigma_v^2 = 0$—that is, if we allow the signal s_{jt} to fully reveal aggregate consumption c_t to intermediate goods firms. Nevertheless, as we see from (30), the firms set their optimal outputs under imperfect information using their signal $s_{jt} = c_t$ because they do not directly observe a_t and z_t separately. So in this case even if aggregate consumption c_t is fully observed by intermediate goods firms, we see from (35) that sentiments z_t still drive aggregate and firm outputs in rational expectations equilibrium.

In Figure 1 we plot the coefficients ϕ for the fundamental stochastic equilibria in Proposition 2, and the corresponding coefficients ϕ for the sentiment-driven equilibria in Proposition 3, against variance of the noise σ_v^2. We calibrate $\theta = 10^9$, $\gamma = 1$, the variance of log A_t at 4.5, and we plot ϕ against feasible σ_v^2 for various variances of sentiments $\sigma_z^2 = \{0,\ 0.25,\ 0.5,\ 1\}$.

Note that $\sigma_z^2 = 0$ (the outmost contour or hyperbola) yields the pairs of ϕ for the two fundamental stochastic equilibria for each value of σ_v^2. Figure 1 thus makes it clear

[9] In typical calibrations $\theta = 10$ implies a markup of about 11%.

that these fundamental equilibria may be viewed as a special case of sentiment-driven equilibria where the variance of sentiments σ_z^2 go to zero. We can also observe in Figure 1 how changing σ_z^2 generates additional pairs of ϕ centering sentiment-driven stochastic equilibria for various values of σ_v^2. For each σ_v^2 we may have up to five types of equilibria: a continuum of pairs of sentiment-driven equilibria indexed by σ_z^2, a pair of stochastic fundamental equilibria where aggregate consumption c_t is driven only by fundamental shocks a_t, and a constant output equilibrium.

5 Persistence

In this section we show that the sentiment-driven equilibria can be serially correlated over time under reasonable information structures and that the persistence in the sentiment-driven equilibria mimics the serial correlation property of the fundamental shocks. Note that so far the noise v_{jt} is not essential for producing sentiment-driven equilibria and, hence, we drop it from the signal. Suppose that the aggregate shock follows

$$a_t = \rho a_{t-1} + \sigma_a \varepsilon_t, \tag{42}$$

where $\rho = 0$ is the special case we considered before. We assume that each firm can observe the entire history of aggregate production (or aggregate demand c_{t-k}, $k = 0, 1, 2, ...$) but not the history of preference shocks separately. Namely,

$$s_{jt} = [c_t, c_{t-1}, c_{t-2}, ...c_{t-\infty}]. \tag{43}$$

In this case, as in the benchmark model, the stochastic fundamental equilibria still exist. It is easy to show that at a fundamental equilibrium, we have $c_{jt} = c_t = \frac{1}{1-\beta} a_t$, where $\beta = 1 - \gamma\theta$.

We conjecture the existence of sentiment-driven equilibria where z_t is serially correlated and aggregate output takes the form

$$c_t = \phi a_t + \sigma_z z_t, \tag{44}$$

where the sentiment z_t follows the same law of motion as the aggregate preference shocks,

$$z_t = \rho z_{t-1} + \varepsilon_{z,t}. \tag{45}$$

Again we have normalized the variance of ε_{zt} to unity. At a sentiment-driven equilibrium the past realizations of aggregate consumption cannot help firms pin down the innovations in fundamental shock $\varepsilon_{t-k}, k = 1, 2, ...$ The history, however, can reveal the sum of ε_{t-k} and $\varepsilon_{z,t-k}$ for $k \geq 1$. So the signal for firm j is

$$s_{jt} = [\phi\varepsilon_t + \sigma_z \varepsilon_{z,t}, ..., \ \phi\varepsilon_{t-k} + \sigma_z \varepsilon_{z,t-k}, ...]. \tag{46}$$

The effective signal for a firm's decision making can be simplified to $s_{jt} = [\phi\varepsilon_t + \sigma_z\varepsilon_{z,t}, \phi a_{t-1} + \sigma_z z_{t-1}]$. Proposition 4 shows the conditions for the existence of serially correlated sentiment-driven equilibria.

Proposition 4 *There exists a continuum of sentiment-driven equilibria indexed by the noise ratio* $\frac{\sigma_z^2}{\sigma_a^a} \in \left(0, \frac{1}{4(1-\beta)^2}\right)$. *For each permissible value of* σ_z^2, *the aggregate price is given by*

$$p_t = \log P_t - \bar{p} = \left(\frac{1}{\theta} - \gamma\phi\right) a_t - \gamma\sigma_z z_t, \tag{47}$$

where $\bar{p} = \log\left(\frac{\theta}{\theta-1}\right) - \gamma\bar{c}$; *and the aggregate production is given by*

$$c_t = \log C_t - \bar{c} = \phi a_t + \sigma_z z_t, \tag{48}$$

where

$$\phi = \frac{1}{2(1-\beta)} \pm \sqrt{\frac{1}{4(1-\beta)^2} - \frac{\sigma_z^2}{\sigma_a^2}} \tag{49}$$

$$\bar{c} = \frac{1}{2\gamma\theta^2} \frac{\sigma_z^2}{\phi^2\sigma_a^2 + \sigma_z^2}. \tag{50}$$

Proof: *See Appendix 4.* ■

6 An Extension

In the baseline model we assumed that all households have the same sentiment z_t. In this section, we show that our results are robust to heterogenous sentiment shocks. We index individual households by $i \in [0, 1]$. Suppose in the beginning of each period households receive a noisy sentiment signal z_{it},

$$z_{it} = z_t + e_{it}, \tag{51}$$

so that the sentiments are correlated across households because of the common component z_t.[10] Suppose consumers choose their consumption expenditure C_{it} on the basis of expected price given their signal z_{it}. As before, suppose each household conjectures that the aggregate price will be determined by

$$\log P_t - \bar{p} = p_t = \phi_a^p a_t + \sigma_z^p z_t, \tag{52}$$

with undetermined coefficients $\{\phi_a^p, \sigma_z^p\}$. In a competitive environment, consumers have the incentive to figure out the aggregate sentiment z_t because it matters for the aggregate price level and the real wage. Each consumer therefore faces a signal extraction problem.[11]

[10] In this case if firms survey a subset of consumers they will obtain a noisy signal (the sample mean) of the average sentiment z_t. In Benhabib, Wang and Wen (2012), in addition to a private signal, we directly introduce a second noisy public signal of the common sentiments. In both cases firms can observe the average sentiment only with noise, so they still face the problem of extracting the separate fundamental and productivity shocks from their signals.

[11] It is easy to see that the fundamental equilibrium is not affected by heterogeneous sentiments: if the aggregate price depends only on the aggregate preference shock, the sentiment shocks will not affect the consumption decision of the households.

The first-order condition for consumers now changes to

$$C_{it} = \left\{ \frac{1}{E(P_t|z_{it})} \exp(a_t/\theta) \left(\frac{\theta}{\theta-1} \right) \right\}^{\frac{1}{\gamma}}. \tag{53}$$

Aggregating across consumers, we obtain the aggregate consumption $c_t = \log C_t = \log(\int_0^1 C_{it} di)$. As before, we assume that each firm receives a noisy signal $\log S_{jt} = c_t + v_{jt}$. The production decision by the firms is given by equation (10) as before, namely,

$$C_{jt} = \left\{ E[P_t C_t^{\frac{1}{\theta}} | S_{jt}](1 - \frac{1}{\theta}) \right\}^{\theta}. \tag{54}$$

An equilibrium of the economy is defined again as in Section 2.3. We have the following Proposition:

Proposition 5 *Suppose $\sigma_v^2 < \frac{1}{4(1-\beta)} \sigma_a^2$ and let $\kappa = \frac{1}{1+\sigma_e^2}$. There exists a continuum of sentiment-driven equilibria indexed by $\sigma_z^2 \in \left(0, \frac{\kappa}{4(1-\beta)^2}\sigma_a^2 - \frac{\kappa \sigma_v^2}{1-\beta}\right)$. At each equilibrium the aggregate price is*

$$p_t = \log P_t - \bar{p} = \phi_a^p a_t + \phi_z^p z_t \equiv \left(\frac{1}{\theta} - \gamma \phi \right) a_t - \frac{\gamma}{\kappa} \sigma_z z_t,$$

and the aggregate consumption (output) is

$$\log C_t - \bar{c} = c_t = \phi a_t + \sigma_z z_t, \tag{55}$$

where

$$\phi = \frac{1}{2(1-\beta)} \pm \sqrt{\frac{1}{4(1-\beta)^2} - \frac{\tilde{\mu}}{1-\beta}} \tag{56}$$

and $\tilde{\mu} = \frac{\sigma_v^2 + \sigma_z^2(1-\beta)/\kappa}{\sigma_a^2}$. Consumers' idiosyncratic consumption demand is

$$\log C_{jt} - \tilde{c} = c_{it} = \phi a_t + \sigma_z (z_t + e_{it}). \tag{57}$$

Each individual firm's optimal production is

$$\log C_{it} - \hat{c} = c_{jt} = \phi a_t + \sigma_z z_t + v_{jt}. \tag{58}$$

The constant terms are given by

$$\bar{p} = \log(\frac{\theta}{\theta-1}) - \frac{\theta-1}{2\theta^2}\sigma_v^2 - \frac{1}{2}\Omega_s \tag{59}$$

$$\bar{c} = \frac{1}{\gamma}[\frac{\theta-1}{2\theta^2}\sigma_v^2 + \frac{1}{2}\Omega_s] - \frac{\gamma}{2}\left(\frac{1}{\kappa}\sigma_z\right)^2 (1-\kappa) + \frac{1}{2}\sigma_z^2 \frac{1-\kappa}{\kappa} \tag{60}$$

$$\hat{c} = \bar{c} - \frac{1}{2}\sigma_z^2 \sigma_e^2, \qquad \tilde{c} = \bar{c} - \frac{1}{2}\frac{\theta-1}{\theta}\sigma_v^2. \tag{61}$$

$$\tilde{c} = \bar{c} - \frac{1}{2}\frac{\theta-1}{\theta}\sigma_v^2. \tag{62}$$

Proof: See Appendix 5. ■

7 Conclusion

We explore the Keynesian idea that sentiments or animal spirits can influence the level of aggregate income and give rise to recurrent boom-bust cycles. We show that in a production economy, pure sentiments (completely unrelated to fundamentals) can indeed affect economic performance and the business cycle even though (i) expectations are fully rational and (ii) there are no externalities or non-convexties or even strategic complementarities. In particular, we show that when consumption and production decisions must be made separately by consumers and firms based on mutual forecasts of each other's actions, the equilibrium outcome can indeed be influenced by animal spirits or sentiments, even though all agents are fully rational.[12] Furthermore the existence of sentiment-driven equilibria is not based on randomizations over the fundamental equilibria studied above. The key to generating our results is a natural friction in information: Although firms can perfectly observe or forecast consumption demand, they cannot separately identify the components of demand stemming from consumer sentiments as opposed to preference shocks (fundamentals). Sentiments matter because they are correlated across households, so they affect aggregate demand and real wages differently than shocks to aggregate productivity (or preferences). Faced with a signal extraction problem, firms make optimal production decisions that depend on the degree of sentiment uncertainty or the variance of sentiment shocks. In our model there exists a continuum of (normal) distributions for sentiment shocks indexed by their variances that give rise to self-fulfilling rational expectations equilibria.

[12] Compare to Townsend (1983).

A Appendix

A. 1 Proof of Proposition 1

Proof: Suppose households conjecture that the aggregate price is given by

$$p_t = \log P_t - \bar{p} = a_t/\theta, \tag{A.1}$$

where P_t satisfies equation (3). Then aggregate consumption must be a constant $\bar{C}$. This implies that the signal $s_{jt} = \log C_t + v_{jt}$ is nothing but pure noise. Hence, by equation (10) each firm's production is also a constant given by $C_{jt} = C_t = \bar{C}$. Equation (11) can be written as

$$C_{jt} = \left\{ E[\exp(a_t/\theta) C_t^{\frac{1}{\theta}-\gamma} | S_{jt}] \right\}^\theta = C_t^{1-\gamma\theta} \left\{ E[\exp(a_t/\theta) | S_{jt}] \right\}^\theta, \tag{A.2}$$

which, under the log-normal assumptions, implies

$$\gamma\theta \log C_t = \theta \log E \exp(a_t/\theta) = \frac{1}{2}\theta \frac{\sigma_a^2}{\theta^2}. \tag{A.3}$$

This implies

$$\log C_t = \log C_{jt} = \bar{c} = \tilde{c} = \frac{1}{2\gamma\theta^2}\sigma_a^2. \tag{A.4}$$

Since the conjecture of the aggregate price is self-fulfilling, the total supply is indeed a constant and all markets clear under the conjectured prices. ■

A. 2 Proof of Proposition 2

Proof: From the definition of μ we obtain

$$\phi(1 - (1-\beta)\phi) = \mu. \tag{A.5}$$

Note that there are two solutions for ϕ if $0 < \mu < \max_\phi \phi(1-(1-\beta)\phi) = \frac{1}{4(1-\beta)}$, given by

$$\phi = \frac{1}{2(1-\beta)} \pm \sqrt{\frac{1}{4(1-\beta)^2} - \frac{\mu}{1-\beta}} > 0. \tag{A.6}$$

It is easy to see that for $\mu > 0$

$$0 < \phi < \frac{1}{1-\beta}. \tag{A.7}$$

Given ϕ, we can calculate the three constants $\tilde{c}, \bar{c}$ and $\bar{p}$ to fully characterize the equilibrium. The fact that aggregate consumption is log-normally distributed implies that we can obtain $\tilde{c}$ from equation (11),

$$\tilde{c} = (1-\theta\gamma)\bar{c} + \frac{\theta}{2}\Omega_s, \tag{A.8}$$

where Ω_s is the conditional variance of $a_t/\theta+(\frac{1}{\theta}-\gamma)\phi a_t$ based on the signal. The variance Ω_s is:

$$\begin{aligned}\Omega_s &= var[(a_t/\theta+(\frac{1}{\theta}-\gamma)\phi a_t)|\phi a_t+v_{jt}] \\ &= \frac{1}{\theta^2}var(a_t+\beta\phi a_t|\phi a_t+v_{jt}) \\ &= \frac{(1+\beta\phi)^2\sigma_a^2-(1+\beta\phi)\phi\sigma_a^2}{\theta^2} \\ &= \frac{(1+\beta\phi)(1-(1-\beta)\phi)\sigma_a^2}{\theta^2}. \end{aligned} \quad (A.9)$$

Finally, notice that $c_{jt}=\phi a_t+v_{jt}$, so the dispersion in the production of the intermediate goods is purely due to the noisy signal. We then obtain

$$\bar{c}=\frac{1}{2}\frac{\theta-1}{\theta}\sigma_v^2+\tilde{c} \quad (A.10)$$

by equation (5). With the two equations and two unknows $\bar{c}$ and $\tilde{c}$, we obtain

$$\bar{c}=\frac{1}{2}\frac{1}{\theta^2\gamma}[(\theta-1)\sigma_v^2+(1+\beta\phi)(1-(1-\beta)\phi)\sigma_a^2]. \quad (A.11)$$

Once we obtain $\bar{c}$, by equation (3) we can obtain $\bar{p}=\log\left(\frac{\theta}{\theta-1}\right)-\gamma\bar{c}$ and

$$\tilde{c}=(1-\theta\gamma)\bar{c}+\frac{1}{2}\frac{(1+\beta\phi)(1-(1-\beta)\phi)\sigma_a^2}{\theta}. \quad (A.12)$$

Since both households and the firms' optimization conditions are satisfied and the planned consumption equals the actual consumption, we have a rational expectations equilibrium. ■

A. 3 Proof of Proposition 3

Proof: Notice that for $\phi>0$ equations (31) and (32) are identical, so we only need to consider equation (32). After re-arranging terms we obtain

$$\phi(1-(1-\beta)\phi)\sigma_a^2=\sigma_v^2+\sigma_z^2(1-\beta). \quad (A.13)$$

Notice that for $\sigma_v^2<\frac{1}{4(1-\beta)}\sigma_a^2$, we can find a continuum of σ_z^2 to satisfy the above equation. Namely, there exists a continuum of sentiment-driven equilibria indexed by $\sigma_z^2\in\left(0,\frac{1}{4(1-\beta)^2}\sigma_a^2-\frac{\sigma_v^2}{1-\beta}\right)$ such that (A.13) is satisfied. Given σ_z^2, we can solve for ϕ as

$$\phi=\frac{1}{2(1-\beta)}\pm\sqrt{\frac{1}{4(1-\beta)^2}-\frac{\tilde{\mu}}{1-\beta}}, \quad (A.14)$$

where $\tilde{\mu}=\frac{\sigma_v^2+\sigma_z^2(1-\beta)}{\sigma_a^2}$. Once we obtain ϕ, we can then solve for $\tilde{c}$ and $\bar{c}$. The production of each firm is given by

$$c_{jt}=\phi a_t+\sigma_z z_t+v_{jt}. \quad (A.15)$$

To solve the constants we first use expression (11) to obtain

$$\tilde{c} = (1-\theta\gamma)\bar{c} + \frac{\theta}{2}\Omega_s, \tag{A.16}$$

where

$$\begin{aligned}
\Omega_s &= var[a_t/\theta + (\frac{1}{\theta} - \gamma)(\phi a_t + \sigma_z z_t)|\phi a_t + \sigma_z z_t + v_{jt}] \\
&= \frac{1}{\theta^2} var(a_t + \beta\phi a_t + \beta\sigma_z z_t|\phi a_t + \sigma_z z_t + v_{jt}) \\
&= \frac{(1+\beta\phi)^2\sigma_a^2 + \beta^2\sigma_z^2 - (1+\beta\phi)\phi\sigma_a^2 - \beta\sigma_z^2}{\theta^2} \\
&= \frac{(1+\beta\phi)(1-(1-\beta)\phi)\sigma_a^2 - \beta\sigma_z^2(1-\beta)}{\theta^2}.
\end{aligned} \tag{A.17}$$

And again we also have

$$\bar{c} = \frac{1}{2}\frac{\theta-1}{\theta}\sigma_v^2 + \tilde{c}, \tag{A.18}$$

or

$$\bar{c} = \frac{1}{2\gamma}\left[\frac{\theta-1}{\theta^2}\sigma_v^2 + \frac{(1+\beta\phi)(1-(1-\beta)\phi)\sigma_a^2 - \beta\sigma_z^2(1-\beta)}{\theta^2}\right]. \tag{A.19}$$

Finally, we have

$$\bar{p} = \log\left(\frac{\theta}{\theta-1}\right) - \gamma\bar{c}, \tag{A.20}$$

so we also have

$$p_t = (\frac{1}{\theta} - \gamma\phi)a_t - \gamma\sigma_z z_t. \tag{A.21}$$

Since all first-order conditions are satisfied and markets clear, we have an equilibrium. ■

A. 4 Proof of Proposition 4

Proof: Notice that without idiosyncratic noise v_{jt}, the production of each individual firm j will the be same. We can write $a_t = \rho a_{t-1} + \varepsilon_t$ and $z_t = \rho z_{t-1} + \varepsilon_{zt}$. Thus,

$$c_{jt} = E_t[\rho a_{t-1} + \varepsilon_t + \beta(\phi a_t + \sigma_z z_t)]|[\phi\varepsilon_t + \sigma_z\varepsilon_{zt}, \phi a_{t-1} + \sigma_z z_{t-1}], \tag{A.22}$$

or we have

$$\begin{aligned}
c_{jt} &= \frac{(\phi+\beta\phi^2)\sigma_a^2 + \beta\sigma_z^2}{\phi^2\sigma_a^2 + \sigma_z^2}(\phi_a\varepsilon_t + \sigma_z\varepsilon_{zt}) \\
&\quad + E[(\rho+\beta\rho\phi)a_{t-1} + \rho\beta\sigma_z z_{t-1}]|[\phi a_{t-1} + \sigma_z z_{t-1}] \\
&= \frac{(\phi+\beta\phi^2)\sigma_a^2 + \beta\sigma_z^2}{\phi^2\sigma_a^2 + \sigma_z^2}(\phi\varepsilon_t + \sigma_z\varepsilon_{zt}) \\
&\quad + \frac{(\rho+\beta\rho\phi)\phi\sigma_a^2 + \beta\rho\sigma_z^2}{\phi^2\sigma_a^2 + \sigma_z^2}(\phi a_{t-1} + \sigma_z z_{t-1}).
\end{aligned} \tag{A.23}$$

Since aggregate production is $c_t = \phi a_t + \sigma_z z_t$, comparing coefficients yields

$$\frac{(\phi + \beta\phi^2)\sigma_a^2 + \beta\sigma_z^2}{\phi^2\sigma_a^2 + \sigma_z^2} = 1 \tag{A.24}$$

or

$$\phi(1 - (1-\beta)\phi)\sigma_a^2 = (1-\beta)\sigma_z^2. \tag{A.25}$$

Solving the above equation gives

$$\phi = \frac{1}{2(1-\beta)} \pm \sqrt{\frac{1}{4(1-\beta)^2} - \frac{\sigma_z^2}{\sigma_a^2}}. \tag{A.26}$$

To obtain the constant, we first notice that

$$\tilde{c} = (1-\theta\gamma)\bar{c} + \frac{\theta}{2}\Omega_s, \tag{A.27}$$

where Ω_s is conditional variance of $a_t/\theta + (\frac{1}{\theta} - \gamma)\phi a_t$ based on the signal and is given by

$$\begin{aligned}
\Omega_s &= \frac{1}{\theta^2} var(a_t + \beta(\phi a_t + \sigma_z z_t)|\phi a_t + \sigma_z z_t) \\
&= \frac{1}{\theta^2} var(a_t|\phi a_t + \sigma_z z_t) \\
&= \frac{1}{\theta^2}[\sigma_a^2 - \frac{\phi^2\sigma_a^4}{\phi^2\sigma_a^2 + \sigma_z^2}] \qquad \text{(A.28)} \\
&= \frac{1}{\theta^2}\frac{\sigma_z^2}{\phi^2\sigma_a^2 + \sigma_z^2}. \qquad \text{(A.29)}
\end{aligned}$$

Finally, following similar steps in the previous proposition, we obtain

$$\bar{c} = \frac{1}{2\gamma}\frac{1}{\theta^2}\frac{\sigma_z^2}{\phi^2\sigma_a^2 + \sigma_z^2}. \tag{A.30}$$

■

A. 5 Proof of Proposition 5

Proof: Denote $\kappa = \frac{1}{1+\sigma_e^2}$. First, taking the log of equation (53) yields

$$a_t/\theta - \gamma c_{it} = \phi_a^p a_t + \sigma_z^p \kappa(z_t + e_{it}). \tag{A.31}$$

Aggregating across consumers we then obtain

$$a_t/\theta - \gamma c_t = \phi_a^p a_t + \sigma_z^p \kappa z_t. \tag{A.32}$$

Since

$$c_t = \phi a_t + \sigma_z z_t, \tag{A.33}$$

we then have

$$\phi_a^p = \frac{1}{\theta} - \gamma\phi,\ \sigma_z^p = -\frac{\gamma}{\kappa}\sigma_z. \tag{A.34}$$

Hence we obtain

$$c_{it} = \phi a_t + \sigma_z(z_t + e_{it}). \tag{A.35}$$

Taking the log of equation (54) gives

$$\begin{aligned} c_{jt} &= E(\theta p_t + c_t)|(c_t + v_{jt}) \\ &= E[a_t + (1-\gamma\theta)\phi a_t + (1 - \frac{\gamma\theta}{\kappa})\sigma_z z_t]|(\phi a_t + \sigma_z z_t + v_{jt}) \\ &= \frac{\phi(1 + (1-\gamma\theta)\phi)\sigma_a^2 + (1 - \frac{\gamma\theta}{\kappa})\sigma_z^2}{\phi^2\sigma_a^2 + \sigma_z^2 + \sigma_v^2}(\phi a_t + \sigma_z z_t + v_{jt}). \end{aligned} \tag{A.36}$$

Aggregating over j yields

$$\frac{\phi(1 + (1-\gamma\theta)\phi)\sigma_a^2 + (1 - \frac{\gamma\theta}{\kappa})\sigma_z^2}{\phi^2\sigma_a^2 + \sigma_z^2 + \sigma_v^2} = 1. \tag{A.37}$$

To be consistent with the production function of final goods (5), we must have

$$\phi^2\sigma_a^2 + \sigma_z^2 + \sigma_v^2 = \phi(1+\beta\phi)\sigma_a^2 + \left(1 - \frac{1-\beta}{\kappa}\right)\sigma_z^2 \tag{A.38}$$

or

$$\phi(1 - (1-\beta)\phi)\sigma_a^2 = \sigma_v^2 + \sigma_z^2\frac{(1-\beta)}{\kappa}. \tag{A.39}$$

Notice that for $\sigma_v^2 < \frac{1}{4(1-\beta)}\sigma_a^2$, there exists a continuum of sentiment-driven equilibria indexed by $\sigma_z^2 \in \left(0, \frac{\kappa}{4(1-\beta)^2}\sigma_a^2 - \frac{\kappa\sigma_v^2}{1-\beta}\right)$. Given any σ_z^2 we have

$$\phi = \frac{1}{2(1-\beta)} \pm \sqrt{\frac{1}{4(1-\beta)^2} - \frac{\tilde{\mu}}{1-\beta}}, \tag{A.40}$$

where $\tilde{\mu} = \frac{\sigma_v^2 + \sigma_z^2(1-\beta)/\kappa}{\sigma_a^2}$. The individual production c_{jt} is hence equal to

$$c_{jt} = \phi a_t + \sigma_z z_t + v_{jt}. \tag{A.41}$$

We still have several remaining constants to be determined. First, by equation (53), we obtain

$$\hat{c} = \frac{1}{\gamma}\log\frac{\theta}{\theta-1} - \frac{1}{\gamma}\bar{p} - \frac{1}{\gamma}\frac{1}{2}\left(\frac{\gamma}{\kappa}\sigma_z\right)^2(1-\kappa). \tag{A.42}$$

Denote

$$\begin{aligned} \Omega_s &= var[\frac{1}{\theta}a_t + \left(\frac{1}{\theta} - \gamma\right)\phi a_t + \left(\frac{1}{\theta} - \frac{\gamma}{\kappa}\right)\sigma_z z_t|\phi a_t + \sigma_z z_t + v_{jt}] \\ &\equiv \frac{1}{\theta^2}var[(a_t + \beta\phi a_t + \tilde{\beta}\sigma_z z_t)|\phi a_t + \sigma_z z_t + v_{jt}] \\ &= \frac{1}{\theta^2}\left[(1+\beta\phi)^2\sigma_a^2 + \tilde{\beta}^2\sigma_z^2 - (1+\beta\phi)\,\phi\sigma_a^2 - \tilde{\beta}\sigma_z^2\right] \\ &= \frac{(1+\beta\phi)(1-(1-\beta)\phi)\sigma_a^2 - \tilde{\beta}\sigma_z^2(1-\tilde{\beta})}{\theta^2}. \end{aligned} \tag{A.43}$$

Then by equation (54) we obtain

$$\tilde{c} = \theta \bar{p} + \bar{c} + \frac{\theta}{2}\Omega_s + \theta \log(1 - \frac{1}{\theta}). \tag{A.44}$$

Finally, from the aggregate production we obtain

$$\bar{c} = \frac{1}{2}\frac{\theta - 1}{\theta}\sigma_v^2 + \tilde{c}. \tag{A.45}$$

We then solve

$$\bar{p} = \log(\frac{\theta}{\theta - 1}) - \frac{\theta - 1}{2\theta^2}\sigma_v^2 - \frac{1}{2}\Omega_s \tag{A.46}$$

and hence

$$\hat{c} = \frac{1}{\gamma}\left[\frac{\theta - 1}{2\theta^2}\sigma_v^2 + \frac{1}{2}\Omega_s - \frac{1}{2}\left(\frac{\gamma}{\kappa}\sigma_z\right)^2 (1 - \kappa)\right]. \tag{A.47}$$

Finally, the relationship between $\hat{c}$ and $\bar{c}$ is

$$\begin{aligned} \bar{c} &= \hat{c} + \frac{1}{2}\sigma_z^2\sigma_e^2 \\ &= \hat{c} + \frac{1}{2}\sigma_z^2\frac{1 - \kappa}{\kappa} \\ &= \frac{1}{\gamma}[\frac{\theta - 1}{2\theta^2}\sigma_v^2 + \frac{1}{2}\Omega_s] - \frac{\gamma}{2}\left(\frac{1}{\kappa}\sigma_z\right)^2 (1 - \kappa) + \frac{1}{2}\sigma_z^2\frac{1 - \kappa}{\kappa}. \end{aligned} \tag{A.48}$$

When $\kappa \to 1$ (or $\sigma_e^2 \to 0$), the above equation reduces to the case with homogenous sentiments. ■

References

[1] Amador, M. and Weill, P. O. 2010. "Learning from Prices: Public Communication and Welfare." *Journal of Political Economy*, 866-907.

[2] Angeletos, G-M., and La'O, J., 2011. "Decentralization, Communication and the Origin of Fluctuations". MIT working Paper 11-09, Cambridge.

[3] Angeletos, G-M and Werning, I., 2006. "Information Aggregation, Multiplicity, and Volatility." *American Economic Review*, Vol. 96, 1720-1736.

[4] Angeletos, G-M , Hellwig, C. and N. Pavan, 2006. "Signaling in a Global Game: Coordination and Policy Traps." *Journal of Political Economy*, 114, 452-484.

[5] Benhabib, J. and Farmer, R., 1994, Indeterminacy and increasing returns, Journal of Economic Theory 63, 19-41.

[6] Benhabib, Jess, Wang, Pengfei and Yi, Wen, 2012. "Sentiments and Aggregate Demand Fluctuations", NBER working paper w18413.

[7] Cass, D. and K. Shell, 1983. "Do Sunspots Matter?" Journal of Political Economy 91, 193-227.

[8] Gaballo, Gaetano, 2012. "Private Uncertainty and Multiplicity." Banque de France, Monetary Policy Research Division, http://www.mwpweb.eu/1/98/resources/document_400_1.pdf

[9] Hellwig, Christian, 2008. "Monetary Business Cycle Models: Imperfect Information." The New Palgrave Dictionary of Economics (2nd edition), London: Palgrave Macmillan.

[10] Hellwig, Christian, Mukherji, A., and A. Tsyvinski, 2006. "Self-Fulfilling Currency Crises: The Role of Interest Rates." *American Economic Review*, 96(5): 1769-87.

[11] Hellwig, Christian and Veldkamp Laura, 2009. "Knowing What Others Know: Coordination Motives in Information Acquisition," Review of Economic Studies, Wiley Blackwell, vol. 76(1), pages 223-251, 01.

[12] Lucas, R. E., Jr. 1972. "Expectations and the Neutrality of Money," *Journal of Economic Theory* 4, 103-124.

[13] Morris, S. and Shin, H. S., 1998. "Unique Equilibrium in a Model of Self-Fulfilling Currency Attacks". *American Economic Review*, Vol. 88, 587-597.

[14] Townsend, R., "Forecasting the Forecast of Others," Journal of Political Economy 91, 546-588.

[15] Wang, P. & Wen, Y. (2007). "Incomplete information and self-fulfilling prophecies," Working Papers 2007-033, Federal Reserve Bank of St. Louis, Revised, 2009.